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$W_0^{1,1}$ SOLUTIONS IN SOME BORDERLINE CASES OF CALDERON-ZYGMUND THEORY

LUCIO BOCCARDO * - THIERRY GALLOUET

1 Old results

Let Ω be a bounded open set in $\mathbb{R}^N$, $N \geq 2$.

1.1 Linear problems

$$\begin{cases} -\text{div}\big(M(x)\nabla u\big) = f(x), & \text{in } \Omega; \\ u = 0, & \text{on } \partial\Omega; \end{cases} \tag{1}$$

A If $M(x)$ is smooth, the Calderon-Zygmund [1] theory states

$$\boxed{f \in L^m(\Omega) \Rightarrow u \in W_0^{1,m^*}(\Omega),\ 1 < m < \infty};$$

B If $M(x)$ is only bounded and elliptic, Guido Stampacchia proved (by duality) the same result for $1 < m < \frac{2N}{N+2}$, that is for infinite energy solutions [2].

1.2 Nonlinear problems, $p = 2$

For nonlinear boundary value problems with differential operators of Leray-Lions type, we proved (last century) that

C $\boxed{\text{there exists a distributional solution } u \text{ such that [B] still holds.}}$

1.3 Nonlinear problems, $p \neq 2$

The simplest example of nonlinear boundary value problem is the Dirichlet problem for the p–Laplace operator, with $1 < p < N$,

$$\begin{cases} -\text{div}\big(|\nabla u|^{p-2}\nabla u\big) = f(x), & \text{in } \Omega; \\ u = 0, & \text{on } \partial\Omega; \end{cases} \tag{2}$$

so that the growth of the differential operator is $p - 1$. The classical theory of nonlinear elliptic equations states that $W_0^{1,p}(\Omega)$ is the natural functional spaces framework to find weak solutions of (2), if the function f belongs to the dual space of $W_0^{1,p}(\Omega)$.

This approach fails if $p = 1$.

On the one hand, if $p > 1$, for the model problem (2), the existence of $W_0^{1,p}(\Omega)$ solutions also fails if the right hand side is a function $f \in L^m(\Omega)$ ($m \geq 1$) which does not belong to the dual space of $W_0^{1,p}(\Omega)$: it is possible to find distributional solutions in function spaces "larger" than $W_0^{1,p}(\Omega)$, but contained in $W_0^{1,1}(\Omega)$ (see [1], [2]).

*Dip. di matematica Guido Castelnuovo, SAPIENZA Univ. di Roma , email: boccardo@mat.uniroma1.it

[1](see also recent results by Haim Brezis)

[2](If $m > \frac{2N}{N+2}$ the result is false)

We present our results only on model problem (2), not for general Leray-Lions operators with p-coercivity; existence of solutions with nonregular right hand side, for general nonlinear problems, are contained in [1], [2]; in particular, we recall the following results.

THEOREM 1.1. *Let $m = 1$ and $2-\frac{1}{N} < p < N$. Then there exists a distributional solution $u \in W_0^{1,q}(\Omega)$, $q < \frac{N(p-1)}{N-1}$, of (2).*

Observe that $\frac{N(p-1)}{N-1} > 1$ if and only if $p > 2 - \frac{1}{N}$.

THEOREM 1.2. *Let $2 - \frac{1}{N} < p < N$. If*

$$\int_\Omega |f| \log(1+|f|) < \infty, \tag{3}$$

then there exists a distributional solution $u \in W_0^{1,\frac{N(p-1)}{N-1}}(\Omega)$ of (2).

THEOREM 1.3 (Calderon-Zygmund theory for infinite energy solutions). *If $f \in L^m(\Omega)$, $\sup\left(1, \frac{N}{N(p-1)+1}\right) < m < \frac{Np}{pN+p-N} = (p^\star)'$,*
$p > 1 + \frac{1}{m} - \frac{1}{N}$, then there exists a distributional solution $u \in W_0^{1,(p-1)m^\star}(\Omega)$ of (2).

REMARK 1.4. *Of course if $p = 2$ the above result is [C] of Subsection 1.2.*

2 New: existence results in $W_0^{1,1}(\Omega)$

Now we will state existence results of $W_0^{1,1}(\Omega)$ distributional solutions. The existence is a consequence of the fact that we improve the existence of Theorem 1.2 and Theorem 1.3 in some borderline cases.

THEOREM 2.1. *Let $f \in L^m(\Omega)$, $m = \frac{N}{N(p-1)+1}$, $1 < p < 2 - \frac{1}{N}$. Then there exists a distributional solution $u \in W_0^{1,1}(\Omega)$ of (2).*

THEOREM 2.2. *Assume (3) and $p = 2 - \frac{1}{N}$. Then there exists a distributional solution $u \in W_0^{1,1}(\Omega)$ of (2).*

References

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