An Introduction to Stochastic Analysis on Manifolds I

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João Pessoa - March, 2017
Overview of the talks

- The mini-course comprises three one-hour talks, which we now describe.

1. This talk introduces the probabilistic concepts needed to precisely formulate Itô’s foundational result on the existence of diffusion processes associated to second order elliptic operators on a smooth manifold.

2. Here we discuss Itô formula, the central result in Itô’s Stochastic Calculus. This is used to prove the result on diffusions mentioned above. We then illustrate the power of these probabilistic methods by discussing basic properties of diffusions on Riemannian manifolds, including the recurrence/transience dichotomy and the Liouville property.

3. This talk is largely based on the preprint arXiv:1607.05646. We prove an integral test describing the exact cut-off between recurrence and transience for normally reflected Brownian motion in certain unbounded domains in a class of warped product manifolds. Besides extending a previous result by R. Pinsky, who treated the case in which the ambient space is flat, our result recovers the classical test for the standard Brownian motion in model spaces. Moreover, it allows us to discuss the recurrence/transience dichotomy for certain generalized tube domains around totally geodesic submanifolds in hyperbolic space. A refinement of the technique is further used to establish similar integral tests for the validity of the Liouville property (absence of non-trivial bounded harmonic functions satisfying Neumann boundary condition) on such domains.
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A few selected quotes

- “The probabilistic approach makes many problems in the theory of differential equations very transparent; it enables one to carry out exact proofs and discover new effects. It is the latter - the possibility of seeing new effects - which seem to us the most significant merit of the probabilistic method” (M. Freidlin in “Functional integration and partial differential equations”)

- “The interplay of probability theory and partial differential equations forms a fascinating part of mathematics. Among the subjects it has inspired are the martingale problems of Stroock and Varandhan, the Harnack inequality of Krylov and Safonov, the theory of symmetric diffusions processes, and the Malliavin calculus” (R. F. Bass. in “Diffusions and elliptic operators”)

- “These equations found applications without much delay: for example as approximations of complicated Markov chains arising in population and ecology models in biology (W. Feller), in electrical engineering where $dW$ models white noise (N. Wiener, I. Gelfand, T. Kailath), in chemical reactions (e.g., L. Arnold), in quantum physics (P. A. Meyer, L. Accardi, etc.), in differential geometry (K. Elworthy, M. Emery), in mathematics (harmonic analysis (Doob), potential theory (G. Hunt, R. Getoor, P. A. Meyer), PDEs, complex analysis, etc.), and, more recently and famously, in mathematical finance (P. Samuelson, F. Black, R. Merton, and M. Scholes)... It is hard to imagine a mathematician whose work has touched so many different areas of applications, other than Isaac Newton and Gottfried Leibniz. The legacy of Kyosı Itô will live on a long, long time.” (P. Protter, in NAMS, 2007)

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Motivation: integrating vector fields

- If $\beta : \mathbb{R}^n \to \mathbb{R}^n$ is a smooth vector field and $x_0 \in \mathbb{R}^n$, then there exists a unique map $\varphi^{x_0} : [0, +\infty) \to \mathbb{R}^n$ such that:

$$\varphi^{x_0}(0) = x_0 \quad \text{and} \quad \frac{d}{dt} \varphi^{x_0}(t) = \beta(\varphi^{x_0}(t)), \quad t \geq 0. \quad (1)$$

- The collection of all such paths defines a flow of diffeomorphisms in $\mathbb{R}^n$:

$$(t, x) \in [0, +\infty) \times \mathbb{R}^n \mapsto \varphi_t(x) = \varphi^x(t) : \mathbb{R}^n \to \mathbb{R}^n.$$

- If $f \in \mathcal{C}^\infty(\mathbb{R}^n)$ then a computation shows that

$$M^{f,x_0}_t := f(\varphi_t(x_0)) - f(x_0) - \int_0^t (\beta(f))(\varphi_s(x_0)) \, ds$$

satisfies

$$M^{f,x_0}_t = 0. \quad (2)$$

This essentially reflects the fact that $\beta$ is a derivation in the sense that

$$\beta(fg) = f\beta(g) + g\beta(f), \quad f, g \in \mathcal{C}^\infty(\mathbb{R}^n).$$

Theorem

A path $t \in [0, +\infty) \mapsto \varphi_t(x_0)$ satisfies (1) if and only if (2) holds for any $f \in \mathcal{C}^\infty(\mathbb{R}^n)$. 

Levi Lopes de Lima (DM–UFC)

Stochastic Analysis I

João Pessoa, March, 2017
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Levi Lopes de Lima (DM–UFC)  
Stochastic Analysis I  
João Pessoa, March, 2017 4 / 18
Motivation: integrating diffusion operators

Definition (Diffusion operator)

A differential operator \( L \) on \( \mathbb{R}^n \) is a diffusion operator if

\[
L = \frac{1}{2} \sum_{i,j=1}^{n} \alpha^{ij}(x) \frac{\partial^2}{\partial x^i \partial x^j} + \sum_{i=1}^{n} \beta^i(x) \frac{\partial}{\partial x^i},
\]

where \( \alpha^{ij} = \alpha^{ji} \) is positive definite.

Notice that

\[
\Gamma_L(f, g) := L(fg) - fL(g) - gL(f) = \sum_{i,j=1}^{n} \alpha^{ij} \frac{\partial f}{\partial x^i} \frac{\partial g}{\partial x^j},
\]

so \( L \) fails to be a derivation.

Nevertheless, we will show that any such \( L \) can be integrated to a stochastic flow \( \varphi_t(x, \omega) \), where \( \omega \) varies in Wiener space.

It turns out that (2) above is replaced by the requirement that the stochastic process

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M_{t, x_0}^f(\omega) := f(\varphi_t(x_0, \omega)) - f(x_0) - \int_0^t (L(f))(\varphi_s(x_0, \omega)) \, ds
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A differential operator $L$ on $\mathbb{R}^n$ is a diffusion operator if

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where $\alpha_{ij} = \alpha_{ji}$ is positive definite.

Notice that

$$\Gamma_L(f,g) := L(fg) - fL(g) - gL(f) = \sum_{i,j=1}^{n} \alpha_{ij} \frac{\partial f}{\partial x^i} \frac{\partial g}{\partial x^j},$$

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**Definition (Random variable)**

Let $(\Omega, \mathcal{F}, P)$ be a probability space. We say that $X : \Omega \to \mathbb{R}^n$ is a random variable if $X$ is $\mathcal{F}$-measurable. The corresponding **distribution** is the probability measure $X_\# P$ on $\mathbb{R}^n$ given by

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Basic probability I

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Basic probability II

Definition (Expectation and Covariance)

If \( X : \Omega \to \mathbb{R}^n \) is a r.v. then its expectation is

\[
E(X) = \int_{\Omega} X \, dP = \int_{\mathbb{R}^n} x \, dX \# P = \int_{\mathbb{R}^n} x \psi_X(x) \, dx,
\]

where \( x = (x^1, \ldots, x^n) \). Also, if \( X, Y : \Omega \to \mathbb{R}^n \) are r.v. then their covariance matrix is

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\text{cov}(X, Y)_{ij} = \text{cov}(X_i, Y_i) := E((X_i - E(X_i))(Y_j - E(Y_j))) = E(X_i Y_j) - E(X_i)E(Y_j).
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Definition (Characteristic function)

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where $A$ is a positive definite, symmetric matrix and $m \in \mathbb{R}^n$. We then write $X \sim \mathcal{N}(m, C)$, where $C = A^{-1}$.

**Theorem**

If $X \sim \mathcal{N}(m, C)$ then the following hold:

1. $\int_{\mathbb{R}^n} \psi_X(x) \, dx = 1$;
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Stochastic processes and the Daniell-Kolmogorov theorem

Definition (Stochastic process)

A stochastic process is a one-parameter family of r.v. \( X_t : \Omega \to \mathbb{R}^n, \ t \geq 0. \)

- We can think of a stochastic process as a collection of paths in \( \mathbb{R}^n \) indexed by \( \Omega. \) In general, the regularity of the typical path defines the regularity of the process.

Definition

Given a stochastic process \( X_t : \Omega \to \mathbb{R}, \) its finite-dimensional distributions in \( \mathbb{R}^k, \ k = 1, 2, \ldots, \) are given by

\[
\mu_{t_1, \ldots, t_k}^X (I_1, \ldots, I_k) = P(X_{t_1} \in I_1, \ldots, X_{t_k} \in I_k), \quad t_1 < \cdots < t_k, \quad I_i \subset \mathbb{R}.
\]

Theorem (Daniell-Kolmogorov extension)

Assume that, for any \( t_1 < \cdots < t_k \) there exists a finite-dimensional distribution \( \nu_{t_1, \ldots, t_k} \) in \( \mathbb{R}^k \) such that:

- \( (K_1) \) \( \nu_{t_\tau(1), \ldots, t_\tau(k)} (I_1 \times \cdots \times I_k) = \nu_{t_1, \ldots, t_k} (I_{\tau^{-1}(1)} \times \cdots \times I_{\tau^{-1}(k)}), \) for any permutation \( \tau. \)
- \( (K_2) \) \( \nu_{t_1, \ldots, t_k, t_{k+1}} (I_1 \times \cdots \times I_k \times \mathbb{R}) = \nu_{t_1, \ldots, t_k} (I_1 \times \cdots \times I_k). \)

Then there exists a probability space \( (\Omega, \mathcal{F}, P) \) and a stochastic process \( X_t : \Omega \to \mathbb{R} \) such that

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Assume that, for any $t_1 < \ldots < t_k$ there exists a finite-dimensional distribution $\nu_{t_1, \ldots, t_k}$ in $\mathbb{R}^k$ such that:

- $(K_1)$ $\nu_{t_\tau(1), \ldots, t_\tau(k)} (I_1 \times \ldots \times I_k) = \nu_{t_1, \ldots, t_k} (I_{\tau^{-1}(1)}, \ldots, I_{\tau^{-1}(k)})$, for any permutation $\tau$.
- $(K_2)$ $\nu_{t_1, \ldots, t_k, t_{k+1}} (I_1 \times \ldots \times I_k \times \mathbb{R})$.

Then there exists a probability space $(\Omega, \mathcal{F}, P)$ and a stochastic process $X_t : \Omega \to \mathbb{R}$ such that

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for any $(t_1, \ldots, t_k)$. 
Stochastic processes and the Daniell-Kolmogorov theorem

**Definition (Stochastic process)**

A stochastic process is a one-parameter family of r.v. $X_t : \Omega \to \mathbb{R}^n, t \geq 0$.

We can think of a stochastic process as a collection of paths in $\mathbb{R}^n$ indexed by $\Omega$. In general, the regularity of the typical path defines the regularity of the process.

**Definition**

Given a stochastic process $X_t : \Omega \to \mathbb{R}$, its finite-dimensional distributions in $\mathbb{R}^k$, $k = 1, 2, \ldots$, are given by

$$\mu_{t_1, \ldots, t_k}^X (I_1, \ldots, I_k) = P(X_{t_1} \in I_1, \ldots, X_{t_k} \in I_k), \quad t_1 < \ldots < t_k, \quad I_i \subset \mathbb{R}.$$

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Levi Lopes de Lima (DM–UFC)
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Brownian motion in $\mathbb{R}^n$

- If $0 < t_1 < \cdots < t_k$ and $x = (x_1, \ldots, x_k) \in \mathbb{R}^k$ define

$$\nu_{t_1, \ldots, t_k}(l_1, \ldots, l_k) = \frac{1}{(2\pi)^{k/2}\sqrt{\det C}} \int_{l_1 \times \cdots \times l_k} e^{-\frac{1}{2}(c^{-1}x, x)} dx,$$ (3)

where $C_{ij} = t_i \wedge t_j$. If $t_1 = 0$ we use instead $\delta_0 \otimes \nu_{t_2, \ldots, t_k}$.

- We can easily check that $(K_1)$ and $(K_2)$ in D-K extension are satisfied. Thus, there exists a probability space $(\Omega, \mathcal{F}, P)$ and a stochastic process $B_t : \Omega \to \mathbb{R}$ so that $P(B_{t_1} \in l_1, \ldots, B_{t_k} \in l_k)$ equals the r.h.s of (3). This is called Brownian motion in $\mathbb{R}$ (starting at 0).

Theorem (Characteristic properties of BM in $\mathbb{R}$)

BM in $\mathbb{R}$ satisfies the following properties:

1. $B_0 = 0$ a.s. and $B_t - B_s \sim \mathcal{N}(0, t - s)$, where $\mathcal{N}(0, t) \sim (2\pi t)^{-1/2} e^{-x^2/2t}$;

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3. for any $0 \leq t_0 < t_1 \cdots < t_k$, $\{B_{t_k} - B_{t_{k-1}}, \ldots, B_{t_1} - B_{t_0}\}$ are independent r.v.’s.;

4. $t \mapsto b_t(\omega)$ is continuous for any $\omega$.

Similarly, we define Brownian motion $B_t$ in $\mathbb{R}^n$. One can always take $\Omega = \mathbb{R}^n := \{\omega : [0, +\infty) \to \mathbb{R}^n; \omega \in C^0, \omega(0) = 0\}$, the Wiener space. In this case, $P$ is called the Wiener measure.
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Regularity properties of BM I

**Theorem**

BM in \( \mathbb{R} \) satisfies

\[
\mathbb{E}((B_s - B_t)^{2k}) = \frac{(2k)!}{2^k k!} (s - t)^k, \quad k \geq 1.
\]

**Proof.**

We have

\[
\mathbb{E} \left( \sum_{k \geq 0} \frac{(i(B_s - B_t)u)^k}{k!} \right) = \sum_{k \geq 0} \frac{(-\frac{1}{2}(s-t)u^2)^k}{k!}, \quad u \in \mathbb{R},
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which follows from the identity \( \mathbb{E}(e^{i(B_s-B_t)u}) = e^{-\frac{1}{2}(s-t)u^2} \).

**Theorem (Kolmogorov-Chentov)**

Any \( X_t : \Omega \to \mathbb{R} \) satisfying

\[
\mathbb{E}(|X_s - X_t|^{\alpha}) \leq C|s - t|^{\beta+1}, \quad s, t \geq 0,
\]

is locally \( C^{\gamma} \), where \( 0 < \gamma < \beta/\alpha \).

- It follows that \( B_t \in C^{1/2-\epsilon} \), \( \epsilon > 0 \). In particular, \( B_t \in C^0 \).
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Theorem (Kolmogorov-Chentov)

Any $X_t : \Omega \to \mathbb{R}$ satisfying

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Theorem

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Regularity properties of BM I

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Regularity properties of BM I

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Levi Lopes de Lima (DM–UFC)  
Stochastic Analysis I  
João Pessoa, March, 2017  
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Regularity properties of BM

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Regularity properties of BM II

Definition

If \( X_t : \Omega \rightarrow \mathbb{R} \) is a stochastic process, its **quadratic variation** is the process

\[
\langle X \rangle_t(\omega) = \lim_{|\Delta^k| \rightarrow 0} \sum_i |X_{t_{i+1}^k}(\omega) - X_{t_i^k}(\omega)|^2, \quad \omega \in \Omega,
\]

where \( \Delta^k = \{0 = t_1^k < \cdots < t_n^k = t\} \) is a partition of \([0, t]\) and \( |\Delta^k| := \sup_i |t_{i+1}^k - t_i^k| \rightarrow 0 \). More generally, if \( X, Y : \Omega \rightarrow \mathbb{R} \) we set

\[
\langle X, Y \rangle_t = \frac{1}{4} (\langle X + Y \rangle_t - \langle X - Y \rangle_t),
\]

so that \( \langle X, X \rangle_t = \langle X \rangle_t \).

Theorem

**BM in \( \mathbb{R} \)** satisfies

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\langle B \rangle_t = t,
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- It is proved that \( B_t = (B_1^t, \cdots, B_n^t) \) is a BM in \( \mathbb{R}^n \) if and only if the components are independent BMs in \( \mathbb{R} \). It follows that \( \langle B^i, B^j \rangle_t = \delta_{ij} t \).
Regularity properties of BM II

Definition

If $X_t : \Omega \to \mathbb{R}$ is a stochastic process, its quadratic variation is the process

$$\langle X \rangle_t(\omega) = \lim_{|\Delta^k| \to 0} \sum_i |X_{t_{i+1}^k}(\omega) - X_{t_i^k}(\omega)|^2, \quad \omega \in \Omega,$$

where $\Delta^k = \{0 = t_1^k < \cdots < t_k^k = t\}$ is a partition of $[0, t]$ and $|\Delta^k| := \sup_i |t_{i+1}^k - t_i^k| \to 0$. More generally, if $X, Y : \Omega \to \mathbb{R}$ we set

$$\langle X, Y \rangle_t = \frac{1}{4} (\langle X + Y \rangle_t - \langle X - Y \rangle_t),$$

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Theorem

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Regularity properties of BM II

Definition

If \( X_t : \Omega \to \mathbb{R} \) is a stochastic process, its **quadratic variation** is the process

\[
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\]

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\[
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Levi Lopes de Lima (DM–UFC)

Stochastic Analysis I

João Pessoa, March, 2017
Theorem

The typical Brownian path fails to be locally $C^\gamma$ for $\gamma > 1/2$.

Proof.

If $|B_{s'} - B_{t'}| \leq K|s' - t'|^\gamma$, $0 \leq t' \leq s' \leq t$, then

$$\sum_i |B_{t_{i+1}}(\omega) - B_{t_i}(\omega)|^2 \leq K^2 t \sup_i |t_{i+1}^k - t_i^k|^2 \gamma^{-1}.$$  

But the supremum goes to $0$ as $k \to +\infty$ if $\gamma > 1/2$, a contradiction.

Corollary

The typical Brownian path fails to be differentiable anywhere. In particular, it fails to have bounded variation in any bounded interval.

- The theorem above holds true for $\gamma = 1/2$, but the proof is a bit harder.
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*The typical Brownian path fails to be differentiable anywhere. In particular, it fails to have bounded variation in any bounded interval.*

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Conditional expectation

- If \( X \in L^2(\Omega, F, P) \) is a r.v. and \( G \subset F \) is a \( \sigma \)-subalgebra, we define the conditional expectation of \( X \) with respect to \( G \) by
  \[
  \mathbb{E}(X|G) := \pi(X),
  \]
  where \( \pi \) is the orthogonal projection over \( L^2(X, G, P|_G) \). Thus, \( \mathbb{E}(X|G) \) is a \( G \)-measurable r.v.

- The definition can be extended to \( X \in L^1(X, F, P) \) by taking limits.

- Notice that \( \mathbb{E}(X|G) \) is characterized by
  \[
  \int_G Y \mathbb{E}(X|G) \, dP = \int_G YX \, dP,
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- (Double expectation formula) Taking \( Y = 1 \) and \( G = \Omega \) we see that \( \mathbb{E}(\mathbb{E}(X|G)) = \mathbb{E}(X) \).

Example (Connection with conditional probability)

If \( (\Omega, F, P) \) is a probability space and \( B \in F \) satisfies \( P(B) > 0 \), consider \( G = \{\emptyset, \Omega, B, B^c\} \). If \( A \in \Omega \) we have
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Martingales

- From now on we will consider the filtered probability space \((\Omega, \mathcal{F}, \mathcal{F}_t^B, P)\) associated to Brownian motion, which means that \(\mathcal{F}_t^B := \mathcal{F}\{B_s\}_{s \leq t}\).

**Definition (Adapted process)**

We say that \(X_t : \Omega \to \mathbb{R}\) is **adapted** if \(X_t\) is \(\mathcal{F}_t^B\)-measurable for any \(t \geq 0\).

**Definition (Martingale)**

We say that a process \(M_t : \Omega \to \mathbb{R}\) is a **martingale** if

- \(M_t\) is adapted;
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It follows easily from the double expectation property that \(\mathbb{E}(M_t) = \mathbb{E}(M_0)\) for any \(t \geq 0\). Thus, martingales are pure fluctuation!

- One checks that \(B_t\), \(B_t + x\) and \(B_t^2 - t\) are martingales. In general, Itô calculus provides a systematic way to produce martingales.
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One checks that \(B_t\), \(B_t + x\) and \(B_t^2 - t\) are martingales. In general, Itô calculus provides a systematic way to produce martingales.
Martingales

• From now on we will consider the **filtered probability space** \((\Omega, \mathcal{F}, \mathcal{F}_t^B, P)\) associated to Brownian motion, which means that \(\mathcal{F}_t^B := \mathcal{F}\{B_s\}_{s \leq t}\).

**Definition (Adapted process)**

We say that \(X_t : \Omega \to \mathbb{R}\) is **adapted** if \(X_t\) is \(\mathcal{F}_t^B\)-measurable for any \(t \geq 0\).

**Definition (Martingale)**

We say that a process \(M_t : \Omega \to \mathbb{R}\) is a **martingale** if

- \(M_t\) is adapted;
- \(\mathbb{E}(|M_t|) < +\infty\);
- \(\mathbb{E}(M_s|\mathcal{F}_t^B) = M_t\) whenever \(t \leq s\).

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Existence of diffusions

Theorem (Itô, 1942)

Let \( L \) be a diffusion operator on \( \mathbb{R}^n \) satisfying suitable regularity assumptions. Then, for each \( x \in \mathbb{R}^n \) there exists an adapted and continuous stochastic process \( X^x_t : \Omega \to \mathbb{R}^n \) satisfying:

- \( X^x_0 = x \);
- For any \( f \in C^\infty(\mathbb{R}^n) \), the process

\[
M^{f,x}_t(\omega) := f(X^x_t(\omega)) - f(x) - \int_0^t (Lf)(X^x_s(\omega)) \, ds, \quad \omega \in \Omega,
\]

is a martingale whose quadratic variation is

\[
\langle M^{f,x} \rangle_t = \int_0^t \left( \sum_{i,j=1}^n \alpha_{ij} \frac{\partial f}{\partial x^i} \frac{\partial f}{\partial x^j} \right)(X^x_s) \, ds.
\]

- The family \( X^x_t(\omega) \) defines an \( L \)-diffusion in \( \mathbb{R}^n \).
- The proof involves solving the SDE

\[
dX^i_t = \beta^i(X_t) \, dt + \sum_j \sigma_{ij}(X_t) \, dB^j_t, \quad X^i_0 = x^i, \quad i = 1, \ldots, n,
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where \( \sigma^2 = \alpha \), and then using Itô calculus to check the properties.
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Some immediate consequences

- If $X_t = X_t^0$ is the diffusion associated to $L = \frac{1}{2} \Delta$ then by choosing $f(x) = x^i \pm x^j$ we have
  \[
  \langle X_t^i \pm X_t^j, X_t^i \pm X_t^j \rangle = \pm 2 \delta_{ij} t,
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  and hence $\langle X^i, X^j \rangle_t = \delta_{ij} t$. By a celebrated theorem due to P. Lévy, this means that $X_t = B_t$. Thus, BM is the diffusion driven by $\frac{1}{2} \Delta$.

- Alternatively, we can take $f(x) = x^i x^j$, which implies that $X_t^i X_t^j - \delta_{ij} t$ is a martingale. Again by Lévy’s theorem, $X_t = B_t$.

- Let $X_t = X_t^x$ be the diffusion associated to a general $L$. It follows that
  \[
  N_t^i := X_t^i - x_i - \int_0^t \beta^i(X_s) \, ds
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