Algunas cuestiones sobre problemas inversos geométricos

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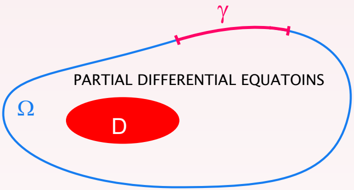
Goal

Essentially, to identify a domain where a PDE is satisfied

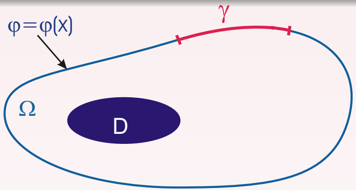


Consider 3 different PDE models:

- Laplace (identify a domain from electrical tests)
- Navier-Stokes and Boussinesq: related to fluid mechanics (identifying a rigid body, immersed in a fluid)
- Wave equation and Lamé systems, motivated by the Electrography (a non-invasive method of tumor detection: when a mechanical compression or vibration is applied, the tumor deforms less than the surrounding tissue)

Outline

1. Laplace equation
   - Uniqueness
   - Sketch of the proof
   - Partial identification of D (an algorithm)
2. Navier-Stokes and Boussinesq
   - Uniqueness
3. Wave equations
   - N-dimensional
   - Lamé systems

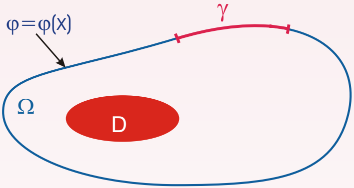


Direct problem

- Given data: D, Ω, φ, $\gamma \subset \partial\Omega$
- Find u

$$\begin{cases} -\Delta u = 0 & \text{in} \quad \Omega \setminus \overline{D} \\ u = \varphi & \text{on} \quad \partial\Omega \\ u = 0 & \text{on} \quad \partial D \end{cases}$$

- Observation: $\alpha = \left.\dfrac{\partial u}{\partial n}\right|_{\gamma}$



Inverse problem

- Given data: Ω, φ, $\gamma \subset \partial\Omega$
- Additional data (observation): $\alpha = \left.\dfrac{\partial u}{\partial n}\right|_{\gamma}$
- Find: D (and then the solution u)

$$\begin{cases} -\Delta u = 0 & \text{in} \quad \Omega \setminus \overline{D} \\ u = \varphi & \text{on} \quad \partial\Omega \\ u = 0 & \text{on} \quad \partial D \end{cases}$$

Uniqueness

Assume: D^i simply connected, ∂D^i Lipshchitz, $\varphi \neq 0$

$$\begin{cases} -\Delta u^i = 0 & \text{in} \quad \Omega \setminus \overline{D^i}, \quad i = 0, 1 \\ u^i = \varphi & \text{on} \quad \partial\Omega \\ u^i = 0 & \text{on} \quad \partial D^i \end{cases}$$

Theorem

$$\frac{\partial u^0}{\partial n} = \frac{\partial u^1}{\partial n} \quad on\, \gamma \subset \partial\Omega \quad \Longrightarrow \quad D^0 = D^1$$

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Consequence of **unique continuation property**

Uniqueness

Assume: D^i simply connected, ∂D^i Lipshchitz, $\varphi \neq 0$

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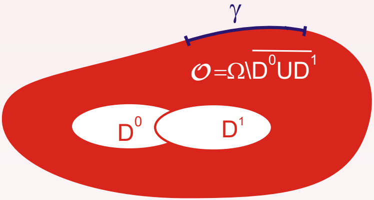
Consequence of **unique continuation property**

Some previous works, several authors...

- Alessandrini & al.
- Andrieux, Abda & Jaoua
- Isakov
- Kavian

(a) In $\mathcal{O} = \Omega \setminus \overline{D^0 \cup D^1}$

$w := u^0 - u^1$



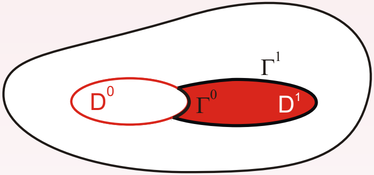
$$\begin{cases} -\Delta w = 0 & \text{in} \quad \mathcal{O} \\ w = 0 & \text{on} \quad \partial\Omega \\ \dfrac{\partial w}{\partial n} = 0, & \text{on} \quad \gamma \end{cases}$$

Unique continuation $\Longrightarrow$ $w = 0$ in $\mathcal{O}$ $\Longleftrightarrow$

$$u^0 = u^1 \quad \text{in } \mathcal{O} = \Omega \setminus \overline{D^0 \cup D^1}$$

Sketch of the proof of uniqueness result

(b) In $D^1 \setminus \overline{D^0} \neq \emptyset$,
$\partial(D^1 \setminus \overline{D^0}) = \Gamma^0 \cup \Gamma^1$



$$\begin{cases} -\Delta u^0 = 0 & \text{in} \quad D^1 \setminus \overline{D^0} \\ u^0 = u^1 = 0 & \text{on} \quad \Gamma^1 \\ u^0 = 0 & \text{on} \quad \Gamma^0 \end{cases}$$

Uniqueness of zero solution $\Rightarrow u^0 = 0$ in $D^1 \setminus \overline{D^0}$

Unique continuation $\Rightarrow u^0 \equiv 0$ in $\Omega \setminus \overline{D^0}$

An absurd!: $u^0 = \varphi \neq 0$ on $\partial\Omega$ on $\partial\Omega$

Then $D^1 \setminus \overline{D^0} = \emptyset$ (analogously $D^0 \setminus \overline{D^1} = \emptyset$) $\Rightarrow D^0 = D^1$

Assume: D is known

$$D + m = \{x + m(x) : x \in D\}, \quad \alpha^m = \frac{\partial u^m}{\partial n}\Big|_\gamma \text{ is known}$$

Find m ?

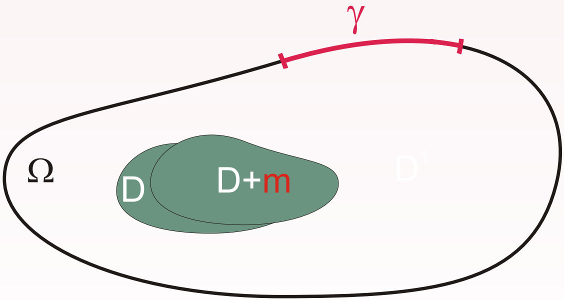


Figure: Deformations of D

Partial identification of D

Assume: $m = 0$ near $\partial\Omega$

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega \setminus \overline{D} \\ u = \varphi & \text{on } \partial\Omega \\ u = 0 & \text{on } \partial D \end{cases} \qquad \begin{cases} -\Delta u^m = 0 & \text{in } \Omega \setminus (\overline{D + m}) \\ u^m = \varphi & \text{on } \partial\Omega \\ u^m = 0 & \text{on } \partial(D + m) \end{cases}$$

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Theorem

Assume: $\varphi \in H^{3/2}(\partial\Omega)$, $\varphi \neq 0$, $\alpha = \left.\dfrac{\partial u}{\partial n}\right|_\gamma$, $\alpha^m = \left.\dfrac{\partial u^m}{\partial n}\right|_\gamma$

Also, assume: $(m \cdot n)|_{\partial D} \in M$, $\mathbf{dim}\, M < \infty$, $M \subset W^{1,\infty}(\partial D)$

Then: $\exists\, G : L^2(\gamma) \mapsto M$ *computable such that*

$$(m \cdot n)|_{\partial D} = G(\alpha^m - \alpha) + o(m) \quad \textit{for small } m$$

Sketch of the proof

In the proof we will use essentially two techniques:

1. Domain variation techniques (F. Murat, J. Simon).

   Allow to write $\alpha^m - \alpha$ *in terms of a linear operator applied to* m*.*

2. Data assimilation techniques (J.P. Puel)

   Allow to identify linear operator using control theory

Consider $u'(m)$:

$$\begin{cases} -\Delta u'(m) = 0 & \text{in } \Omega \setminus \overline{D} \\ u'(m) = 0 & \text{on } \partial\Omega \\ u'(m) = -(m \cdot n)\dfrac{\partial u}{\partial n} & \text{on } \partial D \end{cases}$$

Then:

$$\frac{\partial u'(m)}{\partial n} = \alpha^m - \alpha + o(m) \quad \text{on } \gamma$$

The task: compute $(m \cdot n)|_{\partial D}$ from $\left.\dfrac{\partial u'(m)}{\partial n}\right|_{\gamma}$ ("known")

Goal: find $(m \cdot n)|_{\partial D}$ from $\left.\frac{\partial u'(m)}{\partial n}\right|_{\gamma}$

Compute $(m \cdot n)|_{\partial D} \in M \Leftrightarrow$ compute $\displaystyle\int_{\partial D} (m \cdot n) \frac{\partial u}{\partial n} h \, ds, \ \forall h \in M$

Goal: find $(m \cdot n)|_{\partial D}$ from $\left.\frac{\partial u'(m)}{\partial n}\right|_\gamma$

Compute $(m \cdot n)|_{\partial D} \in M \Leftrightarrow$ compute $\int_{\partial D} (m \cdot n) \frac{\partial u}{\partial n} h\, ds,\ \forall h \in M$

$(m \cdot n)|_{\partial D} \in M$, $\dim M < \infty$, $\Longrightarrow \frac{\partial u'(m)}{\partial n}|_{\partial D} \in E$, $\dim E < \infty$.

Assume the following control problem is solvable $\forall\, h \in M$:

$$\begin{cases} -\Delta\theta_h = 0 & \text{in } \Omega \setminus \overline{D} \\ \theta_h = v 1_\gamma & \text{on } \partial\Omega \\ \dfrac{\partial\theta_h}{\partial n} = h & \text{on } \partial D \end{cases}$$

$P_E(\theta_h|_{\partial D}) = 0 \quad P_E : L^2(\partial\Omega) \mapsto E$ ortogonal projector

Goal: find $(m \cdot n)|_{\partial D}$ from $\left.\frac{\partial u'(m)}{\partial n}\right|_\gamma$

Compute $(m \cdot n)|_{\partial D} \in M \Leftrightarrow$ compute $\int_{\partial D} (m \cdot n) \frac{\partial u}{\partial n} h \, ds, \; \forall h \in M$

$(m \cdot n)|_{\partial D} \in M, \dim M < \infty, \Longrightarrow \frac{\partial u'(m)}{\partial n}|_{\partial D} \in E, \dim E < \infty.$

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$P_E(\theta_h|_{\partial D}) = 0 \quad P_E : L^2(\partial\Omega) \mapsto E$ ortogonal projector

$$-\int_{\partial D} (m \cdot n) \frac{\partial u}{\partial n} h \, ds = \int_\gamma \frac{\partial u'(m)}{\partial n} v \, ds \quad \forall \; h \in M$$

$$\begin{cases} -\Delta u'(m) = 0 & \text{in } \Omega \setminus \overline{D} \\ u'(m) = 0 & \text{on } \partial\Omega \\ u'(m) = -(m \cdot n)\dfrac{\partial u}{\partial n} & \text{on } \partial D \end{cases} \qquad \begin{cases} -\Delta \theta_h = 0 & \text{in } \Omega \setminus \overline{D} \\ \theta_h = v 1_\gamma & \text{on } \partial\Omega \\ \dfrac{\partial \theta_h}{\partial n} = h & \text{on } \partial D \end{cases}$$

$$P_E(\theta_h|_{\partial D}) = 0 \quad P_E : L^2(\partial\Omega) \mapsto E \text{ ortogonal projector}$$

$$\begin{aligned} -\int_{\partial D} (m \cdot n)\frac{\partial u}{\partial n}\, h\, ds &= \int_{\partial D \cup \partial\Omega} u'(m)\frac{\partial \theta_h}{\partial n}\, ds \\ &= \int_{\partial D} \frac{\partial u'(m)}{\partial n} P_E(\theta_h|_{\partial D})\, ds + \int_{\partial\Omega} \frac{\partial u'(m)}{\partial n}\theta_h\, ds \\ &= \int_{\gamma} \frac{\partial u'(m)}{\partial n} v\, ds \qquad \forall\, h \in M \end{aligned}$$

$$\begin{cases} -\Delta\theta_h = 0 & \text{in } \Omega \setminus \overline{D} \\ \theta_h = v 1_\gamma & \text{on } \partial\Omega \\ \dfrac{\partial \theta_h}{\partial n} = h & \text{on } \partial D \end{cases}$$

$$P_E(\theta_h|_{\partial D}) = 0$$

An exact finite-controllability problem

Unique continuation $\Longrightarrow$ Existence

In fact: $\forall\, \varepsilon > 0\ \exists\, v$ such that

$$P_E(\theta_h|_{\partial D}) = 0, \quad ||\theta_h|_{\partial D}||_{L^2} \leq \varepsilon$$

(E. Zuazua)

Algorithm: partial identification of D up to $o(m)$

Assume:

- $\mathcal{I} = \mathbf{dim}\, M$, $\{h^1, \dots, h^{\mathcal{I}}\}$ basis of M,

$$(m \cdot n)|_{\partial D} = \sum_{i=1}^{\mathcal{I}} \lambda_i h^i$$

- v^i, $i = 1, \dots, \mathcal{I}$ controls with $h = h^i$

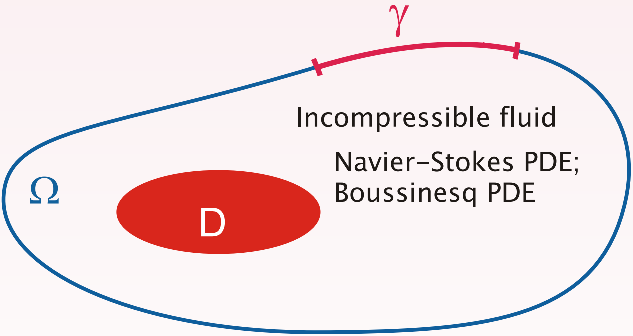
Then: computation of $\lambda_i \iff$ resolution of a linear system:

$$\sum_{i=1}^{\mathcal{I}} H_{ij} \lambda_i = \int_\gamma (\alpha^m - \alpha)\, v^j\, ds, \quad 1 \le j \le \mathcal{I}$$

where

$$H_{ij} = \int_{\partial D} \left(-\frac{\partial u}{\partial n} \right) h^i\, h^j\, ds$$

Geometric inverse problem for fluids



Inverse problem

Find a rigid body D

Inverse problem for stationary Boussinesq system

Find D from Ω, (φ, ψ), (α, β) with

$$\begin{cases} -\nu\Delta u + (u\cdot\nabla)u + \nabla p = \theta g, \quad \nabla\cdot u = 0 & \text{in} \quad \Omega\setminus\overline{D} \\ -\kappa\Delta\theta + u\cdot\nabla\theta = 0 & \text{in} \quad \Omega\setminus\overline{D} \\ u = \varphi, \quad \theta = \psi & \text{on} \quad \partial\Omega \\ u = 0, \quad \theta = 0 & \text{on} \quad \partial D \end{cases}$$

$(u, p, \theta) =$ (velocity, pressure, temperature)
ν =kinematic viscosity; κ =thermal conductivity; g =gravity

Inverse problem for stationary Boussinesq system

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$(u, p, \theta) =$ (velocity, pressure, temperature)
ν =kinematic viscosity; κ =thermal conductivity; g =gravity

$$\left.\begin{array}{l} \sigma(u,p)\cdot n := (-p\,\mathrm{Id} + \nu(\nabla u + {}^t\nabla u))\cdot n = \alpha \\ \kappa\dfrac{\partial\theta}{\partial n} = \beta \end{array}\right\} \quad \text{on } \gamma\subset\Omega$$

Some previous works

Fluids:

- Alvarez, Conca, Friz, Kavian, & Ortega.
- Heck, Uhlmann & Wang
- AD, Fernández-Cara, González-Burgos & Ortega

Assume: $(\varphi, \psi) \neq (0,0)$ given; $\partial\Omega$, $\partial D^i \in C^{1,1}$, $i = 0, 1$

$$\begin{cases} -\nu\Delta u^i + (u^i \cdot \nabla)u^i + \nabla p^i = \theta^i g, \quad \nabla \cdot u^i = 0 & \text{in} \quad \Omega \setminus \overline{D^i} \\ -\kappa\Delta\theta^i + u^i \cdot \nabla\theta^i = 0 & \text{in} \quad \Omega \setminus \overline{D^i} \\ u^i = \varphi, \quad \theta^i = \psi & \text{on} \quad \partial\Omega \\ u^i = 0, \quad \theta^i = 0 & \text{on} \quad \partial D^i \end{cases}$$

and

$$\sigma(u^0, p^0) \cdot n = \sigma(u^1, p^1) \cdot n \quad \text{and} \quad \kappa\frac{\partial\theta^0}{\partial n} = \kappa\frac{\partial\theta^1}{\partial n} \quad \text{on } \gamma \subset \Omega$$

Theorem

$$D^0 = D^1$$

For the proof

Unique continuation property

$a, b \in L^\infty(G)^N, d \in L^\infty(G)$ and $\nabla \cdot a = \nabla \cdot b = 0$ in G

$$\begin{cases} -\nu\Delta v + (a\cdot\nabla)v + (v\cdot\nabla)b + \nabla q = \eta g, \quad \nabla\cdot v = 0 & \text{in} \quad G \\ -\kappa\Delta\eta + a\cdot\nabla\eta + v\cdot\nabla d = 0 & \text{in} \quad G \end{cases}$$

(a generalization of a result by Fabre & Lebeau, 1996)

Theorem

Let $G \subset \mathbb{R}^N$, $\omega \subset G$ (open)

$$v = 0 \text{ and } \eta = 0 \text{ in } \omega \Longrightarrow v \equiv 0 \text{ and } \eta \equiv 0 \ (q \equiv Const.)$$

Corollary

Let $\Gamma \subset \partial G$ (open)

$$\left.\begin{array}{ll} v = 0, \quad \eta = 0 & \text{on } \Gamma \\ \sigma(v,q)\cdot n = 0, \quad \kappa\dfrac{\partial\eta}{\partial n} = 0 & \text{on } \Gamma \end{array}\right\} \Longrightarrow v \equiv 0 \text{ and } \eta \equiv 0$$

Some comments on uniqueness

- More complicated systems: OK (Appropriate cond. on f_i)

$$\begin{cases} -\nu\Delta u + (u\cdot\nabla)u + \nabla p = f_0(\theta,\eta), \quad \nabla\cdot u = 0 & \text{in} \quad \Omega\setminus\overline{D} \\ -\kappa\Delta\theta + u\cdot\nabla\theta = f_1(\theta,\eta) & \text{in} \quad \Omega\setminus\overline{D} \\ -a\Delta\eta + u\cdot\nabla\eta = f_2(\theta,\eta) & \text{in} \quad \Omega\setminus\overline{D}\ldots \end{cases}$$

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- Time-dependent: OK if $u|_{t=0} = 0$, $\theta|_{t=0} = 0$. Other dada?

$$\begin{cases} u_t - \nu\Delta u + (u\cdot\nabla)u + \nabla p = \theta g, \quad \nabla\cdot u = 0 & \text{in} \quad \Omega\setminus\overline{D}\times(0,T) \\ \theta_t - \kappa\Delta\theta + u\cdot\nabla\theta = 0 & \text{in} \quad \Omega\setminus\overline{D}\times(0,T) \\ u(x,0) = u^0(x), \quad u(x,0) = \theta^0(x) & \text{in} \quad \Omega\setminus\overline{D} \end{cases}$$

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- Only one observation $\sigma\cdot n$ on γ ?

The needed unique continuation property: $a, b \in L^\infty(G)^N$, $d \in L^\infty(G)$, $\nabla\cdot a = \nabla\cdot b = 0$ in G

$$\begin{cases} -\nu\Delta v + (a\cdot\nabla)v + (v\cdot\nabla)b + \nabla q = \eta g, \quad \nabla\cdot v = 0 & \text{in} \quad G \\ -\kappa\Delta\eta + a\cdot\nabla\eta + v\cdot\nabla d = 0 & \text{in} \quad G \end{cases}$$

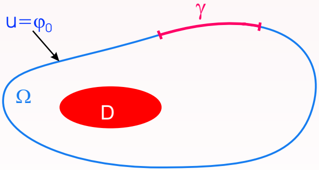
$v = 0$ and $\eta = 0$ in ∂G, $\sigma\cdot n = 0$ on $\gamma \Longrightarrow v \equiv 0$ and $\eta \equiv 0$

joint work in progress with E. Fernández Cara (Univ. de Sevilla)

$$\begin{cases} u_{tt} - \Delta u = 0 & \text{in } \Omega \setminus \overline{D} \times (0,T) \\ u = \varphi_0 & \text{in } \partial\Omega \times (0,T) \\ u = 0 & \text{in } \partial D \times (0,T) \\ u(x,0) = 0, \quad u_t(x,0) = 0 & \text{in } \Omega \setminus \overline{D} \end{cases}$$

Inverse problem

Given data: Ω, T, φ_0, $\gamma \subset \partial\Omega$ (open),
$$\alpha = \frac{\partial u}{\partial n}\Big|_{\gamma \times (0,T)}$$
Find D



Uniqueness

$$\begin{cases} u^i_{tt} - \Delta u^i = 0 & \text{in } \Omega \setminus \overline{D^i} \times (0,T), \quad i = 0,1 \\ u^i = \varphi_0 & \text{in } \partial\Omega \times (0,T) \\ u^i = 0 & \text{in } \partial D \times (0,T) \\ u^i(x,0) = 0, \quad u^i_t(x,0) = 0 & \text{in } \Omega \setminus \overline{D^i} \end{cases}$$

Theorem

Assume D^0, D^1 are open convex, $T > T^*(\Omega,\gamma)$,
$T^*(\Omega,\gamma) = \sup_{x\in\overline{\Omega}} d_c(x,\gamma) + \delta_c(\Omega)$

$$\frac{\partial u^0}{\partial n} = \frac{\partial u^1}{\partial n} \quad \text{on } \gamma \times (0,T) \quad \Longrightarrow \quad D^0 = D^1$$

Fundamental results: Hörmander

Attention: Weaker than the usual geometric condition on $\{\gamma, T\}$
(Only uniqueness, not observability!)

Partial identification of D

Assume: the solution is a ball $D = B(x^0, r)$

$$m = (d, s), \quad D + m = B(x^0 + d, r + s)$$

Known: $\alpha = \frac{\partial u}{\partial n}|_{\gamma\times(0,T)}$ and $\alpha^m = \frac{\partial u^m}{\partial n}|_{\gamma\times(0,T)}$

$$\alpha^m - \alpha = L(d, s) + \frac{1}{2}K((d, s), (d, s)) + o(|(d, s)|^2)$$

$$L(d, s) = \frac{\partial z}{\partial n}|_{\gamma\times(0,T)}, \quad K((d, s), (d, s)) = \frac{\partial w}{\partial n}|_{\gamma\times(0,T)}$$

where...

Partial identification of D

$$\begin{cases} z_{tt} - \Delta z = 0 & \text{in } \Omega \setminus \overline{B(x^0, r)} \times (0, T) \\ z|_{t=0} = z_t|_{t=0} & \text{in } \Omega \setminus \overline{B(x^0, r)} \\ z = 0 & \text{on } \partial\Omega \times (0, T) \\ z = -\frac{\partial u}{\partial n}(d \cdot n - s) & \text{on } \partial B(x^0, r) \times (0, T) \end{cases}$$

$$\begin{cases} w_{tt} - \Delta w = 0 \quad \text{in} \quad \Omega \setminus \overline{B(x^0, r)} \times (0, T) \\ w|_{t=0} = w_t|_{t=0} \quad \text{in} \quad \Omega \setminus \overline{B(x^0, r)} \\ w = 0 \qquad \text{on} \quad \partial\Omega \times (0, T) \\ w = -\nabla z \cdot (d - sn) - \frac{\partial z}{\partial n}(d \cdot n - s) - \frac{\partial^2 u}{\partial x_i \partial x_j}(d_j - sn_j)n_i(d \cdot n - s) \end{cases}$$

Partial identification of D

Algorithm

Known data: α, $\widetilde{\alpha}$, L, K

$$\inf_{(d,s)\in\mathcal{U}} ||(\widetilde{\alpha}-\alpha) - L(d,s) - \frac{1}{2}K((d,s),(d,s))||^2_{L^2(\gamma\times(0,T))}$$

An extremal problem in $\mathbb{R}^N \times \mathbb{R}$ for a polynomial of order 4, iterate until $||\widetilde{\alpha}-\alpha||_{L^2(\gamma\times(0,T))} \leq \varepsilon$

$$\begin{cases} -\nabla \cdot (\mu(x)(\nabla u + \nabla u^t)) - \nabla(\lambda(x)\nabla \cdot u) = 0 & \text{in } \Omega \setminus \overline{D} \\ u = \varphi & \text{on } \partial\Omega \\ u = 0 & \text{on } \partial D \end{cases}$$

Observation: $\alpha = \mu(\nabla u + \nabla u^t) \cdot n + \lambda(\nabla \cdot u)n$ on γ

Key point: **Unique continuation property**

Several results: Dehman, Robbiano, Yamamoto, Weck...

- Lin & Wang (2005): $N = 2$, λ, μ Lipschitz
- Escauriaza (2005): $N = 2$, μ Lipschitz, $\lambda \in L^\infty$
- Alessandrini & Morasi (2001): $N \geq 2$, λ, μ Lipschitz

In that case: **uniqueness** and partial **reconstruction**

Stationary elasticity systems
Anisotropic

$$\begin{cases} \nabla \cdot \sigma(u) = 0, \quad \sigma_{kl}(u) = \sum_{i,j=1}^{3} a_{ijkl}\varepsilon_{ij}(u) \\ \varepsilon_{kl}(u) = \frac{1}{2}(\partial_k u_l + \partial_l u_k) \\ \dots \end{cases}$$

Observation: $\alpha = \sigma(u) \cdot n$ on γ

Key point: **Unique continuation property**

Nakamura & Wang (2006): $N = 2$, a_{ijkl} Lipschitz

Again: **uniqueness**, partial **reconstruction**

Question

Conditions on a_{ijkl} for $N \geq 3$?

$$\begin{cases} u_{tt} - \nabla \cdot (\mu(x)(\nabla u + \nabla u^t)) - \nabla(\lambda(x)\nabla \cdot u) = 0 & \text{in } \Omega \setminus \overline{D} \times (0,T) \\ u = \varphi & \text{on } \partial\Omega \times (0,T) \\ u = 0 & \text{on } \partial D \times (0,T) \\ u(0) = 0, \quad u_t(0) = 0 & \text{in } \Omega \setminus \overline{D} \end{cases}$$

Observation: $\alpha = \mu(\nabla u + \nabla u^t) \cdot n + \lambda(\nabla \cdot u)n$ on $\gamma \times (0,T)$

Key point: **Unique continuation property**

- Uniqueness and partial identification are OK for constant or regular μ, λ (as wave eqution)
- Under consideration: for more general μ, λ