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Graphs with constant mean curvature in the 3–hyperbolic space

PEDRO A. HINOJOSA

CCEN – Departamento de Matemática, Universidade Federal da Paraíba
58051-900 João Pessoa, PB, Brasil and
Pós Graduação em Matemática, Universidade Federal do Ceará, Centro de Ciências
Campus do Pici, Bloco 914, 60455-760 Fortaleza, CE, Brasil

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ABSTRACT

In this work we will deal with disc type surfaces of constant mean curvature in the three dimensional hyperbolic space which are given as graphs of smooth functions over planar domains. From the various types of graphs that could be defined in the hyperbolic space we consider in particular the horizontal and the geodesic graphs. We proved that if the mean curvature is constant, then such graphs are equivalent in the following sense: suppose that M is a constant mean curvature surface in the 3-hyperbolic space such that M is a geodesic graph of a function ρ that is zero at the boundary, then there exist a smooth function f, that also vanishes at the boundary, such that M is a horizontal graph of f. Moreover, the reciprocal is also true.

Key words: hyperbolic space, geodesic and horizontal graphs, constant mean curvature, elliptic partial differential equations.

To describe the hyperbolic space we consider the half-space model, that is, $\mathbb{H}^3 := \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 > 0\}$ with the metric given by $ds^2 = \frac{1}{x_3^2}\left(dx_1^2 + dx_2^2 + dx_3^2\right)$.

Let $\mathbb{P} \subset \mathbb{H}^3$ be a totally geodesic hyperplane and $\Omega \subset \mathbb{P}$ an open, simply connected and bounded domain. We consider $\rho : \overline{\Omega} \to \mathbb{R}$ a real smooth function. Without loss of generality we fix $\mathbb{P} := \{x_2 = 0\}$.

The horizontal graph of ρ in $\mathbb{H}^3$ is the following set

$$G_h(\rho) := \left\{(x_1, \rho(x_1, x_3), x_3)\,, (x_1, x_3) \in \overline{\Omega}\right\}. \tag{1}$$

Correspondence to: Universidade Federal do Ceará
E-mail: hinojosa@mat.ufpb.br

Horizontal graphs with constant mean curvature in $\mathbb{H}^3$ had been studied by Barbosa and Earp (see, for example, (Barbosa and Earp 1997), (Barbosa and Earp 1998a,b)). We know that if $G_h(\rho)$ has constant mean curvature H in $\mathbb{H}^3$, then ρ satisfies the following partial differential equation

$$div\left(\frac{\nabla\rho}{W_\rho}\right) = \frac{2}{x_3}\left(H + \frac{\rho_3}{W_\rho}\right), \tag{2}$$

where $\nabla\rho$ is the euclidean gradient of ρ and $W_\rho^2 = 1 + |\nabla\rho|^2$.

The geodesic graph of ρ in $\mathbb{H}^3$ is defined as the set

$$G_g(\rho) := \left\{(x_1, x_3 \tanh\rho, x_3\, sech\,\rho),\, (x_1, x_3) \in \overline{\Omega}\right\}. \tag{3}$$

Little is known about geodesic graphs with constant mean curvature in $\mathbb{H}^3$. Some results can be found in (Nelli and Semmler 1999) and (Semmler 1997). When $G_g(\rho)$ has mean curvature H in $\mathbb{H}^3$ the function ρ satisfies the following equation

$$div\left(\frac{\nabla\rho}{W}\right) = \frac{2\cosh\rho}{v^2}\left(H + \frac{\sinh\rho}{W}\right), \tag{4}$$

where $\quad W^2 = \cosh^2\rho + v^2\,|\nabla\rho|^2 \qquad$ and $\qquad |\nabla\rho|^2 = \rho_u^2 + \rho_v^2$.

Note that the equations (2) and (4) are elliptic and quasilinear, so the maximum principle is true for both. Consequently, if the mean curvature $H(u, v)$ is different to zero at any point of Ω, then the horizontal and geodesic graphs of functions which are zero at the boundary of Ω lie in only one of the halfspaces determined by the plane of their boundary. In this case, it is possible to choose a sign for ρ.

THEOREM 1. *Let $M \subseteq \mathbb{H}^3$ be a surface of constant mean curvature $H \neq 0$ given as the geodesic graph of a smooth function $\rho : \overline{\Omega} \to \mathbb{R}$ such that $\rho\,|_{\partial\Omega} = 0$, that is, $M = G_g(\rho)$. Then there exists $f : \overline{\Omega} \to \mathbb{R}$ with $f\,|_{\partial\Omega} \equiv 0$ such that $M = G_h(f)$.*

SKETCH OF THE PROOF. If M is not a horizontal graph, then there exists $p \in \mathbb{P}$ such that the horocicle $L_p := \{p + t(0,1,0); t \in \mathbb{R}\}$ intersects M at least in two points q_1 and q_2. But, as observed above, we can consider $\rho \geq 0$, so that q_1 and q_2 will be obtained with $t > 0$. Let $a := max\{x_2 : (x_1, x_2, x_3) \in M\}$ the maximum value of x_2 when $(x_1, x_2, x_3) \in M$. Note that $0 < a < \infty$ since $M \neq \Omega$ and M is compact. Now set

$$\mathcal{W} := \bigcup_{\substack{p \in \Omega \\ 0 < t \leq a}} \{p - t(0,1,0)\}.$$

Note that $(\partial\mathcal{W} \setminus \mathbb{P}) \cup M$ is a topological closed submanifold in $\mathbb{H}^3$, then it bounds an open domain $\mathcal{R}$ in $\mathbb{H}^3$.

For any real number t, lets P_t be a totally geodesic plane in $\mathbb{H}^3$ given by $P_t := \{x_2 = t\}$, $M_t := \{(x_1, x_2, x_3) \in M : x_2 > t\}$ and M_t^* the reflection of M_t with respect to the plane P_t.

It is clear that we have $M_t \neq \phi$ for $0 \leq t < a$. If $t < a$ and $a - t$ is sufficiently small, then $M_t^* \subset \mathcal{R}$. Set $t_0 := inf\{b \in \mathbb{R} : M_t^* \subset \mathcal{R}, \forall t \in (b, a)\}$.

The existence of q_1 and q_2 in $M \cap L_p$ implies $t_0 > 0$. Now we prove that this leads to the existence of a plane of symmetry of M. That is impossible since M is contained in only one of the sides of P_t.

For any $t \geq t_0$, we have $M_t^* \cap P_{-a} = \phi$. In particular, $M_{t_0}^* \cap P_{-a} = \phi$. Moreover, $M_{t_0}^* \cap (\partial\mathcal{W} \setminus (P_{-a} \cup P_0)) = \phi$, since $\rho|_{\partial\Omega} = 0$. Thus we have $M_{t_0}^* \cap M \neq \phi$. Therefore there exists $p_0 \in M_{t_0}$ such that $p_0^* \in M_{t_0}^* \cap M$, where p_0^* denotes the reflection of p_0 with respect to the plane P_{t_0}.

The following are all the possibilities for the relative positions of the points p_0 and p_0^*.

1. $p_0 \in int\left(M_{t_0}\right)$ and $p_0^* \in \partial M$;
2. $p_0 \in int\left(M_{t_0}\right)$ and $p_0^* \in int(M)$;
3. $p_0 = p_0^* \in \partial(M_{t_0})$.

We prove that the above possibilities do not occur. Details can be found in (Hinojosa 2000). So, the assumption that M is not a horizontal graph leads to a contradiction. □

In order to show that a horizontal graph is also a geodesic one we study a Dirichlet problem for the equation 4. The necessary *a priori* estimates for this approach are given in the following results.

Let $\Omega \subset \mathbb{P}$ be a bounded domain, such that the curve $\Gamma := \partial\Omega$ has geodesic curvature greater than or equal to 1. Let $\rho, f : \overline{\Omega} \to \mathbb{R}$ be functions such that $\rho|_{\partial\Omega} = f|_{\partial\Omega} \equiv 0$. Denote by $G := G_g(\rho)$ and $\widetilde{G} := G_h(f)$ their graphs and by h e H their mean curvatures respectively. Represent by $\widetilde{\mathcal{W}}$ the domain of $\mathbb{H}^3$ bounded by $\widetilde{G} \cup \widetilde{G}^*$ where $\widetilde{G}^*$ is the reflection of $\widetilde{G}$ with respect to the plane $\mathbb{P}$. With this notation, we have the following proposition.

PROPOSITION 2. *Suppose that H is a positive constant satisfying $|h| < H$. If h is constant or $|h| < 1$, we have $G \subset \widetilde{\mathcal{W}}$.*

Assuming that the geodesic curvature of $\partial\Omega$ is greater than or equal to one, the previous proposition allows to estimate $|\nabla\rho|$ by $|\nabla f|$. Moreover, it is easy to see that ρ is bounded in $\overline{\Omega}$ by a constant which does not depend on h. In fact, $|\sinh(\rho)| \leq \frac{|f|}{\inf_{\Omega}\{x_3\}}$.

Having the estimate in the boundary given by the above proposition, it is necessary now to get a global estimate in Ω. That is given by the following lemma.

LEMMA 3. *Let $\rho, H : \overline{\Omega} \to \mathbb{R}$ be real functions of class C^3 and C^1, respectively, such that*

$$div\left(\frac{\nabla\rho}{W}\right) = \frac{2\cosh\rho}{v^2}\left(H + \frac{\sinh\rho}{W}\right),$$

where $W^2 = \cosh^2\rho + v^2 |\nabla\rho|^2$ *and* $|\nabla\rho|^2 = \rho_u^2 + \rho_v^2$. *Let us assume that* ρ *is bounded in* Ω *and* $|\nabla\rho|$ *is bounded in* $\partial\Omega$. *Then* $|\nabla\rho|$ *is bounded in* Ω *by a constant that depends only on* $\sup_{\Omega}|\rho|$ *and* $\sup_{\partial\Omega}|\nabla\rho|$.

The proof of this lemma involves a long calculation based upon a suitable change of coordinates suggested by a corresponding result in (Caffarelli et al. 1988). The complete proof of Lemma 3 and Proposition 2 can be found in (Hinojosa 2000).

The theorem below gives a result concerning existence of geodesic graphs. The main corollary of this theorem affirms that, in the case of constant mean curvature, a horizontal graph is also geodesic. This allows to show the mentioned equivalence for these kind of graphs.

THEOREM 4. *Let* $\Omega \subset \mathbb{P}$ *be a bounded domain such that* $\partial\Omega$ *has curvature greater than or equal to 1. Let* $f : \overline{\Omega} \to \mathbb{R}$ *be a smooth function such that* $f|_{\partial\Omega} \equiv 0$ *and* $\widetilde{G} = G_h(f) \subset \mathbb{H}^3$ *be the horizontal graph of* f. *Let us assume that* $\widetilde{G}$ *has constant mean curvature* $H \neq 0$. *Then there exists* $\rho : \overline{\Omega} \to \mathbb{R}$ *such that* $\rho|_{\partial\Omega} \equiv 0$ *and the geodesic graph of* ρ, $G = G_g(\rho)$, *has also mean curvature* H.

COROLLARY 5. *Let* $\widetilde{G} \subset \mathbb{H}^3$ *be a horizontal graph of a smooth function* $f : \overline{\Omega} \subset \mathbb{P} \to \mathbb{R}$ *such that* $f|_{\partial\Omega} \equiv 0$, *where* $\Omega \subset \mathbb{P}$ *is a bounded domain such that* $\partial\Omega$ *has curvature greater than or equal to 1. Let us assume that* $\widetilde{G}$ *has constant mean curvature* $H \neq 0$, *then there exists a smooth function* ρ *that is zero at the boundary* $\partial\Omega$ *and for which* $\widetilde{G} = G_g(\rho)$.

We are going to sketch the proof of the Theorem 4:

The existence of such function $\rho : \overline{\Omega} \to \mathbb{R}$ is equivalent to the existence of a solution for the following Dirichlet problem:

$$\begin{cases} Q(\rho) = div\left(\frac{\nabla\rho}{W}\right) - \frac{2\cosh\rho}{v^2}\left(H + \frac{\sinh\rho}{W}\right) = 0 & \text{in } \Omega \\ \rho = 0 \quad \text{in } \partial\Omega\ , \end{cases} \tag{5}$$

where $W^2 = \cosh^2\rho + v^2 |\nabla\rho|^2$ and $|\nabla\rho|^2 = \rho_u^2 + \rho_v^2$.

In order to apply the Continuity Method as stated in (Gilbarg and Trudinger 1983) we construct a family of quasilinear eliptic operators Q_t, $t \in [0, 1]$ as follows.

The operator Q can be written in the following form

$$Q(\rho) = \frac{1}{W^3}\left(A^{11}\rho_{uu} + A^{12}\rho_{uv} + A^{22}\rho_{vv}\right) + \frac{B}{W^3} - \frac{2\cosh\rho}{v^2}\left(h + \frac{\sinh\rho}{W}\right)$$

where

$$A^{11} = \cosh^2\rho + v^2\rho_v^2, \quad A^{12} = -2v^2\rho_u\rho_v, \quad A^{22} = \cosh^2\rho + v^2\rho_u^2 \quad \text{and}$$
$$B = -(v\rho_v + \sinh\rho\cosh\rho)\,|\nabla\rho|^2.$$

Now we consider the functions $\mathcal{T}_1, \mathcal{T}_2 : [0,1] \to [0,1]$ given by,

$$\mathcal{T}_1(t) := \begin{cases} 2Ht\ , & \text{if } t \in \left[0, \frac{1}{2H}\right] \text{ and } H \geq 1 \\ 1\ , & \text{if } t \in \left[\frac{1}{2H}, 1\right] \text{ and } H \geq 1 \\ t\ , & \text{if } H < 1 \end{cases}$$

and

$$\mathcal{T}_2(t) := \begin{cases} t\ , & \text{if } H \geq 1 \\ 1\ , & \text{if } H < 1, \end{cases}$$

as well as the family of operators Q_t defined by

$$\begin{aligned} Q_t(\rho) &= \frac{1}{W^3}\left(A^{11}\rho_{uu} + A^{12}\rho_{uv} + A^{22}\rho_{vv}\right) \\ &\quad + \mathcal{T}_1(t)\left\{\frac{B}{W^3} - \frac{2\cosh\rho}{v^2}\left(\mathcal{T}_2(t)H + \frac{\sinh\rho}{W}\right)\right\}. \end{aligned}$$

We have also the associated Dirichlet problems

$$\begin{cases} Q_t(\rho) = 0 & \text{in } \Omega \\ \rho = 0 & \text{in } \partial\Omega\,. \end{cases} \tag{6}$$

In this way each solution of the Dirichlet problem given by the equations (6) represents a geodesic graph of mean curvature H_t, where

$$H_t = \mathcal{T}_1(t)\mathcal{T}_2(t)H + (\mathcal{T}_1(t) - 1)\,\frac{\sinh\rho}{W} + (1 - \mathcal{T}_1(t))\,\frac{v^2 B}{2W^3\cosh\rho}\ . \tag{7}$$

Therefore, we have

$$|H_t| \leq \mathcal{T}_1(t)\mathcal{T}_2(t)H + (1 - \mathcal{T}_1(t))\left\{\frac{|\sinh\rho|}{W} + \frac{v^2|B|}{2W^3\cosh\rho}\right\}.$$

Nevertheless, we prove that

$$\frac{|\sinh\rho|}{W} + \frac{v^2|B|}{2W^3\cosh\rho} \leq 1.$$

In this way

$$|H_t| \leq \mathcal{T}_1(t)\mathcal{T}_2(t)H + 1 - \mathcal{T}_1(t). \tag{8}$$

By comparison between the values of H_t and H while t varies we obtain the C^1 bound in the boundary, through the Proposition 2, for the solutions of the Dirichlet problems (6). The Lemma 3 supplies the C^1 uniform estimates in Ω, that is, there exists a constant $C_1 > 0$ such that

$$|\rho|_{1,\overline{\Omega}} := \sup_{\overline{\Omega}} |\rho| + \sup_{\overline{\Omega}} |\nabla\rho| \leq C_1$$

for any solution C^2 to the problem (6), where the constant C_1 is independent of t and H_t.

So, there exists a constant $C > 0$ independent of ρ and t such that any solution $C^{2,\alpha}\left(\overline{\Omega}\right)$ of the Dirichlet problems (6) satisfies *a priori*

$$|\rho|_{C^1} < C\,.$$

The Theorems 11.8 and 13.7 in (Gilbarg and Trudinger 1983) guarantee the existence of $\rho \in C^{2,\alpha}\left(\overline{\Omega}\right)$ solution of the problem

$$\begin{cases} Q_1(\rho) = 0 & \text{in } \Omega \\ \rho = 0 & \text{in } \partial\Omega. \end{cases} \qquad \square$$

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RESUMO

Consideramos superfícies com curvatura média constante no 3-espaço hiperbólico que são dadas como gráfico de uma função suave definida em um aberto limitado e simplesmente conexo contido em um hiperplano totalmente geodésico. Dos vários tipos de gráficos que podemos definir no espaço hiperbólico consideramos em particular o gráfico horizontal e o geodésico. Provamos que se a curvatura média é constante, então tais gráficos são equivalentes no seguinte sentido: suponha que M é uma superfície de curvatura média constante no 3-espaço hiperbólico tal que M é o gráfico geodésico de uma função ρ que se anula no bordo do seu domínio, então existe uma outra função suave f que também se anula no bordo e tal que M é o gráfico horizontal de f. Além disso, a recíproca é verdadeira.

Palavras-chave: espaço hiperbólico, gráfico geodésico e horizontal, curvatura média constante, equações diferenciais parciais elípticas.

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