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p-LAPLACIAN WITH FAST GROWING GRADIENT*

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We consider the existence of positive solutions for the Dirichlet problem in two positive parameters λ and β

$$\begin{cases} -\Delta_p u &= \lambda h(x,u) + \beta f(x,u,\nabla u) \quad in \ \Omega, \\ u &= 0 \qquad\qquad\qquad\qquad\quad on \ \partial\Omega, \end{cases} \tag{P}$$

where λ and β are positive parameters and h, f are continuous nonlinearities satisfying

(H1) $0 \le \omega_1(x)u^{q-1} \le h(x,u) \le \omega_2(x)u^{q-1}$ $(1 < q < p)$;

(H2) $0 \le f(x,u,v) \le \omega_3(x)u^a|v|^b$,

where ω_i, $1 \le i \le 3$, are nonnegative weights in the smooth, bounded and *convex* domain $\Omega \subset \mathbb{R}^N$ $(N \ge 3)$ and $a, b > 0$. We prove the existence of a region $\mathcal{D}$ in the $\lambda\beta$-plane where the Dirichlet problem has at least one positive solution.

Since the nonlinearity in this problem may depend on the gradient with an exponent higher than p, variational methods can not handle directly this kind of problem. Here we show how simple techniques (sub- and super-solution method combined with fixed point theory) are able to solve some problems involving nonlinearities with fast growing gradient.

Our proof rests on the major result of Cianchi and Maz'ya [2], which provides *a priori* bounds for the gradient. In order to simplify our presentation, we suppose that the domain of our problem is smooth, bounded and *convex*. (More general hypotheses are presented in that paper; so, our work is immediately generalizable.)

Theorem [Cianchi and Maz'ya] *Assume that $\Omega \subset \mathbb{R}^N$, $N \ge 3$, is a bounded and convex domain and that $g \in C(\overline{\Omega})$. Then there exists a positive constant $\mathcal{K}$, depending only on p and Ω, such that*

$$\|\nabla u\|_\infty \le \mathcal{K}\,\|g\|_\infty^{\frac{1}{p-1}}$$

where $u \in C^1(\overline{\Omega}) \cap W_0^{1,p}(\Omega)$ is the only weak solution of

$$\begin{cases} -\Delta_p u &= g \quad in \ \Omega \\ u &= 0 \quad on \ \partial\Omega. \end{cases}$$

Let u_1 be the first eigenfunction of the p-Laplacian with weight ω_1, that is,

$$\begin{cases} -\Delta_p u_1 &= \lambda_1\omega_1 u_1^{p-1} \quad \text{in } \Omega, \\ u_1 &= 0 \qquad\qquad\quad \text{on } \partial\Omega, \end{cases}$$

with $\|u_1\|_\infty = 1$ and let $\phi \in W_0^{1,p}(\Omega) \cap C^{1,\alpha}(\overline{\Omega})$ be the solution of the problem

$$\begin{cases} -\Delta_p \phi &= \omega \quad \text{in } \Omega \\ \phi &= 0 \quad \text{on } \partial\Omega, \end{cases}$$

where $\omega(x) := \max_{i\in\{1,2,3\}} w_i(x)$.

We denote

$$\gamma := \frac{\mathcal{K}\,\|\omega\|_\infty^{\frac{1}{p-1}}}{\|\phi\|_\infty},$$

where $\mathcal{K}$ stand for the constant of the result of Cianchi and Maz'ya.

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Our main result is

Theorem 1. *Assume that h and f are continuous and satisfy* (H1) *and* (H2). *There exists a region $\mathcal{D}$ in the $\lambda\beta$-plane such that if $(\lambda,\beta)\in\mathcal{D}$ the Dirichlet problem (P) has at least one positive solution u satisfying, for some positive constants ϵ and M:*

$$\epsilon u_1 \leq u \leq \frac{M\phi}{\|\phi\|_\infty}$$

and

$$\|\nabla u\|_\infty \leq \frac{\mathcal{K}\,\|\omega\|_\infty^{\frac{1}{p-1}}\,M}{\|\phi\|_\infty}.$$

To prove our result we define for each $u\in C^1$,

$$F^u(x,\xi) := \lambda\omega_1(x)\xi^{q-1} + \lambda\left(h(x,u(x)) - \omega_1(x)u(x)^{q-1}\right) + \beta f(x,u(x),\nabla u(x)) \in C(\overline{\Omega}).$$

We observe that $F^u(x,u) = \lambda h(x,u) + \beta f(x,u,\nabla u)$.

We start by considering an auxiliary problem:

Lemma 1. *There exist positive constants ϵ and M, and a region $\mathcal{D}$ in the $\lambda\beta$-plane such that, if $(\lambda,\beta)\in\mathcal{D}$ and*

$$u\in\mathcal{F} := \left\{u\in C^1(\overline{\Omega}) : 0\leq u\leq \frac{M\phi}{\|\phi\|_\infty} \quad \text{and} \quad \|\nabla u\|_\infty \leq \gamma M\right\},$$

then the Dirichlet problem

$$\left\{\begin{array}{rcll} -\Delta_p U &=& F^u(x,U) & \text{in } \Omega \\ U &=& 0 & \text{on } \partial\Omega. \end{array}\right. \tag{P$_a$}$$

has $\underline{u} := \epsilon u_1$ and $\overline{u} = \frac{M\phi}{\|\phi\|_\infty}$ as an ordered pair of sub- and super-solution and a unique positive solution U.

The lemma is proved by applying the sub- and super-solution method, inspired by the seminal paper of Ambrosetti, Brezis and Cerami [1]. The uniqueness of U follows from well-known results proved in [3].

So, the combined effects of the sublinear and superlinear terms make possible the definition of the operator

$$\begin{array}{rcl} T\colon \mathcal{F}\subset C^1(\overline{\Omega}) & \longrightarrow & C^{1,\alpha}(\overline{\Omega})\cap W_0^{1,p}(\Omega)\subset C^1(\overline{\Omega}) \\ u & \longrightarrow & U, \end{array}.$$

The existence of a fixed point of T follows from Schauder's fixed point theorem.

References

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