Universidade Federal da Paraíba
Universidade Federal de Campina Grande
Programa Associado de Pós-Graduação em Matemática
Doutorado em Matemática

Limites assintóticos e estabilidade para o sistema de Mindlin-Timoshenko

por

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João Pessoa - PB
Dezembro/2016

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sob orientação do

Prof. Dr. Fágner Dias Araruna

Tese apresentada ao Corpo Docente do Programa Associado de Pós-Graduação em Matemática - UFPB/UFCG, como requisito parcial para obtenção do título de Doutor em Matemática.

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Resumo

Esta tese aborda a dinâmica do sistema de Mindlin-Timoshenko para vigas e placas. Estudamos questões relacionadas com o limite assintótico em relação aos parâmetros e as taxas de decaimento. No contexto do limite assintótico, como resultado principal, apresentamos uma resposta positiva à conjectura feita por Lagnese e Lions em 1988, onde o modelo de Von-Kármán é obtido como limite singular, quando k tende ao infinito, do sistema de Mindlin-Timoshenko. Introduzindo mecanismos de amortecimento apropriados (internos e de fronteira), também mostramos que, sob certas condições, a energia de solução do sistema de Mindlin-Timoshenko tem propriedades de decaimento exponencial e polinomial com relação aos parâmetros.

Palavras-chave: Sistema de Mindlin-Timoshenko; limite assintótico; estabilização uniforme.

Abstract

This thesis is concerned with the dynamics of Mindlin-Timoshenko system for beams and plates. We study issues relating to the asymptotic limit in relation to the parameters, decay rates and the existence of controls that lead to our solution of the system from an initial state prescribed to a final desired state at a given time positive. In the context of asymptotic limit, as the main result, we present a positive response to the conjecture made by Lagnese and Lions in 1988, where the Von-Kármán model is obtained as singular limit when k tends to infinity, the Mindlin-Timoshenko system. Introducing appropriate damping mechanisms (internal and boundary), we also show that the energy of solutions for the Mindlin-Timoshenko system has decay properties exponential and polynomial, with respect to the parameters.

Keywords: Mindlin-Timoshenko; asymptotic limit; uniform stabilization; Mindlin-Timoshenko.

Agradecimentos

Dedicatória

Sumário

Introdução

Nos últimos anos é crescente o interesse no estudo da estabilização e do controle de sistemas de natureza elástica (envolvendo estruturas flexíveis sujeitas a vibrações), em particular os que modelam a ação de vigas e placas, devido a aplicação à física e à engenharia. Um modelo matemático extremamente usado na descrição de vibrações de vigas e placas finas é o sistema de Mindlin-Timoshenko. Este modelo é considerado um dos mais precisos pelo fato de considerar tanto deformações transversais como também rotacionais (ver Figura 1).

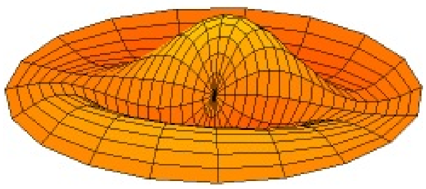

Figura 1

Para descrever esse modelo, consideremos $\Omega \subset \mathbb{R}^2$ um aberto limitado com fronteira Γ suficientemente regular. Sejam $\{\Gamma_0, \Gamma_1\}$ uma partição de Γ e $T > 0$ dado. Consideremos o cilindro $Q = \Omega \times (0, T)$, com fronteira lateral $\Sigma = \Sigma_0 \cup \Sigma_1$, onde $\Sigma_i = \Gamma_i \times (0, T)$, $i = 0, 1$. A ação do sistema de bidimensional de Mindlin-Timoshenko (ver Timoshenko [55], Mindlin [40] e Lagnese- Lions [27]) é dada pelas acopladas equações diferenciais parciais

$$\begin{cases} \dfrac{\rho h^3}{12}\phi_{1tt} - L_1(\phi_1,\phi_2,\psi) = 0, & \text{em} \quad Q, \\ \dfrac{\rho h^3}{12}\phi_{2tt} - L_2(\phi_1,\phi_2,\psi) = 0, & \text{em} \quad Q, \\ \rho h\psi_{tt} - L_3(\phi_1,\phi_2,\psi,\eta_1,\eta_2) = 0, & \text{em} \quad Q, \\ \rho h\eta_{1tt} - L_4(\psi,\eta_1,\eta_2) = 0, & \text{em} \quad Q, \\ \rho h\eta_{2tt} - L_5(\psi,\eta_1,\eta_2) = 0, & \text{em} \quad Q, \end{cases} \tag{1}$$

onde

$$L_1(\phi_1,\phi_2,\psi) = D\left(\phi_{1xx} + \frac{1-\mu}{2}\phi_{1yy} + \frac{1+\mu}{2}\phi_{2xy}\right) - k\left(\phi_1 + \psi_x\right),$$

$$L_2(\phi_1,\phi_2,\psi) = D\left(\phi_{2yy} + \frac{1-\mu}{2}\phi_{2xx} + \frac{1+\mu}{2}\phi_{1xy}\right) - k\left(\phi_2 + \psi_y\right),$$

$$L_3(\phi_1,\phi_2,\psi,\eta_1,\eta_2) = k\left[(\psi_x+\phi_1)_x + (\psi_y+\phi_2)_y\right] + (N_1\psi_x + N_{12}\psi_y)_x + (N_2\psi_y + N_{12}\psi_x)_y,$$

$$L_4(\psi,\eta_1,\eta_2) = N_{1x} + N_{12y},$$

$$L_5(\psi,\eta_1,\eta_2) = N_{2y} + N_{12x},$$

$$N_1 = \frac{Eh}{1-\mu^2}\left(\eta_{1x} + \mu\eta_{2y} + \frac{1}{2}\psi_x^2 + \frac{\mu}{2}\psi_y^2\right),$$

$$N_2 = \frac{Eh}{1-\mu^2}\left(\eta_{2y} + \mu\eta_{1x} + \frac{1}{2}\psi_y^2 + \frac{\mu}{2}\psi_x^2\right),$$

$$N_{12} = \frac{Eh}{2(1+\mu)}\left(\eta_{1y} + \eta_{2x} + \psi_x\psi_y\right).$$

Aqui, como em todo o nosso trabalho, os subscritos representam as derivadas parciais. O vetor $\nu = (\nu_1,\nu_2)$ é o vetor normal exterior a Ω e $\frac{\partial}{\partial\nu}$ representa a derivada normal. Fisicamente, as funções $\phi_1 = \phi_1(x,y,t)$, $\phi_2 = \phi_2(x,y,t)$, representam, respectivamente, os ângulos de rotação da seção transversal da placa $x = const.$, $y = const.$ contendo o filamento que, quando a placa está em equilíbrio, é ortogonal à superfície média no ponto $(x,y,0)$. A função $\psi = \psi(x,y,t)$ é o deslocamento transversal da placa, e $\eta_1 = \eta_1(x,y,t)$, $\eta_2 = \eta_2(x,y,t)$ representam a deformação longitudinal da placa no instante t da seção transversal localizada em (x,y) unidades a partir do ponto final $(x,y) = (0,0)$. A contante positiva h representa a espessura da placa que, para esse modelo, consideramos uniforme e fina. A constante D é o módulo de rigidez à flexão e é dado por $D = Eh^3/12(1-\mu^2)$. A constante ρ é a densidade de massa por unidade de volume e o parâmetro k, que multiplica o acoplamento das equações, é chamado de módulo de elasticidade em torção e é calculado pela fórmula

$$k = \frac{\widehat{k}Eh}{2\,(1+\mu)},$$

onde a contante E é o módulo de Young, μ é o raio de Poisson $\left(0 < \mu < \frac{1}{2}\right)$ e $\widehat{k}$ é chamado coeficiente de correção de torção. Este coeficiente aparece pelo fato de que as deformações sofridas pelas torções não são constantes em toda seção transversal da placa. O módulo k é também inversamente proporcional ao ângulo de rotação da placa. A dedução do modelo (1) pode ser vista em Lagnese-Lions [27].

Consideremos que a placa seja de forma que suas extremidades estejam fixas sobre uma porção Γ_0 da fronteira, enquanto que forças e momentos são aplicados no restante Γ_1 da fronteira. As condições de fronteira associadas a este caso são dadas por

$$\begin{cases} \phi_1 = \phi_2 = \psi = \eta_1 = \eta_2 = 0 & \text{em} \quad \Sigma_0, \\ \{\mathcal{B}_1(\phi_1,\phi_2), \mathcal{B}_2(\phi_1,\phi_2), \mathcal{B}_3(\phi_1,\phi_2,\psi,\eta_1,\eta_2), \mathcal{B}_4(\eta_1,\eta_2), \mathcal{B}_5(\eta_1,\eta_2)\} = \{0,0,0,0,0\} & \text{em} \quad \Sigma_1, \end{cases} \tag{2}$$

onde

$$\mathcal{B}_1(\phi_1,\phi_2) = D\left[\nu_1\phi_{1x} + \mu\nu_1\phi_{2y} + \frac{1-\mu}{2}\left(\phi_{1y}+\phi_{2x}\right)\nu_2\right],$$

$$\mathcal{B}_2(\phi_1,\phi_2) = D\left[\nu_2\phi_{2y} + \mu\nu_2\phi_{1x} + \frac{1-\mu}{2}\left(\phi_{1y}+\phi_{2x}\right)\nu_1\right],$$

$$\mathcal{B}_3(\phi_1,\phi_2,\psi,\eta_1,\eta_2) = k\left(\frac{\partial\psi}{\partial\nu} + \nu_1\phi_1 + \nu_2\phi_2\right) + \left(\nu_1 N_1 + \nu_2 N_{12}\right)\psi_x + \left(\nu_2 N_2 + \nu_1 N_{12}\right)\psi_y,$$

$$\mathcal{B}_4(\psi,\eta_1,\eta_2) = \nu_1 N_1 + \nu_2 N_{12},$$

$$\mathcal{B}_5(\psi,\eta_1,\eta_2) = \nu_2 N_2 + \nu_1 N_{12}.$$

Para completarmos o sistema de Mindlin-Timoshenko, incluamos as condições iniciais

$$\begin{cases} \{\phi_1(\cdot,0), \phi_2(\cdot,0), \psi(\cdot,0), \eta_1(\cdot,0), \eta_2(\cdot,0)\} = \{\phi_{10}, \phi_{20}, \psi_0, \eta_{10}, \eta_{20}\} & \text{em} \quad \Omega, \\ \{\phi_{1t}(\cdot,0), \phi_{2t}(\cdot,0), \psi_t(\cdot,0), \eta_{1t}(\cdot,0), \eta_{2t}(\cdot,0)\} = \{\phi_{11}, \phi_{21}, \psi_1, \eta_{11}, \eta_{21}\} & \text{em} \quad \Omega, \end{cases} \tag{3}$$

A energia deste sistema é definida por

$$\begin{aligned} E_k(t) &= \frac{1}{2}\left\{\frac{\rho h^3}{12}\left[|\phi_{1t}|^2 + |\phi_{2t}|^2\right] + \rho h\left[|\psi_t|^2 + |\eta_{1t}|^2 + |\eta_{2t}|^2\right] + k\left[|\phi_1+\psi_x|^2 + |\phi_2+\psi_y|^2\right]\right. \\ &\quad \left. + F([b_{ij}],[b_{ij}]) + D\left[|\phi_{1x}|^2 + |\phi_{2y}|^2 + \frac{1-\mu}{2}|\phi_{1y}+\phi_{2x}|^2 + 2\mu\int_\Omega \left(\phi_{1x}\phi_{2y}\right)dxdy\right]\right\}, \end{aligned}$$

onde

$$b_{11} = \eta_{1x} + \frac{1}{2}\psi_x^2, \quad b_{22} = \eta_{2y} + \frac{1}{2}\psi_y^2, \quad b_{12} = b_{21} = \eta_{1y} + \eta_{2x} + \psi_x\psi_y,$$

e

$$\begin{aligned}(F([b_{ij}]), [b_{ij}])_{(L^2(\Omega))^4} &= \frac{Eh}{1-\mu^2}\left\{\mu\left|\eta_{1x} + \eta_{2y} + \frac{1}{2}|\nabla\psi|^2\right|^2 + (1-\mu)|b_{11}|^2 + (1-\mu)|b_{22}|^2\right. \\ &\left. + \frac{1-\mu}{2}\left|\eta_{1y} + \eta_{2x} + \psi_x\psi_y\right|^2\right\}.\end{aligned}$$

Este sistema tem um caráter conservativo, isto é,

$$E_k(t) = E_k(0), \quad \forall\, t > 0.$$

Ao supor que o filamento da placa permanece perpendicular à superfície mediana deformada, os efeitos de torção transversais são desprezados (ver Figura 2).

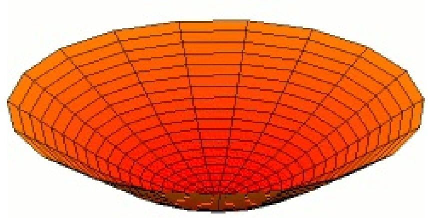

Figura 2

Neste caso, o modelo que descreve dinâmica da placa é o sistema de Von-Kármán (ver [27]), cujas equações são

$$\begin{cases} \rho h\psi_{tt} - \dfrac{\rho h^3}{12}\Delta\psi_{tt} + D\Delta^2\psi - (N_1\psi_x + N_{12}\psi_y)_x - (N_2\psi_y + N_{12}\psi_x)_y = 0 & \text{em} \quad Q, \\ \rho h\eta_{1tt} - (N_{1x} + N_{12y}) = 0 & \text{em} \quad Q, \\ \rho h\eta_{2tt} - (N_{2y} + N_{12x}) = 0 & \text{em} \quad Q. \end{cases} \tag{4}$$

Considerar desprezível o efeito de torção transversal da placa é o mesmo que fazer o módulo k tender ao infinito, visto que, como dissemos anteriormente, este módulo é inversamente proporcional ao ângulo de torção. Dessa forma, vê-se bastante natural a questão proposta por Lagnese e Lions em [27, p. 24], à qual é formulada como segue.

Conjectura (Lagnese-Lions) As soluções do sistema não linear de Mindlin-Timosheko (1) convergem (quando $k \to \infty$) para as soluções do sistema de Von-Kármán (4).

O primeiro trabalho que tentou dar uma resposta a esta conjectura, pelo menos no caso unidimensional, foi [8]. Lá os autores adicionaram um termo regularizante de quarta ordem ao sistema de Mindlin-Timoshenko, isto é, consideraram o sistema

$$\begin{cases} \frac{\rho h^3}{12}\phi_{tt} - D\phi_{xx} + k\left(\phi + \psi_x\right) = 0 & \text{in} \quad Q, \\ \rho h \psi_{tt} - k\left(\phi + \psi_x\right)_x - Eh\left[\psi_x\left(\eta_x + \frac{1}{2}\psi_x^2\right)\right]_x + \frac{1}{k}\psi_{xxxx} = 0 & \text{in} \quad Q, \\ \rho h \eta_{tt} - Eh\left(\eta_x + \frac{1}{2}\psi_x^2\right)_x = 0 & \text{in} \quad Q, \end{cases} \tag{5}$$

e provaram que, quando $k \to \infty$, o sistema (5) converge para o sistema unidimensional de Von-Kármán

$$\begin{cases} \rho h \psi_{tt} - \frac{\rho h^3}{12}\psi_{xxtt} + D\psi_{xxxx} - Eh\left[\psi_x\left(\eta_x + \frac{1}{2}\psi_x^2\right)\right]_x = 0 & \text{em} \quad Q, \\ \rho h \eta_{tt} - Eh\left(\eta_x + \frac{1}{2}\psi_x^2\right)_x = 0 & \text{em} \quad Q. \end{cases} \tag{6}$$

No argumento usado em [8], o uso do termo de regularizante foi indispensável para garantir a compacidade para uma família de soluções e, portanto, possibilitando a passagem ao limite no termo não linear.

Para o caso linear, o sistema bidimensional de Mindlin-Timoshenko é dado por

$$\begin{cases} \frac{\rho h^3}{12}\phi_{1tt} - L_1(\phi_1, \phi_2, \psi) = 0 & \text{em} \quad Q, \\ \frac{\rho h^3}{12}\phi_{2tt} - L_2(\phi_1, \phi_2, \psi) = 0 & \text{em} \quad Q, \\ \rho h \psi_{tt} - \widetilde{L}_3(\phi_1, \phi_2, \psi) = 0 & \text{em} \quad Q, \end{cases} \tag{7}$$

onde L_1, L_2 são como definidos acima e

$$\widetilde{L}_3(\phi_1, \phi_2, \psi) \quad = \quad k\left[\left(\psi_x + \phi_1\right)_x + \left(\psi_y + \phi_2\right)_y\right].$$

Em [27], Lagnese-Lions provaram que a solução do sistema (7) converge, quando $k \to \infty$, para a solução do modelo de Kirchhoff (sujeito a condições de contorno apropriadas)

$$\rho h \psi_{tt} - \frac{\rho h^3}{12}\Delta\psi_{tt} + D\Delta^2\psi = 0. \tag{8}$$

Existe uma extensa literatura no contexto de limites assintóticos em relação a parâmetros singulares. Todavia, vamos mencionar apenas alguns trabalhos que estão relacionados com os sistema hiperbólicos acima descritos. Comecemos com o resultado em [9], o qual os autores estudaram o sistema unidimensional linear de Mindlin-Timoshenko com um controle aplicado na fronteira, e provaram que seu limite assintótico converge, quando $k \to \infty$, para o sistema controlado de Kirchhoff. Menzala e Zuazua em [35] mostraram que o limite do sistema de viga de Timoshenko, quando um parâmetro adequado tende a zero, se aproxima (fracamente) do sistema de Von-Kármán. Em [36], os autores consideraram um modelo dinâmico não linear do sistema de Mindlin-Timoshenko dependendo de um parâmetro $\varepsilon > 0$ e estudaram seu limite fraco quando $\varepsilon \to 0$. Além disso, eles mostraram que, dependendo do tipo de condição de contorno, a não-linearidade deste modelo pode desaparecer ou pode se tornar uma não-linearidade concentrada nos extremos da viga. Em [14], Chueshov e Lasiecka analisaram a dinâmica de uma classe de modelos de placas de Mindlin-Timoshenko com forças de feedback não-lineares, mostrando a existência de um atrator global compacto para o sistema e estudando as propriedades de limite quando o módulo de cisalhamento tende a infinito. Para o sistema não-linear de Mindlin-Timoshenko Rahmani, em [44], considera uma placa reforçada por um reforço fino em uma porção de sua fronteira e estuda esta junção através de um modelo aproximado onde o reforço tem um papel importante em suas condições de contorno.

Estabilização

A estabilização de modelos matemáticos envolvendo estruturas flexíveis sujeitas a vibrações tem sido consideravelmente estimulada pelo número crescente de questões de interesse prático. Dentre esses modelos, podemos destacar aqueles relacionados à engenharia estrutural moderna, que requerem mecanismos de controle ativos para estabilizar estruturas intrinsecamente instáveis ou que possuem um amortecimento natural muito fraco, como por exemplo, os modelos que descrevem os deslocamentos de vigas e placas finas.

Seja $\mathcal{H}$ espaço de Hilbert com norma e produto interno denotados respectivamente por $(\cdot,\cdot)$ e $|\cdot|$. Seja $\mathcal{X}$ um espaço de Hilbert tal que $\mathcal{X} \subset \mathcal{H}$ com imersão densa e contínua. Denotemos a norma e o produto interno em $\mathcal{X}$, respectivamente, por $\|\cdot\|_{\mathcal{X}}$ e $a(\cdot,\cdot)$.

O problema de estabilização pode ser formulado da seguinte forma: Dado um sistema do tipo

$$
\begin{cases}
y_{tt}(t) + \mathcal{A}y(t) + \mathcal{B}(t, y_t(t)) = 0, \quad t \in [0, T], \\
y(0) = y_0, \quad y_t(0) = y_1,
\end{cases}
\tag{9}
$$

onde $T > 0$, $y_0 \in \mathcal{X}$, $y_1 \in \mathcal{H}$, $\mathcal{A} \in \mathcal{L}(\mathcal{X}, \mathcal{X}')$ o único operador tal que $\langle \mathcal{A}u, v\rangle = a(u, v)$ para todo $(u, v) \in \mathcal{X} \times \mathcal{X}$. O domínio do operador $\mathcal{A}$ será denotado por $D(\mathcal{A}) := \{v \in \mathcal{X}; \mathcal{A}v \in \mathcal{H}\}$. Assuma que existe uma constante positiva α_0 tal que

$$(y, \mathcal{A}y) \geq \alpha_0 \|y\|, \quad y \in D(\mathcal{A}).$$

Seja $\{\mathcal{B}(t, \cdot)\}_{t \geq 0}$ uma família de operadores tal que

$$\mathcal{B}(t, 0) = 0, \quad \mathcal{B}(t, y) \in \mathcal{X}' \quad \text{e} \quad (y, \mathcal{B}y) \geq 0,$$

para todo $y \in \mathcal{X}$ e $t \in [0, T]$. Note que a energia do sistema (9), dada por

$$E(t) = \frac{1}{2}\left[|y_t|^2 + |\mathcal{A}^{1/2}y|^2\right],$$

decresce à medida em que t aumenta. Resta-nos saber se essa energia tende a zero, quando $t \to \infty$, e em que taxa isso acontece.

O problema de estabilização (9) pode ser abordado sob diferentes hipóteses em relação à família de operadores $\mathcal{B}$ (ver [20, 28, 30, 31, 47, 59]). Dentre elas podemos considerar $\mathcal{B}$ linear (ver, por exemplo, [46]), não linear com alguma condição de crescimento (ver [59]), $\mathcal{B}$ definido localmente ou no bordo e, para o caso autônomo com uma condição mais forte que a condição de crescimento para o caso não linear (ver [59]).

Há uma vasta literatura sobre problemas de estabilização. Contudo, a fim de sermos mais precisos, citaremos alguns trabalhos que estão relacionados ao tema da tese. Em [8], os autores mostraram que, introduzindo mecanismos de amortecimento interno, a energia das soluções do sistema unidimensional não linear de Mindlin-Timoshenko decai exponencialmente, uniformemente com respeito ao parâmetro k. Os resultados de estabilização para o modelo linear foram obtidos por Lagnese em [25] e Kim e Renard em [22], considerando o amortecimento em ambas as equações, e por Alabau-Boussauira em [1] com um único controle não-linear. Soufyane [52] mostrou a estabilidade exponencial uniforme da viga de Timoshenko usando uma força de controle interno. No caso bidimensional, a estabilização para o modelo linear de Mindlin-Timoshenko foi analisada por Sare em [48] considerando a dissipação atuando sobre uma porção da fronteira. Grobbelaar-Dalsen [19] estudou o decaimento polinomial da solução do modelo de placa Mindlin-Timoshenko com mecanismos de amortecimento interno. Em relação ao sistema de Von-Kármán, a estabilidade exponencial foi estudada por Menzala e Zuazua, em [39], para uma placa termoelástica presa sobre sua fronteira. Em [38], Park e Kang investigaram a estabilidade para as equações de Von-Kármán com memória em domínios não cilíndricos.

A partir de agora, vamos ser mais específicos sobre os problemas que serão abordados nesta tese. Introduziremos os três trabalhos que serão mostrados na ordem seguinte.

Capítulo 1

Asymptotic limits and stabilization for the 2D nonlinear Mindlin-Timoshenko system

Neste capítulo estudamos as propriedades assintóticas para o sistema de bidimensional de Mindlin-Timoshenko. Apresentamos uma resposta positiva à, anteriormente enunciada, conjectura proposta por Lagnese e Lions. Mais precisamente, mostramos rigorosamente que, sob adequadas condições para os dados, o não linear sistema de Mindlin-Timoshenko (1) converge para o sistema Von-Kármán (4), quando o parâmetro k tende ao infinito. Além disso, provamos que adicionando termos de amortecimento apropriados (internos e de fronteira, respectivamente), podemos obter uma propriedade de decaimento exponencial uniforme (em k) das soluções de (1), quando $t \to \infty$.

O principal resultado deste trabalho pode ser enunciado como segue.

Teorema: *Seja* $\left\{\phi_1^k, \phi_2^k, \psi^k, \eta_1^k, \eta_2^k\right\}$ *solução do sistema* (1) *com dados iniciais* $\{\phi_{10}, \phi_{11}, \phi_{20}, \phi_{21}, \psi_0, \psi_1, \eta_{10}, \eta_{11}, \eta_{20}, \eta_{21}\} \in \mathcal{X}$ *satisfazendo*

$$\phi_{10} + \psi_{0x} = 0 \qquad and \qquad \phi_{20} + \psi_{0y} = 0 \qquad in \qquad \Omega. \tag{10}$$

Então, fazendo $k \to \infty$, *obtemos*

$$\{\phi_1^k, \phi_2^k, \psi^k, \eta_1^k, \eta_2^k\} \to \{-\psi_x, -\psi_y, \psi, \eta_1, \eta_2\} \ fraco-* \ em \ L^\infty\left(0, T, [H^1_{\Gamma_0}(\Omega)]^3 \times [L^2(\Omega)]^2\right), \tag{11}$$

onde $\{\psi, \eta_1, \eta_2\}$ *resolve* (4).

A condição (10) e a conservação da energia nos possibilita obter uma limitação das funções num espaço de energia finita e, usando o Teorema de compacidade de Aubin-Lions, obtemos que a solução do sistema de Mindlin-Timoshenko (1) converge na direção da solução do sistema de Von-Kármán (4).

Os próximos resultados estão relacionados ao decaimento da taxa de energia associada a solução do sistema de Mindlin-Timoshenko. Analisamos, inicialmente, o modelo de Mindlin-Timoshenko com amortecimento interno distribuído ao longo da placa. Temos o seguinte resultado.

Teorema: *Seja* $\left\{\phi_1^k, \phi_2^k, \psi^k, \eta_1^k, \eta_2^k\right\}$ *solução do sistema* (1) *com dados iniciais* $\{\phi_{10}, \phi_{11}, \phi_{20}, \phi_{21}, \psi_0, \psi_1, \eta_{10}, \eta_{11}, \eta_{20}, \eta_{21}\} \in \mathcal{X}$. *Então existe constante* $\omega > 0$ *tal que*

$$E_k(t) \leq 4E_k(0)\exp^{-\frac{\omega}{2}t}, \quad \forall\, t > 0.$$

Uma questão natural é o que acontece no caso em que a energia do sistema de Mindlin-Timoshenko é dissipada através de mecanismos de amortecimento de fronteira. A resposta está no terceiro resultado.

Teorema: *Assuma que a condição geométrica*

$$m \cdot \nu \leq 0 \;\; em \;\; \Gamma_0 \quad e \quad m \cdot \nu \geq 0 \;\; em \;\; \Gamma_1$$

é satisfeita. Seja $\left\{\phi_1^k, \phi_2^k, \psi^k, \eta_1^k, \eta_2^k\right\}$ *solução suficientemente regular do sistema* (1). *Então existem constantes positivas* C *and* ω *tais que*

$$E_k(t) \leq CE_k(0)\exp^{-\omega t}, \quad \forall\, t > 0.$$

Nas provas dos resultados anteriores, relacionados com o decaimento exponencial da energia do sistema, usamos um método devido a Zuazua, em [59], o qual consiste em introduzir uma perturbação adequada na energia do sistema de forma a obter inequações diferenciais levando ao decaimento da taxa de energia associada ao sistema.

Como consequência das desigualdades obtidas nos resultados de estabilidade para o sistema de Mindlin-Timoshenko e considerando que os dados iniciais satisfazem a condição (10) obtemos, quando $k \to \infty$, o decaimento exponencial da energia $E(t)$ associado ao sistema (4). Essa taxa de decaimento para o sistema limite está de acordo com os resultados em [43].

Capítulo 2

Asymptotic limits and stabilization for the linear one-dimensional Timoshenko system

O segundo problema abordado nesta tese está relacionado ao modelo de viga linear de Mindlin-Timoshenko:

$$\begin{cases} \frac{\rho h^3}{12}\phi_{tt} - \phi_{xx} + k(\phi + \psi_x) + \alpha\phi_t = 0, & \text{em} \quad Q \\ \rho h\psi_{tt} - k(\phi + \psi_x)_x = 0, & \text{em} \quad Q \\ \phi(0,\cdot) = \phi(L,\cdot) = \psi(0,\cdot) = \psi(L,\cdot) = 0 & \text{em} \quad (0,T), \\ \{\phi(\cdot,0), \psi(\cdot,0)\} = \{\phi_0, \psi_0\} & \text{em} \quad (0,L), \\ \{\phi_t(\cdot,0), \psi_t(\cdot,0)\} = \{\phi_1, \psi_1\} & \text{em} \quad (0,L). \end{cases} \tag{12}$$

onde $\alpha = \alpha(x) > 0$ é uma função dada.

Nosso objetivo neste capítulo é estudar as propriedades assintóticas do sistema de Mindlin-Timoshenko quando as velocidades de propagação para ambas as equações são iguais, isto é,

$$\frac{12}{\rho h^3} = \frac{k}{\rho h}. \tag{13}$$

Baseados nisto, mostramos como um sistema parabólico de quarta ordem do tipo

$$\begin{cases} \phi + \psi_x = 0 & \text{em} \quad Q, \\ \alpha\phi_t - \phi_{xx} = 0 & \text{em} \quad Q, \\ \phi = 0 & \text{em} \quad (0,T), \\ \alpha\phi(\cdot,0) = \alpha\phi_0(\cdot) & \text{em} \quad (0,L), \end{cases} \tag{14}$$

pode ser obtido como um limite singular do sistema de Mindlin-Timoshenko, quando a espessura h e o módulo de elasticidade em cisalhamento k tendem a zero e ao infinito, respectivamente. Mais ainda, provamos que, quando (13) é satisfeita, a dissipação que é produzida por um amortecimento apenas na equação do ângulo de rotação no sistema de Mindlin-Timoshenko, é suficiente para fornecer o decaimento exponencial da energia à medida que $t \to \infty$. Caso contrário, se

$$\frac{12}{\rho h^3} \neq \frac{k}{\rho h},$$

é possível mostrar que a energia de solução do sistema de Mindlin-Timoshenko tem um decaimento polinomial, quando $t \to \infty$.

O principal resultado do trabalho é o seguinte.

Teorema: *Seja* $\{\phi^{h,k}, \psi^{h,k}\}$ *uma sequência de soluções de* (12) *com dados iniciais* $\{\phi_0, \phi_1, \psi_0, \psi_1\}$ *satisfazendo*

$$\phi_0 + \psi_{0x} = 0 \ \ em \ \ (0, L).$$

Seja $\alpha \in L^\infty([0, L])$ *uma função não negativa. Então, fazendo* $(h, k) \to (0, \infty)$, *obtemos*

$$\{\phi^{h,k}, \psi^{h,k}\} \to \{\phi, \psi\} \ \ fraco - * \ em \quad L^\infty \left(0, T, \left[H_0^1(0, L)\right]^2\right)$$

e

$$\sqrt{\alpha}\phi^{h,k} \to \sqrt{\alpha}\phi \quad em \quad C^0\left([0, T], H_0^{1-\theta}(0, L)\right), \ \theta \in \left(0, \frac{1}{2}\right).$$

onde $\{\phi, \psi\}$ *resolve o sistema* (14).

Os próximos resultados consistem em analizar o comportamento assintótico da energia associada à solução do sistema (12) sujeito à condição (13). Assim, temos inicialmente o seguinte resultado.

Teorema: *Assuma que* $\alpha = \alpha(x)$ *é uma função positiva* $C^1([0, L])$ *com*

$$\alpha(x) \geq \alpha_0 > 0 \ \ em \ \ (0, L).$$

Se (13) *é satisfeita em* $(0, L)$, *então existem constante positivas* C_1 *e* C_2 *tais que*

$$E_h(t) \leq C_1 E_h(0) \exp^{-\frac{C_2}{h^3}t}.$$

Este resultado já é conhecido (ver [52], por exemplo), porém há um interesse na dependência da taxa de decaimento da energia associada a este sistema com respeito ao parâmetro h. Sem perda de generalidade assuma $h \in (0, 1)$. Sabendo que a condição (13) é satisfeita, obtemos $k = 12/h^2$ e, por simplicidade, normalizamos todos os outros parâmetros do sistema. Portanto, a fim de estudar o decaimento exponencial da solução, provamos o resultado anterior via técnicas dos multiplicadores e resultados devido a Neves et. al. em [41].

O terceiro resultado consiste em analisar a taxa de decaimento de energia se a condição (13) não for satisfeita. Portanto, temos o seguinte resultado:

Teroema: *Sejam dados iniciais suficientemente regulares. Se a condição* (13) *não for satisfeita, então existe constante* $C > 0$ *tal que para todo* $t > 0$,

$$E_1(t) \leq \frac{C}{h^2}(E_1(0) + E_2(0))t^{-1}.$$

Na prova do resultado anterior usamos técnicas dos multiplicadores de forma a obter inequações diferenciais levando ao decaimento da taxa de energia associada ao sistema. Além disso, usamos a expressão "regularmente suficiente" para as soluções, a fim de assegurar que, sob certas restrições, os resultados se mantém.

Problemas em aberto e trabalhos futuros

Comentaremos brevemente uma série de perguntas e problemas em aberto que os resultados contidos nesta tese produzem.

- Embora conheçamos a dedução física para o sistema não linear de Mindlin-Timoshenko (1) (ver, por exemplo, [27], [44]), não temos conhecimento de resultados relativos a boa colocação e regularidade para todo $k > 0$. No entanto, como no Capítulo 1 o nosso objetivo principal era dar uma resposta positiva à conjectura proposta por Lagnese-Lions, o que podemos dizer é que, para k suficientemente grande e para dados iniciais no espaço $\mathcal{X}$, o sistema (1) se aproxima do sistema de Von-Kármán (4). Por outro lado, existe uma extensa literatura sobre a boa colocação, a regularidade, a estabilidade, etc., para o sistema (4) (ver [17, 25, 26, 29, 43]). Analisamos também o comportamento assintótico (quando $t \to \infty$) para a solução do sistema não linear de Mindlin-Timoshenko com feedback de fronteira. Para isso, tivemos que solicitar uma regularidade adicional para suas soluções. Por esta razão, em todos os resultados dessa seção, usamos a expressão "suficientemente regular" para as soluções, a fim de assegurar que, sob certas restrições, os resultados se mantêm.

- Na prova dos resultados do Capítulo 1 consideramos o caso onde os dados iniciais são fixos. Porém, os mesmos resultados são válidos se considerarmos a dependência em k, desde que assumamos que os dados iniciais $\{\phi_{10}^k, \phi_{11}^k, \phi_{20}^k, \phi_{21}^k, \psi_0^k, \psi_1^k, \eta_{10}^k, \eta_{11}^k, \eta_{20}^k, \eta_{21}^k\}$ sejam tais que a energia $E_k(0)$ permanece limitada e que convergem fracamente para $\{\phi_{10}, \phi_{11}, \phi_{20}, \phi_{21}, \psi_0, \psi_1, \eta_{10}, \eta_{11}, \eta_{20}, \eta_{21}\}$ nos espaços correspondentes.

- Estudamos ainda no Capítulo 1 os resultados de estabilização do sistema bidimensional não linear de Mindlin-Timoshenko utilizando o método de perturbação da energia. Seria interessante analisar se os mesmos resultados de estabilização são válidos considerando os sistemas com menos termos de amortecimento, porém eliminar alguns desses termos de amortecimento é uma tarefa difícil devido às complexas não-linearidades envolvidas.

- Outro problema interessante e difícil é obter o mesmo resultado de estabilização do modelo bidimensional não linear de Mindlin-Timoshenko quando os mecanismos de amortecimento interno atuam em uma região arbitrariamente pequena da placa. A dificuldade para este caso, é claro, consiste em obter um resultado de continuação única para o sistema Mindlin-Timoshenko. Neste contexto, mencionamos [12], [13], [18], [23], [61] que obtiveram taxas de decaimento para a energia de vários sistemas hiperbólicos considerando termos de amortecimento localizados lineares e não-lineares.

- Problemas similares aos do Capítulo 2, podem ser postulados para o sistema não linear unidimensional de Mindlin-Timoshenko. A situação se torna mais difícil quando analisamos os termos não lineares do sistema e, para isto, utilizaremos outros tipos de estratégias para verificar quais hipóteses devemos adicionar no amortecimento de modo que possamos determinar a taxa de decaimento da energia associada ao sistema. Este tema é o alvo de um trabalho em andamento.

Capítulo 1

Asymptotic limits and stabilization for the 2D nonlinear Mindlin-Timoshenko system

Asymptotic limits and stabilization for the 2D nonlinear Mindlin-Timoshenko system

F. D. Araruna, P. Braz e Silva and P. Queiroz-Souza

Abstract: We show how the so called von Kármán model can be obtained as a singular limit of a Mindlin-Timoshenko system when the modulus of elasticity in shear k tends to infinity. This result gives a positive answer to a conjecture by Lagnese-Lions in 1988. Introducing damping mechanisms, we also show that the energy of solutions for this modified Mindlin-Timoshenko system decays exponentially, uniformly with respect to the parameter k. As $k \to \infty$, we obtain the damped Von-Kármán model with associated energy exponentially decaying to zero as well.

1.1 Introduction

The Mindlin-Timoshenko system of equations is a widely used and physically fairly complete mathematical model to describe the dynamics of a plate taking into account transverse shear effects (see, e.g., [27] and the references therein). This model is used, for example, to model aircraft wings (see, for instance, [16]). To describe this model, let $\Omega \subset \mathbb{R}^2$ be an open bounded set whose boundary Γ is regular enough. Consider $\{\Gamma_0, \Gamma_1\}$ to be a partition of Γ. Let $T > 0$ be given and consider the cylinder $Q = \Omega \times (0,T)$, with lateral boundary $\Sigma = \Sigma_0 \cup \Sigma_1$, where $\Sigma_i = \Gamma_i \times (0,T)$, $i = 0, 1$. The two-dimensional Mindlin-Timoshenko system is the following:

$$\begin{cases} \dfrac{\rho h^3}{12}\phi_{1tt} - L_1(\phi_1,\phi_2,\psi) = 0, & \text{in} \quad Q, \\ \dfrac{\rho h^3}{12}\phi_{2tt} - L_2(\phi_1,\phi_2,\psi) = 0, & \text{in} \quad Q, \\ \rho h \psi_{tt} - L_3(\phi_1,\phi_2,\psi,\eta_1,\eta_2) = 0, & \text{in} \quad Q, \\ \rho h \eta_{1tt} - L_4(\psi,\eta_1,\eta_2) = 0, & \text{in} \quad Q, \\ \rho h \eta_{2tt} - L_5(\psi,\eta_1,\eta_2) = 0, & \text{in} \quad Q. \end{cases} \tag{1.1}$$

We complete the system with the boundary conditions

$$\begin{cases} \phi_1 = \phi_2 = \psi = \eta_1 = \eta_2 = 0 & \text{on} \quad \Sigma_0, \\ \{\mathcal{B}_1(\phi_1,\phi_2), \mathcal{B}_2(\phi_1,\phi_2), \mathcal{B}_3(\phi_1,\phi_2,\psi,\eta_1,\eta_2), \mathcal{B}_4(\eta_1,\eta_2), \mathcal{B}_5(\eta_1,\eta_2)\} = \{0,0,0,0,0\} & \text{on} \quad \Sigma_1, \end{cases} \tag{1.2}$$

and initial data

$$\begin{cases} \{\phi_1(\cdot,0), \phi_2(\cdot,0), \psi(\cdot,0), \eta_1(\cdot,0), \eta_2(\cdot,0)\} = \{\phi_{10}, \phi_{20}, \psi_0, \eta_{10}, \eta_{20}\} & \text{in} \quad \Omega, \\ \{\phi_{1t}(\cdot,0), \phi_{2t}(\cdot,0), \psi_t(\cdot,0), \eta_{1t}(\cdot,0), \eta_{2t}(\cdot,0)\} = \{\phi_{11}, \phi_{21}, \psi_1, \eta_{11}, \eta_{21}\} & \text{in} \quad \Omega, \end{cases} \tag{1.3}$$

where

$$L_1(\phi_1,\phi_2,\psi) = D\left(\phi_{1xx} + \frac{1-\mu}{2}\phi_{1yy} + \frac{1+\mu}{2}\phi_{2xy}\right) - k\left(\phi_1 + \psi_x\right),$$

$$L_2(\phi_1,\phi_2,\psi) = D\left(\phi_{2yy} + \frac{1-\mu}{2}\phi_{2xx} + \frac{1+\mu}{2}\phi_{1xy}\right) - k\left(\phi_2 + \psi_y\right),$$

$$L_3(\phi_1,\phi_2,\psi,\eta_1,\eta_2) = k\left[(\psi_x + \phi_1)_x + (\psi_y + \phi_2)_y\right] + (N_1\psi_x + N_{12}\psi_y)_x + (N_2\psi_y + N_{12}\psi_x)_y,$$

$$L_4(\psi,\eta_1,\eta_2) = N_{1x} + N_{12y},$$

$$L_5(\psi,\eta_1,\eta_2) = N_{2y} + N_{12x},$$

$$\mathcal{B}_1(\phi_1,\phi_2) = D\left[\nu_1\phi_{1x} + \mu\nu_1\phi_{2y} + \frac{1-\mu}{2}\left(\phi_{1y} + \phi_{2x}\right)\nu_2\right],$$

$$\mathcal{B}_2(\phi_1,\phi_2) = D\left[\nu_2\phi_{2y} + \mu\nu_2\phi_{1x} + \frac{1-\mu}{2}\left(\phi_{1y} + \phi_{2x}\right)\nu_1\right],$$

$$\mathcal{B}_3(\phi_1,\phi_2,\psi,\eta_1,\eta_2) = k\left(\frac{\partial\psi}{\partial\nu} + \nu_1\phi_1 + \nu_2\phi_2\right) + \left(\nu_1 N_1 + \nu_2 N_{12}\right)\psi_x + \left(\nu_2 N_2 + \nu_1 N_{12}\right)\psi_y,$$

$$\mathcal{B}_4(\psi,\eta_1,\eta_2) = \nu_1 N_1 + \nu_2 N_{12},$$

$$\mathcal{B}_5(\psi,\eta_1,\eta_2) = \nu_2 N_2 + \nu_1 N_{12},$$

$$N_1 = \frac{Eh}{1-\mu^2}\left(\eta_{1x} + \mu\eta_{2y} + \frac{1}{2}\psi_x^2 + \frac{\mu}{2}\psi_y^2\right),$$

$$N_2 = \frac{Eh}{1-\mu^2}\left(\eta_{2y} + \mu\eta_{1x} + \frac{1}{2}\psi_y^2 + \frac{\mu}{2}\psi_x^2\right),$$

$$N_{12} = \frac{Eh}{2(1+\mu)}\left(\eta_{1y} + \eta_{2x} + \psi_x\psi_y\right).$$

In system (1.1), subscripts mean partial derivatives. The vector $\nu = (\nu_1, \nu_2)$ represents the outward unit normal to Ω and $\frac{\partial}{\partial\nu}$ stands for the normal derivative. The unknowns are $\phi_1 = \phi_1(x,y,t)$, $\phi_2 = \phi_2(x,y,t)$, $\psi = \psi(x,y,t)$, $\eta_1 = \eta_1(x,y,t)$, and $\eta_2 = \eta_2(x,y,t)$. Physically, the functions ϕ_1 and ϕ_2 represent, respectively, the angles of rotation of the cross sections $x = constant$, $y = constant$ containing the filament which, when the plate is in equilibrium, is orthogonal to the middle surface at the point $(x, y, 0)$. The function ψ is the vertical displacement, and η_1, η_2 are the in-plane displacement of the plate at time t of the cross section located at (x, y) units from the end-point $(x, y) = (0, 0)$. The positive constant h represents the thickness of the plate which, in this model, is considered to be small and uniform with respect to x. The constant ρ is the mass density per unit volume of the plate and the parameter k is the so called modulus of elasticity in shear. The constant E is the Young's modulus and the constant μ, $0 < \mu < 1/2$, is the Poisson's ratio. The constant D is the modulus of flexural rigidity and is given by $D = Eh^3/12(1-\mu^2)$. The constant k is given by the expression $k = \widehat{k}Eh/2\,(1+\mu)$, where $\widehat{k}$ is a shear correction coefficient. For more details concerning the Mindlin-Timoshenko hypotheses and the governing equations see, for instance, Lagnese-Lions [27].

For the nonlinear system (1.1)–(1.3), in [44] Rahmani considers a plate reinforced by a thin stiffener on a portion of its boundary and models this junction through an approximate model where the stiffener has a role on its boundary conditions.

The linear version of system (1.1)–(1.3) is

$$\begin{cases} \dfrac{\rho h^3}{12}\phi_{1tt} - L_1(\phi_1, \phi_2, \psi) = 0 \quad \text{in} \quad Q, \\ \dfrac{\rho h^3}{12}\phi_{2tt} - L_2(\phi_1, \phi_2, \psi) = 0 \quad \text{in} \quad Q, \\ \rho h \psi_{tt} - \widetilde{L}_3(\phi_1, \phi_2, \psi) = 0 \qquad \text{in} \quad Q, \end{cases} \tag{1.4}$$

where L_1, L_2 are defined above and

$$\widetilde{L}_3(\phi_1, \phi_2, \psi) \;=\; k\left[(\psi_x + \phi_1)_x + (\psi_y + \phi_2)_y\right].$$

There are quite a few works on this system: Lagnese and Lions (see [27]) studied its well-posedness and analyzed its asymptotic limit when the parameter k tends to infinity. In [25], Lagnese studied problems of existence, uniqueness and some other important properties as the asymptotic behavior in time when some damping effects are considered. In [14], Chueshov and Lasiecka studied the dynamics for a class of Mindlin-Timoshenko plate models with nonlinear feedback forces and showed the existence of a compact global attractor for the system. Furthermore they studied its limiting properties when the shear modulus tends to infinity. In [48], Sare investigated system (1.4) with frictional dissipations acting on the equations for the rotation angles and proved that this system is not exponentially stable independent of any relations between the constants of the system. Moreover, he showed that the solution decays polynomially to zero, with rates that can be improved depending on the regularity of the initial data. In 2015, Rahmani (see [45]) studied system (1.4) and obtained results similar to those in [44] for the system (1.1)–(1.3).

If one assumes the filament of the plate to remain orthogonal to the deformed middle surface, the transverse shear effects are neglected, and the resulting model is the so called on Kármán system (see [27])

$$\begin{cases} \rho h \psi_{tt} - \dfrac{\rho h^3}{12}\Delta\psi_{tt} + D\Delta^2\psi - (N_1\psi_x + N_{12}\psi_y)_x - (N_2\psi_y + N_{12}\psi_x)_y = 0 & \text{in} \quad Q, \\ \rho h \eta_{1tt} - (N_{1x} + N_{12y}) = 0 & \text{in} \quad Q, \\ \rho h \eta_{2tt} - (N_{2y} + N_{12x}) = 0 & \text{in} \quad Q, \end{cases} \tag{1.5}$$

with boundary conditions

$$\begin{cases} \psi = \dfrac{\partial \psi}{\partial \nu} = \eta_1 = \eta_2 = 0 & \text{on} \quad \Gamma_0, \\ D\left[\Delta\psi + (1-\mu)\left(2\nu_1\nu_2\psi_{xy} - \nu_1^2\psi_{yy} - \nu_2^2\psi_{xx}\right)\right] = 0 & \text{on} \quad \Sigma_1, \\ D\left[\dfrac{\partial(\Delta\psi)}{\partial\nu} + (1-\mu)\dfrac{\partial}{\partial\tau}\left[\left(\nu_1^2 - \nu_2^2\right)\psi_{xy} + \nu_1\nu_2\left(\psi_{yy} - \psi_{xx}\right)\right]\right] - \dfrac{\rho h^3}{12}\dfrac{\partial \psi_{tt}}{\partial \nu} & \\ \qquad -(\nu_2 N_2 + \nu_1 N_{12})\psi_y = 0 & \text{on} \quad \Gamma_1, \\ \nu_1 N_1 + \nu_2 N_{12} = 0 & \text{on} \quad \Gamma_1, \\ \nu_2 N_2 + \nu_1 N_{12} = 0 & \text{on} \quad \Gamma_1, \end{cases} \tag{1.6}$$

and initial data

$$\begin{cases} \{\psi(\cdot,0), \eta_1(\cdot,0), \eta_2(\cdot,0)\} = \{\psi_0, \eta_{10}, \eta_{20}\} & \text{in} \quad \Omega, \\ \{\psi_t(\cdot,0), \eta_{1t}(\cdot,0), \eta_{2t}(\cdot,0)\} = \{\psi_1, \eta_{11}, \eta_{21}\} & \text{in} \quad \Omega. \end{cases} \tag{1.7}$$

In (1.6), $\tau = (-\nu_2, \nu_1)$ is the tangent vector to Ω and $\frac{\partial}{\partial \tau}$ represents the tangential derivative. System (1.5)–(1.7) has been object of studies for many years. Let us mention some known results about this type of system. Lasiecka [29] and Favini et.al.

[18] studied well-posedness for this problem, as well as the regularity of its solution. Perla Menzala and Zuazua [39] proved exponential decay rates for the energy of the system for a bounded smooth thermoelastic plate clamped on its boundary. A similar result was obtained by Kang [21] for von Kármán equations with a memory term. Finally, for monotonic functions with certain growth properties at the origin and at infinity, Lagnese and Leuring [26] showed that the one-dimensional von Kármán is uniformly asymptotically stable.

Neglecting the shear effects of the plate obtaining system (1.5) is formally equivalent to considering the modulus of elasticity k tending to infinity in system (1.1), since k is inversely proportional to the shear angle. The present paper is devoted to analyze the asymptotic limit of the nonlinear Mindlin-Timoshenko system (1.1) as $k \to \infty$. This problem was mentioned in the 1988 book by Lagnese and Lions [27, p. 24], where it was conjectured that system (1.1) approaches, in some sense, the von Kármán system (1.5), as $k \to \infty$:

> *"One expects that, as $k \to \infty$, solutions of the system* (1.1) *will converge (in some sense) to solution of the von Kármán system* (1.5)*. However, a rigorous proof of convergence is lacking and seems to be a difficult question."*

In this direction, in 1988 Lagnese and Lions proved in [27] (see also [9] for the one-dimensional case) that, in the linear case, the solution of the Mindlin-Timoshenko model (1.4) converges, as $k \to \infty$, towards to the solution of the Kirchhoff model (subject to appropriate boundary conditions)

$$\rho h \psi_{tt} - \frac{\rho h^3}{12} \Delta \psi_{tt} + D \Delta^2 \psi = 0. \tag{1.8}$$

Later on, in [8], the authors studied the following one-dimensional nonlinear Mindlin-Timoshenko system with an extra fourth order regularizing term

$$\begin{cases} \frac{\rho h^3}{12} \phi_{tt} - D\phi_{xx} + k(\phi + \psi_x) = 0 & \text{in } Q, \\ \rho h \psi_{tt} - k(\phi + \psi_x)_x - Eh\left[\psi_x\left(\eta_x + \frac{1}{2}\psi_x^2\right)\right]_x + \frac{1}{k}\psi_{xxxx} = 0 & \text{in } Q, \\ \rho h \eta_{tt} - Eh\left(\eta_x + \frac{1}{2}\psi_x^2\right)_x = 0 & \text{in } Q, \end{cases} \tag{1.9}$$

and showed that, as $k \to \infty$, the system (1.9) converges toward the one-dimensional von Kármám system

$$\begin{cases} \rho h\psi_{tt} - \dfrac{\rho h^3}{12}\psi_{xxtt} + D\psi_{xxxx} - Eh\left[\psi_x\left(\eta_x + \dfrac{1}{2}\psi_x^2\right)\right]_x = 0 & \text{in} \quad Q, \\ \rho h\eta_{tt} - Eh\left(\eta_x + \dfrac{1}{2}\psi_x^2\right)_x = 0 & \text{in} \quad Q. \end{cases} \tag{1.10}$$

In the argument used in [8], the use of the extra fourth order regularizing term was indispensable, since it ensures the compactness of a family of solutions, as $k \to \infty$, allowing one to pass to the limit in the nonlinear term. Here, we study the nonlinear two-dimensional problem without any regularizing term. We prove that the Mindlin-Timoshenko system converges to the von Kármán one, therefore giving a positive answer for the 1988 Lagnese-Lions conjecture. We note that our argument here can be used for the one-dimensional case as well, assuring the conjecture to hold also in the one-dimensional case (as would be expected).

In the context of asymptotic limits, with respect to singular coefficients, Menzala and Zuazua proved in [35] that the one-dimensional von Kármán system of equations approaches (weakly) to a nonlocal beam equation of Timoshenko type as a suitable parameter tends to zero. In [36], the authors considered a dynamical one-dimensional nonlinear von Kármán model depending on one parameter $\varepsilon > 0$ and studied its weak limit as $\varepsilon \to 0$. Furthermore, they proved that, depending on the type of boundary condition, the nonlinearity of Timoshenko model may either vanish or may become a nonlinearity concentrated on the extremes of the beam. In [37], the full nonlinear dynamic von Kármán system of equations was considered and the authors showed how the so-called Timoshenko and Berger models for thin plates may be obtained as singular limits of the von Kármán system when a suitable parameter tends to zero. We also mention the work [43], where the authors obtained the stabilization of Berger-Timoshenko's equation as a limit of the uniform stabilization of the von Kármán system of beams and plates with respect to a singular parameter.

The second part of this work concerns stabilization. Up to our knowledge, exponential stability has not been investigated for the two-dimension nonlinear Mindlin-Timoshenko system, so we study decay properties of its solutions with both internal and boundary damping. More precisely, we show the following: Adding appropriate damping terms, there is a uniform (with respect to k) rate of decay for the total energy of the solutions for (1.1) as $t \to \infty$. As a consequence of this analysis, we obtain a decay rate for the total energy of the solutions for the von Kármán system (as $t \to \infty$) as a singular limit of the uniform (with respect to k) decay rate of the energy of the Mindlin-Timoshenko system.

Let us mention some known results related to the stabilization. In the one-dimensional case, Araruna et. al. showed in [8] the exponential stability of the nonlinear Mindlin-Timoshenko beam under internal damping. Stabilization results for the linear model were obtained in [25, 22] considering damping in both equations, and in [1] with a single nonlinear feedback control. In [5], the system is damped by a memory type term. In the two-dimensional case, the uniform stabilization for linear Mindlin-Timoshenko model was studied in [48] considering frictional dissipations acting on the equations for the rotations angle. Grobbelaar-Dalsen [19] studied the polynomial decay rate of the Mindlin-Timoshenko plate model with thermal dissipation. Stabilization results were obtained in [42] for the multi-dimensional case with nonconstant and nonsmooth coefficients, when the interior dissipation acts either on both equations or only on the elasticity equation. The stabilization of the von Kármán system, in the two-dimensional case, was studied by Menzala and Zuazua in [39], where the energy decreases along trajectories. Bradley-Lasiecka [10] studied the local exponential stabilization for an unstructured perturbation and feedback controls. Kang [21] proved the exponential decay for the nonlinear von Kármán system with memory.

This work is organized as follows. In Section 1.2, we rigorously study the behavior of the Mindlin-Timoshenko system towards the von Kármán system as $k \to \infty$. More precisely, we prove that solutions $\{\phi_1, \phi_2, \psi, \eta_1, \eta_2\}$ of (1.1)–(1.3) converge to $\{-\psi_x, -\psi_y, \psi, \eta_1, \eta_2\}$ as $k \to \infty$, where $\{\psi, \eta_1, \eta_2\}$ solves system (1.5)–(1.7). In Sections 1.3 and 1.4 we prove that, adding appropriate damping terms (internal and boundary, respectively), one can prove an uniform (in k) exponential decay property for the solutions of (1.1)–(1.3). Finally, in Section 1.5, we briefly discuss some related issues and open problems.

1.2 Asymptotic limit

In this section, we study the asymptotic limit of the solutions for the nonlinear Mindlin-Timoshenko system (1.1)–(1.3) as $k \to \infty$. To study this problem, we consider the Hilbert space

$$\mathcal{X} = \left[H^1_{\Gamma_0}(\Omega) \times L^2(\Omega)\right]^2 \times \left[W^{1,4}(\Omega) \cap H^1_{\Gamma_0}(\Omega)\right] \times L^2(\Omega) \times \left[H^1_{\Gamma_0}(\Omega) \times L^2(\Omega)\right]^2, \quad (1.11)$$

where $H^1_{\Gamma_0}(\Omega) = \{\varphi : \varphi \in H^1(\Omega),\ \varphi = 0 \ \text{ on } \ \Gamma_0\}$.

The energy $E_k(t)$ of solutions is given by

$$\begin{aligned} E_k(t) &= \frac{1}{2}\left\{\frac{\rho h^3}{12}\left[|\phi_{1t}|^2 + |\phi_{2t}|^2\right] + \rho h\left[|\psi_t|^2 + |\eta_{1t}|^2 + |\eta_{2t}|^2\right] + k\left[|\phi_1 + \psi_x|^2 + |\phi_2 + \psi_y|^2\right]\right. \quad (1.12) \\ &\left. + F([b_{ij}],[b_{ij}]) + D\left[|\phi_{1x}|^2 + |\phi_{2y}|^2 + \frac{1-\mu}{2}|\phi_{1y} + \phi_{2x}|^2 + 2\mu\int_\Omega (\phi_{1x}\phi_{2y})\,dxdy\right]\right\}, \end{aligned}$$

where

$$b_{11} = \eta_{1x} + \frac{1}{2}\psi_x^2, \quad b_{22} = \eta_{2y} + \frac{1}{2}\psi_y^2, \quad b_{12} = b_{21} = \eta_{1y} + \eta_{2x} + \psi_x\psi_y,$$

and

$$F([b_{ij}]) = \frac{Eh}{1-\mu^2}\left\{\mu\begin{bmatrix} b_{11}+b_{22} & 0 \\ 0 & b_{11}+b_{22}\end{bmatrix} + c(1-\mu)\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22}\end{bmatrix}\right\}.$$

Note that

$$\begin{aligned} (F([b_{ij}]),[b_{ij}])_{(L^2(\Omega))^4} &= \left(\frac{Eh}{1-\mu^2}\left\{\mu\begin{bmatrix} b_{11}+b_{22} & 0 \\ 0 & b_{11}+b_{22}\end{bmatrix} + c(1-\mu)\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22}\end{bmatrix}\right\}, \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22}\end{bmatrix}\right) \\ &= \frac{Eh}{1-\mu^2}\left\{\mu\left|\eta_{1x} + \eta_{2y} + \frac{1}{2}|\nabla\psi|^2\right|^2 + (1-\mu)|b_{11}|^2 + (1-\mu)|b_{22}|^2\right. \\ &\quad \left. + \frac{1-\mu}{2}|\eta_{1y} + \eta_{2x} + \psi_x\psi_y|^2\right\} > 0 \end{aligned}$$

since $\frac{Eh}{1-\mu^2} > 0$ and $0 < \mu < 1$, which shows that F is positive definite. Moreover, we have by [25, Lemma 2.1] that

$$D\left[|\phi_{1x}|^2 + |\phi_{2y}|^2 + \frac{1-\mu}{2}|\phi_{1y} + \phi_{2x}|^2 + 2\mu\int_\Omega (\phi_{1x}\phi_{2y})\right] \geq C\|\phi_1\|^2_{H^1(\Omega)} + \|\phi_2\|^2_{H^1(\Omega)}.$$

So, the energy is positive. Furthermore,

$$E_k(t) = E_k(0), \quad \forall\, t \geq 0. \quad (1.13)$$

The main result of this paper is to give a positive response to a Lagnese-Lions conjecture from [27]. Our result is as follows.

Theorem 1.2.1 *Let* $\{\phi_1^k, \phi_2^k, \psi^k, \eta_1^k, \eta_2^k\}$ *be a solution of system* (1.1)–(1.3) *with initial data* $\{\phi_{10}, \phi_{11}, \phi_{20}, \phi_{21}, \psi_0, \psi_1, \eta_{10}, \eta_{11}, \eta_{20}, \eta_{21}\} \in \mathcal{X}$ *satisfying*

$$\phi_{10} + \psi_{0x} = 0 \qquad \textit{and} \qquad \phi_{20} + \psi_{0y} = 0 \qquad \textit{in} \qquad \Omega. \tag{1.14}$$

Then, letting $k \to \infty$, *one gets*

$$\{\phi_1^k, \phi_2^k, \psi^k, \eta_1^k, \eta_2^k\} \to \{-\psi_x, -\psi_y, \psi, \eta_1, \eta_2\} \;\; \textit{weak}-* \;\; \textit{in} \quad L^\infty\left(0, T, [H^1_{\Gamma_0}(\Omega)]^3 \times [L^2(\Omega)]^2\right),$$

where $\{\psi, \eta_1, \eta_2\}$ *solves* (1.5)–(1.7).

Remark 1.2.2 *The variational formulation of system* (1.5)–(1.7) *is given by*

$$\begin{aligned} &\rho h \frac{d}{dt}(\psi_t, c) + \tfrac{\rho h^3}{12} \frac{d}{dt}(\nabla \psi_t, \nabla c) + \rho h \frac{d}{dt}(\eta_{1t}, d) + \rho h \frac{d}{dt}(\eta_{2t}, e) + (N_1 \psi_x + N_{12}\psi_y, c_x) \\ &+ (N_2 \psi_y + N_{12}\psi_x, c_y) + \left(N_1^k, d_x\right) + \left(N_{12}^k, d_y\right) + \left(N_2^k, e_y\right) + \left(N_{12}^k, e_x\right) + D(\Delta\psi, \Delta c) = 0, \end{aligned} \tag{1.15}$$

for all $\{c, d, e\} \in \left[H^2(\Omega) \cap H^1_{\Gamma_0}(\Omega)\right] \times \left[H^1_{\Gamma_0}(\Omega)\right]^2$ *and the initial conditions* (1.6). *In equation* (1.15), $(\cdot, \cdot)$ *represents the inner product in* $L^2(\Omega)$. *Furthermore, the system* (1.5)–(1.7) *is conservative, that is, its energy*

$$\begin{aligned} E(t) \;=\; \frac{1}{2}\Bigg\{ &\rho h \left[|\psi_t|^2 + |\eta_{1t}|^2 + |\eta_{2t}|^2\right] + \frac{\rho h^3}{12}|\nabla\psi_t|^2 + D|\Delta\psi|^2 + \frac{Eh}{1-\mu}\int_\Omega \left[\eta_{1x} + \frac{1}{2}\psi_x^2\right]^2 \\ &+ \left[\eta_{2y} + \frac{1}{2}\psi_y^2\right]^2 + \left[\eta_{1x} + \eta_{2y} + \frac{1}{2}|\nabla\psi|^2\right]^2 + \tfrac{1-\mu}{2}\left[\eta_{1y} + \eta_{2x} + \psi_x\psi_y\right]^2 \Bigg\} \end{aligned} \tag{1.16}$$

satisfies $E(t) = E(0)$, *for all* $t \in [0, T]$.

Proof of Theorem 1.2.1. For each $k > 0$ fixed, let $\{\phi_1^k, \phi_2^k, \psi^k, \eta_1^k, \eta_2^k\}$ be the solution of system (1.1)–(1.3) with data $\{\phi_{10}, \phi_{11}, \phi_{20}, \phi_{21}, \psi_0, \psi_1, \eta_{10}, \eta_{11}, \eta_{20}, \eta_{21}\} \in \mathcal{X}$. Since the initial data $\{\phi_{10}, \phi_{11}, \phi_{20}, \phi_{21}, \psi_0, \psi_1, \eta_{10}, \eta_{11}, \eta_{20}, \eta_{21}\}$ satisfy the condition (1.14), one has, due to the conservation of energy (1.13),

$$E_k(t) \le C, \quad \forall\, k > 0,\ \forall\, t > 0. \tag{1.17}$$

From now on, the letter C stands for a generic positive constant which may vary from line to line (unless otherwise stated). The estimate (1.17) implies that the sequences (in k)

$$(\phi_{1t}^k),\ (\phi_{2t}^k),\ (\psi_t^k),\ (\eta_{1t}^k),\ (\eta_{2t}^k),\ \sqrt{k}\left(\phi_1^k + \psi_x^k\right),\ \sqrt{k}\left(\phi_2^k + \psi_y^k\right),\ \left(\phi_{1x}^k\right),\left(\phi_{2y}^k\right),\ \left(\phi_{1y}^k + \phi_{2x}^k\right),$$

$$\left(\eta_{1x}^k + \frac{1}{2}\left[\psi_x^k\right]^2\right),\ \left(\eta_{2y}^k + \frac{1}{2}\left[\psi_y^k\right]^2\right),\ \left(\eta_{1x}^k + \eta_{2y}^k + \frac{1}{2}\left[\nabla\psi^k\right]^2\right),\ \left(\eta_{1y}^k + \eta_{2x}^k + \psi_x^k\psi_y^k\right)$$

are bounded in $L^\infty(0,T,L^2(\Omega))$. Furthermore,

$$\left[\phi_{1y}^k\right]_x = \left[\phi_{1x}^k\right]_y \in H^{-1}(\Omega) \quad \text{and} \quad \left[\phi_{2x}^k\right]_y = \left[\phi_{2y}^k\right]_x \in H^{-1}(\Omega),$$

since $\left(\phi_{1x}^k\right)$ and $\left(\phi_{2y}^k\right)$ are bounded in $L^\infty(0,T,L^2(\Omega))$. On the other hand,

$$\left[\phi_{1y}^k\right]_y = \left[\phi_{1y}^k + \phi_{2x}^k\right]_y - \left[\phi_{2x}^k\right]_y = \left[\phi_{1y}^k + \phi_{2x}^k\right]_y - \left[\phi_{2y}^k\right]_x \in H^{-1}(\Omega),$$

which implies that $\left(\phi_{1y}^k\right)$ is bounded in $L^\infty\left(0,T,L^2(\Omega)\right)$. Similarly, one can show that $\left(\phi_{2x}^k\right)$ is bounded in $L^\infty\left(0,T,L^2(\Omega)\right)$. Thus, the sequences (in k) (ϕ_1^k), (ϕ_2^k) and (ψ^k) are bounded in $L^\infty\left(0,T,H^1_{\Gamma_0}(0,L)\right)$.

Since, for each k, η_{1t}^k and η_{2t}^k belong to $C^0\left([0,T],L^2(\Omega)\right)$, we can write

$$\eta_1^k(t) = \eta_{10} + \int_0^t \eta_{1t}^k(s)ds \quad \text{and} \quad \eta_2^k(t) = \eta_{20} + \int_0^t \eta_{2t}^k(s)ds.$$

Therefore, since (η_{1t}^k) is bounded in $L^\infty(0,T,L^2(0,L))$, the sequence $\left(\eta_1^k\right)$ is bounded $L^\infty\left(0,T,L^2(\Omega)\right)$. Indeed,

$$\left|\eta_1^k\right| = \left|\eta_{10} + \int_0^t \eta_{1t}^k\right| \leq C + \int_0^t \left|\eta_{1t}^k\right| \leq C.$$

Analogously, it follows that $\left(\eta_2^k\right)$ is bounded in $L^\infty\left(0,T,L^2(\Omega)\right)$. Therefore, the sequences $\left(\eta_1^k\right)$, $\left(\eta_2^k\right)$ are bounded in $L^\infty\left(0,T,L^2(\Omega)\right)$. Extracting subsequences, without changing notation, one gets

$$\left\{\phi_1^k,\phi_2^k,\psi^k,\eta_1^k,\eta_2^k\right\} \to \{\phi_1,\phi_2,\psi,\eta_1,\eta_2\} \quad \text{weak}-* \quad \text{in} \quad L^\infty\left(0,T;\left[H^1_{\Gamma_0}(\Omega)\right]^3 \times \left[L^2(\Omega)\right]^2\right), \tag{1.18}$$

with

$$\phi_1 + \psi_x = 0 \quad \text{and} \quad \phi_2 + \psi_y = 0, \tag{1.19}$$

$$\left\{\phi_{1t}^k,\phi_{2t}^k,\psi_t^k,\eta_{1t}^k,\eta_{2t}^k\right\} \to \{\phi_{1t},\phi_{2t},\psi_t,\eta_{1t},\eta_{2t}\} \quad \text{weak}-* \quad \text{in} \quad L^\infty\left(0,T;\left[L^2\left(\Omega\right)\right]^5\right), \tag{1.20}$$

$$\eta_{1x}^k + \frac{1}{2}\left[\psi_x^k\right]^2 \to \alpha \quad \text{weak}-* \quad \text{in} \quad L^\infty\left(0,T,L^2(\Omega)\right), \tag{1.21}$$

$$\eta_{2y}^k + \frac{1}{2}\left[\psi_y^k\right]^2 \to \beta \quad \text{weak}-* \quad \text{in} \quad L^\infty\left(0,T,L^2(\Omega)\right), \tag{1.22}$$

and

$$\eta_{1y}^k + \eta_{2x}^k + \psi_x^k\psi_y^k \to \gamma \quad \text{weak}-* \quad \text{in} \quad L^\infty\left(0,T,L^2(\Omega)\right). \tag{1.23}$$

Now, using a compactness theorem due to Aubin-Lions (see [50, Corollary 4]), we obtain

$$\phi_1^k \to \phi_1 \quad \text{strongly} \quad \text{in} \quad L^2(Q) \tag{1.24}$$

and

$$\phi_2^k \to \phi_2 \quad \text{strongly} \quad \text{in} \quad L^2(Q). \tag{1.25}$$

Therefore, given $\varepsilon > 0$, for large enough k one has

$$\left|\psi_x^k + \phi_1\right| \leq \left|\psi_x^k + \phi_1^k\right| + \left|\phi_1^k - \phi_1\right| \leq \frac{C}{\sqrt{k}} + \varepsilon.$$

Consequently,

$$\psi_x^k \to -\phi_1 \quad \text{in} \quad L^2(Q). \tag{1.26}$$

On the other hand, we also have by the convergence (1.18) that

$$\psi_x^k \to \psi_x \quad \text{in} \quad \mathcal{D}'(Q) \tag{1.27}$$

Combining (1.26) and (1.27), we obtain

$$\psi_x = -\phi_1.$$

In a similar way, we get

$$\psi_y = -\phi_2.$$

Therefore,

$$\psi^k \to \psi \quad \text{strongly} \quad \text{in} \quad L^\infty\left(0, T, H^1_{\Gamma_0}(\Omega)\right). \tag{1.28}$$

By the previous convergence we conclude that

$$\left[\psi_x^k\right]^2 \to [\psi_x]^2 \quad \text{strongly} \quad \text{in} \quad L^\infty\left(0, T, L^1(\Omega)\right) \tag{1.29}$$

and

$$\left[\psi_y^k\right]^2 \to [\psi_y]^2 \quad \text{strongly} \quad \text{in} \quad L^\infty\left(0, T, L^1(\Omega)\right). \tag{1.30}$$

On other hand, the sequences $\left(\eta_1^k\right)$, $\left(\eta_2^k\right)$ are bounded in $L^\infty\left(0, T, L^2(\Omega)\right)$ and so

$$\eta_{1x}^k \to \eta_{1x} \quad \text{weak} - * \quad \text{in} \quad L^\infty\left(0, T, H^{-1}(\Omega)\right) \tag{1.31}$$

and

$$\eta_{2y}^k \to \eta_{2y} \quad \text{weak} - * \quad \text{in} \quad L^\infty\left(0, T, H^{-1}(\Omega)\right). \tag{1.32}$$

The same holds for $\left(\eta_{1y}^k\right)$ and $\left(\eta_{2x}^k\right)$. Combining the convergences (1.28)–(1.32), it follows that

$$\alpha = \eta_{1x} + \frac{1}{2}\psi_x^2, \quad \beta = \eta_{2y} + \frac{1}{2}\psi_y^2, \quad \gamma = \eta_{1y} + \eta_{2x} + \psi_x\psi_y,$$

$$N_1^k\psi_x^k + N_{12}^k\psi_y^k \to N_1\psi_x + N_{12}\psi_y \quad \text{weakly in} \quad L^\infty\left(0,T,L^2(\Omega)\right), \tag{1.33}$$

and

$$N_2^k\psi_y^k + N_{12}^k\psi_x^k \to N_2\psi_y + N_{12}\psi_x \quad \text{weakly in} \quad L^\infty\left(0,T,L^2(\Omega)\right). \tag{1.34}$$

For $\{a,b,c,d,e\} \in \left[H^1_{\Gamma_0}(\Omega)\right]^5$ satisfying

$$a + c_x = 0 \qquad \text{and} \qquad b + c_y = 0, \tag{1.35}$$

the variational formulation of problem (1.1)–(1.3) is

$$\begin{aligned}
&\tfrac{\rho h^3}{12}\tfrac{d}{dt}(\phi_{1t}^k,a) + \tfrac{\rho h^3}{12}\tfrac{d}{dt}(\phi_{2t}^k,b) + \rho h\tfrac{d}{dt}(\psi_t^k,c) + \rho h\tfrac{d}{dt}(\eta_{1t}^k,d) + \rho h\tfrac{d}{dt}(\eta_{2t}^k,e) + D\left[\left(\phi_{1x}^k,a_x\right) + \tfrac{1-\mu}{2}\left(\phi_{1y}^k,a_y\right)\right.\\
&\quad + \tfrac{1+\mu}{2}\left(\phi_{2x}^k,a_y\right) + \left(\phi_{2y}^k,b_y\right) + \tfrac{1-\mu}{2}\left(\phi_{2x}^k,b_x\right) + \left.\tfrac{1+\mu}{2}\left(\phi_{1y}^k,b_x\right)\right] + \left(N_1^k\psi_x^k + N_{12}^k\psi_y^k,c_x\right)\\
&\quad + \left(N_1^k,d_x\right) + \left(N_{12}^k,d_y\right) + \left(N_2^k\psi_y^k + N_{12}^k\psi_x^k,c_y\right) + \left(N_2^k,e_y\right) + \left(N_{12}^k,e_x\right) = 0.
\end{aligned} \tag{1.36}$$

Using convergences (1.18), (1.20)–(1.23), (1.33) and (1.34) in equation (1.36), and applying identities (1.19) and (1.35), one obtains the weak formulation of the system (1.5)–(1.7) given in (1.15). To finish the proof, it remains to identify the initial data of the limit system. In view of the convergences (1.18), (1.20), and classical compactness arguments, one has $\{\psi^k,\eta_1^k,\eta_2^k\} \to \{\psi,\eta_1,\eta_2\}$ in $C^0([0,T];[L^2(\Omega)]^3)$. Then, $\left\{\psi^k(\cdot,0),\eta_1^k(\cdot,0),\eta_2^k(\cdot,0)\right\} \to \{\psi(\cdot,0),\eta_1(\cdot,0),\eta_2(\cdot,0)\}$ in $[L^2(\Omega)]^3$, which combined with $(1.3)_1$ guarantees that $\{\psi(\cdot,0),\eta_1(\cdot,0),\eta_2(\cdot,0)\} = \{\psi_0,\eta_{10},\eta_{20}\}$. In order to identify $\{\psi_t(\cdot,0),\eta_{1t}(\cdot,0),\eta_{2t}(\cdot,0)\}$, multiply both sides of equation (1.36) by the function $\theta_\delta \in H^1(0,T)$ defined by

$$\theta_\delta(t) = \begin{cases} -\dfrac{t}{\delta} + 1, & \text{if} \quad t \in [0,\delta], \\ 0, & \text{if} \quad t \in (\delta,T], \end{cases}$$

and integrate by parts to obtain

$$
\begin{aligned}
&-\tfrac{\rho h^3}{12}(\phi_{11},a)+\tfrac{\rho h^3}{12\delta}\int_0^\delta(\phi_{1t}^k,a)dt-\tfrac{\rho h^3}{12}(\phi_{21},b)+\tfrac{\rho h^3}{12\delta}\int_0^\delta(\phi_{2t}^k,b)dt-\rho h(\psi_1,c)+\tfrac{\rho h}{\delta}\int_0^\delta(\psi_t^k,c)dt\\
&-\rho h(\eta_{11},d)+\tfrac{\rho h}{\delta}\int_0^\delta(\eta_{1t}^k,d)dt-\rho h(\eta_{21},e)+\tfrac{\rho h}{\delta}\int_0^\delta(\eta_{2t}^k,e)+\int_0^T D\left[\left(\phi_{1x}^k,a_x\right)+\tfrac{1-\mu}{2}\left(\phi_{1y}^k,a_y\right)\right.\\
&\left.+\tfrac{1+\mu}{2}\left(\phi_{2x}^k,a_y\right)+\left(\phi_{2y}^k,b_y\right)+\tfrac{1-\mu}{2}\left(\phi_{2x}^k,b_x\right)+\tfrac{1+\mu}{2}\left(\phi_{1y}^k,b_x\right)\right]\theta_\delta dt+\int_0^t\left(N_1^k\psi_x^k+N_{12}^k\psi_y^k,c_x\right)\theta_\delta dt\\
&+\int_0^T\left(N_2^k\psi_y^k+N_{12}^k\psi_x^k,c_y\right)\theta_\delta dt-\int_0^T\left(N_{1x}^k+N_{12y}^k,d\right)\theta_\delta dt-\int_0^T\left(N_{2y}^k+N_{12x}^k,e\right)\theta_\delta dt=0.
\end{aligned}
\tag{1.37}
$$

Passing to the limit as $k\to\infty$ in the last equation, and using (1.18), (1.20) – (1.33), one obtains

$$
\begin{aligned}
&-\tfrac{\rho h^3}{12}(\phi_{11},c_x)+\tfrac{\rho h^3}{12\delta}\int_0^\delta(\psi_{xt},c_x)dt+\tfrac{\rho h^3}{12}(\phi_{21},c_y)+\tfrac{\rho h^3}{12\delta}\int_0^\delta(\psi_{yt},c_y)dt-\rho h(\psi_1,c)+\tfrac{\rho h}{\delta}\int_0^\delta(\psi_t,c)dt\\
&-\rho h(\eta_{11},d)+\tfrac{\rho h}{\delta}\int_0^\delta(\eta_{1t},d)dt-\rho h(\eta_{21},e)+\tfrac{\rho h}{\delta}\int_0^\delta(\eta_{2t},e)+D\int_0^T(\Delta\psi,\Delta c)\theta_\delta dt\\
&+\int_0^t\left(N_1\psi_x+N_{12}\psi_y,c_x\right)\theta_\delta dt+\int_0^T\left(N_2\psi_y+N_{12}\psi_x,c_y\right)\theta_\delta dt-\int_0^T\left(N_{1x}+N_{12y},d\right)\theta_\delta dt\\
&-\int_0^T\left(N_{2y}+N_{12x},e\right)\theta_\delta dt=0.
\end{aligned}
$$

On the other hand, multiplying equation (1.15) by θ_δ and integrating in time, we get the identity

$$
\begin{aligned}
&-\tfrac{\rho h^3}{12}(\Delta\psi_t(\cdot,0),c)-\rho h(\psi_t(\cdot,0),c)-\rho h(\eta_{1t}(\cdot,0),d)-\rho h(\eta_{2t},e)\\
&\qquad=-\tfrac{\rho h^3}{12}(\phi_{11x}+\phi_{21y},c)-\rho h(\psi_1,c)-\rho h(\eta_{11},d)-\rho h(\eta_{21},e).
\end{aligned}
\tag{1.38}
$$

Therefore, $\left(-\tfrac{h^2}{12}\Delta\psi+\psi\right)_t(\cdot,0)=\psi_1+\tfrac{h^2}{12}(\phi_{11x}+\phi_{21y})$, $\eta_{1t}(\cdot,0)=\eta_{11}$, and $\eta_{2t}(\cdot,0)=\eta_{21}$. This concludes the proof. ■

Remark 1.2.3 *Note that in order to fully identify the initial data of the solutions of the limit system (1.5)–(1.7) and, more precisely, to determine the initial data of ψ_t, an elliptic equation has to be solved. Namely, the initial datum for the velocity ψ_t in $(1.5)_5$ is determined by solving the elliptic equation*

$$\psi_t(\cdot,0)\in H^1_{\Gamma_0}(\Omega):\quad\left(-\frac{h^2}{12}\Delta\psi+\psi\right)_t(\cdot,0)=\psi_1+\frac{h^2}{12}(\phi_{11x}+\phi_{21y})\quad in\quad\Omega,$$

as the proof of the theorem showed. More precisely, this elliptic equation can be written in the variational form

$$\frac{h^2}{12}(\nabla\psi_t(\cdot,0),\nabla c)+(\psi_t(\cdot,0),c)=(\psi_1,c)-\frac{h^2}{12}(\phi_{11},c_x)-\frac{h^2}{12}(\phi_{21},c_y),$$

where the terms ϕ_{11x} and ϕ_{21y} are not the derived from ϕ_1 and ϕ_2, respectively, in the sense of transposition, but they are rather the linear mappings which, when acting on any element c of $H^1_{\Gamma_0}(\Omega)$, produce $-(\phi_{11},c_x)$ and $-(\phi_{21},c_y)$. The same can be said about $\Delta\psi_t(\cdot,0)$ yielding $-(\nabla\psi_t(\cdot,0),\nabla c)$.

1.3 Stability: Internal feedback

In this section we analyze the plate model with hinged boundary conditions and in the presence of internal damping distributed all along the plate. Consider the system

$$\begin{cases} \dfrac{\rho h^3}{12}\phi_{1tt} - L_1(\phi_1,\phi_2,\psi) + \phi_{1t} = 0 & \text{in} \quad Q, \\ \dfrac{\rho h^3}{12}\phi_{2tt} - L_2(\phi_1,\phi_2,\psi) + \phi_{2t} = 0 & \text{in} \quad Q, \\ \rho h\psi_{tt} - L_3(\phi_1,\phi_2,\psi,\eta_1,\eta_2) + \psi_t = 0 & \text{in} \quad Q, \\ \rho h\eta_{1tt} - L_4(\psi,\eta_1,\eta_2) + \eta_{1t} = 0 & \text{in} \quad Q, \\ \rho h\eta_{2tt} - L_5(\psi,\eta_1,\eta_2) + \eta_{2t} = 0 & \text{in} \quad Q, \end{cases} \tag{1.1}$$

under boundary conditions (1.2) and initial data (1.3). The energy of solutions for (1.1), (1.2), (1.3) decreases in time. Indeed, the energy given by (1.12) obeys the energy dissipation law

$$\frac{d}{dt}E_k(t) = -\left(|\phi_{1t}(t)|^2 + |\phi_{2t}(t)|^2 + |\psi_t(t)|^2 + |\eta_{1t}(t)|^2 + |\eta_{2t}(t)|^2\right). \tag{1.2}$$

The aim of this section is to obtain exponential decay for the energy (1.16) associated to the solution of the von Kármán system

$$\begin{cases} \rho h\psi_{tt} - \dfrac{\rho h^3}{12}\Delta\psi_{tt} + D\Delta^2\psi - [N_1\psi_x + N_{12}\psi_y]_x - [N_2\psi_y + N_{12}\psi_x]_y + \psi_t - \Delta\psi_t = 0 & \text{in} \quad Q, \\ \rho h\eta_{1tt} - [N_{1x} + N_{12y}] + \eta_{1t} = 0 & \text{in} \quad Q, \\ \rho h\eta_{2tt} - [N_{2y} + N_{12x}] + \eta_{2t} = 0 & \text{in} \quad Q, \end{cases} \tag{1.3}$$

with boundary conditions (1.6) and initial data (1.7), as a limit (as $k \to \infty$) of the uniform stabilization of the dissipative Mindlin-Timoshenko system (1.1), (1.2), (1.3).

Analogously to the proof of the Theorem 1.2.1, considering the initial data $\{\phi_{10},\phi_{11},\phi_{20},\phi_{21},\psi_0,\psi_1,\eta_{10},\eta_{11},\eta_{20},\eta_{21}\} \in \mathcal{X}$ satisfying (1.14), system (1.3) can be obtained as a limit, as $k \to \infty$, of system (1.1), (1.2), (1.3).

Since the energy $E_k(t)$ in (1.12) is a non increasing function, we will show that this energy decays exponentially (as $t \to \infty$) uniformly with respect to k. More precisely, the following result holds:

Theorem 1.3.1 *Let* $\{\phi_1, \phi_2, \psi, \eta_1, \eta_2\}$ *be the solution of system* (1.1), (1.2), (1.3) *for data* $\{\phi_{10}, \phi_{11}, \phi_{20}, \phi_{21}, \psi_0, \psi_1, \eta_{10}, \eta_{11}, \eta_{20}, \eta_{21}\} \in \mathcal{X}$. *There exists a constant* $\omega > 0$ *such that*

$$E_k(t) \leq 4E_k(0)\, e^{-\frac{\omega}{2}t}, \quad \forall\, t \geq 0. \tag{1.4}$$

Remark 1.3.2 *As a consequence of inequality (3.4), if the initial data satisfy* (1.14), *letting* $k \to \infty$ *one recovers the exponential decay of the energy* $E(t)$ *associated to system* (1.3) *which is given by* (1.16). *This is in agreement with the results from [43] in the sense that the same decay rate for the solutions of the von Kármám system was obtained.*

Proof of Theorem 1.3.1. For each $k \geq 1$ fixed, let $\{\phi_1^k, \phi_2^k, \psi^k, \eta_1^k, \eta_2^k\}$ be the solution of system (1.1), (1.2), (1.3) with data $\{\phi_{10}, \phi_{20}, \psi_0, \eta_{10}, \eta_{20}\} \in \mathcal{X}$. From now on in this proof, we will omit the index k of the solution to simplify the notation. For an arbitrary $\lambda > 0$, define the perturbed energy

$$G_\lambda(t) := E_k(t) + \lambda F(t), \tag{1.5}$$

where F is the functional

$$F(t) = \theta\left(\frac{\rho h^3}{12}\phi_{1t}, \phi_1\right) + \theta\left(\frac{\rho h^3}{12}\phi_{2t}, \phi_2\right) + \theta(\rho h\psi_t, \psi) + 2\theta(\rho h\eta_{1t}, \eta_1) + 2\theta(\rho h\eta_{2t}, \eta_2), \tag{1.6}$$

where $\theta > 0$ is a constant to be chosen later on. Let us bound each term on the right-hand side of identity (1.6) by an expression involving the energy (1.12).

- Analysis of $\theta\left(\frac{\rho h^3}{12}\phi_{1t}, \phi_1\right) + \theta\left(\frac{\rho h^3}{12}\phi_{2t}, \phi_2\right)$.

  Using Poincaré inequality, one obtains

$$\begin{aligned}
&\theta\left(\frac{\rho h^3}{12}\phi_{1t}, \phi_1\right) + \theta\left(\frac{\rho h^3}{12}\phi_{2t}, \phi_2\right) \\
&\leq C\theta\left(\frac{\rho h^3}{12}|\phi_{1t}|^2 + \frac{\rho h^3}{12}|\phi_{2t}|^2 + |\phi_{1x}|^2 + |\phi_{2y}|^2 + |\phi_{1y} + \phi_{2x}|^2 - 2\int_\Omega \phi_{1x}\phi_{2y}\right) \\
&\leq C\theta E_k(t).
\end{aligned} \tag{1.7}$$

- Analysis of $\theta(\rho h\psi_t(t), \psi(t))$.

  Using Poincaré inequality again, one gets

$$\begin{aligned}
\theta(\rho h\psi_t, \psi) &\leq C\theta\left(\rho h|\psi_t|^2 + |\psi_x|^2 + |\psi_y|^2\right) \\
&\leq C\theta\left(\rho h|\psi_t|^2 + |\phi_1 + \psi_x|^2 + |\phi_2 + \psi_y|^2 + |\phi_1|^2 + |\phi_2|^2\right) \\
&\leq C\theta\left(\rho h|\psi_t|^2 + |\phi_1 + \psi_x|^2 + |\phi_2 + \psi_y|^2 + |\phi_{1x}|^2 + |\phi_{1y}|^2 + |\phi_{2x}|^2 + |\phi_{2y}|^2\right) \\
&\leq C\theta E_k(t).
\end{aligned} \tag{1.8}$$

- Analysis of $2\theta\left(\rho h\eta_{1t},\eta_1\right)+2\theta\left(\rho h\eta_{2t},\eta_2\right)$.

  One has

$$
\begin{aligned}
&2\theta\left(\rho h\eta_{1t},\eta_1\right)+2\theta\left(\rho h\eta_{2t},\eta_2\right)\\
&\leq\ C\theta\left(\rho h|\eta_{1t}|^2+|\eta_{1x}|^2+|\eta_{1y}|^2+\rho h|\eta_{2t}|^2+|\eta_{2x}|^2+|\eta_{2y}|^2\right)\\
&\leq\ C\theta\Bigg(\rho h|\eta_{1t}|^2+\rho h|\eta_{2t}|^2+\left|\eta_{1x}+\frac{1}{2}\psi_x^2\right|^2+\left|\eta_{2y}+\frac{1}{2}\psi_y^2\right|^2\\
&\quad+|\eta_{1y}|^2+|\eta_{2x}|^2+\frac{1}{2}\left|\psi_x^2\right|^2+\frac{1}{2}\left|\psi_y^2\right|^2\Bigg) \qquad (1.9)\\
&\leq\ C\theta\Bigg(\rho h|\eta_{1t}|^2+\rho h|\eta_{2t}|^2+\left|\eta_{1x}+\frac{1}{2}\psi_x^2\right|^2+\left|\eta_{2y}+\frac{1}{2}\psi_y^2\right|^2\\
&\quad+|\eta_{1y}+\eta_{2x}|^2-2\int_\Omega\eta_{1y}\eta_{2x}+|\nabla\psi|^2\Bigg)\\
&\leq\ C\theta E_k(t).
\end{aligned}
$$

According to the bounds (1.7)–(1.9), we conclude that

$$
|F\left(t\right)|\leq CE_k\left(t\right). \qquad (1.10)
$$

Now, using (1.5) and (1.10), one obtains

$$
|G_\lambda\left(t\right)-E_k\left(t\right)|\leq\lambda\left|F\left(t\right)\right|\leq\lambda CE_k\left(t\right),
$$

which is equivalent to

$$
\left(1-\lambda C\right)E_k\left(t\right)\leq G_\lambda\left(t\right)\leq\left(1+\lambda C\right)E_k\left(t\right).
$$

Taking $0<\lambda\leq 1/2C$, one gets

$$
\frac{E_k\left(t\right)}{2}\leq G_\lambda\left(t\right)\leq 2E_k\left(t\right). \qquad (1.11)
$$

Differentiating the functional F and using the equations in (1.1), one obtains

$$
\begin{aligned}
\frac{d}{dt}F(t)\ =\ &-\theta D\left|\phi_{1x}\right|^2-\frac{1-\mu}{2}\theta D\left|\phi_{1y}\right|^2-\theta D\frac{1+\mu}{2}\int_\Omega\phi_{2x}\phi_{1y}-\theta k|\phi_1|^2-\theta k\int_\Omega\psi_x\phi_1\\
&-\theta\int_\Omega\phi_{1t}\phi_1+\theta\frac{\rho h^3}{12}|\phi_{1t}|^2-\theta D\left|\phi_{2y}\right|^2-\theta D\frac{1-\mu}{2}\left|\phi_{2x}\right|^2-\theta D\frac{1+\mu}{2}\int_\Omega\phi_{1y}\phi_{2x}\\
&-\theta k|\phi_2|^2-\theta k\int_\Omega\psi_y\phi_2+\theta\frac{\rho h^3}{12}|\phi_{2t}|^2-\theta\int_\Omega\phi_{2t}\phi_2-\theta k\left|\psi_x\right|^2-\theta k\int_\Omega\phi_1\psi_x \qquad (1.12)\\
&-\theta k\left|\psi_y\right|^2-\theta k\int_\Omega\phi_2\psi_y-\theta\int_\Omega\left[N_1\psi_x+N_{12}\psi_y\right]\psi_x-\theta\int_\Omega\left[N_2\psi_y+N_{12}\psi_x\right]\psi_y\\
&+\theta\rho h|\psi_t|^2-\theta\int_\Omega\psi_t\psi-2\theta\int_\Omega N_1\eta_{1x}-2\theta\int_\Omega N_{12}\eta_{1y}+2\theta\rho h|\eta_{1t}|^2\\
&-2\theta\int_\Omega N_2\eta_{2y}-2\theta\int_\Omega N_{12}\eta_{2x}+2\theta\rho h|\eta_{2t}|^2-2\theta\int_\Omega\eta_{1t}\eta_1-2\theta\int_\Omega\eta_{2t}\eta_2.
\end{aligned}
$$

We bound each term on the right-hand side of identity (1.12) separately.

- Analysis of $-\theta(\phi_{1t}, \phi_1) - \theta(\phi_{2t}, \phi_2)$.

$$
\begin{aligned}
&-\theta(\phi_{1t}, \phi_1) - \theta(\phi_{2t}, \phi_2) \\
&\leq \frac{\theta^2}{2\xi}|\phi_{1t}|^2 + \frac{\xi}{2}|\phi_1|^2 + \frac{\theta^2}{2\xi}|\phi_{2t}|^2 + \frac{\xi}{2}|\phi_2|^2 \\
&\leq \frac{\theta^2}{2\xi}\left[|\phi_{1t}|^2 + |\phi_{2t}|^2\right] + \frac{\xi C}{2}\left(|\phi_{1x}|^2 + |\phi_{1y}|^2 + |\phi_{2x}|^2 + |\phi_{2y}|^2\right) \qquad (1.13) \\
&= \frac{\theta^2}{2\xi}\left[|\phi_{1t}|^2 + |\phi_{2t}|^2\right] + \frac{\xi C}{2}\left(|\phi_{1x}|^2 + |\phi_{2y}|^2 + |\phi_{1y} + \phi_{2x}|^2 - 2\int_\Omega \phi_{1y}\phi_{2x}\right) \\
&\leq \frac{\theta^2}{2\xi}\left[|\phi_{1t}|^2 + |\phi_{2t}|^2\right] + \xi C E_k(t),
\end{aligned}
$$

where $\xi > 0$ is a real number to be appropriately chosen.

- Analysis of $-\theta\left(\psi_t\left(t\right), \psi\left(t\right)\right)$.

$$
\begin{aligned}
-\theta(\psi_t, \psi) &\leq \frac{\theta^2}{2\xi}|\psi_t|^2 + \frac{\xi}{2}|\psi|^2 \\
&\leq \frac{\theta^2}{2\xi}|\psi_t|^2 + \frac{\xi C}{2}\left(|\psi_x|^2 + |\psi_y|^2\right) \\
&\leq \frac{\theta^2}{2\xi}|\psi_t|^2 + \frac{\xi C}{2}\left(|\phi_1 + \psi_x|^2 + |\phi_2 + \psi_y|^2 + |\phi_1|^2 + |\phi_2|^2\right) \qquad (1.14) \\
&\leq \frac{\theta^2}{2\xi}|\psi_t|^2 + \frac{\xi C}{2}\left(|\phi_1 + \psi_x|^2 + |\phi_2 + \psi_y|^2 + |\phi_{1x}|^2 + |\phi_{1y}|^2 + |\phi_{2x}|^2 + |\phi_{2y}|^2\right) \\
&\leq \frac{\theta^2}{2\xi}|\psi_t|^2 + \xi C E_k(t).
\end{aligned}
$$

- Analysis of $-2\theta(\eta_{1t}, \eta_1) - 2\theta(\eta_{1t}, \eta_1)$.

$$
\begin{aligned}
&-2\theta(\eta_{1t},\eta_1) - 2\theta(\eta_{1t},\eta_1) \\
&\quad\le \frac{\theta^2}{2\xi}|\eta_{1t}|^2 + \frac{\xi}{2}|\eta_1|^2 + \frac{\theta^2}{2\xi}|\eta_{2t}|^2 + \frac{\xi}{2}|\eta_2|^2 \\
&\quad\le \frac{\theta^2}{2\xi}\left[|\eta_{1t}|^2 + |\eta_{2t}|^2\right] + \frac{\xi C}{2}\left[|\eta_{1x}|^2 + |\eta_{1y}|^2 + |\eta_{2x}|^2 + |\eta_{2y}|^2\right] \qquad (1.15) \\
&\quad\le \frac{\theta^2}{2\xi}\left[|\eta_{1t}|^2 + |\eta_{2t}|^2\right] + \frac{\xi C}{2}\left[\left|\eta_{1x} + \frac{1}{2}\psi_x^2\right|^2 + \left|\eta_{2y} + \frac{1}{2}\psi_y^2\right|^2 + |\eta_{1y}+\eta_{2x}|^2\right. \\
&\qquad \left. -2\int_\Omega \eta_{1y}\eta_{2x} + \frac{1}{2}\psi_x^2 + \frac{1}{2}\psi_y^2\right] \\
&\quad\le \frac{\theta^2}{2\xi}\left[|\eta_{1t}|^2 + |\eta_{2t}|^2\right] + \frac{\xi C}{2}\left[\left|\eta_{1x} + \frac{1}{2}\psi_x^2\right|^2 + \left|\eta_{2y} + \frac{1}{2}\psi_y^2\right|^2 + |\eta_{1y}+\eta_{2x}|^2\right. \\
&\qquad \left. -2\int_\Omega \eta_{1y}\eta_{2x} + |\nabla\psi|^2\right] \\
&\quad\le \frac{\theta^2}{2\xi}\left[|\eta_{1t}|^2 + |\eta_{2t}|^2\right] + \xi C E_k(t).
\end{aligned}
$$

Using bounds (1.13)–(1.15), one obtains, from (1.12),

$$
\begin{aligned}
\frac{d}{dt}F(t) \le\; & -\theta D\,|\phi_{1x}|^2 - \theta D\,|\phi_{2y}|^2 - \theta k\,|\phi_1+\psi_x|^2 - \theta k\,|\phi_2+\psi_y|^2 - \theta D\left(\frac{1-\mu}{2}\right)|\phi_{1y}+\phi_{2x}|^2 \\
& - 2\theta D\mu\int_\Omega \phi_{1y}\phi_{2x} - 2\theta\frac{Eh}{1-\mu^2}\left(\frac{1-\mu}{2}\right)|\eta_{1y}+\eta_{2x}+\psi_x\psi_y|^2 - 2\theta\left|\eta_{1x}+\frac{1}{2}\psi_x^2\right|^2 \\
& - 2\theta\left|\eta_{2y}+\frac{1}{2}\psi_y^2\right|^2 - 2\mu\theta\left|\eta_{2y}+\frac{1}{2}\psi_y^2\right|^2 - 2\mu\theta\left|\eta_{1x}+\frac{1}{2}\psi_x^2\right|^2 \\
& + 3\xi C E_k(t) + \theta\frac{\rho h^3}{12}\left[|\phi_{1t}|^2+|\phi_{2t}|^2\right] + \theta\rho h\left[|\psi_t|^2 + 2|\eta_{1t}|^2 + 2|\eta_{2t}|^2\right] + \frac{\theta^2}{2\xi}\left[|\phi_{1t}|^2+|\phi_{2t}|^2\right] \\
& + \frac{\theta^2}{2\xi}|\psi_t|^2 + \frac{\theta^2}{2\xi}\left[|\eta_{1t}|^2+|\eta_{2t}|^2\right] \\
\le\; & -(\theta-3\xi C)E_k(t) + \left(\theta\frac{\rho h^3}{12}+\frac{\theta^2}{2\xi}\right)\left[|\phi_{1t}|^2+|\phi_{2t}|^2\right] + \left(\theta\rho h + \frac{\theta^2}{2\xi}\right)|\psi_t|^2 \qquad (1.16) \\
& + \left(2\theta\rho h + \frac{\theta^2}{2\xi}\right)\left[|\eta_{1t}|^2+|\eta_{2t}|^2\right].
\end{aligned}
$$

Therefore,

$$
\frac{d}{dt}F\,(t) \le -(\theta-3\xi C)E_k(t) + C\left[|\phi_{1t}|^2+|\phi_{2t}|^2+|\psi_t|^2+|\eta_{1t}|^2+|\eta_{2t}|^2\right]. \qquad (1.17)
$$

Considering the derivative of the expression (1.5), and observing (1.2) and (1.17), one has

$$
\frac{d}{dt}G_\lambda(t) \le -\lambda(\theta-3\xi C)E_k(t) - (1-\lambda C)\left[|\phi_{1t}|^2+|\phi_{2t}|^2+|\psi_t|^2+|\eta_{1t}|^2+|\eta_{2t}|^2\right].
$$

Choosing $\lambda \leq 1/2C$ and $\xi < \theta/3$, one obtains, according to (1.11),

$$\frac{d}{dt}G_\lambda(t) \leq -\lambda(\theta - 3\xi C)E_k(t) \leq -\frac{w}{2}G_\lambda(t), \ \forall t \geq 0,$$

where $\omega = \lambda\left(\theta - 3\xi C\right)$. Therefore,

$$G_\lambda(t) \leq G_\lambda(0)e^{-\frac{w}{2}t}. \tag{1.18}$$

Combining (1.11) and (1.18), one gets (1.4). This finishes the proof. ■

1.4 Stability: Boundary feedback

In this section we analyze the plate model in the case where the energy of the Mindlin-Timoshenko system is dissipated through boundary feedback mechanisms. Let us assume that $\Gamma_i \neq \emptyset$ $(i = 0, 1)$, and we consider the system (1.1) with boundary conditions

$$\begin{cases} \phi_1 = \phi_2 = \psi = \eta_1 = \eta_2 = 0 & \text{on} \quad \Sigma_0, \\ \{\mathcal{B}_1(\phi_1,\phi_2), \mathcal{B}_2(\phi_1,\phi_2), \mathcal{B}_3(\phi_1,\phi_2,\psi,\eta_1,\eta_2), \mathcal{B}_4(\eta_1,\eta_2), \mathcal{B}_5(\eta_1,\eta_2)\} = -\{\phi_{1t},\phi_{2t},\psi_t,\eta_{1t},\eta_{2t}\} & \text{on} \quad \Sigma_1, \end{cases} \tag{1.1}$$

and initial data (1.3). The energy of this system obeys the following dissipation law:

$$\frac{d}{dt}E_k(t) = -\int_{\Gamma_1}\left[\left(\phi_{1t}^k\right)^2 + \left(\phi_{2t}^k\right)^2 + \left(\psi_t^k\right)^2 + \left(\eta_{1t}^k\right)^2 + \left(\eta_{2t}^k\right)^2\right]d\Gamma.$$

Consequently,

$$E_k(t) \leq E_k(0), \quad \forall\, t \geq 0.$$

We are interested in studying the asymptotic behavior of $E(t)$, as $t \to \infty$.

The variational formulation of (1.1), (1.1), (1.3) is given by

$$\begin{aligned} &\tfrac{\rho h^3}{12}\tfrac{d}{dt}(\phi_{1t}^k, a) + \tfrac{\rho h^3}{12}\tfrac{d}{dt}(\phi_{2t}^k, b) + \rho h\tfrac{d}{dt}(\psi_t^k, c) + \rho h\tfrac{d}{dt}(\eta_{1t}^k, d) + \rho h\tfrac{d}{dt}(\eta_{2t}^k, e) + k\left[\left(\phi_1^k + \psi_x^k, a + c_x\right)\right. \\ &+ \left.\left(\phi_2^k + \psi_y^k, b + c_y\right)\right] + D\left[\left(\phi_{1x}^k, a_x\right) + \tfrac{1-\mu}{2}\left(\phi_{1y}^k, a_y\right) + \tfrac{1+\mu}{2}\left(\phi_{2x}^k, a_y\right) + \left(\phi_{2y}^k, b_y\right) + \tfrac{1-\mu}{2}\left(\phi_{2x}^k, b_x\right)\right. \\ &+ \left.\tfrac{1+\mu}{2}\left(\phi_{1y}^k, b_x\right)\right] + \left(N_1^k\psi_x^k + N_{12}^k\psi_y^k, c_x\right) - \left(N_1^k, d_x\right) + \left(N_{12}^k, d_y\right) + \left(N_2^k\psi_y^k + N_{12}^k\psi_x^k, c_y\right) \\ &- \left(N_2^k, e_y\right) + \left(N_{12}^k, e_x\right) + \textstyle\int_{\Gamma_1}\left[\phi_{1t}^k a + \phi_{2t}^k b + \psi_t^k c + \eta_{1t}^k d + \eta_{2t}^k e\right] = 0, \end{aligned} \tag{1.2}$$

for all $\{a, b, c, d, e\} \in \left[H^1_{\Gamma_0}(\Omega)\right]^5$.

Remark 1.4.1 *Using arguments similar to those in Section 1.2, considering initial data in a suitable class and satisfying* (1.14)*, we can prove that the system* (1.1), (1.1), (1.3) *converges (as* $k \to \infty$*) toward the dissipative von Kármán system* (1.5) *with boundary conditions*

$$\begin{cases} \psi = \frac{\partial \psi}{\partial \nu} = \eta_1 = \eta_2 = 0 & on \quad \Gamma_0, \\ D\left[\Delta\psi + (1-\mu)\left(2\nu_1\nu_2\psi_{xy} - \nu_1^2\psi_{yy} - \nu_2^2\psi_{xx}\right)\right] = -\left(\nu_1\psi_{xt} + \nu_2\psi_{yt}\right) & on \quad \Gamma_1, \\ D\left[\frac{\partial(\Delta\psi)}{\partial\nu} + (1-\mu)\frac{\partial}{\partial\tau}\left[\left(\nu_1^2 - \nu_2^2\right)\psi_{xy} + \nu_1\nu_2\left(\psi_{yy} - \psi_{xx}\right)\right]\right] - \frac{\rho h^3}{12}\frac{\partial \psi_{tt}}{\partial \nu} & \\ \qquad -(\nu_1 N_1 + \nu_2 N_{12})\psi_x - (\nu_2 N_2 + \nu_1 N_{12})\psi_y = \frac{\partial}{\partial\tau}\left(-\nu_1\psi_{yt} + \nu_2\psi_{xt}\right) - \psi_t & on \quad \Gamma_1, \\ \nu_1 N_1 + \nu_2 N_{12} = -\eta_{1t} & on \quad \Gamma_1, \\ \nu_2 N_2 + \nu_1 N_{12} = -\eta_{2t} & on \quad \Gamma_1, \end{cases} \tag{1.3}$$

and initial data (1.7).

In order to establish the uniform asymptotic stability of system (1.1), (1.1), (1.3), some restrictions are needed on the geometry of Ω, Γ_0 and Γ_1. Let us introduce a vector field $m = m(x, y)$ in $\mathbb{R}^2$ defined by

$$m(x, y) = (x, y) - (x_0, y_0),$$

where (x_0, y_0) is a fixed point of $\mathbb{R}^2$. We assume that Γ_0 and Γ_1 are such that there exists $(x_0, y_0) \in \mathbb{R}^2$ such that

$$m \cdot \nu \leq 0 \quad \text{on} \quad \Gamma_0, \qquad m \cdot \nu \geq 0 \quad \text{on} \quad \Gamma_1. \tag{1.4}$$

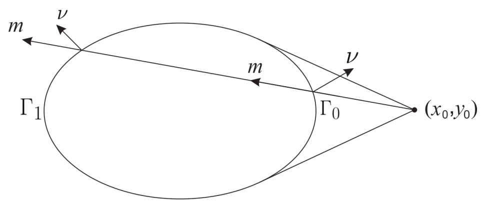

Figura 1.1: Example for which condition (1.4) is satisfied.

Let us consider $G = [g_{ij}]$ the 5×5 matrix such that

$$g_{ij} = 0, \ i \neq j, \quad \text{and} \quad (m \cdot \nu) g_{ii} = 1, \ i = 1, \ldots, 5.$$

Note that $g_{ij} \in C^1(\overline{\Gamma}_1)$. Moreover, there are positive constants g_0 and G_0 such that

$$g_0 |\varsigma|^2 \leq G\varsigma \cdot \varsigma \leq G_0 |\varsigma|^2, \ \forall\, \varsigma \in \mathbb{R}^5, \ \text{ on } \ \Gamma_1. \tag{1.5}$$

Before establishing the main result of this section, we will state and prove the following two lemmas.

Lemma 1.4.2 *Let $\{\phi_1, \phi_2, \psi, \eta_1, \eta_2\}$ and $\{\phi_{1t}, \phi_{2t}, \psi_t, \eta_{1t}, \eta_{2t}\}$ be regular enough. Then*

$$\begin{aligned}
&\int_\Omega [\phi_{1t} L_1(\phi_1, \phi_2, \psi) + \phi_{2t} L_2(\phi_1, \phi_2, \psi) + \psi_t L_3(\phi_1, \phi_2, \psi, \eta_1, \eta_2) + \eta_{1t} L_4(\psi, \eta_1, \eta_2) \\
&\qquad + \eta_{2t} L_5(\psi, \eta_1, \eta_2)]\, dxdy + a(\phi_1, \phi_2, \psi, \eta_1, \eta_2, \phi_{1t}, \phi_{2t}, \psi_t, \eta_{1t}, \eta_{2t}) \qquad (1.6) \\
&= \int_\Gamma [\phi_{1t} \mathcal{B}_1(\phi_1, \phi_2) + \phi_{2t} \mathcal{B}_2(\phi_1, \phi_2) + \psi_t \mathcal{B}_3(\phi_1, \phi_2, \psi, \eta_1, \eta_2) + \eta_{1t} \mathcal{B}_4(\psi, \eta_1, \eta_2) + \eta_{2t} \mathcal{B}_5(\psi, \eta_1, \eta_2)]\, d\Gamma,
\end{aligned}$$

with

$$\begin{aligned}
&a(\phi_1, \phi_2, \psi, \eta_1, \eta_2, \phi_{1t}, \phi_{2t}, \psi_t, \eta_{1t}, \eta_{2t}) \\
&\quad := a_0(\phi_1, \phi_2, \phi_{1t}, \phi_{2t}) + k a_1(\phi_1, \phi_2, \psi, \phi_{1t}, \phi_{2t}, \psi_t) + a_2(\psi, \eta_1, \eta_2, \psi_t, \eta_{1t}, \eta_{2t}),
\end{aligned}$$

where

$$\begin{aligned}
&a_0(\phi_1, \phi_2, \phi_{1t}, \phi_{2t}) \\
&\quad = D \int_\Omega \left[\phi_{1x}\phi_{1tx} + \phi_{2y}\phi_{2ty} + \mu\phi_{1x}\phi_{2ty} + \mu\phi_{1tx}\phi_{2y} + \tfrac{1-\mu}{2}(\phi_{1y} + \phi_{2x})(\phi_{1ty} + \phi_{2tx})\right] dxdy,
\end{aligned}$$

$$a_1(\phi_1, \phi_2, \psi, \phi_{1t}, \phi_{2t}, \psi_t) = \int_\Omega [(\phi_1 + \psi_x)(\phi_{1t} + \psi_{tx}) + (\phi_2 + \psi_y)(\phi_{2t} + \psi_{ty})]\, dxdy,$$

and

$$\begin{aligned}
&a_2(\psi, \eta_1, \eta_2, \psi_t, \eta_{1t}, \eta_{2t}) \\
&= \frac{Eh}{1-\mu^2} \int_\Omega \left[(1-\mu)\left(\eta_{1x} + \frac{1}{2}\psi_x^2\right)(\eta_{1tx} + \psi_x\psi_{tx}) + (1-\mu)\left(\eta_{2y} + \frac{1}{2}\psi_y^2\right)(\eta_{2ty} + \psi_y\psi_{ty})\right. \\
&\quad + \mu\left(\eta_{1x} + \eta_{2y} + \frac{1}{2}|\nabla\psi|^2\right)\Big(\eta_{1tx} + \eta_{2ty} + \nabla\psi\cdot\nabla\psi_t\Big) \\
&\quad \left. + \frac{1-\mu}{2}(\eta_{1y} + \eta_{2x} + \psi_x\psi_y)\Big(\eta_{1ty} + \eta_{2tx} + \psi_x\psi_{ty} + \psi_y\psi_{tx}\Big)\right] dxdy.
\end{aligned}$$

Remark 1.4.3 *Here and elsewhere in this section we use the term "regular enough"to ensure that all integrals are well defined (see Section 1.5 for additional comments on this point).*

Proof of Lemma 1.4.2. By definition of the operators $L_i(\phi_1, \phi_2, \psi, \eta_1, \eta_2)$ $(i = 1, \ldots, 5)$, one has

$$\begin{aligned}
&\int_\Omega [\phi_{1t} L_1(\phi_1, \phi_2, \psi) + \phi_{2t} L_2(\phi_1, \phi_2, \psi) + \psi_t L_3(\phi_1, \phi_2, \psi, \eta_1, \eta_2) + \eta_{1t} L_4(\psi, \eta_1, \eta_2) + \eta_{2t} L_5(\psi, \eta_1, \eta_2)] \\
&\quad = \int_\Omega \left\{\phi_{1t}\left[D\left(\phi_{1xx} + \frac{1-\mu}{2}\phi_{1yy} + \frac{1+\mu}{2}\phi_{2xy}\right) - k(\phi_1 + \psi_x)\right] + \phi_{2t}\left[D\left(\phi_{2yy} + \frac{1-\mu}{2}\phi_{2xx} + \frac{1+\mu}{2}\phi_{1xy}\right)\right.\right. \\
&\quad \left. - k(\phi_2 + \psi_y)\right] + \psi_t\left\{k\left[(\psi_x + \phi_1)_x + (\psi_y + \phi_2)_y\right] + (N_1\psi_x + N_{12}\psi_y)_x + (N_2\psi_y + N_{12}\psi_x)_y\right\} \\
&\quad \left. + \eta_{1t}[N_{1x} + N_{12y}] + \eta_{2t}[N_{2y} + N_{12x}]\right\}.
\end{aligned}$$

Through integration by parts one obtains

$$\int_\Omega \left[\phi_{1t} L_1(\phi_1,\phi_2,\psi) + \phi_{2t} L_2(\phi_1,\phi_2,\psi) + \psi_t L_3(\phi_1,\phi_2,\psi,\eta_1,\eta_2) + \eta_{1t} L_4(\psi,\eta_1,\eta_2) + \eta_{2t} L_5(\psi,\eta_1,\eta_2)\right]$$

$$= -a_0(\phi_1,\phi_2,\phi_{1t},\phi_{2t}) - k a_1(\phi_1,\phi_2,\psi,\phi_{1t},\phi_{2t},\psi_t) - \int_\Omega \left[(N_1\psi_x + N_{12}\psi_y)\psi_{tx} + (N_1\psi_y + N_{12}\psi_x)\psi_{ty}\right.$$

$$\left. + \eta_{1tx}N_1 + N_{1ty}N_{12} + \eta_{2ty}N_2 + \eta_{2tx}N_{12}\right] + \int_\Gamma \left\{ \phi_{1t} D\left[\phi_{1x}\nu_1 + \tfrac{1-\mu}{2}\phi_{1y}\nu_2 + \tfrac{1+\mu}{2}\phi_{2x}\nu_2\right]\right.$$

$$+ \phi_{2t} D\left[\phi_{2y}\nu_2 + \tfrac{1-\mu}{2}\phi_{2x}\nu_1 + \tfrac{1+\mu}{2}\phi_{1y}\nu_1\right] + \psi_t k\left[(\phi_1+\psi_x)\nu_1 + (\phi_2+\psi_y)\nu_2\right] + (N_1\psi_x + N_{12}\psi_y)\nu_1$$

$$\left. + (N_2\psi_y + N_{12}\psi_x)\nu_2 + \eta_{1t}(N_1\nu_1 + N_{12}\nu_2) + \eta_{2t}(N_2\nu_2 + N_{12}\nu_1)\right\}.$$

Finally, using the definition of N_1, N_2 and N_{12}, one has

$$\int_\Omega \left[\phi_{1t} L_1(\phi_1,\phi_2,\psi) + \phi_{2t} L_2(\phi_1,\phi_2,\psi) + \psi_t L_3(\phi_1,\phi_2,\psi,\eta_1,\eta_2) + \eta_{1t} L_4(\psi,\eta_1,\eta_2) + \eta_{2t} L_5(\psi,\eta_1,\eta_2)\right]$$

$$= -a_0(\phi_1,\phi_2,\phi_{1t},\phi_{2t}) - k a_1(\phi_1,\phi_2,\psi,\phi_{1t},\phi_{2t},\psi_t) - a_2(\psi,\eta_1,\eta_2,\psi_t,\eta_{1t},\eta_{2t}) + \int_\Gamma \left\{ \phi_{1t}\mathcal{B}_1(\phi_1,\phi_2)\right.$$

$$\left. + \phi_{2t}\mathcal{B}_2(\phi_1,\phi_2) + \psi_t\mathcal{B}_3(\phi_1,\phi_2,\psi,\eta_1,\eta_2) + \eta_{1t}\mathcal{B}_4(\psi,\eta_1,\eta_2) + \eta_{2t}\mathcal{B}_5(\psi,\eta_1,\eta_2)\right\},$$

which completes the proof. ■

Lemma 1.4.4 *Consider* $\{\phi_1,\phi_2,\psi,\eta_1,\eta_2\}$ *to be regular enough. Then*

$$\begin{aligned}
&\int_\Omega \left[(m\cdot\nabla\phi_1)L_1(\phi_1,\phi_2,\psi) + (m\cdot\nabla\phi_2)L_2(\phi_1,\phi_2,\psi) + (m\cdot\nabla\psi)L_3(\phi_1,\phi_2,\psi) + (m\cdot\nabla\eta_1)L_4(\psi,\eta_1,\eta_2\right. \\
&\left. + (m\cdot\nabla\eta_2)L_5(\psi,\eta_1,\eta_2)\right] dxdt = k\int_\Omega \left[(\phi_1+\psi_x)\,\phi_1 + (\phi_2+\psi_y)\,\phi_2\right] dxdy - \tfrac{1}{2}\int_\Gamma m\cdot\nu\left\{D\left[(\phi_{1x})^2\right.\right. \\
&\left. + (\phi_{2y})^2 + 2\mu\phi_{1x}\phi_{2y} + \tfrac{1-\mu}{2}\left(\phi_{1y}+\phi_{2x}\right)^2\right] + k\left[(\phi_1+\psi_x)^2 + (\phi_2+\psi_y)^2\right] \\
&+ \tfrac{Eh}{1-\mu^2}\left[(1-\mu)\left(\eta_{1x}+\tfrac{1}{2}\psi_x^2\right)^2 + (1-\mu)\left(\eta_{2y}+\tfrac{1}{2}\psi_y^2\right)^2 + \mu\left(\eta_{1x}+\eta_{2y}+\tfrac{1}{2}|\nabla\psi|^2\right)^2\right. \\
&\left.\left. + \tfrac{1-\mu}{2}\left(\eta_{1y}+\eta_{2x}+\psi_x\psi_y\right)^2\right]\right\} d\Gamma + \int_\Gamma \left[(m\cdot\nabla\phi_1)\mathcal{B}_1(\phi_1,\phi_2) + (m\cdot\nabla\phi_2)\mathcal{B}_2(\phi_1,\phi_2)\right. \\
&\left. + (m\cdot\nabla\psi)\mathcal{B}_3(\phi_1,\phi_2,\psi,\eta_1,\eta_2) + (m\cdot\nabla\eta_1)\mathcal{B}_4(\psi,\eta_1,\eta_2) + (m\cdot\nabla\eta_2)\mathcal{B}_5(\psi,\eta_1,\eta_2)\right] d\Gamma.
\end{aligned} \tag{1.7}$$

Proof. Analogously to the proof of Lemma 1.4.2,

$$\begin{aligned}
&\int_\Omega \left[(m\cdot\nabla\phi_1)L_1(\phi_1,\phi_2,\psi) + (m\cdot\nabla\phi_2)L_2(\phi_1,\phi_2,\psi) + (m\cdot\nabla\psi)L_3(\phi_1,\phi_2,\psi) + (m\cdot\nabla\eta_1)L_4(\psi,\eta_1,\eta_2\right. \\
&\left. + (m\cdot\nabla\eta_2)L_5(\psi,\eta_1,\eta_2)\right] dxdt = -a\left(\phi_1,\phi_2,\psi,\eta_1,\eta_2,m\cdot\nabla\phi_1,m\cdot\nabla\phi_2,m\cdot\nabla\psi,m\cdot\nabla\eta_1,m\cdot\nabla\eta_2\right) \\
&+ \int_\Gamma \left[(m\cdot\nabla\phi_1)\mathcal{B}_1(\phi_1,\phi_2) + (m\cdot\nabla\phi_2)\mathcal{B}_2(\phi_1,\phi_2) + (m\cdot\nabla\psi)\mathcal{B}_3(\phi_1,\phi_2,\psi,\eta_1,\eta_2) + (m\cdot\nabla\eta_1)\mathcal{B}_4(\psi,\eta_1,\right. \\
&\left. + (m\cdot\nabla\eta_2)\mathcal{B}_5(\psi,\eta_1,\eta_2)\right] d\Gamma.
\end{aligned} \tag{1.8}$$

In this way, to prove (1.7) we have only to study the term

$$a\left(\phi_1,\phi_2,\psi,\eta_1,\eta_2,m\cdot\nabla\phi_1,m\cdot\nabla\phi_2,m\cdot\nabla\psi,m\cdot\nabla\eta_1,m\cdot\nabla\eta_2\right). \tag{1.9}$$

Note that

$$
\begin{aligned}
&a_0(\phi_1,\phi_2,m\cdot\nabla\phi_1,m\cdot\nabla\phi_2)\\
&= D\int_\Omega [\phi_{1x}(m\cdot\nabla\phi_1)_x+\phi_{2y}(m\cdot\nabla\phi_2)_y+\mu\phi_{1x}(m\cdot\nabla\phi_2)_y+\mu\phi_{2y}(m\cdot\nabla\phi_1)_x\\
&\quad +\frac{1-\mu}{2}(\phi_{1y}+\phi_{2x})\left((m\cdot\nabla\phi_1)_y+(m\cdot\nabla\phi_2)_x\right)\Big]\,dxdy\\
&= \frac{D}{2}\int_\Omega \mathrm{div}\left\{m\left[\phi_{1x}^2+\phi_{2y}^2+2\mu\phi_{1x}\phi_{2y}+\frac{1-\mu}{2}(\phi_{1y}+\phi_{2x})^2\right]\right\}dxdy \qquad (1.10)\\
&= \frac{D}{2}\int_\Gamma m\cdot\nu\left[\phi_{1x}^2+\phi_{2y}^2+2\mu\phi_{1x}\phi_{2y}+\frac{1-\mu}{2}(\phi_{1y}+\phi_{2x})^2\right]d\Gamma,
\end{aligned}
$$

$$
\begin{aligned}
&a_1(\phi_1,\phi_2,\psi,m\cdot\nabla\phi_1,m\cdot\nabla\phi_2,m\cdot\nabla\psi)\\
&= \int_\Omega [(\phi_1+\psi_x)\left((m\cdot\nabla\phi_1)+(m\cdot\nabla\psi)_x\right)+(\phi_2+\psi_y)\left((m\cdot\nabla\phi_2)+(m\cdot\nabla\psi)_y\right)]\,dxdy\\
&= \frac{1}{2}\int_\Omega \mathrm{div}\left\{m\left[(\phi_1+\psi_x)^2+(\phi_2+\psi_y)^2\right]\right\}dxdy-\int_\Omega[(\phi_1+\psi_x)\phi_1+(\phi_2+\psi_y)\phi_2]\,dxdy \qquad (1.11)\\
&= \frac{1}{2}\int_\Gamma \left\{m\cdot\nu\left[(\phi_1+\psi_x)^2+(\phi_2+\psi_y)^2\right]\right\}d\Gamma-\int_\Omega[(\phi_1+\psi_x)\phi_1+(\phi_2+\psi_y)\phi_2]\,dxdy,
\end{aligned}
$$

and

$$
\begin{aligned}
&a_2(\psi,\eta_1,\eta_2,m\cdot\nabla\psi,m\cdot\nabla\eta_1,m\cdot\nabla\eta_2)\\
&= \frac{Eh}{1-\mu^2}\int_\Omega\Bigg[(1-\mu)\left(\eta_{1x}+\frac{1}{2}\psi_x^2\right)\left((m\cdot\nabla\eta_1)_x+\psi_x(m\cdot\nabla\psi)_x\right)\\
&\quad +(1-\mu)\left(\eta_{2y}+\frac{1}{2}\psi_y^2\right)\left((m\cdot\nabla\eta_2)_y+\psi_y(m\cdot\nabla\psi)_y\right)\\
&\quad +\mu\left(\eta_{1x}+\eta_{2y}+\frac{1}{2}|\nabla\psi|^2\right)\Big((m\cdot\nabla\eta_1)_x+(m\cdot\nabla\eta_2)_y+\nabla\psi\cdot\nabla(m\cdot\nabla\psi)\Big)\\
&\quad +\frac{1-\mu}{2}(\eta_{1y}+\eta_{2x}+\psi_x\psi_y)\Big((m\cdot\nabla\eta_1)_y+(m\cdot\nabla\eta_2)_x+\psi_x(m\cdot\nabla\psi)_y+\psi_y(m\cdot\nabla\psi)_x\Big)\Bigg]dxdy\\
&= \frac{Eh}{2(1-\mu^2)}\int_\Omega \mathrm{div}\Bigg\{m\Bigg[(1-\mu)\left(\eta_{1x}+\frac{1}{2}\psi_x^2\right)^2+(1-\mu)\left(\phi_{2y}+\frac{1}{2}\psi_y^2\right)^2 \qquad (1.12)\\
&\quad +\mu\left(\eta_{1x}+\eta_{2y}+\frac{1}{2}|\nabla\psi|^2\right)^2+\frac{1-\mu}{2}(\eta_{1y}+\eta_{2x}+\psi_x\psi_y)^2\Bigg]\Bigg\}dxdy\\
&= \frac{Eh}{2(1-\mu^2)}\int_\Gamma m\cdot\nu\Bigg[(1-\mu)\left(\eta_{1x}+\frac{1}{2}\psi_x^2\right)^2+(1-\mu)\left(\eta_{2y}+\frac{1}{2}\psi_y^2\right)^2\\
&\quad +\mu\left(\eta_{1x}+\eta_{2y}+\frac{1}{2}|\nabla\psi|^2\right)^2+\frac{1-\mu}{2}(\eta_{1y}+\eta_{2x}+\psi_x\psi_y)^2\Bigg]d\Gamma.
\end{aligned}
$$

Plugging (1.10)–(1.12) in (1.9) we get

$$
\begin{aligned}
& a(\phi_1,\phi_2,\psi,\eta_1,\eta_2,m\cdot\nabla\phi_1,m\cdot\nabla\phi_2,m\cdot\nabla\psi,m\cdot\nabla\eta_1,m\cdot\nabla\eta_2) \\
& =\tfrac{1}{2}\int_\Gamma m\cdot\nu\left\{D\left[(\phi_{1x})^2+(\phi_{2y})^2+2\mu\phi_{1x}\phi_{2y}+\tfrac{1-\mu}{2}(\phi_{1y}+\phi_{2x})^2\right]+k\left[(\phi_1+\psi_x)^2+(\phi_2+\psi_y)^2\right]\right. \\
& +\tfrac{Eh}{1-\mu^2}\left[(1-\mu)\left(\eta_{1x}+\tfrac{1}{2}\psi_x^2\right)^2+(1-\mu)\left(\phi_{2y}+\tfrac{1}{2}\psi_y^2\right)^2+\mu\left(\eta_{1x}+\eta_{2y}+\tfrac{1}{2}|\nabla\psi|^2\right)^2\right. \\
& \left.\left.+\tfrac{1-\mu}{2}\left(\eta_{1y}+\eta_{2x}+\psi_x\psi_y\right)^2\right]\right\}d\Gamma-k\int_\Omega\left[(\phi_1+\psi_x)\,\phi_1+(\phi_2+\psi_y)\,\phi_2\right]dxdy.
\end{aligned}
\tag{1.13}
$$

The equation (1.7) follows directly from (1.8) and (1.13). ■

The main result in this section is the following.

Theorem 1.4.5 *Assume the geometric condition* (1.4) *to hold. Let* $\{\phi_1,\phi_2,\psi,\eta_1,\eta_2\}$ *be a regular enough solution of system* (1.1), (1.1), (1.3)*. Then, there exist positive constants* C *and* ω *such that*

$$E_k(t)\le CE_k(0)\,\mathrm{e}^{-\omega t},\quad \forall\, t\ge 0. \tag{1.14}$$

Remark 1.4.6 *For regular enough initial data satisfying* (1.14)*, one obtains, as a consequence of inequality* (2.39)*, exponential decay for the energy* $E(t)$ *associated to system* (1.5), (1.3), (1.7) *as* $k\to\infty$*. This decay rate for the limit system is in agrement with the results from [43].*

Remark 1.4.7 *The case* $\Gamma_0=\emptyset$ *is not considered in this paper. In this case, one cannot assure that the energy decays to zero for every finite energy solution of* (1.1), (1.1), (1.3) *regardless of how the feedbacks are chosen. Indeed, defining*

$$\{\phi_{11},\phi_{21},\psi_1,\eta_{11},\eta_{21}\}=\left\{\alpha,\beta,-\alpha x-\beta y+\gamma,-\frac{1}{2}\alpha^2x-\frac{1}{2}\alpha\beta y+c_1,-\frac{1}{2}\beta^2y-\frac{1}{2}\alpha\beta x+c_2\right\},$$

where α*,* β*,* γ*,* c_1 *and* c_2 *are nonzero constants, and* $\{\phi_{10},\phi_{20},\psi_0,\eta_{10},\eta_{20}\}$ *such that*

$$\begin{cases} L_i(\phi_{10},\phi_{20},\psi_0,\eta_{10},\eta_{20})=0, \quad i=1,\ldots,5, \\ \{\mathcal{B}_1(\phi_{10},\phi_{20}),\mathcal{B}_2(\phi_{10},\phi_{20}),\mathcal{B}_3(\phi_{10},\phi_{20},\psi_0,\eta_{10},\eta_{20}),\mathcal{B}_4(\psi_0,\eta_{10},\eta_{20}),\mathcal{B}_5(\psi_0,\eta_{10},\eta_{20})\}=-\{\phi_{11},\phi_{21},\psi_1,\eta_{11},\eta_{21}\}, \end{cases}$$

it is not difficult to check that

$$\{\phi_1,\phi_2,\psi,\eta_1,\eta_2\}=t\{\phi_{11},\phi_{21},\psi_1,\eta_{11},\eta_{21}\}+\{\phi_{10},\phi_{20},\psi_0,\eta_{10},\eta_{20}\}$$

is a solution of (1.1), (1.1), (1.3)*. However, for this solution,*

$$E(t)=\frac{1}{2}\left[\frac{\rho h^3}{12}\left(|\phi_{11}|^2+|\phi_{21}|^2\right)+\rho h\left(|\psi_1|^2+|\eta_1|^2+|\eta_2|^2\right)\right]=const.>0.$$

Proof of Theorem 1.4.5. We divide the proof into three steps:

Step 1 We apply Lemma 1.4.4 to the solution of (1.1), (1.1), (1.3) and integrate the resulting identity with respect to t from 0 to T to obtain

$$\begin{aligned}
&\rho h\int_0^T\left[\frac{h^2}{12}(\phi_{1tt}, m\cdot\nabla\phi_1)+\frac{h^2}{12}(\phi_{2tt}, m\cdot\nabla\phi_2)+(\psi_{tt}, m\cdot\nabla\psi)+(\eta_{1tt}, m\cdot\nabla\eta_1)+(\eta_{2tt}, m\cdot\nabla\eta_2)\right]dt\\
&\quad -k\int_0^T\int_\Omega[(\phi_1+\psi_x)\phi_1+(\phi_2+\psi_y)\phi_2]\\
&= \quad -\frac{1}{2}\int_0^T\int_\Gamma m\cdot\nu\left\{D\left[(\phi_{1x})^2+(\phi_{2y})^2+2\mu\phi_{1x}\phi_{2y}+\frac{1-\mu}{2}(\phi_{1y}+\phi_{2x})^2\right]+k\left[(\phi_1+\psi_x)^2\right.\right.\\
&\quad \left.+(\phi_2+\psi_y)^2\right]+\frac{Eh}{1-\mu^2}\left[(1-\mu)\left(\eta_{1x}+\frac{1}{2}\psi_x^2\right)^2+(1-\mu)\left(\eta_{2y}+\frac{1}{2}\psi_y^2\right)^2\right. \qquad (1.15)\\
&\quad \left.\left.+\mu\left(\eta_{1x}+\eta_{2y}+\frac{1}{2}|\nabla\psi|^2\right)^2+\frac{1-\mu}{2}(\eta_{1y}+\eta_{2x}+\psi_x\psi_y)^2\right]\right\}\\
&\quad +\int_0^T\int_{\Gamma_0}[(m\cdot\nabla\phi_1)\mathcal{B}_1+(m\cdot\nabla\phi_2)\mathcal{B}_2+(m\cdot\nabla\psi)\mathcal{B}_3+(m\cdot\nabla\eta_1)\mathcal{B}_4+(m\cdot\nabla\eta_2)\mathcal{B}_5]\\
&\quad -\int_0^T\int_{\Gamma_1}[\phi_{1t}(m\cdot\nabla\phi_1)+\phi_{2t}(m\cdot\nabla\phi_2)+\psi_t(m\cdot\nabla\psi)+\eta_{1t}(m\cdot\nabla\eta_1)+\eta_{2t}(m\cdot\nabla\eta_2)] \qquad (1.16)
\end{aligned}$$

All of the integrals on the left-hand side of (1.15) may be interpreted in the $L^2(Q)$ scalar product since $\{\phi_{1tt}, \phi_{2tt}, \psi_{tt}, \eta_{1tt}, \eta_{2tt}\} \in C\left([0,\infty), [L^2(\Omega)]^5\right)$. The first integral on the left hand side may be written as

$$\begin{aligned}
&\rho h\int_0^T\int_\Omega\left\{\frac{h^2}{12}[\phi_{1tt}(m\cdot\nabla\phi_1)+\phi_{2tt}(m\cdot\nabla\phi_2)]+\psi_{tt}(m\cdot\nabla\psi)+\eta_{1tt}(m\cdot\nabla\eta_1)+\eta_{2tt}(m\cdot\nabla\eta_2)\right\} \qquad (1.17)\\
&= \quad \Upsilon_1-\rho h\int_0^T\int_\Omega\left[\frac{h^2}{12}(\phi_{1t}(m\cdot\nabla\phi_{1t})+\phi_{2t}(m\cdot\nabla\phi_{2t}))+\psi_t(m\cdot\nabla\psi_t)+\eta_{1t}(m\cdot\nabla\eta_{1t})+\eta_{2t}(m\cdot\nabla\eta_{2t})\right],
\end{aligned}$$

where

$$\Upsilon_1=\rho h\int_\Omega\left\{\frac{h^2}{12}[\phi_{1t}(m\cdot\nabla\phi_1)+\phi_{2t}(m\cdot\nabla\phi_2)]+\psi_t(m\cdot\nabla\psi)+\eta_{1t}(m\cdot\nabla\eta_1)+\eta_{2t}(m\cdot\nabla\eta_2)\right\}\Bigg|_0^T. \qquad (1.18)$$

A typical term of the last integral in (1.17) is (except for a constant factor)

$$\begin{aligned}
\int_0^T(\phi_{1t}, m\cdot\nabla\phi_{1t})\,dt &= \frac{1}{2}\int_0^T\int_\Omega \operatorname{div}(m\phi_{1t}^2)dxdydt-\int_0^T\int_\Omega\phi_{1t}^2dxdydt\\
&= \frac{1}{2}\int_0^T\int_{\Gamma_1}(m\cdot\nu)\phi_{1t}^2d\Gamma dt-\int_0^T\int_\Omega\phi_{1t}^2dxdydt.
\end{aligned}$$

The other terms of that integral are treated similarly. Thus, it follows that

$$\begin{aligned}
&\rho h\int_0^T\int_\Omega\left\{\tfrac{h^2}{12}[\phi_{1t}(m\cdot\nabla\phi_{1t})+\phi_{2t}(m\cdot\nabla\phi_{2t})]+\psi_t(m\cdot\nabla\psi_t)+\eta_{1t}(m\cdot\nabla\eta_{1t})+\eta_{2t}(m\cdot\nabla\eta_{2t})\right\}dxdydt\\
&\qquad = \tfrac{\rho h}{2}\int_0^T\int_{\Gamma_1}m\cdot\nu\left[\tfrac{h^2}{12}(\phi_{1t}^2+\phi_{2t}^2)+\psi_t^2+\eta_{1t}^2+\eta_{2t}^2\right]d\Gamma dt \qquad (1.1\\
&\qquad -\int_0^T\int_\Omega\rho h\left[\tfrac{h^2}{12}(\phi_{1t}^2+\phi_{2t}^2)+\psi_t^2+\eta_{1t}^2+\eta_{2t}^2\right]dxdydt.
\end{aligned}$$

Combining (1.15), (1.17) and (1.19), one has

$$\begin{aligned}
&\mathrm{Y}_1 + \int_0^T \int_\Omega \rho h \left[\tfrac{h^2}{12}(\phi_{1t}^2+\phi_{2t}^2)+\psi_t^2+\eta_{1t}^2+\eta_{2t}^2\right] - k\int_0^T\int_\Omega \left[(\phi_1+\psi_x)\,\phi_1 + (\phi_2+\psi_y)\,\phi_2\right] \\
&= \mathrm{J}_1 - \mathrm{J}_2 + \int_0^T\int_{\Gamma_0} \left[(m\cdot\nabla\phi_1)\mathcal{B}_1 + (m\cdot\nabla\phi_2)\mathcal{B}_2 + (m\cdot\nabla\psi)\mathcal{B}_3 + (m\cdot\nabla\eta_1)\mathcal{B}_4 + (m\cdot\nabla\eta_2)\mathcal{B}_5\right] d\Gamma dt \\
&- \int_0^T\int_{\Gamma_1} \left[\phi_{1t}(m\cdot\nabla\phi_1) + \phi_{2t}(m\cdot\nabla\phi_2) + \psi_t(m\cdot\nabla\psi) + \eta_{1t}(m\cdot\nabla\eta_1) + \eta_{2t}(m\cdot\nabla\eta_2)\right] d\Gamma dt,
\end{aligned} \tag{1.20}$$

where

$$\mathrm{J}_1 = \frac{\rho h}{2}\int_0^T\int_{\Gamma_1} m\cdot\nu \left[\frac{h^2}{12}(\phi_{1t}^2+\phi_{2t}^2)+\psi_t^2+\eta_{1t}^2+\eta_{2t}^2\right] d\Gamma dt, \tag{1.21}$$

and

$$\begin{aligned}
\mathrm{J}_2 \;=\; & \frac{1}{2}\int_0^T\int_{\Gamma_1} m\cdot\nu \left\{ D\left[(\phi_{1x})^2 + (\phi_{2y})^2 + 2\mu\phi_{1x}\phi_{2y} + \frac{1-\mu}{2}(\phi_{1y}+\phi_{2x})^2\right] + k\left[(\phi_1+\psi_x)^2\right.\right. \\
& \left. + (\phi_2+\psi_y)^2\right] + \frac{Eh}{1-\mu^2}\left[(1-\mu)\left(\eta_{1x}+\frac{1}{2}\psi_x^2\right)^2 + (1-\mu)\left(\eta_{2y}+\frac{1}{2}\psi_y^2\right)^2\right. \\
& \left.\left. + \mu\left(\eta_{1x}+\eta_{2y}+\frac{1}{2}|\nabla\psi|^2\right)^2 + \frac{1-\mu}{2}(\eta_{1y}+\eta_{2x}+\psi_x\psi_y)^2\right]\right\} d\Gamma dt.
\end{aligned} \tag{1.22}$$

Let us examine the integrals on Γ_0 in the right hand side of (1.20). Since $\phi_1 = \phi_2 = \psi = \eta_1 = \eta_2 = 0$ on Γ_0, we have $\nabla\phi_1 = \nu\left(\frac{\partial\phi_1}{\partial\nu}\right)$ on Γ_0 and similarly for the other functions. Therefore,

$$\begin{aligned}
&\int_{\Gamma_0} m\cdot\nu \left\{ D\left[\phi_{1x}^2+\phi_{2y}^2+2\mu\phi_{1x}\phi_{2y}+\frac{1-\mu}{2}(\phi_{1y}+\phi_{2x})^2\right] + k\left[(\phi_1+\psi_x)^2+(\phi_2+\psi_y)\right]\right\} \\
&\quad + \frac{Eh}{1-\mu^2}\left[(1-\mu)\left(\eta_{1x}+\frac{1}{2}\psi_x^2\right)^2 + (1-\mu)\left(\eta_{2y}+\frac{1}{2}\psi_y^2\right)^2 + \mu\left(\eta_{1x}+\eta_{2y}+\frac{1}{2}|\nabla\psi|^2\right)^2\right. \\
&\quad \left. + \frac{1-\mu}{2}(\eta_{1y}+\eta_{2x}+\psi_x\psi_y)^2\right] = \int_{\Gamma_0} m\cdot\nu\left\{D\left[\left(\nu_1\frac{\partial\phi_1}{\partial\nu}+\nu_2\frac{\partial\phi_2}{\partial\nu}\right)^2\right.\right. \\
&\quad \left. - (1-\mu)\nu_1\nu_2\frac{\partial\phi_1}{\partial\nu}\frac{\partial\phi_2}{\partial\nu}\right] + k\left(\frac{\partial\psi}{\partial\nu}\right)^2 + \frac{Eh}{1-\mu^2}\left[(1-\mu)\left(\nu_1\frac{\partial\eta_1}{\partial\nu}+\nu_2\frac{\partial\eta_2}{\partial\nu}+\frac{1}{2}\left(\frac{\partial\psi}{\partial\nu}\right)^2\right)^2\right. \\
&\quad - 2(1-\mu)\left(\nu_1\frac{\partial\eta_1}{\partial\nu}+\frac{1}{2}\left(\nu_1\frac{\partial\psi}{\partial\nu}\right)^2\right)\left(\nu_2\frac{\partial\eta_2}{\partial\nu}+\frac{1}{2}\left(\nu_2\frac{\partial\psi}{\partial\nu}\right)^2\right) \\
&\quad \left.\left. + \mu\left(\nu_1\frac{\partial\eta_1}{\partial\nu}+\nu_2\frac{\partial\eta_2}{\partial\nu}+\frac{1}{2}\left|\left(\frac{\partial\psi}{\partial\nu}\right)\right|^2\right)^2 + \frac{1-\mu}{2}\left(\nu_2\frac{\partial\eta_1}{\partial\nu}+\nu_1\frac{\partial\eta_2}{\partial\nu}+\nu_1\frac{\partial\psi}{\partial\nu}\nu_2\frac{\partial\psi}{\partial\nu}\right)\right]\right\}.
\end{aligned} \tag{1.23}$$

Furthermore,

$$\int_{\Gamma_0} [(m\cdot\nabla\phi_1)\mathcal{B}_1 + (m\cdot\nabla\phi_2)\mathcal{B}_2 + (m\cdot\nabla\psi)\mathcal{B}_3 + (m\cdot\nabla\eta_1)\mathcal{B}_4 + (m\cdot\nabla\eta_2)\mathcal{B}_5] \tag{1.24}$$
$$= \int_{\Gamma_0} m\cdot\nu \left\{ \frac{D}{2}\left[(1-\mu)\left(\left(\frac{\partial\phi_1}{\partial\nu}\right)^2 + \left(\frac{\partial\phi_2}{\partial\nu}\right)^2\right) + (1+\mu)\left(\nu_1\frac{\partial\phi_1}{\partial\nu} + \nu_2\frac{\partial\phi_2}{\partial\nu}\right)^2\right] + k\left(\frac{\partial\psi}{\partial\nu}\right)^2 \right.$$
$$\left. + N_1\left(\nu_1\frac{\partial\psi}{\partial\nu}\right)^2 + 2N_{12}\nu_1\frac{\partial\psi}{\partial\nu}\nu_2\frac{\partial\psi}{\partial\nu} + N_2\left(\nu_2\frac{\partial\psi}{\partial\nu}\right)^2 + N_1\nu_1\frac{\partial\eta_1}{\partial\nu} + N_{12}\nu_2\frac{\partial\eta_1}{\partial\nu} + N_2\nu_2\frac{\partial\eta_2}{\partial\nu} + N_{12}\nu_1\frac{\partial\eta_2}{\partial\nu}\right\}.$$

Since

$$-\frac{1}{2}\left\{ D\left[\phi_{1x}^2 + \phi_{2y}^2 + 2\mu\phi_{1x}\phi_{2y} + \frac{1-\mu}{2}(\phi_{1y}+\phi_{2x})^2\right] + k\left[(\phi_1+\psi_x)^2 + (\phi_2+\psi_y)\right]\right.$$
$$+ \frac{Eh}{1-\mu^2}\left[(1-\mu)\left(\eta_{1x} + \frac{1}{2}\psi_x^2\right)^2 + (1-\mu)\left(\eta_{2y} + \frac{1}{2}\psi_y^2\right)^2 + \mu\left(\eta_{1x}+\eta_{2y}+\frac{1}{2}|\nabla\psi|^2\right)^2\right.$$
$$\left.\left. + \frac{1-\mu}{2}(\eta_{1y}+\eta_{2x}+\psi_x\psi_y)^2\right]\right\} + [(m\cdot\nabla\phi_1)\mathcal{B}_1 + (m\cdot\nabla\phi_2)\mathcal{B}_2 + (m\cdot\nabla\psi)\mathcal{B}_3 + (m\cdot\nabla\eta_1)\mathcal{B}_4 + (m\cdot\nabla\eta_2)\mathcal{B}_5]$$
$$= \frac{1}{2}\left\{ D\left[\left(\nu_1\frac{\partial\phi_1}{\partial\nu} + \nu_2\frac{\partial\phi_2}{\partial\nu}\right)^2 + \frac{1-\mu}{2}\left(\nu_2\frac{\partial\phi_1}{\partial\nu} - \nu_1\frac{\partial\psi_2}{\partial_\nu}\right)^2\right] + k\left(\frac{\partial\psi}{\partial\nu}\right)^2\right.$$
$$+ \frac{Eh}{1-\mu^2}\left[(1-\mu)\left(\nu_1\frac{\partial\eta_1}{\partial\nu} + \frac{1}{2}\left(\nu_1\frac{\partial\psi}{\partial\nu}\right)^2\right)^2 + (1-\mu)\left(\nu_2\frac{\partial\eta_2}{\partial\nu} + \frac{1}{2}\left(\nu_2\frac{\partial\psi}{\partial\nu}\right)^2\right)^2\right. \tag{1.25}$$
$$\left.\left. + \mu\left(\nu_1\frac{\partial\eta_1}{\partial\nu} + \nu_2\frac{\partial\eta_2}{\partial\nu} + \frac{1}{2}|\nabla\psi|^2\right)^2 + \frac{1-\mu}{2}\left(\nu_2\frac{\partial\eta_1}{\partial\nu} + \nu_1\frac{\partial\eta_2}{\partial\nu} + \nu_1\frac{\partial\psi}{\partial\nu}\nu_2\frac{\partial\psi}{\partial\nu}\right)^2\right]\right\},$$

we conclude from (1.20) and (1.25) that

$$\mathrm{Y}_1 + \int_0^T\int_\Omega \rho h\left[\tfrac{h^2}{12}(\phi_{1t}^2+\phi_{2t}^2) + \psi_t^2 + \eta_{1t}^2 + \eta_{2t}^2\right] - k\int_0^T\int_\Omega [(\phi_1+\psi_x)\phi_1 + (\phi_2+\psi_y)\phi_2]$$
$$= \mathrm{J}_0 + \mathrm{J}_1 - \mathrm{J}_2 + \int_0^T\int_{\Gamma_0}[(m\cdot\nabla\phi_1)\mathcal{B}_1 + (m\cdot\nabla\phi_2)\mathcal{B}_2 + (m\cdot\nabla\psi)\mathcal{B}_3 + (m\cdot\nabla\eta_1)\mathcal{B}_4 + (m\cdot\nabla\eta_2)\mathcal{B}_5]\,d\Gamma dt \;\; x$$
$$- \int_0^T\int_{\Gamma_1}[\phi_{1t}(m\cdot\nabla\phi_1) + \phi_{2t}(m\cdot\nabla\phi_2) + \psi_t(m\cdot\nabla\psi) + \eta_{1t}(m\cdot\nabla\eta_1) + \eta_{2t}(m\cdot\nabla\eta_2)]\,d\Gamma dt, \tag{1.26}$$

where

$$\mathrm{J}_0 = \frac{1}{2}\int_0^T\int_{\Gamma_0}\left\{ D\left[\left(\nu_1\frac{\partial\phi_1}{\partial\nu} + \nu_2\frac{\partial\phi_2}{\partial\nu}\right)^2 + \frac{1-\mu}{2}\left(\nu_2\frac{\partial\phi_1}{\partial\nu} - \nu_1\frac{\partial\psi_2}{\partial_\nu}\right)^2\right] + k\left(\frac{\partial\psi}{\partial\nu}\right)^2\right.$$
$$+ \frac{Eh}{1-\mu^2}\left[(1-\mu)\left(\nu_1\frac{\partial\eta_1}{\partial\nu} + \frac{1}{2}\left(\nu_1\frac{\partial\psi}{\partial\nu}\right)^2\right)^2 + (1-\mu)\left(\nu_2\frac{\partial\eta_2}{\partial\nu} + \frac{1}{2}\left(\nu_2\frac{\partial\psi}{\partial\nu}\right)^2\right)^2\right.$$
$$\left.\left. + \mu\left(\nu_1\frac{\partial\eta_1}{\partial\nu} + \nu_2\frac{\partial\eta_2}{\partial\nu} + \frac{1}{2}|\nabla\psi|^2\right)^2 + \frac{1-\mu}{2}\left(\nu_2\frac{\partial\eta_1}{\partial\nu} + \nu_1\frac{\partial\eta_2}{\partial\nu} + \nu_1\frac{\partial\psi}{\partial\nu}\nu_2\frac{\partial\psi}{\partial\nu}\right)^2\right]\right\}.$$

Now, use (1.6) with $\{\phi_1, \phi_2, 0, \eta_1, \eta_2\}$ in the third term on the left-hand side of (1.26) to obtain

$$\rho h\int_\Omega\left[\frac{h^2}{12}(\phi_{1tt}\phi_1 + \phi_{2tt}\phi_2) + \eta_{1tt}\eta_1 + \eta_{2tt}\eta_2\right]dxdy - k\int_\Omega[(\phi_1+\psi_x)\phi_1 + (\phi_2+\psi_y)\phi_2]\,dxdy$$
$$+ a_0(\phi_1,\phi_2) + a_2(\psi,\eta_1,\eta_2,0,\eta_1,\eta_2) = -\int_{\Gamma_1}[\phi_{1t}\phi_1 + \phi_{2t}\phi_2 + \eta_{1t}\eta_1 + \eta_{2t}\eta_2]\,d\Gamma \tag{1.27}$$

Integrate identity (1.27) with respect to t from 0 to T. After an integration by parts in the first term, one obtains

$$\mathrm{Y}_2 - \rho h \int_0^T \int_\Omega \left[\frac{h^2}{12}\left(\phi_{1t}^2 + \phi_{2t}^2\right) + \eta_{1t}^2 + \eta_{2t}^2\right] dxdydt + k\int_0^T \int_\Omega \left[\left(\phi_1 + \psi_x\right)\phi_1 + \left(\phi_2 + \psi_y\right)\phi_2\right] dxdydt$$
$$+ \int_0^T \left[a_0(\phi_1, \phi_2) + a_2(\eta_1, \eta_2)\right] dt = -\int_0^T \int_{\Gamma_1} \left[\phi_{1t}\phi_1 + \phi_{2t}\phi_2 + \eta_{1t}\eta_1 + \eta_{2t}\right] d\Gamma dt \quad (1.28)$$

where

$$\mathrm{Y}_2 = \rho h \int_\Omega \left[\frac{h^2}{12}\left(\phi_{1t}\phi_1 + \phi_{2t}\phi_2\right) + \eta_{1t}\eta_1 + \eta_{2t}\eta_2\right] dxdy \Bigg|_0^T. \quad (1.29)$$

Multiply equation (1.28) by $1-\varepsilon$, with $\varepsilon \in (0,1)$, and add the product to equation (1.26) to get

$$(1-\varepsilon)\rho h \int_0^T \int_\Omega \psi_t^2 dxdydt + \varepsilon \int_0^T \int_\Omega \rho h \left[\frac{h^2}{12}(\phi_{1t}^2 + \phi_{2t}^2) + \psi_t^2 + \eta_{1t}^2 + \eta_{2t}^2\right] dxdydt + (1-\varepsilon)\int_0^T a_0(\phi_1, \phi_2)dt$$
$$(1-\varepsilon)\int_0^T a_2(\psi, \eta_1, \eta_2)dt - \varepsilon k \int_0^T a_1(\phi_1, \phi_2, \psi, \phi_1, \phi_2, 0) + \mathrm{Y}_1 + (1-\varepsilon)\mathrm{Y}_2 = \mathrm{J}_0 + \mathrm{J}_1 - \mathrm{J}_2$$
$$-\int_0^T \int_{\Gamma_1} [\phi_{1t}(m\cdot\nabla\phi_1 + (1-\varepsilon)\phi_1) + \phi_{2t}(m\cdot\nabla\phi_2 + (1-\varepsilon)\phi_2) + \psi_t(m\cdot\nabla\psi) + \eta_{1t}(m\cdot\nabla\eta_1 + (1-\varepsilon)\eta_1)$$
$$+\eta_{2t}(m\cdot\nabla\eta_2 + (1-\varepsilon)\eta_2)]\, d\Gamma dt. \quad (1.30)$$

Now, use (1.6) with $\{0, 0, \psi, 0, 0\}$. After an integration by parts in t one obtains

$$\mathrm{Y}_3 - \rho h \int_0^T \int_\Omega \psi_t^2 dxdydt + k\int_0^T a_1(\phi_1, \phi_2, \psi, 0, 0, \psi)dt$$
$$+ \int_0^T a_2(\psi, \eta_1, \eta_2, \psi, 0, 0) = -\int_0^T \int_{\Gamma_1} \psi\psi_t d\Gamma dt, \quad (1.31)$$

where

$$\mathrm{Y}_3 = \rho h \int_\Omega \psi_t \psi dxdy \Bigg|_0^T. \quad (1.32)$$

Multiply identity (1.31) by ε an add the product to equation (1.30) to obtain

$$(1-2\varepsilon)\rho h \int_0^T \int_\Omega \psi_t^2 dxdydt + \varepsilon \int_0^T \int_\Omega \rho h \left[\frac{h^2}{12}\left(\phi_{1t}^2 + \phi_{2t}^2\right) + \psi_t^2 + \eta_{1t}^2 + \eta_{2t}^2\right] dxdydt$$
$$+ (1-\varepsilon)\int_0^T \left[a_0(\phi_1, \phi_2) + a_2(\psi, \eta_1, \eta_2, 0, \eta_1\eta_2)\right] dt + \varepsilon k \int_0^T a_1(\phi_1, \phi_2, \psi)dt$$
$$- 2\varepsilon k \int_0^T a_1(\phi_1, \phi_2, \psi, \phi_1, \phi_2, 0)dt + \varepsilon \int_0^T a_2(\psi, \eta_1, \eta_2, \psi, 0, 0) + \mathrm{Y}_1 + (1-\varepsilon)\mathrm{Y}_2 + \varepsilon \mathrm{Y}_3$$
$$= \mathrm{J}_0 + \mathrm{J}_1 - \mathrm{J}_2 - \int_0^T \int_{\Gamma_1} [\phi_{1t}(m\cdot\nabla\phi_1 + (1-\varepsilon)\phi_1) + \phi_{2t}(m\cdot\nabla\phi_2 + (1-\varepsilon)\phi_2)$$
$$+ \psi_t(m\cdot\nabla\psi + \varepsilon\psi) + \eta_{1t}(m\cdot\nabla\eta_1 + (1-\varepsilon)\eta_1) + \eta_{2t}(m\cdot\nabla\eta_2 + (1-\varepsilon)\eta_2)]\, d\Gamma dt. \quad (1.33)$$

Step 2 Define the functional

$$\rho_\varepsilon(t) = \rho h \Big[\tfrac{h^2}{12}(\phi_{1t}(t), m\cdot\nabla\phi_1(t)) + \tfrac{h^2}{12}(\phi_{2t}(t), m\cdot\nabla\phi_2(t)) + (\psi_t(t), m\cdot\nabla\psi(t)) + (\eta_{1t}(t), m\cdot\nabla\eta_1(t)) + (\eta_{2t}(t), m\cdot\nabla\eta_2(t))\Big] + (1-\varepsilon)\rho h\Big\{\tfrac{h^2}{12}\left[(\phi_{1t}(t),\phi_1(t)) + (\phi_{2t}(t),\phi_2(t))\right] + (\eta_{1t}(t),\eta_1(t)) + (\eta_{2t}(t),\eta_2(t))\Big\} + \varepsilon\rho h(\psi_t(t),\psi(t)). \tag{1.34}$$

From identities (1.18),(1.29), and (1.32), one sees that

$$\mathrm{Y}_1 + (1-\varepsilon)\mathrm{Y}_2 + \varepsilon\mathrm{Y}_3 = \rho_\varepsilon(T) - \rho_\varepsilon(0). \tag{1.35}$$

Since (1.33) is valid for all $T>0$, we differentiate in T and obtain, writing t in place of T,

$$\begin{aligned}\frac{d}{dt}\rho_\varepsilon(t) \;=\; & \frac{d}{dt}(\mathrm{J}_0 + \mathrm{J}_1 - \mathrm{J}_2) - (1-2\varepsilon)\rho h\int_\Omega \psi_t^2 dxdy - \varepsilon\rho h\int_\Omega\left[\frac{h^2}{12}\left(\phi_{1t}^2+\phi_{2t}^2\right) + \psi_t^2 + \eta_{1t}^2 + \eta_{2t}^2\right]dxdy \\ & - (1-\varepsilon)\left[a_0(\phi_1,\phi_2) + a_2(\psi,\eta_1,\eta_2,0,\eta_1,\eta_2)\right] - \varepsilon k a_1(\phi_1,\phi_2,\psi)dt + 2\varepsilon k a_1(\phi_1\phi_2,\psi,\phi_1,\phi_2,0) \\ & - \varepsilon a_2(\psi,\eta_1,\eta_2,\psi,0,0) - \int_{\Gamma_1}\left[\phi_{1t}(m\cdot\nabla\phi_1 + (1-\varepsilon)\phi_1) + \phi_{2t}(m\cdot\nabla\phi_2 + (1-\varepsilon)\phi_2)\right. \\ & \left. + \psi_t(m\cdot\nabla\psi + \varepsilon\psi) + \eta_{1t}(m\cdot\nabla\eta_1 + (1-\varepsilon)\eta_1) + \eta_{2t}(m\cdot\nabla\eta_2 + (1-\varepsilon)\eta_2)\right]d\Gamma \end{aligned} \tag{1.36}$$

where the right-hand side is evaluated at t. Now, let $\delta>0$ and consider the perturbed energy $F_{\varepsilon,\delta}(t)$ given by

$$F_{\varepsilon,\delta}(t) = E(t) + \delta\rho_\varepsilon(t). \tag{1.37}$$

We are going to prove that for all ε, δ sufficiently small, one has

$$\frac{d}{dt}F_{\varepsilon,\delta}(t) \le -\frac{1}{2}\varepsilon\delta E(t) - \frac{\delta}{2}E_\Gamma(t), \tag{1.38}$$

where

$$\begin{aligned} E_\Gamma(t) \;=\; & \frac{\rho h}{2}\int_{\Gamma_1} m\cdot\nu\left[\frac{h^2}{12}(\phi_{1t}^2+\phi_{2t}^2) + \psi_t^2 + \eta_{1t}^2 + \eta_{2t}^2\right]d\Gamma + \frac{1}{2}\int_\Gamma |m\cdot\nu|\Big\{D\Big[(\phi_{1x})^2 + (\phi_{2y})^2 \\ & + 2\mu\phi_{1x}\phi_{2y} + \frac{1-\mu}{2}(\phi_{1y}+\phi_{2x})^2\Big] + k\left[(\phi_1+\psi_x)^2 + (\phi_2+\psi_y)^2\right] \\ & + \frac{Eh}{1-\mu^2}\Big[(1-\mu)\left(\eta_{1x} + \frac{1}{2}\psi_x^2\right)^2 + (1-\mu)\left(\eta_{2y} + \frac{1}{2}\psi_y^2\right)^2 \\ & + \mu\left(\eta_{1x} + \eta_{2y} + \frac{1}{2}|\nabla\psi|^2\right)^2 + \frac{1-\mu}{2}(\eta_{1y} + \eta_{2x} + \psi_x\psi_y)^2\Big]\Big\}d\Gamma. \end{aligned} \tag{1.39}$$

We begin the proof of inequality (1.38) estimating $\frac{d}{dt}\rho_\varepsilon(t)$. First of all, we bound the term $a_1(\phi_1,\phi_2,\psi,\phi_1\phi_2,0)$ in (1.36). For any $\xi>0$, we have

$$|a_1(\phi_1,\phi_2,\psi,\phi_1,\phi_2,0)| \le \frac{\xi}{2}a_1(\phi_1,\phi_2,\psi) + \frac{1}{2\xi}a_1(\phi_1,\phi_2,0).$$

Since $\Gamma_0 \neq \emptyset$, according to a result due to Lagnese (see [25, Lemma 2.1]) there is a constant γ_0 (depending on the geometry of Ω and on the parameters μ and D) such that

$$a_1(\phi_1, \phi_2, 0) = \|\phi_1\|^2 + \|\phi_2\|^2 \leq \gamma_0 a_0(\phi_1, \phi_2).$$

Therefore,

$$|a_1(\phi_1, \phi_2, \psi, \phi_1, \phi_2, 0)| \leq \frac{\xi}{2} a_1(\phi_1, \phi_2, \psi) + \frac{\gamma_0}{2\xi} a_0(\phi_1, \phi_2). \tag{1.40}$$

Use inequality (1.40) in identity (1.36) to get

$$\begin{aligned}
\frac{d}{dt}\rho_\varepsilon(t) \;\leq\; & \frac{d}{dt}\left(\mathrm{J}_0 + \mathrm{J}_1 - \mathrm{J}_2\right) - (1-2\varepsilon)\rho h \int_\Omega \psi_t^2 dxdy - \varepsilon \rho h \int_\Omega \left[\frac{h^2}{12}\left(\phi_{1t}^2 + \phi_{2t}^2\right) + \psi_t^2 + \eta_{1t}^2 + \eta_{2t}^2\right] dxd \\
& - \left(1 - \varepsilon - \frac{\varepsilon\gamma_0 k}{\xi}\right) a_0(\phi_1, \phi_2) - (1-\varepsilon) a_2(\psi, \eta_1, \eta_2, 0, \eta_1, \eta_2) - \varepsilon k (1-\xi) a_1(\phi_1, \phi_2, \psi) dt \\
& - \varepsilon a_2(\psi, \eta_1, \eta_2, \psi, 0, 0) - \int_{\Gamma_1} \left[\phi_{1t}(m \cdot \nabla\phi_1 + (1-\varepsilon)\phi_1) + \phi_{2t}(m \cdot \nabla\phi_2 + (1-\varepsilon)\phi_2)\right. \\
& \left. + \psi_t(m \cdot \nabla\psi + \varepsilon\psi) + \eta_{1t}(m \cdot \nabla\eta_1 + (1-\varepsilon)\eta_1) + \eta_{2t}(m \cdot \nabla\eta_2 + (1-\varepsilon)\eta_2)\right] d\Gamma.
\end{aligned}$$

Fix $\xi = \frac{1}{2}$, and then choose $\varepsilon > 0$ so that $1 - \varepsilon - 2\varepsilon\gamma_0 k \geq \varepsilon$, that is,

$$0 < \varepsilon \leq \frac{1}{2(1 + \gamma_0 k)}. \tag{1.41}$$

For such ε, one has

$$\begin{aligned}
\frac{d}{dt}\rho_\varepsilon(t) \;\leq\; & \frac{d}{dt}\left(\mathrm{J}_0 + \mathrm{J}_1 - \mathrm{J}_2\right) - (1-2\varepsilon)\rho h \int_\Omega \psi_t^2 dxdy - \varepsilon \rho h \int_\Omega \left[\frac{h^2}{12}\left(\phi_{1t}^2 + \phi_{2t}^2\right) + \psi_t^2 + \eta_{1t}^2 + \eta_{2t}^2\right] dxdy \\
& - \varepsilon a_0(\phi_1, \phi_2) - (1-\varepsilon) a_2(\psi, \eta_1, \eta_2, 0, \eta_1, \eta_2) - \frac{k\varepsilon}{2} a_1(\phi_1, \phi_2, \psi) - \varepsilon a_2(\psi, \eta_1, \eta_2, \psi, 0, 0) \\
& - \int_{\Gamma_1} \left[\phi_{1t}(m \cdot \nabla\phi_1 + (1-\varepsilon)\phi_1) + \phi_{2t}(m \cdot \nabla\phi_2 + (1-\varepsilon)\phi_2) + \psi_t(m \cdot \nabla\psi + \varepsilon\psi)\right. \\
& \left. + \eta_{1t}(m \cdot \nabla\eta_1 + (1-\varepsilon)\eta_1) + \eta_{2t}(m \cdot \nabla\eta_2 + (1-\varepsilon)\eta_2)\right] d\Gamma dt \qquad (1.42) \\
\leq\; & \frac{d}{dt}\left(\mathrm{J}_0 + \mathrm{J}_1 - \mathrm{J}_2\right) - (1-2\varepsilon)\rho h \int_\Omega \psi_t^2 dxdy - \varepsilon E(t) - \frac{\varepsilon}{2}\left\{\rho h \int_\Omega \left[\frac{h^2}{12}\left(\phi_{1t}^2 + \phi_{2t}^2\right) + \psi_t^2\right.\right. \\
& \left.\left. + \;\eta_{1t}^2 + \eta_{2t}^2\right] dxdy + a_0(\phi_1, \phi_2) + a_2(\psi, \eta_1, \eta_2)\right\} - \int_{\Gamma_1} \left[\phi_{1t}(m \cdot \nabla\phi_1 + (1-\varepsilon)\phi_1)\right. \\
& + \phi_{2t}(m \cdot \nabla\phi_2 + (1-\varepsilon)\phi_2) + \psi_t(m \cdot \nabla\psi + \varepsilon\psi) + \eta_{1t}(m \cdot \nabla\eta_1 + (1-\varepsilon)\eta_1) \\
& \left. + \eta_{2t}(m \cdot \nabla\eta_2 + (1-\varepsilon)\eta_2)\right] d\Gamma dt.
\end{aligned}$$

We estimate the last term on the right-hand side of (1.42) as follows:

$$\Big| \int_{\Gamma_1} [\phi_{1t}(m\cdot\nabla\phi_1 + (1-\varepsilon)\phi_1) + \phi_{2t}(m\cdot\nabla\phi_2 + (1-\varepsilon)\phi_2) + \psi_t(m\cdot\nabla\psi + \varepsilon\psi) + \eta_{1t}(m\cdot\nabla\eta_1 + (1-\varepsilon)\eta_1)$$
$$+ \eta_{2t}(m\cdot\nabla\eta_2 + (1-\varepsilon)\eta_2)]\,d\Gamma\Big| \le \frac{1}{2\xi}\int_{\Gamma_1}\left[\phi_{1t}^2 + \phi_{2t}^2 + \psi_t^2 + \eta_{1t}^2 + \eta_{2t}^2\right]d\Gamma + \frac{\xi}{2}\int_{\Gamma_1}\left[(m\cdot\nabla\phi_1 + (1-\varepsilon)\phi_1)^2\right.$$
$$\left. + (m\cdot\nabla\phi_2 + (1-\varepsilon)\phi_2)^2 + (m\cdot\nabla\psi + \varepsilon\psi)^2 + (m\cdot\nabla\eta_1 + (1-\varepsilon)\eta_1)^2 + (m\cdot\nabla\eta_2 + (1-\varepsilon)\eta_2)^2\right]d\Gamma$$
$$= -\frac{1}{2\xi}\frac{d}{dt}E(t) + \frac{\xi}{2}\int_{\Gamma_1}\left[(m\cdot\nabla\phi_1 + (1-\varepsilon)\phi_1)^2 + (m\cdot\nabla\phi_2 + (1-\varepsilon)\phi_2)^2 + (m\cdot\nabla\psi + \varepsilon\psi)^2\right.$$
$$\left. + (m\cdot\nabla\eta_1 + (1-\varepsilon)\eta_1)^2 + (m\cdot\nabla\eta_2 + (1-\varepsilon)\eta_2)^2\right]d\Gamma. \tag{1.43}$$

Looking for the last integral in (1.43), it follows by (1.5) that

$$\int_{\Gamma_1}[(m\cdot\nabla\phi_1 + (1-\varepsilon)\phi_1)^2 + (m\cdot\nabla\phi_2 + (1-\varepsilon)\phi_2)^2 + (m\cdot\nabla\psi + \varepsilon\psi)^2 + (m\cdot\nabla\eta_1 + (1-\varepsilon)\eta_1)^2$$
$$+ (m\cdot\nabla\eta_2 + (1-\varepsilon)\eta_2)^2]\,d\Gamma \le G_0\int_{\Gamma_1} m\cdot\nu\,[(m\cdot\nabla\phi_1 + (1-\varepsilon)\phi_1)^2 + (m\cdot\nabla\phi_2 + (1-\varepsilon)\phi_2)^2$$
$$+ (m\cdot\nabla\psi + \varepsilon\psi)^2 + (m\cdot\nabla\eta_1 + (1-\varepsilon)\eta_1)^2 + (m\cdot\nabla\eta_2 + (1-\varepsilon)\eta_2)^2]\,d\Gamma. \tag{1.44}$$

We now bound the right-hand side of inequality (1.44). Its first term is bounded by

$$\begin{aligned}\int_{\Gamma_1} m\cdot\nu(m\cdot\nabla\phi_1 + (1-\varepsilon)\phi_1)^2 d\Gamma &\le 2\int_{\Gamma_1} m\cdot\nu\left[(m\cdot\nabla\phi_1)^2 + (1-\varepsilon)^2\phi_1^2\right]d\Gamma\\ &\le 2R^2\int_{\Gamma_1} m\cdot\nu|\nabla\phi_1|^2 d\Gamma + 2(1-\varepsilon)^2 R\int_{\Gamma_1}\phi_1^2 d\Gamma,\end{aligned}$$

where $R = \sup_{\Gamma_1} m(x,y)$. The other terms can be bounded analogously. Therefore, one gets

$$\int_{\Gamma_1}[(m\cdot\nabla\phi_1 + (1-\varepsilon)\phi_1)^2 + (m\cdot\nabla\phi_2 + (1-\varepsilon)\phi_2)^2 + (m\cdot\nabla\psi + \varepsilon\psi)^2 + (m\cdot\nabla\eta_1 + (1-\varepsilon)\eta_1)^2$$
$$+ (m\cdot\nabla\eta_2 + (1-\varepsilon)\eta_2)^2]\,d\Gamma \le 2G_0R^2\int_{\Gamma_1} m\cdot\nu\,[|\nabla\phi_1|^2 + |\nabla\phi_2^2 + |\nabla\psi|^2 + |\nabla\eta_1|^2 + |\nabla\eta_2|^2]\,d\Gamma$$
$$+ 2G_0(1-\varepsilon)^2 R\int_{\Gamma_1}[\phi_1^2 + \phi_2^2 + \psi^2 + \eta_1^2 + \eta_2^2]\,d\Gamma. \tag{1.45}$$

For $k \ge k_0 > 0$ we have, according to Lagnese (see [25, Lemma 2.1]) and to trace theory,

$$\int_{\Gamma_1}\left[\phi_1^2 + \phi_2^2 + \psi^2 + \eta_1^2 + \eta_2^2\right]d\Gamma \le \gamma_1\left[a_0(\phi_1,\phi_2) + ka_1(\phi_1,\phi_2,\psi) + a_2(\psi,\eta_1,\eta_2)\right]. \tag{1.46}$$

In addition,

$$\begin{aligned}&\int_{\Gamma_1} m\cdot\nu\left[|\nabla\phi_1|^2 + |\nabla\phi_2|^2 + |\nabla\psi|^2 + |\nabla\eta_1|^2 + |\nabla\eta_2|^2\right]d\Gamma\\ &\le \widetilde{\gamma}_2\left[a_{\Gamma_1}(\phi_1,\phi_2,\psi,\eta_1,\eta_2) + \int_{\Gamma_1}(\phi_1^2 + \phi_2^2 + \eta_1^2 + \eta_2^2)\right]\\ &\le \gamma_2\left[a_{\Gamma_1}(\phi_1,\phi_2,\psi,\eta_1,\eta_2) + a_0(\phi_1,\phi_2) + a_2(\eta_1,\eta_2)\right],\end{aligned} \tag{1.47}$$

where the constants $\gamma_1,\ \gamma_2$ depend only on $\Omega,\ D,\ \mu,$ and k_0, and

$$
\begin{aligned}
&a_{\Gamma_1}(\phi_1,\phi_2,\psi,\eta_1,\eta_2) = 2\frac{d}{dt}\mathrm{J}_2 \\
&= \int_{\Gamma} m\cdot\nu \left\{ D\left[(\phi_{1x})^2+(\phi_{2y})^2+2\mu\phi_{1x}\phi_{2y}+\frac{1-\mu}{2}(\phi_{1y}+\phi_{2x})^2\right] + k\left[(\phi_1+\psi_x)^2+(\phi_2+\psi_y)^2\right]\right. \\
&\quad + \frac{Eh}{1-\mu^2}\left[(1-\mu)\left(\eta_{1x}+\frac{1}{2}\psi_x^2\right)^2+(1-\mu)\left(\eta_{2y}+\frac{1}{2}\psi_y^2\right)^2+\mu\left(\eta_{1x}+\eta_{2y}+\frac{1}{2}|\nabla\psi|^2\right)^2\right. \qquad (1.48) \\
&\quad \left.\left. + \frac{1-\mu}{2}(\eta_{1y}\eta_{2x}+\psi_x\psi_y)^2\right]\right\} d\Gamma.
\end{aligned}
$$

From (1.43)–(1.48), we obtain the estimate

$$
\begin{aligned}
&\left| \int_{\Gamma_1} [\phi_{1t}(m\cdot\nabla\phi_1+(1-\varepsilon)\phi_1)+\phi_{2t}(m\cdot\nabla\phi_2+(1-\varepsilon)\phi_2)+\psi_t(m\cdot\nabla\psi+\varepsilon\psi)\right. \\
&\quad \left. + \eta_{1t}(m\cdot\nabla\eta_1+(1-\varepsilon)\eta_1)+\eta_{2t}(m\cdot\nabla\eta_2+(1-\varepsilon)\eta_2)]\, d\Gamma \right| \\
&\le -\frac{1}{2\xi}\frac{d}{dt}E(t)+\xi G_0R^2\gamma_2\left[a_{\Gamma_1}(\phi_1,\phi_2,\psi,\eta_1,\eta_2)+a_0(\phi_1,\phi_2)+a_2(\psi,\eta_1,\eta_2)\right] \qquad (1.49 \\
&\quad + \xi G_0\gamma_1(1-\varepsilon)^2R\left[a_0(\phi_1,\phi_2)+ka_1(\phi_1,\phi_2,\psi)+a_2(\psi,\eta_1,\eta_2)\right] \\
&\le -\frac{1}{2\xi}\frac{d}{dt}E(t)+\xi G_0R^2\gamma_2\left[a_{\Gamma_1}(\phi_1,\phi_2,\psi,\eta_1,\eta_2)+a_0(\phi_1,\phi_2)+a_2(\psi,\eta_1,\eta_2)\right]+\xi G_0\gamma_1(1-\varepsilon)^2RE(t
\end{aligned}
$$

Using (1.49) in (1.42), it follows that

$$
\begin{aligned}
\frac{d}{dt}\rho_\varepsilon(t) \le\ & \frac{d}{dt}\mathrm{J}_0+\frac{1}{2}c_{\Gamma_1}(\phi_1,\phi_2,\psi,\eta_1,\eta_2)-\frac{1}{2\xi}\frac{d}{dt}E(t)-\left(\frac{1}{2}-\xi\gamma_2G_0R^2\right)a_{\Gamma_1}(\phi_1,\phi_2,\psi,\eta_1,\eta_2) \\
& -\left[\varepsilon-2\xi\gamma_1G_0(1-\varepsilon)^2R\right]E(t)-\left[\frac{\varepsilon}{2}-\xi\gamma_2G_0R^2\right]\left[a_0(\phi_1,\phi_2)+a_2(\psi,\eta_1,\eta_2)\right] \qquad (1.50)
\end{aligned}
$$

where

$$c_{\Gamma_1}(\phi_1,\phi_2,\psi,\eta_1,\eta_2) = 2\frac{d}{dt}J_1 = \rho h\int_{\Gamma_1} m\cdot\nu\left[\frac{h^2}{12}(\phi_{1t}^2+\phi_{2t}^2)+\psi_t^2+\eta_{1t}^2+\eta_{2t}^2\right]d\Gamma.$$

From the definition of J_0 and the first of the geometric assumptions in (1.4), we have

$$
\begin{aligned}
&\frac{d}{dt}\mathrm{J}_0 = \frac{1}{2}\int_0^T\int_{\Gamma_0} m\cdot\nu\left\{D\left[(\phi_{1x})^2+(\phi_{2y})^2+2\mu\phi_{1x}\phi_{2y}+\frac{1-\mu}{2}(\phi_{1y}+\phi_{2x})^2\right]+k\left[(\phi_1+\psi_x)^2\right.\right. \\
&\quad \left. +(\phi_2+\psi_y)^2\right]+\frac{Eh}{1-\mu^2}\left[(1-\mu)\left(\eta_{1x}+\frac{1}{2}\psi_x^2\right)^2+(1-\mu)\left(\eta_{2y}+\frac{1}{2}\psi_y^2\right)^2+\mu\left(\eta_{1x}+\eta_{2y}+\frac{1}{2}|\nabla\psi|^2\right)^2\right. \\
&\quad \left.\left.+\frac{1-\mu}{2}(\eta_{1y}+\eta_{2x}+\psi_x\psi_y)^2\right]\right\}+\frac{1-\mu}{4}\int_{\Gamma_0} m\cdot\nu\left(\nu_2\frac{\partial\phi_1}{\partial\nu}+\nu_1\frac{\partial\phi_2}{\partial\nu}\right)^2 \\
&\le -\frac{1}{2}a_{\Gamma_0}(\phi_1,\phi_2,\psi,\eta_1,\eta_2), \qquad (1.51)
\end{aligned}
$$

where

$$a_{\Gamma_0}(\phi_1,\phi_2,\psi,\eta_1,\eta_2)=\frac{1}{2}\int_0^T\int_{\Gamma_0}|m\cdot\nu|\left\{D\left[(\phi_{1x})^2+(\phi_{2y})^2+2\mu\phi_{1x}\phi_{2y}+\frac{1-\mu}{2}(\phi_{1y}+\phi_{2x})^2\right]+k\left[(\phi_1+\psi_x)^2\right.\right.$$
$$\left.+(\phi_2+\psi_y)^2\right]+\frac{Eh}{1-\mu^2}\left[(1-\mu)\left(\eta_{1x}+\frac{1}{2}\psi_x^2\right)^2+(1-\mu)\left(\eta_{2y}+\frac{1}{2}\psi_y^2\right)^2+\mu\left(\eta_{1x}+\eta_{2y}+\frac{1}{2}|\nabla\psi|^2\right)^2\right.$$
$$\left.\left.+\frac{1-\mu}{2}(\eta_{1y}+\eta_{2x}+\psi_x\psi_y)^2\right]\right\}.$$

Substituting (1.51) in the right-hand side of (1.50), one gets the estimate

$$\begin{aligned}\frac{d}{dt}\rho_\varepsilon(t) &\le +\frac{1}{2}c_{\Gamma_1}(\phi_1,\phi_2,\psi,\eta_1,\eta_2)-\frac{1}{2\xi}\frac{d}{dt}E(t)-\left(\frac{1}{2}-\xi\gamma_2G_0R^2\right)a_{\Gamma_1}(\phi_1,\phi_2,\psi,\eta_1,\eta_2) \\ &\quad-\left[\varepsilon-2\xi\gamma_1G_0(1-\varepsilon)^2R\right]E(t)-\left[\frac{\varepsilon}{2}-\xi\gamma_2G_0R^2\right][a_0(\phi_1,\phi_2)+a_2(\psi,\eta_1,\eta_2)]-\frac{1}{2}a_{\Gamma_0}(\phi_1,\phi_2,\psi,\eta_1,\eta_2),\end{aligned}\tag{1.52}$$

Now, taking $\varepsilon<1/2$ and choosing $\xi>0$ small enough such that

$$2\xi\gamma_1G_0(1-\varepsilon)^2R\le\frac{\varepsilon}{2},\qquad \xi\gamma_2G_0R^2\le\frac{\varepsilon}{4}<\frac{1}{4},$$

we can guarantee from inequality (1.52) that

$$\frac{d}{dt}\rho_\varepsilon(t)\le-\frac{1}{2\xi}\frac{d}{dt}E(t)-\frac{\varepsilon}{2}E(t)+\frac{1}{2}c_{\Gamma_1}(\phi_1,\phi_2,\psi,\eta_1,\eta_2)-\frac{1}{4}a_{\Gamma_1}(\phi_1,\phi_2,\psi,\eta_1,\eta_2)-\frac{1}{4}a_{\Gamma_0}(\phi_1,\phi_2,\psi,\eta_1,\eta_2).$$

Let us consider

$$F_{\varepsilon,\delta}(t)=E(t)+\delta\rho_\varepsilon(t)$$

with $\delta>0$. Therefore,

$$\begin{aligned}\frac{d}{dt}F_{\varepsilon,\delta}(t) &= \frac{d}{dt}E(t)+\delta\frac{d}{dt}\rho_\varepsilon(t) \\ &= \left(1-\frac{\delta}{2\xi}\right)\frac{d}{dt}E(t)-\frac{\delta\varepsilon}{2}E(t)+\frac{\delta}{2}c_{\Gamma_1}(\phi_1,\phi_2,\psi,\eta_1,\eta_2)-\frac{\delta}{4}a_\Gamma(\phi_1,\phi_2,\psi,\eta_1,\eta_2),\end{aligned}$$

where

$$a_\Gamma(\phi_1,\phi_2,\psi,\eta_1,\eta_2)=a_{\Gamma_0}(\phi_1,\phi_2,\psi,\eta_1,\eta_2)+a_{\Gamma_1}(\phi_1,\phi_2,\psi,\eta_1,\eta_2).$$

From (1.5), we get

$$\begin{aligned}\frac{d}{dt}E(t) &= -\int_{\Gamma_1}\left[\phi_{1t}^2+\phi_{2t}^2+\psi_t^2+\eta_{1t}^2+\eta_{2t}^2\right]d\Gamma \\ &\le -g_0\int_{\Gamma_1}m\cdot\nu\left[\phi_{1t}^2+\phi_{2t}^2+\psi_t^2+\eta_{1t}^2+\eta_{2t}^2\right]d\Gamma \\ &\le -\frac{g_0}{\rho h}c_{\Gamma_1}(\phi_1,\phi_2,\psi,\eta_1,\eta_2),\end{aligned}\tag{1.53}$$

provided $\frac{h^2}{12} \leq 1$ (as we may assume). Therefore

$$\begin{aligned}
\frac{d}{dt}F_{\varepsilon,\delta}(t) &= -\left[\frac{g_0}{\rho h}\left(1-\frac{\delta}{2\xi}\right)-\frac{\delta}{2}\right]c_{\Gamma_1}(\phi_1,\phi_2,\psi,\eta_1,\eta_2)-\frac{\varepsilon\delta}{2}E(t)-\frac{\delta}{4}a_{\Gamma_1}(\phi_1,\phi_2,\psi,\eta_1,\eta_2)\\
&\leq -\frac{\varepsilon\delta}{2}E(t)-\frac{\delta}{4}\left[c_{\Gamma_1}(\phi_1,\phi_2,\psi,\eta_1,\eta_2)+a_{\Gamma_1}(\phi_1,\phi_2,\psi,\eta_1,\eta_2)\right] \qquad (1.54)\\
&= -\frac{\varepsilon\delta}{2}E(t)-\frac{\delta}{2}E_\Gamma(t),
\end{aligned}$$

with $\delta > 0$ being chosen such that

$$\frac{g_0}{\rho h}\left(1-\frac{\delta}{2\xi}\right)-\frac{\delta}{2} \geq \frac{\delta}{4}.$$

Step 3 To get the exponential decay of $E(t)$ using inequality (1.54), we need to compare $E(t)$ and $F_{\varepsilon,\delta}(t)$. To carry this out, we use the definition (1.34) of $\rho_\varepsilon(t)$ and a result due to Lagnese (see [25, Lemma 2.1]) to obtain

$$|\rho_\varepsilon(t)| \leq CE(t),$$

where C depends on Ω, D, μ, and K_0 $(K \geq K_0 > 0)$ but not on ε. Consequently

$$|F_{\varepsilon,\delta} - E(t)| = \delta\rho_\varepsilon(t) \leq \delta CE(t).$$

Therefore,

$$(1-\delta C)E(t) \leq F_{\varepsilon,\delta}(t) \leq (1+\delta C)E(t).$$

Moreover, since

$$E(t) + \frac{1}{\varepsilon}E_\Gamma(t) \geq E(t),$$

one gets

$$\frac{d}{dt}F_{\varepsilon,\delta} \leq -\omega F_{\varepsilon,\delta},$$

where $\omega = \frac{\delta\varepsilon}{2(1+\delta C)}$. As a consequence of (1.34), (1.37) and of the choice of ε (see (1.41)), we conclude that there exist positive constants $C > 0$ and $\omega > 0$ such that

$$E(t) \leq CE(0)\mathrm{e}^{-\omega t},$$

for every $t > 0$ and every solution of (1.1), (1.1), (1.3). This completes the proof. ■

1.5 Further comments and open problems

(i) Although we know the physical deduction for the nonlinear Mindlin-Timoshenko system (1.1)–(1.3) (see e.g. [27], [44]), we are not aware of results concerning well-posedness and regularity for all $k > 0$. However, since our main goal was to give a positive response to the Lagnese-Lions conjecture, what we can say is that, for k large enough and for initial data in the space $\mathcal{X}$, the system (1.1)–(1.3) is very close to the known von Kármán system (1.5)–(1.7) (see Theorem 1.2.1). On the other hand, there is an extensive literature dealing with well-posedness, regularity, stability, etc, for system (1.5)–(1.7) (see [17, 25, 26, 29, 43]). In Section 1.4 we have analyzed the asymptotic behavior (as $t \to \infty$) for the solution of the nonlinear Mindlin-Timoshenko system with boundary feedback. To this end, we had to request an additional regularity for their solutions. For this reason, in all results of that section, we have used the expression "regular enough"to the solutions, in order to ensure that, under certain restrictions, the results hold. In our case, for instance, if we consider the solution $\{\phi_1(t), \phi_2(t), \psi(t), \eta_1(t), \eta_2(t)\} \in \left[H^2 \cap H^1_{\Gamma_0}\right]^2 \times \left[H^3 \cap H^1_{\Gamma_0}\right] \times \left[H^2 \cap H^1_{\Gamma_0}\right]^2$, the stability result hold. For the linear Mindlin-Timoshenko system, this issue was treated in [25, Remark 3.1].

(ii) In the proof of Theorems 1.2.1, 1.3.1 and 1.4.5, we have considered the case where the initial data are fixed. The same results hold if we consider the case where they do depend on k, provided we assume the initial data $\left\{\phi^k_{10}, \phi^k_{11}, \phi^k_{20}, \phi^k_{21}, \psi^k_0, \psi^k_1, \eta^k_{10}, \eta^k_{11}, \eta^k_{20}, \eta^k_{21}\right\}$ to be such that the initial energy $E_k(0)$ remains bounded and such that they converge weakly to $\{\phi_{10}, \phi_{11}, \phi_{20}, \phi_{21}, \psi_0, \psi_1, \eta_{10}, \eta_{11}, \eta_{20}, \eta_{21}\}$ in the corresponding spaces.

(iii) It would be interesting to analyze whether the same stabilization results (Theorems 1.3.1, 1.4.5) hold considering the systems (1.1), (1.2), (1.3) and (1.1), (1.1), (1.3) with less damping terms. To eliminate some of these dissipative terms is a difficult task due to the complex nonlinearities involved. In this context, we can mention the works [1]–[3], [6] and [52] which have obtained stability for some hyperbolic systems without damping terms in some of its equations.

(iv) Another interesting and difficult problem is to obtain the same result in Theorem

1.3.1 when the damping mechanisms act in an arbitrary small region of the plate. The difficulty for this case, of course, consists in getting a unique continuation result for the Mindlin-Timoshenko system. On this subject, we mention [12], [13], [18], [23], [61] which have obtained decay rates for the energy of various hyperbolic systems considering both linear and nonlinear localized damping terms.

Capítulo 2

Asymptotic limits and stabilization for the linear one-dimensional Timoshenko system

Asymptotic limits and stabilization for the linear one-dimensional Timoshenko system

F. Ammar-Khodja and P. Queiroz de Souza

Abstract: In this paper we are concerned with a one-dimensional Timoshenko system for beams. We show how a fourth order parabolic system can be obtained as a singular limit of Timoshenko's system when the thickness h and the modulus of elasticity in shear k tends to zero and infinity, respectively. We also show that the Timoshenko system for beam can be uniformly stabilized by one internal damping. The proof is based on multipliers techniques.

2.1 Introduction

The Timoshenko system of equations is widely used and fairly complete mathematical model of beams and elastic systems, so it is very important to investigate the stability of such elastic systems with distributed feedback. More precisely, for a beam of length $L > 0$, a cross section of a beam with a sufficiently smooth boundary Γ, the Timoshenko beam system is described by

$$\begin{cases} \dfrac{\rho h^3}{12}\phi_{tt} - \phi_{xx} + k(\phi + \psi_x) = 0 & \text{in} \quad Q, \\ \rho h \psi_{tt} - k(\phi + \psi_x)_x = 0 & \text{in} \quad Q, \end{cases} \tag{2.1}$$

where $Q = (0, L) \times (0, T)$, the interval $(0, L)$ is the segment occupied by the beam, and T is a given positive time. In system (2.1), subscripts means partial derivatives. The unknowns $\phi = \phi(x, t)$ and $\psi = \psi(x, t)$ represent, respectively, the angle of rotation and the vertical displacement at time t of the cross section located x units from the end-point $x = 0$. The constant $h > 0$ represents the thickness of the beam which, in this model, is considered to be small and uniform with respect to x. The constant ρ is the mass density per unit volume of the beam and the parameter k is the so called modulus of elasticity in shear.

There is a large literature on this model, addressing problems of existence, uniqueness, asymptotic limit and asymptotic behavior in time when some damping effects are considered, as well as some other important properties. Let us mention some known results related to the stabilization of the Timoshenko beam. Kim and Renardy [22] proved the exponential stability of the Timoshenko beam under two boundary controls. Soufyane [52] showed the exponential stability of the uniform Timoshenko beam by using one distributed feedback. Shi et al. [49] considered the case of the uniform Timoshenko beam under two locally distributed feedbacks. Ammar-Khodja et al. [5] studied the stabilization of the uniform Timoshenko beam of memory type. Soufyane and Wehbe [53] proved the uniform stabilization of the Timoshenko beam under one locally distributed feedback. Xu and Yung [57] proved an exponential stability of the uniform Timoshenko beam by two pointwise controls. The first analysis for a Timoshenko beam with variable physical parameters seems to be the one of Taylor [54]. He studied the boundary control of system (2.1) under two feedbacks. Yan et al. [58] studied the case of the nonuniform Timoshenko beam under two locally distributed feedbacks.

In 1988, J.E Lagnese and J.-L. Lions proved in [27] (see also [9]) that, in the linear case, the solution of the Timoshenko model converges, as $k \to \infty$, towards to the solution of the Kirchhoff model (subject to appropriate boundary conditions). More precisely, they proved that, for a beam of length $L > 0$ a cross section of a beam with a sufficiently smooth boundary Γ, the linear Timoshenko system (2.1) converges, as $k \to \infty$, to the Kirchhoff beam model

$$\rho h \psi_{tt} - \frac{\rho h^3}{12}\psi_{xxtt} + \psi_{xxxx} = 0. \tag{2.2}$$

under various boundary conditions. They also proved that in a situations not physically interesting, the behavior of the control as $k \to \infty$. This convergence result for the control but assuming that h is "large"with respect to L - is exactly the opposite of the assumption to obtain the model. In 2010, was proved by Araruna-Braz e Silva-Zuazua (see [8]) that, in the nonlinear one-dimensional case, if a fourth order regularizing term is added in the component ψ, then the nonlinear complete Timoshenko system may be derived as a singular limit of the Von-Kármán system. More precisely, they prove that the nonlinear Timoshenko system

$$\begin{cases} \dfrac{\rho h^3}{12}\phi_{tt} - \phi_{xx} + k\left(\phi + \psi_x\right) = 0 & \text{in} \quad Q, \\ \rho h \psi_{tt} - k\left(\phi + \psi_x\right)_x - \left[\psi_x\left(\eta_x + \dfrac{1}{2}\psi_x^2\right)\right]_x + \dfrac{1}{k}\psi_{xxxx} = 0 & \text{in} \quad Q, \\ \rho h \eta_{tt} - \left(\eta_x + \dfrac{1}{2}\psi_x^2\right)_x = 0 & \text{in} \quad Q, \end{cases} \tag{2.3}$$

approaches, as $k \to \infty$, the Von-Kármán system

$$\begin{cases} \rho h \psi_{tt} - \dfrac{\rho h^3}{12}\psi_{xxtt} + \psi_{xxxx} - \left[\psi_x\left(\eta_x + \dfrac{1}{2}\psi_x^2\right)\right]_x = 0 & \text{in} \quad Q \\ \rho h \eta_{tt} - \left(\eta_x + \dfrac{1}{2}\psi_x^2\right)_x = 0 & \text{in} \quad Q. \end{cases}$$

Furthermore, they proved that, adding appropriate damping term in the three equations of the system (2.3), there is a uniform (with respect to k) rate of decay for the total energy of the solutions of (2.3) as $t \to \infty$.

In this paper we prove that, the asymptotic limit of the Timoshenko system converges to a fourth order parabolic system. Moreover, we investigate the decay properties of solutions of the linear Timoshenko beam under one internal damping and the condition $\frac{12}{\rho h^3} = \frac{k}{\rho h}$.

Let us consider the following damped system:

$$\begin{cases} \dfrac{\rho h^3}{12}\phi_{tt} - \phi_{xx} + k\left(\phi + \psi_x\right) + \alpha \phi_t = 0, \\ \rho h \psi_{tt} - k\left(\phi + \psi_x\right)_x = 0, \end{cases} \tag{2.4}$$

where $\alpha \geq 0$ is a function of the space variable. We consider (2.4) under Dirichlet boundary conditions:

$$\begin{cases} \phi\left(0, \cdot\right) = \phi\left(L, \cdot\right) = 0 & \text{on} \quad (0, T), \\ \psi\left(0, \cdot\right) = \psi\left(L, \cdot\right) = 0 & \text{on} \quad (0, T), \end{cases} \tag{2.5}$$

and the initial data

$$\begin{cases} \{\phi\left(\cdot, 0\right), \psi\left(\cdot, 0\right)\} = \{\phi_0, \psi_0\} & \text{in} \quad (0, L), \\ \{\phi_t\left(\cdot, 0\right), \psi_t\left(\cdot, 0\right)\} = \{\phi_1, \psi_1\} & \text{in} \quad (0, L). \end{cases} \tag{2.6}$$

We can ensure existence and uniqueness of solutions for the system (2.4), (2.5), (2.6) using semigroup theory. More precisely, if we consider the Hilbert space

$$\mathcal{H} = \left[H_0^1(0, L) \times L^2(0, L)\right]^2,$$

equipped with the inner product:

$$\langle Y, \widetilde{Y}\rangle_{\mathcal{H}} = \int_0^L \left[\frac{\rho h^3}{12} v_2 \widetilde{v}_2 + \rho h w_2 \widetilde{w}_2 + k(v_1 + w_{1x})(\widetilde{v}_1 + \widetilde{w}_{1x}) + v_{1x}\widetilde{v}_{1x}\right] dx \tag{2.7}$$

where $Y = (v_1, v_2, w_1, w_2)$, $\widetilde{Y} = (\widetilde{v}_1, \widetilde{v}_2, \widetilde{w}_1, \widetilde{w}_2) \in \mathcal{H}$, then for any $(\phi_0, \phi_1, \psi_0, \psi_1) \in \mathcal{H}$ the problem (2.4), (2.5),(2.6) has a unique weak solution in the class

$$\{\phi, \psi\} \in C^0\left([0, \infty); \left[H_0^1(0, L)\right]^2\right) \cap C^1\left([0, \infty); \left[L^2(0, L)\right]^2\right).$$

Moreover, the energy $E_{h,k}(t)$, give by

$$E_{h,k}(t) = \frac{1}{2}\left[\frac{\rho h^3}{12}\left|\phi_t^{h,k}\right|^2 + \rho h\left|\psi_t^{h,k}\right|^2 + \left|\phi_x^{h,k}\right|^2 + k\left|\phi^{h,k} + \psi_x^{h,k}\right|^2\right] \tag{2.8}$$

obeys the energy dissipation law

$$\frac{d}{dt}E_{h,k}(t) = -\int_0^L \alpha\left(\phi_t^{h,k}\right)^2 \tag{2.9}$$

where $|\cdot|$ denotes the norm in $L^2(0,L)$.

The plan of the paper is the following: In the Section 2.2, we study the asymptotic limit with respect to the parameters k and h proving that in fact the limit of the Timoshenko system converge toward a fourth order parabolic system. In the Section 2.3 the uniform stabilization of (2.1) is proved under the condition $\frac{12}{\rho h^3} = \frac{k}{\rho h}$ on the whole interval, using multipliers techniques and Neves et. al. [41] results.

2.2 Asymptotic limit with respect to the parameters

In this section we analyze the beam model discussed in the introduction in the presence ofan internal localized damping.

2.2.1 The limit system when $(h,k) \to (0,\infty)$

First of all, we will study the asymptotic limit when $(h,k) \to (0,\infty)$.

Theorem 2.2.1 *Let* $\{\phi^{h,k}, \psi^{h,k}\}$ *be the sequence of solutions to* (2.4), (2.5), (2.6) *with data* $\{\phi_0, \phi_1, \psi_0, \psi_1\} \in \mathcal{H}$ *satisfying*

$$\phi_0 + \psi_{0x} = 0 \quad in \quad (0,L). \tag{2.10}$$

Let $\alpha \in L^\infty([0,L])$ *be a nonnegative function. Then, letting* $(h,k) \to (0,\infty)$*, one gets*

$$\{\phi^{h,k}, \psi^{h,k}\} \to \{\phi, \psi\} \ \ weak - * \ in \quad L^\infty\left(0, T, \left[H_0^1(0,L)\right]^2\right)$$

and

$$\sqrt{\alpha}\phi^{h,k} \to \sqrt{\alpha}\phi \quad in \quad C^0\left([0,T], H_0^{1-\theta}(0,L)\right)\ ,\ \theta \in \left(0, \frac{1}{2}\right).$$

where $\{\phi, \psi\}$ *solves the system*

$$\begin{cases} \phi + \psi_x = 0 & in \quad Q, \\ \alpha\phi_t - \phi_{xx} = 0 & in \quad Q, \\ \phi = 0 & in \quad (0,T), \\ \alpha\phi(\cdot,0) = \alpha\phi_0(\cdot) & in \quad (0,L). \end{cases} \tag{2.11}$$

Proof. For each $h,\ k>0$ fixed, let $\{\phi^{h,k},\psi^{h,k}\}$ be the solution of (2.4), (2.5), (2.6) with data in $\mathcal{H}$. According to (2.9) and the assumption on α, we have:

$$E_{h,k}(t)\leq E_{h,k}(0),\ t\in(0,\infty)\,.$$

But, thanks to the assumption (2.10), we first have:

$$\begin{aligned} E_{h,k}(0) &= \frac{1}{2}\left[\frac{\rho h^3}{12}\left|\phi_1\right|^2+\rho h\left|\psi_1\right|^2+\left|\phi_{0,x}\right|^2+k\left|\phi_0+\psi_{0,x}\right|^2\right] \\ &= \frac{1}{2}\left[\frac{\rho h^3}{12}\left|\phi_1\right|^2+\rho h\left|\psi_1\right|^2+\left|\phi_{0,x}\right|^2\right] \\ &\leq C \end{aligned}$$

for, without loss of generality, $h\in(0,1)$. Thus, the following sequences (in h and k)

$$\left\{h^{3/2}\phi_t^{h,k}\right\},\quad \left\{h^{1/2}\psi_t^{h,k}\right\},\quad \left\{\phi_x^{h,k}\right\},\quad \left\{\sqrt{k}\left(\phi^{h,k}+\psi_x^{h,k}\right)\right\}$$

are bounded in $L^\infty(0,T;L^2(0,L))$. We also get that $(\psi_x^{h,k})$ is bounded in $L^\infty(0,T,L^2(0,L))$. In fact,

$$\left|\psi_x^{h,k}\right|\leq\left|\phi^{h,k}+\psi_x^{h,k}\right|+\left|\phi^{h,k}\right|\leq C.$$

Therefore, we conclude that the sequences $(\phi^{h,k})$ and $(\psi^{h,k})$ are bounded in $L^\infty(0,T,H_0^1(0,L))$. Therefore, extracting subsequences (that we still denote by the same index to simplify notations), one gets:

$$\left\{\phi^{h,k},\psi^{h,k}\right\}\to\{\phi,\psi\}\quad \text{weak}-*\text{ in}\quad L^\infty(0,T;H_0^1(0,L)). \tag{2.12}$$

On the other hand, since $\sqrt{k}\left(\phi^{h,k}+\psi_x^{h,k}\right)$ is bounded in $L^\infty(0,T;L^2(0,L))$, we have

$$\phi^{h,k}+\psi_x^{h,k}\to 0,\quad \text{weak}-*\text{ in}\quad L^\infty(0,T;L^2(0,L)). \tag{2.13}$$

It follows from (2.12) and (2.13) that:

$$\phi+\psi_x=0,\quad \text{in}\ \ Q. \tag{2.14}$$

Now, extracting subsequences (that we still denote by the same index to simplify notations), one gets

$$\left\{\sqrt{h^3}\phi_t^{h,k},\sqrt{h}\psi_t^{h,k}\right\}\to\{\mu,\tau\}\quad \text{weak}-*\text{ in}\quad L^\infty\left(0,T,L^2(0,L)\right) \tag{2.15}$$

Moreover, using the convergence (2.15), we obtain

$$\left\{h^3\phi_t^{h,k}, h\psi_t^{h,k}\right\} \to \{0,0\} \text{ weak} - * \text{ in } \quad L^\infty\left(0,T,L^2(0,L)\right). \tag{2.16}$$

For $\{a,b\} \in \left[H_0^1(0,L)\right]^2$ satisfying

$$a + b_x = 0 \tag{2.17}$$

the variational formulation of the Timoshenko system (2.4), (2.5), (2.6) is simply:

$$\frac{\rho h^3}{12}\frac{d}{dt}\left(\phi_t^{h,k}, a\right) + \rho h \frac{d}{dt}\left(\psi_t^{h,k}, b\right) + \left(\phi_x^{h,k}, a_x\right) + \frac{d}{dt}\left(\alpha\phi^{h,k}, a\right) = 0. \tag{2.18}$$

Using convergences (2.15)-(2.16) in equation (2.18), and applying identities (2.10) and (2.17), one obtains the weak formulation of the system (2.11) given in (2.19). To finish the proof, it remains to identify the initial data of the limit system.

Integrating on $(0,T)$ the relation (2.9) gives

$$\int_0^T \alpha\left(\phi_t^{h,k}\right)^2 dt = E_{h,k}(0) - E_{h,k}(t) \le C$$

and the sequence $\left(\sqrt{\alpha}\phi_t^{h,k}\right)$ is bounded in $L^2(Q)$. In view of the convergences (2.12), $\left(\sqrt{\alpha}\phi^{h,k}\right)$ is bounded in $L^\infty(0,T;H_0^1(0,L))$. Thus from the compactness result in Simon [50, Corollary 4, p. 85], $\left(\sqrt{\alpha}\phi^{h,k}\right)$ is in particular compact in $C^0\left([0,T], H_0^{1-\theta}(0,L)\right)$ for any $\theta \in \left(0,\frac{1}{2}\right)$. Thus:

$$\sqrt{\alpha}\phi^{h,k} \to \sqrt{\alpha}\phi \quad \text{in} \quad C^0\left([0,T], H_0^{1-\theta}(0,L)\right)\ ,\ \theta \in \left(0,\frac{1}{2}\right).$$

Then, $\sqrt{\alpha}\phi^{h,k}(\cdot,0) \to \sqrt{\alpha}\phi(\cdot,0)$ in $L^2(0,L)$, which combined with $(2.6)_1$, guarantees that $\sqrt{\alpha}\phi(\cdot,0) = \sqrt{\alpha}\phi_0$ and this ends the proof. ■

Theorem 2.2.2 *For any $\phi_0 \in H_0^1(0,L)$, there exists a unique solution ϕ for the problem* (2.11) *which the variational formulation is given by*

$$\begin{cases} \dfrac{d}{dt}(\alpha\phi, a) + (\phi_x, a_x) = 0, \quad \forall\ a \in H_0^1(0,L), \\ \alpha\phi(\cdot,0) = \alpha\phi_0 \end{cases} \tag{2.19}$$

such that:

$$\phi \in L^\infty(0,T;H_0^1(0,L)), \quad \forall\ T > 0,$$

and

$$\sqrt{\alpha}\phi \in C^0\left([0,\infty); H_0^{1-\theta}(0,L)\right)$$

for any $\theta \in \left(0, \frac{1}{2}\right)$. *Furthermore the energy*

$$E(t) = \frac{1}{2}\int_0^L \alpha\phi^2, \ t \in (0,\infty)$$

in (2.11) *satisfies*

$$E(t) \leq \mathrm{e}^{-\frac{2C}{\alpha_{\max}}t} E\left(0\right), \ t \in (0,\infty),$$

where $C > 0$ *is the Poincaré constant.*

Proof. The existence of solution follows immediately of the previous theorem. Then, we are interested in study, first of all, the uniqueness of solution for the limit problem (2.11).

Let ϕ_1 and ϕ_2 solutions of (2.11) and we define $w = \phi_1 - \phi_2$ such that

$$\begin{cases} \alpha w_t - w_{xx} = 0 & \text{in} \quad Q \\ w = 0 & \text{in} \quad (0,T) \\ w(\cdot,0) = 0 & \text{in} \quad (0,L) \end{cases}$$

By the energy identity of the system

$$\frac{1}{2}\frac{d}{dt}\int_0^L \alpha w^2 + \int_0^L w_x^2 = 0,$$

we infer that

$$\frac{1}{2}\int_0^L \alpha w^2 + \int_0^t\int_0^L w_x^2 = 0, \ t > 0$$

and thus

$$\left\{\begin{array}{lll} \alpha w^2 = 0 \ \Rightarrow \ w\left(\cdot,t\right) \equiv 0 & \text{supp}(\alpha), \\ w_x^2 = 0 \ \Rightarrow \ w = f(t) & (0,L)\times(0,t)\,, \end{array}\right. \quad t > 0.$$

Consequently, we conclude that $w \equiv 0$ in Q and this prove the uniqueness of solution for the problem (2.11).

As in the proof of Theorem 2.2.1, we deduce analogously that

$$\phi \in L^\infty(0,T;H_0^1(0,L)), \quad \forall \ T > 0,$$

and

$$\sqrt{\alpha}\phi \in C^0\left([0,\infty)\,; H_0^{1-\theta}(0,L)\right)$$

for any $\theta \in \left(0, \frac{1}{2}\right)$ are consequence of the Theorem 2.2.1.

Moreover, by the energy of the system (2.11) and using the Poincaré inequality, we can conclude

$$\frac{1}{2}\frac{d}{dt}\int_0^L \alpha\phi^2 = \int_0^L -\phi_x^2 \leq -C\int_0^L \phi^2 \leq -\frac{C}{\alpha_{\max}}\int_0^L \alpha\phi^2$$

which implies

$$\frac{d}{dt}\int_0^L \alpha\phi^2 \leq -\frac{2C}{\alpha_{\max}}\int_0^L \alpha\phi^2.$$

Then,

$$\frac{d}{dt}\left(\mathrm{e}^{\frac{2C}{\alpha_{\max}}t}\int_0^L \alpha\phi^2\right) \leq 0.$$

Integrating the previous equations in the time, one has

$$\int_0^L \alpha\phi^2 \leq \mathrm{e}^{-\frac{2C}{\alpha_{\max}}t}\int_0^L \alpha\phi_0^2$$

and this proof the exponential decay. This ends the proof of the Theorem. ■

Remark 2.2.3 *When $0 < \alpha_0 \leq \alpha(x) \leq \alpha_1$ on $(0, L)$ in the previous theorem, from a result in Ladyženskaja et. al. [24, Theorem 6.1, p. 178] we can ensure the existence and uniqueness of solution to problem (2.29) in $H^1(Q)$ for any initial data $\phi_0 \in H_0^1(0, L)$.*

2.2.2 The limit system as $k \to \infty$ and then $h \to 0$

Now we are interested in studying the asymptotic limit, as $k \to \infty$, of the system (2.4), (2.5), (2.6) when h is fixed.

Theorem 2.2.4 *Let $\alpha \in L^\infty([0, L])$ be a nonnegative function and $\{\phi^k, \psi^k\}$ be the unique solution of (2.4), (2.5), (2.6) with data $\{\phi_0, \phi_1, \psi_0, \psi_1\} \in \mathcal{H}$ satisfying*

$$\phi_0 + \psi_{0x} = 0 \quad in \quad (0, L). \tag{2.20}$$

Then, letting $k \to \infty$ and, later on, $h \to 0$, one gets the following convergence property holds:

$$\{\phi^k, \psi^k\} \to \{-\psi_x, \psi\} \quad weak-* \; in \quad L^\infty\left(0, T, \left[H_0^1(0, L)\right]^2\right)$$

where ψ solves the system (2.11).

Proof. For each $k > 0$ fixed, let $\{\phi^k, \psi^k\}$ be the solution of system (2.4)–(2.6) with $\{\phi_0, \phi_1, \psi_0, \psi_1\} \in \mathcal{H}$. Firstly we will prove that the system (2.4)–(2.6) approaches, as

$k \to \infty$, the kirchhoff system

$$\begin{cases} \rho h \psi_{tt} - \frac{\rho h^3}{12}\psi_{ttxx} + \psi_{xxxx} + (\alpha \psi_{xt})_x = 0 & \text{in} \quad Q \\ \psi = \dfrac{\partial \psi}{\partial \nu} = 0 & \text{in} \quad (0,T) \\ \psi(\cdot,0) = \psi_0(\cdot), \quad \left[\psi_t(\cdot,0) - \dfrac{h^2}{12}\psi_{txx}(\cdot,0)\right] = \psi_1 + \frac{h^2}{12}\phi_{1x} & \text{in} \quad \Omega \end{cases} \tag{2.21}$$

Afterwards, we will show that system (2.21) converges to the system (2.11) as $h \to 0$.

Conforming to definition of the energy of the system (2.4), (2.5), (2.6) in (2.8) satisfying (2.9) we conclude that the following sequences (in k) remains bounded in $L^\infty(0,T,L^2(0,L))$

$$\left(\phi_t^k\right), \quad \left(\psi_t^k\right), \quad \left(\phi_x^k\right), \quad \left(k\left(\phi^k + \phi_x^k\right)\right)$$

and $\left(\psi_t^k\right)$ is bounded in $L^2(0,T,L^2(0,L))$.

Extracting subsequences and keeping the same equation, one gets:

$$\{\phi^k, \psi^k\} \to \{\phi, \psi\} \quad \text{weak} - * \quad L^\infty\left(0, T\left[H_0^1(0,L)\right]^2\right) \tag{2.22}$$

On the other hand, since $\sqrt{k}\left(\phi^{h,k} + \psi_x^{h,k}\right)$ is bounded in $L^\infty(0,T;L^2(0,L))$, we have

$$\phi^{h,k} + \psi_x^{h,k} \to 0 \;\; \text{weak} - * \text{ in } \;\; L^\infty(0,T;L^2(0,L)). \tag{2.23}$$

It follows from (2.22) and (2.23) that:

$$\phi + \psi_x = 0, \quad \text{in} \;\; Q. \tag{2.24}$$

Moreover,

$$\{\phi_t^k, \psi_t^k\} \to \{\phi_t, \psi_t\} \quad \text{in} \quad L^\infty\left(0, T, \left[L^2(0,L)\right]^2\right). \tag{2.25}$$

Using the previous convergences and applying identities (2.17) and (2.24) in (2.18), one obtains the weak formulation of the system (2.21) that can be written in terms os ψ as

$$\rho h \frac{d}{dt}(\psi_t, b) + \frac{\rho h^3}{12}\frac{d}{dt}(\psi_{tx}, b_x) + (\psi_{xx}, b_{xx}) + \frac{d}{dt}(\alpha \psi_x, b_x) = 0. \tag{2.26}$$

To finish the proof, it remains to identify the initial data of the limit system. In view of the convergences (2.22) and (2.25), and classical compactness arguments, one has $\psi^{h,k} \to \psi^h$ in $C^0([0,T],L^2(0,L))$. Then $\psi^{h,k}(\cdot,0) \to \psi^h(\cdot,0)$ in $L^2(0,L)$, which combined with $(2.6)_1$, guarantees that $\psi^h(\cdot,0)=\psi_0(\cdot)$. In order to identify $\psi_t^h(\cdot,0)$, multiply (2.18) by the function $\theta_\delta\in H^1(0,L)$ defined by

$$\begin{cases} -\dfrac{t}{\delta}+1 & \text{if} \quad t\in[0,\delta], \\ 0, & \text{if} \quad t\in(\delta,T], \end{cases}$$

and integrate by parts to obtain

$$-\frac{\rho h^3}{12}(\phi_1,a)+\int_0^T\frac{\rho h^3}{12\delta}\left(\phi_t^k,a\right)-\rho h(\psi_1,b)+\int_0^T\frac{\rho h}{\delta}\left(\psi_t^k,b\right)$$
$$+\int_0^\delta\left(\phi_x^k,a_x\right)\theta_\delta-\alpha(\phi_0,a)+\int_0^T\alpha\left(\phi^k,a\right)=0$$

Passing to the limit as $k\to\infty$ in the last equation and using (2.22) and (2.25), one obtains

$$-\frac{\rho h^3}{12}(\phi_1,a)+\int_0^T\frac{\rho h^3}{12\delta}(\phi_t,a)-\rho h(\psi_1,b)+\int_0^T\frac{\rho h}{\delta}(\psi_t,b)$$
$$+\int_0^\delta(\psi_{xx},b_{xx})\,\theta_\delta+\alpha(\psi_{0x},b_x)+\int_0^T\alpha(\psi_x,b_x)=0$$

On the other hand, multiplying variational formulation of the system (2.32) by θ_δ and integrating by parts in time we get the identity

$$-\rho h(\psi_t(\cdot,0),b)-\frac{\rho h^3}{12}(\psi_{tx}(\cdot,0),b_x)=+\frac{\rho h^3}{12}(\phi_1,b_x)-\rho h(\psi_1,b),$$

for all $b\in H_0^1(0,L)$. In this way, we conclude that $\left(\psi-\dfrac{h^2}{12}\psi_{xx}\right)_t(\cdot,0)=\psi_1+\dfrac{h^2}{12}\phi_{1x}$.

Now, consider the energy of the system (2.21).

$$E_h(t)=\frac{1}{2}\left[\rho h\left|\psi_t^h\right|^2+\frac{\rho h^3}{12}\left|\psi_{tx}^h\right|^2+\left|\psi_{xx}^h\right|^2\right]$$

satisfying

$$\frac{d}{dt}E_h(t)=-\alpha|\psi_{tx}^h|^2.$$

Therefore, the following sequences (in h) remain bounded in $L^\infty\left(0,T,L^2(\Omega)\right)$

$$\left(\sqrt{h}\psi_t^h\right),\quad\left(\sqrt{h^3}\psi_{tx}^h\right),\quad\left(\psi_{xx}^h\right)$$

and the sequence $\left(\psi_t^h\right)$ is bounded in $L^2(0,T,L^2(0,L))$. Extracting subsequences, and keeping the same notation, one gets

$$\left\{\sqrt{h}\psi_t^h,\sqrt{h^3}\psi_{tx}^h\right\}\to\{\mu,\tau\}\quad\text{weak}-*\quad\text{in}\quad L^\infty\left(0,T,\left[L^2(0,L)\right]^2\right)\qquad(2.27)$$

$$\psi^h \to \psi \quad \text{weak} - * \quad \text{in} \quad L^\infty\left(0,T,H_0^2(0,L)\right). \tag{2.28}$$

By the convergence (2.27) we can conclude that $\mu = \tau = 0$ and applying the convergences (2.27) and (2.28) in (2.26), one obtains the weak formulation of the system (2.11) given in (2.19). The initial data are similarly identified. ■

2.2.3 The limit system as $h \to 0$ and then $k \to \infty$

Now, we are interested in studying the asymptotic limit when $h \to 0$ of the sequence $\{\phi^{h,k}, \psi^{h,k}\}$ of solutions to

Theorem 2.2.5 *Let $\alpha \in L^\infty([0,L])$ be a nonnegative function. Let $\{\phi^{h,k}, \psi^{h,k}\}$ the sequence of soluations to (2.4), (2.5), (2.6) with data $\{\phi_0, \phi_1, \psi_0, \psi_1\} \in \mathcal{H}$.*

Then, letting $h \to 0$, one gets

$$\{\phi^{h,k}, \psi^{h,k}\} \to \{\phi^k, \psi^k\} \quad \textit{weak} - * \textit{ in} \quad L^\infty\left(0,T,\left[H_0^1(0,L)\right]^2\right)$$

where $\{\phi^k, \psi^k\}$ solves the system

$$\begin{cases} \alpha\phi_t - \phi_{xx} + k\left(\phi + \psi_x\right) = 0 & \textit{in} \quad Q, \\ \left(\phi + \psi_x\right)_x = 0 & \textit{in} \quad Q, \\ \phi = \psi = 0 & \textit{in} \quad (0,T), \\ \alpha\phi(\cdot,0) = \alpha\phi_0(\cdot) & \textit{in} \quad (0,L). \end{cases} \tag{2.29}$$

Proof. Fix $k > 0$ and for each $h > 0$, let $\{\phi^{h,k}, \psi^{h,k}\}$ be the unique solution of the system (2.4), with initial data $\{\phi_0, \phi_1, \psi_0, \psi_1\} \in \mathcal{H}$. According to (2.9) and the assumption on α,

$$E_{h,k}(t) \leq E_{h,k}(0), \qquad t \in (0,\infty).$$

But, for k fixed, we have

$$E_{h,k}(0) \leq \frac{1}{2}\left[\frac{\rho h^3}{12}|\phi_1|^2 + \rho h|\psi_1|^2 + |\phi_{0,x}|^2 + k|\phi_0|^2 + k|\psi_{0,x}|^2\right] \leq C_k$$

where $C_k > 0$ is a generic constant only depending on k. Thus, we conclude that the following sequences (in h)

$$\left\{\sqrt{h^3}\phi_t^{h,k}\right\}, \quad \left\{\sqrt{h}\psi_t^{h,k}\right\}, \quad \left\{\phi_x^{h,k}\right\}, \quad \left\{\phi^{h,k} + \psi_x^{h,k}\right\}$$

are bounded in $L^\infty(0,T;L^2(0,L))$. We also get that $(\psi_x^{h,k})$ is bounded in $L^\infty(0,T,L^2(0,L))$. In fact,

$$\left|\psi_x^{h,k}\right| \leq \left|\phi^{h,k}+\psi_x^{h,k}\right| + \left|\phi^{h,k}\right| \leq C_k.$$

Therefore, we conclude that the sequences $(\phi^{h,k})$ and $(\psi^{h,k})$ are bounded in $L^\infty(0,T,H_0^1(0,L))$. Extracting subsequences, one gets:

$$\left\{\sqrt{h^3}\phi_t^{h,k}, \sqrt{h}\psi_t^{h,k}\right\} \to \left\{\mu^k,\tau^k\right\} \quad \text{weak} - * \text{ in} \quad L^\infty\left(0,T,L^2(0,L)\right),$$

and

$$\left\{\phi^{h,k},\psi^{h,k}\right\} \to \left\{\phi^k,\psi^k\right\} \quad \text{weak} - * \text{ in} \quad L^\infty\left(0,T,\left[H_0^1(0,L)\right]^2\right).$$

Analogously, one can show that $\mu^k = \tau^k = 0$.

For $\{a,b\} \in [H_0^1(0,L)]^2$, the variational formulation of the Timoshenko system (2.4), (2.5), (2.6) is given by

$$\frac{\rho h^3}{12}\frac{d}{dt}\left(\phi_t^{h,k},a\right) + \rho h\left(\psi_t^{h,k},b\right) + \left(\phi_x^{h,k},a_x\right) + k\left(\phi^{h,k}+\psi_x^{h,k},a+b_x\right) + \frac{d}{dt}\left(\alpha\phi^{h,k},a\right) = 0. \tag{2.30}$$

Applying the previous convergences in (2.30), one obtains the weak formulation of the system (2.29) given by

$$\frac{d}{dt}\left(\alpha\phi^k,a\right) + k\left(\phi^k+\psi_x^k,a+b_x\right) + \left(\phi_x^k,a_x\right) = 0 \tag{2.31}$$

for all $\{a,b\} \in [H_0^1(0,L)]^2$. We conclude similarly that $\alpha\phi^k(\cdot,0) = \alpha\phi_0(\cdot)$ which completes the proof. ■

Proposition 2.2.6 *Let $\alpha \in L^\infty([0,L])$ be a nonnegative function. Let $\{\phi^k,\psi^k\}$ the sequence of solutions to (2.29) with data $\phi_0 \in H_0^1(0,L)$.*

Theorem 2.2.7 *Then, letting $k \to \infty$, one gets*

$$\left\{\phi^k,\psi^k\right\} \to \{\phi,\psi\} \quad \textit{weak in} \quad L^2\left(0,T,H_0^1(0,L)\right)$$

where $\{\phi,\psi\}$ satisfies

$$\begin{cases} \alpha\phi_t - \phi_{xx} = 0 & \textit{in} \quad Q, \\ \phi+\psi_x = 0 & \textit{in} \quad Q, \\ \phi = \psi = 0 & \textit{in} \quad (0,T), \\ \alpha\phi(\cdot,0) = \alpha\phi_0(\cdot) & \textit{in} \quad (0,L). \end{cases} \tag{2.32}$$

Proof. Note that the solution of system (2.29) satisfies

$$\frac{1}{2}\int_0^L \alpha \left|\phi^k(t)\right|^2 + \int_0^t \int_0^L \left(\left|\phi_x^k\right|^2 + k\left|\phi^k + \psi_x^k\right|^2\right) = \frac{1}{2}\int_0^L \alpha \left|\phi_0\right|^2.$$

Therefore, the following sequences (in k) remain uniformly bounded with respect to k in $L^2(Q)$

$$\left(\phi_x^k\right), \quad \left(\sqrt{k}(\phi^k + \psi_x^k)\right)$$

and the sequence $(\sqrt{\alpha}\phi^k)$ is uniformly bounded in $L^\infty(0, T, L^2(0, L))$. Extracting subsequences, and keeping the same notation, one gets

$$\phi^k \to \phi \quad \text{weak in} \quad L^2\left(0, T, H_0^1(0, L)\right)$$

$$\sqrt{k}(\phi^k + \psi_x^k) \to \theta \quad \text{weak in} \quad L^2(Q),$$

Therefore

$$\phi^k \to \phi \quad \text{strong in} \quad L^2(Q)$$

$$(\phi^k + \psi_x^k) \to 0 \quad \text{weak in} \quad L^2(Q),$$

and there exists $\psi \in L^2\left(0, T, H_0^1(0, L)\right)$ such that:

$$\phi^k \to \phi \quad \text{strong in} \quad L^2(Q)$$

$$\psi_x^k \to \psi_x = -\phi \quad \text{weak in} \quad L^2(Q),$$

Analogously, applying the previous convergences in variational formulation

$$\frac{d}{dt}\left(\alpha\phi^k, a\right) + \left(\phi_x^k, a_x\right) = 0$$

with $\{a, b\} \in \left[H_0^1(0, L)\right]^2$ satisfying

$$a + b_x = 0.$$

we obtain the weak formulation of the system (2.11). The initial data for ϕ is similarly identified. ■

The next result ensures the existence of solution for the problem (2.29).

Theorem 2.2.8 *Let $\alpha \in L^\infty([0, L])$ be a nonnegative function. For any $\phi_0 \in H_0^1(0, L)$, there exists a unique solution $\{\phi^k, \psi^k\}$ to the problem* (2.29)

$$\left\{\begin{array}{l} \frac{d}{dt}\left(\alpha\phi^k, a\right) + \left(\phi_x^k, a_x\right) + k\left(\phi^k + \psi_x^k, a + b_x\right) = 0, \quad \forall\ \{a, b\} \in \left[H_0^1(0, L)\right]^2, \\ \alpha\phi^k(\cdot, 0) = \alpha\phi_0 \end{array}\right. \tag{2.33}$$

such that:

$$\{\phi^k, \psi^k\} \in L^\infty\left(0, T; \left[H_0^1(0,L)\right]^2\right), \quad \forall \ T > 0,$$

and

$$\sqrt{\alpha}\phi^k \in C^0\left([0,\infty); H_0^{1-\theta}(0,L)\right)$$

for any $\theta \in \left(0, \frac{1}{2}\right)$. *Furthermore*

$$\int_0^L \alpha \left|\phi^k\right|^2 \leq \mathrm{e}^{-\frac{2C}{\alpha_{\max}}t} \int_0^L \alpha \left|\phi_0\right|^2, \ t \in (0,\infty), \ k > 0.$$

where $C > 0$ *is the Poincaré constant.*

Proof. The existence of solution for the system (2.29) follows as a consequence of the Theorem 2.2.5. Now we are interested to prove the uniqueness of solution. To prove this, let $\{\phi_1, \psi_1\}$ and $\{\phi_2, \psi_2\}$ solutions of (2.29) and we define $\{w, s\} = \{\phi_1 - \phi_2, \psi_1 - \psi_2\}$ such that

$$\begin{cases} \alpha w_t - w_{xx} + k\left(w + s_x\right) = 0 & \text{in} \quad Q \\ k\left(w + s_x\right)_x = 0 & \text{in} \quad Q \\ w = s = 0 & \text{in} \quad (0,T) \\ w(\cdot, 0) = 0 & \text{in} \quad (0,L) \end{cases}$$

By the energy identity of the system (2.29)

$$\frac{1}{2}\frac{d}{dt}\int_0^L \alpha w^2 + \int_0^L \left(w_x^2 + k(w + s_x)^2\right) = 0,$$

consequently,

$$\frac{1}{2}\int_0^L \alpha w^2 + \int_0^t \int_0^L \left(w_x^2 + k(w + s_x)^2\right) = 0, \quad t > 0$$

and thus

$$\begin{cases} \alpha w^2 = 0 & \Rightarrow \quad w(\cdot, t) \equiv 0, & \text{supp}(\alpha), & \\ w_x^2 = 0 & \Rightarrow \ w\left(x,t\right) = f\left(t\right), & (0,L) \times (0,t), & t > 0 \\ w + s_x = 0, & & (0,L) \times (0,t), & \end{cases}$$

hence, $w \equiv 0$ in Q and, from $w + s_x = 0$,it follows that $s\left(x,t\right) = \sigma\left(t\right)$. But $s = 0$ on Σ, therefore, $s = 0$ in Q.

To proof the exponential decay, with respect the component ϕ, for the system (2.29) it's sufficient to see that

$$\frac{1}{2}\frac{d}{dt}\int_0^L \alpha\phi^2 = -\int_0^L \left(\phi_x^2 + k(\phi+\psi_x)^2\right) \leq -\int_0^L \phi_x^2.$$

We can now proceed analogously to the proof of the Theorem 2.2.2 which completes the proof. ∎

Remark 2.2.9 *Note that to study the asymptotic limit in relation to parameters, we get the same limit system, but is important to see that: when we study the asymptotic limit when* $(h,k) \to (0,\infty)$ *in the Theorem 2.2.1 and when we study the limit with* $k \to \infty$ *and later* $k \to \infty$ *In Theorem 2.2.4, we need a restriction in relation to the initial data of type*

$$\phi_0 + \psi_{x0} = 0 \quad in \quad (0, L).$$

On the other hand, to study the asymptotic limit when $h \to 0$ *later on* $k \to \infty$*, in the Theorem 2.2.5 we obtain the system type* (2.11) *without the restricting condition to the initial data (i.e., for any data initial).*

2.3 Asymptotic behavior in time

In this section we are interested in studying the asymptotic behavior for the energy associated to the solution of the (2.4), (2.5), (2.6). To make this, we need to recall some definitions and results in view of the proof of the some our results.

We will introduce the Riemann invariants associated with system (2.4), (2.5), (2.6):

$$\begin{aligned} u_1 &= \phi_t - \sqrt{\tfrac{12}{\rho h^3}}\phi_x, & v_1 &= \phi_t + \sqrt{\tfrac{12}{\rho h^3}}\phi_x, \\ u_2 &= \psi_t - \sqrt{\tfrac{k}{\rho h}}(\phi+\psi_x), & v_2 &= \psi_t + \sqrt{\tfrac{k}{\rho h}}(\phi+\psi_x). \end{aligned}$$

Therefore,

$$u_{1t} = -\sqrt{\frac{12}{\rho h^3}}u_{1x} - 6\sqrt{\frac{k}{\rho h^5}}(v_2 - u_2) - \frac{6}{\rho h^3}\alpha\left(u_1 + v_1\right),$$

$$u_{2t} = -\sqrt{\frac{k}{\rho h}}u_{2x} - \sqrt{\frac{k}{\rho h}}\frac{u_1+v_1}{2}.$$

Analogously, we can deduce that

$$v_{1t} = \sqrt{\frac{12}{\rho h^3}}v_{1x} - 6\sqrt{\frac{k}{\rho h^5}}(v_2 - u_2) - \frac{6}{\rho h^3}\alpha\left(u_1 + v_1\right),$$

and

$$v_{2t} = \sqrt{\frac{k}{\rho h}} v_{2x} - \sqrt{\frac{k}{\rho h}} \frac{u_1 + v_1}{2}.$$

Thus, the problem (2.4), (2.5), (2.6) can be written as

$$\begin{pmatrix} U \\ V \end{pmatrix}_t = M \begin{pmatrix} U \\ V \end{pmatrix}_x + F \begin{pmatrix} U \\ V \end{pmatrix} \quad \text{in } Q, \tag{2.34}$$

where M is the diagonal 4×4 matrix given by

$$M = \operatorname{diag}\left[\ -\sqrt{\tfrac{12}{\rho h^3}},\ \ -\sqrt{\tfrac{k}{\rho h}},\ \ \sqrt{\tfrac{12}{\rho h^3}},\ \ \sqrt{\tfrac{k}{\rho h}}\ \right], \tag{2.35}$$

$$U = \begin{bmatrix} u_1 & u_2 \end{bmatrix}^{\mathrm{T}},\ V = \begin{bmatrix} v_1 & v_2 \end{bmatrix}^{\mathrm{T}}$$

and

$$F = \begin{bmatrix} -\frac{6\alpha}{\rho}\frac{1}{h^3} & 6\sqrt{\frac{k}{\rho h^5}} & -\frac{6\alpha}{\rho}\frac{1}{h^3} & -6\sqrt{\frac{k}{\rho h^5}} \\ -\frac{1}{2}\sqrt{\frac{k}{\rho h}} & 0 & -\frac{1}{2}\sqrt{\frac{k}{\rho h}} & 0 \\ -\frac{6\alpha}{\rho}\frac{1}{h^3} & 6\sqrt{\frac{k}{\rho h^5}} & -\frac{6\alpha}{\rho}\frac{1}{h^3} & -6\sqrt{\frac{k}{\rho h^5}} \\ -\frac{1}{2}\sqrt{\frac{k}{\rho h}} & 0 & -\frac{1}{2}\sqrt{\frac{k}{\rho h}} & 0 \end{bmatrix}.$$

The boundary conditions (2.5) become

$$\begin{cases} U(0,\cdot) = -V(0,\cdot), \\ V(L,\cdot) = -V(L,\cdot), \end{cases} \qquad i = 1,\ 2. \tag{2.36}$$

Let us define, on the new energy space $G = [L^2(0,L)]^4$, the operator

$$\mathcal{B} = M\tfrac{\partial}{\partial x} + F,$$

$$D(\mathcal{B}) = \left\{ \begin{pmatrix} U \\ V \end{pmatrix} \in [H^1(0,L)]^2 \times [H^1(0,L)]^2;\ \text{boundary conditions (2.36)} \right\} \tag{2.37}$$

Is easy to check that solutions of (2.34) in G correspond to solutions of (2.4) in $\mathcal{H}$ and the converse holds true.

Returning to the transformed system (2.34), we note that the entries of M (see (2.35) for its definition) are distinct if, and only if,

$$\frac{12}{\rho h^3} \neq \frac{k}{\rho h} \Leftrightarrow \frac{12}{h^2} \neq k$$

In this case, a result of Neves, Ribeiro and Lopes [41, Theorem A and B] asserts that

$$r_e(\mathrm{e}^{\mathcal{B}t}) = r_e(\mathrm{e}^{\mathcal{B}_0 t}) = \mathrm{e}^{\alpha_0 t},\ t > 0,$$

where

$$\mathcal{B}_0 = M\frac{\partial}{\partial x} + F_0, \qquad D(\mathcal{B}_0) = D(\mathcal{B}),$$

$$F_0 = \operatorname{diag}\left[-\frac{6}{\rho h^3}\alpha,\ \ 0,\ \ -\frac{6}{\rho h^3}\alpha,\ \ 0\right]$$

and $\alpha_0 = s(\mathcal{B}_0) = \sup\{\text{Re}\lambda; \lambda \in \sigma(\mathcal{B}_0)\}$. To compute α_0, we set $U = (u_1, u_2)$, $V = (v_1, v_2)$ and solve the problem:

$$\mathcal{B}_0 \begin{pmatrix} U \\ V \end{pmatrix} = \lambda \begin{pmatrix} U \\ V \end{pmatrix}, \quad \begin{pmatrix} U \\ V \end{pmatrix} \in D(\mathcal{B}_0).$$

A straightforward computation (it is a diagonal differential system) leads us to the following equations for the eigenvalues of $\mathcal{B}_0$

$$\mathrm{e}^{2\left(\lambda + \frac{6}{\rho h^3}\alpha\right)\sqrt{\frac{\rho h^3}{12}}L} = 1 \quad \text{or} \quad \mathrm{e}^{2\lambda\sqrt{\frac{\rho h}{k}}L} = 1.$$

Therefore, the eigenvalues of $\mathcal{B}_0$ are:

$$\lambda_n^1(k,h) = -\frac{6}{\rho h^3}\alpha - \frac{in\pi}{L}\sqrt{\frac{12}{\rho h^3}}, \qquad \lambda_n^2(k,h) = -\frac{in\pi}{L}\sqrt{\frac{k}{\rho h}}$$

and it follows that

$$s(\mathcal{B}_0) = 0,$$

which implies that $\omega(\mathcal{A}) = \omega(\mathcal{B}) = 0$. Therefore, we conclude that not exist a asymptotic stability of the operator concerned.

On the other hand, assume that the coefficients of the system (2.4) satisfy the condition

$$\frac{12}{\rho h^3} = \frac{k}{\rho h}. \tag{2.38}$$

So, if we consider that the condition (2.38) is true, one has

$$\lambda \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} -\sqrt{\sigma} & 0 \\ 0 & -\sqrt{\sigma} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}_x + \begin{pmatrix} -\frac{\alpha\sigma}{2} & \mu \\ -\frac{\sqrt{\sigma}}{2} & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \tag{2.39}$$

where $\sigma = \frac{12}{\rho h^3}$ and $\mu = \frac{144}{\rho h^5}\frac{1}{2\sqrt{\sigma}}$. The characteristic polynomial is

$$\det(P - XI) = X^2 + \frac{1}{\sqrt{\sigma}}\left(\frac{\alpha\sigma}{2} + 2\lambda\right)X + \frac{\lambda}{\sigma}\left(\frac{\alpha\sigma}{2} + \lambda\right) + \frac{\mu}{2\sqrt{\sigma}}.$$

Therefore, the eigenvalues associated with the problem (2.39) are

$$X_1 = \frac{1}{2}\left[-\frac{1}{\sqrt{\sigma}}\left(\frac{\alpha\sigma}{2} + 2\lambda\right) - \sqrt{\Delta}\right] \quad \text{and} \quad X_2 = \frac{1}{2}\left[-\frac{1}{\sqrt{\sigma}}\left(\frac{\alpha\sigma}{2} + 2\lambda\right) + \sqrt{\Delta}\right]$$

where the discriminant is

$$\Delta = \frac{1}{\sigma}\left(\frac{\alpha\sigma}{2}\right)^2 - \frac{2\mu}{\sqrt{\sigma}}.$$

After some computations, we obtain the eigenvectors associated the eigenvalues, so we have

$$V_1 = \begin{pmatrix} -\frac{\alpha\sqrt{\sigma}}{2} - \sqrt{\left(\frac{\alpha\sqrt{\sigma}}{2}\right)^2 - \frac{2\mu}{\sqrt{\sigma}}} \\ 1 \end{pmatrix}$$

and

$$V_2 = \begin{pmatrix} -\frac{\alpha\sqrt{\sigma}}{2} + \sqrt{\left(\frac{\alpha\sqrt{\sigma}}{2}\right)^2 - \frac{2\mu}{\sqrt{\sigma}}} \\ 1 \end{pmatrix}.$$

Thus, we conclude that the eigenfunctions correspond are:

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = a\mathrm{e}^{X_1 x} V_1 + b\mathrm{e}^{X_2 x} V_2.$$

Analogously, we prove

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = a'\mathrm{e}^{X_1 x} V_1 + b'\mathrm{e}^{X_2 x} V_2.$$

When:

- $x = 0$, we have

$$u_i(0,\cdot) = -v_i(0,\cdot) \Rightarrow u_i(0,\cdot) + v_i(0,\cdot) = 0.$$

Then

$$(a + a')V_1 + (b + b')V_2 = 0 \Rightarrow a = -a' \text{ and } b = -b'.$$

- $x = L$,

$$u_i(L,\cdot) = -v_i(L,\cdot) \Rightarrow u_i(L,\cdot) + v_i(L,\cdot) = 0.$$

It follows

$$(a\mathrm{e}^{X_1 L} + a'\mathrm{e}^{-X_1 L})V_1 + (b\mathrm{e}^{X_2 L} + b'\mathrm{e}^{-X_2 L})V_2 = 0$$
$$\Rightarrow (a\mathrm{e}^{X_1 L} - a\mathrm{e}^{-X_1 L})V_1 + (b\mathrm{e}^{X_2 L} - b\mathrm{e}^{-X_2 L})V_2 = 0$$
$$\Rightarrow a(\mathrm{e}^{X_1 L} - \mathrm{e}^{-X_1 L})V_1 + b(\mathrm{e}^{X_2 L} - \mathrm{e}^{-X_2 L}) = 0.$$

Consequently,

$$e^{2X_1L} = 1 \quad \text{or} \quad e^{2X_2L} = 1,$$

so

$$\begin{cases} X_1 = \frac{im\pi}{L}, & m \in \mathbb{Z} \\ X_2 = \frac{in\pi}{L}, & n \in \mathbb{Z}. \end{cases}$$

Thus,

$$-\frac{1}{\sqrt{\sigma}}\left(\frac{\alpha\sigma}{2} + 2\lambda\right) - \sqrt{\Delta} = \frac{2in\pi}{L}, \quad n \in \mathbb{Z} \tag{2.40}$$

and

$$-\frac{1}{\sqrt{\sigma}}\left(\frac{\alpha\sigma}{2} + 2\lambda\right) + \sqrt{\Delta} = \frac{2in\pi}{L}, \quad n \in \mathbb{Z}. \tag{2.41}$$

Analyzing the equations (2.40) and (2.41), we found two families of eigenvalues of the problem (2.4). In fact, first let's look at the equation (2.40),

$$-\frac{1}{\sqrt{\sigma}}\left(\frac{\alpha\sigma}{2} + 2\lambda\right) + \sqrt{\Delta} = \frac{2im\pi}{L}.$$

Consequently,

$$\lambda_h^1 = -\frac{\alpha\sigma}{4} + \frac{\sqrt{\sigma}}{2}\sqrt{\Delta} - \frac{im\pi\sqrt{\sigma}}{L}.$$

Analogously, we obtain

$$\lambda_h^2 = -\frac{\alpha\sigma}{4} - \frac{\sqrt{\sigma}}{2}\sqrt{\Delta} - \frac{in\pi\sqrt{\sigma}}{L}.$$

Remark 2.3.1 *At this moment we can see that the real part is negative. In fact, for h sufficiently small:*

$$\sqrt{\frac{\alpha^2\sigma}{4} - \frac{6}{h^2}} > \frac{\alpha\sqrt{\sigma}}{2} \Leftrightarrow -\frac{\alpha\sigma}{4} + \frac{\sqrt{\sigma}}{2}\sqrt{\frac{\alpha^2\sigma}{4} - \frac{6}{h^2}} > 0$$

Then, we conclude that

$$\mathrm{Re}(\lambda_h^2) = -\frac{\alpha\sigma}{4} - \frac{\sqrt{\sigma}}{2}\sqrt{\Delta} < \mathrm{Re}(\lambda_h^1) = -\frac{\alpha\sigma}{4} + \frac{\sqrt{\sigma}}{2}\sqrt{\Delta} < 0.$$

Since $\mathrm{Re}(\lambda_h^1)$ and $\mathrm{Re}(\lambda_h^2)$ are negatives, we conclude that the essential type of semigroup is strictly negative, i.e.,

$$\omega_e = \mathrm{Re}(\lambda_h^1) < 0.$$

From this we conclude that exist a asymptotic stability of the operator concerned. Finally, it is sufficient to proof that, for this case, the stabilization is uniform.

2.3.1 Uniform stability

In this section we will study the uniformy stability of the system (2.4) when we consider all the cases with respect to the coefficients of the system.

First of all, is important see that the system (2.4), (2.5), (2.6) can be put in the abstract form

$$\begin{cases} Y_t = \mathcal{A}Y, \\ Y(0) = Y_0, \end{cases} \tag{2.42}$$

where

$$\mathcal{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{12}{\rho h^3}\left(\frac{\partial^2}{\partial x^2} - k\right) & -\frac{12}{\rho h^3}\alpha & -\frac{12}{\rho h^3}\left(k\frac{\partial}{\partial x}\right) & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k}{\rho h}\frac{\partial}{\partial x} & 0 & \frac{k}{\rho h}\frac{\partial^2}{\partial x^2} & 0 \end{bmatrix}, \quad Y = \begin{bmatrix} \phi \\ \phi' \\ \psi \\ \psi' \end{bmatrix},$$

and

$$Y_0 = \begin{bmatrix} \phi_0 & \phi_1 & \psi_0 & \psi_1 \end{bmatrix}^T$$

It is easy see that the operator $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$ with domain

$$D(\mathcal{A}) = \{Y \in \mathcal{H}; \mathcal{A}Y \in \mathcal{H}\},$$

is the infinitesimal generator of a semigroup of operators in $\mathcal{H}$. Indeed,

$$\langle -\mathcal{A}Y, Y\rangle_{\mathcal{H}} = -\int_0^L \phi_x\phi_{xt}dx - \int_0^L \phi_{xx}\phi_t dx + k\int_0^L (\phi + \psi_x)\,\phi_t dxdx$$
$$-k\int_0^L (\phi_t + \psi_{xt})(\phi + \psi_x) - k\int_0^L (\phi + \psi_x)_x\,\psi_t dx + \int_0^L \alpha\phi_t^2 \geq 0.$$

Therefore, we proof that $\mathcal{A}$ is maximal dissipative and so, by the Lumer-Philips theorem, it is the infinitesimal generator of a C_0-semigroup $(e^{\mathcal{A}t})$.

We are now ready to state the following theorem:

Theorem 2.3.2 *Assume that $\alpha = \alpha(x)$ is a positive $C^1([0, L])$ function with*

$$\alpha(x) \geq \alpha_0 > 0 \qquad on \qquad (0, L).$$

If

$$\frac{12}{\rho h^3} = \frac{k}{\rho h} \quad on \quad (0, L)$$

then there exist positive constants C_1 and C_2 such that

$$E_h(t) \leq C_1 E_h(0) \exp^{-\frac{C_2}{h^3}t}.$$

This result is already known (see [52] for instance) but we are interested by the dependence of the decay rate of the energy associated with this system with respect to the parameter h. Without loss of generality we will assume $h \in (0;1)$. Knowing that the condition $\frac{12}{\rho h^3} = \frac{k}{h}$ is satisfied, we obtain $k = \frac{12}{h^2}$ and, for simplicity, we will "normalize"all the other parameters of the system. More precisely, we will study the following system

$$\begin{cases} h^3\phi_{tt} - \phi_{xx} + \dfrac{1}{h^2}(\phi + \psi_x) + \alpha\phi_t = 0, \\ h\psi_{tt} - \dfrac{1}{h^2}(\phi + \psi_x)_x = 0, \end{cases} \tag{2.43}$$

under boundary conditions (2.5) and initial data (2.6).

For this system, the energy $E_h(t)$ is given by (2.8) and obeys the dissipation law (2.9)

Proof of Theorem 2.3.2. The main idea of the proof is based on the construction of a Lyapunov functional $\mathcal{L}$ that is a function which has the form

$$\mathcal{L}(t) = V(Y(t)), \quad t \in \mathbb{R}^+$$

where $Y = (\phi, \phi_t, \psi, \psi_t)$ is a solution of (2.43), (2.5), (2.6) and V a functional from $\mathcal{H}$ into $\mathbb{R}^+$, such that $\mathcal{L}$ satisfies the following inequalities:

i) There exist two positive constants $C_1, \;\; C_2$ such that

$$C_1\|Y\|_{\mathcal{H}}^2 \leq V(Y) \leq C_2\|Y\|_{\mathcal{H}}^2, \quad Y \in \mathcal{H}$$

ii) There exists a positive constante C_3 such that

$$\frac{d}{dt}\mathcal{L}(t) \leq -C_3\|Y(t)\|_{\mathcal{H}}^2,$$

for any solution of (2.4), (2.5), (2.6).

To construct this functional we will use the multiplyer technique.

Step 1: We multiply the first equation of (2.43) by $-\phi$ and integrate the resulting equation over $[0, L]$

$$\begin{aligned} \frac{d}{dt}\left(\int_0^L -h^3\phi_t\phi\right) &= \int_0^L -\phi_{xx}\phi + \frac{1}{h^2}(\phi+\psi_x)\phi + \alpha\phi_t\phi - h^3\phi_t^2 \\ &= \int_0^L \phi_x^2 + \frac{1}{h^2}(\phi+\psi_x)\phi + \alpha\phi_t\phi - h^3\phi_t^2 \end{aligned}$$

and multiply the second equation of (2.43) by $-\psi$ and integrate over $[0,L]$

$$\begin{aligned}\frac{d}{dt}\left(\int_0^L -h\psi_t\psi\right) &= \int_0^L -\frac{1}{h^2}(\phi+\psi_x)_x\psi - h\psi_t^2\\ &= \int_0^L \frac{1}{h^2}(\phi+\psi_x)\psi_x - h\psi_t^2.\end{aligned}$$

Hence

$$\frac{d}{dt}\left(\int_0^L -\left(h^3\phi_t\phi + h\psi_t\psi\right)\right) = \int_0^L -h^3\phi_t^2 + \phi_x^2 + \frac{1}{h^2}(\phi+\psi_x)^2 + \alpha\phi_t\phi - h\psi_t^2,$$

denoting

$$I_1 = -\int_0^L \left[h^3\phi_t\phi + h\psi_t\psi\right]$$

and using the Young and Poincaré inequalities, we conclude that

$$\begin{aligned}\frac{d}{dt}I_1 &\leq \int_0^L -h^3\phi_t^2 + \phi_x^2 + \frac{1}{h^2}(\phi+\psi_x)^2 + \varepsilon\phi_t^2 + C_\varepsilon\phi^2 - h\psi_t^2\\ &\leq \int_0^L -h^3\phi_t^2 + \phi_x^2 + \frac{1}{h^2}(\phi+\psi_x)^2 + \varepsilon\phi_t^2 + C_\varepsilon\phi_x^2 - h\psi_t^2 \qquad (2.44)\\ &= \int_0^L -\left(h^3-\varepsilon\right)\phi_t^2 - h\psi_t^2 + C\phi_x^2 + \frac{1}{h^2}(\phi+\psi_x)^2.\end{aligned}$$

Step 2: Note that we have a problem in the two last terms of the inequality (2.44). Then, to solve this problem, we multiply the first equation of (2.43) by ϕ and integrate over $[0,L]$,

$$\frac{d}{dt}\left(\int_0^L h^3\phi_t\phi\right) = \int_0^L -\phi_x^2 - \frac{1}{h^2}(\phi+\psi_x)\phi - \alpha\phi_t\phi + h^3\phi_t^2$$

On other hand, let's consider w the solution to

$$-w_{xx} = \phi_x, \quad w(0) = w(L) = 0 \qquad (2.45)$$

i.e.

$$w(x) = -\int_0^x \phi(y)dx + \frac{x}{L}\int_0^L \phi(y)dx.$$

then multiply the second equation of (2.43) by w and integrate over $[0,L]$, one gets

$$\frac{d}{dt}\int_0^L h\psi_t w dx = \int_0^L \frac{1}{h^2}(\phi+\psi_x)_x w + h\psi_t w_t.$$

Note that

$$\int_0^L \frac{1}{h^2}\left(\phi+\psi_x\right)_x w = \int_0^L \frac{1}{h^2}w_x^2 - \frac{1}{h^2}\psi\phi_x.$$

Therefore, we conclude that

$$\begin{aligned}\frac{d}{dt}\left(\int_0^L h^3\phi_t\phi + h\psi_t w\right) &= \int_0^L h^3\phi_t^2 - \phi_x^2 - \frac{1}{h^2}(\phi+\psi_x)\phi - \alpha\phi_t\phi + h\psi_t w_t + \frac{1}{h^2}(\phi+\psi_x)_x w \\ &= \int_0^L h^3\phi_t^2 - \phi_x^2 - \frac{1}{h^2}\phi^2 - \frac{1}{h^2}\psi_x\phi - \alpha\phi_t\phi + h\psi_t w_t + \frac{1}{h^2}w_x^2 - \frac{1}{h^2}\psi\phi_x \\ &= \int_0^L h^3\phi_t^2 - \phi_x^2 - \frac{1}{h^2}\phi^2 - \alpha\phi_t\phi + h\psi_t w_t + \frac{1}{h^2}w_x^2\end{aligned}$$

In this moment we need observe that by the equality (2.66) we obtain

$$\int_0^L \frac{1}{h^2}w_x^2 - \frac{1}{h^2}\phi^2 = -\frac{1}{Lh^2}\left(\int_0^L \phi\right)^2 < 0.$$

Then, using Young's inequality

$$\begin{aligned}\frac{d}{dt}\left(\int_0^L h^3\phi_t\phi + h\psi_t w\right) &= \int_0^L h^3\phi_t^2 - \phi_x^2 - \alpha\phi_t\phi + h\psi_t w_t \\ &\le \int_0^L \left(h^3 + C_\varepsilon\alpha\right)\phi_t^2 - (1-\varepsilon)\phi_x^2 + \varepsilon h\psi_t^2 + C_\varepsilon w_t^2 \quad (2.46)\end{aligned}$$

But thanks the definition of w, it is easy to check that

$$\int_0^L w^2 \le C\int_0^L \phi^2$$

Since, the equation defining w can be differentiated with respect to t, we have

$$\int_0^L w_t^2 \le C\int_0^L \phi_t^2$$

Consequently,

$$\frac{d}{dt}I_2 \le \int_0^L \left(h^3 + C_\varepsilon\alpha + C\right)\phi_t^2 - (1-\varepsilon)\phi_x^2 + \varepsilon h\psi_t^2 \tag{2.47}$$

where

$$I_2 = \int_0^L h^3\phi_t\phi + h\psi_t w.$$

Step 3: Now we are interested in obtain one estimative for the term $(\phi+\psi_x)$. And to do this we multiply the first equation by $\frac{1}{h^2}(\phi+\psi_x)$ and integrate the resulting equation over $[0,L]$, we get

$$\frac{d}{dt}\left(\int_0^L h\phi_t(\phi+\psi_x)\right) = \int_0^L \frac{1}{h^2}\phi_{xx}(\phi+\psi_x) - \frac{1}{h^4}(\phi+\psi_x)^2 - \frac{1}{h^2}\alpha\phi_t(\phi+\psi_x) + h\phi_t(\phi+\psi_x)_t$$

$$= \frac{1}{h^2}\left[\phi_x\psi_x\right]\Big|_0^L + \int_0^L -\frac{1}{h^2}\phi_x(\phi+\psi_x)_x - \frac{1}{h^4}(\phi+\psi_x)^2 - \frac{1}{h^2}\alpha\phi_t(\phi+\psi_x) + h\phi_t(\phi+\psi_x)_t$$

and multiply the second equation by ϕ_x and integrate over $[0, L]$

$$\frac{d}{dt}\left(\int_0^L h\psi_t\phi_x\right) = \int_0^L \frac{1}{h^2}(\phi+\psi_x)_x\phi_x + h\psi_t\phi_{xt}$$

Thus,

$$\begin{aligned}&\frac{d}{dt}\left(\int_0^L h\phi_t(\phi+\psi_x) + h\psi_t\phi_x\right)\\ &= \frac{1}{h^2}\left[\phi_x\psi_x\right]\Big|_0^L + \int_0^L -\frac{1}{h^2}\phi_x(\phi+\psi_x)_x - \frac{1}{h^4}(\phi+\psi_x)^2 - \frac{1}{h^2}\alpha\phi_t(\phi+\psi_x)\\ &\quad + h\phi_t\left(\phi+\psi_x\right)_t + \frac{1}{h^2}(\phi+\psi_x)_x\phi_x + h\psi_t\phi_{xt}\end{aligned}$$

which implies

$$\begin{aligned}\frac{d}{dt}I_3 &= \frac{1}{h^2}\left[\phi_x\psi_x\right]\Big|_0^L + \int_0^L -\frac{1}{h^4}(\phi+\psi_x)^2 - \frac{1}{h^2}\alpha\phi_t(\phi+\psi_x) + h\phi_t^2\\ &\leq \frac{1}{h^2}\left[\phi_x\psi_x\right]\Big|_0^L + \int_0^L \left[-\left(1-\varepsilon\right)\frac{1}{h^4}(\phi+\psi_x)^2 + \left(h+C_\varepsilon\right)\phi_t^2\right], \qquad (2.48)\end{aligned}$$

where

$$I_3 = \int_0^L h\phi_t(\phi+\psi_x) + h\psi_t\phi_x.$$

Step 4: In order to deal with the boundary terms appearing in (2.48), we consider $b(0) > 0$, $b(L) < 0$ $\left(\text{for example, } b(x) = \frac{L-2x}{2}\right)$. We multiply the first equation of (2.43) by $b\phi_x$ and integrate the resulting equation over $[0, L]$. This gives

$$\begin{aligned}\frac{d}{dt}\left(\int_0^L h^3\phi_t b\phi_x\right) &= \int_0^L h^3\phi_t b\phi_{xt} + \phi_{xx}b\phi_x - \frac{1}{h^2}\left(\phi+\psi_x\right)b\phi_x - \alpha\phi_t b\phi_x\\ &= \int_0^L \frac{h^3}{2}b(\phi_t^2)_x + \frac{1}{2}b(\phi_x^2)_x - \frac{1}{h^2}(\phi+\psi_x)b\phi_x - \alpha\phi_t b\phi_x \qquad (2.49)\\ &= \int_0^L -\frac{h^3}{2}b_x(\phi_t^2) - \frac{1}{2}b_x(\phi_x^2) - \frac{1}{h^2}(\phi+\psi_x)b\phi_x - \alpha\phi_t b\phi_x + \left[\frac{b}{2}\phi_x^2\right]\Big|_0^L\end{aligned}$$

Now we multiply the second equation of (2.43) by $b\psi_x$ and integrate the resulting equation over $[0, L]$, we obtain

$$\begin{aligned}\frac{d}{dt}\left(\int_0^L h\psi_t b\psi_x\right) &= \int_0^L h\psi_t b\psi_{xt} + \frac{1}{h^2}\left(\phi+\psi_x\right)_x b(\phi+\psi_x) - \frac{1}{h^2}\left(\phi+\psi_x\right)_x b\phi\\ &= \int_0^L \frac{h}{2}b(\psi_t^2)_x + \frac{1}{2h^2}b\left((\phi+\psi_x)^2\right)_x - \frac{1}{h^2}\left(\phi+\psi_x\right)_x b\phi \qquad (2.50)\\ &= \int_0^L -\frac{h}{2}b_x\psi_t^2 - \frac{1}{2h^2}b_x(\phi+\psi_x)^2 + \frac{1}{h^2}\left(\phi+\psi_x\right)\left(b_x\phi + b\phi_x\right) + \left[\frac{b}{2}\psi_x^2\right]\Big|_0^L\end{aligned}$$

Adding the equations (2.48), (2.49), (2.50), and considering N_1 large number and N_2 sufficiently small, we deduce

$$\begin{aligned}\frac{d}{dt}\left[N_1 I_3 + \int_0^L \left(h^3\phi_t b\phi_x + N_2 h\psi_t b\psi_x\right)\right] \\ = \int_0^L -N_1(1-\varepsilon)\frac{1}{h^4}\left(\phi+\psi_x\right)^2 + N_1\left(h+C_\varepsilon\right)\phi_t^2 - \frac{h^3}{2}b_x\phi_t^2 - \frac{b_x}{2}\phi_x^2 - \frac{1}{h^2}\left(\phi+\psi_x\right)b\phi_x - \alpha\phi_t b\phi_x \\ -N_2\frac{h}{2}b_x\psi_t^2 - N_2\frac{b_x}{2h^2}\left(\phi+\psi_x\right)^2 + N_2\frac{1}{h^2}\left(\phi+\psi_x\right)\left(b_x\phi+b\phi_x\right)\end{aligned}$$

Using the Young inequality,

$$\begin{aligned}\frac{d}{dt}\left[N_1 I_3 - \int_0^L \left(h^3\phi_t b\phi_x + N_2 h\psi_t b\psi_x\right)\right] \\ \leq \int_0^L -N_1(1-\varepsilon)\frac{1}{h^4}\left(\phi+\psi_x\right)^2 + N_1\left(h+C_\varepsilon\right)\phi_t^2 - \frac{h^3}{2}b_x\phi_t^2 - \frac{b_x}{2}\phi_x^2 + \frac{\varepsilon}{h^4}b\left(\phi+\psi_x\right)^2 + C_\varepsilon b\phi_x^2 \\ +bC_\varepsilon\phi_t^2 + \varepsilon b\phi_x^2 - N_2\frac{h}{2}b_x\psi_t^2 - N_2\frac{b_x}{2h^2}\left(\phi+\psi_x\right)^2 + N_2\frac{\varepsilon}{h^4}\left(\phi+\psi_x\right)^2 + N_2 C_\varepsilon b_x\phi^2 + N_2 C_\varepsilon b\phi_x^2\end{aligned}$$

Then we can conclude

$$\begin{aligned}\frac{d}{dt}I_4 \leq & \int_0^L \left[N_1(h+C_\varepsilon) + bC_\varepsilon - \frac{h^3}{2}b_x\right]\phi_t^2 - \left[N_1(1-\varepsilon) - \varepsilon b + N_2\frac{b_x h^2}{2} - N_2\varepsilon b_x\right]\frac{1}{h^4}\left(\phi+\psi_x\right)^2 \\ & + \int_0^L \left[-\frac{b_x}{2} + C_\varepsilon b + \varepsilon b + N_2 C_\varepsilon b_x + N_2 C_\varepsilon b\right]\phi_x^2 - N_2\frac{h}{2}b_x\psi_t^2 \qquad (2.51)\end{aligned}$$

where

$$I_4 = N_1 I_3 + \int_0^L \left(h^3\phi_t b\phi_x + N_2 h\psi_t b\psi_x\right)$$

Finally, we consider

$$I_5 = I_4 + 2NI_1$$

choosing N large number, then

$$\begin{aligned}\frac{d}{dt}I_5 \leq & \int_0^L \left[N_1(h+C_\varepsilon) + bC_\varepsilon - \frac{h^3}{2}b_x - 2N(h^3-\varepsilon)\right]\phi_t^2 \\ & - \left(N_1(1-\varepsilon) - \varepsilon b + N_2\frac{b_x h^2}{2} - N_2\varepsilon b_x - 2Nh^2\right)\frac{1}{h^4}\left(\phi+\psi_x\right)^2 \qquad (2.52) \\ & - \left(\frac{b_x}{2} - C_\varepsilon b - \varepsilon b - N_2 C_\varepsilon b_x - N_2 C_\varepsilon b - 2NC\right)\phi_x^2 - \left(2N + N_2\frac{b_x}{2}\right)h\psi_t^2\end{aligned}$$

The Lyapunov functional is now defined by

$$\mathcal{L}_h(t) = \widetilde{N}E_h(t) + I_2 + \mu_1 I_5$$

choosing μ_1 sufficiently small and N sufficiently large.

Differentiating the functional $\mathcal{L}$ and using the equations in (2.43), one obtains

$$\begin{aligned}\frac{d}{dt}\mathcal{L}_h(t) \leq & \int_0^L -\left(\widetilde{N}\alpha - \frac{C_\varepsilon}{h^3}\alpha - 1 - \frac{C}{h^3} - \mu_1\frac{N_1}{h^3}(h+C_\varepsilon) - \mu_1\frac{bC_\varepsilon}{h^3} + \mu_1\frac{b_x}{2} + \mu_1\frac{2N}{h^3}(h^3-\varepsilon)\right)h^3\phi_t^2 \\ & -\mu_1\left(N_1(1-\varepsilon) - \varepsilon b + N_2\frac{b_x h^2}{2} - N_2\varepsilon b_x - 2Nh^2\right)\frac{1}{h^4}(\phi+\psi_x)^2 - \left(\mu_1 2N + \mu_1 N_2\frac{b_x}{2} - \varepsilon\right)h\psi_t^2 \\ & -\left[(1-\varepsilon) + \mu_1\left(\frac{b_x}{2} - C_\varepsilon b - \varepsilon b - N_2 C_\varepsilon b_x - N_2 C_\varepsilon b - 2NC\right)\right]\phi_x^2\end{aligned} \tag{2.53}$$

Consequently, by estimating the constants, one gets (choosing ε sufficiently small, $N=\frac{C}{h^2}$ and $\widetilde{N}=\frac{N}{h^3}$)

$$\frac{d}{dt}\mathcal{L}_h(t) \leq -\frac{C}{h^5}E_h(t)$$

where C is a positive constant that not has a dependence in h.

On other hand, for $h\in(0,1)$, we can conclude that

$$|\mathcal{L}_h(t) - \widetilde{N}E_h(t)| \leq \frac{C}{h^2}E_h(t)$$

where C is a positive constant that not has a dependence in h.

Therefore,

$$\left(\widetilde{N} - \frac{C}{h^2}\right)E_h(t) \leq \mathcal{L}(t) \leq \left(\widetilde{N} + \frac{C}{h^2}\right)E_h(t) \tag{2.54}$$

where the constant $C>0$ not has a dependence in h.

Considering the derivative of the expression $\mathcal{L}_h(t)$, and observing (2.54), we conclude

$$\frac{d}{dt}\mathcal{L}_h(t) \leq -\frac{C}{h^3}\mathcal{L}_h(t)$$

which implies

$$\frac{d}{dt}\left(\mathcal{L}_h(t)e^{\frac{C}{h^3}t}\right) \leq 0.$$

Integrating in time and using (2.54), we get

$$CE_h(t) \leq \mathcal{L}_h \leq \mathcal{L}_h(0)e^{-\frac{C}{h^3}t} \leq CE_h(0)e^{-\frac{C}{h^3}t},$$

where C is a positive constant that not has dependence in h. Then we get the uniform stability and this finishes the proof. ■

As in the above theorems, using results due to Soufyane [52] and Neves, Ribeiro and Lopes [41], the Timoshenko system can be uniformly stable or no. In others words the uniform stabilization of (2.4), (2.5), (2.6) is assured under the condition

(2.38) on the interval (0,L). So, if (2.38) not is satisfied $\left(\text{i.e., } \frac{12}{\rho h^3} \neq \frac{k}{\rho h}\right)$, the system is not uniformly stable. However, we can show that the linear Timoshenko system is polynomial stable with the condition $\frac{12}{\rho h^3} \neq \frac{k}{\rho h}$. More precisely, we will show that the energy associated with the solution of the Timoshenko system decay polynomially over time.

Without loss of generality we will assume k fix, $h \in (0,1)$ and, for simplicity, we will "normalize"all the other parameters of the system. Then, we will study the following system

$$\begin{cases} h^3\phi_{tt} - \phi_{xx} + k(\phi + \psi_x) + \alpha\phi_t = 0, \\ h\psi_{tt} - k(\phi + \psi_x)_x = 0, \end{cases} \tag{2.55}$$

under boundary conditions (2.5) and initial data (2.6).

For this system, the energy $E_h(t)$ is given by

$$E_h(t) = \frac{1}{2}\left[h^3\left|\phi_t^h\right|^2 + h\left|\psi_t^h\right|^2 + \left|\phi_x^h\right|^2 + k\left|\phi^h + \psi_x^h\right|^2\right], \tag{2.56}$$

and obeys the dissipation law

$$\frac{d}{dt}E_h(t) = -\int_0^L \alpha\left(\phi_t^h\right)^2.$$

Let

$$E_h(t) = E_h(t,\phi,\psi) = E_1(t)$$

denote the energy defined in (2.56), and let

$$E_2(t) = E_h(t,\phi_t,\psi_t)$$

denote the energy of second order, for a suitability smooth solution. We shall prove the following.

Theorem 2.3.3 *Let the initial data regular enough. Then there is $C > 0$ such that for all $t > 0$:*

$$E_1(t) \leq \frac{C}{h^2}(E_1(0) + E_2(0))t^{-1}.$$

Proof. According with the definition of the energies $E_1(t)$ and $E_2(t)$, we have

$$\frac{d}{dt}E_1(t) = -\int_0^L \alpha\phi_t^2 \quad \text{and} \quad \frac{d}{dt}E_2(t) = -\int_0^L \alpha\phi_{tt}^2.$$

The proof consists in the construction of a appropriated functional using multipliers techniques.

Step 1: We multiply the first equation (2.55) by $(\phi+\psi_x)$ and integrate the resulting equation over $[0,L]$

$$\begin{aligned}\frac{d}{dt}\left(\int_0^L h^3\phi_t(\phi+\psi_x)\right) &= \int_0^L h^3\phi_t(\phi+\psi_x)_t+\phi_{xx}(\phi+\psi_x)-k(\phi+\psi_x)^2-\alpha\phi_t\,(\phi+\psi_x)\\ &= \int_0^L h^3\phi_t(\phi+\psi_x)_t-\phi_x\,(\phi+\psi_x)_x-k\,(\phi+\psi_x)^2-\alpha\phi_t\,(\phi+\psi_x)+[\phi_x\psi_x]\,\big|_0^L\end{aligned}$$

But

$$\int_0^L h^3\phi_t\,(\phi+\psi_x)_t=\int_0^L h^3\phi_t^2+h^3\phi_t\psi_{xt}=\int_0^L h^3\phi_t^2-\int_0^L h^3\psi_x\phi_{tt}+\frac{d}{dt}\left(\int_0^L h^3\phi_t\psi_x\right)$$

Then, we obtain

$$\frac{d}{dt}\left(\int_0^L h^3\phi_t(\phi+\psi_x)-h^3\phi_t\psi_x\right) \tag{2.57}$$

$$=\int_0^L h^3\phi_t^2-\int_0^L h^3\psi_x\phi_{tt}-\phi_x\,(\phi+\psi_x)_x-k\,(\phi+\psi_x)^2-\alpha\phi_t\,(\phi+\psi_x)+[\phi_x\psi_x]\,\big|_0^L$$

Now, we are interested in to cancel the term $\phi_x\,(\phi+\psi_x)_x$. To do this, let's multiply the second equation of (2.55) by $k^{-1}\phi_x$ and integrate the resulting equation over $[0,L]$.

$$\frac{d}{dt}\left(\int_0^L \frac{h}{k}\psi_t\phi_x\right)=\int_0^L \frac{h}{k}\psi_t\phi_{xt}+(\phi+\psi_x)_x\phi_x=\int_0^L \frac{h}{k}\psi_x\phi_{tt}+(\phi+\psi_x)_x\,\phi_x-\frac{d}{dt}\left(\int_0^L \frac{h}{k}\phi_t\psi_x\right)$$

which implies

$$\frac{d}{dt}\left(\int_0^L \frac{h}{k}\psi_t\phi_x+\frac{h}{k}\phi_t\psi_x\right)=\int_0^L \frac{h}{k}\psi_x\phi_{tt}+(\phi+\psi_x)_x\,\phi_x \tag{2.58}$$

Finally we adding the two relations, (2.57) and (2.58), gives

$$\frac{d}{dt}I_1 = \int_0^L h^3\phi_t^2-\left(h^3-\frac{h}{k}\right)\psi_x\phi_{tt}-k\,(\phi+\psi_x)^2-\alpha\phi_t\,(\phi+\psi_x)+[\phi_x\psi_x]\,\big|_0^L \tag{2.59}$$

where

$$I_1=\int_0^L h^3\phi_t(\phi+\psi_x)-\left(h^3-\frac{h}{k}\right)\phi_t\psi_x+\frac{h}{k}\psi_t\phi_x$$

Step 2: In order to deal with the boundary terms appearing (2.59), we consider $b=b(x)\in C^1([0,L])$ be a given function satisfying in addition $b(0)<0,\quad b(L)>0$ $\big($for example, $\quad b(x)=\frac{2x-L}{2}\big)$. We multiply the first equation of (2.55) by $b\phi_x$ and integrate the resulting equation over $[0,L]$. This gives

$$\begin{aligned}\frac{d}{dt}\left(\int_0^L h^3\phi_t b\phi_x\right) &= \int_0^L h^3\phi_t b\phi_{xt}+\phi_{xx}b\phi_x-k\,(\phi+\psi_x)\,b\phi_x-\alpha\phi_t b\phi_x\\ &= \int_0^L \frac{h^3}{2}b(\phi_t^2)_x+\frac{1}{2}b(\phi_x^2)_x-k(\phi+\psi_x)b\phi_x-\alpha\phi_t b\phi_x \qquad (2.60)\\ &= \int_0^L -\frac{h^3}{2}b_x(\phi_t^2)-\frac{1}{2}b_x(\phi_x^2)-k(\phi+\psi_x)b\phi_x-\alpha\phi_t b\phi_x\end{aligned}$$

Now we multiply the second equation of (2.55) by $b\psi_x$ and integrate the resulting equation over $[0, L]$, we obtain

$$\begin{aligned} \frac{d}{dt}\left(\int_0^L h\psi_t b\psi_x\right) &= \int_0^L h\psi_t b\psi_{xt} + k(\phi+\psi_x)_x b(\phi+\psi_x) - k(\phi+\psi_x)_x b\phi \\ &= \int_0^L \frac{h}{2}b(\psi_t^2)_x + \frac{k}{2}b\left((\phi+\psi_x)^2\right)_x - k(\phi+\psi_x)_x b\phi \\ &= \int_0^L -\frac{h}{2}b_x\psi_t^2 - \frac{k}{2}b_x(\phi+\psi_x)^2 + k(\phi+\psi_x)(b_x\phi+b\phi_x) + \left[\frac{1}{2}b\psi_x^2\right]\Big|_0^L \quad (2.61) \end{aligned}$$

Adding the equations (2.59), (2.60) and (2.61), we deduce for N_1 sufficiently large,

$$\begin{aligned} &\frac{d}{dt}\left[N_1 I_1 - \int_0^L \left(h^3\phi_t b\phi_x + h\psi_t b\psi_x\right)\right] \\ &= \int_0^L N_1 h^3\phi_t^2 - N_1\left(h^3 - \frac{h}{k}\right)\psi_x\phi_{tt} - N_1 k(\phi+\psi_x)^2 - N_1\alpha\phi_t(\phi+\psi_x) + \frac{h^3}{2}b_x(\phi_t^2) + \frac{1}{2}b_x(\phi_x^2) \\ &\int_0^L +k(\phi+\psi_x)b\phi_x + \alpha\phi_t b\phi_x + \frac{h}{2}b_x\psi_t^2 + \frac{k}{2}b_x(\phi+\psi_x)^2 - k(\phi+\psi_x)(b_x\phi+b\phi_x) \end{aligned}$$

Using the Young inequality,

$$\begin{aligned} &\frac{d}{dt}\left[I_1 - \int_0^L \left(h^3\phi_t b\phi_x + h\psi_t b\psi_x\right)\right] \\ &\leq \int_0^L N_1\left(h^3 + C_\varepsilon\right)\phi_t^2 + N_1\left|\left(h^3 - \frac{h}{k}\right)\right|^2 C_\varepsilon\phi_{tt}^2 + N_1\varepsilon\psi_x^2 - N_1(k-\varepsilon)(\phi+\psi_x)^2 + C_\varepsilon\alpha\phi_t^2 + \varepsilon(\phi+\psi_x)^2 \\ &+ \int_0^L \frac{h^3}{2}b_x\phi_t^2 + \frac{1}{2}b_x\phi_x^2 + \varepsilon k(\phi+\psi_x)^2 + C_\varepsilon b\phi_x^2 + C_\varepsilon\alpha\phi_t^2 + \varepsilon b\phi_x^2 + \frac{h}{2}b_x\psi_t^2 + \frac{k}{2}b_x(\phi+\psi_x)^2 \\ &+ \varepsilon k(\phi+\psi_x)^2 + C_\varepsilon b_x^2\phi^2 + C_\varepsilon b^2\phi_x^2 \end{aligned}$$

Then we can conclude

$$\begin{aligned} \frac{d}{dt}I_2 \leq &\int_0^L \left(N_1 h^3 + N_1 C_\varepsilon + C_\varepsilon\alpha + \frac{h^3}{2}b_x\right)\phi_t^2 + N_1\left|\left(h^3 - \frac{h}{k}\right)\right|^2 C_\varepsilon\phi_{tt}^2 \quad (2.62) \\ &+ \int_0^L -\left(N_1 k - N_1\varepsilon - \varepsilon k - \frac{k}{2}b_x\right)(\phi+\psi_x)^2 + \left(\frac{1}{2}b_x + C_\varepsilon b + \varepsilon b + C_\varepsilon b_x + C_\varepsilon b\right)\phi_x^2 + \frac{h}{2}b_x\psi_t^2 \end{aligned}$$

where

$$I_2 = N_1 I_1 - \int_0^L \left(h^3\phi_t b\phi_x + h\psi_t b\psi_x\right)$$

Step 3: Note that, for $\varepsilon > 0$ chosen sufficiently small, we have a problem in the two last terms of the inequality (2.62). Then, to solve this problem, we multiply the second equation of (2.55) by ψ and integrate over $[0, L]$.

$$\frac{d}{dt}\left(\int_0^L h\psi_t\psi\right) = \int_0^L h\psi_t^2 + k(\phi+\psi_x)_x\psi = \int_0^L h\psi_t^2 - k(\phi+\psi_x)^2 + k(\phi+\psi_x)\phi \quad (2.63)$$

We can see that by previous multiplier, we obtain a estimation for ψ_t^2.

We need an estimate on the integral of ϕ_x^2 to conclude. For this, we multiply the first equation of (2.55) by ϕ.

$$\begin{aligned}\frac{d}{dt}\left(\int_0^L h^3\phi_t\phi\right) &= \int_0^L h^3\phi_t^2 + \phi_{xx}\phi - k(\phi+\psi_x)\phi - \alpha\phi_t\phi \\ &= \int_0^L h^3\phi_t^2 - \phi_x^2 - k(\phi+\psi_x)\phi - \alpha\phi_t\phi \qquad (2.64)\end{aligned}$$

Finally, let's consider w the solution to

$$-w_{xx} = \phi_x, \quad w(0) = w(L) = 0 \qquad (2.65)$$

i.e.

$$w(x) = -\int_0^x \phi(y)dx + \frac{x}{L}\int_0^L \phi(y)dx.$$

Then one can easily check that

$$w_x = -\phi + \frac{1}{L}\left(\int_0^L \phi\right)$$

so that, we have

$$\int_0^L \phi^2 = \int_0^L w_x^2 + \frac{1}{L}\left(\int_0^L \phi\right)^2. \qquad (2.66)$$

In fact,

$$\phi^2 = w_x^2 - 2w\left(\frac{1}{L}\int_0^L \phi\right) + \frac{1}{L^2}\left(\int_0^L \phi\right)^2$$

Integrating on $(0, L)$, one gets

$$\int_0^L \phi^2 = \int_0^L w_x^2 - \frac{2}{L}\left(\int_0^L \phi\right)\int_0^L w_x + \frac{1}{L^2}\left(\int_0^L \phi\right)^2 \int_0^L dy$$

which implies

$$\int_0^L \phi^2 = \int_0^L w_x^2 - \frac{2}{L}\left(\int_0^L \phi\right)(w(L) - w(0)) + \frac{L}{L^2}\left(\int_0^L \phi\right)^2$$

and that's sufficient to obtain (2.66).

Furthermore by standard elliptic estimates imply that

$$\int_0^L w_x^2 \leq C\int_0^L \phi_x^2$$

where C is a positive constant.

Now, we multiply the second equation of (2.55) by w and integrate over space.

$$\frac{d}{dt}\left(\int_0^L h\psi_t w\right) = \int_0^L h\psi_t w_t + k(\phi+\psi_x)_x w \qquad (2.67)$$

Note that

$$\int_0^L k(\phi+\psi_x)_x w = \int_0^L kw_x^2 - k\psi\phi_x. \tag{2.68}$$

In fact,

$$\int_0^L k(\phi+\psi_x)_x w = \int_0^L k\phi_x w + k\psi_{xx} w = \int_0^L -kw_{xx}w + k\psi w_{xx} = \int_0^L kw_x^2 - k\psi\phi_x.$$

Finally, adding the equations (2.64), (2.67) and using the equation (2.68), on gets

$$\begin{aligned}\frac{d}{dt}\left(\int_0^L h^3\phi_t\phi + h\psi_t w\right) &= \int_0^L h^3\phi_t^2 - \phi_x^2 - k(\phi+\psi_x)\phi + h\psi_t w_t + k(\phi+\psi_x)_x w - \alpha\phi_t\phi \\ &= \int_0^L h^3\phi_t^2 - \phi_x^2 - k\phi^2 - k\psi_x\phi + h\psi_t w_t + kw_x^2 - k\psi\phi_x - \alpha\phi_t\phi \\ &= \int_0^L h^3\phi_t^2 - \phi_x^2 - k\phi^2 + h\psi_t w_t + kw_x^2 - \alpha\phi_t\phi\end{aligned}$$

In this moment we need observe that by the equality (2.66) we obtain

$$\int_0^L kw_x^2 - k\phi^2 = -\frac{k}{L}\left(\int_0^L \phi\right)^2 < 0.$$

Then, using Young's inequality

$$\begin{aligned}\frac{d}{dt}\left(\int_0^L h^3\phi_t\phi + h\psi_t w\right) &= \int_0^L h^3\phi_t^2 - \phi_x^2 + h\psi_t w_t - \alpha\phi_t\phi \\ &\leq \int_0^L \left(h^3 + C_\varepsilon\right)\phi_t^2 - (1-\varepsilon)\phi_x^2 + \varepsilon h\psi_t^2 + C_\varepsilon w_t^2 \qquad (2.69)\end{aligned}$$

But thanks to (2.65), it is easy to check that

$$\int_0^L w^2 \leq C\int_0^L \phi^2$$

Since, the equation defining w can be differentiated with respect to t, we have

$$\int_0^L w_t^2 \leq C\int_0^L \phi_t^2$$

Using this last estimate in (2.69) together with (2.63) , we obtain

$$\begin{aligned}&\frac{d}{dt}\left(\int_0^L -h\psi_t\psi + h^3\phi_t\phi + h\psi_t w\right) \\ &\leq \int_0^L \left(h^3 + C_\varepsilon + C\right)\phi_t^2 - (1-\varepsilon)\phi_x^2 + \varepsilon h\psi_t^2 - h\psi_t^2 + k(\phi+\psi_x)^2 - k(\phi+\psi_x)\phi \\ &= \int_0^L \left(h^3 + C_\varepsilon + C\right)\phi_t^2 - (1-\varepsilon)\phi_x^2 - (1-\varepsilon)h\psi_t^2 + k(\phi+\psi_x)^2 - k(\phi+\psi_x)\phi \qquad (2.70)\end{aligned}$$

Using successively the estimates in (2.62) and (2.70), we obtain

$$
\begin{aligned}
\frac{d}{dt}I_4 \;\leq\; & \int_0^L N_1 \left| \left(h^3 - \frac{h}{k} \right) \right|^2 C_\varepsilon \phi_{tt}^2 + \left(N_2 + \frac{N_2 C}{h^3} + \frac{N_2 C_\varepsilon}{h^3} + N_1 + \frac{N_1 C_\varepsilon}{h^3} + \frac{N_1 b_x}{2} \right) h^3 \phi_t^2 \\
& + \int_0^L - \left(N_2 - N_2\varepsilon - \frac{N_1 b_x}{2} \right) h\psi_t^2 - \left(N_1 - \frac{N_1 \varepsilon}{k} - \varepsilon - \frac{b_x}{2} - N_2 - N_2\varepsilon \right) k\left(\phi + \psi_x\right)^2 \\
& + \int_0^L - \left(N_2 - N_2\varepsilon - \frac{N_2}{2} - \frac{N_1 b_x}{2} - N_1 C_\varepsilon b - N_1 \varepsilon b - N_1 C_\varepsilon b_x^2 - N_1 C_\varepsilon b^2 \right) \phi_x^2
\end{aligned}
$$

where

$$I_4 = I_2 + N_2 I_3$$

for $N_2 > 0$ large and

$$I_3 = \int_0^L -h\psi_t\psi + h^3\phi_t\phi + h\psi_t w + \frac{\alpha}{2}\phi^2$$

Finally, let's define the functional L by

$$L(t) := \widetilde{N} E_1(t) + \widehat{N} E_2(t) + I_4.$$

Then,

$$
\begin{aligned}
\frac{d}{dt}L(t) \;\leq\; & \int_0^L - \left(\widehat{N}\alpha - N_1 \left| \left(h^3 - \frac{h}{k} \right) \right|^2 C_\varepsilon \right) \phi_{tt}^2 - \left(\frac{\widetilde{N}\alpha}{h^3} - N_2 - \frac{N_2 C}{h^3} - \frac{N_2 C_\varepsilon}{h^3} - N_1 - \frac{N_1 C_\varepsilon}{h^3} - \frac{N_1 b_x}{2} \right) h^3 \phi_t^2 \\
& + \int_0^L - \left(N_2 - N_2\varepsilon - \frac{N_1 b_x}{2} \right) h\psi_t^2 - \left(N_1 - \frac{N_1 \varepsilon}{k} - \varepsilon - \frac{b_x}{2} - N_2 - N_2\varepsilon \right) k\left(\phi + \psi_x\right)^2 \\
& + \int_0^L - \left(N_2 - N_2\varepsilon - \frac{N_2}{2} - \frac{N_1 b_x}{2} - N_1 C_\varepsilon b - N_1 \varepsilon b - N_1 C_\varepsilon b_x^2 - N_1 C_\varepsilon b^2 \right) \phi_x^2 \qquad (2.71)
\end{aligned}
$$

On other hand, for $h \in (0,1)$ we can conclude that

$$\left[\widetilde{N} - \frac{C}{h^2} \right] E_1(t) + \widehat{N} E_2(t) \leq L(t) \leq \left[\widetilde{N} + \frac{C}{h^2} \right] E_1(t) + \widehat{N} E_2(t).$$

then choosing $\widetilde{N} = \frac{C}{h^2}$, we obtain

$$\frac{C}{h^2}\left(E_1(t) + E_2(t)\right) \leq L(t) \leq \frac{C}{h^2}\left(E_1(t) + E_2(t)\right), \qquad (2.72)$$

where C is a positive constant.

Returning to the equation (2.71), using the definition of N, choosing ε sufficiently small and if we identify $\widehat{N} = Ch^2$, one gets

$$\frac{d}{dt}L(t) \;\leq\; -Ch^2 E_2(t) - C\left(\frac{\widetilde{N}}{h^3} + 1 \right) E_1(t) \leq -Ch^2\left(E_1(t) + E_2(t)\right)$$

Using the inequality (2.72) we can conclude that

$$\frac{d}{dt}L(t) = -CE_1(t).$$

Integrating in time, one gets

$$L(t) - L(0) \leq -C\int_0^L E_1(t)$$

which implies

$$\int_0^L E_1(t) \leq C\left(L(0) - L(t)\right). \tag{2.73}$$

Therefore, using again (2.72), one has

$$\begin{aligned}\int_0^t E_1(t) &\leq C\left(L(0) - L(t)\right)\\ &\leq C\left[\frac{C}{h^2}\left(E_1(0) + E_2(0)\right) - CE_1(t)\right]\\ &\leq \frac{C}{h^2}\left(E_1(0) + E_2(0)\right).\end{aligned}$$

Furthermore,

$$\frac{d}{dt}(tE_1(t)) \leq E_1(t).$$

Then, integrating in time and using the previous estimative, we obtain

$$\int_0^t \frac{d}{dt}\left(tE_1(t)\right) \leq \int_0^t E_1(t)$$

which implies

$$tE_1(t) \leq \int_0^t E_1 \leq \frac{C}{h^2}\left(E_1(0) + E_2(0)\right).$$

Consequently,

$$E_1(t) \leq \frac{C}{h^2}\left(E_1(0) + E_2(0)\right)t^{-1}$$

and this finishes the proof. ■

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