Lectures on Point Residues

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To Helena

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Introduction

These notes were written to complement a series of lectures to be delivered at IMCA, Instituto de Matemática y Ciencias Afines, Lima, Perú, in July 2002. Our aim was to present, in a form as elementary as possible, the definition and basic properties of point residues from a geometric point of view. This concept was introduced by Alexander Grothendieck around 1957 and an extensive account of it was given by R. Hartshorne in [Ha]. Since our point of view was to present it in a geometric fashion, we were very much guided by the works [Gr], [G-H] and [A-V-GZ].

Throughout these notes we will sometimes refer, without proof, to results on Differential Topology, Commutative Algebra, Several Complex Variables and Algebraic Topology. In each section we quote basic references on these subjects and we urge the reader, in case he (she) is not familiarized with them, to have this bibliography at hand.

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Chapter 1

A brief view of Cauchy's theory

1.1 Index of a point relative to a path

We start with some very basic definitions. Let $U \subset \mathbb{C}$ be an open set and $f : U \to \mathbb{C}$ be a function. The derivative of f at a point $p \in U$, noted $f'(p)$, is

$$\lim_{z \to p} \frac{f(z) - f(p)}{z - p}$$

provided this limit exists. f is *holomorphic* on U if $f'(p)$ exists for all $p \in U$.

A *domain* in $\mathbb{C}$ is an open connected set $U \subset \mathbb{C}$.

A *path* in $\mathbb{C}^n$ is a continuous mapping $\gamma : J \to \mathbb{C}^n$, where $J = [a, b] \subset \mathbb{R}$ and $a < b$. $\gamma(a)$ and $\gamma(b)$ are called the *initial point* and the *end point* of γ, respectively. γ is said to be *closed* if $\gamma(a) = \gamma(b)$. We will denote by $\underline{\gamma}$ the image of the interval J by γ, that is, $\underline{\gamma} = \gamma(J) \subset \mathbb{C}$. γ is *differentiable* if γ' exists and is continuous throughout J (note that, at the end points of J, we have only one-sided derivatives).

If γ_1 and γ_2 are two paths such that the end point of γ_1 is the initial point of γ_2, we can form the path $\gamma_1 \cdot \gamma_2$, called the *juxtaposition* of γ_1 and γ_2, as follows: let $[a_i, b_i]$ be the interval of definition of γ_i. Choose C^1 diffeomorphisms h_1, h_2, preserving orientations, $h_1 : [0, 1/2] \to [a_1, b_1]$, $h_2 : [1/2, 1] \to [a_2, b_2]$ and define $\gamma_1 \cdot \gamma_2$ by

$$\gamma_1 \cdot \gamma_2(t) = \begin{cases} \gamma_1 \circ h_1(t) & \text{, if } t \in [0, 1/2] \\ \\ \gamma_2 \circ h_2(t) & \text{, if } t \in [1/2, 1]. \end{cases}$$

Clearly $\underline{\gamma_1 \cdot \gamma_2} = \underline{\gamma_1} \cup \underline{\gamma_2}$ and, similarly, we can form the juxtaposition of a finite number of paths.

Finally, γ is a *piecewise differentiable path* if it is the juxtaposition of a finite number of differentiable paths. The *reverse* path γ^- of a path γ is defined by $\gamma^-(t) = \gamma(a+b-t)$. Observe that the initial point and the end point of γ^- are the end point and the initial point of γ, respectively, and that $\underline{\gamma} = \underline{\gamma^-}$.

Consider a differentiable path $\gamma : J \to \mathbb{C}$ and let f be a continuous complex valued function defined on $\underline{\gamma}$. The *integral* of f along γ is defined by:

$$\int_\gamma f = \int_\gamma f(z)\,dz = \int_a^b f(\gamma(t))\,\gamma'(t)\,dt.$$

Remark 1 The following properties hold:
(i) The path integral is independent of the parametrization of γ. This means the following: let $h : [a', b'] \to [a, b]$ be a C^1 diffeomorphism preserving orientation, that is, $h(a') = a$, $h(b') = b$ and let $\lambda = \gamma \circ h$. Then

$$\int_\lambda f = \int_{a'}^{b'} f(\lambda(s))\,\lambda'(s)\,ds =$$
$$\int_{a'}^{b'} f(\gamma \circ h(s))\,\gamma'(h(s))\,h'(s)\,ds = \int_a^b f(\gamma(t))\,\gamma'(t)\,dt =$$
$$\int_\gamma f.$$

(ii) The path integral is "sensitive to the orientation" (exercise):

$$\int_{\gamma^-} f = -\int_\gamma f.$$

(iii) Let $M \geq \sup_{\underline{\gamma}} |f|$, then (exercise)

$$\left|\int_\gamma f\right| \leq M \int_a^b |\gamma'(t)|\,dt$$

where $\int_a^b |\gamma'(t)|\,dt$ is, by definition, the *lenght* of the path γ.

(iv) Let $f : U \to \mathbb{C}$ be a continuous function. Recall that a *primitive* of f is a function $F : U \to \mathbb{C}$ such that $F'(z) = f(z)$ for all $z \in U$. Note that F is necessarily holomorphic. Suppose f admits a primitive in U and let γ be a path in U with initial point z_1 and end point z_2, then (exercise)

$$\int_\gamma f = F(z_2) - F(z_1).$$

In particular, if γ is closed we get $\int_\gamma f = 0$. ♢

More generally, for a piecewise-differentiable path $\gamma = \gamma_1 \cdot \ldots \cdot \gamma_k$ and a continuous f whose domain of definition contains $\underline{\gamma_1 \cdot \ldots \cdot \gamma_k}$, we set

$$\int_{\gamma_1 \cdot \ldots \cdot \gamma_k} f = \int_{\gamma_1} f + \cdots + \int_{\gamma_k} f.$$

From now on, unless explicitly stated, by a path we shall mean a piecewise differentiable path.

We are now in a position to start exploiting the Cauchy kernel $\dfrac{dw}{w-z}$.

Consider a path γ in $\mathbb{C}$. Its image $\underline{\gamma}$ is a compact subset of the plane and therefore is limited. Choose a disc D containing $\underline{\gamma}$. The complement $\mathbb{C} \setminus D$ is connected, not bounded and contained in $\mathbb{C} \setminus \underline{\gamma}$ hence, $\mathbb{C} \setminus D$ is contained in a connected component of $\mathbb{C} \setminus \underline{\gamma}$ and we conclude that $\mathbb{C} \setminus \underline{\gamma}$ has precisely one unbounded component.

Now let γ be a closed path in $\mathbb{C}$ and $z \in \mathbb{C} \setminus \underline{\gamma}$. Define the *index* of the point z with respect to γ by

$$\mathcal{I}_\gamma(z) = \frac{1}{2\pi \boldsymbol{i}} \int_\gamma \frac{dw}{w-z}.$$

We have the following integrality result:

Theorem 1.1.1 *For each $z \in \mathbb{C} \setminus \underline{\gamma}$ the number $\mathcal{I}_\gamma(z)$ is an integer, that is, we have a function $\mathcal{I}_\gamma : \mathbb{C} \setminus \underline{\gamma} \longrightarrow \mathbb{Z}$. Moreover, this function is continuous, hence constant in each connected component of $\mathbb{C} \setminus \underline{\gamma}$ and furthermore, it assumes the value zero in the unbounded component of $\mathbb{C} \setminus \underline{\gamma}$.*

Proof: Let $\gamma : [a, b] \to \mathbb{C}$ and $z \in \mathbb{C} \setminus \underline{\gamma}$. By definition,

$$\mathcal{I}_\gamma(z) = \frac{1}{2\pi \boldsymbol{i}} \int_a^b \frac{\gamma'(t)}{\gamma(t) - z}\, dt.$$

Consider the function $\psi : [a, b] \to \mathbb{C}$ given by

$$\psi(t) = \exp\left(\int_a^t \frac{\gamma'(s)}{\gamma(s) - z}\, ds \right).$$

We have $\psi'(t) = \left(\frac{\gamma'(t)}{\gamma(t) - z} \right) \psi(t)$ except at the finite set of points $\{t_i\}_{i=1,\dots,m}$, where the path γ is not differentiable. Hence,

$$\frac{\psi'(t)}{\psi(t)} = \frac{\gamma'(t)}{\gamma(t) - z} \tag{1}$$

in $[a, b] \setminus \{t_i\}_{i=1,\dots,m}$. Look at the function $\varphi(t) = \frac{\psi(t)}{\gamma(t) - z}$. It is continuous in $[a, b]$ and its derivative at any $t \in [a, b] \setminus \{t_i\}_{i=1,\dots,m}$ is, by (1),

$$\varphi'(t) = \frac{\psi'(t)(\gamma(t) - z) - \psi(t)\gamma'(t)}{(\gamma(t) - z)^2} = 0.$$

It follows that φ is constant in $[a, b]$ and, since $\varphi(a) = \frac{1}{\gamma(a) - z}$, we have

$$\psi(t) = \frac{\gamma(t) - z}{\gamma(a) - z} \qquad \forall\ t \in [a, b].$$

Since $\gamma(a) = \gamma(b)$ we get $\psi(b) = 1$. Therefore,

$$\exp\left(\int_a^b \frac{\gamma'(s)}{\gamma(s) - z}\, ds \right) = 1$$

and we conclude that $\int_a^b \frac{\gamma'(s)}{\gamma(s) - z}\, ds = 2\pi \boldsymbol{i} k$, with k an integer. This shows $\mathcal{I}_\gamma(z) \in \mathbb{Z}$. The continuity of $\mathcal{I}_\gamma$ will follow from the

Lemma 1.1.2 *The function $\mathcal{I}_\gamma$ admits a power series expansion around each $\zeta \in \mathbb{C} \setminus \underline{\gamma}$.*

Proof: Fix $\zeta \in \mathbb{C} \setminus \underline{\gamma}$ and let $D(\zeta; r)$ be an open disc contained in $\mathbb{C} \setminus \underline{\gamma}$ and centered at ζ. Now, for any $t \in [a, b]$ we have $|\gamma(t) - \zeta| \geq r$ and then

$$\left| \frac{z - \zeta}{\gamma(t) - \zeta} \right| \leq \frac{|z - \zeta|}{r} < 1$$

for any $z \in D(\zeta; r)$. Hence, for fixed z, the series

$$\sum_{i=0}^{\infty} \frac{(z-\zeta)^i}{(\gamma(t)-\zeta)^i}$$

converges uniformly on $[a,b]$. Since

$$\frac{1}{\gamma(t)-z} = \frac{1}{\gamma(t)-\zeta+\zeta-z} = \frac{1}{\gamma(t)-\zeta}\,\frac{1}{\left(1-\dfrac{z-\zeta}{\gamma(t)-\zeta}\right)}$$

$$= \frac{1}{\gamma(t)-\zeta}\sum_{i=0}^{\infty} \frac{(z-\zeta)^i}{(\gamma(t)-\zeta)^i} = \sum_{i=0}^{\infty} \frac{(z-\zeta)^i}{(\gamma(t)-\zeta)^{i+1}}$$

we conclude

$$\mathcal{I}_\gamma(z) = \frac{1}{2\pi \boldsymbol{i}} \int_a^b \sum_{i=0}^{\infty} \frac{\gamma'(t)}{(\gamma(t)-\zeta)^{i+1}}(z-\zeta)^i \, dt$$

$$= \frac{1}{2\pi \boldsymbol{i}} \sum_{i=0}^{\infty} \left[\int_a^b \frac{\gamma'(t)}{(\gamma(t)-\zeta)^{i+1}}\, dt\right](z-\zeta)^i$$

because we can interchange summation and integration. The lemma is proved.

□

The lemma shows $\mathcal{I}_\gamma$ is a continuous function. It remains to show that $\mathcal{I}_\gamma(z) = 0$ for z in the unbounded component of $\mathbb{C} \setminus \underline{\gamma}$. To this end choose a point z in this component which satisfies

$$\inf_{t\in[a,b]} |z-\gamma(t)| > \int_a^b |\gamma'(t)|\, dt.$$

It follows from (iii) of Remark 1 that

$$|\mathcal{I}_\gamma(z)| \le \frac{1}{\inf\limits_{t\in[a,b]} |z-\gamma(t)|} \int_a^b |\gamma'(t)|\, dt < 1$$

and since $\mathcal{I}_\gamma$ is integer-valued and continuous it must be identically zero in the unbounded component. This finishes the proof of the theorem.

□

Example 1.1.3 Let γ be a circle centered at a point $z \in \mathbb{C}$, $\gamma(t) = z + re^{2\pi \boldsymbol{i} t}$, $r > 0$, $0 \leq t \leq 1$. Then

$$\mathcal{I}_\gamma(z) = \frac{1}{2\pi \boldsymbol{i}} \int_\gamma \frac{1}{w-z}\, dw = \frac{1}{2\pi \boldsymbol{i}} \int_0^1 \frac{2\pi \boldsymbol{i}\, re^{2\pi \boldsymbol{i} t}}{re^{2\pi \boldsymbol{i} t}}\, dt = \int_0^1 dt = 1.$$

We leave to the reader the task to convince himself that the index, $\mathcal{I}_\gamma(z)$, measures the effective number of turns that the plane vector $\gamma(t)$ describes around the point z, as t varies in the interval of definition of γ.

1.2 Holomorphic functions

In this section we present the structure of Cauchy's theory on holomorphic functions. The first step is the simple result (recall (iv) of Remark 1):

Proposition 1.2.1 *Let $f : U \to \mathbb{C}$ be a continuous function defined in the domain $U \subset \mathbb{C}$. Then the following properties are equivalent:*
(i) f admits a primitive in U.
(ii) $\int_\gamma f = 0$ for any closed path γ in U.
(iii) $\int_\lambda f$ depends only on the initial and end points of any path λ in U.

Proof: Exercise or see [Soares].

□

Next we have the fundamental result

Theorem 1.2.2 (Cauchy-Goursat) *Let $f : U \to \mathbb{C}$ be a holomorphic function defined in the domain $U \subset \mathbb{C}$. Assume T is a closed triangular region entirely contained in U and denote by Δ its boundary. Then*

$$\int_\Delta f = 0.$$

Proof: See [Soares].

□

We exploit this result for a particular type of open sets in the plane. Suppose $U \subset \mathbb{C}$ is open. U is a *starlike domain* if there exists a point

$z_0 \in U$ with the property that, given any point $z \in U$ the line segment $\overline{z_0\, z}$ is entirely contained in U. Any convex open set is an example of such a domain. We then have:

Corollary 1.2.3 *Let $U \subset \mathbb{C}$ be a starlike domain and $f : U \to \mathbb{C}$ a holomorphic function. Then f admits a primitive in U.*

Proof: See [Soares].

□

Using this corollary we imediately have

Corollary 1.2.4 (Cauchy-Goursat revisited) *Let $U \subset \mathbb{C}$ be a starlike domain and $f : U \to \mathbb{C}$ a holomorphic function. If γ is a closed path in U then*

$$\int_\gamma f = 0.$$

Proof: Exercise.

□

Corollary 1.2.4 allow us to prove the

Theorem 1.2.5 (Local Cauchy's integral formula) *Let $U \subset \mathbb{C}$ be a domain and $f : U \to \mathbb{C}$ a holomorphic function. Let $\overline{D}(z_0, r_0) \subset U$ be a closed disc and Γ its boundary, oriented counterclockwise. If z is any point in $D(z_0, r_0)$ then,*

$$f(z) = \frac{1}{2\pi \boldsymbol{i}} \int_\Gamma \frac{f(w)}{w - z} dw.$$

Proof: See [Soares].

□

This fundamental theorem unveils the local nature of holomorphic functions because, by manipulating the integrand we deduce the following facts: (i) holomorphic functions have derivatives of all orders at all points of their domains and

$$f^{(k)}(z) = \frac{1}{2\pi \boldsymbol{i}} \int_\Gamma \frac{f(w)}{(w - z)^{k+1}} dw, \quad k \geq 0.$$

Note that the derivatives of a holomorphic function are also holomorphic.
(ii) holomorphic functions are analytic, that is, if ζ belongs to the domain of f then

$$f(z) = \sum_{i=0}^{\infty} \frac{f^{(i)}(\zeta)}{i!} (z - \zeta)^i$$

and this series has positive radius of convergence.

From (i) we deduce the

Proposition 1.2.6 (Cauchy's estimates) *Let f be holomorphic on the disc $D(\zeta, r)$ and $|f(z)| \leq M$ for all $z \in D(\zeta, r)$. Then*

$$|f^{(k)}(\zeta)| \leq \frac{k!M}{r^k}.$$

Proof: Exercise.

□

This last proposition furnishes the

Theorem 1.2.7 (Liouville's theorem) *Let $f : \mathbb{C} \to \mathbb{C}$ be holomorphic (such a function is called an* entire function*). If $|f|$ is bounded then f is constant.*

Proof: Suppose $|f(z)| \leq M \quad \forall\, z \in \mathbb{C}$. Let $\zeta \in \mathbb{C}$. The Cauchy estimate $|f'(\zeta)| < M/r$ holds for all $r > 0$. Hence $f'(\zeta) = 0 \quad \forall\, \zeta \in \mathbb{C}$ and f is constant.

□

A partial converse to theorem 1.2.2 is the

Theorem 1.2.8 (Morera's theorem) *Let $U \subset \mathbb{C}$ be a domain and $f : U \to \mathbb{C}$ a continuous function. Suppose $\int_{\Delta} f = 0$ for every triangular path $\Delta \subset U$. Then f is holomorphic in U.*

Proof: Let $\zeta \in U$ and choose a disc $D(\zeta, r) \subset U$, $r > 0$. Use the hypothesis to show that f admits a primitive F in $D(\zeta, r)$. Since F is holomorphic and $F' = f$ in $D(\zeta, r)$, we conclude that f is holomorphic.

□

We now introduce some objects of homological nature and then proceed to present the global theorem of Cauchy.

A *chain* σ is a formal sum of a finite number of paths in the plane, $\sigma = \gamma_1 + \cdots + \gamma_k$. If f is a continuous function defined in $\underline{\sigma} = \underline{\gamma_1} \cup \cdots \cup \underline{\gamma_k}$ we define

$$\int_{\sigma} f = \sum_{i=1}^{k} \int_{\gamma_i} f.$$

If $\underline{\sigma}$ is contained in a domain $U \subset \mathbb{C}$, we say that σ is a *chain in U*.

Let $\sigma = \gamma_1 + \cdots + \gamma_k$ be a chain. If each γ_i is replaced by its reverse γ_i^-, then the chain so obtained is denoted by $-\sigma$ and

$$\int_{-\sigma} f = -\int_{\sigma} f.$$

In this way chains can be added and subtracted.

Observe that a chain σ can be expressed in several ways as a sum of paths and, in case $\sigma = \gamma_1 + \cdots + \gamma_k = \alpha_1 + \cdots + \alpha_m$, we have

$$\sum_{i=1}^{k} \int_{\gamma_i} f = \sum_{j=1}^{m} \int_{\alpha_j} f$$

for any f which is continuous and defined in $\underline{\gamma_1} \cup \cdots \cup \underline{\gamma_k} \cup \underline{\alpha_1} \cup \cdots \cup \underline{\alpha_m}$.

If the chain $\sigma = \gamma_1 + \cdots + \gamma_k$ is such that all paths γ_i are closed, then σ is called a *cycle*. Since the representation of a chain as a sum of paths is not unique, a cycle may be represented by a sum of paths that are not closed.

Let $\sigma = \gamma_1 + \cdots + \gamma_k$ be a cycle. If $z \in \mathbb{C} \setminus \underline{\sigma}$ then we set

$$\mathcal{I}_\sigma(z) = \sum_{i=1}^{k} \mathcal{I}_{\gamma_i}(z).$$

Note that $\mathcal{I}_{-\sigma}(z) = -\mathcal{I}_\sigma(z)$. With this at hand we have the main result of the theory:

Theorem 1.2.9 (Cauchy's theorem) *Let $U \subset \mathbb{C}$ be a domain and $f : U \to \mathbb{C}$ a holomorphic map. Suppose σ is a cycle in U satisfying*

$$\mathcal{I}_\sigma(\zeta) = 0 \qquad \forall\, \zeta \notin U.$$

Then,

$$\mathcal{I}_\sigma(z)\, f(z) = \frac{1}{2\pi \boldsymbol{i}} \int_\sigma \frac{f(w)}{w - z}\, dw \quad \text{for } z \in U \setminus \underline{\sigma}, \tag{I}$$

$$\int_\sigma f(z)\, dz = 0. \tag{II}$$

Moreover, if σ_0 and σ_1 are cycles in U such that $\mathcal{I}_{\sigma_0}(\zeta) = \mathcal{I}_{\sigma_1}(\zeta)$ for all $\zeta \notin U$ then,

$$\int_{\sigma_0} f(z)\, dz = \int_{\sigma_1} f(z)\, dz. \tag{III}$$

Proof: The proof of this global version of Cauchy's theorem is due to J.Dixon [Dixon]. Consider the function $g : U \times U \to \mathbb{C}$ defined by

$$g(z,w) = \begin{cases} \dfrac{f(w)-f(z)}{w-z} & \text{, if } w \neq z \\ f'(z) & \text{, if } w = z. \end{cases}$$

Lemma 1.2.10 *g is continuous.*

Proof: It's immediate that g is continuous in $U \times U \setminus \{(\zeta,\zeta) : \zeta \in U\}$. Let us show its continuity at a point (ζ,ζ). Given $\epsilon > 0$ choose $\delta > 0$ such that $|\ell - \zeta| < \delta \Rightarrow |f'(\ell) - f'(\zeta)| < \epsilon$. Let z and w belong to the open set $D(\zeta,\delta) \cap U$. If $w = z$ we get $|g(z,z) - g(\zeta,\zeta)| < \epsilon$. If $z \neq w$ consider the line segment joining them, $\ell(t) = (1-t)z + tw$, $0 \leq t \leq 1$. We have

$$f(w) - f(z) = f(\ell(1)) - f(\ell(0)) =$$

$$\int_0^1 f'(\ell(t))\, \ell'(t)\, dt =$$

$$\int_0^1 f'(\ell(t))\, (w-z)\, dt$$

so that

$$g(z,w) = \int_0^1 f'(\ell(t))\, dt.$$

Since $g(\zeta,\zeta) = f'(\zeta) = \int_0^1 f'(\zeta)\, dt$ we obtain

$$g(z,w) - g(\zeta,\zeta) = \int_0^1 \left[f'(\ell(t)) - f'(\zeta)\right]\, dt.$$

By (iii) of Remark 1

$$|g(z,w) - g(\zeta,\zeta)| \leq \sup_{t \in [0,1]} |f'(\ell(t)) - f'(\zeta)| < \epsilon$$

and the lemma is proved.

□

Next we consider, for fixed $w \in U$, the function $g_w : U \to \mathbb{C}$ defined by $g_w(z) = g(z,w)$. This function is clearly holomorphic in $U \setminus \{w\}$. We claim the

Lemma 1.2.11 *g_w is holomorphic in U.*

Proof: By the previous lemma, g_w is continuous at w and $g_w(w) = f'(w)$. Put $\phi(z) = (z-w)g_w(z)$. ϕ is continuous in U, holomorphic in $U \setminus \{w\}$ and $\phi(w) = 0$. Now,

$$\lim_{z\to w} \frac{\phi(z)-\phi(w)}{z-w} = \lim_{z\to w} \frac{(z-w)g_w(z)}{z-w} = \lim_{z\to w} g_w(z) = f'(w),$$

so ϕ is differentiable at w and therefore holomorphic in U. Around w it has a convergent power series expansion

$$\phi(z) = f'(w)\,(z-w) + \sum_{j=2}^{\infty} a_j (z-w)^j =$$

$$(z-w)[f'(w) + \sum_{j=2}^{\infty} a_j (z-w)^{j-1}].$$

We conclude $g_w(z) = f'(w) + \sum\limits_{j=2}^{\infty} a_j(z-w)^{j-1}$ and the lemma is proved. w is called a removable (or fake) singularity (which will be considered later).

□

Returning to the proof of the theorem, we let $\varphi : U \to \mathbb{C}$ be defined by

$$\varphi(z) = \frac{1}{2\pi \boldsymbol{i}} \int_{\sigma} g(z,w)\,dw.$$

We claim that φ is continuous. In fact, let $(z_n) \to z$ be a sequence in U, convergent to $z \in U$. The set $(\{z_n\}_{n=1}^{\infty} \cup \{z\}) \times \underline{\sigma}$ is a compact subset of $U \times U$. Hence, g is uniformly continuous in this set and therefore $g_w(z_n) \to g_w(z)$ uniformly on w. This shows the continuity of φ.

Let us prove that φ is holomorphic in U. Consider a closed triangular region $T \subset U$ with boundary Δ. Then,

$$\int_{\Delta} \varphi(z)\,dz = \frac{1}{2\pi \boldsymbol{i}} \int_{\sigma} \left[\int_{\Delta} g(z,w)\,dz \right] dw =$$

$$\frac{1}{2\pi \boldsymbol{i}} \int_{\sigma} \left[\int_{\Delta} g_w(z)\,dz \right] dw = 0$$

because $\int\limits_{\Delta} g_w(z)\,dz = 0$ since g_w is holomorphic. Invoking Morera's theorem we conclude that φ is holomorphic.

Set $V = \{z \in \mathbb{C} : \mathcal{I}_\sigma(z) = 0\}$. By hypothesis, $\mathbb{C} \setminus U \subset V$ and by theorem 1.1.1, the unbounded component of $\mathbb{C} \setminus \underline{\sigma}$ is also contained in V. Define the holomorphic function $\psi : V \to \mathbb{C}$ by

$$\psi(z) = \frac{1}{2\pi \boldsymbol{i}} \int_\sigma \frac{f(w)}{w - z}\, dw.$$

If $z \in U \cap V$ then

$$\varphi(z) = \frac{1}{2\pi \boldsymbol{i}} \int_\sigma \frac{f(w) - f(z)}{w - z}\, dw =$$

$$\frac{1}{2\pi \boldsymbol{i}} \int_\sigma \frac{f(w)}{w - z}\, dw - \frac{f(z)}{2\pi \boldsymbol{i}} \int_\sigma \frac{1}{w - z}\, dw =$$

$$\frac{1}{2\pi \boldsymbol{i}} \int_\sigma \frac{f(w)}{w - z}\, dw - f(z)\, \mathcal{I}_\sigma(z) =$$

$$\frac{1}{2\pi \boldsymbol{i}} \int_\sigma \frac{f(w)}{w - z}\, dw = \psi(z).$$

Hence, there exist a holomorphic function $\Psi : U \cup V \to \mathbb{C}$ such that $\Psi_{|U} = \varphi$ and $\Psi_{|V} = \psi$. Since V contains the complement of U we have that Ψ is an entire function. Now,

$$\lim_{|z| \to \infty} \Psi(z) = \lim_{|z| \to \infty} \psi(z) = 0$$

and we conclude that $|\Psi|$ is bounded. By Liouville's theorem $\Psi(z) = 0$ for all $z \in \mathbb{C}$. It follows that $\varphi(z) = 0$ for all $z \in U$. Hence, for $z \in U \setminus \underline{\sigma}$,

$$0 = \varphi(z) = \frac{1}{2\pi \boldsymbol{i}} \int_\sigma \frac{f(w) - f(z)}{w - z}\, dw =$$

$$\frac{1}{2\pi \boldsymbol{i}} \int_\sigma \frac{f(w)}{w - z}\, dw - \frac{f(z)}{2\pi \boldsymbol{i}} \int_\sigma \frac{1}{w - z}\, dw =$$

$$\frac{1}{2\pi \boldsymbol{i}} \int_\sigma \frac{f(w)}{w - z}\, dw - f(z)\, \mathcal{I}_\sigma(z)$$

and (I) is proved.

To prove (II) we use (I) as follows: choose $\zeta \in U \setminus \underline{\sigma}$ and let $F(z) = (z-\zeta)\, f(z)$. Since $F(\zeta) = 0$ we get

$$\int_\sigma f(z)\, dz = \int_\sigma \frac{F(z)}{z-\zeta}\, dz = 2\pi \boldsymbol{i}\, \mathcal{I}_\sigma(\zeta)\, F(\zeta) = 0.$$

Finally, let σ_0 and σ_1 be cycles in U such that $\mathcal{I}_{\sigma_0}(\zeta) = \mathcal{I}_{\sigma_1}(\zeta)$ for all $\zeta \notin U$. Consider the cycle $\sigma_0 - \sigma_1$. Then $\mathcal{I}_{\sigma_0-\sigma_1}(\zeta) = \mathcal{I}_{\sigma_0}(\zeta) - \mathcal{I}_{\sigma_1}(\zeta) = 0$. By (II),

$$0 = \int_{\sigma_0-\sigma_1} f(z)\, dz = \int_{\sigma_0} f(z)\, dz - \int_{\sigma_1} f(z)\, dz.$$

This proves (III) and finishes the proof of the theorem.

□

1.3 Meromorphic functions

The *annulus* $A(\zeta; R_1, R_2)$ with center $\zeta \in \mathbb{C}$ and radii R_1, R_2 where $0 \le R_1 < R_2 \le \infty$, is the open set

$$A(\zeta; R_1, R_2) = \{z \in \mathbb{C} : R_1 < |z-\zeta| < R_2\}.$$

Holomorphic functions defined in an annulus have a representation by power series as follows:

Theorem 1.3.1 (Laurent's expansion) *Consider a holomorphic function $f : A(\zeta; R_1, R_2) \to \mathbb{C}$. Then*

$$f(z) = \sum_{m=1}^{\infty} b_m \frac{1}{(z-\zeta)^m} + \sum_{n=0}^{\infty} a_n\, (z-\zeta)^n,$$

where the series $\sum_{m=1}^{\infty} b_m \frac{1}{(z-\zeta)^m}$ converges for $|z-\zeta| > R_1$ and the series $\sum_{n=0}^{\infty} a_n\, (z-\zeta)^n$ converges for $|z-\zeta| < R_2$. Moreover, this expansion is unique and the coefficients b_m and a_n are given by:

$$b_m = \frac{1}{2\pi \boldsymbol{i}} \int_\gamma f(z)\, (z-\zeta)^{m-1}\, dz, \quad m \ge 1$$

$$a_n = \frac{1}{2\pi \boldsymbol{i}} \int_\gamma \frac{f(z)}{(z-\zeta)^{n+1}}\, dz, \quad n \ge 0.$$

Proof: See [Soares].

□

Definition 1.3.2 *Let $f : U \to \mathbb{C}$ be a holomorphic function defined in the domain U. A point $\zeta \in \mathbb{C} \setminus U$ is an* isolated singularity *of f if there exists a positive R such that the annulus $A(\zeta; 0, R) \subset U$.*

Invoke the Laurent expansion of f in $A(\zeta; 0, R)$:

$$f(z) = \sum_{m=1}^{\infty} \frac{b_m}{(z-\zeta)^m} + \sum_{n=0}^{\infty} a_n (z-\zeta)^n.$$

We have the following mutually exclusive possibilities:
(1) $b_m = 0$ for all $m \geq 1$. In this case we say that ζ is a *removable singularity* of f. By setting $f(\zeta) = a_0$ we have that f admits a holomorpic extension to the disc $D(\zeta, R)$.
(2) There exist a $k \geq 1$ such that $b_k \neq 0$ and $b_m = 0$ for all $m > k$. In this case we say that ζ is a *pole of order k* of f, or simply a *pole* of f. Observe that for $z \in A(\zeta; 0, R)$ we have:

$$f(z) = \frac{b_k}{(z-\zeta)^k} + \cdots + \frac{b_1}{(z-\zeta)} + \sum_{n=0}^{\infty} a_n (z-\zeta)^n.$$

The rational function

$$Q(z) = \frac{b_k}{(z-\zeta)^k} + \cdots + \frac{b_1}{(z-\zeta)}$$

is called the *principal part* of f at the pole ζ. It follows from (1) that the function $g(z) = (z-\zeta)^k f(z)$ has a removable singularity at ζ and that $g(\zeta) = b_k \neq 0$. Hence,

$$\lim_{z \to \zeta} f(z) = \lim_{z \to \zeta} \frac{g(z)}{(z-\zeta)^k} = \infty.$$

(3) $b_m \neq 0$ for infinite values of m. In this case we say that ζ is an *essential singularity* of f.

Another caracterization of isolated singularities is the following:

Proposition 1.3.3 *Let ζ be an isolated singularity of f. Then,*

(1) *ζ is a removable singularity if, and only if, $|f|$ is bounded in some annulus $A(\zeta; 0, R) \subset U$.*

(2) *ζ is a pole of f if, and only if, $\lim_{z \to \zeta} f(z) = \infty$.*

(3) *ζ is an essential singularity of f if, and only if, for every $R > 0$ such that $A(\zeta; 0, R) \subset U$, $f(A(\zeta; 0, R))$ is dense in $\mathbb{C}$.*

Proof: See [Soares]

□

Definition 1.3.4 *A function f is said to be* meromorphic *on an open set U if there is a subset* P *of U such that:*
(i) P *is discrete.*
(ii) f is holomorphic in $U \setminus$ P.
(iii) f has a pole at each point of P.

Note that the possibility $\mathrm{P} = \emptyset$ is allowed and so holomorphic functions are also meromorphic.

Definition 1.3.5 *Let f be a meromorphic function on the open set U and $\zeta \in \mathrm{P}$. Invoke the Laurent expansion of f in an annulus $A(\zeta; 0, R) \subset U$,*

$$f(z) = \frac{b_k}{(z-\zeta)^k} + \cdots + \frac{b_1}{(z-\zeta)} + \sum_{n=0}^{\infty} a_n (z-\zeta)^n.$$

The Cauchy residue *of f at ζ, noted $Res(f, \zeta)$, is the coefficient b_1.*

Let us point out that $Res(f, \zeta)$ is *not* invariant by changes of coordinates. For instance, if $f(z) = 1/z$, then $Res(f, 0) = 1$. Let $h(w) = w/(w-1)$. Then $f \circ h(w) = 1 - 1/w$ and $Res(f \circ h, 0) = -1$.

Consider the principal part $Q(z) = \dfrac{b_k}{(z-\zeta)^k} + \cdots + \dfrac{b_1}{(z-\zeta)}$ of f at ζ and let σ be a cycle in $\mathbb{C}$ such that $\zeta \notin \underline{\sigma}$. Applying (I) of Cauchy's theorem to the constant (entire) functions $f_j(z) \equiv b_j$, $1 \le j \le k$, we get

$$\frac{1}{2\pi \boldsymbol{i}} \int_\sigma \frac{b_j}{(z-\zeta)^j}\, dz = \begin{cases} \mathcal{I}_\sigma(\zeta)\ f_j^{(j-1)}(\zeta) = 0, & \text{for } 2 \le j \le k \\ \mathcal{I}_\sigma(\zeta)\ f_1(\zeta) = \mathcal{I}_\sigma(\zeta)\, b_1, & \text{for } j = 1. \end{cases}$$

Therefore,

$$\frac{1}{2\pi \boldsymbol{i}} \int_\sigma Q(z)\, dz = \mathcal{I}_\sigma(\zeta)\ Res(Q, \zeta). \qquad (\star)$$

We have the

Theorem 1.3.6 (Cauchy's residue theorem) *Let f be a meromorphic function on the domain U and* P *be its set of poles. If σ is a cycle in $U \setminus$* P *such that $\mathcal{I}_\sigma(w) = 0$ for all $w \notin U$ then,*

$$\frac{1}{2\pi \boldsymbol{i}} \int_\sigma f(z)\, dz = \sum_{\zeta \in \mathrm{P}} \mathcal{I}_\sigma(\zeta)\ Res(f, \zeta).$$

Proof: By theorem 1.1.1 we know that $\mathcal{I}_\sigma$ is constant in each connected component $\mathcal{C}$ of $\mathbb{C} \setminus \underline{\sigma}$. If $\mathcal{C}$ is unbounded, or if $\mathcal{C} \cap (\mathbb{C} \setminus U) \neq \emptyset$ then, by theorem 1.1.1, or by the hypothesis, $\mathcal{I}_\sigma(w) = 0, \forall\, w \in \mathcal{C}$. Since the set P is discrete, we conclude that the set $\mathrm{P}^* = \{z \in \mathrm{P} : \mathcal{I}_\sigma(z) \neq 0\}$ is finite (it could as well be empty). Hence the summation above is actually over a finite number of points $\zeta \in \mathrm{P}$.

Let $\mathrm{P}^* = \{\zeta_1, \ldots, \zeta_m\}$ and $Q_1, \ldots, Q_m$ be the principal parts of f at $\zeta_1, \ldots, \zeta_m$, respectively. Set $g = f - (Q_1 + \cdots + Q_m)$. The points $\zeta_1, \ldots, \zeta_m$ are all removable singularities of g and therefore g is holomorphic on the open set $U \setminus (\mathrm{P} \setminus \mathrm{P}^*)$. By hypothesis, $\mathcal{I}_\sigma(w) = 0$ for all $w \notin U \setminus (\mathrm{P} \setminus \mathrm{P}^*)$, so that we can apply (II) of Cauchy's theorem 1.2.9 to the function g and obtain

$$0 = \int_\sigma g(z)\, dz = \int_\sigma f(z)\, dz - \int_\sigma (Q_1(z) + \cdots + Q_m(z))\, dz.$$

But this gives, using ($\star$),

$$\frac{1}{2\pi \boldsymbol{i}} \int_\sigma f(z)\, dz = \sum_{i=1}^{m} \frac{1}{2\pi \boldsymbol{i}} \int_\sigma Q_i(z)\, dz = \sum_{i=1}^{m} \mathcal{I}_\sigma(\zeta_i)\ Res(Q_i, \zeta_i).$$

Since $Res(Q_i, \zeta_i) = Res(f, \zeta_i)$ we get

$$\frac{1}{2\pi \boldsymbol{i}} \int_\sigma f(z)\, dz = \sum_{\zeta \in \mathrm{P}} \mathcal{I}_\sigma(\zeta)\ Res(f, \zeta).$$

□

The next two results are very useful consequences of the residue theorem. Before stating them let's recall the multiplicity of a zero of a holomorphic function of one variable. Suppose $f : U \to \mathbb{C}$ is a holomorphic function defined in a neighborhood $U \subset \mathbb{C}$ of a point ζ and such that $f(\zeta) = 0$. Expanding f in power series around ζ we get

$$f(z) = \sum_{k=\mu}^{\infty} a_k\, (z - \zeta)^k = (z - \zeta)^\mu\, g(z)$$

where $a_\mu = g(\zeta) \neq 0$, g is holomorphic and $g(z) = \sum_{j=0}^{\infty} a_{\mu+j}\, (z - \zeta)^j$. The number $\mu = \mu(f, \zeta)$ is the *multiplicity of the zero* ζ of f.

Remark 2 Let f be a meromorphic function in U and $L'f$ be the function $L'f(z) = \dfrac{f'(z)}{f(z)}$. We claim that the poles of $L'f$ are the zeros and poles of f. To see this let $\zeta \in U$. If $f(\zeta) \neq 0$ then $L'f$ is holomorphic in a neighborhood of ζ and $Res(L'f, \zeta) = 0$. If ζ is a zero of multiplicity μ of f, then $f(z) = (z-\zeta)^{\mu}\, g(z)$ in a neighborhood of ζ, $g(\zeta) \neq 0$ and

$$L'f(z) = \frac{f'(z)}{f(z)} = \frac{\mu}{z-\zeta} + \frac{g'(z)}{g(z)},$$

so that $L'f$ has a pole of order 1 at ζ with $Res(L'f, \zeta) = \mu$. Now if ζ is a pole of order m of f then, in an annulus $A(\zeta; 0, \epsilon) \subset U$ we have $f(z) = (z-\zeta)^{-m}\, h(z)$, where h is holomorphic in this annulus with $h(\zeta) \neq 0$. Hence

$$L'f(z) = \frac{f'(z)}{f(z)} = \frac{-m}{z-\zeta} + \frac{h'(z)}{h(z)}$$

and $L'f$ has a pole of order 1 at ζ with $Res(L'f, \zeta) = -m$. Summarizing

$$Res(L'f, \zeta) = 0 \Longleftrightarrow f \text{ is holomorphic at } \zeta \text{ and } f(\zeta) \neq 0.$$

$$Res(L'f, \zeta) = \mu > 0 \Longleftrightarrow \zeta \text{ is a zero of multiplicity } \mu \text{ of } f.$$

$$Res(L'f, \zeta) = -m < 0 \Longleftrightarrow \zeta \text{ is a pole of order } m \text{ of } f.$$

$\diamondsuit$

Let's now make the following convention. If f is a meromorphic function on U, denote by Z and P its sets of zeros and poles, respectively. The number of zeros and poles of f in $V \subset U$, $\mathrm{Z}(f;V)$, $\mathrm{P}(f;V)$, counted with multiplicities is, by definition:

$$\mathrm{Z}(f;V) = \sum_{\zeta \in V \cap \mathrm{Z}} \mu(f, \zeta)$$

$$\mathrm{P}(f;V) = \sum_{\zeta \in V \cap \mathrm{P}} m(f, \zeta)$$

where $m(f, \zeta)$ is the order of the pole ζ of f. With this at hand we have the

Theorem 1.3.7 (Argument Principle) *Let $U \subset \mathbb{C}$ be a domain and γ a closed path in U such that $\mathcal{I}_\gamma(\zeta) = 0$ for all $\zeta \notin U$. Assume $\mathcal{I}_\gamma(\zeta) = 0$ or 1 for all $\zeta \in U \setminus \underline{\gamma}$ and let $U^* = \{\zeta \in \mathbb{C} : \mathcal{I}_\gamma(\zeta) = 1\}$. Suppose f is a meromorphic function on U and that f has neither zeros nor poles on $\underline{\gamma}$. Then*

$$\mathrm{Z}(f;U^*) - \mathrm{P}(f;U^*) = \frac{1}{2\pi \boldsymbol{i}} \int_\gamma \frac{f'(z)}{f(z)}\, dz = \mathcal{I}_\Gamma(0)$$

where $\Gamma = f \circ \gamma$.

Proof: We start by proving the last equality.

$$\mathcal{I}_\Gamma(0) = \frac{1}{2\pi\boldsymbol{i}} \int\limits_\Gamma \frac{dz}{z} = \frac{1}{2\pi\boldsymbol{i}} \int\limits_0^1 \frac{\Gamma'(t)}{\Gamma(t)}\, dt =$$

$$\frac{1}{2\pi\boldsymbol{i}} \int\limits_0^1 \frac{(f \circ \gamma)'(t)}{(f \circ \gamma)\,(t)}\, \gamma'(t)\, dt = \frac{1}{2\pi\boldsymbol{i}} \int\limits_\gamma \frac{f'(z)}{f(z)}\, dz.$$

To prove the first equality let us look at the function $L'f$. It is meromorphic in U and by the hypotheses and remark 2, it has no poles on $\underline{\gamma}$. Let $P(L'f)$ denote its set of poles. Invoking the Residue theorem 1.3.6 and remark 2 again we get

$$\frac{1}{2\pi\boldsymbol{i}} \int\limits_\gamma \frac{f'(z)}{f(z)}\, dz = \frac{1}{2\pi\boldsymbol{i}} \int\limits_\gamma L'f(z)\, dz =$$

$$\sum_{\zeta \in P(L'f)} Res(L'f, \zeta) = \mathrm{Z}(f; U^*) - \mathrm{P}(f; U^*).$$

□

Theorem 1.3.8 (Rouché's principle) *Let $U \subset \mathbb{C}$ be a domain and γ a closed path in U such that $\mathcal{I}_\gamma(\zeta) = 0$ for all $\zeta \notin U$. Assume $\mathcal{I}_\gamma(\zeta) = 0$ or 1 for all $\zeta \in U \setminus \underline{\gamma}$ and let $U^* = \{\zeta \in \mathbb{C} : \mathcal{I}_\gamma(\zeta) = 1\}$. Let f be holomorphic on U, with no zeros on $\underline{\gamma}$. If g is holomorphic on U and satisfies*

$$|f(z) - g(z)| < |f(z)| \quad \forall\, z \in \underline{\gamma}$$

then

$$\mathrm{Z}(g; U^*) = \mathrm{Z}(f; U^*).$$

Proof: The inequality above implies that g has no zeros on $\underline{\gamma}$. Hence the previous theorem 1.3.7 holds for g and we get $\mathrm{Z}(g; U^*) = \mathcal{I}_\Lambda(0)$ where Λ is the closed path $\Lambda = g \circ \gamma$. On the other hand we also have by theorem 1.3.7, $\mathcal{I}_\Gamma(0) = \mathrm{Z}(f; U^*)$ with $\Gamma = f \circ \gamma$. It remains to show that $\mathcal{I}_\Lambda(0) = \mathcal{I}_\Gamma(0)$. We have by hypothesis

$$|\Gamma(t) - \Lambda(t)| < |\Gamma(t)| \quad \forall\, t \in [0, 1].$$

Note that this gives $\Gamma(t) \neq 0$ and $\Lambda(t) \neq 0$ for all $t \in [0, 1]$. Set $\xi(t) = \dfrac{\Lambda(t)}{\Gamma(t)}$. Then $|1 - \xi(t)| < 1$ which gives $\underline{\xi} \subset D(1, 1)$, so that 0 lies in the unbounded component of $\mathbb{C} \setminus \underline{\xi}$ and we conclude $\mathcal{I}_\xi(0) = 0$. Since

$$\frac{\xi'(t)}{\xi(t)} = \frac{\Lambda'(t)}{\Lambda(t)} - \frac{\Gamma'(t)}{\Gamma(t)}$$

we get

$$0 = \mathcal{I}_\xi(0) = \frac{1}{2\pi \boldsymbol{i}} \int_0^1 \frac{\xi'(t)}{\xi(t)}\, dt =$$
$$\frac{1}{2\pi \boldsymbol{i}} \int_0^1 \frac{\Lambda'(t)}{\Lambda(t)}\, dt - \frac{1}{2\pi \boldsymbol{i}} \int_0^1 \frac{\Gamma'(t)}{\Gamma(t)}\, dt = \mathcal{I}_\Lambda(0) - \mathcal{I}_\Gamma(0)$$

and the theorem is proved.

□

Rouché's principle can be used to prove the Fundamental Theorem of Algebra (exercise).

Chapter 2

The Index and the Multiplicity

In this chapter we introduce the topological index of Poincaré Hopf and the algebraic multiplicity, which became known as Milnor number. These concepts are fundamental and extremely useful and we shall exploit them when we talk about residues.

2.1 The Poincaré Hopf index

2.1.1 The Brouwer degree

The basic references for this section are the books by E. Lima [Lima 1] and J. Milnor [Milnor].

We will be mainly concerned with problems of local nature, so it suffices for our purposes to consider only manifolds which are embedded in euclidean spaces. The first tool we need is the

Theorem 2.1.1 (Sard's theorem) *Let $U \subset \mathbb{R}^m$ be an open set and $f : U \to \mathbb{R}^n$ be a smooth map. Denote by Σ the set of critical points of f, that is, $\Sigma = \{p \in U : \text{rank} f'(p) < n\}$. Then the image $f(\Sigma) \subset \mathbb{R}^n$ has Lebesgue measure zero.*

Proof: See [Milnor]. □

Without difficulty we deduce from this the (exercise)

Corollary 2.1.2 (Brown's theorem) *Let X and Y be smooth ma- nifolds and $f : X \to Y$ be a smooth map. Then the set of regular values of f, $Y \setminus f(\Sigma)$, is everywhere dense in Y.*

□

In order to fix notations let us recall the concept of orientable manifolds.

An *orientation* for a finite dimensional real vector space is an equivalence class of *ordered* bases, the relation been defined by: the ordered basis $\mathcal{B}$ determines the *same orientation* as the ordered basis $\mathcal{B}'$ if the isomorphism changing $\mathcal{B}$ into $\mathcal{B}'$ has *positive* determinant. $\mathcal{B}$ and $\mathcal{B}'$ determine *opposite orientations* if the isomorphism changing $\mathcal{B}$ into $\mathcal{B}'$ has *negative* determinant. It follows that each non-trivial vector space has precisely two orientations. In case the vector space is zero dimensional we define orientations by the symbols $+1$ and -1. For $\mathbb{R}^N$ the standard orientation is the one corresponding to the ordered canonical basis

$$\mathcal{B} = \{e_1, \ldots, e_N\} \text{ where } e_i = \underbrace{(0, \ldots, 0, 1, 0, \ldots, 0)}_{\text{i−th position}}.$$

Now let X be a connected manifold, $\dim X = n \geq 1$ (with boundary or not). X is *orientable* if we can choose an orientation for each tangent space T_pX in such a way that the following holds: given $p \in X$, there exist a neighborhood $p \in U \subset X$ and a diffeomorphism $\varphi : U \to V$, $\varphi(U) = V$, where $V \subset \mathbb{R}^n$ is open ($V \subset \{x \in \mathbb{R}^n : x_n \geq 0\}$ is open, in case X has boundary) which *preserves orientation*, that is, $\varphi'(p)$ carries the chosen orientation for T_pX into the standard orientation for $\mathbb{R}^n$.

An orientation for a connected manifold can also be given in terms of differential forms. More precisely, X is orientable if there is a *nowhere zero continuous n-form* ω on X. Two such forms, say ω_1, ω_2, are said to define the same orientation if $\omega_2 = \rho\,\omega_1$ with ρ a positive continuous function on X (see [Lima1] for details).

If X has a boundary and is orientable, an orientation for X induces an orientation for ∂X as follows: given $p \in \partial X$, choose a positive basis $\mathcal{B} = \{v_1, v_2, \ldots, v_n\}$ for T_pX with the following property: $\{v_2, \ldots, v_n\}$ generate $T_p\partial X$ and v_1 is an outward vector. Then $\mathcal{B}' = \{v_2, \ldots, v_n\}$ determines the positive orientation for $T_p\partial X$. If $\dim X = 1$, to each boundary point p is assigned the orientation -1 or $+1$ depending on whether a positively oriented vector at p points inward or outward.

Before introducing the concept of degree recall that a continous map $f : X \to Y$ between two manifolds is *proper* provided the inverse image $f^{-1}(K) \subset X$ is compact whenever $K \subset Y$ is compact.

Let X and Y be oriented manifolds, both of dimension n, Y connected and $f : X \to Y$ a smooth, proper map. Pick a regular point $p \in X$ of f. Then the tangent map $f'(p) : T_pX \to T_{f(p)}Y$ is a linear isomorphism between oriented vector spaces. Define the *sign* of $f'(p)$ by

$$\operatorname{sgn} f'(p) = \begin{cases} +1 & \text{, if } f'(p) \text{ preserves orientation} \\ -1 & \text{, if } f'(p) \text{ reverses orientation} \end{cases}$$

Now, if $q \in Y$ is a regular value of f set $\deg(f,q) = \sum_{p \in f^{-1}(q)} \operatorname{sgn} f'(p)$. The remarkable fact about $\deg(f,q)$ is

Theorem 2.1.3 *The integer* $\deg(f,q)$ *does not depend on the regular value* $q \in Y$*.*

Proof: See [Lima 1].

□

Hence we have the

Definition 2.1.4 *The degree of the map* f *is* $\deg f = \deg(f,q)$ *where* $q \in Y$ *is a regular value of* f*.*

Recall that a smooth *homotopy* between two maps $f, g : X \to Y$ is a smooth map $F : X \times [0,1] \to Y$ such that $F(0,.) \equiv f$ and $F(1,.) \equiv g$. The degree is invariant under homotopy, more precisely:

Theorem 2.1.5 *If* f *is smoothly homotopic to* g*, then* $\deg f = \deg g$*.*

Proof: See [Lima 1].

□

We shall need the following useful result: suppose that X^{n+1} is a compact oriented manifold with boundary ∂X and Y^n is connected and oriented. Let $f : \partial X \to Y$ be a smooth map (note that f is necessarily proper).

Proposition 2.1.6 *If* f *admits a smooth extension* $F : X \to Y$*, then* $\deg f = 0$*.*

Proof: See [Milnor].

□

2.1.2 Holomorphic maps

In this section we'll be interested in maps $f : \mathbb{C}^n \to \mathbb{C}^m$ and in map germs.

Let $U \subset \mathbb{C}^n$ be a domain (open and connected set). Recall that if $n = 1$, then a function $f : U \to \mathbb{C}$ is holomorphic provided $f'(z)$ exists for every

$z \in U$. If we identify $\mathbb{C} \approx \mathbb{R}^2$, $z = x + \boldsymbol{i}y$, $\bar{z} = x - \boldsymbol{i}y$, $f(z) = u + \boldsymbol{i}v$ and introduce the derivations

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - \boldsymbol{i}\frac{\partial}{\partial y}\right), \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + \boldsymbol{i}\frac{\partial}{\partial y}\right),$$

then f holomorphic is equivalent to:

$$f' = \frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - \boldsymbol{i}\frac{\partial f}{\partial y}\right)$$

and is also equivalent to: f is continuous and its partial derivatives with respect to x and y exist and satisfy the Cauchy-Riemann differential equation

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + \boldsymbol{i}\frac{\partial f}{\partial y}\right) = 0.$$

This last equivalence is a difficult theorem of Loomann and Menchof (see [Na]). It is easy to show this equivalence in case the partial derivatives of f are continuous (exercise).

Consider now a function $f : U \to \mathbb{C}$, $U \subset \mathbb{C}^n$.

Definition 2.1.7 *f is called* partially holomorphic *if, for each point $(p_1, \ldots, p_n) \in U$ and each $j = 1, \ldots, n$, the function of one variable defined by*

$$z_j \longmapsto f(p_1, \ldots, p_{j-1}, z_j, p_{j+1}, \ldots, p_n)$$

is holomorphic. A continuous partially holomorphic function is called holomorphic.

A nontrivial theorem due to Hartogs states that a partially holomorphic function is necessarily continuous (see [Hö], theorem 2.2.8), so we could skip the word continuous in the above definition.

Let $\mathcal{O}(U)$ be the set of holomorphic functions on U. Then,

Proposition 2.1.8 *$\mathcal{O}(U)$ is an algebra whose set of units $\mathcal{O}^*(U)$ consists of the holomorphic functions on U which vanish nowhere.*

Proof: Exercise.

□

Exercise 1 Let $U \subset \mathbb{C}^n$ be a domain and denote by $\dim_{\mathbb{C}} \mathcal{O}(U)$ its dimension as a $\mathbb{C}$-linear space. Show that

$$\dim_{\mathbb{C}} \mathcal{O}(U) < \infty \Longleftrightarrow \dim_{\mathbb{C}} \mathcal{O}(U) = 1 \Longleftrightarrow n = 0. \qquad \diamondsuit$$

Identify $\mathbb{C}^n \approx \mathbb{R}^{2n}$ by

$$(z_1, \ldots, z_n) = (x_1 + \boldsymbol{i} y_1, \ldots, x_n + \boldsymbol{i} y_n) \approx (x_1, y_1, \ldots, x_n, y_n).$$

In $\mathbb{C}^n$ we introduce the derivations

$$\frac{\partial}{\partial z_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} - \boldsymbol{i}\frac{\partial}{\partial y_j}\right), \quad \frac{\partial}{\partial \bar{z}_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} + \boldsymbol{i}\frac{\partial}{\partial y_j}\right)$$

for $j = 1, \ldots, n$.

Invoking the theorems of Loomann-Menchof and of Hartogs we see that a function f is holomorphic if, and only if, it has partial derivatives and they satisfy the Cauchy-Riemann equations:

$$\frac{\partial f}{\partial \bar{z}_j} = \frac{1}{2}\left(\frac{\partial f}{\partial x_j} + \boldsymbol{i}\frac{\partial f}{\partial y_j}\right) = 0, \qquad 1 \le j \le n.$$

Exercise 2 Show that

$$\overline{\left(\frac{\partial f}{\partial z_j}\right)} = \frac{\partial \bar{f}}{\partial \bar{z}_j} \quad \text{and} \quad \overline{\left(\frac{\partial f}{\partial \bar{z}_j}\right)} = \frac{\partial \bar{f}}{\partial z_j}. \qquad \diamondsuit$$

Definition 2.1.9 *A map $f = (f_1, \ldots, f_m) : U \to \mathbb{C}^m$, where U is a domain in $\mathbb{C}^n$, is* holomorphic *if each component f_j is a holomorphic function. If also f is a bijection and f^{-1} is holomorphic, then f is a* biholomorphism *or* biholomorphic*.*

We now treat questions of orientation. Let $\mathbb{C}^n \approx \mathbb{R}^{2n}$ with the identification given above. Consider the *complexified* of $\mathbb{R}^{2n}$, that is, $\mathbb{R}^{2n} \otimes \mathbb{C}$. The meaning of this is that we consider $\mathbb{R}^{2n}$ as a complex vector space, so the scalar field is now $\mathbb{C}$ and $\dim_{\mathbb{C}}(\mathbb{R}^{2n} \otimes \mathbb{C}) = 2n$. We have the following bases of $\mathbb{R}^{2n} \otimes \mathbb{C}$:

$$\mathcal{B}_1 = \left\{\frac{\partial}{\partial x_1}, \frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial x_n}, \frac{\partial}{\partial y_n}\right\}.$$

Of course this is a basis of $\mathbb{R}^{2n}$ (as real vector space) and determines the standard orientation.

$$\mathcal{B}_2 = \left\{\frac{\partial}{\partial z_1}, \frac{\partial}{\partial \bar{z}_1}, \ldots, \frac{\partial}{\partial z_n}, \frac{\partial}{\partial \bar{z}_n}\right\}$$

and

$$\mathcal{B}_3 = \left\{\frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_n}, \frac{\partial}{\partial \bar{z}_1}, \ldots, \frac{\partial}{\partial \bar{z}_n}\right\}.$$

Let us use these bases to show the

Proposition 2.1.10 *Biholomorphic maps preserve orientation.*

Proof: We will show that if we consider f as a smooth map from $\mathbb{R}^{2n}$ into itself, then the derivative $f'(p)$ preserves the orientation determined by $\mathcal{B}_1$.

Let $f = (f_1, \ldots, f_n)$ be a biholomorphism, which we write in coordinates as $f(x_1, y_1, \ldots, x_n, y_n) = (u_1, v_1, \ldots, u_n, v_n)$. Then the derivative $f'(p)$ is represented by the matrix

$$[f'(p)] = \begin{pmatrix} \dfrac{\partial(u_1, v_1)}{\partial(x_1, y_1)} & \cdots & \dfrac{\partial(u_1, v_1)}{\partial(x_n, y_n)} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial(u_n, v_n)}{\partial(x_1, y_1)} & \cdots & \dfrac{\partial(u_n, v_n)}{\partial(x_n, y_n)} \end{pmatrix}_{|p}$$

relative to the basis $\mathcal{B}_1$, where

$$\frac{\partial(u_j, v_j)}{\partial(x_k, y_k)}_{|p} = \begin{pmatrix} \dfrac{\partial u_j}{\partial x_k} & \dfrac{\partial u_j}{\partial y_k} \\ \dfrac{\partial v_j}{\partial x_k} & \dfrac{\partial v_j}{\partial y_k} \end{pmatrix}_{|p}, \quad 1 \leq j, k \leq n.$$

The change from the basis

$$\left\{ \frac{\partial}{\partial x_j}, \frac{\partial}{\partial y_j} \right\} \quad \text{to the basis} \quad \left\{ \frac{\partial}{\partial z_j}, \frac{\partial}{\partial \bar{z}_j} \right\}$$

is given by the matrix

$$P = \begin{pmatrix} 1/2 & 1/2 \\ -\boldsymbol{i}/2 & \boldsymbol{i}/2 \end{pmatrix} \text{ with } P^{-1} = \begin{pmatrix} 1 & \boldsymbol{i} \\ 1 & -\boldsymbol{i} \end{pmatrix}.$$

Hence, passing from the basis

$$\left\{ \frac{\partial}{\partial x_1}, \frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial x_n}, \frac{\partial}{\partial y_n} \right\}$$

to the basis

$$\left\{ \frac{\partial}{\partial z_1}, \frac{\partial}{\partial \bar{z}_1}, \ldots, \frac{\partial}{\partial z_n}, \frac{\partial}{\partial \bar{z}_n} \right\}$$

the matrix representing $f'(p)$ becomes

$$\begin{pmatrix} P^{-1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & P^{-1} \end{pmatrix} \begin{pmatrix} \frac{\partial(u_1,v_1)}{\partial(x_1,y_1)} & \cdots & \frac{\partial(u_1,v_1)}{\partial(x_n,y_n)} \\ \vdots & & \vdots \\ \vdots & \ddots & \vdots \\ \vdots & & \vdots \\ \frac{\partial(u_n,v_n)}{\partial(x_1,y_1)} & \cdots & \frac{\partial(u_n,v_n)}{\partial(x_n,y_n)} \end{pmatrix}_{|p} \begin{pmatrix} P & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & P \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\partial f_1}{\partial z_1} & 0 & \cdots & \frac{\partial f_1}{\partial z_n} & 0 \\ 0 & \overline{\frac{\partial f_1}{\partial z_1}} & & 0 & \overline{\frac{\partial f_1}{\partial z_n}} \\ \vdots & & \ddots & \vdots & \\ \frac{\partial f_n}{\partial z_1} & 0 & \cdots & \frac{\partial f_n}{\partial z_n} & 0 \\ 0 & \overline{\frac{\partial f_n}{\partial z_1}} & & 0 & \overline{\frac{\partial f_n}{\partial z_n}} \end{pmatrix}_{|p} .$$

Changing now from the basis

$$\left\{ \frac{\partial}{\partial z_1}, \frac{\partial}{\partial \bar{z}_1}, \dots, \frac{\partial}{\partial z_n}, \frac{\partial}{\partial \bar{z}_n} \right\}$$

to the basis

$$\left\{ \frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_n}, \frac{\partial}{\partial \bar{z}_1}, \dots, \frac{\partial}{\partial \bar{z}_n} \right\}$$

this last matrix transforms into

$$\begin{pmatrix} \frac{\partial f_1}{\partial z_1} & \cdots & \frac{\partial f_1}{\partial z_n} & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial z_1} & \cdots & \frac{\partial f_n}{\partial z_n} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \overline{\frac{\partial f_1}{\partial z_1}} & \cdots & \overline{\frac{\partial f_1}{\partial z_n}} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \overline{\frac{\partial f_n}{\partial z_1}} & \cdots & \overline{\frac{\partial f_n}{\partial z_n}} \end{pmatrix}_{|p} ,$$

hence is of the form

$$[f'(p)] \;=\; \begin{pmatrix} Jf(p) & 0 \\ 0 & \overline{Jf(p)} \end{pmatrix},$$

where

$$Jf(p) \;=\; \left(\frac{\partial f_i}{\partial z_j}(p) \right)_{1 \le i,j \le n} .$$

In particular

$$\det[f'(p)] = \det Jf(p) \det \overline{Jf(p)} = \det Jf(p) \overline{\det Jf(p)} = |\det Jf(p)|^2 > 0$$

and the proposition is proved.

□

We finish this section with the

Definition 2.1.11 *Let $p \in \mathbb{C}^n$. A* map germ *(smooth or holomorphic) or* germ *at p is an equivalence class of maps (smooth or holomorphic) where two maps are equivalent if they agree on a neighborhood of p. We adopt the notation $f : (\mathbb{C}^n, p) \to (\mathbb{C}^m, q)$ to denote the germ of f at p with $f(p) = q$.*

2.1.3 The index

We denote by $|z|$ the hermitian norm in $\mathbb{C}^n$, $|z| = \sqrt{\sum_{j=1}^{n} z_j \bar{z}_j}$. Consider map germs $f : (\mathbb{C}^n, p) \to (\mathbb{C}^n, q)$. Without loss of generality we shall assume $f(p) = q = 0$ and we also refer to p as a *root* of $f = 0$.

Definition 2.1.12 *Let $f : (\mathbb{C}^n, p) \to (\mathbb{C}^n, 0)$ be a holomorphic map germ with $f^{-1}(0) = \{p\}$. The* index *or* Poincaré Hopf index *of f at p, noted $\mathcal{I}_p(f)$, is the degree of the smooth map*

$$\frac{f}{|f|} : S_\epsilon^{2n-1}(p) \longrightarrow S_1^{2n-1}$$

where $S_\epsilon^{2n-1}(p)$ is the euclidean sphere of radius $\epsilon > 0$, $S_\epsilon^{2n-1}(p) = \{z \in \mathbb{C}^n : |z - p| = \epsilon\}$ and S_1^{2n-1} is the unit sphere centered at $0 \in \mathbb{C}^n$.

Remark that if ϵ is sufficiently small then the index is well defined and, by Proposition 2.1.6, it does not depend on ϵ (exercise).

To illustrate this concept we have the

Proposition 2.1.13 *If $f : (\mathbb{C}^n, p) \to (\mathbb{C}^n, 0)$ is the germ of a biholomorphism, then $\mathcal{I}_p(f) = 1$.*

Proof: We need the auxiliary

Lemma 2.1.14 *Let U be an open convex subset of $\mathbb{C}^n$, $p \in U$, and $\phi : U \to \mathbb{C}$ holomorphic. Then there exist holomorphic functions $g_1, \ldots, g_n : U \to \mathbb{C}$ such that*

$$\phi(z) = \phi(p) + \sum_{j=1}^{n} g_j(z)\,(z_j - p_j)$$

where $p = (p_1, \ldots, p_n)$. Moreover, $g_j(p) = \dfrac{\partial \phi}{\partial z_j}(p)$.

Proof: Fix $z \in U$ and define $h(t) = \phi(p + t(z-p))$. Since U is convex h is well defined on the interval $[0,1]$. We have

$$\phi(z) - \phi(p) = h(1) - h(0) = \int_0^1 h'(t)dt.$$

By the chain rule $h'(t) = \sum_{j=1}^{n} \dfrac{\partial \phi}{\partial z_j}(p + t(z-p))\,(z_j - p_j)$. Put

$$g_j(z) = \int_0^1 \frac{\partial \phi}{\partial z_j}(p + t(z-p))\,dt.$$

□

Now to the proof of the theorem. By using a translation (which is necessarily orientation preserving) we may assume $p = 0$. The derivative of f at 0 is given by

$$f'(0).z = \lim_{t \to 0} \frac{f(tz)}{t}$$

hence we let

$$F(z,t) = \begin{cases} \dfrac{f(tz)}{t} & \text{, for } 0 < t \leq 1 \\ f'(0).z & \text{, for } t = 0. \end{cases}$$

To see the smoothness of F we invoke the above lemma:

$$F(z,t) = \left(\sum_{j=1}^{n} g_{1\,j}(tz)\,z_j, \ldots, \sum_{j=1}^{n} g_{n\,j}(tz)\,z_j \right) \quad \forall\, t \in [0,1].$$

Now, $F(z,t) \neq 0$ for all $t \in [0,1]$ because f is bijective and then

$$\frac{F(z,t)}{|F(z,t)|}$$

gives a smooth homotopy between $f/|f|$ and $f'(0)/|f'(0)|$. This linear isomorphism preserves orientation, since $\mathrm{GL}\,(n;\mathbb{C})$ is connected, and we get $1 = \mathcal{I}_0\,(f'(0)) = \mathcal{I}_0\,(f)$.

□

Choose a closed euclidean ball centered at p, $\bar{B}_\epsilon(p)$, of radius ϵ small enough so that the only solution of $f(z) = 0$ in $\bar{B}_\epsilon(p)$ is p.

Proposition 2.1.15 *$\mathcal{I}_p\,(f)$ is the number of points of the set $f^{-1}(\zeta) \cap B_\epsilon(p)$ where ζ is a regular value of f sufficiently close to* 0.

Proof: Let $\delta = \inf_{S_\epsilon^{2n-1}(p)} |f| > 0$. Then $|f(z) - t\,\zeta| \geq \delta - t\,|\zeta| > 0$ for all $t \in [0,1]$, $z \in S_\epsilon^{2n-1}(p)$ and ζ a regular value sufficiently close to 0. It follows that $f^{-1}(t\,\zeta) \cap S_\epsilon^{2n-1}(p) = \emptyset$ for all $0 \leq t \leq 1$. We then have that

$$F(z,t) = \frac{f(z) - t\,\zeta}{|f(z) - t\,\zeta|}$$

gives a smooth homotopy between $\dfrac{f-\zeta}{|f-\zeta|}$ and $\dfrac{f}{|f|}$. Hence, $\mathcal{I}_p\,(f) = \deg \dfrac{f-\zeta}{|f-\zeta|}$.

Let $\{\xi_1, \ldots, \xi_k\} = f^{-1}(\zeta) \cap B_\epsilon(p)$. Choose two by two disjoint small spheres $S_{\delta_j}^{2n-1}(\xi_j)$, centered at ξ_j and satisfying $S_{\delta_j}^{2n-1}(\xi_j) \cap S_\epsilon^{2n-1}(p) = \emptyset$. Consider the oriented manifold

$$X = \bar{B}_\epsilon(p) \setminus \cup_{j=1}^k B_{\delta_j}(\xi_j).$$

Its boundary is the disjoint union

$$\partial X = S_\epsilon^{2n-1}(p) \amalg S_{\delta_j}^{2n-1}(\xi_j) \amalg \cdots \amalg S_{\delta_k}^{2n-1}(\xi_k).$$

The map $\varphi = \dfrac{f-\zeta}{|f-\zeta|} : \partial X \to S_1^{2n-1}(0)$ admits the obvious smooth extension $\dfrac{f-\zeta}{|f-\zeta|}$ to all of X. By proposition 2.1.6 we get $\deg\varphi = 0$ but, due to the orientation of X,

$$\deg\varphi = \mathcal{I}_p\,(f) - \mathcal{I}_{\xi_1}\,(f-\zeta) - \cdots - \mathcal{I}_{\xi_k}\,(f-\zeta).$$

Hence, $\mathcal{I}_p\,(f) = \mathcal{I}_{\xi_1}\,(f-\zeta) + \cdots + \mathcal{I}_{\xi_k}\,(f-\zeta) = k$ since f is biholomorphic at each ξ_j and then $\mathcal{I}_{\xi_k}\,(f-\zeta) = 1$ by proposition 2.1.13.

□

Example 2.1.16 Let $f(z_1, z_2) = (z_1^2, z_1 + z_2^3)$. Then $f^{-1}(0) = \{0\}$ and the index $\mathcal{I}_0(f)$ is given by the number of solutions of the equations $z_1^2 = \zeta_1$ and $z_1 + z_2^3 = \zeta_2$ where $0 < |(\zeta_1, \zeta_2)| << 1$. We immediately obtain $\mathcal{I}_0(f) = 6$.

More generally we have the

Theorem 2.1.17 *Let $X \subset \mathbb{C}^n$ be a compact and connected smooth manifold with boundary, $\dim_{\mathbb{R}} X = 2n$. Let f be a holomorphic map $f : U \to \mathbb{C}^n$ where U is a domain containing X, $p \in X \setminus \partial X$, $f(p) = 0$ and $f^{-1}(0) \cap \partial X = \emptyset$. Suppose the degree of the map*

$$\varphi = \frac{f}{|f|} : \partial X \longrightarrow S_1^{2n-1}(0)$$

is k. Then, the equation $f = 0$ has a finite number of solutions in the interior of X and the sum of the indices of f at these points is precisely k.

Proof: Assume we have $k+1$ distinct points $\xi_1, \dots, \xi_{k+1}$ in the interior of X satisfying $f(\xi_j) = 0$. Choose two by two disjoint small spheres $S_{\delta_j}^{2n-1}(\xi_j)$, centered at ξ_j and satisfying $S_{\delta_j}^{2n-1}(\xi_j) \cap \partial X = \emptyset$. Consider the oriented manifold

$$\tilde{X} = X \setminus \cup_{j=1}^{k+1} B_{\delta_j}(\xi_j).$$

Its boundary is the disjoint union

$$\partial \tilde{X} = \partial X \amalg S_{\delta_j}^{2n-1}(\xi_j) \amalg \cdots \amalg S_{\delta_{k+1}}^{2n-1}(\xi_{k+1}).$$

The map $\tilde{\varphi} : \partial \tilde{X} \to S_1^{2n-1}(0)$, $\tilde{\varphi} = f / |f|$, extends smoothly as $f / |f| : \tilde{X} \to S_1^{2n-1}(0)$ and so, by 2.1.6, $\deg \tilde{\varphi} = 0$. But, keeping in mind the orientation of $\tilde{X}$, $\deg \tilde{\varphi} = \deg \varphi - \mathcal{I}_{\xi_1}(f) - \cdots - \mathcal{I}_{\xi_{k+1}}(f)$. Hence, $\deg \varphi = \mathcal{I}_{\xi_1}(f) + \cdots + \mathcal{I}_{\xi_{k+1}}(f)$. Now note that the above proposition 2.1.15 tells us that $\mathcal{I}_{\xi_j}(f)$ is a positive integer, because it is the number of elements of a finite non empty set. We conclude $\deg \varphi \geq k+1$ which is absurd. Therefore, we have at most k solutions of the equation $f = 0$ in the interior of X, and the reasoning above shows that the sum of the indices of f at these points is exactly k.

□

From this we derive the

Theorem 2.1.18 (Additive character of the Poincaré Hopf index) *Suppose we have a holomorphic map germ f from $\mathbb{C}^n$ to $\mathbb{C}^n$ and p an isolated root of $f = 0$. Consider a holomorphic deformation f_λ of the germ $f =$*

f_0, depending on the complex parameter λ. Then, as λ varies in a small neighborhood of 0, the root p decomposes into a finite number of roots of f_λ and the sum of the indices of f_λ at these roots is equal to the index of f_0 at p.

Proof: Suppose $p = 0$ and take a ball $B_\delta(0)$ with δ so small that f has no zeros on the sphere $\partial B_\delta(0)$. Let $\delta_1 > 0$ be such that if $|\lambda| \leq \delta_1$, then f_λ has no zeros on the sphere $\partial B_\delta(0)$. Put

$$\inf_{\substack{|\lambda| \leq \delta_1 \\ z \in \partial B_\delta(0)}} |f_\lambda(z)| = K > 0.$$

Given $\epsilon < K$ there exists $\delta_2 > 0$ such that if $|\lambda| \leq \delta_2$, then

$$\sup_{\partial B_\delta(0)} |f(z) - f_\lambda(z)| < \epsilon.$$

Let $\delta_3 = \min\{\delta_1, \delta_2\}$. We claim that, for $|\lambda| < \delta_3$ the maps

$$\frac{f_\lambda}{|f_\lambda|} : \partial B_\delta(0) \longrightarrow S_1^{2n-1}(0)$$

are homotopic. It's enough to show they are homotopic to $f = f_0$. Consider $\varphi_t = (1-t)\,f + t\,f_\lambda$ and suppose there are $t_0 \in (0,1)$ and $z_0 \in \partial B_\delta(0)$ such that $\varphi_{t_0}(z_0) = 0$. This gives

$$f(z_0) = \frac{-t_0}{1-t_0} f_\lambda(z_0).$$

But then

$$\epsilon > |f(z_0) - f_\lambda(z_0)| = \frac{1}{1-t_0} |f_\lambda(z_0)| \geq \frac{K}{1-t_0} > K$$

a contradiction. Hence, $\varphi_t(z)$ never vanishes and gives the desired homotopy $\dfrac{\varphi_t(z)}{|\varphi_t(z)|}$.

Now,

$$\mathcal{I}_0\,(f) = \deg \frac{f}{|f|} = \deg \frac{f_\lambda}{|f_\lambda|} = \sum_{\xi_i \in f_\lambda^{-1}(0)} \mathcal{I}_{\xi_i}\,(f_\lambda).$$

□

Definition 2.1.19 *Let* $f, g : (\mathbb{C}^n, p) \to \mathbb{C}^n$ *be two holomorphic map germs.* f *and* g *are* algebraically equivalent*, or* A-equivalent*, if there is a holomorphic map germ* $A : (\mathbb{C}^n, p) \to \mathrm{GL}\,(n; \mathbb{C})$ *such that*

$$f(z) = A(z)\, g(z).$$

The Poincaré Hopf index is invariant under A-equivalence, more precisely:

Proposition 2.1.20 *If* $f, g : (\mathbb{C}^n, p) \to \mathbb{C}^n$ *are A-equivalent and* $f^{-1}(f(p)) = \{p\}$*, then* $\mathcal{I}_p\,(f) = \mathcal{I}_p\,(g)$.

Proof: First of all recall that $\mathrm{GL}\,(n; \mathbb{C})$ is open, dense and connected in $\mathrm{M}\,(n; \mathbb{C})$ (this is so since $\mathrm{GL}\,(n; \mathbb{C}) = \mathrm{M}\,(n; \mathbb{C}) \setminus \det^{-1}(0)$ and $\det = 0$ defines a real codimension two subvariety of $\mathrm{M}\,(n; \mathbb{C})$). Let $V \subset \mathrm{GL}\,(n; \mathbb{C})$ be a small contractible open neighborhood of $A(p)$. Then there is a smooth homotopy $G(z, t)$ such that $G(z, 0) = A(z) \in V$ and $G(z, 1) = A(p)$. It follows that

$$\frac{G(z,t)\, g(z)}{|G(z,t)\, g(z)|}$$

is a smooth homotopy between $\dfrac{f(z)}{|f(z)|} = \dfrac{A(z)\, g(z)}{|A(z)\, g(z)|}$ and $\dfrac{A(p)\, g(z)}{|A(p)\, g(z)|}$. Now choose a smooth real path γ in $\mathrm{GL}\,(n; \mathbb{C})$ such that $\gamma(0) = A(p)$, $\gamma(1) = I$. Then $\dfrac{\gamma(t)\, g(z)}{|\gamma(t)\, g(z)|}$ gives a smooth homotopy between $\dfrac{A(p)\, g(z)}{|A(p)\, g(z)|}$ and $\dfrac{g(z)}{|g(z)|}$. □

2.2 The Milnor number

2.2.1 First results on the multiplicity

We start by introducting some notations:

$\mathcal{O}_p$ denotes the (local) ring of germs of holomorphic functions at $p \in \mathbb{C}^n$. $\mathcal{O}_p$ is a $\mathbb{C}$-algebra.

$\mathfrak{M}_p$ denotes the maximal ideal of $\mathcal{O}_p$ that is,

$$\mathfrak{M}_p = \{h \in \mathcal{O}_p : h(p) = 0\}.$$

Given $f : (\mathbb{C}^n, p) \to \mathbb{C}^k$, $f = (f_1, \ldots, f_k)$, we denote by $\mathfrak{I}_f$ the ideal in $\mathcal{O}_p$ generated by $f_1, \ldots, f_k$ that is,

$$\mathfrak{I}_f = \{h_1\, f_1 + \cdots + h_k\, f_k \,: h_j \in \mathcal{O}_p\} = \langle f_1, \ldots, f_k \rangle_{\mathcal{O}_p}.$$

Definition 2.2.1 *Let $f : (\mathbb{C}^n, p) \to (\mathbb{C}^n, 0)$ be a holomorphic map germ. The* local algebra of f at p *is the quotient $\mathbb{C}$-algebra*

$$\mathcal{Q}_f = \mathcal{O}_p \,/\, \mathfrak{I}_f .$$

A germ of biholomorphism $\psi : (\mathbb{C}^n, p) \hookleftarrow$ induces a $\mathbb{C}$-algebra isomorphism $\psi^* : \mathcal{O}_p \to \mathcal{O}_p$ by $\psi^*(f) = f \circ \psi$ hence, $\mathcal{Q}_f$ is independent of the choice of coordinates.

Definition 2.2.2 *Let $f : (\mathbb{C}^n, p) \to (\mathbb{C}^n, 0)$ be a holomorphic map germ. The* multiplicity of f at p, *or* Milnor number of f at p, *noted $\mu_p(f)$, is the dimension of the $\mathbb{C}$-linear space $\mathcal{Q}_f$.*

Example 2.2.3 Let $f = (f_1, f_2) = (z_1^2, z_1 + z_2^3)$, $p = 0$ (recall example 2.1.16). We have $z_1^2 = f_1 \in \mathfrak{I}_f$, $z_1 z_2^3 = z_1 f_2 - f_1 \in \mathfrak{I}_f$ and $z_2^6 = z_2^3 f_2 - z_1 z_2^3 \in \mathfrak{I}_f$. On the other hand, $z_2^3 \equiv -z_1 \mod \mathfrak{I}_f$, $z_1 z_2 \equiv -z_2^4 \mod \mathfrak{I}_f$ and $z_1 z_2^2 \equiv -z_2^5 \mod \mathfrak{I}_f$. Hence, a basis of the $\mathbb{C}$-linear space $\mathcal{Q}_f$ is given by $\{1, z_1, z_2, z_1 z_2, z_2^2, z_1 z_2^2\}$ and we get $\mu_0(f) = \dim_{\mathbb{C}} \mathcal{Q}_f = 6$.

Lemma 2.2.4 *Let $f : (\mathbb{C}^n, p) \to (\mathbb{C}^n, 0)$ be a holomorphic map germ of multiplicity μ at p. Given any collection of μ germs of functions in $\mathfrak{M}_p$, $h_1, \ldots, h_\mu$, their product $h_1 \cdots h_\mu$ lies in $\mathfrak{I}_f$.*

Proof: Consider the $\mu + 1$ germs $H_1 = 1$, $H_2 = h_1$, $H_3 = h_1 \cdot h_2$, ... , $H_{\mu+1} = h_1 \cdots h_\mu$. Since $\dim_{\mathbb{C}} \mathcal{Q}_f = \mu$, their classes in $\mathcal{Q}_f$ are linearly dependent and so there are complex numbers $a_0, \ldots, a_\mu$ such that

$$a_0 + a_1 H_2 + \cdots + a_\mu H_{\mu+1} \in \mathfrak{I}_f .$$

Let k be the smallest integer such that $a_k \neq 0$. Then

$$a_k H_{k+1} + a_{k+1} H_{k+2} + \cdots + a_\mu H_{\mu+1} =$$

$$H_{k+1} \left(a_k + a_{k+1} \frac{H_{k+2}}{H_{k+1}} + \cdots + a_\mu \frac{H_{\mu+1}}{H_{k+1}} \right) \in \mathfrak{I}_f .$$

But the factor $a_k + a_{k+1} \dfrac{H_{k+2}}{H_{k+1}} + \cdots + a_\mu \dfrac{H_{\mu+1}}{H_{k+1}}$ is a *unit* in $\mathcal{O}_p$ (which means it is algebraically invertible) and therefore $H_{k+1} \in \mathfrak{I}_f$. It follows that $H_{\mu+1} = H_{k+1} \, h_{k+1} \, h_{k+2} \cdots h_\mu \in \mathfrak{I}_f$.

□

The usefulness of this lemma will be exploited below, but first recall that a holomorphic function $F(z_1, \ldots, z_n)$, defined around $p = (p_1, \ldots, p_n)$, is expressible in the form

$$F = F_m + F_{m+1} + \cdots + F_{m+\ell} + \cdots \qquad F_m \not\equiv 0$$

where F_j is a homogeneous polynomial of degree j in the variables $z_1 - p_1, \ldots, z_n - p_n$. The number m is called the *order of* F *at* p .

Proposition 2.2.5 *Let* $f, g : (\mathbb{C}^n, p) \to (\mathbb{C}^n, 0)$ *be holomorphic map germs where* f *has multiplicity* μ. *Suppose each component of the difference* $g - f$ *has an expansion of the form* $g_i - f_i = F_{i\,\mu+r_i} + F_{i\,\mu+r_i+1} + \cdots$ *with* $r_i \geq 1$. *Then* f *and* g *are* A*-equivalent.*

Proof: Write $F_{i\,\mu+\ell}$ as

$$F_{i\,\mu+\ell} = \sum_J a_{i\,J}\,(z_1 - p_1)^{j_1} \cdots (z_n - p_n)^{j_n}$$

with $j_1 + \cdots + j_n = |J| = \mu + \ell$. Hence, each term is a product of $\mu + \ell > \mu$ functions in $\mathfrak{M}_p$ and by lemma 2.2.4 we can write each one as

$$a_{i\,J}\,(z_1 - p_1)^{j_1} \cdots (z_n - p_n)^{j_n} = g_{J\,1}\, f_1 + \cdots + g_{J\,n}\, f_n.$$

Observe that the functions $g_{J\,k}$ lie in $\mathfrak{M}_p$ because the left side is of degree $\mu + \ell > \mu$. We conclude $F_{i\,\mu+\ell} = \sum_j b_{i\,j}^{(\mu+\ell)}\, f_j$ with $b_{i\,j}^{(\mu+\ell)} \in \mathfrak{M}_p$. Summing over ℓ we get $g_i - f_i = \sum_j c_{i\,j}\, f_j$, $c_{i\,j} \in \mathfrak{M}_p$.

This gives $g = (I + C)\, f$, $C = (c_{i\,j})$. Since $C(p) = 0$ the matrix $I + C$ is invertible in a neighborhood of p and the proposition is proved.

□

Proposition 2.2.6 *If* f *and* g *are holomorphic* A*-equivalent map germs then, they have the same multiplicity at* p.

Proof: Since $f(z) = A(z)\, g(z)$ we have $\mathfrak{I}_f \subset \mathfrak{I}_g$ and, because $A(z)$ is invertible, $\mathfrak{I}_g \subset \mathfrak{I}_f$ so that $\mathfrak{I}_f = \mathfrak{I}_g$.

□

Exercise 3 Let $T : \mathbb{C}^n \to \mathbb{C}^n$ be an invertible linear transformation. Show that $\mu_0(T) = 1$.

Proposition 2.2.7 *If $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ is the germ of a biholomorphism then, $\mu_0(f) = 1$.*

Proof: In fact, we have $f(z) = f'(0).z + F_2(z) + F_3(z) + \cdots$ and so $f(z) - f'(0).z = F_2(z) + F_3(z) + \cdots$. By the above exercise $\mu_0(f'(0)) = 1$, by proposition 2.2.5 f and $f'(0)$ are A-equivalent and, by proposition 2.2.6, $\mu_0(f) = \mu_0(f'(0))$.

□

Definition 2.2.8 *A* Pham map (see [Pham]) *is a map $\Upsilon : \mathbb{C}^n \to \mathbb{C}^n$ of the form*

$$\Upsilon^J(z_1, \ldots, z_n) = (z_1^{j_1}, z_2^{j_2}, \ldots, z_n^{j_n})$$

where $J = (j_1, j_2, \ldots, j_n) \in \mathbb{N}^n$, $j_k \geq 1$, $\forall k$.

Lemma 2.2.9 $\mathcal{I}_0(\Upsilon^J) = \mu_0(\Upsilon^J)$.

Proof: This is shown by direct calculation. By 2.1.15, $\mathcal{I}_0(\Upsilon^J)$ is the number of solutions of $z_1^{j_1} = \xi_1$, ..., $z_n^{j_n} = \xi_n$, for $(\xi_1, \ldots, \xi_n)$ a regular value of Υ^J, which is $j_1 j_2 \cdots j_n$. On the other hand, a basis for the local algebra of Υ^J at 0 is formed by the classes of the monomials

$$z_1^{m_1} \cdots z_n^{m_n}, \qquad 0 \leq m_1 < j_1, \ \ldots \ , 0 \leq m_n < j_n.$$

There are $j_1 j_2 \cdots j_n$ of such.

□

Proposition 2.2.10 *Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ be a germ with multiplicity μ at 0. Consider the Pham map*

$$\Upsilon^{[\mu+1]}, \qquad [\mu+1] = \underbrace{(\mu+1, \ldots, \mu+1)}_{\text{n components}}$$

and the holomorphic deformation $\Upsilon_\lambda^{[\mu+1]} = \Upsilon^{[\mu+1]} + \lambda f$, λ in a small neighborhood of 0 in $\mathbb{C}$. Then f is A-equivalent to $\Upsilon_\lambda^{[\mu+1]}$ for $\lambda \neq 0$.

Proof: Note that $\Upsilon_\lambda^{[\mu+1]} - \lambda f = \Upsilon^{[\mu+1]}$ and all components of $\Upsilon^{[\mu+1]}$ have degree $> \mu$. By proposition 2.2.5, $\Upsilon_\lambda^{[\mu+1]}$ is A-equivalent to λf. Since λf is obviously A-equivalent to f the result follows.

□

Before we proceed to consider the question of additivity of the Milnor number (as we did for the Poincaré Hopf index) let us give a result which is very helpful in understanding the multiplicity.

Theorem 2.2.11 *Let $f : (\mathbb{C}^n, p) \to (\mathbb{C}^n, 0)$ be a holomorphic map germ. $\mu_p(f)$ is finite if, and only if, p is an isolated point in $f^{-1}(0)$.*

Proof: Suppose $\mu_p(f) < \infty$. Invoke lemma 2.2.4 to write, for $i = 1, \ldots, n$,

$$(z_i - p_i)^\mu = \sum_j g_{i\,j}\, f_j.$$

If we had a sequence $(p_k) = ((p_{1\,k}, \ldots, p_{n\,k})) \to p$ with $p_k \neq p$ and $f(p_k) = 0$ then, since the $g_{i\,j}$ are defined in a neighborhood of p, we would have $p_{i\,k} - p_i = 0$ for all i, which is absurd.

To prove the converse we invoke Hilbert's zero-theorem (see [Gu], p. 53). Suppose p is isolated in $f^{-1}(0)$. Then, there exist $m_i \geq 1$ such that the germ $(z_i - p_i)^{m_i} \in \mathfrak{T}_f$, $i = 1, \ldots, n$. It follows that $\mu_p(f) < \infty$.

□

2.2.2 The preparation theorem

Definition 2.2.12 *A* Weierstrass polynomial *of degree $k > 0$ is an element $h \in \mathcal{O}_{0,\,n-1}[z_n]$ of the form*

$$h = z_n^k + a_1\, z_n^{k-1} + \ldots + a_{k-1}\, z_n + a_k$$

where the coefficients a_j are germs at $0 \in \mathbb{C}^{n-1}$ which vanish at 0, that is, $a_j \in \mathfrak{M}_{0,\,n-1} \subset \mathcal{O}_{0,\,n-1}$, $1 \leq j \leq k$.

Let $f : (\mathbb{C}^n, 0) \to \mathbb{C}$ be a holomorphic function germ. f is regular of order k in z_n *if $f(0, \ldots, 0, z_n) = c_k z_n^k + \cdots$, where $c_k \neq 0$, that is, $f(0, \ldots, 0, z_n)$ has a zero of order k at $0 \in \mathbb{C}$.*

The following is a fundamental result:

Theorem 2.2.13 (Weierstrass preparation theorem) *Suppose $f \in \mathcal{O}_{0,\,n}$ is regular of order k in z_n. Then, there is a unique Weierstrass polynomial $h \in \mathcal{O}_{0,\,n-1}[z_n]$, of degree k in z_n, such that $f = u\,h$, where $u \in \mathcal{O}_{0,\,n}$ is a unit.*

Proof: The proof is given in the remark below.

□

Example 2.2.14 The holomorphic version of the implicit function theorem follows immediately from 2.2.13. Suppose $f(0) = 0$ and $\partial f/\partial z_n(0) \neq 0$ (this is the same as to say f is regular of order 1 in z_n). Then, in a neighborhood of 0 we have $f(z_1, \ldots, z_n) = u(z)(z_n + a_1(z_1, \ldots, z_{n-1}))$, $u(0) \neq 0$ and a_1 unique. Hence, the level set $f = 0$ is described by $z_n = -\,a_1(z_1, \ldots, z_{n-1})$.

Theorem 2.2.13 is a consequence of the more general

Theorem 2.2.15 (Weierstrass division theorem) *Suppose* $h \in \mathcal{O}_{0,n-1}[z_n]$ *is a Weierstrass polynomial of degree* k. *Then, any* $f \in \mathcal{O}_{0,n}$ *can be written uniquely in the form*

$$f = g\,h + R$$

where $g \in \mathcal{O}_{0,n}$ *and* $R \in \mathcal{O}_{0,n-1}[z_n]$ *is a polynomial in* z_n *of degree* $< k$. *Moreover, if* $f \in \mathcal{O}_{0,n-1}[z_n]$, *then* $g \in \mathcal{O}_{0,n-1}[z_n]$.

Proof: See [Gu].

□

Remark 3 To see why theorem 2.2.15 implies theorem 2.2.13 we do as follows: let f be regular of order k and consider $H(z_1, \dots, z_n) = z_n^k$. By the division theorem $f = g\,H + R$ which reads

$$f = g\,z_n^k + a_1\,z_n^{k-1} + \ldots + a_{k-1}\,z_n + a_k$$

with $a_j \in \mathcal{O}_{0\,n-1}$. If $k = 0$ then $f = a_0$ and 2.2.13 holds. If $k \geq 1$ then, since $f(0) = 0$, we have $a_k(0) = 0$ and thus $a_k \in \mathfrak{M}_{0\,n-1}$. Successive differentiation with respect to z_n and evaluation at $z_n = 0$ shows that $a_j \in \mathfrak{M}_{0\,n-1}$, for $j = 1, \dots, k-1$. Now, $f(0, z_n) = g(0, z_n)\,z_n^k$ and therefore $g(0, z_n)$ is a non zero constant. It follows that g is a unit and 2.2.13 is proved.

◇

Using the above theorem it can be shown that: *$\mathcal{O}_p$ is a unique factorization domain* and *$\mathcal{O}_p$ is a Noetherian ring* (see [Gu]).

We will derive another form, much more general, for this theorem. But first we consider a result from Commutative Algebra.

Let $\mathfrak{R}$ be a commutative ring with identity and $\mathfrak{G}$ an abelian group. $\mathfrak{G}$ is an $\mathfrak{R}$-module if we can define an action of $\mathfrak{R}$ in $\mathfrak{G}$:

$$\begin{array}{c} \mathfrak{R} \times \mathfrak{G} \longrightarrow \mathfrak{G} \\ \\ (x, \alpha) \longmapsto x\alpha \end{array} \qquad \text{such that} \qquad \begin{cases} (x+y)\alpha = x\alpha + y\alpha \\ (xy)\alpha = x(y\alpha) \\ x(\alpha+\beta) = x\alpha + x\beta \\ 1.\alpha = \alpha \end{cases}$$

$\mathfrak{G}$ is *finitely generated* over $\mathfrak{R}$ if there is a finite number of elements $\alpha_1, \ldots, \alpha_n$ such that every element $\beta \in \mathfrak{G}$ can be written as a linear combination of the α_j with coefficients in $\mathfrak{R}$, $\beta = x_1\alpha_1 + \cdots, +x_n\alpha_n$. We have the

Lemma 2.2.16 (Nakayama's lemma) *Let $\mathfrak{R}$ be a commutative local ring, $\mathfrak{M} \subset \mathfrak{R}$ its maximal ideal and $\mathfrak{G}$ an $\mathfrak{R}$-module. Suppose*
(i) $\mathfrak{G}$ *is finitely generated.*
(ii) $\mathfrak{G} = \mathfrak{M}\,\mathfrak{G}$.
Then $\mathfrak{G} = \{0\}$.

Proof: Let $e_1, \ldots, e_n$ be a set of generators for $\mathfrak{G}$ over $\mathfrak{R}$. Since $\mathfrak{G} = \mathfrak{M}\,\mathfrak{G}$, each e_k can be written as $e_k = x_1\alpha_1 + \cdots + x_m\alpha_m$ with $x_i \in \mathfrak{M}$. Using the fact that the e_i generate $\mathfrak{G}$ we have $\alpha_i = \sum_{j=1}^{n} y_{i,j}e_j$. Hence, $e_k = \sum_{j=1}^{n} z_{k,j}e_j$ with $z_{k,j} = \sum_{i=1}^{m} x_i\, y_{i,j} \in \mathfrak{M}$. This amounts to

$$(I - Z)\, e = 0 \qquad (\star)$$

where I is the identity $n \times n$ matrix, $Z = (z_{k,j})$, $1 \leq k, j \leq n$, and $e = (e_1, \ldots, e_n)$.

Now, $\mathfrak{M}$ is precisely the set of non-invertible elements of $\mathfrak{R}$. To see this suppose $x \in \mathfrak{M}$ were invertible. Then, $x\,x^{-1} = 1 \in \mathfrak{M} \Rightarrow \mathfrak{M} = \mathfrak{R}$ which is absurd. Conversely, if $x \notin \mathfrak{M}$ then, the ideal $\mathfrak{A}$ generated by x is not contained in $\mathfrak{M}$ and, by maximality, $\mathfrak{A} = \mathfrak{R}$. Thus, there is an element $y \in \mathfrak{R}$ such that $xy = 1$ and x is invertible. It follows that $\mathfrak{R}/\mathfrak{M}$ is a field.

Returning to system $(\star)$, the determinant of the the matrix $I - Z$ is of the form $\det(I - Z) = 1 + x$ with $x \in \mathfrak{M}$. Hence it is invertible and the only solution of the system is $e = 0$.

□

Corollary 2.2.17 *Let $\mathfrak{G}$ is a finitely generated $\mathfrak{R}$-module. Then, $\mathfrak{G}/\,\mathfrak{M}\,.\,\mathfrak{G}$ is a finite dimensional vector space over the field $\mathfrak{R}/\mathfrak{M}$. Let $\mathfrak{p} : \mathfrak{G} \to \mathfrak{G}/\,\mathfrak{M}\,.\,\mathfrak{G}$ be the projection onto the quocient and $u_1, \ldots, u_n$ be a basis for $\mathfrak{G}/\,\mathfrak{M}\,.\,\mathfrak{G}$. Choose elements $e_1, \ldots, e_n \in \mathfrak{G}$ such that $\mathfrak{p}(e_i) = u_i$. Then the elements of the set $\{e_1, \ldots, e_n\}$ generate $\mathfrak{G}$ over $\mathfrak{R}$.*

Proof: To see that $\mathfrak{G}/\,\mathfrak{M}\,.\,\mathfrak{G}$ is a vector space over $\mathfrak{R}/\mathfrak{M}$ is an exercise. Now let $\{\alpha_1, \ldots, \alpha_\ell\}$ be a set of generators of $\mathfrak{G}$ over $\mathfrak{R}$. Given $u \in \mathfrak{G}/\,\mathfrak{M}\,.\,\mathfrak{G}$ there exists $\beta \in \mathfrak{G}$ such that $\mathfrak{p}(\beta) = u$. Write $\beta = x_1\alpha_1 + \cdots + x_\ell\alpha_\ell$. Then

$$u = \mathfrak{p}(\beta) = \widetilde{x_1}\,\mathfrak{p}(\alpha_1) + \cdots + \widetilde{x_\ell}\,\mathfrak{p}(\alpha_\ell)$$

where $\widetilde{x_j}$ is the class of x_j in $\mathfrak{R}/\mathfrak{M}$. This shows $\{\mathfrak{p}(\alpha_1), \ldots, \mathfrak{p}(\alpha_\ell)\}$ is a basis for $\mathfrak{G}/\,\mathfrak{M}\,.\,\mathfrak{G}$ and so it is finite dimensional.

Suppose now $\{u_1, \ldots, u_n\}$ is a basis for $\mathfrak{G}/\,\mathfrak{M}\,.\,\mathfrak{G}$ and $\{e_1, \ldots, e_n\}$ as in the statement. Consider the submodule $\mathfrak{B}$ of $\mathfrak{G}$ generated by $\{e_1, \ldots, e_n\}$ and let $\mathfrak{C}$ be the quocient module $\mathfrak{C} = \mathfrak{G}/\mathfrak{B}$. Since $\mathfrak{G}$ is finitely generated, the same holds for $\mathfrak{C}$.

Let $\alpha \in \mathfrak{G}$. Then, $\mathfrak{p}(\alpha) = \widetilde{x_1}\, u_1 + \cdots + \widetilde{x_n}\, u_n$ and thus $\alpha = x_1 e_1 + \cdots + x_n e_n + t$, where $t \in \mathfrak{M}\,\mathfrak{G}$. Hence we have $\mathfrak{G} = \mathfrak{B} + \mathfrak{M}\,\mathfrak{G}$. But then

$$\mathfrak{C} = \mathfrak{G}/\mathfrak{B} = (\mathfrak{B} + \mathfrak{M}\,.\,\mathfrak{G})/\mathfrak{B} = \mathfrak{M}\,.\,(\mathfrak{G}/\mathfrak{B}) = \mathfrak{M}\,.\,\mathfrak{C}.$$

By Nakayama's lemma, $\mathfrak{C} = 0$ and thus $\mathfrak{G} = \mathfrak{B}$.

□

Returning to our local ring of interest, let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^m, 0)$ be a holomorphic map germ and $\mathfrak{G}$ a $\mathcal{O}_{0\,n}$-module. The germ f allow us to consider $\mathfrak{G}$ as an $\mathcal{O}_{0\,m}$-module as follows: it induces a ring homomorphism, the pull-back f^*, defined by $f^*h = h \circ f$ and then we have an action

$$\mathcal{O}_{0\,m} \times \mathfrak{G} \longrightarrow \mathfrak{G}$$

$$(h, \alpha) \longmapsto (f^*h)\,\alpha = (h \circ f)\,\alpha.$$

The next theorem is nontrivial and is a cornerstone of the theory of singulatities of maps. It holds in the real C^∞ situation as well, where it is known as the *Malgrange-Mather preparation theorem* (see [Mather]). For our purposes it is enough to present it in the following particular form:

Theorem 2.2.18 (Preparation theorem) *Consider a holomorphic map germ $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^m, 0)$ and let $\mathfrak{G}$ be a finitely generated $\mathcal{O}_{0\,n}$-module. Then:*
$\mathfrak{G}$ is a finitely generated $\mathcal{O}_{0\,m}$-module (via f^*) if, and only if, the $\mathbb{C}$-linear space $\mathfrak{G}/\,(f^*\mathfrak{M}_{0,m}\,.\,\mathfrak{G})$ is finite dimensional.

Proof: Suppose $\mathfrak{G}$ is finitely generated as $\mathcal{O}_{0\,m}$-module (via f^*). Let $\{e_1, \ldots, e_k\}$ be a set of generators and choose an element $u \in \mathfrak{G}/\,(f^*\mathfrak{M}_{0,m}\,.\,\mathfrak{G})$. If $\mathfrak{p} : \mathfrak{G} \to \mathfrak{G}/\,(f^*\mathfrak{M}_{0,m}\,.\,\mathfrak{G})$ is the natural projection, then $u = \mathfrak{p}(\alpha)$ for some $\alpha \in \mathfrak{G}$. Now, α can be written as $\alpha = (h_1 \circ f)\, e_1 + \cdots + (h_k \circ f)\, e_k$. Each $h_j \in \mathcal{O}_{0\,m}$ has an expansion $h_j = c_j + H_j$ where $c_j \in \mathbb{C}$ and $H_j \in \mathfrak{M}_{0\,m}$. Thus, $h_j \circ f = c_j + \varphi$ with $\varphi \in f^*\mathfrak{M}_{0,m}$. We have

$$u = \mathfrak{p}(\alpha) = c_1\mathfrak{p}(e_1) + \cdots + c_n\mathfrak{p}(e_n)$$

and the elements $\mathfrak{p}(e_1), \ldots, \mathfrak{p}(e_n)$ generate $\mathfrak{G}/\,(f^*\mathfrak{M}_{0,m}\,.\,\mathfrak{G})$.

The other direction is the nontrivial one and will be proved in three steps.

Case of a submersion. Suppose $n = m+1$ and $f : (\mathbb{C}\times\mathbb{C}^m, 0) \to (\mathbb{C}^m, 0)$ is the projection $f(w,z) = z$. Note that $f^*\mathfrak{M}_{0\,m}$ coincides with $\mathfrak{M}_{0\,m}$ as a subset of $\mathfrak{M}_{0\,m+1}$. Choose $e_1, \ldots, e_k \in \mathfrak{G}$ such that $\{\mathfrak{p}(e_1), \ldots, \mathfrak{p}(e_k)\}$ is a basis for $\mathfrak{G}/\,(f^*\mathfrak{M}_{0,m}\,.\,\mathfrak{G})$ as a complex vector space. Now, $f^*\mathfrak{M}_{0\,m} \subset \mathfrak{M}_{0\,m+1}$ and there is a natural surjection

$$\mathfrak{q} : \mathfrak{G}/\,(f^*\mathfrak{M}_{0,m}\,.\,\mathfrak{G}) \longrightarrow \mathfrak{G}/\,\mathfrak{M}_{0,m+1}\,.\,\mathfrak{G},$$

thus $\mathfrak{q}(\mathfrak{p}(e_1)), \ldots, \mathfrak{q}(\mathfrak{p}(e_k))$ is a set of generators of $\mathfrak{G}/\,\mathfrak{M}_{0,m+1}\,.\,\mathfrak{G}$. By corollary 2.2.17 we have that

$$e_1, \ldots, e_k \ \ \textit{generate} \ \ \mathfrak{G} \ \ \textit{as an} \ \ \mathcal{O}_{0\,m+1}\textit{-module.} \tag{I}$$

Next we show that:

$$\begin{gathered} \textit{All elements of } \mathfrak{G} \textit{ have the form } \sum_{j=1}^{k} (c_j\, e_j + h_j\, e_j) \\ \textit{with } c_j \in \mathbb{C} \textit{ and } h_j \in \mathfrak{M}_{0\,m}.\,\mathcal{O}_{0\,m+1}. \end{gathered} \tag{II}$$

To see this observe that, since $\{\mathfrak{p}(e_1), \ldots, \mathfrak{p}(e_k)\}$ form a basis for $\mathfrak{G}/\,(f^*\mathfrak{M}_{0,m}\,.\,\mathfrak{G})$, every element $\alpha \in \mathfrak{G}$ can be written as $\alpha = c_1\, e_1 + \cdots + c_k\, e_k + \widetilde{\beta}$ with $\widetilde{\beta} \in f^*\mathfrak{M}_{0,m}\,.\,\mathfrak{G}$. Hence, $\widetilde{\beta} = \sum_{i=1}^{\ell} g_i\, \sigma_i$ where $g_i \in \mathfrak{M}_{0\,m}$ and $\sigma_i \in \mathfrak{G}$. By (I), $\sigma_i = \sum_{s=1}^{k} \varphi_s\, e_s$ with $\varphi_s \in \mathcal{O}_{0\,m+1}$. Thus, $\widetilde{\beta} = \sum_{s=1}^{k} \left(\sum_{i=1}^{\ell} g_i\, \varphi_s \right) e_s$. Put $h_j = \sum_{i=1}^{\ell} g_i\, \varphi_j$ and (II) is proved.

Apply (II) to the elements $w\, e_i$, $i = 1, \ldots, k$. We get

$$w\, e_i = \sum_{j=1}^{k} (c_{i\,j}\, e_j + h_{i\,j}\, e_j), \quad c_{i\,j} \in \mathbb{C}, \quad h_{i\,j} \in \mathfrak{M}_{0\,m}.\,\mathcal{O}_{0\,m+1}.$$

If $(\delta_{i\,j})$ is the identity matrix, then these equations take the form:

$$(w\, \delta_{i\,j} - c_{i\,j} - h_{i\,j}).\, e = 0$$

where $e = (e_1, \ldots, e_k)$. Let $(b_{i\,j})$ be the matrix whose entries are $b_{i\,j} = w\, \delta_{i\,j} - c_{i\,j} - h_{i\,j}$. If $(B_{i\,j})$ is the transpose of the matrix of the cofactors of $(b_{i\,j})$ (Cramer's rule) then,

$$(B_{i\,j}).(b_{i\,j}) = \det(b_{i\,j}).(\delta_{i\,j}).$$

Set $P(w,z) = \det(b_{ij})$. It follows that $P(w,z)\,e_i = 0$ for each i. Since $h_{ij} \in \mathfrak{M}_{0\,m}.\mathcal{O}_{0\,m+1}$ we have that $P(w,0) = \det(w\,\delta_{ij} - c_{ij})$ is a polynomial in w of order $d \leq k$. Thus, $P(w,0) = u(w)\,w^d$ with $u(0) \neq 0$ and $P(z,w)$ is regular of order d at $(0,0)$. By the Weierstrass preparation theorem 2.2.13, $P = v\,H$. Given $\alpha \in \mathfrak{G}$, by (II) again we can write α as $\alpha = \sum_{i=1}^{k}(c_i\,e_i + \rho_i\,e_i)$ with $c_i \in \mathbb{C}$ and $\rho_i \in \mathfrak{M}_{0\,m}.\mathcal{O}_{0\,m+1}$. By the Weierstrass division theorem 2.2.15

$$\rho_i = q_i\,H + \sum_{j=0}^{d-1} R_{ij}(z_1,\ldots,z_m)\,w^j.$$

But then,

$$\rho_i = \left(\frac{q_i}{v}\right)(v\,H) + \sum_{j=0}^{d-1} R_{ij}(z_1,\ldots,z_m)\,w^j = \left(\frac{q_i}{v}\right)P + \sum_{j=0}^{d-1} R_{ij}(z_1,\ldots,z_m)\,w^j.$$

Since $P\,e_i = 0$, we have that $\rho_i\,e_i = \sum_{j=0}^{d-1} R_{ij}(z_1,\ldots,z_m)\,w^j\,e_i$ and therefore

$$\alpha = \sum_{i=1}^{k}(c_i\,e_i + \rho_i\,e_i) = \sum_{i=1}^{k}\left(c_i\,e_i + \sum_{j=0}^{d-1} R_{ij}(z_1,\ldots,z_m)\,w^j\,e_i\right)$$

and we conclude that $\mathfrak{G}$ is generated by the kd elements $e_1,\ldots,e_k$, $w\,e_1,\ldots,w\,e_k$, ..., $w^{d-1}\,e_1,\ldots,w^{d-1}\,e_k$ as an $\mathcal{O}_{0\,m}$-module because $R_{ij} \in \mathcal{O}_{0\,m}$.

Case of an immersion. Let $f : (\mathbb{C}^n,0) \to (\mathbb{C}^m,0)$ be a holomorphic map germ of rank n. By the rank theorem we have that, up to changes of coordinates, f is written as

$$(z_1,\ldots,z_n) \longmapsto (z_1,\ldots,z_n,0,\ldots,0).$$

Now, any germ $g : (\mathbb{C}^n,0) \to \mathbb{C}$ extends holomorphically to $(\mathbb{C}^m,0)$ (simply define $g(z_1,\ldots,z_n,z_{n+1},\ldots,z_m) = g(z_1,\ldots,z_n)$). This mea- ns that the map $f^* : \mathcal{O}_{0\,m} \to \mathcal{O}_{0\,n}$ is a surjection. But then any finite set of generators of $\mathfrak{G}$ as an $\mathcal{O}_{0\,n}$-module is also a set of generators for $\mathfrak{G}$ as an $\mathcal{O}_{0\,m}$-module.

General case. Given $f : (\mathbb{C}^n,0) \to (\mathbb{C}^m,0)$ define

$$F : (\mathbb{C}^n,0) \longrightarrow (\mathbb{C}^n,0) \times (\mathbb{C}^m,0) \quad \text{by}$$

$$\xi \longmapsto (\xi, f(\xi)).$$

Denoting by $\pi_i : \mathbb{C}^i \times \mathbb{C}^m \to \mathbb{C}^{i-1} \times \mathbb{C}^m$ the projection

$$\pi_i(z_1, \ldots, z_i, w) = (z_2, \ldots, z_i, w)$$

we have $f = \pi_1 \circ \cdots \circ \pi_n \circ F$. Since F is an immersion we see that $\mathfrak{G}$ is a finitely generated $\mathcal{O}_{(0,0)\, n\times m}$-module. Suppose now that $\mathfrak{G} / \mathfrak{M}_{0\, m} \,.\, \mathfrak{G}$ is a finite dimensional complex vector space. Since $\mathfrak{M}_{0\, m} \subset \mathfrak{M}_{(0,0)\, n-1\times m}$ we have a surjection

$$\mathfrak{G} / \mathfrak{M}_{0\, m} \,.\, \mathfrak{G} \longrightarrow \mathfrak{G} / \mathfrak{M}_{(0,0)\, n-1\times m} \,.\, \mathfrak{G}$$

and this last vector space is finite dimensional. Since π_n is a submersion, we conclude that $\mathfrak{G}$ is a finitely generated $\mathcal{O}_{(0,0)\, n-1\times m}$-module. Look now at

$$\pi_{n-1}^* : \mathcal{O}_{(0,0)\, n-2\times m} \longrightarrow \mathcal{O}_{(0,0)\, n-1\times m}$$

and apply the reasoning of the submersive case. We get $\mathfrak{G}$ a finitely generated $\mathcal{O}_{(0,0)\, n-2\times m}$-module. Continuing this way, with π_{n-2}^* and so on, we obtain the result. The theorem is proved.

□

To see this theorem in action, consider a holomorphic map germ $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ of finite multiplicity μ at 0 and let $\mathfrak{G} = \mathcal{O}_{0\, n}$. We have

$$f^* : \mathcal{O}_{0\, n} \longrightarrow \mathcal{O}_{0\, n}$$

and remark that $f^*\mathfrak{M}_{0\, n} \,.\, \mathcal{O}_{0\, n} = \mathfrak{T}_f$. The complex vector space $\mathcal{O}_{0\, n} \,/\, f^*\mathfrak{M}_{0\, n} \,.\, \mathcal{O}_{0\, n}$ is finite dimensional because $\dim_{\mathbb{C}} \mathcal{O}_{0\, n} \,/\, \mathfrak{T}_f = \mu$. By the preparation theorem we have that $\mathcal{O}_{0\, n}$ is a finitely generated $\mathcal{O}_{0\, n}$-module via f^*. Moreover, by corollary 2.2.17, $\mathcal{O}_{0\, n}$ is generated by μ elements (via f^*). This means the following:

Given $g \in \mathcal{O}_{0\, n}$ we can write

$$g(z) = h_1(f(z))\, e_1(z) + \cdots + h_\mu(f(z))\, e_\mu(z)$$

with h_j and e_j in $\mathcal{O}_{0\, n}$.

We exploit this prepared form of the germ g as follows:

Lemma 2.2.19 *Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ be a holomorphic map germ of finite multiplicity μ at 0. There exist neighborhoods of 0, U in the domain and V in the target, such that all germs appearing in the preparation of all polynomials are defined in U and V.*

Proof: Consider the finite collection of functions: 1, z_k and e_j, for $1 \leq k \leq n$ and $1 \leq j \leq \mu$. Write each one as

$$f^*(h_1(w))\, e_1(z) + \cdots + f^*(h_\mu(w))\, e_\mu(z).$$

Let V be an open set in the target $\mathbb{C}^n$ such that all functions h_ℓ appearing in the preparation of this collection are defined. Let $U \subset f^{-1}(V) \subset \mathbb{C}^n$ be a neighborhood of 0 in which all functions e_j are defined. We now proceed by induction on the degree of the polynomials. If P has degree 0 then $P = c \cdot 1$, $c \in \mathbb{C}$. Any polynomial of degree d can be written as

$$P(z) = \sum z_j Q_j + c \cdot 1$$

where the degree of the polynomials Q_j is smaller than d. Assuming the lemma to hold for the Q_j, it holds also for $z_j Q_j$ and therefore for P.

□

2.3 Relation between $\mathcal{I}$ and μ

In this section we show that the Poincaré Hopf index and the Milnor number coincide. First some definitions.

Let $U \subset \mathbb{C}^n$ be a domain and denote by $\mathcal{O}(U)$ the $\mathbb{C}$-algebra of holomorphic functions defined in U. Let $\mathfrak{I}_f$ be the ideal of $\mathcal{O}(U)$ generated by the components of a holomorphic map $f : U \to \mathbb{C}^n$.

Definition 2.3.1 *The* algebra $\mathcal{Q}_f\,(U)$ *is the quocient $\mathbb{C}$-algebra*

$$\mathcal{O}(U) \,/\, \mathfrak{I}_f.$$

The polynomial subalgebra $\mathcal{Q}_f\,[U]$ *is the image of the polynomial algebra* $\mathbb{C}[z_1, \ldots, z_n]_{|U}$ *by the quocient map* $\mathfrak{q} : \mathcal{O}(U) \to \mathcal{Q}_f\,(U)$.

Suppose we have a holomorphic map germ $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ of finite multiplicity μ at 0. Consider a holomorphic deformation f_λ of f, $\lambda \in \mathbb{C}^m$, $f_0 = f$.

Lemma 2.3.2 *Let* $F : (\mathbb{C}^n \times \mathbb{C}^m, 0) \to (\mathbb{C}^n \times \mathbb{C}^m, 0)$ *be defined by* $F(z, \lambda) = (f_\lambda(z), \lambda)$. *Then the $\mathbb{C}$-algebras $\mathcal{Q}_f$ and $\mathcal{Q}_F$ are isomorphic. Moreover, if* $e_1, \ldots, e_\mu$ *form a basis for* $\mathcal{Q}_f$ *then, they also form a basis for* $\mathcal{Q}_F$.

Proof: Write $F = (F_1, \ldots, F_n, \lambda_1, \ldots, \lambda_m)$ with $F_j = f_{j\,\lambda}$. Then, the ideal generated by the components of F is the same as the ideal $\mathfrak{J}$ generated by $f_1, \ldots, f_n, \lambda_1, \ldots, \lambda_n$. But $\mathcal{O}_{n\times m} / \mathfrak{J} \approx \mathcal{O}_n / \mathfrak{I}_f$ and thus $\mathcal{Q}_F \approx \mathcal{Q}_f$. Suppose now that $e_1, \ldots, e_n$ form a basis for the $\mathbb{C}$-linear space $\mathcal{Q}_f$. Since $\mathcal{Q}_F \approx \mathcal{Q}_f$ these give also a basis for $\mathcal{Q}_F$.

□

Lemma 2.3.3 *There exists a neighborhood $U_1 \subset \mathbb{C}^n$ of 0 such that, for all $|\lambda|$ sufficiently small, the $\mathbb{C}$-linear space generated by the images of $e_1, \ldots, e_\mu$ in the algebra $\mathcal{Q}_{f_\lambda}(U_1)$ contains the polynomial subalgebra $\mathcal{Q}_{f_\lambda}[U_1]$.*

Proof: By lemma 2.2.19 we can find a neighborhood $U_1 \times U_2 \subset \mathbb{C}^n \times \mathbb{C}^m$ of 0 and a neighborhood $V \subset \mathbb{C}^n \times \mathbb{C}^m$ of 0, which we may suppose convex, with $F(U_1 \times U_2) \subset V$, such that every polynomial, when restricted to $U_1 \times U_2$, can be written in the form

$$P(z) = \sum_{j=1}^{\mu} g_j(w, \lambda)\, e_j(z), \qquad w = f_\lambda(z).$$

By lemma 2.1.14 each g_j has an expansion of the form

$$g_j(w, \lambda) = G_j(\lambda) + \sum_{i=1}^{n} w_i\, g_{j\,i}(w, \lambda).$$

Substituting into the expression for P we get

$$P(z) = \sum_{j=1}^{\mu} G_j(\lambda)\, e_j(z) + \sum_{i=1}^{n} w_i\, h_i(z, \lambda), \qquad w = f_\lambda(z).$$

Now, $\sum_{i=1}^{n} f_{i\,\lambda}(z)\, h_i(z, \lambda)$ lies in the ideal $\mathfrak{I}_{f_\lambda}(U_1)$ provided $|\lambda|$ is small enough (require $\lambda \in U_2$). The lemma is proved.

□

With this at hand we have the

Proposition 2.3.4 *Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ be a holomorphic map germ of finite multiplicity μ at 0. Consider a holomorphic deformation f_λ of f, $\lambda \in \mathbb{C}^m$, $f_0 = f$. There exists a neighborhood $U \subset \mathbb{C}^n$ of 0 such that, for $|\lambda|$ sufficiently small, the dimension of the $\mathbb{C}$-linear space $\mathcal{Q}_{f_\lambda}[U]$ is at most μ.*

Proof: By lemma 2.3.3, $\dim_{\mathbb{C}} \mathcal{Q}_{f_\lambda}[U] \le \dim_{\mathbb{C}} \mathcal{Q}_{f_\lambda}(U)$ and, by lemma 2.3.2, $\dim_{\mathbb{C}} \mathcal{Q}_{f_\lambda}(U) \le \mu$.

□

Lemma 2.3.5 *Suppose we have a holomorphic map* $f : U \to \mathbb{C}^n$, $U \subset \mathbb{C}^n$ *a domain, such that* $\dim_{\mathbb{C}} \mathcal{Q}_f\,[U] < \infty$. *Then, each zero of* f *in* U *has finite multiplicity. Moreover, the number of solutions of the equation* $f = 0$ *in* U *(counted without multiplicities) is bounded by* $\dim_{\mathbb{C}} \mathcal{Q}_f\,[U]$.

Proof: Denote by ν the $\dim_{\mathbb{C}} \mathcal{Q}_f\,[U]$ and let $\xi \in U$ be such that $f(\xi) = 0$. Let ℓ_i, $i = 1, \dots, \nu$, be linear functions vanishing at ξ and consider the $\nu + 1$ functions, 1, ℓ_1, $\ell_1\,\ell_2$,..., $\ell_1\,\ell_2 \cdots \ell_\nu$. If $\mathfrak{p}$ is the quocient map $\mathfrak{p} : \mathbb{C}[z_1, \dots, z_n]_{|U} \to \mathcal{Q}_f\,[U]$ then, the classes $\mathfrak{p}(1)$,..., $\mathfrak{p}(\ell_1\,\ell_2 \cdots \ell_\nu)$ are linearly dependent. By repeating the same argument as in the proof of lemma 2.2.4, we conclude that there is an element $u \in \mathcal{O}(U)$, $u(\xi) \neq 0$, such that $u\,\ell_1\,\ell_2 \cdots \ell_\nu \in \mathfrak{I}_f(U)$. Then,

$$u^{-1}\,(u\,\ell_1\,\ell_2 \cdots \ell_\nu) = \ell_1\,\ell_2 \cdots \ell_\nu \in \mathfrak{I}_{\xi\,f}.$$

We've shown that any collection of ν linear functions in $\mathfrak{M}_{\xi\,n}$ have their product in $\mathfrak{I}_{\xi\,f}$. Hence, $\mathfrak{M}^{\nu}_{\xi\,n} \subset \mathfrak{I}_{\xi\,f}$ and therefore

$$\dim_{\mathbb{C}}\, \mathcal{O}_\xi \,/\, \mathfrak{I}_{\xi\,f} \leq \dim_{\mathbb{C}}\, \mathcal{O}_\xi \,/\, \mathfrak{M}^{\nu}_{\xi\,n} < \infty.$$

This shows the first part of the lemma. Suppose now we had $\nu + 1$ solutions in U of the equation $f = 0$, say $\xi_0, \dots, \xi_\nu$. For each $j = 0, \dots, \nu$ choose a polynomial P_j such that

$$P_j(\xi_i) = \begin{cases} 1 & \text{, if } i = j \\ 0 & \text{, if } i \neq j. \end{cases}$$

Consider a linear combination of the P_j satisfying:

$$c_0\,P_0 + \cdots + c_\nu\,P_\nu = 0.$$

Evaluating at ξ_i gives $c_i = 0$ and hence the classes $\mathfrak{p}(P_j)$, $0 \leq j \leq \nu$, are linearly independent in $\mathcal{Q}_f\,[U]$, which is an absurd.

□

Consider a holomorphic map $f : U \to \mathbb{C}^n$, $U \subset \mathbb{C}^n$ a domain, and suppose that $\xi_1, \dots, \xi_k$ are all the solutions of the equation $f = 0$ in U. Look at its germs at the points $\xi_1, \dots, \xi_k$ and consider the corresponding local algebras $\mathcal{Q}_{\xi_i\,f}$. The sum

$$\bigoplus_{i=1}^{k} \mathcal{Q}_{\xi_i\,f}$$

is called the *multilocal algebra* of f in U. We define a homomorphism of $\mathbb{C}$-algebras

$$\aleph : \mathcal{O}(U) \to \bigoplus_{i=1}^{k} \mathcal{Q}_{\xi_i f}$$

as follows: given $g \in \mathcal{O}(U)$ take its germs at the points ξ_i, g_{ξ_i}, and look at their images $\widetilde{g_{\xi_i}} \in \mathcal{Q}_{\xi_i f}$. In other words, $\aleph(g) = (\widetilde{g_{\xi_1}}, \ldots, \widetilde{g_{\xi_k}})$.

Before exploiting $\aleph$ we introduce some notation. Let $g \in \mathcal{O}(U)$ and $\xi \in U$. The Taylor polynomial of degree ℓ of g at ξ is noted $T^{\ell}_{\xi} g$.

Lemma 2.3.6 *Given a finite number of distinct points in U, say $\xi_1, \ldots, \xi_k$, and a polynomial P_i of degree d_i, centered at ξ_i, there exists a polynomial Q such that $T^{d_i}_{\xi_i} Q = P_i$.*

Proof: Let $Q = Q_0 + Q_1 + \cdots + Q_N$, a sum of homogeneous polynomials whose coefficients are to be determined. We first solve the system

$$\begin{aligned} Q(\xi_1) &= P_1(\xi_1) \\ &\vdots \\ Q(\xi_k) &= P_k(\xi_k) \end{aligned} \tag{$\star_0$}$$

which is possible if N is large enough. Next we have the systems

$$\begin{aligned} \frac{\partial Q}{\partial z_j}(\xi_1) &= \frac{\partial P_1}{\partial z_j}(\xi_1) \\ &\vdots \\ \frac{\partial Q}{\partial z_j}(\xi_k) &= \frac{\partial P_k}{\partial z_j}(\xi_k) \end{aligned} \tag{$\star_1$}$$

By enlarging N we can solve ($\star_1$) without interfering with the solution of ($\star_0$). Continuing this way we obtain the polynomial Q.

□

We have the

Lemma 2.3.7 *Suppose* $\dim_{\mathbb{C}} \mathcal{Q}_f[U] < \infty$. *Then*

$$\aleph\left(\mathbb{C}[z_1, \ldots, z_n]_{|U}\right) = \bigoplus_{i=1}^{k} \mathcal{Q}_{\xi_i f}.$$

Proof: By lemma 2.3.5 the number of solutions in U of the equation $f = 0$ is finite, say $\xi_1, \ldots, \xi_k$, and each solution ξ_i is of finite multiplicity μ_i. If $g \in \mathcal{O}(U)$ then, g and its Taylor polynomial of degree μ_i at ξ_i, $T^{\mu_i}_{\xi_i} g$, are mapped into the same element of $\mathcal{Q}_{\xi_i f}$. Choose a polynomial Q such that $T_{\xi_i}^{\mu_i} Q = T_{\xi_i}^{\mu_i} g$ (this possible by lemma 2.3.6). Then $\aleph(Q) = \aleph(g)$ and the lemma is proved.

$\square$

We can now prove the

Proposition 2.3.8 *The number of solutions in U, counting multiplicities, of the equation $f = 0$ is bounded by* $\dim_{\mathbb{C}} \mathcal{Q}_f[U]$.

Proof: Write $f = (f_1, \ldots, f_n)$. Then $\aleph(f_j) = 0$ and thus the ideal $\mathfrak{I}_f(U)$ is mapped to 0 by $\aleph$. We then have an induced homomorphism of $\mathbb{C}$-algebras

$$\widetilde{\aleph} : \mathcal{Q}_f[U] \to \bigoplus_{i=1}^{k} \mathcal{Q}_{\xi_i f}$$

which is surjective by the previous lemma 2.3.7. Hence,

$$\dim_{\mathbb{C}} \mathcal{Q}_f[U] \geq \sum_{i=1}^{k} \dim_{\mathbb{C}} \mathcal{Q}_{\xi_i f}.$$

$\square$

Proposition 2.3.9 *Suppose $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ is a holomorphic map germ such that $\mu_0(f) < \infty$. Then $\mu_0(f) \geq \mathcal{I}_0(f)$.*

Proof: By theorem 2.2.11, 0 is isolated in $f^{-1}(0)$ and by proposition 2.1.15, $\mathcal{I}_0(f)$ is the number of solutions of the equation $f_\lambda = f - \lambda = 0$, λ a regular value of f with $|\lambda| << 1$, in a small neighborhood U of 0. By lemma 2.3.5

$$\dim_{\mathbb{C}} \mathcal{Q}_{f_\lambda}[U] \geq \mathcal{I}_0(f)$$

and by proposition 2.3.4, $\dim_{\mathbb{C}} \mathcal{Q}_{f_\lambda}[U]$ is finite and

$$\mu_0(f) \geq \dim_{\mathbb{C}} \mathcal{Q}_{f_\lambda}[U].$$

The proposition is proved.

$\square$

We finally have the

Theorem 2.3.10 *Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ be a holomorphic map germ. If $\mu_0(f)$ is finite, then $\mu_0(f) = \mathcal{I}_0(f)$.*

Proof: This is now a matter of putting together all bits and pieces we've deduced so far. We start by considering a Pham map $\Upsilon^{[\mu+1]}$, where $\mu = \mu_0(f)$. By proposition 2.2.10 the deformation $\Upsilon_\lambda^{[\mu+1]} = \Upsilon^{[\mu+1]} + \lambda f$, λ in a small neighborhood of 0 in $\mathbb{C}$, is A-equivalent to f.

By proposition 2.1.20

$$\mathcal{I}_0\,(\Upsilon_\lambda^{[\mu+1]}) = \mathcal{I}_0\,(f)$$

and by propositions 2.2.6 and 2.2.10

$$\mu_0(\Upsilon_\lambda^{[\mu+1]}) = \mu_0(f).$$

We now exploit the properties of the Pham map and of its deformation. Fix a ball $B_\epsilon(0)$ and a value of the parameter λ in such a way that proposition 2.3.4 holds for $\Upsilon_\lambda^{[\mu+1]}$. Let $\{\xi_i\}$ be the solutions in $B_\epsilon(0)$ of the equation $\Upsilon_\lambda^{[\mu+1]} = 0$.

By proposition 2.3.4,

$$\mu_0(\Upsilon^{[\mu+1]}) \geq \dim_{\mathbb{C}} \mathcal{Q}_{\Upsilon_\lambda^{[\mu+1]}}\,[B_\epsilon(0)].$$

By proposition 2.3.8,

$$\dim_{\mathbb{C}} \mathcal{Q}_{\Upsilon_\lambda^{[\mu+1]}}\,[B_\epsilon(0)] \geq \sum_i \mu_{\xi_i}(\Upsilon_\lambda^{[\mu+1]}).$$

By proposition 2.3.9,

$$\mu_{\xi_i}(\Upsilon_\lambda^{[\mu+1]}) \geq \mathcal{I}_{\xi_i}\,(\Upsilon_\lambda^{[\mu+1]}).$$

By theorem 2.1.17,

$$\sum_i \mathcal{I}_{\xi_i}\,(\Upsilon_\lambda^{[\mu+1]}) = \deg \frac{\Upsilon_\lambda^{[\mu+1]}}{|\Upsilon_\lambda^{[\mu+1]}|}$$

where this last map is restricted to the sphere $\partial B_\epsilon(0)$.

By theorem 2.1.18,

$$\deg \frac{\Upsilon_\lambda^{[\mu+1]}}{|\Upsilon_\lambda^{[\mu+1]}|} = \deg \frac{\Upsilon^{[\mu+1]}}{|\Upsilon^{[\mu+1]}|} = \mathcal{I}_0\,(\Upsilon^{[\mu+1]})$$

By lemma 2.2.9,

$$\mathcal{I}_0\,(\Upsilon^{[\mu+1]}) = \mu_0(\Upsilon^{[\mu+1]}).$$

It follows that

$$\sum_i \mu_{\xi_i}(\Upsilon_\lambda^{[\mu+1]}) = \sum_i \mathcal{I}_{\xi_i}(\Upsilon_\lambda^{[\mu+1]}).$$

Since all terms involved are positive and

$$\mu_{\xi_i}(\Upsilon_\lambda^{[\mu+1]}) \geq \mathcal{I}_{\xi_i}(\Upsilon_\lambda^{[\mu+1]}),$$

we conclude

$$\mu_{\xi_i}(\Upsilon_\lambda^{[\mu+1]}) = \mathcal{I}_{\xi_i}(\Upsilon_\lambda^{[\mu+1]}) \quad \forall\, i.$$

But 0 is one of the solutions ξ_i of the equation $\Upsilon_\lambda^{[\mu+1]} = 0$ and thus

$$\mu_0(f) = \mu_0(\Upsilon_\lambda^{[\mu+1]}) = \mathcal{I}_0(\Upsilon_\lambda^{[\mu+1]}) = \mathcal{I}_0(f).$$

The theorem is proved.

□

Chapter 3

Grothendieck residues

In this chapter we introduce the concept of point residue due to A. Grothendieck. It embodies the Poincaré Hopf index, the Milnor number, the intersection number of n divisors in $\mathbb{C}^n$, which intersect properly, and has many uses in deep results such as the Baum-Bott theorem, which is a generalization of both the Poincaré Hopf theorem and the Gauss Bonnet theorem in the complex realm. We hope the reader will appreciate such a mathematical construction.

3.1 The Trace map

In this section we prove the *Trace theorem*, which is a basic result in the understanding of point residues and has its origins in a theorem of Abel. The reference for it is the work of P. Griffiths in [Gr]. The reader is assumed to have some familiarity with differential forms.

We start by looking at a holomorphic map $f : U \subset \mathbb{C}^n \to \mathbb{C}^n$, $f(0) = 0$, with finite multiplicity μ at 0. By theorems 2.1.18 and 2.3.10 we see that f satisfies the following property: there is a connected open neighborhood V of 0 such that, for $\zeta \in V$, $f^{-1}(\zeta)$ is a finite set and the sum of the multiplicities

$$\sum_{\xi \in f^{-1}(\zeta)} \mu_\xi(f - \zeta) = \mu$$

Redefining $U = f^{-1}(V)$ we have that $f : U \to V$ satisfies:
(i) f is surjective.
(ii) f is open.
(iii) f is proper.

(iv) for $\zeta \in V$, $f^{-1}(\zeta)$ is a finite set and the sum of the multiplicities of the zeros of the map $f - \zeta$ is constant throughout V.

Such a map is called a *finite map*. This is equivalent to saying that $f : U \to V$ is a ramified holomorphic covering of degree μ.

Let $f : U \to V$ be as above and η a holomorphic n-form on U, $\eta = g(z)\, dz_1 \wedge \cdots \wedge dz_n$, $g \in \mathcal{O}(U)$.

Definition 3.1.1 *The* trace *or* push forward *of η by f, noted $f_!(\eta)$, is the holomorphic n-form defined on the open set $V_{\mathrm{reg}} \subset V$ of regular values of f, obtained by the following procedure:*

Let ζ be a regular value of f and $f^{-1}(\zeta) = \{\xi_1, \ldots, \xi_\mu\}$. Given an n-vector

$$(v_1, \ldots, v_n) \in \underbrace{\mathbb{C}^n \times \cdots \times \mathbb{C}^n}_{\text{n factors}},$$

for each component $v_j \in \mathbb{C}^n$ there is a unique vector $u_{i\,j} \in \mathbb{C}^n$ such that $v_j = f'(\xi_i).u_{i\,j}$. Set

$$f_!(\eta)_{\,\zeta}.(v_1, \ldots, v_n) = \sum_{i=1}^{\mu} \eta_{\,\xi_i}.(u_{i\,1}, \ldots, u_{i\,n}).$$

This amounts to do the following: if ζ is a regular value of f and $f^{-1}(\zeta) = \{\xi_1, \ldots, \xi_\mu\}$, then there is a neighborhood V_ζ of ζ and neighborhoods U_{ξ_i} of ξ_i such that $f_{|U_{\xi_i}} : U_{\xi_i} \to V_\zeta$ is a biholomorphism. Let f_i^{-1} denote the inverse maps $(f_{|U_{\xi_i}})^{-1}$. Then

$$f_!(\eta)_{\,|V_\zeta} = \sum_{i=1}^{\mu} (f_i^{-1})^* \, \eta_{\,|U_{\xi_i}}.$$

Let us derive a local expression for $f_!(\eta)$. In the target V we take coordinates $w = (w_1, \ldots, w_n)$, write $f = (f_1, \ldots, f_n)$,

$$\eta = g(z)\, dz_1 \wedge \cdots \wedge dz_n$$

and

$$f_!(\eta) = \mathrm{tr}(w)\, dw_1 \wedge \cdots \wedge dw_n.$$

If $f_{|U_{\xi_i}} : U_{\xi_i} \to V_\zeta$ is as above we take coordinates $(f_1, \ldots, f_n)$ in U_{ξ_i}. Denoting by $g_i\, df_1 \wedge \cdots \wedge df_n$ the expression of η in these coordinates we have that, for $p \in U_{\xi_i}$,

$$(\eta_{\,|U_{\xi_i}})_p = g_i(f_1(p), \ldots, f_n(p))\, (df_1)_p \wedge \cdots \wedge (df_n)_p.$$

Now, if $\{(e_1)_p, \ldots, (e_n)_p\}$ is the basis dual to $\{(df_1)_p, \ldots, (df_n)_p\}$ then,

$$(\eta_{|U_{\xi_i}})_p.((e_1)_p, \ldots, (e_n)_p) = g_i(f_1(p), \ldots, f_n(p))$$

and hence

$$\mathrm{tr}_{|V_\zeta}(w) = \sum_{i=1}^{\mu} g_i(w).$$

Remark 4 In U_{ξ_i},

$$g_i = \frac{g}{\det Jf} \quad \text{where} \quad Jf \ = \ \left(\frac{\partial f_i}{\partial z_j}\right)_{1\leq i,j\leq n}.$$

We've shown that the function

$$\mathrm{tr}(\eta) : V_{\mathrm{reg}} \to \mathbb{C}$$

is holomorphic. The *Trace theorem* asserts that this function admits a holomorphic extension to an open neighborhood of 0 in V. There are two proofs of this fact, one makes use of Remmert's theorem and of Hartogs' extension theorem (see [Gr]) and the other, given in [A-V-GZ], uses Cauchy's integral formula and hence exhibits an integral representation of $\mathrm{tr}(\eta)$. We follow this last one since it will be very useful in the definition of the residue. But before the proof we need some topological preliminaries (see [D-N-F]).

Consider the maps

$$|f| : U \longrightarrow \mathbb{R}^n$$

$$|f|(z) = (|f_1(z)|, \ldots, |f_n(z)|)$$

and

$$|f-w| : U \longrightarrow \mathbb{R}^n, \quad w \in V$$

$$|f-w|(z) = (|f_1(z)-w_1|, \ldots, |f_n(z)-w_n|)$$

Let $\mathbb{D}(0,\epsilon)$ be a polydisc of multiradius $\epsilon = (\epsilon_1, \ldots, \epsilon_n)$, centered at $0 \in V$ and $\mathbb{T}_\epsilon$ its distinguished boundary $|w_1| = \epsilon_1, \ldots, |w_n| = \epsilon_n$. Set $\Gamma_\epsilon = |f|^{-1}(\epsilon)$. Γ_ϵ is a compact real n-cycle in U, which is also a smooth submanifold of U if we take ϵ a regular value of $|f|$. We have as coordinate functions for Γ_ϵ, in an open dense set of Γ_ϵ, the arguments $\arg f_i$ and, from now on, we adopt as orientation for Γ_ϵ the one determined by $d\arg f_1 \wedge \cdots \wedge d\arg f_n$.

Fix $w \in V$ and let $\rho = (\rho_1, \ldots, \rho_n)$, $\rho_i > 0$, be a regular value of the map $|f - w|$. Then,

$$\Gamma_{w,\rho} = \{z \in U : |f_1(z) - w_1| = \rho_1, \ldots, |f_n(z) - w_n| = \rho_n\} \subset U$$

is a smooth real submanifold of dimension n. The orientation for $\Gamma_{w,\rho}$ is obtained in the same way as that for Γ_ϵ.

Now let ζ be a regular value of f and $f^{-1}(\zeta) = \{\xi_1, \ldots, \xi_\mu\}$. By choosing ρ sufficiently small so that the torus

$$\{u \in V : |u_1 - \zeta_1| = \rho_1, \ldots, |u_n - \zeta_n| = \rho_n\}$$

is contained in V_ζ we have that $\Gamma_{\zeta,\rho}$ consists of precisely μ tori $\mathbb{T}_{\zeta\, i}$, corresponding to ξ_i. Consider the meromorphic n-form on U, depending on $w \in V$,

$$\eta_w = \frac{\eta}{\prod_{j=1}^{n} (f_j - w_j)}.$$

Lemma 3.1.2 *Let ζ be a regular value of f. There exists a neighborhood $W_\zeta \subset V_\zeta$ of ζ such that, for $w \in W_\zeta$,*

$$\operatorname{tr}(\eta)(w) = \left(\frac{1}{2\pi \boldsymbol{i}}\right)^n \int_{\Gamma_{w,\rho}} \eta_w .$$

Proof: Choose W_ζ in such a way that $\Gamma_{w,\rho} \subset f^{-1}(V_\zeta)$. We've seen that, in V_ζ, the local expression of $\operatorname{tr}(\eta)$ is $\sum_{i=1}^{\mu} g_i(w)$. Now, by Cauchy's integral formula

$$g_i(w) = \left(\frac{1}{2\pi \boldsymbol{i}}\right)^n \int_{\mathbb{T}_{w\,i}} \frac{g_i(f_1, \ldots, f_n)\, df_1 \wedge \cdots \wedge df_n}{\prod_{j=1}^{n} (f_j - w_j)}$$

and thus

$$\operatorname{tr}(\eta)(w) =$$

$$\left(\frac{1}{2\pi \boldsymbol{i}}\right)^n \sum_{i=1}^{\mu} \int_{\mathbb{T}_{w\,i}} \frac{g_i(f_1, \ldots, f_n)\, df_1 \wedge \cdots \wedge df_n}{\prod_{j=1}^{n} (f_j - w_j)} =$$

$$\left(\frac{1}{2\pi \boldsymbol{i}}\right)^n \int_{\Gamma_{w,\rho}} \eta_w .$$

□

Lemma 3.1.3 *Let ζ be a regular value of f sufficiently close to $0 \in V$ and ρ appropiately small. Then the cycles Γ_ϵ and $\Gamma_{\zeta,\rho}$ are homologous in $U \setminus f^{-1}(\zeta)$ and so $[\Gamma_\epsilon] = [\Gamma_{\zeta,\rho}] \in H_n(U \setminus f^{-1}(\zeta); \mathbb{Z})$.*

Proof: The map $F(z,t) = |f - t\zeta|$, $0 \leq t \leq 1$, induces a smooth homotopy $\Gamma_{t\zeta,\epsilon}$ between $\Gamma_\epsilon = \Gamma_{0,\epsilon}$ and $\Gamma_{\zeta,\epsilon}$, provided $|\zeta| << 1$. On the other hand, for ρ sufficiently small, $\Gamma_t = \Gamma_{\zeta, t\rho+\epsilon}$ exhibits a smooth homotopy between $\Gamma_{\zeta,\epsilon}$ and $\Gamma_{\zeta,\rho+\epsilon}$. Now consider the map $G(z,t) = |f - \zeta| - t\epsilon$. If ρ is a regular value of G such that $\Gamma_{\zeta,\rho}$ consists of μ tori, then $G^{-1}(\rho)$ is a submanifold Δ of real dimension $n+1$ of $U \times \mathbb{R}$. The projection $\pi : U \times \mathbb{R} \to U$ sends $\Delta \cap (U \times [0,1])$ over an n+1-cycle whose boundary is $\Gamma_{\zeta,\epsilon+\rho} \cup \Gamma_{\zeta,\rho}$. Thus, $\Gamma_{\zeta,\epsilon+\rho}$ is homologous to $\Gamma_{\zeta,\rho}$. Since the homotopies above are smooth, we have that Γ_ϵ is homologous to $\Gamma_{\zeta,\rho}$. Noticing that all the procedures were carried out in $U \setminus f^{-1}(\zeta)$ we have the assertion of the lemma.

□

We then have the

Theorem 3.1.4 (Trace theorem) *The holomorphic function*

$$\mathrm{tr}(\eta) : V_{\mathrm{reg}} \to \mathbb{C}$$

admits a holomorphic extension to an open neighborhood of 0 in V.

Proof: We will show that if ϵ is a regular value of the map $|f|$, sufficiently close to 0 and such that both $\mathbb{D}(0,\epsilon)$ and $\mathbb{T}_\epsilon$ are contained in V then, for $w \in \mathbb{D}(0,\epsilon)$, the function

$$\Psi(w) = \left(\frac{1}{2\pi \boldsymbol{i}}\right)^n \int_{\Gamma_\epsilon} \frac{\eta}{\prod_{j=1}^{n} (f_j - w_j)}$$

is the desired extension of $\mathrm{tr}(\eta)$. Write $\eta = g(z)\, dz_1 \wedge \cdots \wedge dz_n$. Then

$$\Psi(w) = \left(\frac{1}{2\pi \boldsymbol{i}}\right)^n \int_{\Gamma_\epsilon} \frac{g(z)\, dz_1 \wedge \cdots \wedge dz_n}{\prod_{j=1}^{n} (f_j - w_j)}$$

is holomorphic for $w \in \mathbb{D}(0,\epsilon)$. Let ζ be a regular value of f and ρ small enough so that lemmas 3.1.2 and 3.1.3 hold. Then,

$$\mathrm{tr}(\eta)(w) = \left(\frac{1}{2\pi \boldsymbol{i}}\right)^n \int_{\Gamma_{w,\rho}} \eta_w$$

in a small neighborhood of W_ζ contained in $\mathbb{D}(0,\epsilon)$ by 3.1.2. By 3.1.3,

$$\mathrm{tr}(\eta)(w) = \left(\frac{1}{2\pi \boldsymbol{i}}\right)^n \int_{\Gamma_{w,\rho}} \eta_w = \left(\frac{1}{2\pi \boldsymbol{i}}\right)^n \int_{\Gamma_\epsilon} \eta_w = \Psi(w)$$

in W_ζ. The theorem is proved.

□

3.2 The Residue

Let $f = (f_1, \ldots, f_n) : U \to V$ be a finite holomorphic map of multiplicity μ and $g \in \mathcal{O}(U)$. Suppose ζ is a regular value of f and let $f^{-1}(\zeta) = \{\xi_1, \ldots, \xi_\mu\}$.

Consider the sum

$$\sum_{i=1}^{\mu} \frac{g(\xi_i)}{\det Jf(\xi_i)}$$

where

$$Jf(\xi_i) = \left(\frac{\partial f_i}{\partial z_j}(\xi_i)\right)_{1\le i,j\le n}.$$

Definition 3.2.1 *The* residue *at* 0 *of* g *relative to* f *is the limit*

$$\mathrm{Res}_0(g,f) = \lim_{\zeta\to 0} \sum_{i=1}^{\mu} \frac{g(\xi_i)}{\det Jf(\xi_i)}.$$

Of course we must show the limit exists. This is a job for the Trace theorem 3.1.4.

Theorem 3.2.2 *Let* $\epsilon = (\epsilon_1, \ldots, \epsilon_n)$, $\epsilon_i > 0$, *and consider the real n-cycle* $\Gamma_\epsilon = \{z \in U : |f_i(z)| = \epsilon_i, 1 \le i \le n\}$ *with orientation prescribed by the n-form* $d\arg f_1 \wedge \cdots \wedge d\arg f_n$. *If* ϵ *is sufficiently close to* 0 *then,*

$$\mathrm{Res}_0(g,f) = \left(\frac{1}{2\pi \boldsymbol{i}}\right)^n \int_{\Gamma_\epsilon} \frac{g\, dz_1 \wedge \cdots \wedge dz_n}{f_1 \cdots f_n}.$$

Proof: Consider the holomorphic n-form $\eta = g\, dz_1 \wedge \cdots \wedge dz_n$. In an open and dense subset of U we have

$$g\, dz_1 \wedge \cdots \wedge dz_n = \frac{g}{\det Jf}\, df_1 \wedge \cdots \wedge df_n$$

and, due to the manner in which the cycle Γ_ϵ is oriented,

$$\left(\frac{1}{2\pi\boldsymbol{i}}\right)^n \int_{\Gamma_\epsilon} \frac{g\,dz_1 \wedge \cdots \wedge dz_n}{f_1 \cdots f_n} = \left(\frac{1}{2\pi\boldsymbol{i}}\right)^n \int_{\Gamma_\epsilon} \frac{g}{\det Jf} \frac{df_1 \wedge \cdots \wedge df_n}{f_1 \cdots f_n}$$

Over the open and dense set of regular values of f, the trace of the form η is, by remark 4,

$$\mathrm{tr}(\eta)(w) = \sum_{i=1}^{\mu} \frac{g(f_i^{-1}(w))}{\det Jf(f_i^{-1}(w))}.$$

By theorem 3.1.4,

$$\mathrm{tr}(\eta)(w) = \left(\frac{1}{2\pi\boldsymbol{i}}\right)^n \int_{\Gamma_\epsilon} \frac{g\,dz_1 \wedge \cdots \wedge dz_n}{\prod_{j=1}^{n} (f_j - w_j)}.$$

It follows that

$$\lim_{w \to 0} \mathrm{tr}(\eta)(w) = \left(\frac{1}{2\pi\boldsymbol{i}}\right)^n \int_{\Gamma_\epsilon} \frac{g\,dz_1 \wedge \cdots \wedge dz_n}{f_1 \cdots f_n}.$$

□

Exercise 4 Show that, when $n = 1$ this reduces to the classical residue for meromorphic functions, introduced by Cauchy.

Properties of the Residue

Property 1 *If $a, b \in \mathbb{C}$ and $g, h \in \mathcal{O}(U)$ then,*

$$\mathrm{Res}_0(ag + bh, f) = a\,\mathrm{Res}_0(g, f) + b\,\mathrm{Res}_0(h, f).$$

Moreover, $\mathrm{Res}_0(g, f)$ *is* alternating *in the components $f_1, \ldots, f_n$ of f due to the orientation prescribed for the cycle Γ_ϵ.*

Property 2 $\mathrm{Res}_0(\det Jf, f) = \mu_0(f) = \mathcal{I}_0(f)$.

To see this simply note that, if ζ is a regular value of f and $f^{-1}(\zeta) = \{\xi_1, \ldots, \xi_\mu\}$, then the sum $\displaystyle\sum_{i=1}^{\mu} \frac{\det Jf(\xi_i)}{\det Jf(\xi_i)}$ is constant and equal to $\mu_0(f)$. Hence,

$$\mathrm{Res}_0(\det Jf, f) = \lim_{\zeta \to 0} \sum_{i=1}^{\mu} \frac{\det Jf(\xi_i)}{\det Jf(\xi_i)} = \mu_0(f).$$

Property 3 *If f is a biholomorphism, then*

$$\operatorname{Res}_0(g,f) = \frac{g(0)}{\det Jf(0)}.$$

In this case every point ζ in V is a regular value of f and $f^{-1}(\zeta) = \{\xi\}$. Thus,

$$\operatorname{Res}_0(g,f) = \lim_{\zeta\to 0} \frac{g(\xi)}{\det Jf(\xi)} = \lim_{\xi\to 0} \frac{g(\xi)}{\det Jf(\xi)} = \frac{g(0)}{\det Jf(0)}.$$

Property 4 *If $g \in \mathfrak{I}_f$, then* $\operatorname{Res}_0(g,f) = 0$.

Write $g = h_1 f_1 + \cdots + h_n f_n$. By property 1 it's enough to show $\operatorname{Res}_0(h_1 f_1, f) = 0$. Look at the n-form

$$\omega = \frac{h_1 f_1\, dz_1 \wedge \cdots \wedge dz_n}{f_1 \cdots f_n} = \frac{h_1\, dz_1 \wedge \cdots \wedge dz_n}{f_2 \cdots f_n}.$$

Let $D_i = \{z \in U : f_i(z) = 0\}$. ω is holomorphic in the open set $U' = U \setminus (D_2 \cup \cdots \cup D_n) \supset U \setminus (D_1 \cup \cdots \cup D_n)$. The chain $\Delta_\epsilon = \{z \in U : |f_1(z)| \leq \epsilon_1, |f_i(z)| = \epsilon_i, 2 \leq i \leq n\}$ is contained in U' and $\partial\Delta_\epsilon = \pm\Gamma_\epsilon$. By Stokes theorem

$$\operatorname{Res}_0(h_1 f_1, f) = \int_{\Gamma_\epsilon} \omega = \pm \int_{\Delta_\epsilon} d\omega = 0.$$

We then have

Theorem 3.2.3 $\det Jf \notin \mathfrak{I}_f$.

Proof: By property 2, $\operatorname{Res}_0(\det Jf, f) = \mu_0(f) \neq 0$. Hence, by property 4, $\det Jf \notin \mathfrak{I}_f$.

□

Exercise 5 Show that, if $\mu_0(f) = 1$ then f is a biholomorphism.

Property 5 (Transformation law) *Suppose $g : U \to V$ is a holomorphic map with $g^{-1}(0) = \{0\}$ and that $g(z) = A(z)f(z)$, where $A(z) = (a_{ij}(z))$ is a matrix with holomorphic entries. Then,*

$$\operatorname{Res}_0(h,f) = \operatorname{Res}_0(h \det A, g).$$

The condition $g(z) = A(z)f(z)$ tells us that $\mathfrak{T}_g \subseteq \mathfrak{T}_f$. We start by proving the transformation law in case f and g are A-equivalent, so $\mathfrak{T}_f = \mathfrak{T}_g$. Consider the holomorphic deformation $f_\lambda = f - \lambda$ and the corresponding deformation of g, $g_\lambda = A(z)f_\lambda$. By shrinking V, if necessary, we have that g_λ and $f - \lambda$ have the same zeros $\xi_1, \ldots, \xi_\mu$, for λ a regular value of f. At each point ξ_i, $Jg_\lambda(\xi_i) = A(\xi_i) \cdot Jf(\xi_i)$. Hence,

$$\det Jg_\lambda(\xi_i) = \det A(\xi_i) \cdot \det Jf(\xi_i)$$

and so

$$\sum_{i=1}^{\mu} \frac{h(\xi_i)}{\det Jf(\xi_i)} = \sum_{i=1}^{\mu} \frac{h(\xi_i)\, \det A(\xi_i)}{\det Jg_\lambda(\xi_i)},$$

which gives

$$\operatorname{Res}_0(h, f) = \lim_{\lambda \to 0} \sum_{i=1}^{\mu} \frac{h(\xi_i)}{\det Jf(\xi_i)} =$$

$$\lim_{\lambda \to 0} \sum_{i=1}^{\mu} \frac{h(\xi_i)\, \det A(\xi_i)}{\det Jg_\lambda(\xi_i)} = \operatorname{Res}_0(h \det A, g).$$

Now for the general case. Choose a smooth family of holomorphic matrices $A_t(z)$ with $A_0(z) = A(z)$ and $\det A_t(0) \neq 0$ for $t \neq 0$. Put $g_t(z) = A_t(z)f(z)$. Then, for $t \neq 0$, g_t and f are A-equivalent and by the previous case

$$\operatorname{Res}_0(h, f) = \operatorname{Res}_0(h \det A_t, g_t)\,, \qquad \forall\, t \neq 0.$$

But

$$\operatorname{Res}_0(h \det A_t, g_t) = \left(\frac{1}{2\pi \boldsymbol{i}}\right)^n \int_{\Gamma_{t\,\epsilon}} \frac{h \det A_t\, dz_1 \wedge \cdots \wedge dz_n}{g_{t\,1} \cdots g_{t\,n}}$$

where the n-cycle $\Gamma_{t\,\epsilon} = \{z : |g_t(z)| = \epsilon\}$. By choosing ϵ a regular value of $|g| = |g_0|$, we have that $\Gamma_{0\,\epsilon}$ is a smooth manifold and that ϵ is a regular value of $|g_t|$ for $|t| << 1$. Hence, $\Gamma_{t\,\epsilon}$ can be realized as a small deformation of the zero section $\Gamma_{0\,\epsilon}$ in a tubular neighborhood of $\Gamma_{0\,\epsilon}$. It follows that $\Gamma_{0\,\epsilon}$ and $\Gamma_{t\,\epsilon}$ are homologous and thus

$$\left(\frac{1}{2\pi \boldsymbol{i}}\right)^n \int_{\Gamma_{t\,\epsilon}} \frac{h \det A_t\, dz_1 \wedge \cdots \wedge dz_n}{g_{t\,1} \cdots g_{t\,n}} =$$

$$\left(\frac{1}{2\pi \boldsymbol{i}}\right)^n \int_{\Gamma_{0\,\epsilon}} \frac{h \det A_t\, dz_1 \wedge \cdots \wedge dz_n}{g_{t\,1} \cdots g_{t\,n}}$$

so that

$$\operatorname{Res}_0(h,f) = \lim_{t\to 0} \operatorname{Res}_0(h \det A_t, g_t) =$$

$$\lim_{t\to 0} \left(\frac{1}{2\pi \boldsymbol{i}}\right)^n \int_{\Gamma_{0\,\epsilon}} \frac{h \det A_t\, dz_1 \wedge \cdots \wedge dz_n}{g_{t\,1} \cdots g_{t\,n}} = \operatorname{Res}_0(h \det A, g).$$

3.3 Local duality

This is perhaps the most interesting property of the residue and deserves a special treatment. Let $f : U \to V$, $f^{-1}(0) = \{0\}$, be as before and consider its local algebra $\mathcal{Q}_f$ at $0 \in \mathbb{C}^n$. By property 1, $\operatorname{Res}_0(g,f)$ is linear in g and, by property 4, $\operatorname{Res}_0(g,f)$ depends only on the class $\widetilde{g}$ of g in $\mathcal{Q}_f$. Moreover, by theorem 3.2.3, the class of $\det Jf$ defines a nonzero element of $\mathcal{Q}_f$. Therefore, the residue induces a $\mathbb{C}$-linear functional,

$$\mathcal{R}es_{\,0\,f} : \mathcal{Q}_f \longrightarrow \mathbb{C}$$

$$\widetilde{g} \longmapsto \operatorname{Res}_0(g,f)$$

which in turn induces a $\mathbb{C}$-bilinear form

$$\mathfrak{B}_{0\,f} : \mathcal{Q}_f \times \mathcal{Q}_f \longrightarrow \mathbb{C}$$

$$(\widetilde{g}, \widetilde{h}) \longmapsto \mathcal{R}es_{\,0\,f}(\widetilde{g}\,\widetilde{h}).$$

We now present the Local duality theorem. For a smooth version of this result we refer the reader to [E-L].

Theorem 3.3.1 (Local duality) *The bilinear form*

$$\mathfrak{B}_{0\,f} : \mathcal{Q}_f \times \mathcal{Q}_f \longrightarrow \mathbb{C}$$

is nondegenerate.

Proof: (we follow [G-H]) This assertion can be rephrased as: if $\operatorname{Res}_0(g\,h,f) = 0$ for all $h \in \mathcal{O}_{0\,n}$, then $g \in \mathfrak{I}_f$. Also, the fact that the map germ f is finite is equivalent to the following property of its components: the germs $f_1, \ldots, f_n \in \mathcal{O}_{0\,n}$ form a *regular sequence* (see [Gu]). This means that

$$f_i \quad \text{is not a zero divisor in} \quad \mathcal{O}_{0\,n}/\langle f_1, \ldots, f_{i-1}\rangle_{\mathcal{O}_{0\,n}}, \quad 1 \le i \le n.$$

By Hilbert's zero-theorem we know that there exist $m_1, \ldots, m_n$ such that $z_i^{m_i} \in \mathfrak{I}_f$, $1 \le i \le n$. Consider then the Pham map

$$\Upsilon(z) = (z_1^{m_1+1}, \ldots, z_n^{m_n+1}).$$

Lemma 3.3.2 $\mathfrak{B}_{0\,\Upsilon}$ *is nondegenerate.*

Proof: This is done by direct calculation. Suppose

$$h(z) = z_1^{j_1} \cdots z_n^{j_n}$$

and expand g in power series

$$g(z) = \sum \alpha_{i_1 \ldots i_n}\, z_1^{i_1} \cdots z_n^{i_n}.$$

Then,

$$\mathrm{Res}_{\,0}(g\,h, \Upsilon) = \left(\frac{1}{2\pi \boldsymbol{i}}\right)^n \int\limits_{\Gamma_\epsilon} \frac{h\, g\, dz_1 \wedge \cdots \wedge dz_n}{z_1^{m_1+1} \cdots z_n^{m_n+1}} =$$

$$\left(\frac{1}{2\pi \boldsymbol{i}}\right)^n \sum_{i_1 \ldots i_n} \alpha_{i_1 \ldots i_n} \int\limits_{\Gamma_\epsilon} \frac{z_1^{i_1+j_1} \cdots z_n^{i_n+j_n}\, dz_1 \wedge \cdots \wedge dz_n}{z_1^{m_1+1} \cdots z_n^{m_n+1}} =$$

$$\left(\frac{1}{2\pi \boldsymbol{i}}\right)^n \sum_{i_1 \ldots i_n} \alpha_{i_1 \ldots i_n} \int\limits_{\Gamma_\epsilon} \frac{dz_1 \wedge \cdots \wedge dz_n}{z_1^{m_1+1-i_1-j_1} \cdots z_n^{m_n+1-i_n-j_n}}.$$

Noticing that $\Gamma_\epsilon = \{z : |z_1| = \epsilon_1, \ldots, |z_n| = \epsilon_n\}$ is just a n-torus we have, by Cauchy's integral formula,

$$\left(\frac{1}{2\pi \boldsymbol{i}}\right)^n \sum_{i_1 \ldots i_n} \alpha_{i_1 \ldots i_n} \int\limits_{\Gamma_\epsilon} \frac{dz_1 \wedge \cdots \wedge dz_n}{z_1^{m_1+1-i_1-j_1} \cdots z_n^{m_n+1-i_n-j_n}} =$$

$$\alpha_{m_1-j_1, \ldots, m_n-j_n}.$$

Hence, to say that $\mathrm{Res}_{\,0}(g\,h, \Upsilon) = 0$ for all h is the same as to say that $\alpha_{i_1 \ldots i_n} = 0$ for $0 \le i_1 \le m_1, \ldots, 0 \le i_n \le m_n$. We conclude $g \in \langle z_1^{m_1+1}, \ldots, z_n^{m_n+1} \rangle_{\mathcal{O}_{0\,n}} \subset \mathfrak{T}_f$.

□

Lemma 3.3.3 *Let* $\varphi \in \mathcal{O}_{0\,n}$ *be such that:*
(i) *The map* $\Phi = (\varphi, f_2, \ldots, f_n)$ *satisfies* $\Phi^{-1}(0) = \{0\}$, *where* $f = (f_1, \ldots, f_n)$.
(ii) $\varphi \in \mathfrak{T}_f$, *so that* $\mathfrak{T}_\Phi \subset \mathfrak{T}_f$.

If $\mathfrak{B}_{0\,\Phi}$ *is nondegenerate, then* $\mathfrak{B}_{0\,f}$ *is also nondegenerate.*

Proof: Write $\varphi = \sum\limits_{i=1}^{n} a_i\, f_i$, so that

$$\begin{pmatrix} \varphi \\ f_2 \\ \vdots \\ f_n \end{pmatrix} = \begin{pmatrix} a_1 & a_2 & \ldots & a_n \\ 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \\ 0 & 0 & \ldots & 1 \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix}$$

Given $g = \sum_{i=1}^{n} b_i\, f_i \in \mathfrak{I}_f$ we have

$$a_1\, g = b_1 \left(\sum_{i=1}^{n} a_i\, f_i \right) + \sum_{i\geq 2} (a_1 b_i - b_1 a_i)\, f_i =$$
$$b_1\, \varphi + \sum_{i\geq 2} (a_1 b_i - b_1 a_i)\, f_i \in \mathfrak{I}_\Phi$$

and thus we have morphisms in both directions,

$$\psi : \mathcal{O}_{0\,n} \,/\, \mathfrak{I}_f \longrightarrow \mathcal{O}_{0\,n} \,/\, \mathfrak{I}_\Phi$$

induced by multiplication by a_1, and the natural projection

$$\pi : \mathcal{O}_{0\,n} \,/\, \mathfrak{I}_\Phi \longrightarrow \mathcal{O}_{0\,n} \,/\, \mathfrak{I}_f .$$

Since $\det \begin{pmatrix} a_1 & a_2 & \dots & a_n \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \\ 0 & 0 & \dots & 1 \end{pmatrix} = a_1$ we have, by the Transformation Law (property 5),

$$\mathfrak{B}_{0\,f}(\widetilde{g}, \widetilde{h}) = \mathfrak{B}_{0\,\Phi}(\widetilde{a_1\, g}, \widetilde{h}) \qquad \forall\, g, h \in \mathcal{O}_{0\,n}.$$

If $\mathfrak{B}_{0\,f}(\widetilde{g}, \widetilde{h}) = 0$ for all h, then $\operatorname{Res}_0((a_1\, g)\, h, \Phi) = 0$ for all h and, by hypothesis, $a_1\, g \in \mathfrak{I}_\Phi$. Write

$$a_1\, g = c_1 \varphi + \sum_{i\geq 2} c_i\, f_i = a_1\, c_1\, f_1 + \sum_{i\geq 2} (c_1 a_i + c_i) f_i.$$

It follows that

$$a_1\, (\, g - c_1\, f_1) \equiv 0 \mod \langle f_2, \dots, f_n \rangle_{\mathcal{O}_{0\,n}}.$$

We have then two possibilities: either $g - c_1\, f_1 \in \langle f_2, \dots, f_n \rangle_{\mathcal{O}_{0\,n}}$ or a_1 is a zero divisor in $\mathcal{O}_{0\,n} \,/\, \langle f_2, \dots, f_n \rangle_{\mathcal{O}_{0\,n}}$. If a_1 is a zero divisor, then so is $a_1\, f_1$ and the same holds for $\varphi = a_1\, f_1 + (a_2\, f_2 + \dots + a_n\, f_n)$. But this is impossible since $\varphi, f_2, \dots, f_n$ is a regular sequence. We are left with $g - c_1\, f_1 \in \langle f_2, \dots, f_n \rangle_{\mathcal{O}_{0\,n}}$ and thus $g \in \mathfrak{I}_f$.

$\square$

Now for the proof of the theorem. The fact that $f_1, f_2, \dots, f_n$ form a regular sequence is equivalent to the following geometric fact: choose any k distinct integers in $\{1, \dots, n\}$ and look at the map

$$\phi(z) = (f_{j_1}(z), f_{j_2}(z), \dots, f_{j_k}(z)).$$

Then, the set $W = \phi^{-1}(0)$ is a subvariety of $\mathbb{C}^n$ of dimension $n-k$, that is, W_{reg}, which is the set of points in W at which the derivative $\phi'(z)$ attains its maximal rank, is a complex manifold of dimension $n-k$.

With this at hand we do as follows: consider the map $F_1 = (f_2, \dots, f_n)$. The variety $F_1^{-1}(0)$ is an analytic curve through the origin in $\mathbb{C}^n$. Choose a hyperplane H_1, passing through 0, such that $\{H_1 = 0\} \cap F_1^{-1}(0) = \{0\}$. Change coordinates in $\mathbb{C}^n$ by setting $H_1 = z_1$. Then the map

$$\Psi_1 = (z_1^{m_1+1}, f_2, \dots, f_n), \qquad m_1 \geq 0,$$

is a finite map.

By repeating this procedure with the map Ψ_1 and so on, we obtain finite maps

$$\Psi_j = (z_1^{m_1+1}, z_2^{m_2+1}, \dots, z_j^{m_j+1}, f_{j+1}, \dots, f_n), \qquad m_1, \dots, m_j \geq 0.$$

Remark that $\Psi_0 = f$ and $\Psi_n = \Upsilon$. Invoking Hilbert's zero-theorem we choose $m_1, \dots, m_n$ such that $z_j^{m_j} \in \mathfrak{I}_{\Psi_{j-1}}$.

By lemma 3.3.2, $\mathfrak{B}_{0\,\Upsilon}$ is nondegenerate and hence, by lemma 3.3.3, $\mathfrak{B}_{0\,\Psi_{n-1}}$ is also nondegenerate. Repeated application of lemma 3.3.3 give $\mathfrak{B}_{0\,f}$ nondegenerate. The theorem is proved.

□

Chapter 4

Residues and Kernels

In this chapter we will introduce the Bochner-Martinelli kernel, show that it provides a generalization of Cauchy's integral formula and, after, we indicate how the point residue can be seen through this kernel.

4.1 Complex valued differential forms

Let U be a domain in $\mathbb{C}^n$. Recall the notations of section 2.1.2: we identify $\mathbb{C}^n \approx \mathbb{R}^{2n}$ by

$$(z_1, \dots, z_n) = (x_1 + \boldsymbol{i}y_1, \dots, x_n + \boldsymbol{i}y_n) \approx (x_1, y_1, \dots, x_n, y_n)$$

and consider the *complexified* of $\mathbb{R}^{2n}$, $\mathbb{R}^{2n} \otimes \mathbb{C}$. We have the following bases of $\mathbb{R}^{2n} \otimes \mathbb{C}$:

$$\mathcal{B}_1 = \left\{ \frac{\partial}{\partial x_1}, \frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial x_n}, \frac{\partial}{\partial y_n} \right\}$$

and

$$\mathcal{B}_3 = \left\{ \frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_n}, \frac{\partial}{\partial \bar{z}_1}, \dots, \frac{\partial}{\partial \bar{z}_n} \right\}.$$

We observe that $\mathcal{B}_3$ induces a decomposition of $\mathbb{R}^{2n} \otimes \mathbb{C}$ as a direct sum of complex n-dimensional subspaces,

$$\mathbb{R}^{2n} \otimes \mathbb{C} = \mathbb{V} \oplus \overline{\mathbb{V}}$$

where

$$\mathbb{V} = \left\langle \frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_n} \right\rangle_{|\mathbb{C}}, \quad \overline{\mathbb{V}} = \left\langle \frac{\partial}{\partial \bar{z}_1}, \dots, \frac{\partial}{\partial \bar{z}_n} \right\rangle_{|\mathbb{C}}$$

At each point $\xi \in U$, the tangent space $T_\xi \mathbb{R}^{2n} \approx \mathbb{R}^{2n}$ has as basis (defining the canonical orientation)

$$\left\{ \frac{\partial}{\partial x_1}(\xi), \frac{\partial}{\partial y_1}(\xi), \dots, \frac{\partial}{\partial x_n}(\xi), \frac{\partial}{\partial y_n}(\xi) \right\}$$

with dual basis $\{dx_{1\xi}, dy_{1\xi}, \dots, dx_{n\xi}, dy_{n\xi}\}$. Hence, with the identification $\mathbb{C}^n \approx \mathbb{R}^{2n}$, the complexified tangent space $T_\xi^{\mathbb{C}}\,\mathbb{C}^n \approx \mathbb{R}^{2n} \otimes \mathbb{C}$ admits the decomposition

$$T_\xi^{\mathbb{C}}\,\mathbb{C}^n = \mathbb{V}_\xi \oplus \overline{\mathbb{V}}_\xi$$

where

$$\mathbb{V}_\xi = \left\langle \frac{\partial}{\partial z_1}(\xi), \dots, \frac{\partial}{\partial z_n}(\xi) \right\rangle_{|\mathbb{C}}, \quad \overline{\mathbb{V}}_\xi = \left\langle \frac{\partial}{\partial \bar{z}_1}(\xi), \dots, \frac{\partial}{\partial \bar{z}_n}(\xi) \right\rangle_{|\mathbb{C}}$$

with the corresponding dual bases

$$\check{\mathbb{V}}_\xi = \langle dz_{1\xi}, \dots, dz_{n\xi} \rangle_{|\mathbb{C}}, \quad \check{\overline{\mathbb{V}}}_\xi = \langle d\bar{z}_{1\xi}, \dots, d\bar{z}_{n\xi} \rangle_{|\mathbb{C}}.$$

A C^∞ p-form ω on U is given by a sum of terms of the types $f_I dx_I$, $g_J dy_J$ and $h_K d(x,y)_K$, where $dx_I = dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_p}$, $dy_J = dy_{j_1} \wedge dy_{j_2} \wedge \cdots \wedge dy_{j_p}$, $d(x,y)_K$ is a product of p-forms of types dx_i and dy_j, and f_I, g_J, h_K are smooth complex valued functions.

Now, $dx_i = (1/2)(dz_i + d\bar{z}_i)$ and $dy_i = (1/2\boldsymbol{i})(dz_i - d\bar{z}_i)$. Expressing the terms in ω by using dz_i and $d\bar{z}_i$ we arrive at

$$\omega \;=\; \sum k_{i_1,\dots,i_r,j_1,\dots,j_s} dz_{i_1} \wedge \cdots \wedge dz_{i_r} \wedge d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_s},$$

which we abbreviate as $\omega = \sum k_{I,J}\, dz_I \wedge d\bar{z}_J$. We say that each term of this sum is a p-form of type (r,s), $r+s=p$. It follows that a p-form ω has a unique expression as a sum

$$\omega \;=\; \omega^{(p,0)} + \omega^{(p-1,1)} + \cdots + \omega^{(0,p)},$$

where $\omega^{(r,s)}$ is of type (r,s).

Let $\mathcal{A}^0(U)$ be the $\mathbb{C}$-algebra $C^\infty(U,\mathbb{C})$ and $\mathcal{A}^p(U)$ the $\mathcal{A}^0(U)$-module of C^∞ complex p-forms on U. The decomposition above induces a decomposition

$$\mathcal{A}^p(U) \;=\; \mathcal{A}^{(p,0)}(U) \oplus \mathcal{A}^{(p-1,1)}(U) \oplus \cdots \oplus \mathcal{A}^{(0,p)}(U).$$

We have the exterior differential $d : \mathcal{A}^p(U) \to \mathcal{A}^{p+1}(U)$ (see [Lima2]). For $f \in \mathcal{A}^0(U)$, using the derivations defined in section 2.1.2, we have

$$df = \sum_{i=1}^n \frac{\partial f}{\partial z_i} dz_i + \sum_{i=1}^n \frac{\partial f}{\partial \bar{z}_i} d\bar{z}_i.$$

Define, on the level of functions,

$$\partial f = \sum_{i=1}^{n} \frac{\partial f}{\partial z_i} dz_i \quad \text{and} \quad \overline{\partial} f = \sum_{i=1}^{n} \frac{\partial f}{\partial \bar{z}_i} d\bar{z}_i. \tag{1}$$

On the level of forms, if

$$\omega^{(r,s)} = \sum k_{i_1,\ldots,i_r,j_1,\ldots,j_s} dz_{i_1} \wedge \cdots \wedge dz_{i_r} \wedge d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_s},$$

we let

$$\partial \omega^{(r,s)} = \sum \partial k_{i_1,\ldots,i_r,j_1,\ldots,j_s} \wedge dz_{i_1} \wedge \cdots \wedge dz_{i_r} \wedge d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_s} \tag{2}$$

a form of type $(r+1, s)$ and

$$\overline{\partial} \omega^{(r,s)} = \sum \overline{\partial} k_{i_1,\ldots,i_r,j_1,\ldots,j_s} \wedge dz_{i_1} \wedge \cdots \wedge dz_{i_r} \wedge d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_s} \tag{3}$$

of type $(r, s+1)$. We are left with

$$d\omega^{(r,s)} = \partial \omega^{(r,s)} + \overline{\partial} \omega^{(r,s)}. \tag{4}$$

For an arbitrary p-form $\omega = \sum\limits_{r+s=p} \omega^{(r,s)}$, we put

$$\partial \omega = \sum_{r+s=p} \partial \omega^{(r,s)} \quad \text{and} \quad \overline{\partial} \omega = \sum_{r+s=p} \overline{\partial} \omega^{(r,s)}. \tag{5}$$

It follows that $d = \partial + \overline{\partial}$ and the following properties hold (exercise):

$$\partial(\omega^p \wedge \eta) = \partial \omega^p \wedge \eta + (-1)^p \omega^p \wedge \partial \eta,$$

$$\overline{\partial}(\omega^p \wedge \eta) = \overline{\partial} \omega^p \wedge \eta + (-1)^p \omega^p \wedge \overline{\partial} \eta.$$

Moreover, (exercise)

$$\partial\partial\omega^{(r,s)} + \overline{\partial}\partial\omega^{(r,s)} + \partial\overline{\partial}\omega^{(r,s)} + \overline{\partial}\,\overline{\partial}\omega^{(r,s)} = dd\omega^{(r,s)} = 0.$$

By comparing the form types in the above summation we conclude that

$$\partial^2 = \partial\,\partial = 0 \ , \ \partial\,\overline{\partial} + \overline{\partial}\,\partial = 0 \ , \ \overline{\partial}^2 = \overline{\partial}\,\overline{\partial} = 0. \tag{6}$$

A $(p,0)$-form $\omega^{(p,0)} = \sum f_{i_1,\ldots,i_p} dz_{i_1} \wedge \cdots \wedge dz_{i_p}$ is *holomorphic* if the coefficients $f_{i_1,\ldots,i_p}$ are holomorphic functions. In this case,

$$\overline{\partial} \omega = \sum \overline{\partial} f_{i_1,\ldots,i_p} \wedge dz_{i_1} \wedge \cdots \wedge dz_{i_p} = 0.$$

Conversely, if $\overline{\partial} \omega^{(p,0)} = 0$, then ω has holomorphic coefficients. For holomorphic forms we have $\partial \omega = d\omega$.

4.2 Volume forms and the Hodge $*$-operator

With the identification $\mathbb{C}^n \approx \mathbb{R}^{2n}$, the usual inner product on $\mathbb{R}^{2n}$ extends to a Hermitian product on $T^{\mathbb{C}}_\xi \mathbb{C}^n$,

$$< av, bw >_\xi = a\,\bar{b} < v, w >_\xi, \;\; a, b \in \mathbb{C}, \;\; v, w \in T^{\mathbb{C}}_\xi \mathbb{C}^n. \tag{7}$$

The basis

$$\left\{ \frac{\partial}{\partial z_1}(\xi), \ldots, \frac{\partial}{\partial z_n}(\xi), \frac{\partial}{\partial \bar{z}_1}(\xi), \ldots, \frac{\partial}{\partial \bar{z}_n}(\xi) \right\}$$

is orthogonal and

$$\left\langle \frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_i} \right\rangle_\xi = \left\langle \frac{\partial}{\partial \bar{z}_i}, \frac{\partial}{\partial \bar{z}_i} \right\rangle_\xi = \frac{1}{2} \tag{8}$$

for $1 \le i \le n$ (exercise). It follows that the decomposition

$$T^{\mathbb{C}}_\xi \mathbb{C}^n = \mathbb{V}_\xi \oplus \overline{\mathbb{V}}_\xi$$

is orthogonal.

On the other hand, this Hermitian product induces naturally a Hermitian inner product on the algebra of complex valued forms at a point ξ, which is characterized by the property that: if $\{v_1, \ldots, v_{2n}\}$ is a basis for $T^{\mathbb{C}}_\xi \mathbb{C}^n$ and $\{u_1, \ldots, u_{2n}\}$ is its dual basis, then

$$u_{j_1} \wedge \cdots \wedge u_{j_r}, \;\; 1 \le j_1 < \cdots < j_r \le 2n, \;\; 1 \le r \le 2n$$

is ortonormal. It follows that two forms of different bidegree are orthogonal and, for two (r, s)-forms $\omega = \sum a_{I,J}\, dz_I \wedge d\bar{z}_J$ and $\eta = \sum b_{I,J}\, dz_I \wedge d\bar{z}_J$,

$$< \omega, \eta >_\xi = 2^{r+s} \sum_{I,J} a_{I,J}(\xi)\, \bar{b}_{I,J}(\xi). \tag{9}$$

The factor 2^{r+s} is because

$$< dz_i, dz_i >_\xi = < d\bar{z}_i, d\bar{z}_i >_\xi = 2, \qquad 1 \le i \le n, \tag{10}$$

since the dual basis satisfies (8). The norm of ω at ξ is defined by

$$|\omega|_\xi = \sqrt{< \omega, \omega >_\xi}. \tag{11}$$

A *volume form* $d\mathcal{V}$ on U is a real, continuous 2n-form on U with $|d\mathcal{V}|_\xi = 1$, for all $\xi \in U$. Such a form clearly defines an orientation of U (see [Lima1]) and conversely, if U is oriented, then there is a unique volume form on U which defines this orientation. In $\mathbb{R}^{2n} \approx \mathbb{C}^n$ the volume form is

$$d\mathcal{V} = dx_1 \wedge dy_1 \wedge \cdots \wedge dx_n \wedge dy_n. \tag{12}$$

This translates into

$$d\mathcal{V} = \left(\frac{\boldsymbol{i}}{2}\right)^n dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_n. \tag{13}$$

We leave to the reader, as an exercise, to show that equivalently,

$$d\mathcal{V} = \begin{cases} \dfrac{1}{n!}\varsigma^n, \text{ where } \varsigma = \dfrac{\boldsymbol{i}}{2}\displaystyle\sum_{i=1}^{n} dz_i \wedge d\bar{z}_i \\ \dfrac{(-1)^{n(n-1)/2}}{(2\boldsymbol{i})^n} d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_n \wedge dz_1 \wedge \cdots \wedge dz_n. \end{cases} \tag{13 bis}$$

The volume of U is, by definition, $vol(U) = \int_U d\mathcal{V}$.

Consider now continuous differential forms on U with *compact support*. We have a Hermitian inner product defined by:

$$< \omega, \eta >_U = \int_U < \omega, \eta >_\xi \; d\mathcal{V}. \tag{14}$$

The associated norm is

$$|\omega|_{L^2} = \sqrt{< \omega, \omega >_U}. \tag{15}$$

Suppose now that $X \subset \mathbb{R}^N$ is a manifold of dimension N with boundary ∂X. The inner product in $\mathbb{R}^N$ induces, by restriction, an inner product in $T_\xi \partial X \subset T_\xi X$. We have then a unique volume element on ∂X, $d\mathcal{S}$, which defines the induced orientation of ∂X. As before, the integral of a function g along ∂X is $\int_{\partial X} g\, d\mathcal{S}$ and the volume of ∂X is $\int_{\partial X} d\mathcal{S}$ (this is not as obvious as in the case of a domain U. Here we must use the Riesz representation theorem which states that there is a unique positive Borel measure ν on ∂X, such that $\int_{\partial X} g\, d\mathcal{S} = \int_{\partial X} g\, d\nu$, g compactly supported).

Let $\mathbf{i} : \partial X \to \mathbb{R}^N$ be the inclusion map. If f is a defining function for X in a neighborhood of $\xi \in \partial X$ such that $|df|_\xi = 1$ then, by choosing $N-1$ continuous real 1-forms $\eta_2, \ldots, \eta_N$ in such a way that

$$\{df_\xi, \eta_{2\xi}, \ldots, \eta_{N\xi}\}$$

is a positively oriented orthonormal basis of the cotangent space $\check{T}_\xi \mathbb{R}^N$, we have that

$$d\mathcal{S} = \mathbf{i}^*(\eta_2 \wedge \cdots \wedge \eta_N). \tag{16}$$

We are now ready to introduce the Hodge $*$-operator. First some notations. Consider the set $\{1, \ldots, N\}$. If $A \subset \{1, \ldots, N\}$ we let $|A|$ denote its cardinality and $A' = \{1, \ldots, N\} \setminus A$, with the order induced by the order of $\{1, \ldots, N\}$. Given $A, B \subset \{1, \ldots, N\}$ we let

$$\delta_B^A = \begin{cases} \operatorname{sgn} \sigma, & \text{if } A = B \text{ as sets and } \sigma \text{ is a permutation} \\ & \text{taking } A \text{ onto } B. \\ 0, & \text{in all other cases.} \end{cases} \tag{17}$$

Exercise 6 Show that $\delta_B^A = \delta_A^B$, $\delta_C^A = \delta_B^A \, \delta_C^B$, $\delta_{BA}^{AB} = (-1)^{rs}$ where $|A| = r$, $|B| = s$.

Theorem 4.2.1 *Let $d\mathcal{V}$ be a volume form for the domain $U \subset \mathbb{C}^n$. There exists a unique operator*

$$* : \mathcal{A}^p(U) \longrightarrow \mathcal{A}^{2n-p}(U)$$

satisfying:

$$*(a\omega_\xi + b\eta_\xi) = a(*\omega_\xi) + b(*\eta_\xi), \;\; a, b \in \mathbb{C}. \tag{18}$$

that is, $*$ is $\mathbb{C}$-linear.

$$* \text{ is real}, \;\; *\bar{\omega} = \overline{*\omega}. \tag{19}$$

$$* * \omega = (-1)^{(2n-p)p}\omega, \qquad \omega \in \mathcal{A}^p(U). \tag{20}$$

$$*1 = d\mathcal{V}_\xi \,, \quad *d\mathcal{V}_\xi = 1. \tag{21}$$

$$\omega_\xi \wedge *\bar{\eta}_\xi = <\omega, \eta>_\xi \; d\mathcal{V}_\xi. \tag{22}$$

Proof: Choose an orthonormal basis $\{u_1, \ldots, u_{2n}\}$ for $\check{T}_\xi^{\mathbb{C}}\, \mathbb{C}^n$ such that $u_1 \wedge \cdots \wedge u_{2n} = d\mathcal{V}_\xi$. Let $u_J = u_{j_1} \wedge \cdots \wedge u_{j_p}$.

By linearity, it's enough to show that the properties above determine $*u_J$ for each p-tuple $J \subset \{1, \ldots, 2n\}$. By (21) it is only necessary to consider $1 \leq p \leq 2n-1$. By (18), $*u_J$ is a (2n-p)-form and thus $*\bar{u}_J = \sum_{|K|=2n-p} a_K \, u_K$, with $a_K \in \mathbb{C}$ and the sum extends over all strictly increasing (2n-p)-tuples $K \subset \{1, \ldots, 2n\}$. For one fixed such K we have

$$u_{K'} \wedge *\bar{u}_J = a_K \, u_{K'} \wedge u_K = a_K \, \delta_{\{1,\ldots,2n\}}^{K'\,K} \, d\mathcal{V}_\xi. \tag{23}$$

By (22),

$$u_{K'} \wedge *\bar{u}_J = < u_{K'}, u_J > \; d\mathcal{V}_\xi = \delta_{K'}^J \, d\mathcal{V}_\xi. \tag{24}$$

(23) and (24) give $a_K = \delta^J_{K'}\,\delta^{K'\,K}_{\{1,\ldots,2n\}} = \delta^{J\,K}_{\{1,\ldots,2n\}}$ and we've discovered the face of $*$:

$$*u_J = \delta^{J\,J'}_{\{1,\ldots,2n\}}\,\bar{u}_{J'}. \tag{25}$$

This shows uniqueness.

For the existence, choose any orthonormal basis $\{u_1,\ldots,u_{2n}\}$ for $\check{T}^{\mathbb{C}}_\xi\,\mathbb{C}^n$ such that $u_1\wedge\cdots\wedge u_{2n} = d\mathcal{V}_\xi$. Define $*u_J$ by (25) and extend it by demanding $\mathbb{C}$-linearity. We leave to the reader the task to verify that $*$ so defined satisfies all the stated properties. Notice that we've shown that $*$ does not depend on the choice of the orthonormal basis.

□

Specializing further to (r,s)-forms we have

Proposition 4.2.2

$$\omega^{(r,s)} \in \mathcal{A}^{r,s}(U) \Longrightarrow *\omega^{(r,s)} \in \mathcal{A}^{n-s,n-r}(U). \tag{26}$$

$$\omega^{(r,s)} \in \mathcal{A}^{r,s}(U) \Longrightarrow **\,\omega^{(r,s)} = (-1)^{r+s}\omega^{(r,s)}. \tag{27}$$

For $J \subset \{1,\ldots,n\}$ and $|J| = s$,

$$*dz_J = \frac{(-1)^{s(s-1)/2}}{2^{n-s}\boldsymbol{i}^n} dz_J \wedge \left(\bigwedge_{i\in J'} d\bar{z}_i \wedge dz_i\right). \tag{28}$$

Proof: For $\omega^{(r,s)} \in \mathcal{A}^{r,s}(U)$ we have $<\omega^{(r,s)},\eta>_\xi \neq 0$ only when $\eta \in \mathcal{A}^{r,s}(U)$. (22) implies that $\overline{*\omega^{(r,s)}} \in \mathcal{A}^{n-r,n-s}(U)$ and so $*\omega^{(r,s)} \in \mathcal{A}^{n-s,n-r}(U)$. (27) follows from (20). Now,

$$dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_n = (-1)^{s(s-1)/2}\,dz_J \wedge d\bar{z}_J \wedge (dz\wedge d\bar{z})_{J'}.$$

By (22), $dz_J \wedge \overline{*dz_J} = 2^s d\mathcal{V}$. Replacing $d\mathcal{V}$ by its expression (13) we get

$$dz_J \wedge \overline{*dz_J} = 2^s\left(\frac{\boldsymbol{i}}{2}\right)^n (-1)^{s(s-1)/2} dz_J \wedge d\bar{z}_J \wedge (dz\wedge d\bar{z})_{J'}.$$

It follows that

$$\overline{*dz_J} = \frac{\boldsymbol{i}^n(-1)^{s(s-1)/2}}{2^{n-s}}\,d\bar{z}_J \wedge (dz\wedge d\bar{z})_{J'}$$

which is the conjugated of (28).

□

Returning to volume forms let us now consider a real manifold $X \in \mathbb{R}^n$, of dimension n, with boundary ∂X. Suppose f is a defining function for X in a neighborhood of a point $\xi \in \partial X$. Then

Proposition 4.2.3

$$d\mathcal{S}_\xi = \mathbf{i}^* \left(\frac{*df_\xi}{|df_\xi|} \right). \tag{29}$$

Proof: Choose $u_2, \ldots, u_n \in \check{T}_\xi \mathbb{R}^n$ such that $\dfrac{df_\xi}{|df_\xi|}, u_2, \ldots, u_n$ is a positive orthonormal basis. By (25),

$$* \left(\frac{df_\xi}{|df_\xi|} \right) = u_2 \wedge \cdots \wedge u_n.$$

The proposition follows from (16).

□

Exercise 7 Show that

$$*df = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \, dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n.$$

Also, calculate the area of the unit sphere in $\mathbb{R}^n$.

In the complex situation we have the

Corollary 4.2.4 *Let $U \in \mathbb{C}^n$ be a domain, with boundary ∂U a smooth manifold. Suppose f is a defining function for U in a neighborhood of a point $\xi \in \partial U$. Then,*

$$d\mathcal{S} = 2\, \mathbf{i}^* \left(\frac{*\partial f}{|df|} \right). \tag{30}$$

Proof: We have, $*df = *(\partial + \overline{\partial}) f = *\partial f + \overline{*\partial f}$, by (19). Now,

$$\partial f = \sum_{j=1}^{n} \frac{\partial f}{dz_j} \, dz_j$$

and then, by (28),

$$*\partial f = \sum_{j=1}^{n} \frac{\partial f}{dz_j} * dz_j = \frac{1}{2^{n-1} \boldsymbol{i}^n} \sum_{j=1}^{n} \frac{\partial f}{dz_j} \, dz_j \wedge \left(\bigwedge_{i \neq j} d\bar{z}_i \wedge dz_i \right) = \frac{1}{\boldsymbol{i}} \, \partial f \wedge \frac{\varsigma^{n-1}}{(n-1)!} \tag{31}$$

where ς is the form given in (13 bis). Conjugate this expression to get $\overline{*\partial f}$. We are left with

$$*df = \left(\frac{1}{i}\,\partial f - \frac{1}{i}\,\overline{\partial} f\right) \wedge \frac{\varsigma^{n-1}}{(n-1)!}. \tag{32}$$

Now, $\mathbf{i}^*df = 0$ because $f = 0$ defines ∂U around ξ. Hence, $-\mathbf{i}^*(\partial f) = \mathbf{i}^*(\overline{\partial} f)$. It follows from (32) and (31) that

$$\mathbf{i}^*(*df) = 2\mathbf{i}^*\left(\frac{1}{i}\,\partial f \wedge \frac{\varsigma^{n-1}}{(n-1)!}\right) = 2\mathbf{i}^*(*\partial f). \tag{33}$$

(30) is then a consequence of (29).

□

4.3 The Bochner-Martinelli kernel

In order to define and exploit the Bochner-Martinelli kernel we must first use integration by parts to derive a *formal adjoint* of $\overline{\partial}$. The procedure is the same as in Riemannian geometry, where the formal adjoint of d is used to define the Laplace-Beltrami operator.

Let $\omega \in \mathcal{A}^{r,s}(\mathbb{C}^n)$, $\eta \in \mathcal{A}^{r,s+1}(\mathbb{C}^n)$ and suppose at least one of them has compact support. Recall the inner product defined in (14). We have $\overline{\partial}\omega \in \mathcal{A}^{r,s+1}(\mathbb{C}^n)$ and so

$$< \overline{\partial}\omega, \eta >_{\mathbb{C}^n} = \int_{\mathbb{C}^n} < \overline{\partial}\omega, \eta >_\xi \; d\mathcal{V}.$$

Now $*\bar{\eta} \in \mathcal{A}^{n-r,n-s-1}(\mathbb{C}^n)$ by (26). Thus, $\overline{\partial}\omega \wedge *\bar{\eta} \in \mathcal{A}^{n,n}(\mathbb{C}^n)$ and $< \overline{\partial}\omega, \eta >_\xi \; d\mathcal{V}_\xi = (\overline{\partial}\omega \wedge *\bar{\eta})_\xi$. Choose a closed euclidean ball $\bar{B}$ containing the support of the pertinent form in its interior. Then

$$< \overline{\partial}\omega, \eta >_{\mathbb{C}^n} = \int_{\mathbb{C}^n} \overline{\partial}\omega \wedge *\bar{\eta} = \int_{\bar{B}} \overline{\partial}\omega \wedge *\bar{\eta}.$$

We have $d(\omega \wedge *\bar{\eta}) = \partial\omega \wedge *\bar{\eta} + \overline{\partial}\omega \wedge *\bar{\eta} + (-1)^{r+s}\omega \wedge d * \bar{\eta}$. But $\partial\omega \wedge *\bar{\eta} = 0$ since $\omega \wedge *\bar{\eta} \in \mathcal{A}^{n,n-1}(\mathbb{C}^n)$ and we are left with

$$d(\omega \wedge *\bar{\eta}) = \overline{\partial}\omega \wedge *\bar{\eta} + (-1)^{r+s}\omega \wedge d * \bar{\eta}.$$

On the other hand, $\omega \wedge d * \bar{\eta} = \omega \wedge \partial * \bar{\eta} + \omega \wedge \overline{\partial} * \bar{\eta}$. But $\omega \wedge \partial * \bar{\eta} = 0$ because $\partial * \bar{\eta}$ is of type $(n-r+1, n-s-1)$ and so

$$d(\omega \wedge *\bar{\eta}) = \overline{\partial}\omega \wedge *\bar{\eta} + (-1)^{r+s}\omega \wedge \overline{\partial} * \bar{\eta}.$$

We conclude

$$< \overline{\partial}\omega, \eta >_{\mathbb{C}^n} = \int\limits_{\bar{B}} \overline{\partial}\omega \wedge *\bar{\eta} = \int\limits_{\bar{B}} d(\omega \wedge *\bar{\eta}) - \int\limits_{\bar{B}} (-1)^{r+s}\omega \wedge \overline{\partial} * \bar{\eta}.$$

By Stokes theorem,

$$\int\limits_{\bar{B}} d(\omega \wedge *\bar{\eta}) = \int\limits_{\partial\bar{B}} \omega \wedge *\bar{\eta} = 0$$

since $\omega \wedge *\bar{\eta} \equiv 0$ on $\partial\bar{B}$ and thus

$$< \overline{\partial}\omega, \eta >_{\mathbb{C}^n} = \int\limits_{\bar{B}} \overline{\partial}\omega \wedge *\bar{\eta} = -\int\limits_{\bar{B}} (-1)^{r+s}\omega \wedge \overline{\partial} * \bar{\eta}.$$

By (27), $(-1)^{r+s}\,\overline{\partial} * \bar{\eta} = * * \overline{\partial} * \bar{\eta}$. By (19), $* * \overline{\partial} * \bar{\eta} = \overline{* * \partial * \eta}$. Therefore,

$$\begin{aligned} < \overline{\partial}\omega, \eta >_{\mathbb{C}^n} = -\int\limits_{\bar{B}} \omega \wedge \overline{* * \partial * \eta} = \\ \int\limits_{\bar{B}} \omega \wedge \overline{*(- * \partial * \eta)} = < \omega, - * \partial * \eta >_{\mathbb{C}^n} \end{aligned} \tag{34}$$

and $- * \partial *$ is the *formal adjoint* of $\overline{\partial}$.

Suppose now that neither ω nor η have compact support. We then do as follows: let $U \subset \mathbb{C}^n$ be a limited domain whose boundary ∂U is a smooth manifold and assume ω and η are smooth in a neighborhood of the closure $\bar{U}$. Then, proceeding exactly as in the deduction of (34) we arrive at

$$< \overline{\partial}\omega, \eta >_U = < \omega, - * \partial * \eta >_U + \int\limits_{\bar{U}} d(\omega \wedge *\bar{\eta}).$$

Using Stokes theorem we get

$$< \overline{\partial}\omega, \eta >_U = < \omega, - * \partial * \eta >_U + \int\limits_{\partial U} \omega \wedge *\bar{\eta}. \tag{35}$$

Exercise 8 Show that

$$< - * \partial * \eta, \omega >_U = < \eta, \overline{\partial}\omega >_U - \int\limits_{\partial U} \bar{\omega} \wedge *\eta. \tag{36}$$

We now introduce a kernel in $\mathbb{C}^n \times \mathbb{C}^n$, which is the complex analogue of the *Newtonian potential* in $\mathbb{R}^n \times \mathbb{R}^n$:

$$\mathsf{G}(w,z) = \begin{cases} -\dfrac{1}{2\pi} \log |w-z|^2 & \text{for } n = 1 \\ \dfrac{(n-2)!}{2\pi^n} |w-z|^{2-2n} & \text{for } n \geq 2. \end{cases} \tag{37}$$

In what follows, w will denote the variable of integration and z will be a parameter and we let

$$\alpha_{2n-1} = \frac{2\pi^n}{(n-1)!} \qquad \text{and} \qquad \Lambda = |w-z|^2. \tag{38}$$

Notice that, since the area of the sphere $S_R^{2n-1} \subset \mathbb{C}^n$ of radius R is $\alpha_{2n-1}\, R^{2n-1}$, α_{2n-1} is just the area of the unit sphere S_1^{2n-1}.

Definition 4.3.1 *The* Bochner-Martinelli kernel *(for functions) is the double form*

$$\mathsf{K}(w,z) = -*\partial_w \mathsf{G}(w,z)$$

of type $(n, n-1)$ *in* w *and type* $(0,0)$ *in* z.

Lemma 4.3.2 $\mathsf{K}(w,z)$ *is represented by the form*

$$\mathsf{K} = \frac{(n-1)!}{(2\pi \boldsymbol{i})^n |w-z|^{2n}} \sum_{i=1}^n (\bar{w}_i - \bar{z}_i)\, dw_i \wedge \left(\bigwedge_{j \neq i} d\bar{w}_j \wedge dw_j \right). \tag{39}$$

Proof: We have

$$\partial_w \mathsf{G}(w,z) = -\frac{(n-1)!}{2\pi^n |w-z|^{2n}} \sum_{i=1}^n (\bar{w}_i - \bar{z}_i)\, dw_i$$

and so

$$-*\partial_w \mathsf{G}(w,z) = \frac{(n-1)!}{2\pi^n |w-z|^{2n}} \sum_{i=1}^n (\bar{w}_i - \bar{z}_i) * dw_i. \tag{40}$$

By (28),

$$*dw_i = \frac{1}{2^{n-1} \boldsymbol{i}^n}\, dw_i \wedge \left(\bigwedge_{j \neq i} d\bar{w}_j \wedge dw_j \right).$$

Substituting this into (40) gives (39).

□

Notice that for $n = 1$ (39) reads

$$\mathsf{K}(w,z) = \frac{1}{2\pi \boldsymbol{i}} \frac{dw}{w-z},$$

which is just the *Cauchy kernel* in one variable.

Lemma 4.3.3 *$\overline{\partial}_w \mathsf{K}(w,z) = 0$ on $\mathbb{C}^n \times \mathbb{C}^n \setminus \{w = z\}$.*

Proof: To simplify the writting put $C_n = \dfrac{(n-1)!}{(2\pi \boldsymbol{i})^n}$. Then (39) assumes the form

$$\mathsf{K} = C_n \Lambda^{-n} \sum_{i=1}^{n} (\bar{w}_i - \bar{z}_i)\, dw_i \,\wedge \left(\bigwedge_{j \neq i} d\bar{w}_j \wedge dw_j \right).$$

Thus,

$$\begin{aligned} \overline{\partial}_w \mathsf{K} = C_n \overline{\partial}_w \Lambda^{-n} \wedge \sum_{i=1}^{n} (\bar{w}_i - \bar{z}_i)\, dw_i \,\wedge \left(\bigwedge_{j \neq i} d\bar{w}_j \wedge dw_j \right) + \\ C_n \Lambda^{-n}\, \overline{\partial}_w \left[\sum_{i=1}^{n} (\bar{w}_i - \bar{z}_i)\, dw_i \,\wedge \left(\bigwedge_{j \neq i} d\bar{w}_j \wedge dw_j \right) \right]. \end{aligned} \tag{41}$$

Now, $\overline{\partial}_w \Lambda^{-n} = -n\Lambda^{-n-1} \sum_{k=1}^{n} (w_k - z_k)\, d\bar{w}_k$ and hence

$$\begin{aligned} &\overline{\partial}_w \Lambda^{-n} \wedge \sum_{i=1}^{n} (\bar{w}_i - \bar{z}_i)\, dw_i \,\wedge \left(\bigwedge_{j \neq i} d\bar{w}_j \wedge dw_j \right) = \\ &-n\Lambda^{-n-1} \sum_{i=1}^{n} (w_i - z_i)(\bar{w}_i - \bar{z}_i)\, d\bar{w}_i \wedge dw_i \,\wedge \left(\bigwedge_{j \neq i} d\bar{w}_j \wedge dw_j \right) = \\ &-n\Lambda^{-n}\, d\bar{w}_1 \wedge dw_1 \wedge \cdots \wedge d\bar{w}_n \wedge dw_n. \end{aligned}$$

Also,

$$\begin{aligned} \overline{\partial}_w \left[\sum_{i=1}^{n} (\bar{w}_i - \bar{z}_i)\, dw_i \,\wedge \left(\bigwedge_{j \neq i} d\bar{w}_j \wedge dw_j \right) \right] = \\ n\, d\bar{w}_1 \wedge dw_1 \wedge \cdots \wedge d\bar{w}_n \wedge dw_n \end{aligned}$$

and the two terms in (41) cancel each other.

□

K normalizes the area of spheres, more precisely,

Lemma 4.3.4 *Let $B_\epsilon(z)$ denote the euclidean ball centered at z and with radius ϵ. Then,*

$$\int_{\partial B_\epsilon(z)} \mathsf{K}(w,z) = 1$$

for all $z \in \mathbb{C}^n$ and for all $\epsilon > 0$.

Proof: We have

$$\partial_w \mathsf{G}(w,z) = \frac{-1}{\alpha_{2n-1}} \frac{\partial\Lambda}{\Lambda^n}.$$

Now, along the sphere $\partial B_\epsilon(z)$, $\Lambda = \epsilon^2$ and then,

$$- * \partial_w \mathsf{G}(w,z) = \frac{1}{\alpha_{2n-1}\epsilon^{2n}} * \partial\Lambda \tag{42}$$

on $\partial B_\epsilon(z)$.

On the other hand, $f(w) = |w-z|^2 - \epsilon^2 = \Lambda - \epsilon^2$ is a defining function for $B_\epsilon(z)$ satisfying $\partial f = \partial_w \Lambda$ and

$$df = d_w\Lambda = \sum_{i=1}^n (\bar{w}_i - \bar{z}_i)\, dw_i.$$

By (9),

$$|df|^2_\xi = < df, df >_\xi = 2^{1+1} \sum_{i=1}^n (\bar{\xi_i} - \bar{z}_i)(\xi_i - z_i) = 2^{1+1}\epsilon^2 = 4\epsilon^2$$

where $\xi = (\xi_1, \ldots, \xi_n) \in \partial B_\epsilon(z)$, so that $|df| = 2\epsilon$ on the sphere. Invoking (30) we see that

$$*\partial_w\Lambda = *\partial f = \epsilon\, d\mathcal{S}$$

on $\partial B_\epsilon(z)$ and (42) becomes,

$$- * \partial_w \mathsf{G}(w,z) = \frac{1}{\alpha_{2n-1}\epsilon^{2n}}\, \epsilon\, d\mathcal{S} = \frac{1}{\alpha_{2n-1}\epsilon^{2n-1}}\, d\mathcal{S}. \tag{43}$$

Integration gives

$$\int_{\partial B_\epsilon(z)} \mathsf{K}(w,z) = \int_{\partial B_\epsilon(z)} - * \partial_w \mathsf{G}(w,z) = \frac{1}{\alpha_{2n-1}\epsilon^{2n-1}} \int_{\partial B_\epsilon(z)} d\mathcal{S} = 1.$$

□

Theorem 4.3.5 *Let $U \subset \mathbb{C}^n$ be a limited domain whose boundary ∂U is a smooth manifold. Suppose f is a smooth complex function defined in a neighborhood of $\bar{U}$. Then, for $z \in U$,*

$$f(z) = < \overline{\partial} f, \overline{\partial}_w \mathsf{G} >_U - \int_{\partial U} f \wedge * \partial_w \mathsf{G}.$$

Proof: Given $z \in U$ choose ϵ small enough so that $B_\epsilon(z) \subset U$. Invoking (35) we have, taking due attention to the orientation,

$$< \overline{\partial} f, \overline{\partial}_w \mathsf{G} >_{U \setminus B_\epsilon(z)} =$$

$$< f, - * \partial_w * \overline{\partial}_w \mathsf{G} >_{U \setminus B_\epsilon(z)} + \int_{\partial U} f \wedge * \overline{\overline{\partial}_w \mathsf{G}} - \int_{\partial B_\epsilon(z)} f \wedge * \overline{\overline{\partial}_w \mathsf{G}}. \tag{44}$$

Since $\mathsf{G} = \overline{\mathsf{G}}$,

$$\partial_w * \overline{\partial}_w \mathsf{G} = \partial_w * \overline{\partial_w \mathsf{G}} = \partial_w (\overline{* \partial_w \mathsf{G}}) = \overline{\overline{\partial}_w (* \partial_w \mathsf{G})} = -\overline{\overline{\partial}_w \mathsf{K}} = 0$$

by lemma 4.3.3. Therefore (44) assumes the form

$$< \overline{\partial} f, \overline{\partial}_w \mathsf{G} >_{U \setminus B_\epsilon(z)} - \int_{\partial U} f \wedge * \partial_w \mathsf{G} = - \int_{\partial B_\epsilon(z)} f \wedge * \partial_w \mathsf{G}. \tag{45}$$

We now let $\epsilon \to 0$. The left side of (45) tends to

$$< \overline{\partial} f, \overline{\partial}_w \mathsf{G} >_U - \int_{\partial U} f \wedge * \partial_w \mathsf{G}$$

and it remains to show

$$\lim_{\epsilon \to 0} - \int_{\partial B_\epsilon(z)} f \wedge * \partial_w \mathsf{G} = f(z) \qquad (46).$$

To see why this holds we do as follows:

$$- \int_{\partial B_\epsilon(z)} f(w) * \partial_w \mathsf{G}(w, z) = - \int_{\partial B_\epsilon(z)} [f(z) + f(w) - f(z)] * \partial_w \mathsf{G}(w, z).$$

Now,

$$\begin{aligned} - \int_{\partial B_\epsilon(z)} [f(z) + f(w) - f(z)] * \partial_w \mathsf{G}(w, z) = \\ - \int_{\partial B_\epsilon(z)} f(z) * \partial_w \mathsf{G}(w, z) - \int_{\partial B_\epsilon(z)} [f(w) - f(z)] * \partial_w \mathsf{G}(w, z). \end{aligned} \tag{47}$$

But,

$$-\int\limits_{\partial B_\epsilon(z)} f(z) * \partial_w \mathsf{G}(w,z) = f(z) \int\limits_{\partial B_\epsilon(z)} - * \partial_w \mathsf{G}(w,z) =$$
$$f(z) \int\limits_{\partial B_\epsilon(z)} \mathsf{K}(w,z) = f(z)$$

by lemma 4.3.4 and (47) reads

$$-\int\limits_{\partial B_\epsilon(z)} f \wedge *\partial_w \mathsf{G} = f(z) - \int\limits_{\partial B_\epsilon(z)} [f(w) - f(z)] * \partial_w \mathsf{G}(w,z).$$

Since f is continuous, $\sup\limits_{w \in \partial B_\epsilon(z)} |f(w) - f(z)| \to 0$ as $\epsilon \to 0$ and, by the proof of lemma 4.3.4

$$*\partial_w \mathsf{G}(w,z) = \frac{-1}{\alpha_{2n-1}\epsilon^{2n-1}}\, d\mathcal{S}$$

on $\partial B_\epsilon(z)$. Hence,

$$\left| \int\limits_{\partial B_\epsilon(z)} [f(w) - f(z)] * \partial_w \mathsf{G}(w,z) \right| \leq$$
$$\sup_{w \in \partial B_\epsilon(z)} |f(w) - f(z)| \cdot \frac{1}{\alpha_{2n-1}\epsilon^{2n-1}} \int\limits_{\partial B_\epsilon(z)} d\mathcal{S} =$$
$$\sup_{w \in \partial B_\epsilon(z)} |f(w) - f(z)| \underset{\epsilon \to 0}{\longrightarrow} 0$$

and (46) is true. The theorem is proved.

□

Notice that, when $n = 1$ theorem 4.3.5 reads

$$f(z) = \frac{1}{2\pi \boldsymbol{i}} \int\limits_{\partial U} \frac{f(w)}{w - z}\, dw - \frac{1}{2\pi \boldsymbol{i}} \int\limits_{U} \frac{\partial f}{\partial \bar{w}}(w)\, \frac{d\bar{w} \wedge dw}{w - z},$$

which is the classical *generalized Cauchy integral formula* for smooth functions.

Theorem 4.3.6 (The Bochner-Martinelli integral formula) *Let $U \subset \mathbb{C}^n$ be a limited domain whose boundary ∂U is a smooth manifold. Suppose $f : \bar{U} \to \mathbb{C}$ is continuous and f is holomorphic in U. Then,*

$$\int\limits_{\partial U} f(w)\, \mathsf{K}(w,z) = \begin{cases} f(z) & \text{for } z \in U \\ \\ 0 & \text{for } z \notin U. \end{cases}$$

Proof: Suppose first that f is smooth in a neighborhood of $\bar{U}$ and $\overline{\partial} f = 0$. If $z \in U$, the result follows from theorem 4.3.5. If $z \notin \bar{U}$ then,

$$d(f\,\mathsf{K}) = \overline{\partial}(f\,\mathsf{K}) = \overline{\partial} f \wedge \mathsf{K} + f \wedge \overline{\partial}_w \mathsf{K} = 0$$

by lemma 4.3.3. By Stokes theorem,

$$0 = \int_U d(f\,\mathsf{K}) = \int_{\partial U} f\,\mathsf{K}.$$

The proof of the theorem now proceeds by constructing an exhaustion of U by relatively compact domains U_k, whose boundaries are smooth manifolds, $U = \cup_{k\geq 1} \bar{U}_k$, $\bar{U}_k \subset U_{k+1}$, and passing to the limit $k \to \infty$.

□

Let us now consider $\mathsf{B}_0(w) = \mathsf{K}(w, 0)$. We have, by (39),

$$\mathsf{B}_0(w) = \frac{(n-1)!}{(2\pi \boldsymbol{i})^n |w|^{2n}} \sum_{i=1}^n \bar{w}_i\, dw_i \wedge \left(\bigwedge_{j\neq i} d\bar{w}_j \wedge dw_j \right).$$

A manipulation shows that (exercise),

$$\mathsf{B}_0(w) = (-1)^{n(n-1)/2} \frac{(n-1)!}{(2\pi \boldsymbol{i})^n |w|^{2n}} \sum_{i=1}^n \overline{\vartheta_i(w)} \wedge \vartheta(w), \tag{48}$$

where

$$\vartheta(w) = dw_1 \wedge \cdots \wedge dw_n$$

and

$$\overline{\vartheta_i(w)} = (-1)^{i-1} \bar{w}_i\, d\bar{w}_1 \wedge \cdots \wedge \widehat{d\bar{w}_i} \wedge \cdots d\bar{w}_n.$$

Let

$$\ell_n = (-1)^{n(n-1)/2} \frac{(n-1)!}{(2\pi \boldsymbol{i})^n}.$$

We now define

$$\mathsf{B}(z,\zeta) = \ell_n \frac{\sum_{i=1}^n \overline{\vartheta_i(z-\zeta)} \wedge \vartheta(\zeta)}{|z-\zeta|^2}. \tag{49}$$

This is the same as

$$\mathsf{B}(z,\zeta) = \ell_n \frac{\sum_{i=1}^n (-1)^{i-1} (\bar{z}_i - \bar{\zeta}_i) \bigwedge_{j\neq i} (d\bar{z}_j - d\bar{\zeta}_j) \wedge d\zeta_1 \wedge \cdots \wedge d\zeta_n}{|z-\zeta|^2}. \tag{50}$$

$\mathsf{B}(z,\zeta)$ is related to the Grothendieck residues, as we will indicate in the next section.

4.4 Dolbeault cohomology

We now want to introduce a cohomology theory for the $\overline{\partial}$ operator. Recall that De Rham's cohomology on U is defined by (see [Lima2]): let $Z^p(U) = \{\omega \in \mathcal{A}^p(U) : d\,\omega = 0\}$, $p \geq 0$, (closed p-forms) and $B^p(U) = d\,(\mathcal{A}^{p-1}(U))$ if $p \geq 1$ and $B^0(U) = \{0\}$ (exact p-forms). Since $d^2 = 0$, $B^p(U)$ is a subspace of $Z^p(U)$. The quocient spaces

$$H^p_{DR}(U) = \frac{Z^p(U)}{B^p(U)} \qquad p \geq 0$$

measure the obstruction to solving the equation $d\theta = \omega$ on U, that is, given ω such that $d\omega = 0$, find θ satisfying $d\theta = \omega$. Notice that the differentiable structure of U is clearly involved in the definition of the groups $H^p_{DR}(U)$. Locally, the necessary condition $d\omega = 0$ is also sufficient to solve $d\theta = \omega$ (Poincaré's lemma). A deep theorem by De Rham shows that in fact the groups $H^p_{DR}(U)$ depend only on the topology of U, since this result exhibits an isomorphism $H^p_{DR}(U) \approx H^p_s(U;\mathbb{C})$ (singular cohomology with coefficients in $\mathbb{C}$).

For example, the kernel B_0 restricts, by (43), to a positive multiple of the area element of the sphere $S^{2n-1}_\delta(0)$. B_0 is then a generator of $H^{2n-1}_{DR}(S^{2n-1}_\delta(0);\mathbb{C})$.

Now for the Dolbeault cohomology. Given $\omega \in \mathcal{A}^{(r,s)}(U)$, $s \geq 1$, one wants to find a solution $\theta \in \mathcal{A}^{(r,s-1)}(U)$ of the equation $\overline{\partial}\theta = \omega$. Again, because $\overline{\partial}^2 = 0$, a necessary condition is that $\overline{\partial}\omega = 0$. "Locally", this necessary condition is also sufficient, more precisely, for *polydiscs* in $\mathbb{C}^n$ a solution can be found. This is the content of the Bochner-Dolbeault-Grothendieck lemma (see [Gu]). On the other hand, the solvability of this equation globally, even for $U \subset \mathbb{C}^n$ a domain, is a much more involved question and depends on the global complex analytic properties of U.

The definition of the Dolbeault groups is: let

$$Z_{r\,s}(U) = \{\omega \in \mathcal{A}^{(r,s)}(U) : \overline{\partial}\omega = 0\}$$

($\overline{\partial}$ closed (r,s)-forms) and

$$B_{r\,s}(U) = \overline{\partial}\left(\mathcal{A}^{(r,s-1)}(U)\right) \quad \text{if} \quad s \geq 1 \quad \text{and} \quad B_{r\,0}(U) = \{0\}.$$

The quotient

$$H^{r\,s}_{\overline{\partial}}(U) = \frac{Z_{r\,s}(U)}{B_{r\,s}(U)}$$

is the (r,s) Dolbeault cohomology group of U. Remark that $\mathcal{O}(U) = H^{0\,0}_{\overline{\partial}}(U)$.

Let us briefly indicate how the Bochner-Martinelli kernel is related to point residues. We cannot present complete arguments here since, to do so, we would have to develop sheaf cohomology theory. We refer the reader to [G-H].

For X a complex manifold of dimension n and $\mathcal{a}^p$ the sheaf of complex valued smooth p-forms on X, the *Dolbeault theorem* gives an isomorphism

$$H^q(X, \mathcal{a}^p) \approx H^{p\,q}_{\overline{\partial}}(X).$$

To define point residues we considered meromorphic forms of the form $\eta = g\,\dfrac{dz_1 \wedge \cdots \wedge dz_n}{f_1 \cdots f_n}$, with $f = (f_1, \ldots, f_n)$ a finite holomorphic map. We may assume f defined in a small euclidean ball B centered at $0 \in \mathbb{C}^n$. Recall that $f^{-1}(0) = \{0\}$.

Since $d = \overline{\partial}$ on forms of type (n,q), we have a natural map

$$H^{n-1}(B \setminus \{0\}, \mathcal{a}^n) \approx H^{n\,n-1}_{\overline{\partial}}(B \setminus \{0\}) \longrightarrow H^{2n-1}_{DR}(B \setminus \{0\}; \mathbb{C}).$$

$B \setminus \{0\}$ is homotopically the sphere $S^{2n-1}_{\delta}(0)$ and B_0 is a generator of $H^{2n-1}_{DR}(S^{2n-1}_{\delta}(0); \mathbb{C}) \approx \mathbb{C}$. The arrow above is then just integration over the sphere and the above sequence of spaces and maps means

$$\left(\frac{1}{2\pi \boldsymbol{i}}\right)^n \eta \xrightarrow[\approx]{\text{Dolbeault theorem}} \varpi_\eta \longrightarrow \int\limits_{S^{2n-1}_{\delta}(0)} \varpi_\eta.$$

ϖ_η is called the *distinguished Dolbeault representative* of $\dfrac{\eta}{(2\pi \boldsymbol{i})^n}$.

Consider the map

$$F : B \to \mathbb{C}^n \times \mathbb{C}^n$$

defined by

$$F(z) = (z + f(z), z).$$

It can be shown that $\varpi_\eta = g\,F^*\mathsf{B}$ (recall (50)) and that

$$\operatorname{Res}_0(g,f) = \frac{1}{(2\pi \boldsymbol{i})^n} \int\limits_{\Gamma_\epsilon} g(z)\,\frac{dz_1 \wedge \cdots \wedge dz_n}{f_1(z) \cdots f_n(z)} = \int\limits_{S^{2n-1}_{\delta}(0)} g(z)\,F^*\mathsf{B}(z,\zeta).$$

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