Universidade Federal da Paraíba
Universidade Federal de Campina Grande
Programa Associado de Pós-Graduação em Matemática
Doutorado em Matemática

Controlabilidade para alguns modelos da mecânica dos fluidos

por

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João Pessoa - PB
Março/2014

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sob orientação do

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Tese apresentada ao Corpo Docente do Programa Associado de Pós-Graduação em Matemática - UFPB/UFCG, como requisito parcial para obtenção do título de Doutor em Matemática.

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Resumo

Esta tese se preocupa com a controlabilidade de alguns modelos da mecânica dos fluidos, relacionados às equações de Navier-Stokes. O objetivo é provar a existência de controles que conduzem a solução de nosso sistema de um estado inicial prescrito à um estado final desejado em um tempo positivo dado.

Os dois primeiros Capítulos preocupam-se com a controlabilidade dos modelos de Burgers-α e Leray-α. O modelo de Leray-α é uma variante regularizada do sistema de Navier-Stokes (α é um parâmetro positivo pequeno) que pode também ser visto como um modelo de fluxos turbulentos; o modelo Burgers-α pode ser visto como um modelo simplificado. Nós provamos que as equações de Leray-α e Burgers-α são localmente controláveis a zero, com controles limitados independentes de α. Nós também provamos que, se os dados iniciais são suficientemente pequenos, o controle nulo das equações de Leray-α (resp. a equação de Burgers-α) converge quando $\alpha \to 0^+$ a um controle nulo das equações de Navier-Stokes equations (resp. a equação de Burgers).

O terceiro Capítulo preocupa-se com a controlabilidade nula de fluidos incompressíveis invíscidos para os quais efeitos térmicos são importantes. Ele serão modelados através da então chamada Aproximação de Boussinesq. No caso em que o calor tem difusão zero, adaptando e extendendo algumas idéias de J.-M. Coron [13, 15] e O. Glass [43, 44, 45], nós estabelecemos a controlabilidade exata global simultaneamente do campo velocidade e da temperatura para os caso 2D e 3D. Quando o coeficiente de difusão do calor é positivo, nós apresentamos alguns resultados adicionais sobre a controlabilidade exata para o campo velocidade e controlabilidade nula local para a temperatura.

O quarto Capítulo é dedicado à provar a controlabilidade exata local às trajetórias para um sistema acoplado, do tipo Boussinesq, com um número reduzido de controles. Nesse sistema, as variáveis desconhecidas são o campo velocidade e a pressão do fluido $(\mathbf{y}, p)$, a temperatura θ e uma variavél adicional c que pode ser vista como a concentração de um soluto contaminante. Nós provamos vários resultados, que essencialmente mostram que é suficiente atuar localmente no espaço sobre as equações satisfeitas por θ e c.

Palavras-chave: desigualdade de Carleman; controlabilidade nula; sistema Burgers-α; sistema de Boussinesq invíscido; sistema Leray-α; sistemas acoplados do tipo Boussinesq.

Abstract

This thesis deals with the controllability of some fluid mechanic models, related to the Navier-Stokes equations. The goal is prove the existence of controls that drive the solution of our system from a prescribed initial state to a desired final state at a given positive time.

The two first Chapters deal with the controllability of the Burgers-α and Leray-α models. The Leray-α model is a regularized variant of the Navier-Stokes system (α is a small positive parameter) that can also be viewed as a model for turbulent flows; the Burgers-α model can be viewed as a related toy model. We prove that the Leray-α and Burgers-α equations are locally null controllable, with controls bounded independently of α. We also prove that, if the initial data are sufficiently small, the control of the Leray-α equations (resp. the Burgers-α equation) converge as $\alpha \to 0^+$ to a null control of the Navier-Stokes equations (resp. the Burgers equation).

The third Chapter deals with the boundary controllability of inviscid incompressible fluids for which thermal effects are important. They will be modeled through the so called Boussinesq approximation. In the zero heat diffusion case, by adapting and extending some ideas from J.-M. Coron [13, 15] and O. Glass [43, 44, 45], we establish the simultaneous global exact controllability of the velocity field and the temperature for 2D and 3D flows. When the heat diffusion coefficient is positive, we present some additional results concerning exact controllability for the velocity field and local null controllability of the temperature.

The fourth Chapter is devoted to prove the local exact controllability to the trajectories for a coupled system, of the Boussinesq kind, with a reduced number of controls. In the state system, the unknowns are the velocity field and pressure of the fluid $(\mathbf{y}, p)$, the temperature θ and an additional variable c that can be viewed as the concentration of a contaminant solute. We prove several results, that essentially show that it is sufficient to act locally in space on the equations satisfied by θ and c.

Keywords: Carleman inequality; null controllability; Burgers-α system; inviscid Boussinesq system; Leray-α system; coupled systems of Boussinesq type.

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Dedicatória

À minha família e aos meus amigos.

Sumário

Introdução

Generalidades

O objetivo principal desta tese é analisar e controlar (em um sentido que se dirá) alguns problemas de valores iniciais e de contorno para equações diferenciais parciais com origem na mecânica dos fluidos. Ao longo desta introdução, apresentaremos uma dedução pouco profunda de tais equações, que nos permitirá, à continuação, descrever com precisão os problemas que trataremos.

Podemos dizer que a teoria matemática da dinâmica dos fluidos começa no século XVII com os trabalhos de Isaac Newton, que foi o primeiro a aplicar suas leis da mecânica ao movimento dos fluidos. Pouco tempo depois, Leonhard Paul Euler escreveu (1755) as equações diferenciais que descrevem o movimento de um fluido ideal, isto é, um fluido ausente de dissipação devido à interação entre as moléculas, essas equações são conhecidas como *equações de Euler*). E finalmente, Claude-Louis Henri Navier (1822) e, independentemente, George Gabriel Stokes (1845) introduziram no modelo o termo de viscosidade e chegaram às equações que hoje denominamos *Equações de Navier-Stokes*. O modelo unidimensional dessas equações foi proposto por Andrew Russell Forsyth (1906) e foi estudada por Harry Bateman (1915). Devido aos vários trabalhos de Johannes Martinus Burgers (1948) o modelo unidimensional das equações de Navier-Stokes é conhecido como *equação de Burgers*.

As equações de Burgers, Euler e Navier-Stokes formam um dos mais úteis conjuntos de equações diferenciais, pois descrevem fisicamente um grande número de fenômenos de interesses econômicos e da natureza. Por exemplo, as equações de Euler e Navier-Stokes são usadas para modelar o clima, correntes oceânicas, fluxos da água em oceanos, estuários, lagos e rios, movimentos das estrelas dentro e fora da galáxia, fluxo ao redor de aerofólios de automóveis e de aviões, propagação de fumaça em incêndios e em chaminés industriais (dispersão). Também são usadas diretamente nos projetos de aeronaves e carros, nos estudos do fluxo sangüíneo (hemodinâmica), no projeto de usinas hidrelétricas, nos projetos de hidráulica marítima, na análise dos efeitos da poluição hídrica em rios, mares, lagos, oceanos e da dispersão da poluição atmosférica, etc. Já a equação de Burgers aparece em várias áreas da matemática aplicada, tais como em modelos da dinâmica dos gases e no fluxo de tráfego.

Quase todos os fluxos que encontramos no dia-a-dia são turbulentos. Típicos exemplos são os fluxos que rodeiam carros, aviões e edifícios. Podemos dizer que a *turbulência* é o comportamento caótico que varia ao longo do tempo observado em muitos fluidos. A solução numérica das equações de Navier-Stokes para um fluxo turbulento é extremamente difícil e, devido às diferentes escalas de comprimento de mistura, que estão relacionadas ao fluxo turbulento, a solução estável deste fluxo requer uma resolução do problema numérico em uma malha muito fina, tornando o tempo computacional inviável para o cálculo (simulação numérica direta). Isto quer dizer que simular um fenômeno de turbulência usando as equações de Navier-Stokes é bastante complicado visto que requer computadores super potentes. Para resolver essa situação, as equações média de tempo tais como as *equações médias de Reynolds*, complementadas com modelos de turbulência, são utilizadas como aplicações da *dinâmica de fluidos computacional* para a modelagem de fluxos turbulentos. Outra técnica para resolver numericamente as equações de Navier-Stokes é a simulação de grandes escalas. Esta abordagem é computacionalmente mais custosa do que o método com equações médias de Reynolds (em tempo de cálculo e memória do computador), mas produz melhores resultados desde que as escalas turbulentas maiores são resolvidas explicitamente.

Antes de entrar nos detalhes da dedução das equações de Navier-Stokes, é necessário fazer várias suposições sobre fluidos. A primeira é que um fluido é um *meio continuo.* Isto significa que ele não contém vazios, como por exemplo, bolhas dissolvidas no gás, ou que ele não consiste de partículas como a neblina. Outra hipótese necessária é que todas as variáveis de interesse tais como pressão, velocidade, densidade, temperatura, etc., são diferenciáveis.

Estas equações são obtidas de princípios básicos de conservação da massa e momento. Para este objetivo, algumas vezes é necessário considerar um volume arbitrariamente finito, chamado de *volume de controle*, sobre o qual estes princípios possam ser facilmente aplicados. Este volume é representado por Ω e sua superfície de confinamento por Γ. O volume de controle permanece fixo no espaço ou pode mover-se como o fluido. Isto conduz, contudo, à considerações especiais, como mostraremos a seguir (seguiremos um esquema parecido ao das referências [11, 27]).

a) O problema fundamental em mecânica dos contínuos

Para fixar as idéias, vamos primeiramente adotar o ponto de vista de um físico. Assim, seja $\Omega \subset \mathbb{R}^3$ um conjunto aberto conexo e limitado e seja $T > 0$ dado. Assumimos que um contínuo ocupa o conjunto durante o intervalo de tempo $(0, T)$. Isto significa que o meio que estudaremos é composto de partículas e que existem funções suficientemente regulares ρ e $\mathbf{y}$ verificando:

- A *massa* das partículas do meio cuja posição no tempo t são pontos do conjunto aberto $W \subset \Omega$ é dada por
$$m(W, t) = \int_W \rho(\mathbf{x}, t)\, d\mathbf{x}.$$
- O *momento linear* associado às partículas em $W \subset \Omega$ no tempo t é
$$\mathbf{L}(W, t) = \int_W (\rho\mathbf{y})(\mathbf{x}, t)\, d\mathbf{x}.$$

As funções ρ e $\mathbf{y}$ são chamadas de *densidade de massa* e *campo velocidade*, respectivamente. Para cada t, as funções $\rho(\cdot, t)$ e $\mathbf{y}(\cdot, t)$ fornecem uma completa descrição do estado mecânico do meio no tempo t. O problema fundamental da mecânica dos contínuos (PFM) é o seguinte:

Assumamos que, para um dado meio, o estado mecânico no tempo $t = 0$ e as propriedades físicas para todo t são conhecidas. Então, o problema é determinar o estado mecânico deste meio para todo t.

b) Coordenadas Lagrangianas e o Lema do transporte

Para qualquer $\mathbf{x} \in \Omega$, consideramos o problema de Cauchy:

$$\begin{cases} \mathbf{X}_t(\mathbf{x}, t) = \mathbf{y}(\mathbf{X}(\mathbf{x}, t), t), \\ \mathbf{X}(\mathbf{x}, 0) = \mathbf{x}. \end{cases}$$

Dizemos que $t \mapsto \mathbf{X}(\mathbf{x}, t)$ é a trajetória da partícula do fluido que inicialmente ($t = 0$) estava em $\mathbf{x}$. A trajetória $\mathbf{X}$ é conhecida como a *função fluxo* do meio. Vamos supor que para cada $t \in (0, T)$, $\mathbf{X}(\cdot, t)(\Omega) \subset \Omega$ e $\mathbf{X}(\cdot, t)$ é bijetiva.

Para qualquer conjunto aberto $W \subset \Omega$, o conjunto

$$W_t := \{\mathbf{X}(\mathbf{x}, t) \,:\, \mathbf{x} \in W\}$$

deve ser visto como o conjunto das posições, no tempo t, das partículas do meio que estavam em um ponto de W no tempo $t = 0$.

O seguinte resultado é conhecido como *Lema do transporte.*

Lema (Lema do transporte)**.** *Suponhamos que $f = f(\mathbf{x}, t)$ está dada, com $f \in C^1(\Omega \times (0, T))$. Seja $W \subset \Omega$ e definamos*

$$F(t) := \int_{W_t} f(\mathbf{x}, t)\, d\mathbf{x}$$

para todo $t \in [0, T]$. Então $F : [0, T] \mapsto \mathbb{R}$ é uma função C^1 bem definida. Além disso,

$$\frac{dF}{dt}(t) = \int_{W_t} (f_t + \nabla \cdot (f\mathbf{y}))(\mathbf{x}, t)\, d\mathbf{x} \quad \forall t \in [0, T].$$

c) Leis de conservação

- **Conservação da massa:** Seja $W \subset \Omega$ um conjunto aberto. Então,

  $$\frac{d}{dt}\left(\int_{W_t} \rho(\mathbf{x}, t)\, d\mathbf{x}\right) = 0 \quad \forall t \in [0, T].$$

  Graças ao Lema do transporte, podemos deduzir a seguinte equação

  $$\rho_t + \nabla \cdot (\rho \mathbf{y}) = 0 \quad \text{in} \quad \Omega \times (0, T).$$

  Esta é a primeira EDP para ρ e $\mathbf{y}$. É frequentemente chamada de *equação de continuidade.*

- **Conservação do momento linear:** Seja $W \subset \Omega$ um conjunto aberto. Então,

  $$\frac{d}{dt}\left(\int_{W_t} (\rho \mathbf{y})(\mathbf{x}, t)\, d\mathbf{x}\right) = \mathbf{F}(W_t, t), \quad \forall t \in [0, T],$$

onde $\mathbf{F}(W_t,t)$ é, por definição, a resultante das forças agindo sobre as partículas que estão em W_t no tempo t. Na verdade, esta lei é a versão em mecânica dos contínuos da famosa *segunda lei de Newton*.

A força resultante $\mathbf{F}(W_t,t)$ pode ser decomposta como soma de das funções vetoriais: $\mathbf{F} := \mathbf{F}_{ten} + \mathbf{F}_{ext}$, onde $\mathbf{F}_{ten}$ é a resultante das forças tensoriais (i.e. as forças exercidas pelas partículas localizadas fora de W_t sobre as partículas localizadas dentro de W_t) e F_{ext} é a resultante de todas as forças externas.

Usualmente, temos que

$$\mathbf{F}_{ten}(W_t,t) = \int_{\partial W_t} \sigma \cdot \mathbf{n}\, d\gamma,$$

onde $\sigma = \sigma(\mathbf{x},t)$ é uma função matricial de classe C^1, usualmente chamada de *tensor tensão*.

Também, podemos ver que

$$\mathbf{F}_{ext}(W_t,t) = \int_{W_t} (\rho\mathbf{f})(\mathbf{x},t)\, d\mathbf{x},$$

para alguma $\mathbf{f} = \mathbf{f}(\mathbf{x},t)$.

Graças ao Lema do transporte, podemos deduzir a seguinte equação

$$(\rho\mathbf{y})_t + \nabla\cdot(\rho\mathbf{y}\otimes\mathbf{y}) = \nabla\cdot\sigma + \rho\mathbf{f} \quad \text{in} \quad \Omega\times(0,T).$$

Esta é a segunda EDP para ρ e $\mathbf{y}$. É frequentemente chamada de *equação do movimento*.

Deste modo, obtemos $N+1$ equações (a equação de continuidade e as N equações que compõem a equação do movimento) para N^2+N+1 funções desconhecidas (o escalar ρ, as N componentes de $\mathbf{y}$ e as N^2 componentes de σ. É Claro que somente essas equações não são suficientes por si só para fornecer uma descrição completa do comportamento do meio.

A fim de obter uma descrição completa, devemos particularizar e introduz algumas leis adicionais.

d) Um conjunto particular de leis constitutivas

- **Conservação do volume:** Seja $W \subset \Omega$ um conjunto aberto. Então,

$$\frac{d}{dt}\left(\int_{W_t} d\mathbf{x}\right) = 0, \quad \forall t\in[0,T].$$

Graças ao Lema do transporte, podemos deduzir a seguinte equação

$$\nabla\cdot\mathbf{y} = 0 \quad \text{in} \quad \Omega\times(0,T).$$

Esta equação é frequentemente chamada de *equação de incompressibilidade*.

- **Lei Newtoniana:** O campo velocidade $\mathbf{y}$ é de classe $\mathbf{C}^2$ e satisfaz

$$\sigma = -p\mathrm{Id} + \mu(\nabla\mathbf{y} + \nabla\mathbf{y}^t) - \frac{2}{3}\mu(\nabla\cdot\mathbf{y})\mathrm{Id}$$

para alguma função p de classe C^1 (a *pressão*) e uma constante positiva μ (a *viscosidade do fluido*).

Assim, fazendo o uso de todas as leis descritas acima, podemos chegar ao seguinte conjunto de equações:

$$\begin{cases} \rho_t + \nabla \cdot (\rho \mathbf{y}) = 0 & \text{em} \quad \Omega \times (0,T), \\ (\rho \mathbf{y})_t + \nabla \cdot (\rho \mathbf{y} \otimes \mathbf{y}) = \mu \Delta \mathbf{y} - \nabla p + \rho \mathbf{f} & \text{em} \quad \Omega \times (0,T), \\ \nabla \cdot \mathbf{y} = 0 & \text{em} \quad \Omega \times (0,T). \end{cases}$$

Essas são as equações de *Navier-Stokes para fluidos incompressíveis não-hom ogêneas* (ou equações de Navier-Stokes com densidade variável). Assim, encontramos um sistema de $N+2$ EDPs para $N+2$ variáveis desconhecidas.

d) Turbulência

A palavra *turbulência* tem diferentes significados, sempre indicando que a turbulência é um fenômeno complicado e multifásico. Um dos principais objetivos de realizar pesquisas sobre turbulência é obter simulações computacionais confiáveis e segura dos fluxos turbulentos.

As principais questões relacionadas com a turbulência foram levantadas desde o início do século XX, e um grande número de resultados empíricos e heurísticas foram obtidos, motivados principalmente por aplicações à engenharia.

Ao mesmo tempo, em matemática, aparecem os trabalhos pioneiros de Jean Leray de 1933-1934 (veja [64, 65, 66]) sobre as equações de Navier-Stokes. Leray especulou que a turbulência aparece devido à formação de ponto ou "linhas de vórtices" no qual alguma componente da velocidade torna-se infinita.

Para poder lidar com tal situação, Leray sugeriu o conceito de soluções fracas, soluções não-clássicas para as equações de Navier-Stokes, e este tornou-se o ponto de partida da teoria matemática das equações de Navier-Stokes. É importante ressaltar que as idéias de Leray serviram também como ponto de partida para vários elementos importantes da teoria moderna das equações diferenciais parciais. Ainda hoje, apesar de muito esforço, a conjectura de Jean Leray sobre o aparecimento de singularidades nos fluxos turbulentos no caso tridimensional não foi nem provada nem refutada.

Para mais detalhes, consideremos um fluido newtoniano, incompressível e homogêneo governado pelo sistema:

$$\begin{cases} \mathbf{y}_t + (\mathbf{y} \cdot \nabla)\mathbf{y} = \nu \Delta \mathbf{y} - \nabla p & \text{em} \quad \Omega \times (0,T), \\ \nabla \cdot \mathbf{y} = 0 & \text{em} \quad \Omega \times (0,T), \\ \mathbf{y} = \mathbf{0} & \text{sobre} \quad \partial\Omega \times (0,T), \end{cases} \tag{1}$$

onde se tomou, por simplicidade, $\rho \equiv 1$ and $\nu = \mu > 0$.

Um fluido pode fluir (evoluir) de duas maneiras completamente diferentes:

- Para viscosidade cinemática ν "grande", as partículas fluidas seguem trajetórias mais ou menos ordenadas. Então dizemos que o fluido é *laminar*.

- Para ν suficientemente pequeno, a velocidade e a pressão exibem variações extremamente rápidas e oscilações em tempo e espaço, observando assim um comportamento caótico no movimento das partículas. Neste caso, dizemos que o fluxo é *turbulento*.

Usualmente, nos fluxos turbulentos dividimos o campo velocidade em uma parte média $\overline{\mathbf{y}}$ e uma parte flutuante $\mathbf{y}'$ tal que $\mathbf{y} := \overline{\mathbf{y}} + \mathbf{y}'$. Assim,

$$\begin{cases} \overline{\mathbf{y}}_t + (\overline{\mathbf{y}} \cdot \nabla)\overline{\mathbf{y}} + \nabla \cdot (\overline{\mathbf{y}' \otimes \mathbf{y}'}) = \nu \Delta \overline{\mathbf{y}} - \nabla \overline{p} & \text{em } \ \Omega \times (0,T), \\ \nabla \cdot \overline{\mathbf{y}} = 0 & \text{em } \ \Omega \times (0,T), \\ \overline{\mathbf{y}} = \mathbf{0} & \text{sobre } \ \partial\Omega \times (0,T). \end{cases} \tag{2}$$

Nota-se a presença de um termo adicional de esforços devido a turbulência (velocidades flutuantes) e este é desconhecido. Necessitamos uma expressão para o termo $\nabla \cdot (\overline{\mathbf{y}' \otimes \mathbf{y}'})$ a fim de fechar o sistema de equações acima. Portanto, os modelos de turbulência baseiam-se em fazer hipóteses sobre o termo $\nabla \cdot (\overline{\mathbf{y}' \otimes \mathbf{y}'})$ a fim de fechar o problema e, assim, poder resolvê-lo.

Por definição, o tensor de Reynolds é $\mathbf{R} := -\overline{\mathbf{y}' \otimes \mathbf{y}'}$. Existem (ao menos) duas maneiras de fechar o problema:

- Em primeiro lugar, fazendo o uso da *hipótese de Boussinesq*, isto é, escrevendo que o tensor $\mathbf{R}$ é da forma:

$$\mathbf{R} = \nu_T D\mathbf{y},$$

  onde $D\mathbf{y} = \nabla \mathbf{y} + \nabla \mathbf{y}^t$ é a parte simétrica do gradiente espacial de $\mathbf{y}$ (também chamado *tensor de deformações*) e ν_T é uma função (mais ou menos complicada) de $\mathbf{y}$. Isto conduz aos distintos modelos de Reynolds (modelos algébricos do tipo *Smagorinski*, modelos com N equações, modelo $k - \varepsilon$, etc.).

- Outra forma de fechar o problema consiste em supor que

$$\overline{\mathbf{y} \otimes \mathbf{y}} = (\mathbf{z} \cdot \nabla)\overline{\mathbf{y}},$$

  onde $\mathbf{z}$ é um novo campo velocidade obtido regularizando $\mathbf{y}$. Este modelo será estudado nesta tese.

Para mais detalhes sobre turbulência, podemos citar as referências [35, 70, 73, 74, 75].

Agora, vamos apresentar alguns modelos particulares relacionados às equações de Navier-Stokes:

- Equação de Burgers:

$$y_t + y y_x = \nu y_{xx} + f \quad \text{em} \quad \Omega \times (0,T).$$

- Equações de Euler:

$$\begin{cases} \mathbf{y}_t + (\mathbf{y} \cdot \nabla)\mathbf{y} = -\nabla p + \mathbf{f} & \text{em} \quad \Omega \times (0,T), \\ \nabla \cdot \mathbf{y} = 0 & \text{em} \quad \Omega \times (0,T). \end{cases}$$

- Equações de Navier-Stokes (densidade constante):

$$\begin{cases} \mathbf{y}_t + (\mathbf{y} \cdot \nabla)\mathbf{y} = \nu \Delta \mathbf{y} - \nabla p + \mathbf{f} & \text{em} \quad \Omega \times (0,T), \\ \nabla \cdot \mathbf{y} = 0 & \text{em} \quad \Omega \times (0,T). \end{cases}$$

- Equações de Boussinesq:

$$\begin{cases} \mathbf{y}_t + (\mathbf{y}\cdot\nabla)\mathbf{y} = \nu\Delta\mathbf{y} - \nabla p + \mathbf{f} + \theta\mathbf{e}_N & \text{em} \quad \Omega\times(0,T), \\ \nabla\cdot\mathbf{y} = 0 & \text{em} \quad \Omega\times(0,T), \\ \theta_t - \kappa\Delta\theta + \mathbf{y}\cdot\nabla\theta = g & \text{em} \quad \Omega\times(0,T). \end{cases}$$

  Neste sistema, a variável θ representa a temperatura do fluido, $\mathbf{e}_N$ é o n-ésimo vetor da base canônica de $\mathbb{R}^N$, $\kappa \geq 0$ é o coeficiente de difusão e g representa uma fonte de calor.

- Sistema Burgers-α:

$$\begin{cases} y_t + zy_x = \nu y_{xx} + f & \text{em} \quad \Omega\times(0,T), \\ z - \alpha^2 z_{xx} = y & \text{em} \quad \Omega\times(0,T). \end{cases}$$

- Sistema Leray-α:

$$\begin{cases} \mathbf{y}_t + (\mathbf{z}\cdot\nabla)\mathbf{y} = \nu\Delta\mathbf{y} - \nabla p + \mathbf{f} & \text{em} \quad \Omega\times(0,T), \\ \mathbf{z} - \alpha^2\Delta\mathbf{z} + \nabla\pi = \mathbf{y} & \text{em} \quad \Omega\times(0,T), \\ \nabla\cdot\mathbf{y} = \nabla\cdot\mathbf{y} = 0 & \text{em} \quad \Omega\times(0,T). \end{cases}$$

  Esses dois últimos são motivados pelo fenômeno de turbulência e serão tratados neste trabalho.

Em geral, para sistemas como os anteriores, o objetivo consistirá em encontrar uma função *controle* v que, atuando de algum modo sobre o sistema, conduzirá a solução deste sistema a um comportamento desejado no instante de tempo final T.

Diremos que temos a controlabilidade aproximada do sistema se a solução, partindo de um estado inicial arbitrário, pode ser conduzida arbitrariamente próximo (com respeito a uma determinada norma) à um estado desejado arbitrário no instante final.

Por outro lado, a *controlabilidade exata* indicará que a solução pode ser conduzida exatamente a todo estado desejado no instante final de tempo.

Como caso particular da controlabilidade exata, se diz que o sistema possui a propriedade de *controlabilidade nula* se, partindo de um estado inicial arbitrário, a solução pode ser conduzida ao estado nulo no tempo final.

Finalmente, outro caso interessante de controlabilidade exata é a *controlabilidade exata às trajetórias*, que indica que podemos fazer que a solução do nosso sistema controlado coincida, no tempo final, com uma trajetória do mesmo sistema, isto é, uma solução não controlada.

Para provar resultados de controlabilidade para problemas lineares, a principal ferramenta é provar uma *desigualdade de observabilidade.* Esta desigualdade surge quando formulamos o problema de controlabilidade de maneira abstrata e utilizamos clássicos resultados da análise funcional. Outra importante ferramenta é a *desigualdade global de Carleman.* Essas desigualdades de Carleman são estimações de normas L^2 ponderadas globais por normas L^2 ponderadas locais.

No entanto, para provar resultados de controlabilidade de problemas não-lineares os argumentos são bem mais complicados. Em geral, as técnicas se baseiam em algum dos dois seguintes argumentos: aplicar um teorema de ponto fixo ou aplicar um teorema de função inversa. Uma questão interessante é que a maioria dos resultados de controlabilidade para problemas não-lineares são do tipo locais.

Resultados anteriores

Nesta tese, uma das principais ferramentas para obter resultados de controlabilidade será a desigualdade de Carleman cujo uso popularizou-se graças aos trabalhos de Fursikov e Imanuvilov, veja [40]. Também, podemos mencionar o trabalho de Lebeau e Robbiano [63] para a equação do calor linear no qual combinaram o método de Russell, transformada integral e uma desigualdade de Carleman para equações elípticas para obter a controlabilidade nula para a equação do Calor. Para a controlabilidade aproximada da equação do calor semilinear, onde o termo não-linear satisfaz uma condição de crescimento sublinear, veja o trabalho [25], de Fabre, Puel e Zuazua.

Lembremos que, no contexto das equações de Navier-Stokes, Lions conjecturou em [67] a controlabilidade aproximada fronteira e distribuída global; desde então, a controlabilidade dessas equações tem sido intensivamente estudadas, mas até o momento somente resultados parciais são conhecidos.

Em [38], Fursikov e Imanuvilov provaram a controlabilidade exata local à trajetórias C^∞ do sistema de Navier-Stokes, usando uma desigualdade de Carleman e um teorema de função inversa. Posteriormente, Fernández-Cara, Guerrero, Imanuvilov e Puel, em [29], melhoraram este resultado provando o mesmo resultado para trajetórias L^∞. Em seguida, inspirado em [29, 38], Guerrero prova, em [49], a controlabilidade exata local às trajetórias do sistema de Boussinesq. E por último, usando os resultados dados em [29], Fernández-Cara, Guerrero, Imanuvilov e Puel provaram, sob algumas condições geométricas, a controlabilidade exata local para as trajetórias dos sistemas N-dimensionais de Navier-Stokes e Boussinesq com uma quantidade reduzida de controles escalares, veja [30]. Vamos mencionar também [7, 18, 19, 30], onde resultados análogos são obtidos com um número reduzido de controles escalares.

Em relação aos trabalhos sobre as equações de Euler podemos destacar [13, 15] de Coron. Nestes trabalhos, Coron provou a controlabilidade exata na fronteira global para a equação de Euler bidimensional usando o *método do retorno*. Daí, estendeu esses resultados para provar um resultado de controlabilidade aproximada global para o sistema de Navier-Stokes bidimensional com condições de fronteira do tipo Navier slip, veja [14]. Além disso, combinando resultados sobre controlabilidade global e local, a controlabilidade nula global para as equações de Navier-Stokes sobre uma variedade bidimensional sem fronteira foi estabelecida por Coron e Fursikov em [17]; Veja também Guerrero *et al.* [51] para um outro resultado de controlabilidade global para Navier-Stokes. Posteriormente Glass, nos trabalhos [43, 44], provou a controlabilidade exata na fronteira global para a equação de Euler tridimensional.

Os resultados de controlabilidade nula local para a equação de Burgers foram obtidos por Fursikov e Imanuvilov em [39]. Podemos citar também outros trabalhos que trataram a controlabilidade da equação de Burgers, veja [8, 22, 28, 40, 50, 55]. Destes trabalhos podemos destacar [28], em que os autores provam um resultado ótimo para a controlabilidade nula da equação de Burgers.

Descrição dos resultados

Nesta tese, apresentaremos resultados locais e globais de controlabilidade para problemas não-lineares com origem em mecânica dos fluidos. Todos os resultados constam em artigos publicados, aceitos e em preparação. Mencionaremos a referência precisa ao final da descrição de cada capítulo.

Capítulo 1: Controlabilidade do sistema Burgers-α

Sejam $L > 0$ e $T > 0$ números reais positivos. Seja $(a,b) \subset (0,L)$ um subconjunto aberto não-vazio que será chamado de *domínio de controle*.

Consideramos a seguinte equação de Burgers controlada:

$$\begin{cases} y_t - y_{xx} + yy_x = v1_{(a,b)} & \text{em} \quad (0,L)\times(0,T), \\ y(0,\cdot) = y(L,\cdot) = 0 & \text{sobre} \quad (0,T), \\ y(\cdot,0) = y_0 & \text{em} \quad (0,L). \end{cases} \tag{3}$$

Em (3), a função $y = y(x,t)$ pode ser interpretada como uma velocidade unidimensional de um fluido e $y_0 = y_0(x)$ é uma velocidade inicial. A função $v = v(x,t)$ (usualmente em $L^2((a,b)\times(0,T)))$ é o controle atuando sobre o sistema e $1_{(a,b)}$ denota a função característica de (a,b).

Neste capítulo da tese vamos considerar um sistema semelhante a (3), em que o termo de transporte é da forma zy_x e z é a solução de uma equação elíptico governado por y. A saber, nós consideramos a seguinte versão *regularizada* de (3), com $\alpha > 0$:

$$\begin{cases} y_t - y_{xx} + zy_x = v1_{(a,b)} & \text{em} \quad (0,L)\times(0,T), \\ z - \alpha^2 z_{xx} = y & \text{em} \quad (0,L)\times(0,T), \\ y(0,\cdot) = y(L,\cdot) = z(0,\cdot) = z(L,\cdot) = 0 & \text{sobre} \quad (0,T), \\ y(\cdot,0) = y_0 & \text{em} \quad (0,L). \end{cases} \tag{4}$$

Como já foi dito antes, este sistema é chamado de sistema Burgers-α.

A seguir, introduzimos a propriedade de controlabilidade nula para (3) e (4) no tempo $T > 0$:

> *Para qualquer $y_0 \in H_0^1(0,L)$, encontrar $v \in L^2((a,b)\times(0,T))$ tal que a solução associada a* (3) (*resp.* (4)) *satisfaz*
>
> $$y(\cdot,T) = 0 \quad \textit{em} \quad (0,L). \tag{5}$$

Notemos que (4) é diferente de (3) pelo menos em dois aspectos: primeiro, o aparecimento de termos não-lineares que são não-locais na variável espacial; segundo, o fato que o parâmetro α aparece.

Nossos primeiros resultados principais são os seguintes:

Teorema (Controlabilidade nula local uniforme). *Para cada $T > 0$, o sistema* (4) *é localmente controlável a zero no tempo T. Mais precisamente, existe $\delta > 0$ (independente de α) tal que, para qualquer $y_0 \in H_0^1(0,L)$ com $\|y_0\|_\infty \leq \delta$, existem controles $v_\alpha \in L^\infty((a,b)\times(0,T))$ e estados associados (y_α, z_α) satisfazendo* (5)*. Além disso, temos*

$$\|v_\alpha\|_\infty \leq C, \quad \forall \alpha > 0. \tag{6}$$

Teorema (Controlabilidade para tempo grande). *Para cada $y_0 \in H_0^1(0,L)$ com $\|y_0\|_\infty < \pi/L$, o sistema* (4) *é controlável a zero para tempo grande. Em outras palavras, existe $T > 0$ (independente de α), controles $v_\alpha \in L^\infty((a,b)\times(0,T))$ e estados associados (y_α, z_α) satisfazendo* (5) *e* (6).

Lembremos que π/L é a raiz quadrada do primeiro autovalor do operador Laplaciano.

Por outro lado, notemos que esses resultados fornecem controles em $L^\infty((a,b)\times(0,T))$ e não somente em $L^2((a,b)\times(0,T))$. De fato, isto é muito conveniente não somente em (3) e (4), mas também em alguns problemas intermediários que aparecem nas demonstrações, desde que desta maneira obtemos melhores estimativas para os estados, tornando as afirmações de convergência e existência facilmente estabelecidas.

A principal novidade desses resultados é que eles garantem o controle de um tipo de equações parabólicas não-lineares que são não-locais. Na análise da controlabilidade de EDPs, este tipo de equações não são freqüentes. De fato, em geral quando tratamos com não-linearidades não locais, não parece ser fácil transmitir a informação fornecida por controles localmente suportados ao domínio inteiro de maneira satisfatória.

Também vamos provar um resultado relacionado à controlabilidade nula local no limite, quando $\alpha \to 0^+$. Mais precisamente, o seguinte resultado é válido:

Teorema (Controlabilidade local no limite). *Seja $T > 0$ dado e seja $\delta > 0$ a constante fornecida pelo Teorema da controlabilidade nula local uniforme. Suponhamos que $y_0 \in H_0^1(0,L)$, com $\|y_0\|_{L^\infty} \leq \delta$, seja v_α um controle nulo para* (4) *satisfazendo* (6) *e seja (y_α, z_α) um estado associado satisfazendo* (5)*. Então, ao menos para uma subseqüência, temos*

$$\begin{gathered} v_\alpha \to v \text{ fraco-}* \text{ em } L^\infty((a,b)\times(0,T)), \\ z_\alpha \to y \text{ e } y_\alpha \to y \text{ fraco-}* \text{ em } L^\infty((0,L)\times(0,T)) \end{gathered} \tag{7}$$

quando $\alpha \to 0^+$, onde (v,y) é um par controle-estado para (3) *que verifica* (5).

Estes resultados foram publicados em [3].

Capítulo 2: Controle nulo local uniforme do modelo Leray-α

Seja $\Omega \subset \mathbb{R}^N (N = 2,3)$ um conjunto aberto e conexo cuja fronteira Γ é de classe C^2. Sejam $\omega \subset \Omega$ um conjunto aberto não-vazio, $\gamma \subset \Gamma$ um subconjunto aberto não-vazio de Γ e

$T > 0$. Usaremos as notações $Q = \Omega \times (0, T)$ e $\Sigma = \Gamma \times (0, T)$, e denotaremos por $\mathbf{n} = \mathbf{n}(\mathbf{x})$ o vetor normal exterior a Ω nos pontos $\mathbf{x} \in \Gamma$; Os espaços de funções vetoriais, assim como os seus elementos, serão representados por letras em negrito.

O sistema de Navier-Stokes para um fluido incompressível viscoso homogêneo (com densidade unitária e viscosidade cinemática unitária) sujeito às condições de fronteira do tipo Dirichlet homogênea é dado por

$$\begin{cases} \mathbf{y}_t - \Delta \mathbf{y} + (\mathbf{y} \cdot \nabla)\mathbf{y} + \nabla p = \mathbf{f} & \text{em} \quad Q, \\ \nabla \cdot \mathbf{y} = 0 & \text{em} \quad Q, \\ \mathbf{y} = \mathbf{0} & \text{sobre} \quad \Sigma, \\ \mathbf{y}(0) = \mathbf{y}_0 & \text{em} \quad \Omega, \end{cases} \tag{8}$$

onde $\mathbf{y}$ (o campo velocidade) e p (a pressão) são as variáveis desconhecidas, $\mathbf{f} = \mathbf{f}(\mathbf{x}, t)$ é uma termo força e $\mathbf{y}_0 = \mathbf{y}_0(\mathbf{x})$ é um campo velocidade inicial dado.

Para provar a existência de uma solução para o sistema de Navier-Stokes, Leray em [66] teve a idéia de criar um modelo *fechado* de turbulência sem aumentar a dissipação viscosa. Assim, ele introduziu uma variante "regularizada" de (8) modificando o termo não-linear como segue:

$$\begin{cases} \mathbf{y}_t - \Delta \mathbf{y} + (\mathbf{z} \cdot \nabla)\mathbf{y} + \nabla p = \mathbf{f} & \text{em} \quad Q, \\ \nabla \cdot \mathbf{y} = 0 & \text{em} \quad Q, \end{cases}$$

onde $\mathbf{y}$ e $\mathbf{z}$ estão relacionados por

$$\mathbf{z} = \phi_\alpha * \mathbf{y}$$

e ϕ_α é um núcleo regular. Ao menos formalmente, as equações de Navier-Stokes são recuperadas no limite quando $\alpha \to 0^+$, $\mathbf{z} \to \mathbf{y}$.

Aqui, consideraremos um núcleo regular especial, associado ao operador do tipo Stokes $\mathbf{Id} + \alpha^2 \mathbf{A}$, onde $\mathbf{A}$ é o operador de Stokes. Isto leva às seguintes equações de Navier-Stokes modificadas, chamadas sistema Leray-α (veja [10]):

$$\begin{cases} \mathbf{y}_t - \Delta \mathbf{y} + (\mathbf{z} \cdot \nabla)\mathbf{y} + \nabla p = \mathbf{f} & \text{em} \quad Q, \\ \mathbf{z} - \alpha^2 \Delta \mathbf{z} + \nabla \pi = \mathbf{y} & \text{em} \quad Q, \\ \nabla \cdot \mathbf{y} = 0, \ \nabla \cdot \mathbf{z} = 0 & \text{em} \quad Q, \\ \mathbf{y} = \mathbf{z} = \mathbf{0} & \text{sobre} \quad \Sigma, \\ \mathbf{y}(0) = \mathbf{y}_0 & \text{em} \quad \Omega. \end{cases} \tag{9}$$

Neste capítulo, vamos trabalhar com os seguintes sistemas de controlabilidade:

$$\begin{cases} \mathbf{y}_t - \Delta \mathbf{y} + (\mathbf{z} \cdot \nabla)\mathbf{y} + \nabla p = \mathbf{v} 1_\omega & \text{em} \quad Q, \\ \mathbf{z} - \alpha^2 \Delta \mathbf{z} + \nabla \pi = \mathbf{y} & \text{em} \quad Q, \\ \nabla \cdot \mathbf{y} = 0, \ \nabla \cdot \mathbf{z} = 0 & \text{em} \quad Q, \\ \mathbf{y} = \mathbf{z} = \mathbf{0} & \text{sobre} \quad \Sigma, \\ \mathbf{y}(0) = \mathbf{y}_0 & \text{em} \quad \Omega, \end{cases} \tag{10}$$

e

$$\begin{cases} \mathbf{y}_t - \Delta \mathbf{y} + (\mathbf{z} \cdot \nabla)\mathbf{y} + \nabla p = \mathbf{0} & \text{em} \quad Q, \\ \mathbf{z} - \alpha^2 \Delta \mathbf{z} + \nabla \pi = \mathbf{y} & \text{em} \quad Q, \\ \nabla \cdot \mathbf{y} = 0, \ \nabla \cdot \mathbf{z} = 0 & \text{em} \quad Q, \\ \mathbf{y} = \mathbf{z} = \mathbf{h} 1_\gamma & \text{sobre} \quad \Sigma, \\ \mathbf{y}(0) = \mathbf{y}_0 & \text{em} \quad \Omega, \end{cases} \tag{11}$$

onde $\mathbf{v} = \mathbf{v}(\mathbf{x}, t)$ (respectivamente $\mathbf{h} = \mathbf{h}(\mathbf{x}, t)$) representa o controle, atuando somente em um pequeno conjunto ω (respectivamente sobre γ) durante todo o intervalo de tempo $(0, T)$. O simbolo 1_ω (respectivamente 1_γ) representa a função característica de ω (respectivamente de γ).

Nas aplicações, o *controle interno* $\mathbf{v}1_\omega$ pode ser visto como um campo gravitacional ou eletromagnético. Enquanto, o *controle fronteira* $\mathbf{h}1_\gamma$ é o traço do campo velocidade sobre Σ.

Observação. É natural supor que $\mathbf{y}$ e $\mathbf{z}$ satisfazem as mesmas condições de fronteira sobre Σ desde que, no limite, deveríamos ter $\mathbf{z} = \mathbf{y}$. Consequentemente, vamos supor que o controle fronteira $\mathbf{h}1_\gamma$ atua simultaneamente sobre ambas variáveis $\mathbf{y}$ e $\mathbf{z}$.

Relembrando algumas definições usuais de alguns espaços no contexto de fluidos incompressíveis:

$$\mathbf{H} = \{\, \mathbf{u} \in \mathbf{L}^2(\Omega) : \nabla \cdot \mathbf{u} = 0 \text{ em } \Omega,\ \mathbf{u} \cdot \mathbf{n} = 0 \text{ sobre } \Gamma \,\},$$
$$\mathbf{V} = \{\, \mathbf{u} \in \mathbf{H}_0^1(\Omega) : \nabla \cdot \mathbf{u} = 0 \text{ em } \Omega \,\}.$$

Notemos que, para todo $\mathbf{y}_0 \in \mathbf{H}$ e todo $\mathbf{v} \in \mathbf{L}^2(\omega \times (0, T))$, existe uma única solução $(\mathbf{y}, p, \mathbf{z}, \pi)$ para (10) que satisfaz (entre outras coisas)

$$\mathbf{y}, \mathbf{z} \in C^0([0, T]; \mathbf{H}).$$

Isto está em desacordo com a falta de unicidade do sistema de Navier-Stokes quando $N = 3$.

Os principais objetivos desde capítulo são analisar as propriedades de controlabilidade de (10) e (11) e determinar a forma que eles dependem de α quando $\alpha \to 0^+$.

O problema de controlabilidade nula para (10) no tempo $T > 0$ é a seguinte:

Para qualquer $\mathbf{y}_0 \in \mathbf{H}$, *encontrar* $\mathbf{v} \in \mathbf{L}^2(\omega \times (0, T))$ *tal que o correspondente estado* (*a correspondente solução para* (10)) *satisfaz*

$$\mathbf{y}(T) = \mathbf{0} \quad \textit{em} \quad \Omega. \tag{12}$$

O problema de controlabilidade nula para (11) no tempo $T > 0$ é a seguinte:

Para qualquer $\mathbf{y}_0 \in \mathbf{H}$, *encontrar* $\mathbf{h} \in \mathbf{L}^2(0, T; \mathbf{H}^{-1/2}(\gamma))$ *com* $\int_\gamma \mathbf{h} \cdot \mathbf{n} \, d\Gamma = 0$ *e um estado associado* (*a correspondente solução para* (11)) *satisfazendo*

$$\mathbf{y}, \mathbf{z} \in C^0([0, T]; \mathbf{L}^2(\Omega))$$

e (12).

Nosso primeiro resultado principal deste capítulo é:

Teorema (Controlabilidade interna nula local uniforme)**.** *Existe* $\epsilon > 0$ (*independente de* α) *tal que, para cada* $\mathbf{y}_0 \in \mathbf{H}$ *com* $\|\mathbf{y}_0\| \leq \epsilon$, *existem controles* $\mathbf{v}_\alpha \in L^\infty(0, T; \mathbf{L}^2(\omega))$ *tais que as soluções associadas para* (10) *verificam* (12). *Além disso, esses controles podem ser encontrados satisfazendo a estimativa*

$$\|\mathbf{v}_\alpha\|_{L^\infty(\mathbf{L}^2)} \leq C, \tag{13}$$

onde C *é também independente de* α.

Nosso segundo resultado principal é o análogo ao Teorema anterior no âmbito da controlabilidade fronteira:

Teorema (Controlabilidade fronteira nula local uniforme)**.** *Existe* $\delta > 0$ (*independente de* α) *tal que, para cada* $\mathbf{y}_0 \in \mathbf{H}$ *com* $\|\mathbf{y}_0\| \leq \delta$*, existem controles* $\mathbf{h}_\alpha \in L^\infty(0,T;\mathbf{H}^{-1/2}(\gamma))$ *com* $\int_\gamma \mathbf{h}_\alpha \cdot \mathbf{n}\, d\Gamma = 0$ *e soluções associadas para* (11) *que verificam* (12)*. Além disso, esses controles podem ser encontrados satisfazendo a estimativa*

$$\|\mathbf{h}_\alpha\|_{L^\infty(H^{-1/2})} \leq C, \tag{14}$$

onde C *é também independente de* α*.*

As provas se baseiam em argumentos de ponto fixo. A idéia base tem sido aplicada em muitos outros problemas de controle não-linear. No entanto, nos presentes casos, encontramos duas dificuldades específicas:

- Para encontrar espaços e aplicações de ponto fixo adequadas para o Teorema do ponto fixo de Schauder, o dado inicial $\mathbf{y}_0$ deve ser suficientemente regular. Consequentemente, devemos estabelecer *propriedades de regularidade* para (10) e (11).
- Para a prova das estimativas uniformes (13) e (14), cuidadosas estimativas dos controles nulos e dos estados associados de alguns problemas lineares serão necessárias.

Também provaremos resultados relacionados à controlabilidade no limite, quando $\alpha \to 0^+$. Será mostrado que os controles nulos para (10) podem ser escolhidos de tal maneira que eles convergem para controles nulos do sistema de Navier-Stokes

$$\begin{cases} \mathbf{y}_t - \Delta\mathbf{y} + (\mathbf{y}\cdot\nabla)\mathbf{y} + \nabla p = \mathbf{v}1_\omega & \text{em} \quad Q, \\ \nabla\cdot\mathbf{y} = 0 & \text{em} \quad Q, \\ \mathbf{y} = \mathbf{0} & \text{sobre} \quad \Sigma, \\ \mathbf{y}(0) = \mathbf{y}_0 & \text{em} \quad \Omega. \end{cases} \tag{15}$$

Também, será visto que os controles nulos para (11) podem ser escolhidos tais que eles convergem a controles nulos fronteira do sistema de Navier-Stokes

$$\begin{cases} \mathbf{y}_t - \Delta\mathbf{y} + (\mathbf{y}\cdot\nabla)\mathbf{y} + \nabla p = \mathbf{0} & \text{em} \quad Q, \\ \nabla\cdot\mathbf{y} = 0 & \text{em} \quad Q, \\ \mathbf{y} = \mathbf{h}1_\gamma & \text{sobre} \quad \Sigma, \\ \mathbf{y}(0) = \mathbf{y}_0 & \text{em} \quad \Omega. \end{cases} \tag{16}$$

Mais precisamente, temos os seguintes resultados:

Teorema (Convergência no caso da controlabilidade interna)**.** *Seja* $\epsilon > 0$ *fornecido pelo Teorema sobre controlabilidade interna nula local uniforme do sistema Leray-*α*. Suponhamos que* $\mathbf{y}_0 \in \mathbf{H}$ *e* $\|\mathbf{y}_0\| \leq \epsilon$*, seja* $\mathbf{v}_\alpha$ *um controle nulo para* (10) *satisfazendo* (13) *e seja* $(\mathbf{y}_\alpha, p_\alpha, \mathbf{z}_\alpha, \pi_\alpha)$ *o estado associado. Então, ao menos para um subseqüência, temos*

$$\mathbf{v}_\alpha \to \mathbf{v} \text{ fraco-}* \text{ em } L^\infty(0,T;\mathbf{L}^2(\omega)),$$
$$\mathbf{z}_\alpha \to \mathbf{y} \text{ e } \mathbf{y}_\alpha \to \mathbf{y} \text{ fortemente em } \mathbf{L}^2(Q),$$

quando $\alpha \to 0^+$*, onde* $(\mathbf{y}, \mathbf{v})$ *é, junto com alguma pressão* p*, um par estado-controle para* (15) *satisfazendo* (12)*.*

Teorema (Convergência no caso da controlabilidade fronteira)**.** *Seja* $\delta > 0$ *fornecido pelo Teorema sobre a Controlabilidade fronteira nula local uniforme do sistema de Leray-*α*. Suponhamos que* $\mathbf{y}_0 \in \mathbf{H}$ *e* $\|\mathbf{y}_0\| \leq \delta$*, seja* $\mathbf{h}_\alpha$ *um controle nulo para* (11) *satisfazendo* (14) *e seja* $(\mathbf{y}_\alpha, p_\alpha, \mathbf{z}_\alpha, \pi_\alpha)$ *o estado associado. Então, ao menos para um subseqüência, temos*

$$\mathbf{h}_\alpha \to \mathbf{h} \text{ fraco-}* \text{ em } L^\infty(0,T; H^{-1/2}(\gamma)),$$
$$\mathbf{z}_\alpha \to \mathbf{y} \text{ e } \mathbf{y}_\alpha \to \mathbf{y} \text{ fortemente em } \mathbf{L}^2(Q),$$

quando $\alpha \to 0^+$*, onde* $(\mathbf{y}, \mathbf{h})$ *é, junto com alguma pressão* p*, um par estado-controle para* (16) *satisfazendo* (12)*.*

Estes resultados encontram-se em [2].

Capítulo 3: Sobre a controlabilidade fronteira de fluidos Euler incompressíveis com efeitos de calor Boussinesq

Seja Ω um subconjunto aberto, limitado e não-vazio de $\mathbb{R}^N$ de classe C^∞ ($N = 2$ ou $N = 3$). Suponhamos que Ω é conexo e (por simplicidade) simplesmente conexo. Seja Γ_0 um subconjunto aberto e não-vazio da fronteira Γ de Ω.

Neste capítulo, estamos preocupados com a controlabilidade fronteira do sistema:

$$\begin{cases} \mathbf{y}_t + (\mathbf{y}\cdot\nabla)\mathbf{y} = -\nabla p + \vec{\mathbf{k}}\theta & \text{em} \quad \Omega\times(0,T), \\ \nabla\cdot\mathbf{y} = 0 & \text{em} \quad \Omega\times(0,T), \\ \theta_t + \mathbf{y}\cdot\nabla\theta = 0 & \text{em} \quad \Omega\times(0,T), \\ \mathbf{y}(\mathbf{x},0) = \mathbf{y}_0(\mathbf{x}),\ \theta(\mathbf{x},0) = \theta_0(\mathbf{x}) & \text{em} \quad \Omega, \end{cases} \tag{17}$$

onde:

- $\mathbf{y}$ e a função escalar p representam o campo velocidade e a pressão de um fluido incompressível invíscido em $\Omega \times (0,T)$.
- A função θ fornece a distribuição de temperatura de um fluido.
- $\vec{\mathbf{k}}\theta$ pode ser visto como a densidade da *força de flutuação* ($\vec{\mathbf{k}}$ é um vetor não nulo de $\mathbb{R}^N$).

O sistema (17) é chamado de *Boussinesq invíscido incompressível.*

De agora em diante, suponhamos que $\alpha \in (0,1)$ e definamos

$$\mathbf{C}(m,\alpha,\Gamma_0) := \{\, \mathbf{u} \in \mathbf{C}^{m,\alpha}(\overline{\Omega};\mathbb{R}^N) \,:\, \nabla\cdot\mathbf{u} = 0 \text{ em } \overline{\Omega} \text{ e } \mathbf{u}\cdot\mathbf{n} = 0 \text{ sobre } \Gamma\backslash\Gamma_0 \,\}, \tag{18}$$

onde $\mathbf{C}^{m,\alpha}(\overline{\Omega};\mathbb{R}^N)$ denota o espaço das funções em $\mathbf{C}^m(\overline{\Omega};\mathbb{R}^N)$ em que as derivadas de ordem m são Hölder-contínuas com expoente α.

O problema de controlabilidade fronteira exata para (17) pode ser formulado como segue:

Dados $\mathbf{y}_0$, $\mathbf{y}_1 \in \mathbf{C}(2,\alpha,\Gamma_0)$ *e* θ_0, $\theta_1 \in C^{2,\alpha}(\overline{\Omega};\mathbb{R})$, *encontrar* $\mathbf{y} \in C^0([0,T];\mathbf{C}(1,\alpha,\Gamma_0))$, $\theta \in C^0([0,T];C^{1,\alpha}(\overline{\Omega};\mathbb{R}))$ *e* $p \in \mathcal{D}'(\Omega \times (0,T))$ *tal que* (17) *vale e*

$$\mathbf{y}(\mathbf{x},T) = \mathbf{y}_1(\mathbf{x}), \quad \theta(\mathbf{x},T) = \theta_1(\mathbf{x}) \quad em \quad \Omega. \tag{19}$$

Se é sempre possível encontrar $\mathbf{y}$, θ e p verificando (17) e (19), dizemos que o sistema de Boussinesq invíscido incompressível é*exatamente controlável* para (Ω,Γ_0) no tempo T.

Observação. Para determinar, sem ambigüidade, uma única solução regular local no tempo para (17), é suficiente prescrever a componente normal do campo velocidade sobre a fronteira de uma região fluida e todo o campo $\mathbf{y}$ e a temperatura θ somente sobre a seção de entrada de fluido, i.e. somente onde $\mathbf{y}\cdot\mathbf{n} < 0$, veja por exemplo [68, 59]. Assim, em (17), podemos supor que os controles são dados por:

$$\begin{cases} \mathbf{y}\cdot\mathbf{n} \text{ sobre } \Gamma_0 \times (0,T), \text{ com } \displaystyle\int_{\Gamma_0} \mathbf{y}\cdot\mathbf{n}\, d\Gamma = 0; \\ \mathbf{y} \text{ em qualquer ponto de } \Gamma_0 \times (0,T]) \text{ satisfazendo } \mathbf{y}\cdot\mathbf{n} < 0; \\ \theta \text{ em qualquer ponto de } \Gamma_0 \times (0,T) \text{ satisfazendo } \mathbf{y}\cdot\mathbf{n} < 0. \end{cases}$$

□

O significado das propriedades de controlabilidade exata é que, quando valem, podemos dirigir exatamente um fluido, atuando somente sobre uma parte arbitrariamente pequena Γ_0 da fronteira durante um intervalo de tempo arbitrariamente pequeno $(0,T)$, de qualquer estado inicial $(\mathbf{y}_0,\theta_0)$ a qualquer estado final $(\mathbf{y}_1,\theta_1)$.

O sistema de Boussinesq é potencialmente relevante para o estudo da turbulência atmosférica e oceanográfica, bem como para outras situações astrofísicas onde a rotação e estratificação desempenham um papel dominante (ver, e.g. [71]). Em mecânica dos fluidos, (17) é utilizado no campo de fluxo orientado a flutuabilidade. Ele descreve o movimento de um fluido viscoso incompressível sujeito a transferência de calor por convecção sob a influência de forças gravitacionais, veja [69].

A controlabilidade de sistemas governados por EDPs (lineares e não-lineares) tem sido foco de atenção de muitos pesquisadores nas últimas décadas. Alguns resultados relacionados podem ser encontrados em [16, 47, 62, 81]. No contexto de fluidos incompressíveis ideais, este assunto tem sido principalmente investigado por Coron [13, 15] e Glass [43, 44, 45].

Neste capítulo, vamos adaptar as técnicas e argumentos de [15] e [45] às situações modeladas por (17).

O principal resultado é o seguinte:

Teorema. *O sistema de Boussinesq invíscido incompressível* (17) *é exatamente controlável para* (Ω,Γ_0) *no tempo* $T > 0$.

A prova deste Teorema se baseia nos métodos de extensão e do retorno. Esses métodos tem sido aplicados em vários contextos diferentes para estabelecer controlabilidade; veja o pioneiro trabalho [72] e as contribuições [13, 15, 43, 44].

Vamos dar um esboço da estratégia:

- Primeiro, construímos uma "boa" trajetória conectando **0** a **0**.
- Apliquemos o método de extensão de David L. Russell [72].
- Usamos um teorema de ponto-fixo para obter um resultado de controlabilidade local.
- Finalmente, usamos um argumento de mudança de escala apropriado para deduzir o resultado global desejado.

De fato, o Teorema acima é uma consequência do seguinte resultado:

Proposição. *Existe* $\delta > 0$ *tal que, para qualquer* $\theta_0 \in C^{2,\alpha}(\overline{\Omega};\mathbb{R})$ *e* $\mathbf{y}_0 \in \mathbf{C}(2,\alpha,\Gamma_0)$ *com*

$$\max\{\|\mathbf{y}_0\|_{2,\alpha}, \|\theta_0\|_{2,\alpha}\} < \delta,$$

exitem $\mathbf{y} \in C^0([0,1];\mathbf{C}(1,\alpha,\Gamma_0))$, $\theta \in C^0([0,1];C^{1,\alpha}(\overline{\Omega};\mathbb{R}))$ *e* $p \in \mathcal{D}'(\overline{\Omega}\times[0,1])$ *satisfazendo* (17) (*para* $T=1$) *em* $\Omega\times(0,1)$ *e*

$$\mathbf{y}(\mathbf{x},1) = \mathbf{0},\ \theta(\mathbf{x},1) = 0 \quad em \quad \Omega. \tag{20}$$

Estes resultados podem ser vistos em [31].

Capítulo 4: Sobre o controle de alguns sistemas acoplados do tipo Boussinesq com poucos controles

Seja $\Omega \subset \mathbb{R}^N$ um conjunto aberto conexo e limitado cuja fronteira $\partial\Omega$ é suficientemente regular (por exemplo de classe C^2; $N=2$ ou $N=3$). Seja $\mathcal{O} \subset \Omega$ um subconjunto aberto não-vazio e suponha que $T > 0$. Usaremos a notação $Q = \Omega \times (0,T)$ e $\Sigma = \partial\Omega \times (0,T)$ e denotaremos por $\mathbf{n} = \mathbf{n}(\mathbf{x})$ o vetor normal exterior a Ω em qualquer ponto $\mathbf{x} \in \partial\Omega$.

Vamos tratar com o seguinte sistema de controle

$$\begin{cases} \mathbf{y}_t - \Delta\mathbf{y} + (\mathbf{y}\cdot\nabla)\mathbf{y} + \nabla p = \mathbf{v}1_{\mathcal{O}} + \mathbf{F}(\theta,c) & \text{em} & Q, \\ \nabla\cdot\mathbf{y} = 0 & \text{em} & Q, \\ \theta_t - \Delta\theta + \mathbf{y}\cdot\nabla\theta = w_1 1_{\mathcal{O}} + f_1(\theta,c) & \text{em} & Q, \\ c_t - \Delta c + \mathbf{y}\cdot\nabla c = w_2 1_{\mathcal{O}} + f_2(\theta,c) & \text{em} & Q, \\ \mathbf{y} = \mathbf{0},\ \theta = c = 0 & \text{sobre} & \Sigma, \\ \mathbf{y}(0) = \mathbf{y}_0,\ \theta(0) = \theta_0,\ c(0) = c_0 & \text{em} & \Omega, \end{cases} \tag{21}$$

onde $\mathbf{v} = \mathbf{v}(\mathbf{x},t)$, $w_1 = w_1(\mathbf{x},t)$ e $w_2 = w_2(\mathbf{x},t)$ representam as funções controle. Elas atuam sobre o pequeno conjunto $\mathcal{O}$ durante todo o intervalo de tempo $(0,T)$. O símbolo $1_{\mathcal{O}}$ representa a função característica de $\mathcal{O}$. Vamos assumir que as funções $\mathbf{F} = (F_1,\ldots,F_N)$, f_1 e f_2 satisfazem:

$$\begin{cases} F_i, f_1, f_2 \in C^1(\mathbb{R}^2;\mathbb{R}),\ \text{com } \nabla F_i,\ \nabla f_1,\ \nabla f_2 \in \mathbf{L}^\infty(\mathbb{R}^2;\mathbb{R}^2) \text{ e} \\ F_i(0,0) = f_1(0,0) = f_2(0,0) = 0\ (1 \le i \le N). \end{cases} \tag{22}$$

Em (21), $\mathbf{y}$ e p podem ser interpretadas, respectivamente, como o campo velocidade e a pressão de um fluido. A função θ (resp. c) pode ser vista como a temperatura de um fluido (resp. a concentração de um soluto contaminante). Por outro lado, $\mathbf{v}$, w_1 e w_2 devem ser vistos como termos de força, localmente suportados no espaço, respectivamente para as EDPs satisfeitas por $(\mathbf{y}, p)$, θ e c.

Do ponto de vista da teoria do controle, $(\mathbf{v}, w_1, w_2)$ é o controle e $(\mathbf{y}, p, \theta, c)$ é o estado. Nos problemas considerados neste capítulo, o principal objetivo sempre estará relacionado a escolher $(\mathbf{v}, w_1, w_2)$ tal que $(\mathbf{y}, p, \theta, c)$ satisfaça uma propriedade desejada em $t = T$.

Mais precisamente, apresentaremos alguns resultados que mostram que o sistema (21) pode ser controlado, ao menos localmente, com somente N controles escalares em $L^2(\mathcal{O} \times (0, T))$. Também veremos que, quando $N = 3$, (21) pode ser controlado, ao menos sobre algumas suposições geométricas, com somente 2 (i.e. $N - 1$) controles escalares.

Assim, vamos introduzir os espaços $\mathbf{H}$, $\mathbf{E}$ e $\mathbf{V}$, com

$$\begin{aligned} &\mathbf{V} = \{ \ \varphi \in \mathbf{H}_0^1(\Omega) : \nabla \cdot \varphi = 0 \text{ em } \Omega \ \}, \\ &\mathbf{H} = \{ \ \varphi \in \mathbf{L}^2(\Omega) : \nabla \cdot \varphi = 0 \text{ em } \Omega \text{ e } \varphi \cdot \mathbf{n} = 0 \text{ sobre } \partial\Omega \ \} \\ \text{e} \\ &\mathbf{E} = \begin{cases} \mathbf{H}, & \text{se } N = 2, \\ \mathbf{L}^4(\Omega) \cap \mathbf{H}, & \text{se } N = 3 \end{cases} \end{aligned} \tag{23}$$

e vamos fixar uma *trajetória* $(\overline{\mathbf{y}}, \overline{p}, \overline{\theta}, \overline{c})$, isto é, uma solução suficientemente regular para o sistema não-controlável:

$$\begin{cases} \overline{\mathbf{y}}_t - \Delta\overline{\mathbf{y}} + (\overline{\mathbf{y}} \cdot \nabla)\overline{\mathbf{y}} + \nabla\overline{p} = \mathbf{F}(\overline{\theta}, \overline{c}) & \text{em} \quad Q, \\ \nabla \cdot \overline{\mathbf{y}} = 0 & \text{em} \quad Q, \\ \overline{\theta}_t - \Delta\overline{\theta} + \overline{\mathbf{y}} \cdot \nabla\overline{\theta} = f_1(\overline{\theta}, \overline{c}) & \text{em} \quad Q, \\ \overline{c}_t - \Delta\overline{c} + \overline{\mathbf{y}} \cdot \nabla\overline{c} = f_2(\overline{\theta}, \overline{c}) & \text{em} \quad Q, \\ \overline{\mathbf{y}} = \mathbf{0}, \ \overline{\theta} = \overline{c} = 0 & \text{sobre} \quad \Sigma, \\ \overline{\mathbf{y}}(0) = \overline{\mathbf{y}}_0, \ \overline{\theta}(0) = \overline{\theta}_0, \ \overline{c}(0) = \overline{c}_0 & \text{em} \quad \Omega. \end{cases} \tag{24}$$

Vamos supor que

$$\overline{y}_i, \ \overline{\theta}, \ \overline{c} \in L^\infty(Q) \text{ e } (\overline{y}_i)_t, \overline{\theta}_t, \overline{c}_t \in L^2\big(0, T; L^\kappa(\Omega)\big), \ 1 \leq i \leq N, \tag{25}$$

com

$$\kappa > \begin{cases} 1, & \text{se } N = 2, \\ 6/5, & \text{se } N = 3. \end{cases} \tag{26}$$

Notemos que, se os dados iniciais em (24) satisfazem condições de regularidade adequadas e $(\overline{\mathbf{y}}, \overline{p}, \overline{\theta}, \overline{c})$ resolve (24) (por exemplo no sentido fraco usual) e $\overline{y}_i, \overline{\theta}, \overline{c} \in L^\infty(Q)$, então temos (25). Por exemplo, se $\overline{\mathbf{y}}_0 \in \mathbf{V}$ e $\overline{\theta}_0, \ \overline{c}_0 \in H_0^1(\Omega)$, temos da teoria de regularidade parabólica que $(\overline{y}_i)_t, \ \overline{\theta}_t, \ \overline{c}_t \in L^2(Q)$.

No nosso primeiro resultado principal, vamos supor o seguinte:

$$f_1 \equiv f_2 \equiv 0 \text{ e } \mathbf{F}(a_1, a_2) = a_1\mathbf{e}_N + a_2\overrightarrow{\mathbf{h}}, \text{ onde:} \tag{27}$$

- $\mathbf{e}_N$ é o N-ésimo vetor da base canônica de $\mathbb{R}^N$ e
- $\mathbf{e}_N$ e $\overrightarrow{\mathbf{h}}$ são linearmente independentes.

Então, temos o seguinte resultado:

Teorema. *Suponhamos que* $T > 0$ *é dado e que* (24)–(27) *são satisfeitas. Então existe* $\delta > 0$ *tal que, sempre que* $(\mathbf{y}_0,\ \theta_0,\ c_0) \in \mathbf{E} \times L^2(\Omega) \times L^2(\Omega)$ *e*

$$\|(\mathbf{y}_0,\ \theta_0,\ c_0) - (\overline{\mathbf{y}}_0,\ \overline{\theta}_0,\ \overline{c}_0)\| \leq \delta,$$

podemos encontrar controles L^2 $\mathbf{v}$, w_1 *e* w_2 *com* $\mathbf{v}_i \equiv \mathbf{v}_N \equiv 0$ *para algum* $i < N$ *e estados associados* $(\mathbf{y}, p, \theta, c)$ *satisfazendo*

$$\mathbf{y}(T) = \overline{\mathbf{y}}(T),\ \theta(T) = \overline{\theta}(T) \quad e \quad c(T) = \overline{c}(T). \tag{28}$$

No nosso segundo resultado, vamos considerar funções mais gerais (e talvez não-lineares) $\mathbf{F}$. Denotaremos por $\mathbf{G}$ e $\mathbf{L}$ as derivadas parciais de $\mathbf{F}$ com respeito a θ e a c:

$$\mathbf{G} = \frac{\partial \mathbf{F}}{\partial \theta}, \quad \mathbf{L} = \frac{\partial \mathbf{F}}{\partial c}.$$

Suporemos o seguinte:

$$\begin{array}{l} \text{Existe um conjunto aberto não-vazio } \mathcal{O}_* \subset \mathcal{O} \text{ tal que} \\ \mathbf{G}(\overline{\theta}, \overline{c}) \text{ e } \mathbf{L}(\overline{\theta}, \overline{c}) \text{ são contínuas e linearmente independentes em } \mathcal{O}_* \times (0, T). \end{array} \tag{29}$$

Então, conseguimos um generalização do Teorema anterior:

Teorema. *Suponhamos que* $T > 0$ *é dado e que* (24)–(26) *and* (29) *são satisfeitas. Então existe* $\delta > 0$ *tal que, sempre quando* $(\mathbf{y}_0,\ \theta_0,\ c_0) \in \mathbf{E} \times L^2(\Omega) \times L^2(\Omega)$ *e*

$$\|(\mathbf{y}_0,\ \theta_0,\ c_0) - (\overline{\mathbf{y}}_0,\ \overline{\theta}_0,\ \overline{c}_0)\| \leq \delta,$$

podemos encontrar controles L^2 $\mathbf{v}$, w_1 *e* w_2 *com* $\mathbf{v}_i \equiv \mathbf{v}_j \equiv 0$ *para algum* $i \neq j$ *e estados associados* $(\mathbf{y}, p, \theta, c)$ *satisfazendo* (28).

No caso tridimensional, podemos melhorar o primeiro Teorema apresentado se adicionamos às hipóteses uma apropriada condição geométrica sobre $\mathcal{O}$. Mais precisamente, vamos supor que

$$\exists x^0 \in \partial\Omega,\ \exists a > 0 \ \text{ tal que } \ \overline{\mathcal{O}} \cap \partial\Omega \supset B_a(x^0) \cap \partial\Omega \tag{30}$$

(aqui, $B_a(x^0)$ é a bola centrada em x^0 de raio a).

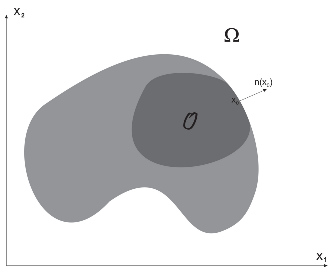

Figura 1: O domínio Ω

Então o seguinte vale:

Teorema. *Suponhamos que* $N = 3$, $T > 0$ *é dado, as suposições do primeiro Teorema são satisfeitas,* (30) *vale e que*

$$\mathbf{h}_1\mathbf{n}_2(x^0) - \mathbf{h}_2\mathbf{n}_1(x^0) \neq 0. \tag{31}$$

Então, a conclusão do primeiro Teorema continua válida com controles L^2 $\mathbf{v}$, w_1 *e* w_2 *tais que* $\mathbf{v} \equiv \mathbf{0}$.

Estes resultados foram publicados em [32].

Comentários adicionais e trabalhos futuros

a) Controlabilidade fronteira para o sistema Burgers-α

Podemos usar um argumento de extensão para provar resultados de controlabilidade fronteira para o sistema Burgers-α.

Assim, vamos introduzir o sistema

$$\begin{cases} y_t - y_{xx} + zy_x = 0 & \text{em} \quad (0,L)\times(0,T), \\ z - \alpha^2 z_{xx} = y & \text{em} \quad (0,L)\times(0,T), \\ z(0,\cdot) = y(0,\cdot) = 0,\ z(L,\cdot) = y(L,\cdot) = u & \text{sobre} \quad (0,T), \\ y(\cdot,0) = y_0 & \text{em} \quad (0,L), \end{cases} \tag{32}$$

onde $u = u(t)$ representa a função controle e $y_0 \in H_0^1(0,L)$ está dado.

Seja a, b e $\tilde{L}$ dados, com $L < a < b < \tilde{L}$. Então, vamos definir $\tilde{y}_0 : [0,\tilde{L}] \mapsto \mathbb{R}$, com $\tilde{y}_0 := y_0 1_{[0,L]}$. Pode ser provado que existe $(\tilde{y}, z, \tilde{v})$, com $\tilde{v} \in L^\infty((a,b)\times(0,T))$,

$$\begin{cases} \tilde{y}_t - \tilde{y}_{xx} + z1_{[0,L]}\,\tilde{y}_x = \tilde{v}1_{(a,b)} & \text{em} \quad (0,\tilde{L})\times(0,T), \\ z - \alpha^2 z_{xx} = \tilde{y} & \text{em} \quad (0,L)\times(0,T), \\ \tilde{y}(0,\cdot) = z(0,\cdot) = \tilde{y}(\tilde{L},\cdot) = 0,\ z(L,\cdot) = \tilde{y}(L,\cdot) & \text{sobre} \quad (0,T), \\ \tilde{y}(\cdot,0) = \tilde{y}_0 & \text{em} \quad (0,\tilde{L}), \end{cases}$$

e $\tilde{y}(x,T) \equiv 0$. Então, $y := \tilde{y}1_{(0,L)}$, z e $u(t) := \tilde{y}(L,t)$ satisfaz (32).

b) Controlabilidade nula global para o sistema Burgers-α?

Para nosso conhecimento, não se sabe se um resultado de controlabilidade nula global geral vale para (4).

Note que não podemos esperar que (4) seja globalmente controlável a zero com controles limitados independentemente de α, desde que o problema limite (3) não é globalmente controlável a zero, veja [28, 50].

c) Sobre as propriedades globais de controlabilidade para o sistema Leray-α

Não se sabe se um resultado de controlabilidade nula global geral vale para (10). Isto não é uma surpresa, desde que a mesma questão está também aberta para as equações de Navier-Stokes.

O que se pode provado (bem como para as equações de Navier-Stokes) é a controlabilidade nula para tempo grande: para qualquer $\mathbf{y}_0 \in \mathbf{H}$, existe $T_* = T_*(\|\mathbf{y}_0\|)$ tal que (10) pode ser dirigido exatamente a zero com controles $\mathbf{v}_\alpha$ uniformemente limitados em $L^\infty(0, T_*; \mathbf{L}^2(\omega))$.

Observações análogas podem ser feitas para o sistema (11).

d) Controlando (10) **e** (11) **com poucos controles**

A controlabilidade nula local com $N-1$ ou menos controles escalares é também uma questão interessante.

Em vista dos resultados de [7] e [19] para as equações de Navier-Stokes, é razoável esperar que resultados semelhantes aos do capítulo 2 valem com controles $\mathbf{v}$ tais que $v_i \equiv 0$ para algum i; Sob algumas restrições geométricas no domínio de controle, é também esperado que a controlabilidade exata local às trajetórias vale com controles do mesmo tipo, veja [30].

e) É possivel obter para as soluções de (17) **a mesma regularidade dos dados?**

Mais precisamente, dados $\mathbf{y}_0$, $\mathbf{y}_1 \in \mathbf{C}(2, \alpha, \Gamma_0)$ e θ_0, $\theta_1 \in C^{2,\alpha}(\overline{\Omega}; \mathbb{R})$, encontrar $\mathbf{y} \in C^0([0,T]; \mathbf{C}(2, \alpha, \Gamma_0))$, $\theta \in C^0([0,T]; C^{2,\alpha}(\overline{\Omega}; \mathbb{R}))$ e $p \in \mathcal{D}'(\Omega \times (0,T))$ tais que (17) vale e

$$\mathbf{y}(\mathbf{x}, T) = \mathbf{y}_1(\mathbf{x}), \quad \theta(\mathbf{x}, T) = \theta_1(\mathbf{x}) \quad \text{em} \quad \Omega.$$

Esta questão será tratada em um trabalho futuro.

f) O que podemos dizer sobre a controlabilidade do sistema de Boussinesq invíscido, calor difusivo, incompressível?

Mais precisamente, o sistema de Boussinesq invíscido com calor difusivo e incompressível é dado por:

$$\begin{cases} \mathbf{y}_t + (\mathbf{y} \cdot \nabla)\mathbf{y} = -\nabla p + \vec{\mathbf{k}}\theta & \text{em} \quad \Omega \times (0,T), \\ \nabla \cdot \mathbf{y} = 0 & \text{em} \quad \Omega \times (0,T), \\ \theta_t + \mathbf{y} \cdot \nabla\theta = \kappa\, \Delta\theta & \text{em} \quad \Omega \times (0,T), \\ \mathbf{y}(\mathbf{x},0) = \mathbf{y}_0(\mathbf{x}), \ \theta(\mathbf{x},0) = \theta_0(\mathbf{x}) & \text{em} \quad \Omega, \end{cases} \tag{33}$$

onde $\kappa > 0$ pode ser visto com um coeficiente de difusão.

Esta questão será tratada em um trabalho futuro.

g) O caso $N = 2$ do sistema (21)

Os resultados que apresentamos para o sistema (21), no caso $N = 2$, mesmo sem impor condição geométrica sobre $\mathcal{O}$, tal como (4.10), valem com somente dois controles escalares w_1 e w_2. Em outras palavras, neste caso, temos controles atuando somente nas EDPs satisfeitas por θ e c (isto que dizer que para controlar este sistema não é necessária uma ação puramente mecânica).

Uma questão natural é se esses resultados valem com somente um controle escalar, mesmo que seja necessário impor uma condição geométrica ou qualquer outra.

h) Controlabilidade nula para (21) sem condições geométricas

Os teoremas do capítulo 4, supondo que a trajetória é $(\overline{\mathbf{y}}, \overline{p}, \overline{\theta}, \overline{c}) \equiv (\mathbf{0}, 0, 0, 0)$. Usando os argumentos de [18], a controlabilidade nula local com controles L^2 $\mathbf{v} \equiv \mathbf{0}$, w_1 and w_2, sem qualquer condição sobre $\mathcal{O}$, pode ser obtida.

Capítulo 1

On the control of the Burgers-alpha model

On the control of the Burgers-alpha model

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Abstract. This work is devoted to prove the local null controllability of the Burgers-α model. The state is the solution to a regularized Burgers equation, where the transport term is of the form zy_x, $z = (Id - \alpha^2 \frac{\partial^2}{\partial x^2})^{-1} y$ and $\alpha > 0$ is a small parameter. We also prove some results concerning the behavior of the null controls and associated states as $\alpha \to 0^+$.

1.1 Introduction and main results

Let $L > 0$ and $T > 0$ be positive real numbers. Let $(a, b) \subset (0, L)$ be a (small) nonempty open subset which will be referred as the control domain.

We will consider the following controlled system for the Burgers equation:

$$\begin{cases} y_t - y_{xx} + yy_x = v1_{(a,b)} & \text{in} \quad (0, L) \times (0, T), \\ y(0, \cdot) = y(L, \cdot) = 0 & \text{on} \quad (0, T), \\ y(\cdot, 0) = y_0 & \text{in} \quad (0, L). \end{cases} \tag{1.1}$$

In (1.1), the function $y = y(x, t)$ can be interpreted as a one-dimensional velocity of a fluid and $y_0 = y_0(x)$ is an initial datum. The function $v = v(x, t)$ (usually in $L^2((a, b) \times (0, T))$) is the control acting on the system and $1_{(a,b)}$ denotes the characteristic function of (a, b).

In this paper, we will also consider a system similar to (1.1), where the transport term is of the form zy_x, where z is the solution to an elliptic problem governed by y. Namely, we consider the following *regularized* version of (1.1), where $\alpha > 0$:

$$\begin{cases} y_t - y_{xx} + zy_x = v1_{(a,b)} & \text{in} \quad (0, L) \times (0, T), \\ z - \alpha^2 z_{xx} = y & \text{in} \quad (0, L) \times (0, T), \\ y(0, \cdot) = y(L, \cdot) = z(0, \cdot) = z(L, \cdot) = 0 & \text{on} \quad (0, T), \\ y(\cdot, 0) = y_0 & \text{in} \quad (0, L). \end{cases} \tag{1.2}$$

This will be called in this paper the Burgers-α system. It is a particular case of the systems introduced in [54] to describe the balance of convection and stretching in the dynamics of one-dimensional nonlinear waves in a fluid with small viscosity. It can also be viewed as a simplified 1D version of the so called Leray-α system, introduced to describe turbulent flows as an alternative to the classical averaged Reynolds models, see [34]; see also [10]. By considering a special kernel associated to the Green's function for the Helmholtz operator, this model compares successfully with empirical data from turbulent channel and pipe flows for a wide range of Reynolds numbers, at least for periodic boundary conditions, see [10] (the Leray-α system is also closely related to the systems treated by Leray in [66] to prove the existence of solutions to the Navier-Stokes equations; see [53]).

Other references concerning systems of the kind (1.2) in one and several dimensions are [9, 41] and [76, 80], respectively for numerical and optimal control issues.

Let us present the notations used along this work. The symbols C, $\hat{C}$ and C_i, $i = 0, 1, \dots$ stand for generic positive constants (usually depending on a, b, L and T). For any $r \in [1, \infty]$ and any given Banach space X, $\|\cdot\|_{L^r(X)}$ will denote the usual norm in $L^r(0, T; X)$. In particular, the norms in $L^r(0, L)$ and $L^r((0, L) \times (0, T))$ will be denoted by $\|\cdot\|_r$. We will also need the Hilbert space $K^2(0, L) := H^2(0, L) \cap H_0^1(0, L)$.

The null controllability problems for (1.1) and (1.2) at time $T > 0$ are the following:

For any $y_0 \in H_0^1(0, L)$, find $v \in L^2((a, b) \times (0, T))$ such that the associated solution to (1.1) (*resp.* (1.2)) *satisfies*

$$y(\cdot, T) = 0 \quad in \quad (0, L). \tag{1.3}$$

Recently, important progress has been made in the controllability analysis of linear and semilinear parabolic equations and systems. We refer to the works [23, 26, 33, 40, 82, 83]. In particular, the controllability of the Burgers equation has been analyzed in [8, 22, 28, 40, 50, 55]. Consequently, it is natural to try to extend the known results to systems like (1.2). Notice that (1.2) is different from (1.1) at least in two aspects: first, the occurrence of non-local in space nonlinearities; secondly, the fact that a small paramter α appears.

Our first main results are the following:

Theorem 1. *For each $T > 0$, the system* (1.2) *is locally null-controllable at time T. More precisely, there exists $\delta > 0$ (independent of α) such that, for any $y_0 \in H_0^1(0, L)$ with $\|y_0\|_\infty \leq \delta$, there exist controls $v_\alpha \in L^\infty((a, b) \times (0, T))$ and associated states (y_α, z_α) satisfying* (2.7). *Moreover, one has*

$$\|v_\alpha\|_\infty \leq C \quad \forall \alpha > 0. \tag{1.4}$$

Theorem 2. *For each $y_0 \in H_0^1(0, L)$ with $\|y_0\|_\infty < \pi/L$, the system* (1.2) *is null-controllable at large time. In other words, there exist $T > 0$ (independent of α), controls $v_\alpha \in L^\infty((a, b) \times (0, T))$ and associated states (y_α, z_α) satisfying* (2.7) *and* (2.8).

Recall that π/L is the square root of the first eigenvalue of the Dirichlet Laplacian in this case. On the other hand, notice that these results provide controls in $L^\infty((a, b) \times (0, T))$ and not only in $L^2((a, b) \times (0, T))$. In fact, this is very convenient not only in (1.1) and (1.2), but also in some intermediate problems arising in the proofs, since this way we obtain better estimates for the states and the existence and convergence assertions are easier to establish.

The main novelty of these results is that they ensure the control of a kind of nonlocal nonlinear parabolic equations. This makes the difference with respect to other previous works, such as [26] or [23, 33]. This is not frequent in the analysis of the controllability of PDEs. Indeed, in general when we deal with nonlocal nonlinearities, it does not seem easy to transmit the information furnished by locally supported controls to the whole domain in a satisfactory way.

We will also prove a result concerning the controllability in the limit, as $\alpha \to 0^+$. More precisely, the following holds:

Theorem 3. *Let $T > 0$ be given and let $\delta > 0$ be the constant furnished by Theorem 1. Assume that $y_0 \in H_0^1(0,L)$ with $\|y_0\|_{L^\infty} \leq \delta$, let v_α be a null control for (1.2) satisfying (2.8) and let (y_α, z_α) be an associated state satisfying (2.7). Then, at least for a subsequence, one has*

$$\begin{aligned} v_\alpha \to v \text{ weakly-}* \text{ in } L^\infty((a,b) \times (0,T)), \\ z_\alpha \to y \text{ and } y_\alpha \to y \text{ weakly-}* \text{ in } L^\infty((0,L) \times (0,T)) \end{aligned} \tag{1.5}$$

as $\alpha \to 0^+$, where (v, y) is a control-state pair for (1.1) that verifies (2.7).

The rest of this paper is organized as follows. In Section 4.2, we prove some results concerning the existence, uniqueness and regularity of the solution to (1.6). Sections 4.3, 4.4, and 4.5 deal with the proofs of Theorems 1, 2 and 3, respectively. Finally, in Section 4.6, we present some additional comments and questions.

1.2 Preliminaries

In this Section, we will first establish a result concerning global existence and uniqueness for the Burgers-α system

$$\begin{cases} y_t - y_{xx} + z y_x = f & \text{in} \quad (0,L) \times (0,T), \\ z - \alpha^2 z_{xx} = y & \text{in} \quad (0,L) \times (0,T), \\ y(0,\cdot) = y(L,\cdot) = z(0,\cdot) = z(L,\cdot) = 0 & \text{on} \quad (0,T), \\ y(\cdot,0) = y_0 & \text{in} \quad (0,L). \end{cases} \tag{1.6}$$

It is the following:

Proposition 1. *Assume that $\alpha > 0$. Then, for any $f \in L^\infty((0,L)\times(0,T))$ and $y_0 \in H_0^1(0,L)$, there exists exactly one solution (y_α, z_α) to (1.6), with*

$$\begin{gathered} y_\alpha \in L^2(0,T;H^2(0,L)) \cap C^0([0,T];H_0^1(0,L)), \\ z_\alpha \in L^2\big(0,T;H^4(0,L)\big) \cap L^\infty\big(0,T;H_0^1(0,L) \cap H^3(0,L)\big), \\ (y_\alpha)_t \in L^2((0,L)\times(0,T)),\ (z_\alpha)_t \in L^2\big(0,T;H^2(0,L)\big). \end{gathered}$$

Furthermore, the following estimates hold:

$$\begin{aligned} \|(y_\alpha)_t\|_2 + \|y_\alpha\|_{L^2(H^2)} + \|y_\alpha\|_{L^\infty(H_0^1)} &\leq C(\|y_0\|_{H_0^1} + \|f\|_2) e^{C(M(T))^2}, \\ \|z_\alpha\|^2_{L^\infty(L^2)} + 2\alpha^2 \|z_\alpha\|^2_{L^\infty(H_0^1)} &\leq \|y_\alpha\|^2_{L^\infty(L^2)}, \\ 2\alpha^2 \|(z_\alpha)_x\|^2_{L^\infty(L^2)} + \alpha^4 \|(z_\alpha)_{xx}\|^2_{L^\infty(L^2)} &\leq \|y_\alpha\|^2_{L^\infty(L^2)}, \\ \|y_\alpha\|_\infty &\leq M(T), \\ \|z_\alpha\|_\infty &\leq M(T), \end{aligned} \tag{1.7}$$

where $M(t) := \|y_0\|_\infty + t\|f\|_\infty$.

Demonstração. A) EXISTENCE: We will reduce the proof to the search of a fixed point of an appropriate mapping Λ_α.

Thus, for each $\overline{y} \in L^\infty((0,L)\times(0,T))$, let $z = z(x,t)$ be the unique solution to

$$\begin{cases} z - \alpha^2 z_{xx} = \overline{y}, & \text{in} \quad (0,L)\times(0,T), \\ z(0,\cdot) = z(L,\cdot) = 0 & \text{on} \quad (0,T). \end{cases} \tag{1.8}$$

Since $\overline{y} \in L^\infty((0,L)\times(0,T))$, it is clear that $z \in L^\infty(0,T;K^2(0,L))$. Then, thanks to the Sobolev embedding, we have $z,\ z_x \in L^\infty((0,L)\times(0,T))$ and the following is satisfied:

$$\begin{aligned} \|z\|^2_{L^\infty(L^2)} + 2\alpha^2\|z\|^2_{L^\infty(H^1_0)} &\leq \|\overline{y}\|^2_{L^\infty(L^2)}, \\ 2\alpha^2\|z_x\|^2_{L^\infty(L^2)} + \alpha^4\|z_{xx}\|^2_{L^\infty(L^2)} &\leq \|\overline{y}\|^2_{L^\infty(L^2)}, \\ \|z\|_\infty &\leq \|\overline{y}\|_\infty. \end{aligned} \tag{1.9}$$

From this z, we can obtain y as the unique solution to the linear problem

$$\begin{cases} y_t - y_{xx} + z y_x = f & \text{in} \quad (0,L)\times(0,T), \\ y(0,\cdot) = y(L,\cdot) = 0 & \text{on} \quad (0,T), \\ y(\cdot,0) = y_0 & \text{in} \quad (0,L). \end{cases} \tag{1.10}$$

Since $z, f \in L^\infty((0,L)\times(0,T))$ and $y_0 \in H^1_0(0,L)$, it is *clear* that

$$y \in L^2(0,T;K^2(0,L)) \cap C^0([0,T];H^1_0(0,L)),$$
$$y_t \in L^2((0,L)\times(0,T))$$

and we have the following estimate:

$$\|y_t\|_2 + \|y\|_{L^2(H^2)} + \|y\|_{L^\infty(H^1_0)} \leq C(\|y_0\|_{H^1_0} + \|f\|_2)e^{C\|z\|^2_\infty}. \tag{1.11}$$

Indeed, this can be easily deduced, for instance, from a standard Galerkin approximation and Gronwall's Lemma; see for instance [21].

We will use the following result, whose proof is given below, after the proof of this Theorem.

Lemma 1. *The solution y to* (1.10) *satisfies*

$$\|y\|_\infty \leq M(T). \tag{1.12}$$

Now, we introduce the Banach space

$$W = \{w \in L^\infty(0,T;H^1_0(0,L)) : w_t \in L^2((0,L)\times(0,T))\}, \tag{1.13}$$

the closed ball

$$K = \{w \in L^\infty((0,L)\times(0,T)) : \|w\|_\infty \leq M(T)\}$$

and the mapping $\tilde{\Lambda}_\alpha$, with $\tilde{\Lambda}_\alpha(\overline{y}) = y$ for all $\overline{y} \in L^\infty((0,L)\times(0,T))$. Obviously $\tilde{\Lambda}_\alpha$ is well defined and, in view of Lemma 1, maps the whole space $L^\infty((0,L)\times(0,T))$ into $W \cap K$.

Let us denote by Λ_α the restriction to K of $\tilde{\Lambda}_\alpha$. Then, thanks to Lemma 1, Λ_α maps K into itself. Moreover, it is clear that $\Lambda_\alpha : K \mapsto K$ satisfies the hypotheses of Schauder's Theorem. Indeed, this nonlinear mapping is continuous and compact (the latter is a consequence of the fact that, if B is bounded in $L^\infty((0,L)\times(0,T))$, then $\Lambda_\alpha(B)$ is bounded in W and therefore it is relatively compact in the space $L^\infty((0,L)\times(0,T))$, in view of the classical results of the Aubin-Lions' kind, see for instance [77]). Consequently, Λ_α possesses at least one fixed point in K.

This immediately achieves the proof of existence.

B) Uniqueness: Let (z'_α, y'_α) be another solution to (1.6) and let us introduce $u := y_\alpha - y'_\alpha$ and $m := z_\alpha - z'_\alpha$. Then

$$\begin{cases} u_t - u_{xx} + z_\alpha u_x = -m(y'_\alpha)_x & \text{in} \quad (0,L)\times(0,T), \\ m - \alpha^2 m_{xx} = u & \text{in} \quad (0,L)\times(0,T), \\ u(0,\cdot) = u(L,\cdot) = m(0,\cdot) = m(L,\cdot) = 0 & \text{on} \quad (0,T), \\ u(\cdot,0) = 0 & \text{in} \quad (0,L). \end{cases}$$

Since $y'_\alpha \in L^2(0,T;H^2(0,L))$, thanks to the Sobolev embedding, we have $y'_\alpha \in L^2(0,T;C^1[0,L])$. Therefore, we easily get from the first equation of the previous system that

$$\frac{1}{2}\frac{d}{dt}\|u\|_2^2 + \|u_x\|_2^2 \le \|z_\alpha\|_\infty \|u_x\|_2 \|u\|_2 + \|(y'_\alpha)_x\|_\infty \|m\|_2 \|u\|_2.$$

Since $\|m\|_2 \le \|u\|_2$, we have

$$\frac{d}{dt}\|u\|_2^2 + \|u_x\|_2^2 \le \Big(\|z_\alpha\|_\infty^2 + 2\|(y'_\alpha)_x\|_\infty\Big)\|u\|_2^2.$$

Therefore, in view of Gronwall's Lemma, we necessarily have $u \equiv 0$. Accordingly, we also obtain $m \equiv 0$ and uniqueness holds. □

Let us now return to Lemma 1 and establish its proof.

Proof of Lemma 1. Let y be the solution to (1.10) and let us set $w = (y - M(t))_+$. Notice that $w(x,0) \equiv 0$ and $w(0,t) \equiv w(L,t) \equiv 0$.

Let us multiply the first equation of (1.10) by w and let us integrate on $(0,L)$. Then we obtain the following for all t:

$$\int_0^L (y_t w + z y_x w)\,dx + \int_0^L y_x w_x\,dx = \int_0^L f w\,dx.$$

This can also be written in the form

$$\int_0^L (w_t w + z w_x w)\,dx + \int_0^L |w_x|^2\,dx = \int_0^L (f - M_t) w\,dx$$

and, consequently, we obtain the identity

$$\frac{1}{2}\frac{d}{dt}\|w\|_2^2 + \|w_x\|_2^2 - \frac{1}{2}\int_0^L z_x |w|^2\,dx = \int_0^L (f - \|f\|_\infty) w\,dx$$

and, therefore,

$$\frac{1}{2}\frac{d}{dt}\|w\|_2^2 + \|w_x\|_2^2 - \frac{1}{2}\int_0^L z_x |w|^2\,dx \le 0. \tag{1.14}$$

Since $z_x \in L^\infty((0,L)\times(0,T))$, it follows by (1.14) that

$$\frac{d}{dt}\|w\|_2^2 \le \|z_x\|_\infty \|w\|_2^2.$$

Then, using again Gronwall's Lemma, we see that $w \equiv 0$.

Analogously, if we introduce $\widetilde{w} = (y + M(t))_-$, similar computations lead to the identity $\widetilde{w} \equiv 0$. Therefore, y satisfies (1.12) and the Lemma is proved.

□

We will now see that, when f is fixed and $\alpha \to 0^+$, the solution to (1.6) converges to the solution to the Burgers system

$$\begin{cases} y_t - y_{xx} + y y_x = f & \text{in} \quad (0,L)\times(0,T), \\ y(0,\cdot) = y(L,\cdot) = 0 & \text{on} \quad (0,T), \\ y(\cdot,0) = y_0 & \text{in} \quad (0,L). \end{cases} \tag{1.15}$$

Proposition 2. *Assume that $y_0 \in H_0^1(0,L)$ and $f \in L^\infty((0,L)\times(0,T))$ are given. For each $\alpha > 0$, let (y_α, z_α) be the unique solution to* (1.6). *Then*

$$z_\alpha \to y \text{ and } y_\alpha \to y \text{ strongly in } L^2(0,T;H_0^1(0,L)) \tag{1.16}$$

as $\alpha \to 0^+$, where y is the unique solution to (1.15).

Demonstração. Since (y_α, z_α) is the solution to (1.6), we have (1.7). Therefore, there exists y such that, at least for a subsequence, we have

$$\begin{aligned} y_\alpha &\to y \text{ weakly in } L^2(0,T;H^2(0,L)), \\ y_\alpha &\to y \text{ weakly-}* \text{ in } L^\infty(0,T;H_0^1(0,L)), \\ (y_\alpha)_t &\to y_t \text{ weakly in } L^2((0,L)\times(0,T)). \end{aligned} \tag{1.17}$$

The Hilbert space

$$Y = \{\, w \in L^2(0,T;K^2(0,L)) : w_t \in L^2((0,L)\times(0,T)) \,\}$$

is compactly embedded in $L^2(0,T;H_0^1(0,L))$. Consequently,

$$y_\alpha \to y \text{ strongly in } L^2(0,T;H_0^1(0,L)). \tag{1.18}$$

Let us see that y is the unique solution to (1.15).

Using the second equation in (1.6), we have

$$(z_\alpha - y) - \alpha^2 (z_\alpha - y)_{xx} = (y_\alpha - y) + \alpha^2 y_{xx}.$$

Multiplying this equation by $-(z_\alpha - y)_{xx}$ and integrating in $(0,L)\times(0,T)$, we obtain

$$
\begin{aligned}
\int_0^T\!\!\int_0^L |(z_\alpha - y)_x|^2\,dx\,dt + \alpha^2 \int_0^T\!\!\int_0^L |(z_\alpha - y)_{xx}|^2\,dx\,dt \\
= \int_0^T\!\!\int_0^L (y_\alpha - y)_x (z_\alpha - y)_x\,dx\,dt \\
- \alpha^2 \int_0^T\!\!\int_0^L y_{xx}(z_\alpha - y)_{xx}\,dx\,dt,
\end{aligned}
$$

whence

$$
\int_0^T\!\!\int_0^L |(z_\alpha - y)_x|^2\,dx\,dt \le \int_0^T\!\!\int_0^L |(y_\alpha - y)_x|^2\,dx\,dt + \alpha^2 \|y_{xx}\|_2^2.
$$

This shows that

$$
z_\alpha \to y \text{ strongly in } L^2(0,T;H_0^1(0,L)) \tag{1.19}
$$

and, consequently,

$$
z_\alpha (y_\alpha)_x \to y y_x, \text{ strongly in } L^1((0,L)\times(0,T)). \tag{1.20}
$$

Finally, for each $\psi \in L^\infty(0,T;H_0^1(0,L))$, we have

$$
\int_0^T\!\!\int_0^L \left((y_\alpha)_t\psi + (y_\alpha)_x\psi_x + z_\alpha (y_\alpha)_x \psi\right)\,dx\,dt = \int_0^T\!\!\int_0^L f\psi\,dx\,dt. \tag{1.21}
$$

Using (1.17) and (1.20), we can take limits in all terms and find that

$$
\int_0^T\!\!\int_0^L \left(y_t\psi + y_x\psi_x + y y_x \psi\right)\,dx\,dt = \int_0^T\!\!\int_0^L f\psi\,dx\,dt, \tag{1.22}
$$

that is, y is the unique solution to (1.15).

This proves that (1.16) holds at least for a subsequence. But, in view of uniqueness, not only a subsequence but the whole sequence converges. □

Remark 1. In fact, a result similar to Proposition 2 can also be established with varying f and y_0. More precisely, if we introduce data f_α and $(y_0)_\alpha$ with

$$
f_\alpha \to f \text{ weakly-}* \text{ in } L^\infty((0,L)\times(0,T))
$$

and

$$
(y_0)_\alpha \to y_0 \text{ weakly-}* \text{ in } L^\infty(0,L),
$$

then we find that the associated solutions (y_α, z_α) satisfy again (1.16). □

To end this Section, we will now recall a result dealing with the null controllability of general parabolic linear systems of the form

$$
\begin{cases}
y_t - y_{xx} + A y_x = v 1_{(a,b)} & \text{in} \quad (0,L)\times(0,T),\\
y(0,\cdot) = y(L,\cdot) = 0 & \text{on} \quad (0,T),\\
y(\cdot,0) = y_0 & \text{in} \quad (0,L).
\end{cases} \tag{1.23}
$$

where $y_0 \in L^2(0,L)$, $A \in L^\infty((0,L)\times(0,T))$ and $v \in L^2((a,b)\times(0,T))$.

It is well known that there exists exactly one solution y to (1.23), with

$$y \in C^0([0,T]; L^2(0,L)) \cap L^2(0,T; H_0^1(0,L)).$$

Related to controllabilty result, we have the following:

Theorem 4. *The linear system* (1.23) *is null controllable at any time* $T > 0$*. In other words, for each* $y_0 \in L^2(0,L)$ *there exists* $v \in L^2((a,b)\times(0,T))$ *such that the associated solution to* (1.23) *satisfies* (2.7)*. Furthermore, the extremal problem*

$$\begin{cases} \textit{Minimize } \; \dfrac{1}{2}\displaystyle\int_0^T\!\!\int_a^b |v|^2\,dx\,dt \\ \textit{Subject to: } v \in L^2((a,b)\times(0,T)),\ (1.23),\ (2.7) \end{cases} \tag{1.24}$$

possesses exactly one solution $\hat{v}$ *satisfying*

$$\|\hat{v}\|_2 \le C_0 \|y_0\|_2, \tag{1.25}$$

where

$$C_0 = e^{C_1(1+1/T+(1+T)\|A\|_\infty^2)}$$

and C_1 *only depends on* a*,* b *and* L*.*

The proof of this result can be found in [58].

1.3 Local null controllability of the Burgers-α model

In this Section, we present the proof of Theorem 1.

Roughly speaking, we fix $\overline{y}$, we solve (1.8), we control exactly to zero the linear system (1.23) with $A = z$ and we set $\Lambda_\alpha(\overline{y}) = y$. Then the task is to solve the fixed point equation $y = \Lambda_\alpha(y)$.

Several fixed point theorems can be applied. In this paper, we have preferred to use Schauder's Theorem, although other results also lead to the good conclusion; for instance, an argument relying on Kakutani's Theorem, like in [23], is possible.

As mentioned above, in order to get good properties for Λ_α, it is very appropriate that the control belongs to L^∞. This can be achieved by several ways; for instance, using an "improved" observability estimate for the solutions to the adjoint of (1.23) and arguing as in [23]. We have preferred here to use other techniques that rely on the regularity of the states and were originally used in [5]; see also [6].

Let $y_0 \in H_0^1(0,L)$ and a', a'', b' and b'' be given, with $0 < a < a' < a'' < b'' < b' < b < L$. Let θ and η satisfy

$$\theta \in C^\infty([0,T]),\ \ \theta \equiv 1 \text{ in } [0,T/4],\ \ \theta \equiv 0 \text{ in } [3T/4,T],$$

$$\eta \in \mathcal{D}(a,b),\ \ \eta \equiv 1 \text{ in a neighborhood of } [a',b'],\ \ 0 \le \eta \le 1.$$

As in the proof of Proposition 1, we can associate to each $\overline{y} \in L^\infty((0,L)\times(0,T))$ the function z through (1.8). Recall that $z, z_x \in L^\infty((0,L)\times(0,T))$ and the inequalities (1.9) are satisfied. In view of Theorem 4, we can associate to z the null control $\hat{v}$ of minimal norm in $L^2((a'',b'')\times(0,T))$, that is, the solution to (1.23)–(2.31) with a, b and A respectively replaced by a'', b'' and z. Let us denote by $\hat{y}$ the corresponding solution to (1.23).

Then, we can write that $\hat{y} = \theta(t)\hat{u} + \hat{w}$, where $\hat{u}$ and $\hat{w}$ are the unique solutions to the linear systems

$$\begin{cases} \hat{u}_t - \hat{u}_{xx} + z\hat{u}_x = 0 & \text{in} \quad (0,L)\times(0,T), \\ \hat{u}(0,\cdot) = \hat{u}(L,\cdot) = 0 & \text{on} \quad (0,T), \\ \hat{u}(\cdot,0) = y_0 & \text{in} \quad (0,L) \end{cases} \tag{1.26}$$

and

$$\begin{cases} \hat{w}_t - \hat{w}_{xx} + z\hat{w}_x = \hat{v}1_{(a'',b'')} - \theta_t\hat{u} & \text{in} \quad (0,L)\times(0,T), \\ \hat{w}(0,\cdot) = \hat{w}(L,\cdot) = 0 & \text{on} \quad (0,T), \\ \hat{w}(\cdot,0) = 0, \ \hat{w}(\cdot,T) = 0 & \text{in} \quad (0,L), \end{cases} \tag{1.27}$$

respectively.

If we now set $w := (1-\eta(x))\hat{w}$, then we have that w is the unique solution of the parabolic system

$$\begin{cases} w_t - w_{xx} + zw_x = v - \theta_t\hat{u} & \text{in} \quad (0,L)\times(0,T), \\ w(0,\cdot) = w(L,\cdot) = 0 & \text{on} \quad (0,T), \\ w(\cdot,0) = 0, \ w(\cdot,T) = 0 & \text{in} \quad (0,L), \end{cases} \tag{1.28}$$

where $v := \eta\theta_t\hat{u} - \eta_x z\hat{w} + 2\eta_x\hat{w}_x + \eta_{xx}\hat{w} + (1-\eta(x))\hat{v}1_{(a'',b'')}$.

Notice that $(1-\eta)\hat{v}1_{(a'',b'')} \equiv 0$, since $\eta \equiv 1$ in $[a',b']$. Therefore, one has

$$v = \eta\theta_t\hat{u} - \eta_x z\hat{w} + 2\eta_x\hat{w}_x + \eta_{xx}\hat{w} \tag{1.29}$$

and then *supp* $v \subset (a,b)$.

Let us prove that $v \in L^\infty((a,b)\times(0,T))$ and

$$\|v\|_\infty \le \hat{C}\|y_0\|_\infty, \tag{1.30}$$

for some

$$\hat{C} = e^{C(a,b,L)(1+1/T+(1+T)\|\overline{y}\|_\infty^2)}. \tag{1.31}$$

First, note that $\hat{u} \in L^\infty((0,L)\times(0,T))$ and $\|\hat{u}\|_\infty \le \|y_0\|_\infty$. Defining

$$G = (a,a') \cup (b',b),$$

we see that it suffices to check that $\eta_x z\hat{w}$, $\eta_x\hat{w}_x$ and $\eta_{xx}\hat{w}$ belong to $L^\infty(G\times(0,T))$, with norms in $L^\infty(G\times(0,T))$ bounded by a constant times the L^2-norm of $\hat{v}$ and the L^∞-norm of y_0, since η_x and η_{xx} are identically zero in a neighborhood of $[a',b']$.

From the usual parabolic estimates for (1.27) and the estimate (1.9), we first obtain that

$$\|\hat{w}_t\|_{L^2(L^2)} + \|\hat{w}\|_{L^2(H^2)} + \|\hat{w}\|_{L^\infty(H_0^1)} \le \|\hat{v}1_{(a'',b'')} - \theta_t\hat{u}\|_{L^2(L^2)} e^{C\|\overline{y}\|_\infty^2}. \tag{1.32}$$

In particular, we have $\hat{w} \in L^\infty((a,b)\times(0,T))$, with appropriate estimates.

On the other hand, $\theta_t \hat{u} \in L^\infty((0,L)\times(0,T))$ and, from the equation satisfied by $\hat{w}$, we have

$$\hat{w}_t - \hat{w}_{xx} + z\hat{w}_x = -\theta_t \hat{u} \ \text{ in } \ [(0,a'')\cup(b'',L)]\times(0,T).$$

Hence, from standard (local in space) parabolic estimates, we deduce that $\hat{w}$ belongs to the space $X^p(0,T;G) = \{\hat{w} \in L^p(0,T;W^{2,p}(G)) : \hat{w}_t \in L^p(0,T;L^p(G))\}$ for all $2 < p < +\infty$.

Then, using Lemma 3.3 (p. 80) of [61], we can take $p > 3$ to get the embedding $X^p(0,T;G) \hookrightarrow C^0([0,T];C^1(\overline{G}))$ and $\hat{w}_x \in C^0(\overline{G}\times[0,T])$. This proves that $\hat{w}_x \in L^\infty(G)$, again with the appropriate estimates.

Therefore, if we define $y := \theta(t)\hat{u} + w$, one has

$$\begin{cases} y_t - y_{xx} + zy_x = v1_{(a,b)} & \text{in} \quad (0,L)\times(0,T), \\ y(0,\cdot) = y(L,\cdot) = 0 & \text{on} \quad (0,T), \\ y(\cdot,0) = y_0 & \text{in} \quad (0,L), \end{cases} \tag{1.33}$$

and (2.7). Moreover, the control v satisfies (1.30)–(1.31).

Let us set $\Lambda_\alpha(\overline{y}) = y$. In this way, we have been able to introduce a mapping

$$\Lambda_\alpha : L^\infty((0,L)\times(0,T)) \mapsto L^\infty((0,L)\times(0,T))$$

for which the following properties are easy to check:

a) Λ_α is continuous and compact. The compactness can be explained as follows: if $B \subset L^\infty((0,L)\times(0,T))$ is bounded, then $\Lambda_\alpha(B)$ is bounded in the space W in (1.13) and, therefore, it is relatively compact in $L^\infty((0,L)\times(0,T))$, in view of classical results of the Aubin-Lions' kind, see for instance [77]).

b) If $R > 0$ and $\|y_0\|_\infty \le \varepsilon(R)$ (independent of α!), then Λ_α maps the ball $B_R := \{\,\overline{y} \in L^\infty((0,L)\times(0,T)) : \|\overline{y}\|_\infty \le R\,\}$ into itself.

The consequence is that, again, Schauder's Theorem can be applied and there exists controls $v_\alpha \in L^\infty((0,L)\times(0,T))$ such that the corresponding solutions to (1.2) satisfy (2.7). This achieves the proof of Theorem 1.

1.4 Large time null controllability of the Burgers-α system

The proof of Theorem 2 is similar. It suffices to replace the assumption "y_0 is small" by an assumption imposing that T is large enough. Again, this makes it possible to apply a fixed point argument.

More precisely, let us accept that, if $y_0 \in H_0^1(0,L)$ and $\|y_0\|_\infty < \pi/L$, then the associted uncontrolled solution y_α to (1.2) satisfies

$$\|y_\alpha(\cdot\,,t)\|_{H_0^1} \le C(y_0)e^{-\frac{1}{2}((\pi/L)^2 - \|y_0\|_\infty^2)t} \tag{1.34}$$

where $C(y_0)$ is a constant only depending on $\|y_0\|_\infty$ and $\|y_0\|_{H_0^1}$. Then, if we first take $v \equiv 0$, the state $y_\alpha(\cdot\,,t)$ becomes small for large t. In a second step, when $\|y_\alpha(\cdot\,,t)\|_{H_0^1}$ is sufficiently small, we can apply Theorem 1 and drive the state exactly to zero.

Let us now see that (1.34) holds. Arguing as in the proof of Proposition 1, we see that

$$\frac{d}{dt}\|y_\alpha\|_2^2 + \|(y_\alpha)_x\|_2^2 \le \|y_0\|_\infty^2 \|y_\alpha\|_2^2 \tag{1.35}$$

and, using Poincaré's inequality, we obtain:

$$\frac{d}{dt}\|y_\alpha\|_2^2 + (\pi/L)^2\|y_\alpha\|_2^2 \le \|y_0\|_\infty^2 \|y_\alpha\|_2^2.$$

Let us introduce $r = \frac{1}{2}((\pi/L)^2 - \|y_0\|_\infty^2)$. It then follows that

$$\|y_\alpha(\cdot\,,t)\|_2^2 \le \|y_0\|_2^2 e^{-2rt}. \tag{1.36}$$

Hence, by combining (1.35) and (1.36), it is easy to see that

$$\frac{d}{dt}\left(e^{rt}\|y_\alpha\|_2^2\right) + e^{rt}\|(y_\alpha)_x\|_2^2 \le (r + \|y_0\|_\infty^2)\|y_0\|_2^2\, e^{-rt}.$$

Integrating from 0 to t yields

$$\int_0^t e^{r\sigma}\|(y_\alpha)_x\|_2^2\,d\sigma \le \left(2 + \frac{\|y_0\|_\infty^2}{r}\right)\|y_0\|_2^2. \tag{1.37}$$

Now, we take the L^2-inner product of (1.6) and $-(y_\alpha)_{xx}$ and get

$$\frac{d}{dt}\|(y_\alpha)_x\|_2^2 \le \|y_0\|_\infty^2 \|(y_\alpha)_x\|^2.$$

Multiplying this inequality by e^{rt}, we deduce that

$$\frac{d}{dt}\left(e^{rt}\|(y_\alpha)_x\|_2^2\right) \le (r + \|y_0\|_\infty^2)\, e^{rt}\, \|(y_\alpha)_x\|_2^2$$

and, consequently, we see from (1.37) that

$$\|(y_\alpha)_x(\cdot\,,t)\|_2^2 \le \left[(r + \|y_0\|_\infty^2)\left(2 + \frac{\|y_0\|_\infty^2}{r}\right)\|y_0\|_2^2 + \|y_0\|_{H_0^1}^2\right]e^{-rt},$$

which implies (1.34).

Remark 2. To our knowledge, it is unknown what can be said when the smallness assumption $\|y_0\|_\infty < \pi/L$ is not satisfied. In fact, it is not clear whether or not the solutions to (1.2) with large initial data and $v \equiv 0$ decay as $t \to +\infty$. □

1.5 Controllability in the limit

In this Section, we are going to prove Theorem 3.

For the null controls v_α furnished by Theorem 1 and the associated solutions (y_α, z_α) to (1.2), we have the uniform estimates (1.30) and (1.7) with $f = v_\alpha 1_{(a,b)}$. Then, there exists

$y \in L^2(0,T;K^2(0,L))$, with $y_t \in L^2((0,L)\times(0,T))$, and $v \in L^\infty((a,b)\times(0,T))$ such that, at least for a subsequence, one has:

$$\begin{aligned} &y_\alpha \to y \text{ weakly in } L^2(0,T;K^2(0,L)),\\ &(y_\alpha)_t \to y_t \text{ weakly in } L^2((0,L)\times(0,T))\\ &v_\alpha \to v \text{ weakly} - * \text{ in } L^\infty((a,b)\times(0,T)). \end{aligned} \tag{1.38}$$

As before, the Aubin-Lions' Lemma implies that

$$y_\alpha \to y \text{ strongly in } L^2(0,T;H_0^1(0,L)). \tag{1.39}$$

Using the second equation in (1.2), we see that

$$(z_\alpha - y) - \alpha^2(z_\alpha - y)_{xx} = (y_\alpha - y) + \alpha^2 y_{xx}.$$

Multiplying this equation by $-(z_\alpha - y)_{xx}$ and integrating in $(0,L)\times(0,T)$, we deduce

$$\begin{aligned} \int_0^T\int_0^L |(z_\alpha - y)_x|^2\,dx\,dt + \alpha^2 \int_0^T\int_0^L |(z_\alpha - y)_{xx}|^2\,dx\,dt \\ = \int_0^T\int_0^L (y_\alpha - y)_x (z_\alpha - y)_x\,dx\,dt \\ - \alpha^2 \int_0^T\int_0^L y_{xx}(z_\alpha - y)_{xx}\,dx\,dt. \end{aligned}$$

Whence,

$$\int_0^T\int_0^L |(z_\alpha - y)_x|^2\,dx\,dt \le \int_0^T\int_0^L |(y_\alpha - y)_x|^2\,dx\,dt + \alpha^2\|y_{xx}\|_2^2.$$

This shows that

$$z_\alpha \to y \text{ strongly in } L^2(0,T;H_0^1(0,L)). \tag{1.40}$$

and the transport terms in (1.2) satisfy

$$z_\alpha (y_\alpha)_x \to y y_x \text{ strongly in } L^1((0,L)\times(0,T)). \tag{1.41}$$

In this way, for each $\psi \in L^\infty(0,T;H_0^1(0,L))$, we obtain

$$\int_0^T\int_0^L \left((y_\alpha)_t\psi + (y_\alpha)_x\psi_x + z_\alpha (y_\alpha)_x\psi\right)\,dx\,dt = \int_0^T\int_0^L v_\alpha 1_{(a,b)}\psi\,dx\,dt. \tag{1.42}$$

Using (1.38) and (1.41), we can pass to the limit, as $\alpha \to 0^+$, in all the terms of (1.42) to find

$$\int_0^T\int_0^L \left(y_t\psi + y_x\psi_x + y y_x\psi\right)\,dx\,dt = \int_0^T\int_0^L v 1_{(a,b)}\psi\,dx\,dt, \tag{1.43}$$

that is, y is the unique solution of (1.1) and y satisfies (2.7).

1.6 Additional comments and questions

1.6.1 A boundary controllability result

We can use an extension argument to prove local boundary controllability results similar to those above.

For instance, let us see that the analog of Theorem 1 remains true. Thus, let us introduce the controlled system

$$\begin{cases} y_t - y_{xx} + zy_x = 0 & \text{in} \quad (0,L)\times(0,T), \\ z - \alpha^2 z_{xx} = y & \text{in} \quad (0,L)\times(0,T), \\ z(0,\cdot) = z(L,\cdot) = y(0,\cdot) = 0, \ y(L,\cdot) = u & \text{on} \quad (0,T), \\ y(\cdot,0) = y_0 & \text{in} \quad (0,L), \end{cases} \tag{1.44}$$

where $u = u(t)$ stands for the control function and $y_0 \in H_0^1(0,L)$ is given.

Let a, b and $\tilde{L}$ be given, with $L < a < b < \tilde{L}$. Then, let us define $\tilde{y}_0 : [0,\tilde{L}] \mapsto \mathbb{R}$, with $\tilde{y}_0 := y_0 1_{[0,L]}$. Arguing as in Theorem 1, it can be proved that there exists $(\tilde{y},\tilde{v})$, with $\tilde{v} \in L^\infty((a,b)\times(0,T))$,

$$\begin{cases} \tilde{y}_t - \tilde{y}_{xx} + z1_{[0,L]}\,\tilde{y}_x = \tilde{v}1_{(a,b)} & \text{in} \quad (0,\tilde{L})\times(0,T), \\ z - \alpha^2 z_{xx} = \tilde{y} & \text{in} \quad (0,L)\times(0,T), \\ \tilde{y}(0,\cdot) = z(0,\cdot) = z(L,\cdot) = \tilde{y}(\tilde{L},\cdot) = 0 & \text{on} \quad (0,T), \\ \tilde{y}(\cdot,0) = \tilde{y}_0 & \text{in} \quad (0,\tilde{L}), \end{cases}$$

and $\tilde{y}(x,T) \equiv 0$. Then, $y := \tilde{y}1_{(0,L)}$, z and $u(t) := \tilde{y}(L,t)$ satisfy (1.44).

Notice that the control that we have obtained satisfies $u \in C^0([0,T])$, since it can be viewed as the lateral trace of a strong solution of the heat equation with a L^∞ right hand side.

1.6.2 No global null controllability?

To our knowledge, it is unknown whether a general global null controllability result holds for (1.2). We can prove global null controllability "for large α".

More precisely, the following holds:

Theorem 5. *Let $y_0 \in H_0^1(0,L)$ and $T > 0$ be given. There exists $\alpha_0 = \alpha_0(y_0,T)$ such that (1.2) can be controlled to zero for all $\alpha > \alpha_0$.*

Demonstração. The main idea is, again, to apply a fixed point argument in $L^\infty(0,T;L^2(0,L))$.

For each $\overline{y} \in L^\infty(0,T;L^2(0,L))$, we introduce the solution z to (1.8). We notice that z satisfies

$$\|z\|_2^2 + 2\alpha^2\|z_x\|_2^2 \le \|\overline{y}\|_2^2,$$
$$2\alpha^2\|z_x\|_2^2 + \alpha^4\|z_{xx}\|_2^2 \le \|\overline{y}\|_2^2.$$

Then, as in the proof of Theorem 1, we consider the solution (y,v) to the system

$$\begin{cases} y_t - y_{xx} + zy_x = v1_{(a,b)} & \text{in} \quad (0,L)\times(0,T), \\ y(0,\cdot) = y(L,\cdot) = 0 & \text{on} \quad (0,T), \\ y(\cdot,0) = y_0 & \text{in} \quad (0,L), \end{cases} \tag{1.45}$$

where we assume that y satisfies (2.7) and v satisfies the estimate

$$\|v\|_\infty \leq \hat{C}\|y_0\|_\infty, \tag{1.46}$$

with

$$\hat{C} = e^{C(a,b,L)(1+1/T+(1+T)\|z\|_\infty^2)}.$$

It is then clear that

$$\|y_t\|_2 + \|y\|_{L^2(H^2)} + \|y\|_{L^\infty(H_0^1)} \leq C\|y_0\|_{H_0^1} e^{C(a,b,L)(1+1/T+(1+T)\|z\|_\infty^2)}. \tag{1.47}$$

Since $\|z\|_\infty^2 \leq \frac{C}{\alpha^2}\|\overline{y}\|_2^2$, we have

$$\|y_t\|_2 + \|y\|_{L^2(H^2)} + \|y\|_{L^\infty(H_0^1)} \leq C\|y_0\|_{H_0^1} e^{C(a,b,L)\left(1+1/T+(1+T)\frac{1}{\alpha^2}\|\overline{y}\|_{L^\infty(L^2)}^2\right)}.$$

We can check that there exist R and α_0 such that

$$C\|y_0\|_{H_0^1} e^{C(a,b,L)\left(1+1/T+(1+T)\frac{1}{\alpha^2}R^2\right)} < R,$$

for all $\alpha > \alpha_0$. Therefore, we can apply the fixed point argument in the ball B_R of $L^2((0,L)\times(0,T))$ for these α. This ends the proof. □

Notice that we cannot expect (1.2) to be globally null-controllable with controls bounded independently of α, since the limit problem (1.1) is not globally null-controllable, see [28, 50]. More precisely, let $y_0 \in H_0^1(0,L)$ and $T > 0$ be given and let us denote by $\hat{\alpha}(y_0,T)$ the infimum of all α_0 furnished by Theorem 5. Then, either $\hat{\alpha}(y_0,T) > 0$ or the associated cost of null controllability grows to infinity as $\alpha \to 0$, i.e. the null controls of minimal norm v_α satisfy

$$\limsup_{\alpha\to 0^+} \|v_\alpha\|_{L^\infty((a,b)\times(0,T))} = +\infty.$$

1.6.3 The situation in higher spatial dimensions. The Leray-α system

Let $\Omega \subset \mathbb{R}^N$ be a bounded connected and regular open set ($N = 2$ or $N = 3$) and let $\omega \subset \Omega$ be a (small) open set. We will use the notation $Q := \Omega \times (0,T)$ and $\Sigma := \partial\Omega \times (0,T)$ and we will use bold symbols for vector-valued functions and spaces of vector-valued functions.

For any $\mathbf{f}$ and any $\mathbf{y}_0$ in appropriate spaces, we will consider the Navier-Stokes system

$$\begin{cases} \mathbf{y}_t - \Delta\mathbf{y} + (\mathbf{y}\cdot\nabla)\mathbf{y} + \nabla p = \mathbf{f} & \text{in} \quad Q, \\ \nabla\cdot\mathbf{y} = 0 & \text{in} \quad Q, \\ \mathbf{y} = \mathbf{0} & \text{on} \quad \Sigma, \\ \mathbf{y}(0) = \mathbf{y}_0 & \text{in} \quad \Omega. \end{cases} \tag{1.48}$$

As before, we will also introduce a smoothing kernel and a related modification of (1.48). More precisely, the following so called Leray-α model will be of interest:

$$\begin{cases} \mathbf{y}_t - \Delta\mathbf{y} + (\mathbf{z}\cdot\nabla)\mathbf{y} + \nabla p = \mathbf{f} & \text{in} \quad Q, \\ \nabla\cdot\mathbf{y} = \nabla\cdot\mathbf{z} = 0 & \text{in} \quad Q, \\ \mathbf{z} - \alpha^2\Delta\mathbf{z} + \nabla\pi = \mathbf{y} & \text{in} \quad Q, \\ \mathbf{y} = \mathbf{z} = \mathbf{0} & \text{on} \quad \Sigma, \\ \mathbf{y}(0) = \mathbf{y}_0 & \text{in} \quad \Omega. \end{cases} \tag{1.49}$$

Let us recall the definitions of some function spaces that are frequently used in the analysis of incompressible fluids:

$$\mathbf{H} = \left\{ \boldsymbol{\varphi} \in \mathbf{L}^2(\Omega) : \nabla \cdot \boldsymbol{\varphi} = 0 \text{ in } \Omega, \quad \boldsymbol{\varphi} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega \right\},$$
$$\mathbf{V} = \left\{ \boldsymbol{\varphi} \in \mathbf{H}_0^1(\Omega) : \nabla \cdot \boldsymbol{\varphi} = 0 \text{ in } \Omega \right\}.$$

It is not difficult to prove that, for any $\alpha > 0$, under some reasonable conditions on $\mathbf{f}$ and $\mathbf{y}_0$, (2.3) possesses a unique global weak solution. This is stated rigorously in the following proposition, that we present without proof (the arguments are similar to those in [79]; the detailed proof will appear in a forthcoming paper):

Proposition 3. *Assume that $\alpha > 0$. Then, for any $\mathbf{f} \in L^2(0,T;\mathbf{H}^{-1}(\Omega))$ and any $\mathbf{y}_0 \in \mathbf{H}$, there exists exactly one solution $(\mathbf{y}_\alpha, p_\alpha, \mathbf{z}_\alpha, \pi_\alpha)$ to (2.3), with*

$$\mathbf{y}_\alpha \in L^2(0,T;\mathbf{V}) \cap C^0([0,T];\mathbf{H}), \ (\mathbf{y}_\alpha)_t \in L^1(0,T;\mathbf{V}'),$$

$$\mathbf{z}_\alpha \in L^2\big(0,T;\mathbf{H}^2(\Omega) \cap \mathbf{V}\big) \cap L^\infty\big(0,T;\mathbf{H}\big).$$

Furthermore, the following estimates hold:

$$\begin{aligned} \|(\mathbf{y}_\alpha)_t\|_{L^1(\mathbf{V}')} + \|\mathbf{y}_\alpha\|_{L^2(\mathbf{V})} + \|\mathbf{y}_\alpha\|_{L^\infty(\mathbf{H})} &\leq C(\|\mathbf{y}_0\|_2 + \|\mathbf{f}\|_{L^2(\mathbf{H}^{-1})}), \\ \|\mathbf{z}_\alpha\|^2_{L^\infty(\mathbf{H})} + 2\alpha^2\|\mathbf{z}_\alpha\|^2_{L^\infty(\mathbf{V})} &\leq \|\mathbf{y}_\alpha\|^2_{L^\infty(\mathbf{H})}, \\ 2\alpha^2\|(\mathbf{z}_\alpha)_x\|^2_{L^\infty(\mathbf{H})} + \alpha^4\|\Delta(\mathbf{z}_\alpha)\|^2_{L^\infty(\mathbf{H})} &\leq \|\mathbf{y}_\alpha\|^2_{L^\infty(\mathbf{H})}. \end{aligned} \tag{1.50}$$

In view of the estimates (2.14), there exists $\mathbf{y} \in L^2\left(0,T;\mathbf{V}\right))$ with $\mathbf{y}_t \in L^1(0,T;\mathbf{V}')$ such that, at least for a subsequence,

$$\begin{aligned} &\mathbf{y}_\alpha \to \mathbf{y} \text{ weakly in } L^2\left(0,T;\mathbf{V}\right)), \\ &(\mathbf{y}_\alpha)_t \to \mathbf{y}_t \text{ weakly-* in } L^1(0,T;\mathbf{V}'). \end{aligned} \tag{1.51}$$

Thanks to the Aubin-Lions' Lemma, the Hilbert space

$$W = \{w \in L^2\left(0,T;\mathbf{V}\right); w_t \in L^1(0,T;\mathbf{V}')\}$$

is compactly embedded in $\mathbf{L}^2(Q)$ and we thus have

$$\mathbf{y}_\alpha \to \mathbf{y} \text{ strongly in } \mathbf{L}^2(Q). \tag{1.52}$$

Also, using the second equation in (2.3) we see that

$$(\mathbf{z}_\alpha - \mathbf{y}) - \alpha^2\Delta(\mathbf{z}_\alpha - \mathbf{y}) + \nabla\pi = (\mathbf{y}_\alpha - \mathbf{y}) + \alpha^2\Delta\mathbf{y}.$$

Therefore, after some computations, we deduce that

$$\mathbf{z}_\alpha \to \mathbf{y} \text{ strongly in } \mathbf{L}^2(Q). \tag{1.53}$$

This proves that we can find p such that $(\mathbf{y}, p)$ is solution to (1.48).

In other words, at least for a subsequence, the solutions to the Leray-α system converge (in the sense of (1.51)) towards a solution to the Navier-Stokes system.

Let us now consider the following controlled systems for the Navier-Stokes and Leray-α systems:

$$\begin{cases} \mathbf{y}_t - \Delta\mathbf{y} + (\mathbf{y}\cdot\nabla)\mathbf{y} + \nabla p = \mathbf{v}1_\omega & \text{in} \quad Q, \\ \nabla\cdot\mathbf{y} = 0 & \text{in} \quad Q, \\ \mathbf{y} = \mathbf{0} & \text{on} \quad \Sigma, \\ \mathbf{y}(0) = \mathbf{y}_0 & \text{in} \quad \Omega \end{cases} \tag{1.54}$$

and

$$\begin{cases} \mathbf{y}_t - \Delta\mathbf{y} + (\mathbf{z}\cdot\nabla)\mathbf{y} + \nabla p = \mathbf{v}1_\omega & \text{in} \quad Q, \\ \nabla\cdot\mathbf{y} = \nabla\cdot\mathbf{z} = 0 & \text{in} \quad Q, \\ \mathbf{z} - \alpha^2\Delta\mathbf{z} + \nabla\pi = \mathbf{y} & \text{in} \quad Q, \\ \mathbf{y} = \mathbf{z} = \mathbf{0} & \text{on} \quad \Sigma, \\ \mathbf{y}(0) = \mathbf{y}_0 & \text{in} \quad \Omega, \end{cases} \tag{1.55}$$

where $\mathbf{v} = \mathbf{v}(x,t)$ stands for the control function.

With arguments similar to those in [29], it can be proved that, for any $T > 0$, there exists $\varepsilon > 0$ such that, if $\|\mathbf{y}_0\| < \varepsilon$, for each $\alpha > 0$ we can find controls $\mathbf{v}_\alpha \in \mathbf{L}^2(\omega\times(0,T))$ and associate states $(\mathbf{y}_\alpha, p_\alpha, \mathbf{z}_\alpha, \pi_\alpha)$ satisfying

$$\mathbf{y}_\alpha(x,T) = \mathbf{0} \quad \text{in} \quad \Omega.$$

In a forthcoming paper, we will show that these null controls $\mathbf{v}_\alpha$ can be bounded independently of α and a result similar to Theorem 3 holds for (2.5).

Capítulo 2

Uniform local null control of the Leray-α model

Uniform local null control of the Leray-α model

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Abstract. This paper deals with the distributed and boundary controllability of the so called Leray-α model. This is a regularized variant of the Navier-Stokes system (α is a small positive parameter) that can also be viewed as a model for turbulent flows. We prove that the Leray-α equations are locally null controllable, with controls bounded independently of α. We also prove that, if the initial data are sufficiently small, the controls converge as $\alpha \to 0^+$ to a null control of the Navier-Stokes equations. We also discuss some other related questions, such as global null controllability, local and global exact controllability to the trajectories, etc.

Keywords: Null controllability, Carleman inequalities, Leray-α model, Navier-Stokes equations.

Mathematics Subject Classification: 93B05, 35Q35, 35G25, 93B07.

2.1 Introduction. The main results

Let $\Omega \subset \mathbb{R}^N (N = 2, 3)$ be a bounded connected open set whose boundary Γ is of class C^2. Let $\omega \subset \Omega$ be a (small) nonempty open set, let $\gamma \subset \Gamma$ be a (small) nonempty open subset of Γ and assume that $T > 0$. We will use the notation $Q = \Omega \times (0, T)$ and $\Sigma = \Gamma \times (0, T)$ and we will denote by $\mathbf{n} = \mathbf{n}(\mathbf{x})$ the outward unit normal to Ω at the points $\mathbf{x} \in \Gamma$; spaces of $\mathbb{R}^N$-valued functions, as well as their elements, are represented by boldface letters.

The Navier-Stokes system for a homogeneous viscous incompressible fluid (with unit density and unit kinematic viscosity) subject to homogeneous Dirichlet boundary conditions is given by

$$\begin{cases} \mathbf{y}_t - \Delta \mathbf{y} + (\mathbf{y} \cdot \nabla)\mathbf{y} + \nabla p = \mathbf{f} & \text{in} \quad Q, \\ \nabla \cdot \mathbf{y} = 0 & \text{in} \quad Q, \\ \mathbf{y} = \mathbf{0} & \text{on} \quad \Sigma, \\ \mathbf{y}(0) = \mathbf{y}_0 & \text{in} \quad \Omega, \end{cases} \tag{2.1}$$

where $\mathbf{y}$ (the velocity field) and p (the pressure) are the unknowns, $\mathbf{f} = \mathbf{f}(\mathbf{x}, t)$ is a forcing term and $\mathbf{y}_0 = \mathbf{y}_0(\mathbf{x})$ is a prescribed initial velocity field.

In order to prove the existence of a solution to the Navier-Stokes system, Leray in [66] had the idea of creating a turbulence *closure* model without enhancing viscous dissipation. Thus, he introduced a "regularized" variant of (2.1) by modifying the nonlinear term as follows:

$$\begin{cases} \mathbf{y}_t - \Delta \mathbf{y} + (\mathbf{z} \cdot \nabla)\mathbf{y} + \nabla p = \mathbf{f} & \text{in} \quad Q, \\ \nabla \cdot \mathbf{y} = 0 & \text{in} \quad Q, \end{cases}$$

where $\mathbf{y}$ and $\mathbf{z}$ are related by

$$\mathbf{z} = \phi_\alpha * \mathbf{y} \tag{2.2}$$

and ϕ_α is a smoothing kernel. At least formally, the Navier-Stokes equations are recovered in the limit as $\alpha \to 0^+$, so that $\mathbf{z} \to \mathbf{y}$.

In this paper, we will consider a special smoothing kernel, associated to the Stokes-like operator $\mathbf{Id} + \alpha^2 \mathbf{A}$, where $\mathbf{A}$ is the Stokes operator (see Section 4.2). This leads to the following modification of the Navier-Stokes equations, called the Leray-α system (see [10]):

$$\begin{cases} \mathbf{y}_t - \Delta \mathbf{y} + (\mathbf{z} \cdot \nabla)\mathbf{y} + \nabla p = \mathbf{f} & \text{in} \quad Q, \\ \mathbf{z} - \alpha^2 \Delta \mathbf{z} + \nabla \pi = \mathbf{y} & \text{in} \quad Q, \\ \nabla \cdot \mathbf{y} = 0, \ \nabla \cdot \mathbf{z} = 0 & \text{in} \quad Q, \\ \mathbf{y} = \mathbf{z} = \mathbf{0} & \text{on} \quad \Sigma, \\ \mathbf{y}(0) = \mathbf{y}_0 & \text{in} \quad \Omega. \end{cases} \tag{2.3}$$

In almost all previous works found in the literature, Ω is either the N-dimensional torus and the PDE's in (2.3) are completed with periodic boundary conditions or the whole space $\mathbb{R}^N$. Then, $\mathbf{z}$ satisfies an equation of the kind

$$\mathbf{z} - \alpha^2 \Delta \mathbf{z} = \mathbf{y} \tag{2.4}$$

and the model is (apparently) slightly different from (2.3). However, since $\nabla \cdot \mathbf{y} = 0$, it is easy to see that (2.4), in these cases, is equivalent to the equation satisfied by $\mathbf{z}$ and π in (2.3).

It has been shown in [10] that, at least for periodic boundary conditions, the numerical solution of the equations in (2.3) matches successfully with empirical data from turbulent channel and pipe flows for a wide range of Reynolds numbers. Accordingly, the Leray-α system has become preferable to other turbulence models, since the associated computational cost is lower and no introduction of *ad hoc* parameters is required.

In [42], the authors have compared the numerical solutions of three different α-models useful in turbulence modelling (in terms of the Reynolds number associated to a Navier-Stokes velocity field). The results improve as one passes from the Navier-Stokes equations to these models and clearly show that the Leray-α system has the best performance. Therefore, it seems quite natural to carry out a theoretical analysis of the solutions to (2.3).

We will be concerned with the following controlled systems

$$\begin{cases} \mathbf{y}_t - \Delta \mathbf{y} + (\mathbf{z} \cdot \nabla)\mathbf{y} + \nabla p = \mathbf{v} 1_\omega & \text{in} \quad Q, \\ \mathbf{z} - \alpha^2 \Delta \mathbf{z} + \nabla \pi = \mathbf{y} & \text{in} \quad Q, \\ \nabla \cdot \mathbf{y} = 0, \ \nabla \cdot \mathbf{z} = 0 & \text{in} \quad Q, \\ \mathbf{y} = \mathbf{z} = \mathbf{0} & \text{on} \quad \Sigma, \\ \mathbf{y}(0) = \mathbf{y}_0 & \text{in} \quad \Omega, \end{cases} \tag{2.5}$$

and

$$\begin{cases} \mathbf{y}_t - \Delta \mathbf{y} + (\mathbf{z} \cdot \nabla)\mathbf{y} + \nabla p = \mathbf{0} & \text{in} \quad Q, \\ \mathbf{z} - \alpha^2 \Delta \mathbf{z} + \nabla \pi = \mathbf{y} & \text{in} \quad Q, \\ \nabla \cdot \mathbf{y} = 0, \ \nabla \cdot \mathbf{z} = 0 & \text{in} \quad Q, \\ \mathbf{y} = \mathbf{z} = \mathbf{h} 1_\gamma & \text{on} \quad \Sigma, \\ \mathbf{y}(0) = \mathbf{y}_0 & \text{in} \quad \Omega, \end{cases} \tag{2.6}$$

where $\mathbf{v} = \mathbf{v}(\mathbf{x}, t)$ (respectively $\mathbf{h} = \mathbf{h}(\mathbf{x}, t)$) stands for the control, assumed to act only in the (small) set ω (respectively on γ) during the whole time interval $(0, T)$. The symbol 1_ω (respectively 1_γ) stands for the characteristic function of ω (respectively of γ).

In the applications, the *internal control* $\mathbf{v}1_\omega$ can be viewed as a gravitational or electromagnetic field. The *boundary control* $\mathbf{h}1_\gamma$ is the trace of the velocity field on Σ.

Remark 3. It is completely natural to suppose that $\mathbf{y}$ and $\mathbf{z}$ satisfy the same boundary conditions on Σ since, in the limit, we should have $\mathbf{z} = \mathbf{y}$. Consequently, we will assume that the boundary control $\mathbf{h}1_\gamma$ acts simultaneously on both variables $\mathbf{y}$ and $\mathbf{z}$.

In what follows, $(\cdot\,,\cdot)$ and $\|\cdot\|$ denote the usual L^2 scalar products and norms (in $L^2(\Omega)$, $\mathbf{L}^2(\Omega)$, $L^2(Q)$, etc.) and K, C, C_1, C_2, ... denote various positive constants (usually depending on ω, Ω and T). Let us recall the definitions of some usual spaces in the context of incompressible fluids:

$$\mathbf{H} = \{\, \mathbf{u} \in \mathbf{L}^2(\Omega) : \nabla \cdot \mathbf{u} = 0 \text{ in } \Omega,\ \mathbf{u} \cdot \mathbf{n} = 0 \text{ on } \Gamma \,\},$$
$$\mathbf{V} = \{\, \mathbf{u} \in \mathbf{H}_0^1(\Omega) : \nabla \cdot \mathbf{u} = 0 \text{ in } \Omega \,\}.$$

Note that, for every $\mathbf{y}_0 \in \mathbf{H}$ and every $\mathbf{v} \in \mathbf{L}^2(\omega \times (0, T))$, there exists a unique solution $(\mathbf{y}, p, \mathbf{z}, \pi)$ for (2.5) that satisfies (among other things)

$$\mathbf{y}, \mathbf{z} \in C^0([0, T]; \mathbf{H});$$

see Proposition 6 below. This is in contrast with the lack of uniqueness of the Navier-Stokes system when $N = 3$.

The main goals of this paper are to analyze the controllability properties of (2.5) and (2.6) and determine the way they depend on α as $\alpha \to 0^+$.

The null controllability problem for (2.5) at time $T > 0$ is the following:

> *For any* $\mathbf{y}_0 \in \mathbf{H}$*, find* $\mathbf{v} \in \mathbf{L}^2(\omega \times (0, T))$ *such that the corresponding state* (*the corresponding solution to* (2.5)) *satisfies*
>
> $$\mathbf{y}(T) = \mathbf{0} \quad \textit{in} \quad \Omega. \tag{2.7}$$

The null controllability problem for (2.6) at time $T > 0$ is the following:

> *For any* $\mathbf{y}_0 \in \mathbf{H}$*, find* $\mathbf{h} \in \mathbf{L}^2(0, T; \mathbf{H}^{-1/2}(\gamma))$ *with* $\int_\gamma \mathbf{h} \cdot \mathbf{n}\, d\Gamma = 0$ *and an associated state (the corresponding solution to* (2.6)*) satisfying*
>
> $$\mathbf{y}, \mathbf{z} \in C^0([0, T]; \mathbf{L}^2(\Omega))$$
>
> *and* (2.7).

Recall that, in the context of the Navier-Stokes equations, J.-L. Lions conjectured in [67] the global distributed and boundary approximate controllability; since then, the controllability of these equations has been intensively studied, but for the moment only partial results are known.

Thus, the global approximate controllability of the two-dimensional Navier-Stokes equations with Navier slip boundary conditions was obtained by Coron in [14]. Also, by combining results concerning global and local controllability, the global null controllability for the Navier-Stokes system on a two-dimensional manifold without boundary was established in Coron and Fursikov [17]; see also Guerrero *et al.* [51] for another global controllability result.

The local exact controllability to bounded trajectories has been obtained by Fursikov and Imanuvilov [40, 38], Imanuvilov [56] and Fernández-Cara *et al.* [29] under various circumstances; see Guerrero [49] and González-Burgos *et al.* [48] for similar results related to the Boussinesq system. Let us also mention [7, 18, 19, 30], where analogous results are obtained with a reduced number of scalar controls.

For the (simplified) one-dimensional viscous Burgers model, positive and negative results can be found in [28, 46, 50]; see also [24], where the authors consider the one-dimensional compressible Navier-Stokes system.

Our first main result in this paper is the following:

Theorem 6. *There exists* $\epsilon > 0$ *(independent of* α*) such that, for each* $\mathbf{y}_0 \in \mathbf{H}$ *with* $\|\mathbf{y}_0\| \le \epsilon$*, there exist controls* $\mathbf{v}_\alpha \in L^\infty(0,T;\mathbf{L}^2(\omega))$ *such that the associated solutions to* (2.5) *fulfill* (2.7)*. Furthermore, these controls can be found satisfying the estimate*

$$\|\mathbf{v}_\alpha\|_{L^\infty(\mathbf{L}^2)} \le C, \tag{2.8}$$

where C *is also independent of* α*.*

Our second main result is the analog of Theorem 6 in the framework of boundary controllability. It is the following:

Theorem 7. *There exists* $\delta > 0$ *(independent of* α*) such that, for each* $\mathbf{y}_0 \in \mathbf{H}$ *with* $\|\mathbf{y}_0\| \le \delta$*, there exist controls* $\mathbf{h}_\alpha \in L^\infty(0,T;\mathbf{H}^{-1/2}(\gamma))$ *with* $\int_\gamma \mathbf{h}_\alpha \cdot \mathbf{n}\, d\Gamma = 0$ *and associated solutions to* (2.6) *that fulfill* (2.7)*. Furthermore, these controls can be found satisfying the estimate*

$$\|\mathbf{h}_\alpha\|_{L^\infty(H^{-1/2})} \le C, \tag{2.9}$$

where C *is also independent of* α*.*

The proofs rely on suitable fixed-point arguments. The underlying idea has applied to many other nonlinear control problems. However, in the present cases, we find two specific difficulties:

- In order to find spaces and fixed-point mappings appropriate for Schauder's Theorem, the initial state $\mathbf{y}_0$ must be regular enough. Consequently, we have to establish *regularizing properties* for (2.5) and (2.6); see Lemmas 2 and 5 below.

- For the proof of the uniform estimates (2.8) and (2.9), careful estimates of the null controls and associated states of some particular linear problems are needed.

We will also prove results concerning the controllability in the limit, as $\alpha \to 0^+$. It will be shown that the null-controls for (2.5) can be chosen in such a way that they converge to null-controls for the Navier-Stokes system

$$\begin{cases} \mathbf{y}_t - \Delta \mathbf{y} + (\mathbf{y}\cdot\nabla)\mathbf{y} + \nabla p = \mathbf{v}1_\omega & \text{in} \quad Q, \\ \nabla\cdot\mathbf{y} = 0 & \text{in} \quad Q, \\ \mathbf{y} = \mathbf{0} & \text{on} \quad \Sigma, \\ \mathbf{y}(0) = \mathbf{y}_0 & \text{in} \quad \Omega. \end{cases} \tag{2.10}$$

Also, it will be seen that the null-controls for (2.6) can be chosen such that they converge to boundary null-controls for the Navier-Stokes system

$$\begin{cases} \mathbf{y}_t - \Delta \mathbf{y} + (\mathbf{y}\cdot\nabla)\mathbf{y} + \nabla p = \mathbf{0} & \text{in} \quad Q, \\ \nabla\cdot\mathbf{y} = 0 & \text{in} \quad Q, \\ \mathbf{y} = \mathbf{h}1_\gamma & \text{on} \quad \Sigma, \\ \mathbf{y}(0) = \mathbf{y}_0 & \text{in} \quad \Omega. \end{cases} \tag{2.11}$$

More precisely, our third and fourth main results are the following:

Theorem 8. *Let $\epsilon > 0$ be furnished by Theorem 6. Assume that $\mathbf{y}_0 \in \mathbf{H}$ and $\|\mathbf{y}_0\| \le \epsilon$, let $\mathbf{v}_\alpha$ be a null control for (2.5) satisfying (2.8) and let $(\mathbf{y}_\alpha, p_\alpha, \mathbf{z}_\alpha, \pi_\alpha)$ be the associated state. Then, at least for a subsequence, one has*

$$\mathbf{v}_\alpha \to \mathbf{v} \ \textit{weakly-}* \ \textit{in} \ L^\infty(0,T;\mathbf{L}^2(\omega)),$$
$$\mathbf{z}_\alpha \to \mathbf{y} \ \textit{and} \ \mathbf{y}_\alpha \to \mathbf{y} \ \textit{strongly in} \ \mathbf{L}^2(Q),$$

as $\alpha \to 0^+$, where $(\mathbf{y}, \mathbf{v})$ is, together with some p, a state-control pair for (2.10) satisfying (2.7).

Theorem 9. *Let $\delta > 0$ be furnished by Theorem 7. Assume that $\mathbf{y}_0 \in \mathbf{H}$ and $\|\mathbf{y}_0\| \le \delta$, let $\mathbf{h}_\alpha$ be a null control for (2.6) satisfying (2.9) and let $(\mathbf{y}_\alpha, p_\alpha, \mathbf{z}_\alpha, \pi_\alpha)$ be the associated state. Then, at least for a subsequence, one has*

$$\mathbf{h}_\alpha \to \mathbf{h} \ \textit{weakly-}* \ \textit{in} \ L^\infty(0,T;H^{-1/2}(\gamma)),$$
$$\mathbf{z}_\alpha \to \mathbf{y} \ \textit{and} \ \mathbf{y}_\alpha \to \mathbf{y} \ \textit{strongly in} \ \mathbf{L}^2(Q),$$

as $\alpha \to 0^+$, where $(\mathbf{y}, \mathbf{h})$ is, together with some p, a state-control pair for (2.11) satisfying (2.7).

The rest of this paper is organized as follows. In Section 4.2, we will recall some properties of the Stokes operator and we will prove some results concerning the existence, uniqueness and regularity of the solution to (2.3). Section 4.3 deals with the proofs of Theorems 6 and 8. Section 4.4 deals with the proofs of Theorems 7 and 9. Finally, in Section 4.5, we present some additional comments and open questions.

2.2 Preliminaries

In this section, we will recall some properties of the Stokes operator. Then, we will prove that the Leray-α system is well-posed. Also, we will recall the Carleman inequalities and null controllability properties of the Oseen system.

2.2.1 The Stokes operator

Let $\mathbf{P} : \mathbf{L}^2(\Omega) \mapsto \mathbf{H}$ be the orthogonal projector, usually known as the *Leray Projector*. Recall that $\mathbf{P}$ maps $\mathbf{H}^s(\Omega)$ into $\mathbf{H}^s(\Omega) \cap \mathbf{H}$ for all $s \geq 0$.

We will denote by $\mathbf{A}$ the *Stokes operator*, i.e. the self-adjoint operator in $\mathbf{H}$ formally given by $\mathbf{A} = -\mathbf{P}\Delta$. For any $\mathbf{u} \in D(\mathbf{A}) := \mathbf{V} \cap \mathbf{H}^2(\Omega)$ and any $\mathbf{w} \in \mathbf{H}$, the identity $\mathbf{A}\mathbf{u} = \mathbf{w}$ holds if and only if

$$(\nabla \mathbf{u}, \nabla \mathbf{v}) = (\mathbf{w}, \mathbf{v}) \quad \forall\, \mathbf{v} \in \mathbf{V}.$$

It is well known that $\mathbf{A} : D(\mathbf{A}) \mapsto \mathbf{H}$ can be inverted and its inverse $\mathbf{A}^{-1}$ is self-adjoint, compact and positive. Consequently, there exists a nondecreasing sequence of positive numbers λ_j and an associated orthonormal basis of $\mathbf{H}$, denoted by $(\mathbf{w}_j)_{j=1}^{\infty}$, such that

$$\mathbf{A}\mathbf{w}_j = \lambda_j \mathbf{w}_j \quad \forall j \geq 1.$$

Accordingly we can introduce the real powers of the Stokes operator. Thus, for any $r \in \mathbb{R}$, we set

$$D(\mathbf{A}^r) = \{\, \mathbf{u} \in \mathbf{H} : \mathbf{u} = \sum_{j=1}^{\infty} u_j \mathbf{w}_j, \text{ with } \sum_{j=1}^{\infty} \lambda_j^{2r} |u_j|^2 < +\infty \,\}$$

and

$$\mathbf{A}^r \mathbf{u} = \sum_{j=1}^{\infty} \lambda_j^r u_j \mathbf{w}_j, \quad \forall\, \mathbf{u} = \sum_{j=1}^{\infty} u_j \mathbf{w}_j \in D(\mathbf{A}^r).$$

Let us present a result concerning the domains of the powers of the Stokes operator.

Theorem 10. *Let $r \in \mathbb{R}$ be given, with $-\frac{1}{2} < r < 1$. Then*

$$D(\mathbf{A}^{r/2}) = \mathbf{H}^r(\Omega) \cap \mathbf{H} \;\; \textit{whenever} \;\; -\frac{1}{2} < r < \frac{1}{2},$$
$$D(\mathbf{A}^{r/2}) = \mathbf{H}_0^r(\Omega) \cap \mathbf{H} \;\; \textit{whenever} \;\; \frac{1}{2} \leq r \leq 1.$$

Moreover, $\mathbf{u} \mapsto (\mathbf{u}, \mathbf{A}^r \mathbf{u})^{1/2}$ is a Hilbertian norm in $D(\mathbf{A}^{r/2})$, equivalent to the usual Sobolev $\mathbf{H}^r$-norm. In other words, there exist constants $c_1(r), c_2(r) > 0$ such that

$$c_1(r) \|\mathbf{u}\|_{\mathbf{H}^r} \leq (\mathbf{u}, \mathbf{A}^r \mathbf{u})^{1/2} \leq c_2(r) \|\mathbf{u}\|_{\mathbf{H}^r} \quad \forall \mathbf{u} \in D(\mathbf{A}^{r/2}).$$

The proof of Theorem 10 can be found in [37]. Notice that, in view of the interpolation K-method of Lions and Peetre, we have $D(\mathbf{A}^{r/2}) = D((-\Delta)^{r/2}) \cap \mathbf{H}$. Hence, thanks to an explicit description of $D((-\Delta)^{r/2})$, the stated result holds.

Now, we are going to recall an important property of the semigroup of contractions $e^{-t\mathbf{A}}$ generated by $\mathbf{A}$, see [36]:

Theorem 11. *For any $r > 0$, there exists $C(r) > 0$ such that*

$$\|\mathbf{A}^r e^{-t\mathbf{A}}\|_{\mathcal{L}(\mathbf{H};\mathbf{H})} \leq C(r)\, t^{-r} \quad \forall\, t > 0. \tag{2.12}$$

In order to prove (2.12), it suffices to observe that, for any $\mathbf{u} = \sum_{j=1}^{+\infty} u_j \mathbf{w}_j \in \mathbf{H}$, one has

$$\mathbf{A}^r e^{-t\mathbf{A}}\mathbf{u} = \sum_{j=1}^{+\infty} \lambda_j^r e^{-t\lambda_j} u_j \mathbf{w}_j.$$

Consequently,

$$\|\mathbf{A}^r e^{-t\mathbf{A}}\mathbf{u}\|^2 = \sum_{j=1}^{+\infty} \left|\lambda_j^r e^{-t\lambda_j} u_j\right|^2 \leq \left(\max_{\lambda\in\mathbb{R}} \lambda^r e^{-t\lambda}\right)^2 \|\mathbf{u}\|^2$$

and, since $\max\limits_{\lambda\in\mathbb{R}} \lambda^r e^{-t\lambda} = (r/e)^r\, t^{-r}$, we get easily (2.12).

2.2.2 Well-posedness for the Leray-α system

Let us see that, for any $\alpha > 0$, under some reasonable conditions on $\mathbf{f}$ and $\mathbf{y}_0$, the Leray-α system (2.3) possesses a unique global weak solution. Before this, let us introduce σ_N given by

$$\sigma_N = \begin{cases} 2 & \text{if } N = 2, \\ 4/3 & \text{if } N = 3. \end{cases}$$

Then, we have the following result:

Proposition 4. *Assume that $\alpha > 0$ is fixed. Then, for any $\mathbf{f} \in L^2(0,T;\mathbf{H}^{-1}(\Omega))$ and any $\mathbf{y}_0 \in \mathbf{H}$, there exists exactly one solution $(\mathbf{y}_\alpha, p_\alpha, \mathbf{z}_\alpha, \pi_\alpha)$ to (2.3), with*

$$\begin{gathered} \mathbf{y}_\alpha \in L^2(0,T;\mathbf{V}) \cap C^0([0,T];\mathbf{H}),\ (\mathbf{y}_\alpha)_t \in L^2(0,T;\mathbf{V}'), \\ \mathbf{z}_\alpha \in L^2(0,T;D(\mathbf{A}^{3/2})) \cap C^0([0,T];D(\mathbf{A})). \end{gathered} \tag{2.13}$$

Furthermore, the following estimates hold:

$$\begin{aligned} \|\mathbf{y}_\alpha\|_{L^2(\mathbf{V})} + \|\mathbf{y}_\alpha\|_{C^0([0,T];\mathbf{H})} &\leq CB_0(\mathbf{y}_0,\mathbf{f}), \\ \|(\mathbf{y}_\alpha)_t\|_{L^{\sigma_N}(\mathbf{V}')} &\leq CB_0(\mathbf{y}_0,\mathbf{f})(1 + B_0(\mathbf{y}_0,\mathbf{f})), \\ \|\mathbf{z}_\alpha\|^2_{L^\infty(\mathbf{H})} + 2\alpha^2\|\mathbf{z}_\alpha\|^2_{L^\infty(\mathbf{V})} &\leq CB_0(\mathbf{y}_0,\mathbf{f})^2, \\ 2\alpha^2\|\mathbf{z}_\alpha\|^2_{L^\infty(\mathbf{V})} + \alpha^4\|\mathbf{z}_\alpha\|^2_{L^\infty(D(\mathbf{A}))} &\leq CB_0(\mathbf{y}_0,\mathbf{f})^2. \end{aligned} \tag{2.14}$$

Here, C is independent of α and we have introduced the notation

$$B_0(\mathbf{y}_0,\mathbf{f}) := \|\mathbf{y}_0\| + \|\mathbf{f}\|_{L^2(\mathbf{H}^{-1})}.$$

Demonstração. The proof follows classical and rather well known arguments; see for instance [20, 79]. For completeness, they will be recalled.

• Existence: We will reduce the proof to the search of a fixed point of an appropriate mapping Λ_α. [1]

[1] Alternatively, we can prove the existence of a solution by introducing adequate Galerkin approximations and applying (classical) compactness arguments.

Thus, for each $\overline{\mathbf{y}} \in L^2(0,T;\mathbf{H})$, let $(\mathbf{z},\pi)$ be the unique solution to

$$\begin{cases} \mathbf{z} - \alpha^2\Delta\mathbf{z} + \nabla\pi = \overline{\mathbf{y}} & \text{in} \quad Q, \\ \nabla\cdot\mathbf{z} = 0 & \text{in} \quad Q, \\ \mathbf{z} = \mathbf{0} & \text{on} \quad \Sigma. \end{cases}$$

It is clear that $\mathbf{z} \in L^2(0,T;D(\mathbf{A}))$ and then, thanks to the Sobolev embedding, we have $\mathbf{z} \in L^2(0,T;\mathbf{L}^\infty(\Omega))$. Moreover, the following estimates are satisfied:

$$\begin{aligned} \|\mathbf{z}\|^2 + 2\alpha^2\|\mathbf{z}\|^2_{L^2(\mathbf{V})} &\leq \|\overline{\mathbf{y}}\|^2, \\ 2\alpha^2\|\mathbf{z}\|^2_{L^2(\mathbf{V})} + \alpha^4\|\mathbf{z}\|^2_{L^2(D(\mathbf{A}))} &\leq \|\overline{\mathbf{y}}\|^2. \end{aligned}$$

From this $\mathbf{z}$, we can obtain the unique solution $(\mathbf{y},p)$ to the linear system of the Oseen kind

$$\begin{cases} \mathbf{y}_t - \Delta\mathbf{y} + (\mathbf{z}\cdot\nabla)\mathbf{y} + \nabla p = \mathbf{f} & \text{in} \quad Q, \\ \nabla\cdot\mathbf{y} = 0 & \text{in} \quad Q, \\ \mathbf{y} = \mathbf{0} & \text{on} \quad \Sigma, \\ \mathbf{y}(0) = \mathbf{y}_0 & \text{in} \quad \Omega. \end{cases}$$

Since $\mathbf{f} \in L^2(0,T;\mathbf{H}^{-1}(\Omega))$ and $\mathbf{y}_0 \in \mathbf{H}$, it is clear that

$$\mathbf{y} \in L^2(0,T;\mathbf{V}) \cap C^0([0,T];\mathbf{H}), \quad \mathbf{y}_t \in L^2(0,T;\mathbf{V}')$$

and the following estimates hold:

$$\begin{aligned} \|\mathbf{y}\|_{C^0([0,T];\mathbf{H})} + \|\mathbf{y}\|_{L^2(\mathbf{V})} &\leq C_1 B_0(\mathbf{y}_0,\mathbf{f}), \\ \|\mathbf{y}_t\|_{L^2(\mathbf{V}')} \leq C_2(1+\|\mathbf{z}\|_{L^2(D(\mathbf{A}))})B_0(\mathbf{y}_0,\mathbf{f}) &\leq C_2(1+\alpha^{-2}\|\overline{\mathbf{y}}\|)B_0(\mathbf{y}_0,\mathbf{f}). \end{aligned} \tag{2.15}$$

Now, we introduce the Banach space

$$\mathbf{W} = \{\mathbf{w} \in L^2(0,T;\mathbf{V}) : \mathbf{w}_t \in L^2(0,T;\mathbf{V}')\},$$

the closed ball

$$\mathbf{K} = \{\, \overline{\mathbf{y}} \in L^2(0,T;\mathbf{H}) : \|\overline{\mathbf{y}}\| \leq C_1\sqrt{T}B_0(\mathbf{y}_0,\mathbf{f}) \,\}$$

and the mapping $\tilde{\Lambda}_\alpha$, with $\tilde{\Lambda}_\alpha(\overline{\mathbf{y}}) = \mathbf{y}$, for all $\overline{\mathbf{y}} \in L^2(0,T;\mathbf{H})$. Obviously $\tilde{\Lambda}_\alpha$ is well defined and maps continuously the whole space $L^2(0,T;\mathbf{H})$ into $\mathbf{W} \cap \mathbf{K}$.

Notice that any bounded set of $\mathbf{W}$ is relatively compact in the space $L^2(0,T;\mathbf{H})$, in view of the classical results of the Aubin-Lions kind, see for instance [77].

Let us denote by Λ_α the restriction to $\mathbf{K}$ of $\tilde{\Lambda}_\alpha$. Then, thanks to (2.15), Λ_α maps $\mathbf{K}$ into itself. Moreover, it is clear that $\Lambda_\alpha : \mathbf{K} \mapsto \mathbf{K}$ satisfies the hypotheses of Schauder's Theorem. Consequently, Λ_α possesses at least one fixed point in $\mathbf{K}$.

This immediately achieves the proof of the existence of a solution satisfying (2.13).

The estimates $(2.14)_a$, $(2.14)_c$ and $(2.14)_d$ are obvious. On the other hand,

$$\begin{aligned}
\|(\mathbf{y}_\alpha)_t\|_{L^{\sigma_N}(\mathbf{V}')} &\le C\left(\|\mathbf{f}\|_{L^2(\mathbf{H}^{-1})} + \|\mathbf{y}_\alpha\|_{L^2(\mathbf{V})} + \|(\mathbf{z}_\alpha\cdot\nabla)\mathbf{y}_\alpha\|_{L^{\sigma_N}(\mathbf{H}^{-1})}\right)\\
&\le C\left(B_0(\mathbf{y}_0,\mathbf{f}) + \|\mathbf{z}_\alpha\|_{L^{s_N}(\mathbf{L}^4)}\|\mathbf{y}_\alpha\|_{L^{s_N}(\mathbf{L}^4)}\right)\\
&\le C\left[B_0(\mathbf{y}_0,\mathbf{f}) + \left(\|\mathbf{z}_\alpha\|_{L^\infty(\mathbf{H})} + \|\mathbf{z}_\alpha\|_{L^2(\mathbf{V})}\right)\left(\|\mathbf{y}_\alpha\|_{L^\infty(\mathbf{H})} + \|\mathbf{y}_\alpha\|_{L^2(\mathbf{V})}\right)\right]\\
&\le CB_0(\mathbf{y}_0,\mathbf{f})(1+B_0(\mathbf{y}_0,\mathbf{f})),
\end{aligned}$$

where $s_N = 2\,\sigma_N$. Here, the third inequality is a consequence of the continuous embedding

$$L^\infty(0,T;\mathbf{H})\cap L^2(0,T;\mathbf{V}) \hookrightarrow L^{s_N}(0,T;\mathbf{L}^4(\Omega)).$$

This estimate completes the proof of (2.14).

• Uniqueness: Let $(\mathbf{y}_\alpha, p_\alpha, \mathbf{z}_\alpha, \pi_\alpha)$ and $(\mathbf{y}'_\alpha, {p'}_\alpha, \mathbf{z}'_\alpha, {\pi'}_\alpha)$ be two solutions to (2.3) and let us introduce $\mathbf{u} := \mathbf{y}_\alpha - \mathbf{y}'_\alpha$, $q = p_\alpha - p'_\alpha$, $\mathbf{m} := \mathbf{z}_\alpha - \mathbf{z}'_\alpha$ and $h = \pi_\alpha - \pi'_\alpha$. Then

$$\begin{cases}
\mathbf{u}_t - \Delta\mathbf{u} + (\mathbf{z}_\alpha\cdot\nabla)\mathbf{u} + \nabla q = -(\mathbf{m}\cdot\nabla)\mathbf{y}'_\alpha & \text{in} \quad Q,\\
\mathbf{m} - \alpha^2\Delta\mathbf{m} + \nabla h = \mathbf{u} & \text{in} \quad Q,\\
\nabla\cdot\mathbf{u} = 0,\ \nabla\cdot\mathbf{m} = 0 & \text{in} \quad Q,\\
\mathbf{u} = \mathbf{m} = \mathbf{0} & \text{on} \quad \Sigma,\\
\mathbf{u}(0) = \mathbf{0} & \text{in} \quad \Omega.
\end{cases}$$

Since $\mathbf{u}\in L^\infty(0,T;\mathbf{H})$, we have $\mathbf{m}\in L^\infty(0,T;D(\mathbf{A}))$ (where the estimate of this norm depends on α). Therefore, we easily deduce from the first equation of the previous system that

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{u}\|^2 + \|\nabla\mathbf{u}\|^2 \le \|\mathbf{m}\|_\infty\|\nabla\mathbf{y}'_\alpha\|\|\mathbf{u}\|$$

for all t. Since $\|\mathbf{m}\|_\infty \le C\|\mathbf{m}\|_{D(\mathbf{A})} \le C\alpha^{-2}\|\mathbf{u}\|$, we get

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{u}\|^2 + \|\nabla\mathbf{u}\|^2 \le C\alpha^{-2}\|\nabla\mathbf{y}'_\alpha\|\|\mathbf{u}\|^2.$$

Therefore, in view of Gronwall's Lemma, we see that $\mathbf{u}\equiv 0$. Accordingly, we also have $\mathbf{m}\equiv\mathbf{0}$ and uniqueness holds. □

We are now going to present some results concerning the existence and uniqueness of a strong solution. We start with a global result in the two-dimensional case.

Proposition 5. *Assume that $N = 2$ and $\alpha > 0$ is fixed. Then, for any $\mathbf{f}\in L^2(0,T;\mathbf{L}^2(\Omega))$ and any $\mathbf{y}_0\in\mathbf{V}$, there exists exactly one solution $(\mathbf{y}_\alpha, p_\alpha, \mathbf{z}_\alpha, \pi_\alpha)$ to* (2.3)*, with*

$$\begin{gathered}
\mathbf{y}_\alpha \in L^2(0,T;D(\mathbf{A}))\cap C^0([0,T];\mathbf{V}),\ (\mathbf{y}_\alpha)_t\in L^2(0,T;\mathbf{H}),\\
\mathbf{z}_\alpha\in L^2(0,T;D(\mathbf{A}^2))\cap C^0([0,T];D(\mathbf{A}^{3/2})).
\end{gathered}\tag{2.16}$$

Furthermore, the following estimates hold:

$$\begin{aligned}\|(\mathbf{y}_\alpha)_t\| + \|\mathbf{y}_\alpha\|_{C^0([0,T];\mathbf{V})} + \|\mathbf{y}_\alpha\|_{L^2(D(\mathbf{A}))} &\le B_1(\|\mathbf{y}_0\|_{\mathbf{V}}, \|\mathbf{f}\|),\\ \|\mathbf{z}_\alpha\|^2_{C^0([0,T];\mathbf{V})} + 2\alpha^2\|\mathbf{z}_\alpha\|^2_{C^0([0,T];D(\mathbf{A}))} &\le \|\mathbf{y}_\alpha\|^2_{C^0([0,T];\mathbf{V})},\end{aligned} \tag{2.17}$$

where we have introduced the notation

$$B_1(r,s) := (r+s)\left[1+(r+s)^2\right] e^{C(r^2+s^2)^2}.$$

Demonstração. First, thanks to Proposition 4, we see that there exists a unique weak solution $(\mathbf{y}_\alpha, p_\alpha, \mathbf{z}_\alpha, \pi_\alpha)$ satisfying (2.13)–(2.14). In particular, $\mathbf{z}_\alpha \in L^2(0,T;\mathbf{V})$ and we have

$$\|\mathbf{z}_\alpha(t)\| \le \|\mathbf{y}_\alpha(t)\| \quad \text{and} \quad \|\mathbf{z}_\alpha(t)\|_{\mathbf{V}} \le \|\mathbf{y}_\alpha(t)\|_{\mathbf{V}}, \quad \forall t \in [0,T].$$

As usual, we will just check that good estimates can be obtained for $\mathbf{y}_\alpha$, $(\mathbf{y}_\alpha)_t$ and $\mathbf{z}_\alpha$. Thus, we assume that it is possible to multiply by $-\Delta\mathbf{y}_\alpha$ the motion equation satisfied by $\mathbf{y}_\alpha$. Taking into account that $N=2$, we obtain:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\nabla\mathbf{y}_\alpha\|^2 + \|\Delta\mathbf{y}_\alpha\|^2 &= -(\mathbf{f},\Delta\mathbf{y}_\alpha) + ((\mathbf{z}_\alpha\cdot\nabla)\mathbf{y}_\alpha, \Delta\mathbf{y}_\alpha)\\ &\le \|\mathbf{f}\|^2 + \frac{1}{4}\|\Delta\mathbf{y}_\alpha\|^2 + \|\mathbf{z}_\alpha\|^{1/2}\|\mathbf{z}_\alpha\|^{1/2}_{\mathbf{V}}\|\mathbf{y}_\alpha\|^{1/2}_{\mathbf{V}}\|\Delta\mathbf{y}_\alpha\|^{3/2}\\ &\le \|\mathbf{f}\|^2 + \frac{1}{2}\|\Delta\mathbf{y}_\alpha\|^2 + C\|\mathbf{z}_\alpha\|^2\|\mathbf{z}_\alpha\|^2_{\mathbf{V}}\|\mathbf{y}_\alpha\|^2_{\mathbf{V}}.\end{aligned}$$

Therefore,

$$\frac{d}{dt}\|\nabla\mathbf{y}_\alpha\|^2 + \|\Delta\mathbf{y}_\alpha\|^2 \le C\left[\|\mathbf{f}\|^2 + (\|\mathbf{y}_\alpha\|^2\|\mathbf{y}_\alpha\|^2_{\mathbf{V}})\|\nabla\mathbf{y}_\alpha\|^2\right].$$

In view of Gronwall's Lemma and the estimates in Proposition 4, we easily deduce (2.16) and (2.17). □

Notice that, in this two-dimensional case, the strong estimates for $\mathbf{y}_\alpha$ in (2.17) are independent of α; obviously, we cannot expect the same when $N=3$.

In the three-dimensional case, what we obtain is the following:

Proposition 6. *Assume that $N=3$ and $\alpha>0$ is fixed. Then, for any $\mathbf{f}\in L^2(0,T;\mathbf{L}^2(\Omega))$ and any $\mathbf{y}_0\in\mathbf{V}$, there exists exactly one solution $(\mathbf{y}_\alpha, p_\alpha, \mathbf{z}_\alpha, \pi_\alpha)$ to (2.3), with*

$$\begin{gathered}\mathbf{y}_\alpha \in L^2(0,T;D(\mathbf{A}))\cap C^0([0,T];\mathbf{V}),\ (\mathbf{y}_\alpha)_t \in L^2(0,T;\mathbf{H}),\\ \mathbf{z}_\alpha \in L^2(0,T;D(\mathbf{A}^2))\cap C^0([0,T];D(\mathbf{A}^{3/2})).\end{gathered}$$

Furthermore, the following estimates hold:

$$\begin{aligned}\|\mathbf{y}_\alpha\|_{L^\infty(\mathbf{V})} + \|\mathbf{y}_\alpha\|_{L^2(D(\mathbf{A}))} + \|(\mathbf{y}_\alpha)_t\| &\le B_2(\|\mathbf{y}_0\|_{\mathbf{V}}, \|\mathbf{f}\|, \alpha),\\ \|\mathbf{z}_\alpha\|^2_{L^\infty(\mathbf{V})} + 2\alpha^2\|\mathbf{z}_\alpha\|^2_{L^\infty(D(\mathbf{A}))} &\le \|\mathbf{y}_\alpha\|^2_{L^\infty(\mathbf{V})},\end{aligned} \tag{2.18}$$

where we have introduced

$$B_2(r,s,\alpha) := C(r+s)e^{C\alpha^{-4}(r+s)^2}.$$

Demonstração. Thanks to Proposition 4, there exists a unique weak solution $(\mathbf{y}_\alpha, p_\alpha, \mathbf{z}_\alpha, \pi_\alpha)$ satisfying (2.13) and (2.14).

In particular, we obtain that $\mathbf{z}_\alpha \in \mathbf{L}^\infty(Q)$, with

$$\|\mathbf{z}_\alpha\|_\infty \leq \frac{C}{\alpha^2}\left(\|\mathbf{y}_0\|_{\mathbf{H}} + \|\mathbf{f}\|_{L^2(\mathbf{H}^{-1})}\right).$$

On the other hand, $\mathbf{y}_0 \in \mathbf{V}$. Hence, from the usual (parabolic) regularity results for Oseen systems, the solution to (2.3) is more regular, i.e. $\mathbf{y}_\alpha \in L^2(0,T;D(\mathbf{A})) \cap C^0([0,T];\mathbf{V})$ and $(\mathbf{y}_\alpha)_t \in L^2(0,T;\mathbf{H})$. Moreover, $\mathbf{y}_\alpha$ verifies the first estimate in (2.18). This achieves the proof. □

Let us now provide a result concerning three-dimensional strong solutions corresponding to small data, with estimates independent of α:

Proposition 7. *Assume that $N = 3$. There exists $C_0 > 0$ such that, for any $\alpha > 0$, any $\mathbf{f} \in L^\infty(0,T;\mathbf{L}^2(\Omega))$ and any $\mathbf{y}_0 \in \mathbf{V}$ with*

$$M := \max\left\{\|\nabla \mathbf{y}_0\|^2,\ \|\mathbf{f}\|^{2/3}_{L^\infty(\mathbf{L}^2)}\right\} < \frac{1}{\sqrt{2(1+C_0)T}}, \tag{2.19}$$

the Leray-α system (2.3) possesses a unique solution $(\mathbf{y}_\alpha, p_\alpha, \mathbf{z}_\alpha, \pi_\alpha)$ satisfying

$$\mathbf{y}_\alpha \in L^2(0,T;D(\mathbf{A})) \cap C^0([0,T];\mathbf{V}),\ (\mathbf{y}_\alpha)_t \in L^2(0,T;\mathbf{H}),$$
$$\mathbf{z}_\alpha \in L^2(0,T;D(\mathbf{A})) \cap C^0([0,T];\mathbf{V}).$$

Furthermore, in that case, the following estimates hold:

$$\begin{gathered}\|\mathbf{y}_\alpha\|^2_{C^0([0,T];\mathbf{V})} + \|\mathbf{y}_\alpha\|^2_{L^2(D(\mathbf{A}))} \leq B_3(M,T),\\ \|\mathbf{z}_\alpha\|^2_{C^0([0,T];\mathbf{V})} + 2\alpha^2\|\mathbf{z}_\alpha\|^2_{L^2(D(\mathbf{A}))} \leq \|\mathbf{y}_\alpha\|^2_{L^\infty(\mathbf{V})},\end{gathered} \tag{2.20}$$

where we have introduced

$$B_3(M,T) := 2\left[M^3 + M + C_0T\left(\frac{M}{\sqrt{1-2(1+C_0)M^2T}}\right)^3\right].$$

Demonstração. The proof is very similar to the proof of the existence of a local in time strong solution to the Navier-Stokes system; see for instance [12, 79].

As before, there exists a unique weak solution $(\mathbf{y}_\alpha, p_\alpha, \mathbf{z}_\alpha, \pi_\alpha)$ and this solution satisfies (2.13) and (2.14).

By multiplying by $\Delta\mathbf{y}_\alpha$ the motion equation satisfied by $\mathbf{y}_\alpha$, we see that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\nabla\mathbf{y}_\alpha\|^2 + \|\Delta\mathbf{y}_\alpha\|^2 &= (\mathbf{f}, \Delta\mathbf{y}_\alpha) - ((\mathbf{z}_\alpha\cdot\nabla)\mathbf{y}_\alpha, \Delta\mathbf{y}_\alpha)\\ &\leq \frac{1}{2}\|\mathbf{f}\|^2 + \frac{1}{2}\|\Delta\mathbf{y}_\alpha\|^2 + \|\mathbf{z}_\alpha\|_{\mathbf{L}^6}\|\nabla\mathbf{y}_\alpha\|_{\mathbf{L}^3}\|\Delta\mathbf{y}_\alpha\|\\ &\leq \frac{1}{2}\|\mathbf{f}\|^2 + \frac{1}{2}\|\Delta\mathbf{y}_\alpha\|^2 + C\|\mathbf{z}_\alpha\|_{\mathbf{V}}\|\mathbf{y}_\alpha\|^{1/2}_{\mathbf{V}}\|\Delta\mathbf{y}_\alpha\|^{3/2}.\end{aligned}$$

Then,

$$\frac{d}{dt}\|\nabla\mathbf{y}_\alpha\|^2 + \frac{1}{2}\|\Delta\mathbf{y}_\alpha\|^2 \le \|\mathbf{f}\|^2 + C_0\|\nabla\mathbf{y}_\alpha\|^6, \tag{2.21}$$

for some $C_0 > 0$.

Let us see that, under the assumption (2.19), we have

$$\|\nabla\mathbf{y}_\alpha\|^2 \le \frac{M}{\sqrt{1-2(1+C_0)M^2T}}, \quad \forall t\in[0,T]. \tag{2.22}$$

Indeed, let us introduce the real-valued function ψ given by

$$\psi(t) = \max\left\{M, \|\nabla\mathbf{y}_\alpha(t)\|^2\right\}, \quad \forall t\in[0,T].$$

Then, ψ is almost everywhere differentiable and, in view of (2.19) and (2.21), one has

$$\frac{d\psi}{dt} \le (1+C_0)\psi^3, \quad \psi(0) = M.$$

Therefore,

$$\psi(t) \le \frac{M}{\sqrt{1-2(1+C_0)M^2t}} \le \frac{M}{\sqrt{1-2(1+C_0)M^2T}}$$

and, since $\|\nabla\mathbf{y}_\alpha\|^2 \le \psi$, (2.22) holds. From this estimate, it is very easy to deduce (2.20). □

The following lemma is inspired by a result by Constantin and Foias for the Navier-Stokes equations, see [12]:

Lemma 2. *There exists a continuous function $\phi : \mathbb{R}_+ \mapsto \mathbb{R}_+$, with $\phi(s) \to 0$ as $s \to 0^+$, satisfying the following properties:*

a) For $\mathbf{f} = \mathbf{0}$, any $\mathbf{y}_0 \in \mathbf{H}$ and any $\alpha > 0$, there exist arbitrarily small times $t^* \in (0, T/2)$ such that the corresponding solution to (2.3) *satisfies $\|\mathbf{y}_\alpha(t^*)\|^2_{D(\mathbf{A})} \le \phi(\|\mathbf{y}_0\|)$.*

b) The set of these t^* has positive measure.

Demonstração. We are only going to consider the three-dimensional case; the proof in the two-dimensional case is very similar and even easier.

The proof consists of several steps:

- Let us first see that, for any $k > 3/2$ and any $\tau \in (0, T/2]$, the set

  $$R_\alpha(k,\tau) := \{\, t\in[0,\tau] : \|\nabla\mathbf{y}_\alpha(t)\|^2 \le \frac{k}{\tau}\,\|\mathbf{y}_0\|^2 \,\}$$

  is non-empty and its measure $|R_\alpha(k,\tau)|$ satisfies $|R_\alpha(k,\tau)| \ge \tau/k$.

  Obviously, we can assume that $\mathbf{y}_0 \ne \mathbf{0}$. Now, if we suppose that $|R_\alpha(k,\tau)| < \tau/k$, we have:

  $$\begin{aligned}\int_0^\tau \|\nabla\mathbf{y}_\alpha(t)\|^2\,dt &\ge \int_{R_\alpha(k,\tau)^c} \|\nabla\mathbf{y}_\alpha(t)\|^2\,dt \ge \left(\tau - \frac{\tau}{k}\right)\frac{k}{\tau}\|\mathbf{y}_0\|^2 \\ &= (k-1)\|\mathbf{y}_0\|^2 > \frac{1}{2}\|\mathbf{y}_0\|^2.\end{aligned}$$

But, since $\mathbf{f} = 0$ in (2.3), we also have the following estimate:

$$\int_0^\tau \|\nabla \mathbf{y}_\alpha(t)\|^2\,dt \le \frac{1}{2}\|\mathbf{y}_\alpha(\tau)\|^2 + \int_0^\tau \|\nabla \mathbf{y}_\alpha(t)\|^2\,dt = \frac{1}{2}\|\mathbf{y}_0\|^2.$$

So, we get a contradiction and, necessarily, $|R_\alpha(k,\tau)| \ge \tau/k$.

- Let us choose $\tau \in (0, T/2]$, $k > 3/2$, $t_{0,\alpha} \in R_\alpha(k,\tau)$ and

$$\overline{T}_\alpha \in \left[t_{0,\alpha} + \frac{\tau^2}{4((1+C_0)k^2\|\mathbf{y}_0\|^4}, t_{0,\alpha} + \frac{3\tau^2}{8((1+C_0)k^2\|\mathbf{y}_0\|^4}\right],$$

where C_0 is the constant furnished by Proposition 7. Since $\|\nabla \mathbf{y}_\alpha(t_{0,\alpha})\|^2 \le \frac{k}{\tau}\|\mathbf{y}_0\|^2$, there exists exactly one strong solution to (2.3) in $[t_{0,\alpha}, \overline{T}_\alpha]$ starting from $\mathbf{y}_\alpha(t_{0,\alpha})$ at time $t_{0,\alpha}$ and satisfying

$$\|\nabla \mathbf{y}_\alpha(t)\|^2 \le \frac{2k}{\tau}\|\mathbf{y}_0\|^2, \quad \forall t \in [t_{0,\alpha}, \overline{T}_\alpha].$$

Obviously, it can be assumed that $\overline{T}_\alpha < T$.

Let us introduce the set

$$G_\alpha(t_{0,\alpha}, k, \tau) := \{\, t \in [t_{0,\alpha}, \overline{T}_\alpha] : \ \ \|\Delta \mathbf{y}_\alpha(t)\|^2 \le 65(1+C_0)\left(\frac{k}{\tau}\right)^3 \|\mathbf{y}_0\|^6 \,\}.$$

Then, again $G_\alpha(t_{0,\alpha}, k, \tau)$ is non-empty and possesses positive measure. More precisely, one has

$$|G_\alpha(t_{0,\alpha}, k, \tau)| \ge \frac{\tau^2}{8(1+C_0)k^2\|\mathbf{y}_0\|^4}\,. \tag{2.23}$$

Indeed, otherwise we would get

$$\begin{aligned}\frac{1}{2}\int_{t_{0,\alpha}}^{\overline{T}_\alpha} \|\Delta \mathbf{y}_\alpha(t)\|^2\,dt &\ge \frac{1}{2}\int_{G_\alpha(t_{0,\alpha},k,\tau)^c} \|\Delta \mathbf{y}_\alpha(t)\|^2\,dt \\ \ge 65\left(\overline{T}_\alpha - t_{0,\alpha} - \frac{\tau^2}{8(1+C_0)k^2\|\mathbf{y}_0\|^4}\right)&(1+C_0)\left(\frac{k}{\tau}\right)^3\|\mathbf{y}_0\|^6 \\ &\ge \frac{65k}{16\tau}\|\mathbf{y}_0\|^2 > 4\frac{k}{\tau}\|\mathbf{y}_0\|^2.\end{aligned}$$

However, arguing as in the proof of Proposition 7, we also have

$$\begin{aligned}\frac{1}{2}\int_{t_{0,\alpha}}^{\overline{T}_\alpha} \|\Delta \mathbf{y}_\alpha(t)\|^2\,dt &\le \|\nabla \mathbf{y}_\alpha(\overline{T}_\alpha)\|^2 + \frac{1}{2}\int_{t_{0,\alpha}}^{\overline{T}_\alpha} \|\Delta \mathbf{y}_\alpha(t)\|^2\,dt \\ &\le \|\nabla \mathbf{y}_\alpha(t_{0,\alpha})\|^2 + C_0\int_{t_{0,\alpha}}^{\overline{T}_\alpha} \|\nabla \mathbf{y}_\alpha(t)\|^6\,dt \\ &\le \frac{k}{\tau}\|\mathbf{y}_0\|^2 + 8\left(\frac{k}{\tau}\|\mathbf{y}_0\|^2\right)^3(\overline{T}_\alpha - t_{0,\alpha}) \le 4\frac{k}{\tau}\|\mathbf{y}_0\|^2.\end{aligned}$$

Consequently, we arrive again to a contradiction and this proves (2.23).

- Let us fix $\tau \in (0, T/2]$ and $k > 3/2$. We can now define $\phi : \mathbb{R}_+ \mapsto \mathbb{R}_+$ as follows:

$$\phi(s) := 65(1 + C_0)\frac{k^3}{\tau} s^6.$$

  Then, as a consequence of the previous steps, the set

$$\{ t^* \in [0, T/2] : \|\mathbf{A}\mathbf{y}_\alpha(t^*)\|^2 \leq \phi(\|\mathbf{y}_0\|) \}$$

  is non-empty and it measure is bounded from below by a positive quantity independent of α. This ends the proof. □

We will end this section with some estimates:

Lemma 3. *Let $s \in [1, 2]$ be given, and let us assume that $\mathbf{f} \in \mathbf{H}^s(\Omega)$. Then there exist unique functions $\mathbf{u} \in D(\mathbf{A}^{s/2})$ and $\pi \in H^{s-1}$ (π is unique up to a constant) such that*

$$\begin{cases} \mathbf{u} - \alpha^2 \Delta \mathbf{u} + \nabla \pi = \alpha^2 \Delta \mathbf{f} & in \quad \Omega, \\ \nabla \cdot \mathbf{u} = 0 & in \quad \Omega, \\ \mathbf{u} = \mathbf{0} & on \quad \Gamma \end{cases} \tag{2.24}$$

and there exists a constant $C = C(s, \Omega)$ independent of α such that

$$\|\mathbf{u}\|_{D(\mathbf{A}^{s/2})} \leq C \|\mathbf{f}\|_{\mathbf{H}^s(\Omega)}. \tag{2.25}$$

Moreover, by interpolation arguments, $\mathbf{f} \in \mathbf{H}^s(\Omega)$, $s \in (m, m+1)$ then there exist unique functions $\mathbf{u} \in D(\mathbf{A}^{s/2})$ and $\pi \in H^{s-1}(\Omega)$ (π is unique up to a constant) which are solution of the problem above and there exists a constant $C = C(m, \Omega)$ such that

$$\|\mathbf{u}\|_{D(\mathbf{A}^{s/2})} \leq C \|\mathbf{f}\|_{\mathbf{H}^s(\Omega)}. \tag{2.26}$$

When s is an integer ($s = 1$ or $s = 2$), the proof can be obtained by adapting the proof of Proposition 2.3 in [79]. For other values of s, it suffices to use a classical interpolation argument (see [78]).

2.2.3 Carleman inequalities and null controllability

In this subsection, we will recall some Carleman inequalities and a null controllability result for the Oseen system

$$\begin{cases} \mathbf{y}_t - \Delta \mathbf{y} + (\mathbf{h} \cdot \nabla)\mathbf{y} + \nabla p = \mathbf{v} 1_\omega & \text{in} \quad Q, \\ \nabla \cdot \mathbf{y} = 0 & \text{in} \quad Q, \\ \mathbf{y} = \mathbf{0} & \text{on} \quad \Sigma, \\ \mathbf{y}(0) = \mathbf{y}_0 & \text{in} \quad \Omega, \end{cases} \tag{2.27}$$

where $\mathbf{h} = \mathbf{h}(\mathbf{x}, t)$ is given. The null controllability problem for (2.27) at time $T > 0$ is the following:

For any $\mathbf{y}_0 \in \mathbf{H}$, *find* $\mathbf{v} \in \mathbf{L}^2(\omega \times (0,T))$ *such that the associated solution to* (2.27) *satisfies* (2.7).

We have the following result from [29] (see also [56]):

Theorem 12. *Assume that* $\mathbf{h} \in \mathbf{L}^\infty(Q)$ *and* $\nabla \cdot \mathbf{h} = 0$. *Then, the linear system* (2.27) *is null-controllable at any time* $T > 0$. *More precisely, for each* $\mathbf{y}_0 \in \mathbf{H}$ *there exists* $\mathbf{v} \in L^\infty(0,T;\mathbf{L}^2(\omega))$ *such that the corresponding solution to* (2.27) *satisfies* (2.7). *Furthermore, the control* $\mathbf{v}$ *can be chosen satisfying the estimate*

$$\|\mathbf{v}\|_{L^\infty(\mathbf{L}^2(\omega))} \leq e^{K(1+\|\mathbf{h}\|_\infty^2)}\|\mathbf{y}_0\|, \tag{2.28}$$

where K *only depends on* Ω, ω *and* T.

The proof is a consequence of an appropriate Carleman inequality for the adjoint system of (2.27).

More precisely, let us consider the backwards in time system

$$\begin{cases} -\boldsymbol{\varphi}_t - \Delta\boldsymbol{\varphi} - (\mathbf{h}\cdot\nabla)\boldsymbol{\varphi} + \nabla q = \mathbf{G} & \text{in} \quad Q, \\ \nabla\cdot\boldsymbol{\varphi} = 0 & \text{in} \quad Q, \\ \boldsymbol{\varphi} = \mathbf{0} & \text{on} \quad \Sigma, \\ \boldsymbol{\varphi}(T) = \boldsymbol{\varphi}_0, & \text{in} \quad \Omega. \end{cases} \tag{2.29}$$

The following result is established in [29]:

Proposition 8. *Assume that* $\mathbf{h} \in \mathbf{L}^\infty(Q)$ *and* $\nabla \cdot \mathbf{h} = 0$. *There exist positive continuous functions* α, α^*, $\hat{\alpha}$, ξ, ξ^* *and* $\hat{\xi}$ *and positive constants* $\hat{s}$, $\hat{\lambda}$ *and* $\widehat{C}$, *only depending on* Ω *and* ω, *such that, for any* $\boldsymbol{\varphi}_0 \in \mathbf{H}$ *and any* $\mathbf{G} \in \mathbf{L}^2(Q)$, *the solution to the adjoint system* (2.29) *satisfies:*

$$\begin{aligned} &\iint_Q e^{-2s\alpha}\left[s^{-1}\xi^{-1}(|\boldsymbol{\varphi}_t|^2 + |\Delta\boldsymbol{\varphi}|^2) + s\xi\lambda^2|\nabla\boldsymbol{\varphi}|^2 + s^3\xi^3\lambda^4|\boldsymbol{\varphi}|^2\right] d\mathbf{x}\,dt \\ &\qquad \leq \widehat{C}(1+T^2)\Bigg(s^{15/2}\lambda^{20}\iint_Q e^{-4s\hat{\alpha}+2s\alpha^*}\xi^{*15/2}|\mathbf{G}|^2\,d\mathbf{x}\,dt \\ &\qquad\qquad + s^{16}\lambda^{40}\iint_{\omega\times(0,T)} e^{-8s\hat{\alpha}+6s\alpha^*}\xi^{*16}|\boldsymbol{\varphi}|^2\,d\mathbf{x}\,dt\Bigg), \end{aligned} \tag{2.30}$$

for all $s \geq \hat{s}(T^4+T^8)$ *and for all* $\lambda \geq \hat{\lambda}\left(1 + \|\mathbf{h}\|_\infty + e^{\hat{\lambda}T\|\mathbf{h}\|_\infty^2}\right)$.

Now, we are going to construct the a null-control for (2.27) like in [29]. First, let us introduce the auxiliary extremal problem

$$\begin{cases} \text{Minimize } \dfrac{1}{2}\left\{\displaystyle\iint_Q \hat{\rho}^2|\mathbf{y}|^2\,d\mathbf{x}\,dt + \iint_{\omega\times(0,T)} \hat{\rho}_0^2|\mathbf{v}|^2\,d\mathbf{x}\,dt\right\} \\ \text{Subject to } (\mathbf{y},\mathbf{v}) \in \mathcal{M}(\mathbf{y}_0,T), \end{cases} \tag{2.31}$$

where the linear manifold $\mathcal{M}(\mathbf{y}_0, T)$ is given by

$$\mathcal{M}(\mathbf{y}_0, T) = \{ (\mathbf{y}, \mathbf{v}) : \mathbf{v} \in \mathbf{L}^2(\omega \times (0,T)),\ (\mathbf{y}, p) \text{ solves (2.27)} \}$$

and $\hat\rho$, $\hat\rho_0$ are respectively given by

$$\hat\rho = s^{-15/4}\lambda^{-10}e^{2s\hat\alpha - s\alpha^*}\xi^{*-15/4}, \quad \hat\rho_0 = s^{-8}\lambda^{-20}e^{4s\hat\alpha-3s\alpha^*}\xi^{*-8}.$$

It can be proved that (2.31) possesses exactly one solution $(\mathbf{y}, \mathbf{v})$ satisfying

$$\|\mathbf{v}\|_{L^2(\mathbf{L}^2(\omega))} \le e^{K(1+\|\mathbf{h}\|_\infty^2)}\|\mathbf{y}_0\|,$$

where K only depends on Ω, ω and T.

Moreover, thanks to the Euler-Lagrange characterization, the solution to the extremal problem (2.31) is given by

$$\mathbf{y} = \hat\rho^{-2}(-\boldsymbol{\varphi}_t - \Delta\boldsymbol{\varphi} - (\mathbf{h}\cdot\nabla)\boldsymbol{\varphi} + \nabla q) \text{ and } \mathbf{v} = -\hat\rho_0^{-2}\boldsymbol{\varphi}1_\omega.$$

From the Carleman inequality (2.30), we can conclude that $\rho_2^{-1}\boldsymbol{\varphi} \in L^\infty(0,T;\mathbf{L}^2(\Omega))$ and

$$\|\rho_2^{-1}\boldsymbol{\varphi}\|_{L^\infty(\mathbf{L}^2)} \le C\|\hat\rho_0^{-1}\boldsymbol{\varphi}\|_{L^2(\mathbf{L}^2(\omega))},$$

where $\rho_2 = s^{1/2}\xi^{1/2}e^{s\alpha}$.
Hence,

$$\mathbf{v} = -(\hat\rho_0)^{-2}\boldsymbol{\varphi}1_\omega = -(\hat\rho_0^{-2}\rho_2)(\rho_2^{-1}\boldsymbol{\varphi}1_\omega) \in L^\infty(0,T;\mathbf{L}^2(\Omega))$$

and, therefore,

$$\|\mathbf{v}\|_{L^\infty(\mathbf{L}^2(\omega))} \le C\|\mathbf{v}\|_{L^2(\mathbf{L}^2(\omega))} \le e^{K(1+\|\mathbf{h}\|_\infty^2)}\|\mathbf{y}_0\|.$$

2.3 The distributed case: Theorems 6 and 8

This section is devoted to prove the local null controllability of (2.5) and the uniform controllability property in Theorem 8.

Proof of Theorem 6: We will use a fixed point argument. Contrarily to the case of the Navier-Stokes equations, it is not sufficient to work here with controls in $\mathbf{L}^2(\omega \times (0,T))$. Indeed, we need a space $\mathbf{Y}$ for $\mathbf{y}$ that ensures $\mathbf{z}$ in $\mathbf{L}^\infty(Q)$ and a space $\mathbf{X}$ for $\mathbf{v}$ guaranteeing that the solution to (2.27) with $\mathbf{h} = \mathbf{z}$ belongs to a compact set of $\mathbf{Y}$. Furthermore, we want estimates in $\mathbf{Y}$ and $\mathbf{X}$ independent of α.

In view of Lemma 2, in order to prove Theorem 6, we just need to consider the case in which the initial state $\mathbf{y}_0$ belongs to $D(\mathbf{A})$ and possesses a sufficiently small norm in $D(\mathbf{A})$.

Let us fix σ with $N/4 < \sigma < 1$. Then, for each $\tilde{\mathbf{y}} \in L^\infty(0,T;D(\mathbf{A}^\sigma))$, let $(\mathbf{z}, \pi)$ be the unique solution to

$$\begin{cases} \mathbf{z} - \alpha^2\Delta\mathbf{z} + \nabla\pi = \tilde{\mathbf{y}} & \text{in} \quad Q, \\ \nabla\cdot\mathbf{z} = 0 & \text{in} \quad Q, \\ \mathbf{z} = \mathbf{0} & \text{on} \quad \Sigma. \end{cases}$$

Since $\tilde{\mathbf{y}} \in L^\infty(0,T;D(\mathbf{A}^\sigma))$, it is clear that $\mathbf{z} \in L^\infty(0,T;D(\mathbf{A}^\sigma))$. Then, thanks to Theorem 10, we have $\mathbf{z} \in \mathbf{L}^\infty(Q)$ and the following is satisfied:

$$\begin{aligned} \|\mathbf{z}\|^2_{L^\infty(0,T;D(\mathbf{A}^\sigma))} + 2\alpha^2\|\mathbf{z}\|^2_{L^\infty(D(\mathbf{A}^{1/2+\sigma}))} &\leq \|\tilde{\mathbf{y}}\|^2_{L^\infty(0,T;D(\mathbf{A}^\sigma))}, \\ 2\alpha^2\|\mathbf{z}\|^2_{L^\infty(D(\mathbf{A}^{1/2+\sigma}))} + \alpha^4\|\mathbf{z}\|^2_{L^\infty(D(\mathbf{A}^{1+\sigma}))} &\leq \|\tilde{\mathbf{y}}\|^2_{L^\infty(0,T;D(\mathbf{A}^\sigma))}. \end{aligned} \tag{2.32}$$

In particular, we have:

$$\|\mathbf{z}\|_{L^\infty(0,T;D(\mathbf{A}^\sigma))} \leq \|\tilde{\mathbf{y}}\|_{L^\infty(0,T;D(\mathbf{A}^\sigma))}.$$

Let us consider the system (2.27) with $\mathbf{h}$ replaced by $\mathbf{z}$. In view of Theorem 12, we can associate to $\mathbf{z}$ the null control $\mathbf{v}$ of minimal norm in $L^\infty(0,T;\mathbf{L}^2(\omega))$ and the corresponding solution $(\mathbf{y},p)$ to (2.27).

Since $\mathbf{y}_0 \in D(\mathbf{A})$, $\mathbf{z} \in \mathbf{L}^\infty(Q)$ and $\mathbf{v} \in L^\infty(0,T;\mathbf{L}^2(\omega))$, we have

$$\mathbf{y} \in L^2(0,T;D(\mathbf{A})) \cap C^0([0,T];\mathbf{V}),\ \mathbf{y}_t \in L^2(0,T;\mathbf{H})$$

and the following estimate holds:

$$\|\mathbf{y}_t\|^2_{L^2(\mathbf{H})} + \|\mathbf{y}\|^2_{L^2(D(\mathbf{A}))} + \|\mathbf{y}\|^2_{L^\infty(\mathbf{V})} \leq C(\|\mathbf{y}_0\|^2_{\mathbf{V}} + \|\mathbf{v}\|^2_{L^\infty(\mathbf{L}^2(\omega))})e^{C\|\mathbf{z}\|^2_\infty}. \tag{2.33}$$

We will use the following result:

Lemma 4. *One has $\mathbf{y} \in L^\infty(0,T;D(\mathbf{A}^{\sigma'}))$, for all $\sigma' \in (\sigma,1)$, with*

$$\|\mathbf{y}\|_{L^\infty(D(\mathbf{A}^{\sigma'}))} \leq C(\|\mathbf{y}_0\|_{D(\mathbf{A})} + \|\mathbf{v}\|_{L^\infty(\mathbf{L}^2(\omega))})e^{C\|\tilde{\mathbf{y}}\|^2_{L^\infty(D(\mathbf{A}^\sigma))}}.$$

Demonstração. In view of (2.27), $\mathbf{y}$ solves the following abstract initial value problem:

$$\begin{cases} \mathbf{y}_t = -\mathbf{A}\mathbf{y} - \mathbf{P}((\mathbf{z}\cdot\nabla)\mathbf{y}) + \mathbf{P}(\mathbf{v}1_\omega) \quad \text{in} \quad [0,T], \\ \mathbf{y}(0) = \mathbf{y}_0. \end{cases}$$

This system can be rewritten as the nonlinear integral equation

$$\mathbf{y}(t) = e^{-t\mathbf{A}}\mathbf{y}_0 - \int_0^t e^{-(t-s)\mathbf{A}}\mathbf{P}((\mathbf{z}\cdot\nabla)\mathbf{y})(s)\, ds + \int_0^t e^{-(t-s)\mathbf{A}}\mathbf{P}(\mathbf{v}1_\omega)(s)\, ds.$$

Consequently, applying the operator $\mathbf{A}^{\sigma'}$ to both sides, we have

$$\mathbf{A}^{\sigma'}\mathbf{y}(t) = \mathbf{A}^{\sigma'}e^{-t\mathbf{A}}\mathbf{y}_0 + \int_0^t \mathbf{A}^{\sigma'}e^{-(t-s)\mathbf{A}}\left[-P((\mathbf{z}\cdot\nabla)\mathbf{y})(s) + P(\mathbf{v}1_\omega)(s)\right]\, ds.$$

Taking norms in both sides and using Theorem 11, we see that

$$\begin{aligned} \|\mathbf{A}^{\sigma'}\mathbf{y}\|(t) &\leq \|\mathbf{y}_0\|_{D(A^{\sigma'})} + \int_0^t (t-s)^{-\sigma'}\left[\|\mathbf{z}(s)\|_\infty\|\nabla\mathbf{y}(s)\| + \|\mathbf{v}(s)1_\omega\|\right]\, ds \\ &\leq C\|\mathbf{y}_0\|_{D(\mathbf{A})} + (\|\mathbf{z}\|_\infty\|\mathbf{y}\|_{L^\infty(\mathbf{V})} + \|\mathbf{v}\|_{L^\infty(\mathbf{L}^2(\omega))})\int_0^t (t-s)^{-\sigma'}\, ds. \end{aligned}$$

Now, using (2.32) and (2.33) and taking into account that $\sigma' < 1$, we easily obtain that

$$\|\mathbf{A}^{\sigma'}\mathbf{y}\|(t) \leq C(\|\mathbf{y}_0\|_{D(\mathbf{A})} + \|\mathbf{v}\|_{L^\infty(\mathbf{L}^2(\omega))})\left[1 + \|\tilde{\mathbf{y}}\|_{L^\infty(D(\mathbf{A}^\sigma))}e^{C\|\tilde{\mathbf{y}}\|^2_{L^\infty(D(\mathbf{A}^\sigma))}}\right].$$

This ends the proof. □

Now, let us set

$$\mathbf{W} = \{\, \mathbf{w} \in L^\infty(0,T;D(\mathbf{A}^{\sigma'})) : \mathbf{w}_t \in L^2(0,T;\mathbf{H}) \,\}$$

and let us consider the closed ball

$$\mathbf{K} = \{\, \tilde{\mathbf{y}} \in L^\infty(0,T;D(\mathbf{A}^{\sigma})) : \|\tilde{\mathbf{y}}\|_{L^\infty(D(\mathbf{A}^\sigma))} \le 1 \,\}$$

and the mapping $\tilde{\Lambda}_\alpha$, with $\tilde{\Lambda}_\alpha(\tilde{\mathbf{y}}) = \mathbf{y}$ for all $\tilde{\mathbf{y}} \in L^\infty(0,T;D(\mathbf{A}^\sigma))$. Obviously, $\tilde{\Lambda}_\alpha$ is well defined; furthermore, in view of Lemma 4 and (2.33), it maps the whole space $L^\infty(0,T;D(\mathbf{A}^\sigma))$ into $\mathbf{W}$.

Notice that, if $\mathbf{U}$ is bounded set of $\mathbf{W}$ then it is relatively compact in the space $L^\infty(0,T;D(\mathbf{A}^\sigma))$, in view of the classical results of the Aubin-Lions kind, see for instance [77].

Let us denote by Λ_α the restriction to $\mathbf{K}$ of $\tilde{\Lambda}_\alpha$. Then, thanks to Lemma 4 and (2.28), if $\|\mathbf{y}_0\|_{D(\mathbf{A})} \le \varepsilon$ (independent of α!) Λ_α maps $\mathbf{K}$ into itself. Moreover, it is clear that $\Lambda_\alpha : \mathbf{K} \mapsto \mathbf{K}$ satisfies the hypotheses of Schauder's Theorem. Indeed, this nonlinear mapping is continuous and compact (the latter is a consequence of the fact that, if $\mathbf{B}$ is bounded in $L^\infty(0,T;D(\mathbf{A}^\sigma))$, then $\tilde{\Lambda}_\alpha(\mathbf{B})$ is bounded in $\mathbf{W}$). Consequently, Λ_α possesses at least one fixed point in $\mathbf{K}$, and this ends the proof of Theorem 6. □

Proof of Theorem 8: Let $\mathbf{v}_\alpha$ be a null control for (2.5) satisfying (2.8) and let $(\mathbf{y}_\alpha, p_\alpha, \mathbf{z}_\alpha, \pi_\alpha)$ be the state associated to $\mathbf{v}_\alpha$. From (2.8) and the estimates (2.14) for the solutions $\mathbf{y}_\alpha$, there exist $\mathbf{v} \in L^\infty\left(0,T;\mathbf{L}^2(\omega)\right)$ and $\mathbf{y} \in L^\infty(0,T;\mathbf{H}) \cap L^2(0,T;\mathbf{V})$ with $\mathbf{y}_t \in L^{\sigma_N}(0,T;\mathbf{V}')$ such that, at least for a subsequence

$$\begin{aligned}
\mathbf{v}_\alpha &\to \mathbf{v} \text{ weakly-}\star \text{ in } L^\infty\left(0,T;\mathbf{L}^2(\omega)\right), \\
\mathbf{y}_\alpha &\to \mathbf{y} \text{ weakly-}\star \text{ in } L^\infty\left(0,T;\mathbf{H}\right) \text{ and weakly in } L^2\left(0,T;\mathbf{V}\right), \\
(\mathbf{y}_\alpha)_t &\to \mathbf{y}_t \text{ weakly in } L^{\sigma_N}(0,T;\mathbf{V}').
\end{aligned}$$

Since $\mathbf{W} := \{\mathbf{m} \in L^2\left(0,T;\mathbf{V}\right) : \ \mathbf{m}_t \in L^{\sigma_N}(0,T;\mathbf{V}')\}$ is continuously and compactly embedded in $\mathbf{L}^2(Q)$, we have that

$$\mathbf{y}_\alpha \to \mathbf{y} \text{ in } \mathbf{L}^2(Q) \text{ and a.e.}$$

This is sufficient to pass to the limit in the equations satisfied by $\mathbf{y}_\alpha$, $\mathbf{v}_\alpha$ and $\mathbf{z}_\alpha$. We conclude that $\mathbf{y}$ is, together with some pressure p, a solution to the Navier-Stokes equations associated to a control $\mathbf{v}$ and satisfies (2.7). □

2.4 The boundary case: Theorems 7 and 9

This section is devoted to prove the local boundary null controllability of (2.6) and the uniform controllability property in Theorem 9.

Proof of Theorem 7: Again, we will use a fixed point argument. Contrarily to the case of distributed controllability, we will have to work in a space $\tilde{\mathbf{Y}}$ of functions defined in an extended domain.

Let $\widetilde{\Omega}$ be given, with $\Omega \subset \widetilde{\Omega}$ and $\partial\widetilde{\Omega} \cap \Gamma = \Gamma \setminus \gamma$ such that $\partial\widetilde{\Omega}$ is of class C^2 (see Fig. 2.1). Let $\omega \subset \widetilde{\Omega} \setminus \overline{\Omega}$ be a non-empty open subset and let us introduce $\widetilde{Q} := \widetilde{\Omega} \times (0, T)$ and $\widetilde{\Sigma} := \partial\widetilde{\Omega} \times (0, T)$. The spaces and operators associate to the domain $\widetilde{\Omega}$ will be denoted by $\widetilde{\mathbf{H}}$, $\widetilde{\mathbf{V}}$, $\widetilde{\mathbf{A}}$, etc.

Remark 4. In view of Lemma 2, for the proof of Theorem 7 we just need to consider the case in which the initial state $\mathbf{y}_0$ belongs to $\mathbf{V}$ and possesses a sufficiently small norm in $\mathbf{V}$. Indeed, we only have to take initially $\mathbf{h}_\alpha \equiv \mathbf{0}$ and apply Lemma 2 to the solution to (2.6). □

Let $\mathbf{y}_0 \in \mathbf{V}$ be given and let us introduce the extension by zero $\tilde{\mathbf{y}}_0$ of $\mathbf{y}_0$. Then $\tilde{\mathbf{y}}_0 \in \widetilde{\mathbf{V}}$.

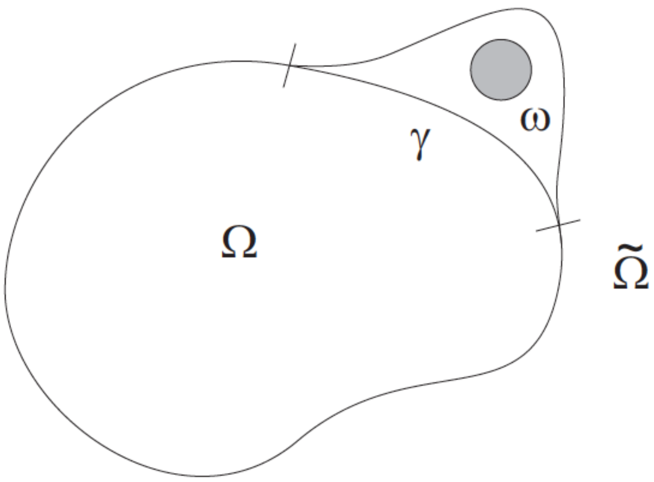

Figura 2.1: The domain $\widetilde{\Omega}$

We will use the following result, similar to Lemma 2, whose proof is postponed to the end of the section:

Lemma 5. *There exists a continuous function $\phi : \mathbb{R}_+ \mapsto \mathbb{R}_+$ satisfying $\phi(s) \to 0$ as $s \to 0^+$ with the following property:*

a) For any $\mathbf{y}_0 \in \mathbf{V}$ and any $\alpha > 0$, there exist times $T_0 \in (0, T)$, controls $\mathbf{h}_\alpha \in L^2(0, T_0; \mathbf{H}^{1/2}(\Gamma))$ with $\int_\gamma \mathbf{h}_\alpha \cdot \mathbf{n}\, d\Gamma \equiv 0$, associated solutions $(\mathbf{y}_\alpha, p_\alpha, \mathbf{z}_\alpha, \pi_\alpha)$ to (2.6) in $\Omega \times (0, T_0)$ and arbitrarily small times $t^* \in (0, T/2)$ such that the $\mathbf{y}_\alpha$ can be extended to $\widetilde{\Omega} \times (0, T_0)$ and the extensions satisfy $\|\tilde{\mathbf{y}}_\alpha(t^*)\|^2_{D(\widetilde{\mathbf{A}})} \le \phi(\|\mathbf{y}_0\|_{\mathbf{V}})$.

b) The set of these t^* has positive measure.

c) The controls $\mathbf{h}_\alpha$ are uniformly bounded, i.e.

$$\|\mathbf{h}_\alpha\|_{L^\infty(0,T_0;\mathbf{H}^{1/2}(\gamma))} \le C.$$

In view of Lemma 5, for the proof of Theorem 7, we just need to consider the case in which the initial state $\mathbf{y}_0$ is such that its extension $\tilde{\mathbf{y}}_0$ to $\widetilde{\Omega}$ belongs to $D(\tilde{\mathbf{A}})$ and possesses a sufficiently small norm in $D(\tilde{\mathbf{A}})$.

We will prove that there exists $(\tilde{\mathbf{y}}_\alpha, \tilde{p}_\alpha, \mathbf{z}_\alpha, \pi_\alpha, \tilde{\mathbf{v}})$, with $\tilde{\mathbf{v}} \in L^\infty(0,T;\mathbf{L}^2(\omega))$, satisfying

$$\begin{cases} \tilde{\mathbf{y}}_t - \Delta\tilde{\mathbf{y}} + (\tilde{\mathbf{z}}\cdot\nabla)\tilde{\mathbf{y}} + \nabla\tilde{p} = \tilde{\mathbf{v}}1_\omega & \text{in} \quad \widetilde{Q}, \\ \mathbf{z} - \alpha^2\Delta\mathbf{z} + \nabla\pi = \tilde{\mathbf{y}} & \text{in} \quad Q, \\ \nabla\cdot\tilde{\mathbf{y}} = 0 & \text{in} \quad \widetilde{Q}, \\ \nabla\cdot\mathbf{z} = 0 & \text{in} \quad Q, \\ \tilde{\mathbf{y}} = \mathbf{0} & \text{on} \quad \widetilde{\Sigma}, \\ \mathbf{z} = \tilde{\mathbf{y}} & \text{on} \quad \Sigma, \\ \tilde{\mathbf{y}}(0) = \tilde{\mathbf{y}}_0 & \text{in} \quad \widetilde{\Omega} \end{cases} \tag{2.34}$$

and $\tilde{\mathbf{y}}(T) = \mathbf{0}$ in $\widetilde{\Omega}$, where $\tilde{\mathbf{z}}$ is the extension by zero of $\mathbf{z}$. Obviously, if this were the case, the restriction $(\mathbf{y}, p)$ of $(\tilde{\mathbf{y}}, \tilde{p})$ to Q, the couple $(\mathbf{z}, \pi)$ and the lateral trace $\mathbf{h} := \tilde{\mathbf{y}}|_{\gamma\times(0,T)}$ would satisfy (2.6) and (2.7).

Let us fix σ with $N/4 < \sigma < \beta < 1$. Then, for each $\overline{\mathbf{y}} \in L^\infty(0,T;D(\tilde{\mathbf{A}}^\sigma))$, let $\mathbf{w} = \mathbf{w}(\mathbf{x},t)$ and $\pi = \pi(\mathbf{x},t)$ be the unique solution to

$$\begin{cases} \mathbf{w} - \alpha^2\Delta\mathbf{w} + \nabla\pi = \alpha^2\Delta\overline{\mathbf{y}} & \text{in} \quad Q, \\ \nabla\cdot\mathbf{w} = 0 & \text{in} \quad Q, \\ \mathbf{w} = \mathbf{0} & \text{on} \quad \Sigma. \end{cases}$$

Since $\overline{\mathbf{y}} \in L^\infty(0,T;D(\tilde{\mathbf{A}}^\sigma))$, its restriction to Q belongs to $L^\infty(0,T;\mathbf{H}^{2\sigma}(\Omega))$. Then, Lemma 3 implies $\mathbf{w} \in L^\infty(0,T;D(\mathbf{A}^\sigma))$ and, thanks to Theorem 10, we also have $\mathbf{w} \in \mathbf{L}^\infty(Q)$ and

$$\|\mathbf{w}\|^2_{L^\infty(0,T;D(\mathbf{A}^\sigma))} \le C\|\overline{\mathbf{y}}\|^2_{L^\infty(0,T;D(\tilde{\mathbf{A}}^\sigma))},$$

where C is independent of α.

Let $\tilde{\mathbf{w}}$ be the extension by zero of $\mathbf{w}$ and let us set $\tilde{\mathbf{z}} := \overline{\mathbf{y}} + \tilde{\mathbf{w}}$. Let us consider the system (2.27) with $\mathbf{h}$ replaced by $\tilde{\mathbf{z}}$ and Ω replaced by $\widetilde{\Omega}$. In view of Theorem 12, we can associate to $\tilde{\mathbf{z}}$ the null control $\tilde{\mathbf{v}}$ of minimal norm in $L^\infty(0,T;\mathbf{L}^2(\tilde{\omega}))$ and the corresponding solution $(\tilde{\mathbf{y}}, \tilde{p})$ to (2.27). Since $\tilde{\mathbf{y}}_0 \in D(\tilde{\mathbf{A}})$, $\tilde{\mathbf{z}} \in \mathbf{L}^\infty(\widetilde{Q})$ and $\tilde{\mathbf{v}} \in L^\infty(0,T;\mathbf{L}^2(\tilde{\omega}))$, we have

$$\tilde{\mathbf{y}} \in L^2(0,T;D(\tilde{\mathbf{A}})) \cap C^0([0,T];\tilde{\mathbf{V}}), \ \tilde{\mathbf{y}}_t \in L^2(0,T;\tilde{\mathbf{H}})$$

and the following estimate holds:

$$\|\tilde{\mathbf{y}}_t\|^2_{L^2(\tilde{\mathbf{H}})} + \|\tilde{\mathbf{y}}\|^2_{L^2(D(\tilde{\mathbf{A}}))} + \|\tilde{\mathbf{y}}\|^2_{L^\infty(\tilde{\mathbf{V}})} \le C(\|\tilde{\mathbf{y}}_0\|^2_{\tilde{\mathbf{V}}} + \|\tilde{\mathbf{v}}\|^2_{L^\infty(\mathbf{L}^2(\tilde{\omega}))})e^{C\|\tilde{\mathbf{z}}\|^2_\infty}. \tag{2.35}$$

Also, in account of Lemma 4, one has $\tilde{\mathbf{y}} \in L^\infty(0,T;D(\tilde{\mathbf{A}}^\beta))$ and

$$\|\tilde{\mathbf{y}}\|_{L^\infty(D(\tilde{\mathbf{A}}^\beta))} \le C(\|\tilde{\mathbf{y}}_0\|_{D(\tilde{\mathbf{A}})} + \|\tilde{\mathbf{v}}\|_{L^\infty(\mathbf{L}^2(\tilde{\omega}))})e^{C\|\overline{\mathbf{y}}\|_{L^\infty(0,T;D(\tilde{\mathbf{A}}^\sigma))}}).$$

Now, let us set

$$\mathbf{W} = \{\, \mathbf{m} \in L^\infty(0,T;D(\tilde{\mathbf{A}}^\beta)) : \mathbf{m}_t \in L^2(0,T;\tilde{\mathbf{H}}) \,\},$$

and let us consider the closed ball

$$\mathbf{K} = \{\, \overline{\mathbf{y}} \in L^\infty(0,T; D(\tilde{\mathbf{A}}^\sigma)) : \|\overline{\mathbf{y}}\|_{L^\infty(D(\tilde{\mathbf{A}}^\sigma))} \le 1 \,\}$$

and the mapping $\tilde{\Lambda}_\alpha$, with $\tilde{\Lambda}_\alpha(\overline{\mathbf{y}}) = \tilde{\mathbf{y}}$ for all $\overline{\mathbf{y}} \in L^\infty(0,T; D(\tilde{\mathbf{A}}^\sigma))$. Obviously, $\tilde{\Lambda}_\alpha$ is well defined and maps the whole space $L^\infty(0,T; D(\tilde{\mathbf{A}}^\sigma))$ into $\mathbf{W}$. Furthermore, any bounded set $\mathbf{U} \subset \mathbf{W}$ then it is relatively compact in $L^\infty(0,T; D(\tilde{\mathbf{A}}^\sigma))$.

Let us denote by Λ_α the restriction to $\mathbf{K}$ of $\tilde{\Lambda}_\alpha$. Thanks to Lemma 4 and (2.28), there exists $\varepsilon > 0$ (independent of α) such that if $\|\tilde{\mathbf{y}}_0\|_{D(\tilde{\mathbf{A}})} \le \varepsilon$, Λ_α maps $\mathbf{K}$ into itself and it is clear that $\Lambda_\alpha : \mathbf{K} \mapsto \mathbf{K}$ satisfies the hypotheses of Schauder's Theorem. Consequently, Λ_α possesses at least one fixed point in $\mathbf{K}$ and (2.34) possesses a solution.This ends the proof of Theorem 7. □

Proof of Theorem 9. The proof is easy, in view of the previous uniform estimates. It suffices to adapt the argument in the proof of Theorem 8 and deduce the existence of subsequences that converge (in an appropriate sense) to a solution to (2.11) satisfying (2.7). For brevity, we omit the details. □

Proof of Lemma 5. For instance, let us only consider the case $N = 3$. We will reduce the proof to the search of a fixed point of another mapping Φ_α.

For any $\mathbf{y}_0 \in \mathbf{V}$, any $T_0 \in (0,T)$ and any $\overline{\mathbf{y}} \in L^4(0,T_0; \widetilde{\mathbf{V}}))$, let $(\mathbf{w},\pi)$ be the unique solution to

$$\begin{cases} \mathbf{w} - \alpha^2\Delta\mathbf{w} + \nabla\pi = \alpha^2\Delta\overline{\mathbf{y}} & \text{in} \quad \Omega \times (0,T_0), \\ \nabla\cdot\mathbf{w} = 0 & \text{in} \quad \Omega \times (0,T_0), \\ \mathbf{w} = \mathbf{0} & \text{on} \quad \Gamma \times (0,T_0), \end{cases}$$

let $\tilde{\mathbf{w}}$ be the extension by zero of $\mathbf{w}$, let us set $\tilde{\mathbf{z}} := \overline{\mathbf{y}} + \tilde{\mathbf{w}}$ and let us introduce the Oseen system

$$\begin{cases} \tilde{\mathbf{y}}_t - \Delta\tilde{\mathbf{y}} + (\tilde{\mathbf{z}}\cdot\nabla)\tilde{\mathbf{y}} + \nabla\tilde{p} = \mathbf{0} & \text{in} \quad \widetilde{\Omega} \times (0,T_0), \\ \nabla\cdot\tilde{\mathbf{y}} = 0 & \text{in} \quad \widetilde{\Omega} \times (0,T_0), \\ \tilde{\mathbf{y}} = \mathbf{0} & \text{on} \quad \partial\widetilde{\Omega} \times (0,T_0), \\ \tilde{\mathbf{y}}(0) = \tilde{\mathbf{y}}_0 & \text{in} \quad \widetilde{\Omega}. \end{cases}$$

It is clear that the restriction of $\overline{\mathbf{y}}$ to $\Omega \times (0,T_0)$ belongs to $L^4(0,T_0; \mathbf{H}^1(\Omega))$, whence we have from Lemma 3 that $\mathbf{w} \in L^4(0,T_0; \mathbf{V})$ and

$$\|\mathbf{w}\|_{L^4(0,T_0;\mathbf{V})} \le C\|\overline{\mathbf{y}}\|_{L^4(0,T_0;\widetilde{\mathbf{V}})}.$$

It is also clear that we can get estimates like those in the proof of Proposition 7 for $\widetilde{\mathbf{y}}$. In other words, for any $\mathbf{y}_0 \in \mathbf{V}$, we can find a sufficiently small $T_0 > 0$ such that

$$\widetilde{\mathbf{y}} \in L^2(0,T_0; D(\widetilde{\mathbf{A}})) \cap C^0([0,T_0]; \widetilde{\mathbf{V}}), \quad \widetilde{\mathbf{y}}_t \in L^2(0,T_0; \widetilde{\mathbf{H}})$$

and

$$\|\widetilde{\mathbf{y}}\|_{L^2(0,T_0;D(\widetilde{\mathbf{A}}))} + \|\widetilde{\mathbf{y}}\|_{C^0([0,T_0];\widetilde{\mathbf{V}})} + \|\widetilde{\mathbf{y}}_t\|_{L^2(0,T_0;\widetilde{\mathbf{H}})} \le C\left(T_0, \|\mathbf{y}_0\|_{\mathbf{V}}, \|\overline{\mathbf{y}}\|_{L^4(0,T_0;\widetilde{\mathbf{V}})}\right),$$

where C is nondecreasing with respect to all arguments and goes to zero as $\|\mathbf{y}_0\|_{\mathbf{V}} \to 0$.

Now, let us introduce the mapping $\Phi_\alpha : L^4(0, T_0; \widetilde{\mathbf{V}}) \mapsto L^4(0, T_0; \widetilde{\mathbf{V}})$, with $\Phi_\alpha(\overline{\mathbf{y}}) = \widetilde{\mathbf{y}}$ for all $\overline{\mathbf{y}} \in L^4(0, T; \widetilde{\mathbf{V}})$. This is a continuous and compact mapping. Indeed, from well known interpolation results, we have that the embedding

$$L^2(0, T_0; D(\widetilde{\mathbf{A}})) \cap L^\infty(0, T_0; \widetilde{\mathbf{V}}) \hookrightarrow L^4(0, T_0; D(\widetilde{\mathbf{A}}^{3/4}))$$

is continuous and this shows that, if $\widetilde{\mathbf{y}}$ is bounded in $L^2(0, T_0; D(\widetilde{\mathbf{A}})) \cap C^0([0, T_0]; \widetilde{\mathbf{V}})$ and $\widetilde{\mathbf{y}}_t$ is bounded in $L^2(0, T_0; \widetilde{\mathbf{H}})$, then $\widetilde{\mathbf{y}}$ belongs to a compact set of $L^4(0, T_0; \widetilde{\mathbf{V}})$.

Then, as in the proofs of Theorems 6 and 7, we immediately deduce that, whenever $\|\mathbf{y}_0\|_{\mathbf{V}} \le \delta$ (for some δ independent of α), Φ_α possesses at least one fixed point. This shows that the nonlinear system (2.34) is solvable for $\tilde{\mathbf{v}} \equiv 0$ and $\|\mathbf{y}_0\|_{\mathbf{V}} \le \delta$.

Now, the argument in the proof of Lemma 2 can be applied in this framework and, as a consequence, we easily deduce Lemma 5. □

2.5 Additional comments and questions

2.5.1 Controllability problems for semi-Galerkin approximations

Let $\{ \mathbf{w}^1, \mathbf{w}^2, \dots \}$ be a basis of the Hilbert space $\mathbf{V}$. For instance, we can consider the orthogonal base formed by the eigenvectors of the Stokes operator $\mathbf{A}$. Together with (2.5), we can consider the following semi-Galerkin approximated problems:

$$\begin{cases} \mathbf{y}_t - \Delta \mathbf{y} + (\mathbf{z}^m \cdot \nabla)\mathbf{y} + \nabla p = \mathbf{v} 1_\omega & \text{in } Q, \\ (\mathbf{z}^m(t), \mathbf{w}) + \alpha^2 (\nabla \mathbf{z}^m(t), \mathbf{w}) = (\mathbf{y}(t), \mathbf{w}) \ \forall \mathbf{w} \in \mathbf{V}_m, \ \mathbf{z}^m(t) \in \mathbf{V}_m, & t \in (0, T), \\ \nabla \cdot \mathbf{y} = 0, & \text{in } Q, \\ \mathbf{y} = \mathbf{0} & \text{on } \Sigma, \\ \mathbf{y}(0) = \mathbf{y}_0 & \text{in } \Omega, \end{cases} \tag{2.36}$$

where $\mathbf{V}_m$ denotes the space spanned by $\mathbf{w}^1, \dots, \mathbf{w}^m$.

Arguing as in the proof of Theorem 6, it is possible to prove a local null controllability result for (2.36). More precisely, for each $m \ge 1$, there exists $\varepsilon_m > 0$ such that, if $\|\mathbf{y}_0\| \le \varepsilon_m$, we can find controls $\mathbf{v}^m$ and associated states $(\mathbf{y}^m, p^m, \mathbf{z}^m)$ satisfying (2.7). Notice that, in view of the equivalence of norms in $\mathbf{V}_m$, the fixed point argument can be applied in this case without any extra regularity assumption on $\mathbf{y}_0$; in other words, Lemma 2 is not needed here.

On the other hand, it can also be checked that the maximal ε_m are bounded from below by some positive quantity independent of m and α and the controls $\mathbf{v}^m$ can be found uniformly bounded in $L^\infty(0, T; \mathbf{L}^2(\omega))$. As a consequence, at least for a subsequence, the controls converge weakly-$*$ in that space to a null control for (2.5).

However, it is unknown whether the problems (2.36) are *globally* null-controllable; see below for other considerations concerning global controllability.

2.5.2 Another strategy: applying an inverse function theorem

There is another way to prove the local null controllability of (2.5) that relies on *Liusternik's Inverse Function Theorem,* see for instance [1]. This strategy has been introduced in [40] and has been applied successfully to the controllability of many semilinear and nonlinear PDE's. In the framework of (2.5), the argument is as follows:

1. Introduce an appropriate Hilbert space $\mathbf{Y}$ of *state-control pairs* $(\mathbf{y}_\alpha, p_\alpha, \mathbf{z}_\alpha, \pi_\alpha, \mathbf{v}_\alpha)$ satisfying (2.7).

2. Introduce a second Hilbert space $\mathbf{Z}$ of right hand sides and initial data and a well-defined mapping $\mathbf{F} : \mathbf{Y} \mapsto \mathbf{Z}$ such that the null controllability of (2.5) with state-controls in $\mathbf{Y}$ is equivalent to the solution of the nonlinear equation

$$\mathbf{F}(\mathbf{y}_\alpha, p_\alpha, \mathbf{z}_\alpha, \pi_\alpha, \mathbf{v}_\alpha) = (\mathbf{0}, \mathbf{y}_0), \quad (\mathbf{y}_\alpha, p_\alpha, \mathbf{z}_\alpha, \pi_\alpha, \mathbf{v}_\alpha) \in \mathbf{Y}. \tag{2.37}$$

3. Prove that $\mathbf{F}$ is C^1 in a neighborhood of $(\mathbf{0}, 0, \mathbf{0}, 0, \mathbf{0})$ and $\mathbf{F}'(\mathbf{0}, 0, \mathbf{0}, 0, \mathbf{0})$ is onto.

Arguing as in [29], all this can be accomplished satisfactorily. As a result, (2.37) can be solved for small initial data $\mathbf{y}_0$ and the local null controllability of (2.5) holds.

2.5.3 On global controllability properties

It is unknown whether a general global null controllability result holds for (2.5). This is not surprising, since the same question is also open for the Navier-Stokes system.

What can be proved (as well as for the Navier-Stokes system) is the null controllability for large time: for any given $\mathbf{y}_0 \in \mathbf{H}$, there exists $T_* = T_*(\|\mathbf{y}_0\|)$ such that (2.5) can be driven exactly to zero with controls $\mathbf{v}_\alpha$ uniformly bounded in $L^\infty(0, T_*; \mathbf{L}^2(\omega))$.

Indeed, let ε be the constant furnished by Theorem 6 corresponding to the time $T = 1$ (for instance). Let us first take $\mathbf{v}_\alpha \equiv \mathbf{0}$. Then, since the solution to (2.3) with $\mathbf{f} = \mathbf{0}$ satisfies $\|\mathbf{y}_\alpha(t)\| \searrow 0$, there exists T_0 (depending on $\|\mathbf{y}_0\|$ but not on α) such that $\|\mathbf{y}_\alpha(T_0)\| \le \varepsilon$. Therefore, there exist controls $\mathbf{v}'_\alpha \in L^\infty(T_0, T_0+1; \mathbf{L}^2(\omega))$ such that the solution to (2.5) that starts from $\mathbf{y}_\alpha(T_0)$ at time T_0 satisfies $\mathbf{y}_\alpha(T_0+1) = \mathbf{0}$. Hence, the assertion is fulfilled with $T_* = T_0 + 1$ and

$$\mathbf{v}_\alpha = \begin{cases} \mathbf{0} & \text{for} \quad 0 \le t < T_0, \\ \mathbf{v}'_\alpha & \text{for} \quad T_0 \le t \le T_*. \end{cases}$$

A similar argument leads to the null controllability of (2.5) for large α. In other words, it is also true that, for any given $\mathbf{y}_0 \in \mathbf{H}$ and $T > 0$, there exists $\alpha_* = \alpha_*(\|\mathbf{y}_0\|, T)$ such that, if $\alpha \ge \alpha_*$, then (2.5) can be driven exactly to zero at time T.

2.5.4 The Burgers-α system

There exist similar results for a regularized version of the Burgers equation, more precisely the Burgers-α system

$$\begin{cases} y_t - y_{xx} + zy_x = v1_{(a,b)} & \text{in} \quad (0,L)\times(0,T), \\ z - \alpha^2 z_{xx} = y & \text{in} \quad (0,L)\times(0,T), \\ y(0,t) = y(L,t) = z(0,t) = z(L,t) = 0 & \text{on} \quad (0,T), \\ y(x,0) = y_0(x) & \text{in} \quad (0,L). \end{cases} \tag{2.38}$$

These have been proved in [3].

This system can be viewed as a toy or preliminary model of (2.5). There are, however, several important differences between (2.5) and (2.38):

- The solution to (2.38) satisfies a maximum principle that provides a useful L^∞-estimate.
- There is no apparent energy decay for the uncontrolled solutions. As a consequence, the large time null controllability of (2.38) is unknown.
- It is known that, in the limit $\alpha = 0$, i.e. for the Burgers equation, global null controllability does not hold; consequently, in general, the null controllability of (2.38) with controls bounded independently of α is impossible.

We refer to [3] for further details.

2.5.5 Local exact controllability to the trajectories

It makes sense to consider not only null controllability but also *exact to the trajectories* controllability problems for (2.5). More precisely, let $\hat{\mathbf{y}}_0 \in \mathbf{H}$ be given and let $(\hat{\mathbf{y}}, \hat{p}, \hat{\mathbf{z}}, \hat{\pi})$ a sufficiently regular solution to (2.3) for $\mathbf{f} \equiv \mathbf{0}$ and $\mathbf{y}_0 = \hat{\mathbf{y}}_0$. Then the question is whether, for any given $\mathbf{y}_0 \in \mathbf{H}$, there exist controls $\mathbf{v}$ such that the associated states, i.e. the associated solutions to (2.5), satisfy

$$\mathbf{y}(T) = \hat{\mathbf{y}}(T) \text{ in } \Omega.$$

The change of variables

$$\mathbf{y} = \hat{\mathbf{y}} + \mathbf{u}, \ \ \mathbf{z} = \hat{\mathbf{z}} + \mathbf{w},$$

allows to rewrite this problem as the null controllability of a system similar, but not identical, to (2.5). It is thus reasonable to expect that a local result holds.

2.5.6 Controlling with few scalar controls

The local null controllability with $N-1$ or even less scalar controls is also an interesting question.

In view of the achievements in [7] and [19] for the Navier-Stokes equations, it is reasonable to expect that results similar to Theorems 6 and 8 hold with controls $\mathbf{v}$ such that $v_i \equiv 0$ for some i; under some geometrical restrictions, it is also expectable that local exact controllability to the trajectories holds with controls of the same kind, see [30].

2.5.7 Other related controllability problems

There are many other interesting questions concerning the controllability of (2.5) and related systems.

For instance, we can consider questions like those above for the Leray-α equations completed with other boundary conditions: Navier, Fourier or periodic conditions for $\mathbf{y}$ and $\mathbf{z}$, conditions of different kinds on different parts of the boundary, etc. We can also consider Boussinesq-α systems, i.e. systems of the form

$$\begin{cases} \mathbf{y}_t - \Delta\mathbf{y} + (\mathbf{z}\cdot\nabla)\mathbf{y} + \nabla p = \theta\mathbf{k} + \mathbf{v}1_\omega & \text{in} \quad Q, \\ \theta_t - \Delta\theta + \mathbf{z}\cdot\nabla\theta = \mathbf{w}1_\omega & \text{in} \quad Q, \\ \mathbf{z} - \alpha^2\Delta\mathbf{z} + \nabla\pi = \mathbf{y} & \text{in} \quad Q, \\ \nabla\cdot\mathbf{y} = 0, \ \nabla\cdot\mathbf{z} = 0 & \text{in} \quad Q, \\ \mathbf{y} = \mathbf{z} = \mathbf{0}, \ \theta = 0 & \text{on} \quad \Sigma, \\ \mathbf{y}(0) = \mathbf{y}_0, \ \theta(0) = \theta_0 & \text{in} \quad \Omega. \end{cases}$$

Some of these results will be analyzed in a forthcoming paper.

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Capítulo 3

On the boundary controllability of incompressible Euler fluids with Boussinesq heat effects

On the boundary controllability of incompressible Euler fluids with Boussinesq heat effects

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Abstract. This paper deals with the boundary controllability of inviscid incompressible fluids for which thermal effects are important and are modelled through the Boussinesq approximation. Almost all our results deal with zero heat diffusion. By adapting and extending some ideas from Coron and Glass, we establish the simultaneous global exact controllability of the velocity field and the temperature for 2D and 3D flows.

3.1 Introduction

Let Ω be a nonempty bounded open subset of $\mathbb{R}^N$ of class C^∞ ($N = 2$ or $N = 3$). We assume that Ω is connected and (for simplicity) simply connected. Let Γ_0 be a nonempty open subset of the boundary $\Gamma = \partial\Omega$. Bold letters will denote vector-valued functions; for instance, the vector function $\mathbf{v} \in \mathbf{C}^0(\overline{\Omega};\mathbb{R}^N)$ will be of the form $\mathbf{v} = (v_1,\ldots,v_N)$, where $v_1,\ldots,\ v_N \in C^0(\overline{\Omega};\mathbb{R})$. Let us denote by $\mathbf{n}(\mathbf{x})$ the outward unit normal vector to Ω at any point $\mathbf{x} \in \Gamma$.

In this work, we are concerned with the boundary controllability of the system:

$$\begin{cases} \mathbf{y}_t + (\mathbf{y}\cdot\nabla)\mathbf{y} = -\nabla p + \vec{\mathbf{k}}\theta & \text{in} \quad \Omega\times(0,T), \\ \nabla\cdot\mathbf{y} = 0 & \text{in} \quad \Omega\times(0,T), \\ \theta_t + \mathbf{y}\cdot\nabla\theta = 0 & \text{in} \quad \Omega\times(0,T), \\ \mathbf{y}(\mathbf{x},0) = \mathbf{y}_0(\mathbf{x}),\ \theta(\mathbf{x},0) = \theta_0(\mathbf{x}) & \text{in} \quad \Omega, \end{cases} \tag{3.1}$$

where:

- $\mathbf{y}$ and the scalar function p stand for the velocity field and the pressure of an inviscid incompressible fluid in $\Omega\times(0,T)$.
- The function θ provides the temperature distribution of the fluid.
- $\vec{\mathbf{k}}\theta$ can be viewed as the *buoyancy force* density ($\vec{\mathbf{k}}$ is a nonzero vector of $\mathbb{R}^N$).

The system (3.76) is called the *incompressible inviscid Boussinesq* system.

For now on, we assume that $\alpha \in (0,1)$ and we set

$$\mathbf{C}(m,\alpha,\Gamma_0) := \{\, \mathbf{u} \in \mathbf{C}^{m,\alpha}(\overline{\Omega};\mathbb{R}^N) \,:\, \nabla\cdot\mathbf{u} = 0 \text{ in } \overline{\Omega} \text{ and } \mathbf{u}\cdot\mathbf{n} = 0 \text{ on } \Gamma\backslash\Gamma_0 \,\}, \tag{3.2}$$

where $\mathbf{C}^{m,\alpha}(\overline{\Omega};\mathbb{R}^N)$ denotes the space of functions in $\mathbf{C}^m(\overline{\Omega};\mathbb{R}^N)$ whose m-th order derivatives are Hölder-continuous with exponent α.

The exact boundary controllability problem for (3.76) can be stated as follows:

Given $\mathbf{y}_0$, $\mathbf{y}_1 \in \mathbf{C}(2,\alpha,\Gamma_0)$ *and* θ_0, $\theta_1 \in C^{2,\alpha}(\overline{\Omega};\mathbb{R})$, *find* $\mathbf{y} \in C^0([0,T];\mathbf{C}(1,\alpha,\Gamma_0))$, $\theta \in C^0([0,T];C^{1,\alpha}(\overline{\Omega};\mathbb{R}))$ *and* $p \in \mathcal{D}'(\Omega\times(0,T))$ *such that* (3.76) *holds and*

$$\mathbf{y}(\mathbf{x},T) = \mathbf{y}_1(\mathbf{x}), \quad \theta(\mathbf{x},T) = \theta_1(\mathbf{x}) \quad in \quad \Omega. \tag{3.3}$$

If it is always possible to find $\mathbf{y}$, θ and p, we say that the incompressible inviscid Boussinesq system is *exactly controllable* for (Ω,Γ_0) at time T.

Remark 5. In order to determine without ambiguity a unique local in time regular solution to (3.76), it is sufficient to prescribe the normal component of the velocity vector on the boundary of the flow region and the full field $\mathbf{y}$ and the temperature θ only on the inflow section, i.e. only where $\mathbf{y}\cdot\mathbf{n} < 0$, see for instance [68]. Hence, in (3.76), we can assume that the controls are given by:

$$\begin{cases} \mathbf{y}\cdot\mathbf{n} \text{ on } \Gamma_0\times(0,T), \text{ with } \displaystyle\int_{\Gamma_0} \mathbf{y}\cdot\mathbf{n}\,d\Gamma = 0; \\ \mathbf{y} \text{ at any point of } \Gamma_0\times(0,T]) \text{ satisfying } \mathbf{y}\cdot\mathbf{n} < 0; \\ \theta \text{ at any point of } \Gamma_0\times(0,T) \text{ satisfying } \mathbf{y}\cdot\mathbf{n} < 0. \end{cases}$$

□

The meaning of the exact controllability property is that, when it holds, we can drive exactly the fluid, acting only on an arbitrary small part Γ_0 of the boundary during an arbitrary small time interval $(0,T)$, from any initial state $(\mathbf{y}_0,\theta_0)$ to any final state $(\mathbf{y}_1,\theta_1)$.

The Boussinesq system is potentially relevant to the study of atmospheric and oceanographic turbulence, as well as to other astrophysical situations where rotation and stratification play a dominant role (see e.g. [71]). In fluid mechanics, (3.76) is used in the field of buoyancy-driven flow. It describes the motion of an incompressible inviscid fluid subject to convective heat transfer under the influence of gravitational forces, see [69].

The controllability of systems governed by (linear and nonlinear) PDEs has focused the attention of a lot of researchers the last decades. Some related results can be found in [16, 47, 62, 81]. In the context of incompressible ideal fluids, this subject has been mainly investigated by Coron [13, 15] and Glass [43, 44, 45].

In this paper, we are going to adapt the techniques and arguments of [15] and [45] to the situations modelled by (3.76).

The first main result is the following:

Theorem 13. *The incompressible inviscid Boussinesq system* (3.76) *is exactly controllable for* (Ω,Γ_0) *at any time* $T > 0$.

The proof of Theorem 13 relies on the extension and return methods. These have been applied in several different contexts to establish controllability; see the seminal work [72] and the contributions [13, 15, 43, 44].

Let us give a sketch of the strategy:

- First, we construct a “good"trajectory connecting $\mathbf{0}$ to $\mathbf{0}$ (see sections 3.2.1 and 3.2.2).
- We apply the extension method of David L. Russell [72].
- We use a Fixed-Point Theorem to obtain a local exact controllability result.
- Finally, we use an appropriate scaling argument to deduce the desired global result.

In fact, Theorem 13 is a iconsequence of the following result:

Proposition 9. *There exists $\delta > 0$ such that, for any $\theta_0 \in C^{2,\alpha}(\overline{\Omega};\mathbb{R})$ and $\mathbf{y}_0 \in \mathbf{C}(2,\alpha,\Gamma_0)$ with*

$$\max\{\|\mathbf{y}_0\|_{2,\alpha}, \|\theta_0\|_{2,\alpha}\} < \delta,$$

there exist $\mathbf{y} \in C^0([0,1];\mathbf{C}(1,\alpha,\Gamma_0))$, $\theta \in C^0([0,1];C^{1,\alpha}(\overline{\Omega};\mathbb{R}))$ and $p \in \mathcal{D}'(\overline{\Omega}\times[0,1])$ satisfying (3.76) in $\Omega\times(0,1)$ and

$$\mathbf{y}(\mathbf{x},1) = \mathbf{0},\ \ \theta(\mathbf{x},1) = 0 \ \text{in}\ \Omega. \tag{3.4}$$

The rest of this paper is organized as follows. In Section 4.2, we recall the results needed to prove Theorem 9. In Section 4.3, we give the proof of Theorem 13. In Section 4.4, we prove Proposition 9 in the 2D case. It will be seen that the main ingredients of the proof are the construction of a nontrivial trajectory that starts and ends at $\mathbf{0}$ and a Fixed-Point Theorem (the key ideas of the return method). In Section 4.5, we give the proof of Theorem 9 in the 3D case. Finally, in Section 4.6, we present some additional comments and open questions.

3.2 Preliminary results

In this section, we are going to recall some results used in the proofs of the main results. Also, we are going to indicate how to construct an appropriate trajectory in order to apply the return method.

The first one is a immediate consequence of the well known Banach Fixed Point Theorem.

Theorem 14. *Let $(B_1,\|\cdot\|_1)$ and $(B_2,\|\cdot\|_2)$ Banach spaces with B_2 continuously embedded in B_1. Let B be a subset of B_2 and let $G : B \to B$ be a mapping such that*

$$\|G(u)-G(v)\|_1 \le \gamma\|u-v\|_1 \ \ \forall\ u,v\in B,\ \text{for some}\ 0<\gamma<1.$$

Then, G has a unique fixed point in $\overline{B}^{\|\cdot\|_1}$.

The following lemma will be very important to deduce later appropriate estimates. The proof can be found in [4].

Lemma 6. *Let m be a positive integer. Assume that $u, g \in C^0([0,T];C^{m+1,\alpha}(\overline{\Omega};\mathbb{R}))$ and $\mathbf{v}\in C^0([0,T];\mathbf{C}^{m+1,\alpha}(\overline{\Omega};\mathbb{R}^N))$ are given, with $\mathbf{v}\cdot\mathbf{n}=0$ on $\Gamma\times(0,T)$ and*

$$\frac{\partial u}{\partial t} + \mathbf{v}\cdot\nabla u = g \ \ \text{in}\ \ \Omega\times(0,T). \tag{3.5}$$

Then,

$$\frac{d}{dt^+}\|u(\cdot,t)\|_{m,\alpha} \le \|g(\cdot,t)\|_{m,\alpha} + K\|\mathbf{v}(\cdot,t)\|_{m,\alpha}\|u(\cdot,t)\|_{m,\alpha} \quad in \quad (0,T), \tag{3.6}$$

where K is a constant only depending on α and m.

In the case $m = 0$, we have:

$$\frac{d}{dt^+}\|u(\cdot,t)\|_{0,\alpha} \le \|g(\cdot,t)\|_{0,\alpha} + K\|\mathbf{v}(\cdot,t)\|_{1,\alpha}\|u(\cdot,t)\|_{0,\alpha} \quad in \quad (0,T).$$

In the following sections, we will frequently use a technical lemma whose proof can be found in [43]:

Lemma 7. *Let us assume that*

$$\mathbf{w}_0 \in \mathbf{C}^{1,\alpha}(\bar{\Omega};\mathbb{R}^N), \quad \nabla\cdot\mathbf{w}_0 = 0 \quad in \quad \Omega,$$

$$\mathbf{u} \in C^0([0,T];\mathbf{C}^{1,\alpha}(\bar{\Omega};\mathbb{R}^N)), \quad \mathbf{u}\cdot\mathbf{n} = 0 \quad on \quad \Gamma\times(0,T),$$

$$\mathbf{g} \in C^0([0,T];\mathbf{C}^{0,\alpha}(\bar{\Omega},\mathbb{R}^N)), \quad \nabla\cdot\mathbf{g} = 0 \quad in \quad \Omega\times(0,T).$$

Let $\mathbf{w}$ be a function in $C^0([0,T];\mathbf{C}^{1,\alpha}(\overline{\Omega};\mathbb{R}^N))$ satisfying

$$\begin{cases} \mathbf{w}_t + (\mathbf{u}\cdot\nabla)\mathbf{w} = (\mathbf{w}\cdot\nabla)\mathbf{u} - (\nabla\cdot\mathbf{u})\mathbf{w} + \mathbf{g} & in \quad \Omega\times(0,T), \\ \mathbf{w}(\cdot,0) = \mathbf{w}_0 & in \quad \Omega. \end{cases}$$

Then

$$\nabla\cdot\mathbf{w} = 0 \quad in \quad \Omega\times(0,T).$$

Moreover, there exists $\mathbf{v} \in C^0([0,T];\mathbf{C}^{2,\alpha}(\overline{\Omega};\mathbb{R}^N))$ such that

$$\mathbf{w} = \nabla\times\mathbf{v} \quad in \quad \Omega\times(0,T).$$

3.2.1 Construction of a trajectory when $N = 2$

Our trajectory will be associated to a domain $\Omega \subset \mathbb{R}^2$. To do this, we will argue as in [15]. Thus, let Ω_1 be a bounded, Lipschitz contractible subset of $\mathbb{R}^2$ whose boundary consists of two disjoint closed line segments Γ^- and Γ^+ and two disjoint closed curves Σ' and Σ'' of class C^∞ such that $\partial\Sigma' \cup \partial\Sigma'' = \partial\Gamma^- \cup \partial\Gamma^+$. We also impose that there is a neighborhood U^- of Γ^- (resp. U^+ of Γ^+) such that $\Omega_1 \cap U^-$ (resp. $\Omega_1 \cap U^+$) coincides with the intersection of U^- (resp. U^+), an open semi-plane limited by the line containing Γ^- (resp. Γ^+) and the band limited by the two straight lines orthogonal to Γ^- (resp. Γ^+) and passing through $\partial\Gamma^-$ (resp. $\partial\Gamma^+$); see Figure 3.1.

Let φ be the solution to

$$\begin{cases} \Delta\varphi = 0 & \text{in} \quad \Omega_1, \\ \varphi = 1 & \text{on} \quad \Gamma^+, \\ \varphi = -1 & \text{on} \quad \Gamma^-, \\ \dfrac{\partial\varphi}{\partial\mathbf{n}_1} = 0 & \text{on} \quad \Sigma, \end{cases} \tag{3.7}$$

where $\mathbf{n}_1$ is the outward unit normal vector field on $\partial\Omega_1$ and $\Sigma = \Sigma' \cup \Sigma''$. Then we have the following result from J.-M. Coron [15]:

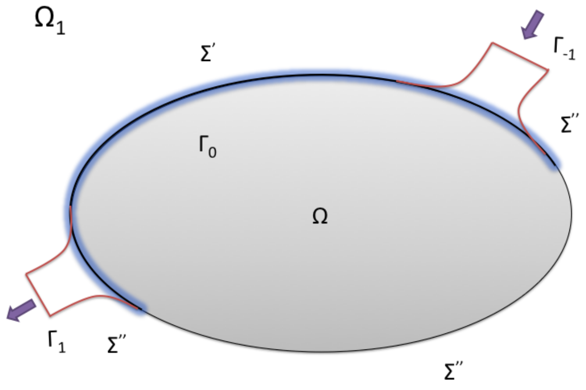

Figura 3.1: The domain Ω_1

Lemma 8. *One has* $\varphi \in C^\infty(\bar{\Omega}_1;\mathbb{R})$, $-1 < \varphi(\mathbf{x}) < 1$ *for all* $\mathbf{x} \in \Omega_1$ *and*

$$\nabla\varphi(\mathbf{x}) \neq \mathbf{0} \quad \forall \mathbf{x} \in \overline{\Omega}_1. \tag{3.8}$$

Let $\gamma \in C^\infty([0,1];\mathbb{R}^+)$ be a non-zero function such that *supp* $\gamma \subset (0,1/2) \cup (1/2,1)$ and the subsets *supp* $\gamma \cap (0,1/2)$ and *supp* $\gamma \cap (1/2,1)$ are non-empty.

Let $M > 0$ (it will be chosen below) and set $\overline{\mathbf{y}}(\mathbf{x},t) := M\gamma(t)\nabla\varphi(\mathbf{x})$, $\overline{p}(\mathbf{x},t) := -M\gamma_t(t)\varphi(\mathbf{x}) - \frac{M^2}{2}\gamma(t)^2|\nabla\varphi(\mathbf{x})|^2$ and $\overline{\theta} \equiv 0$. Then, (3.76) is satisfied for $\mathbf{y} = \overline{\mathbf{y}}$, $p = \overline{p}$, $\theta = \overline{\theta}$, $T = 1$, $\mathbf{y}_0 = \mathbf{0}$ and $\theta_0 = 0$. The solution $(\overline{\mathbf{y}},\ \overline{p},\ \overline{\theta})$ is thus a trajectory of (3.76) that connects the zero state to itself.

Let Ω_3 be a bounded open set of class C^∞ such that $\Omega_1 \subset\subset \Omega_3$. We extend φ to $\overline{\Omega}_3$ as a C^∞ function with compact support in Ω_3 and we still denote this extension by φ. Let us introduce $\mathbf{y}^*(\mathbf{x},t) := M\gamma(t)\nabla\varphi(\mathbf{x})$ (observe that $\overline{\mathbf{y}}$ is the restriction of $\mathbf{y}^*$ to $\overline{\Omega} \times [0,1]$). Also, we consider the associated *flux function* $\mathbf{Y}^* : \overline{\Omega}_3 \times [0,1] \times [0,1] \mapsto \overline{\Omega}_3$, defined as follows:

$$\begin{cases} \mathbf{Y}^*_t(\mathbf{x},t,s) = \mathbf{y}^*(\mathbf{Y}^*(\mathbf{x},t,s),t) \\ \mathbf{Y}^*(\mathbf{x},s,s) = \mathbf{x}. \end{cases} \tag{3.9}$$

The mapping $\mathbf{Y}^*$ contains all the information on the trajectories of the particles transported by the velocity field $\mathbf{y}^*$. It is customary to label the particles with the positions $\mathbf{x}$ at the beginning of observation ($t = s$). This system of ODEs express the fact that the particles travel with velocity $\mathbf{y}^*$. The flux $\mathbf{Y}^*$ is of class C^∞ in $\overline{\Omega}_3 \times [0,1] \times [0,1]$. Furthermore, $\mathbf{Y}^*(\cdot,t,s)$ is a diffeomorphism of $\overline{\Omega}_3$ onto itself for each $s,t \in [0,1]$. Finally,

$$(\mathbf{Y}^*(\cdot,t,s))^{-1} = \mathbf{Y}^*(\cdot,s,t),\ \forall s,t \in [0,1].$$

Remark 6. From the definition of $\mathbf{Y}^*$ and the boundary conditions on Ω_1 satisfied by φ, we observe that:

- The condition $\mathbf{y}^* \cdot \mathbf{n}_1 = 0$ on Σ means that the particles cannot cross Σ;

- Since φ is constant on Γ^+, the gradient $\nabla\varphi$ is parallel to the normal vector on Γ^+. As φ attains a maximum at any point of Γ^+, we have $\nabla\varphi \cdot \mathbf{n}_1 > 0$ on Γ^+, whence, $\mathbf{y}^* \cdot \mathbf{n}_1 \geq 0$ on $\Gamma^+ \times [0,1]$. Similary, $\mathbf{y}^* \cdot \mathbf{n} \leq 0$ on $\Gamma^- \times [0,1]$.

Consequently, the particles moving with velocity field $\mathbf{y}^*$ can leave Ω_1 only through Γ^+ and can enter Ω_1 only through Γ^-. □

The following lemma shows that the particles that travel with velocity $\mathbf{y}^*$ and are inside $\overline{\Omega}_1$ at time $t = 0$ (resp. $t = 1/2$) will be outside $\overline{\Omega}_1$ at time $t = 1/2$ (resp. $t = 1$).

Lemma 9. *There exist $M > 0$ (large enough) and a bounded open subset Ω_2 satisfying $\Omega_1 \subset\subset \Omega_2 \subset\subset \Omega_3$ and*

$$\begin{aligned} \mathbf{x} \in \overline{\Omega}_2 \quad &\Longrightarrow \quad \mathbf{Y}^*(\mathbf{x}, 1/2, 0) \notin \overline{\Omega}_2, \\ \mathbf{x} \in \overline{\Omega}_2 \quad &\Longrightarrow \quad \mathbf{Y}^*(\mathbf{x}, 1, 1/2) \notin \overline{\Omega}_2. \end{aligned} \tag{3.10}$$

The proof is given in [15] and relies on the properties of $\mathbf{y}^*$ and, more precisely, on the fact that $t \mapsto \varphi(\mathbf{Y}^*(\mathbf{x}, t, s))$ is nondecreasing.

The next step is to introduce appropriate extension mappings from Ω to Ω_3. More precisely, we have the following result from [52]:

Lemma 10. *For $i = 1, 2$, there exists $\pi_i : \mathbf{C}^0(\overline{\Omega}; \mathbb{R}^i) \mapsto \mathbf{C}^0(\overline{\Omega}_3; \mathbb{R}^i)$, linear and continuous, such that*

$$\begin{cases} \pi_i(\mathbf{f}) = \mathbf{f} \quad \textit{in} \quad \overline{\Omega}, \ \forall \mathbf{f} \in \mathbf{C}^0(\overline{\Omega}; \mathbb{R}^i), \\ \textit{supp } \pi_i(\mathbf{f}) \subset \Omega_2 \quad \forall \mathbf{f} \in \mathbf{C}^0(\overline{\Omega}; \mathbb{R}^i), \\ \pi_i \textit{ maps continuously } \mathbf{C}^{n,\lambda}(\overline{\Omega}; \mathbb{R}^i) \textit{ into } \mathbf{C}^{n,\lambda}(\overline{\Omega}_3; \mathbb{R}^i) \quad \forall n \geq 0, \quad \forall \lambda \in (0,1). \end{cases} \tag{3.11}$$

The next lemma says that (3.10) holds not only for $\mathbf{y}^*$ but also for any appropriate extension of any flow $\mathbf{z}$ sufficiently close to $\overline{\mathbf{y}}$:

Lemma 11. *For each $\mathbf{z} \in C^0([0,1]; C^{1,\alpha}(\overline{\Omega}; \mathbb{R}^2))$, let us set $\mathbf{z}^* = \mathbf{y}^* + \pi_2(\mathbf{z} - \overline{\mathbf{y}})$. There exists $\nu > 0$ such that, if $\|\mathbf{z} - \overline{\mathbf{y}}\|_0 \leq \nu$ and $\mathbf{Z}^*$ is the flux function associated to $\mathbf{z}^*$, then*

$$\begin{aligned} \mathbf{x} \in \bar{\Omega}_2 \quad &\Longrightarrow \quad \mathbf{Z}^*(\mathbf{x}, 1/2, 0) \notin \overline{\Omega}_2 \\ \mathbf{x} \in \bar{\Omega}_2 \quad &\Longrightarrow \quad \mathbf{Z}^*(\mathbf{x}, 1, 1/2) \notin \overline{\Omega}_2. \end{aligned} \tag{3.12}$$

Demonstração. Let us set

$$\mathbf{F} = \left\{ \mathbf{Y}^*(\mathbf{x}, 1/2, 0) \, : \, \mathbf{x} \in \overline{\Omega}_2 \right\} \cup \left\{ \mathbf{Y}^*(\mathbf{x}, 1, 1/2) \, : \, \mathbf{x} \in \overline{\Omega}_2 \right\}.$$

Since $\mathbf{F}$ and $\overline{\Omega}_2$ are compact subsets of $\mathbb{R}^2$ and, in view of Lemma 9, $\mathbf{F} \cap \overline{\Omega}_2 = \emptyset$, we have $d := \textit{dist}\left(\mathbf{F}, \ \overline{\Omega}_2\right) > 0$.

Let us introduce $\mathbf{W} := \mathbf{Y}^* - \mathbf{Z}^*$. Then, in view of the Mean Value Theorem and the properties of π_2, we have:

$$
\begin{aligned}
|\mathbf{W}(\mathbf{x},t,s)| &\leq M \int_s^t \gamma(\sigma) |\nabla\varphi(\mathbf{Y}^*(\mathbf{x},\sigma,s)) - \nabla\varphi(\mathbf{Z}^*(\mathbf{x},\sigma,s))| \, d\sigma \\
&\quad + \int_s^t |\pi_2(\mathbf{z} - \overline{\mathbf{y}})(\mathbf{Z}^*(\mathbf{x},\sigma,s),\sigma)| \, d\sigma \\
&\leq M\|\nabla\varphi\|_0 \int_s^t \gamma(\sigma)|\mathbf{W}(\mathbf{x},\sigma,s)| \, d\sigma + \int_s^t \|\pi_2(\mathbf{z}-\overline{\mathbf{y}})\|_0(\sigma) d\sigma \\
&\leq M\|\nabla\varphi\|_0 \int_s^t \gamma(\sigma)|\mathbf{W}(\mathbf{x},\sigma,s)| \, d\sigma + C\int_s^t \|\mathbf{z}-\overline{\mathbf{y}}\|_0(\sigma) d\sigma,
\end{aligned}
$$

where $(\mathbf{x},t,s) \in \overline{\Omega}_3 \times [0,1] \times [0,1]$.

Hence, from Gronwall's Lemma, we see that

$$
\begin{aligned}
|\mathbf{W}(\mathbf{x},t,s)| &\leq C \left(\int_s^t \|\mathbf{z}-\overline{\mathbf{y}}\|_0(\sigma) d\sigma \right) \exp\left(M\|\nabla\varphi\|_0 \int_s^t \gamma(\sigma)\, d\sigma \right) \\
&\leq C e^{M\|\nabla\varphi\|_0 \|\gamma\|_0} \|\mathbf{z}-\overline{\mathbf{y}}\|_0
\end{aligned}
$$

and, therefore, there exists $\nu > 0$ such that, if $\|\mathbf{z}-\overline{\mathbf{y}}\|_0 \leq \nu$, then

$$
|\mathbf{W}(\mathbf{x},t,s)| \leq \frac{d}{2}, \quad \forall (\mathbf{x},t,s) \in \overline{\Omega}_3 \times [0,1] \times [0,1]. \tag{3.13}
$$

Thanks to Lemma 9 and (3.13), we necessarily have (3.12) and the proof is done. □

Now, we will introduce a mollifier and we will state a result that will be useful in Sections 4.4 and 4.5. Let $\rho \in C^\infty(\mathbb{R}^2;\mathbb{R})$ be given by

$$
\rho(\mathbf{x}) := \begin{cases} e^{\frac{|\mathbf{x}|^2}{1-|\mathbf{x}|^2}} & \text{if } |\mathbf{x}| < 1, \\ 0 & \text{if } |\mathbf{x}| \geq 1 \end{cases}
$$

and let the $\rho_m \in C^\infty(\mathbb{R}^2;\mathbb{R})$ be defined as follows:

$$
\rho_m(\mathbf{x}) := m^2 \rho(m\mathbf{x}).
$$

Lemma 12. *For each* $\mathbf{v}_0 \in \mathbf{C}^{2,\alpha}(\overline{\Omega};\mathbb{R}^i)$*, let us set*

$$
\mathbf{v}_0^m := (\rho_m * \pi_i(\mathbf{v}_0))\,\big|_{\overline{\Omega}}.
$$

Then $\mathbf{v}_0^m \to \mathbf{v}_0$ *in* $\mathbf{C}^{2,\alpha}(\overline{\Omega};\mathbb{R}^i)$ *as* $m \to +\infty$ *and*

$$
\|\mathbf{v}_0^m\|_{2,\alpha} \leq C\|\mathbf{v}_0\|_{2,\alpha} \quad \|\mathbf{v}_0^m\|_{3,\alpha} \leq mC\|\mathbf{v}_0\|_{2,\alpha}.
$$

3.2.2 Construction of a trajectory when $N = 3$

In this section, we will follow [45]. As in the two-dimensional case, $\overline{\mathbf{y}}$ will be of the potential form "$\nabla\varphi$", with the property that any particle travelling with velocity $\overline{\mathbf{y}}$ must leave $\overline{\Omega}$ at an appropriate time. The main difference is that, in this three-dimensional case, "$\nabla\varphi$" can no longer be chosen independent of t.

We first recall a lemma:

Lemma 13. *Let G be a regular bounded open set, with $G \supset \overline{\Omega}$. For each $\mathbf{a} \in \overline{\Omega}$, there exists $\phi^{\mathbf{a}} \in C^\infty(\overline{G} \times [0,1]; \mathbb{R})$ such that supp $\phi^{\mathbf{a}} \subset G \times (1/4, 3/4)$,*

$$\begin{cases} \Delta\phi^{\mathbf{a}} = 0 & \textit{in} \quad \overline{\Omega} \times [0,1], \\ \dfrac{\partial \phi^{\mathbf{a}}}{\partial \eta} = 0 & \textit{on} \quad (\Gamma \setminus \Gamma_0) \times [0,1], \end{cases} \tag{3.14}$$

and

$$\mathbf{X}^{\mathbf{a}}(\mathbf{a}, 1, 0) \in G \setminus \overline{\Omega},$$

where $\mathbf{X}^{\mathbf{a}} = \mathbf{X}^{\mathbf{a}}(\mathbf{x}, t, s)$ is the flux associated to $\nabla\phi^{\mathbf{a}}$, that is, the unique function that satisfies

$$\begin{cases} \dfrac{\partial}{\partial t}\mathbf{X}^{\mathbf{a}}(\mathbf{x}, t, s) = \nabla\Phi^{\mathbf{a}}(\mathbf{X}^{\mathbf{a}}(\mathbf{x}, t, s), t), \\ \mathbf{X}^{\mathbf{a}}(\mathbf{x}, s, s) = \mathbf{x}. \end{cases} \tag{3.15}$$

The proof is given in [45].

With the help of these $\mathbf{X}^{\mathbf{a}}$, we can construct a vector field $\mathbf{y}^*$ in $G \times (0,1)$ that drives the particles out of Ω and then makes then come back the same way.

Indeed, from the continuity of the functions $\mathbf{X}^{\mathbf{a}}$ and the compactness of $\overline{\Omega}$, we can find $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_k$ in $\bar{\Omega}$, real numbers $r_1, \ldots, r_k$, smooth functions $\Phi^1 := \phi^{\mathbf{a}_1}, \ldots, \Phi^k := \phi^{\mathbf{a}_k}$ satisfying Lemma 13 and a bounded open set G_0 with $\Omega \subset\subset G_0 \subset\subset G$, such that

$$\overline{\Omega} \subset \bigcup_{i=1}^{k} B^i \subset\subset G_0 \quad \text{and} \quad \mathbf{X}^i(\overline{B^i}, 1, 0) \subset G \setminus \bar{G}_0, \tag{3.16}$$

where $B^i := B(\mathbf{a}_i; r_i)$ and $\mathbf{X}^i := \mathbf{X}^{\mathbf{a}_i}$ for $i = 1, \ldots, k$.

As in [45], the definition of $\mathbf{y}^*$ is as follows: let the t_i be given by

$$\begin{aligned} t_i &= \frac{1}{4} + \frac{i}{4k}, \quad i = 0, \ldots, 2k, \\ t_{i+\frac{1}{2}} &= \frac{1}{4} + (i + \frac{1}{2})\frac{1}{4k}, \quad i = 0, \ldots, 2k-1 \end{aligned} \tag{3.17}$$

and let us set

$$\Phi(\mathbf{x}, t) = \begin{cases} 0, & (\mathbf{x}, t) \in \bar{G} \times ([0, 1/4] \times [3/4, 1]), \\ 8k\Phi^j(\mathbf{x}, 8k(t - t_{j-1})), & (\mathbf{x}, t) \in \bar{G} \times \left[t_{j-1}, t_{j-\frac{1}{2}}\right], \\ -8k\Phi^j(\mathbf{x}, 8k(t_j - t)), & (\mathbf{x}, t) \in \bar{G} \times \left[t_{j-\frac{1}{2}}, t_j\right] \end{cases} \tag{3.18}$$

for $j=1,\dots,2k$; then, we set $\mathbf{y}^*=\nabla\Phi$, $\overline{\mathbf{y}}=\mathbf{y}^*|_{\overline{\Omega}\times[0,1]}$ and we denote by $\mathbf{X}^*$ the flux associated to $\nabla\Phi$.

If we set $\bar{p}(\mathbf{x},t):=-\Phi_t(\mathbf{x},t)-\frac{1}{2}|\nabla\Phi(\mathbf{x},t)|^2$ and $\overline{\theta}\equiv 0$, then (3.76) is verified for $\mathbf{y}=\overline{\mathbf{y}}$, $p=\overline{p}$, $\theta=\overline{\theta}$, $T=1$, $\mathbf{y}_0=\mathbf{y}_1=\mathbf{0}$ and $\theta_0=\theta_1=0$.

Thanks to (3.16) and (3.18), the particles of the fluid that travel with velocity $\mathbf{y}^*$ and are located in the ball B^i at $t=0$ go out of $\bar{G}_0$ at $t=t_{i-\frac{1}{2}}$ and again at $t=t_{k+i-\frac{1}{2}}$. More precisely, one has:

Lemma 14. *The following properties hold for all $i=1,...,k$:*

$$\begin{aligned} \mathbf{x}\in\bar{B}^i &\implies \mathbf{X}^*(\mathbf{x},t_{i-\frac{1}{2}},0)\in G\setminus\bar{G}_0,\\ \mathbf{x}\in\bar{B}^i &\implies \mathbf{X}^*(\mathbf{x},t_{i+k-\frac{1}{2}},1/2)\in G\setminus\bar{G}_0. \end{aligned}\tag{3.19}$$

For the proof, it suffices to notice that $\mathbf{Y}^*=\mathbf{X}^0$ in $\bar{G}\times[1/4,3/4]\times[1/4,3/4]$, where

$$\mathbf{X}^0(\mathbf{x},t,s):=\begin{cases} \mathbf{X}^i(\mathbf{x},8k(t-t_{j-1}),8k(s-t_{l-1}) & \text{if } (\mathbf{x},t,s)\in\bar{G}\times[t_{j-1},t_{j-\frac{1}{2}}]\times[t_{l-1},t_{l-\frac{1}{2}}],\\ \mathbf{X}^i(\mathbf{x},8k(t-t_{j-1}),8k(t_l-s)) & \text{if } (\mathbf{x},t,s)\in\bar{G}\times[t_{j-1},t_{j-\frac{1}{2}}]\times[t_{l-\frac{1}{2}},t_l],\\ \mathbf{X}^i(\mathbf{x},8k(t_j-t),8k(s-t_{l-1})) & \text{if } (\mathbf{x},t,s)\in\bar{G}\times[t_{j-\frac{1}{2}},t_j]\times[t_{l-1},t_{l-\frac{1}{2}}],\\ \mathbf{X}^i(\mathbf{x},8k(t_j-t),8k(t_l-s)) & \text{if } (\mathbf{x},t,s)\in\bar{G}\times[t_{j-\frac{1}{2}},t_j]\times[t_{l-\frac{1}{2}},t_l] \end{cases}$$

for $l,j=1,\dots,2k$. Indeed, one has the following for each $\mathbf{x}\in\overline{B^i}$:

$$\mathbf{X}^*(\mathbf{x},t_{i-\frac{1}{2}},0)=\mathbf{X}^*(\mathbf{x},t_{i-\frac{1}{2}},t_0)=\mathbf{X}^i(\mathbf{x},1,0)\in G\setminus\bar{G}_0$$

and

$$\mathbf{X}^*\left(\mathbf{x},t_{k+i-\frac{1}{2}},\frac{1}{2}\right)=\mathbf{X}^*(\mathbf{x},t_{k+i-\frac{1}{2}},t_k)=\mathbf{X}^i(\mathbf{x},1,0).\in G\setminus\bar{G}_0.$$

A result similar to Lemma 10 also holds here:

Lemma 15. *There exist π_1 and π_3 with*

$$\begin{cases} \pi_i(\mathbf{f})(\mathbf{x})=\mathbf{f}(\mathbf{x})\ \forall\mathbf{x}\in\overline{\Omega},\ \forall\mathbf{f}\in C^0(\overline{\Omega};\mathbb{R}^i),\\ \textit{supp}\ \pi_i(\mathbf{f})\subset G_0\quad\forall\mathbf{f}\in C^0(\overline{\Omega};\mathbb{R}^i),\\ \pi_i \textit{ maps continuously } C^{n,\lambda}(\overline{\Omega};\mathbb{R}^i) \textit{ into } C^{n,\lambda}(\bar{G};\mathbb{R}^i)\quad\forall n\geq 0,\quad\forall\lambda\in(0,1). \end{cases}\tag{3.20}$$

Finally, we also have a property similar to (16) for any flux corresponding to a velocity field sufficiently close to $\overline{\mathbf{y}}$:

Lemma 16. *For each $\mathbf{y}\in C([0,1],C^{1,\alpha}(\overline{\Omega};\mathbb{R}^2))$, let us set $\widetilde{\mathbf{y}}=\mathbf{y}^*+\pi_3(\mathbf{y}-\overline{\mathbf{y}})$. There exists $\nu>0$ such that, if $\|\mathbf{y}-\overline{\mathbf{y}}\|_0\leq\nu$, $\tilde{\mathbf{X}}$ is the flux associated to $\tilde{\mathbf{y}}$ and $1\leq i\leq k$, one has:*

$$\begin{aligned} \mathbf{x}\in\bar{B}^i &\implies \tilde{\mathbf{X}}^*(\mathbf{x},t_{i-\frac{1}{2}},0) &&\in G\setminus\bar{G}_0,\\ \mathbf{x}\in\bar{B}^i &\implies \tilde{\mathbf{X}}^*(\mathbf{x},t_{i+k-\frac{1}{2}},1/2) &&\in G\setminus\bar{G}_0. \end{aligned}$$

Demonstração. It is easy to see that

$$\begin{aligned}
\left|\widetilde{\mathbf{X}}(\mathbf{x},t,s)-\mathbf{Y}^*(\mathbf{x},t,s)\right| &= \left|\int_s^t (\widetilde{\mathbf{y}}(\widetilde{\mathbf{X}}(\mathbf{x},\tau,s),\tau)-\mathbf{y}^*(\mathbf{Y}^*(\mathbf{x},\tau,s),\tau))\,d\tau\right| \\
&\leq \int_s^t |\mathbf{y}^*(\widetilde{\mathbf{X}}((\mathbf{x},\tau,s),\tau)-\mathbf{y}^*(\mathbf{Y}^*(\mathbf{x},\tau,s),\tau))|\,d\tau \\
&\quad + \int_s^t |\pi_3(\mathbf{y}-\overline{\mathbf{y}})(\widetilde{\mathbf{X}}(\mathbf{x},\tau,s),\tau))|\,d\tau \\
&\leq C\int_s^t |\widetilde{\mathbf{X}}(\mathbf{x},\tau,s)-\mathbf{Y}^*(\mathbf{x},\tau,s)|\,d\tau + C\|\mathbf{y}-\overline{\mathbf{y}}\|_0.
\end{aligned}$$

For Gronwall's Lemma, we have

$$\left|\widetilde{\mathbf{X}}(\mathbf{x},t,s)-\mathbf{X}^*(\mathbf{x},t,s)\right| \leq C\|\mathbf{y}-\overline{\mathbf{y}}\|_0.$$

Let us set

$$\mathbf{F}=\bigcup_{i=1}^{k}\left(\widetilde{\mathbf{X}}(\overline{B^i},t_{i-\frac{1}{2}},0)\cup\widetilde{\mathbf{X}}\left(\overline{B^i},t_{k+i-\frac{1}{2}},\frac{1}{2}\right)\right).$$

Since $\mathbf{F}$ and $\bar{G}_0$ are compact and $\mathbf{F}\cap\bar{G}=\emptyset$, we have $d' := dist\left(\mathbf{F},\ \overline{\overline{\Omega}_1}\right) > 0$. There exists $\nu > 0$ such that, if $\|\mathbf{y}-\overline{\mathbf{y}}\|_0 \leq \nu$, then

$$\left|\widetilde{\mathbf{X}}(\mathbf{x},t,s)-\mathbf{Y}^*(\mathbf{x},t,s)\right| \leq \frac{d'}{2} \quad \forall(\mathbf{x},t,s)\in\bar{G}\times[0,1]\times[0,1]. \tag{3.21}$$

Consequently, for this ν we get the desired result. □

3.3 Proof of theorem 13

This section is devoted to prove the exact controllability result Theorem 13, we use a scaling argument from Proposition 9.

Let $T>0$, θ_0, $\theta_1\in C^{2,\alpha}(\overline{\Omega};\mathbb{R})$ and $\mathbf{y}_0$, $\mathbf{y}_1\in\mathbf{C}(2,\alpha,\Gamma_0)$ be given. Let us see that, if

$$\|\mathbf{y}_0\|_{2,\alpha}+\|\mathbf{y}_1\|_{2,\alpha}+\|\theta_0\|_{2,\alpha}+\|\theta_1\|_{2,\alpha}$$

is small enough, we can construct a triplet $(\mathbf{y},p,\theta)$ satisfying (3.76) and (3.75).

If $\varepsilon\in(0,T/2)$ is sufficiently small to have

$$\{\varepsilon\|\mathbf{y}_0\|_{2,\alpha},\varepsilon^2\|\theta_0\|_{2,\alpha}\}<\delta \ \ (\text{resp. } \{\varepsilon\|\mathbf{y}_1\|_{2,\alpha},\ \varepsilon^2\|\theta_1\|_{2,\alpha}\}\leq\delta),$$

then, thanks to Proposition 9, there exist $(\mathbf{y}^0,\ \theta^0)$ in $C^0([0,1];\mathbf{C}^{1,\alpha}(\overline{\Omega};\mathbb{R}^{N+1}))$ and a pressure p^0 (resp. $(\mathbf{y}^1,\ \theta^1)$ and p^1) solving (3.76), with $\mathbf{y}^0(\mathbf{x},0)\equiv\varepsilon\mathbf{y}_0(\mathbf{x})$ and $\theta^0(\mathbf{x},0)\equiv\varepsilon^2\theta_0$ (resp. $\mathbf{y}^1(\mathbf{x},0)\equiv-\varepsilon\mathbf{y}_1(\mathbf{x})$ and $\theta^1(\mathbf{x},0)=\varepsilon^2\theta_1(\mathbf{x})$) and satisfying (3.4).

Let us choose ε of this form and let us introduce $\mathbf{y} : \overline{\Omega} \times [0,T] \mapsto \mathbb{R}^N$, $p : \overline{\Omega} \times [0,T] \mapsto \mathbb{R}$ and $\theta : \overline{\Omega} \times [0,T] \mapsto \mathbb{R}$ as follows:

$$\begin{cases} \mathbf{y}(\mathbf{x},t) = \varepsilon^{-1}\mathbf{y}^0(\mathbf{x},\varepsilon^{-1}t), \\ p(\mathbf{x},t) = \varepsilon^{-2}p^0(\mathbf{x},\varepsilon^{-1}t), & (\mathbf{x},t) \in \overline{\Omega} \times [0,\varepsilon], \\ \theta(\mathbf{x},t) = \varepsilon^{-2}\theta^0(\mathbf{x},\varepsilon^{-1}t), \end{cases}$$
$$\begin{cases} \mathbf{y}(\mathbf{x},t) = \mathbf{0}, \\ p(\mathbf{x},t) = 0, & (\mathbf{x},t) \in \overline{\Omega} \times (\varepsilon, T-\varepsilon), \\ \theta(\mathbf{x},t) = 0, \end{cases}$$

$$\begin{cases} \mathbf{y}(\mathbf{x},t) = -\varepsilon^{-1}\mathbf{y}^1(\mathbf{x},\varepsilon^{-1}(T-t)), \\ p(\mathbf{x},t) = \varepsilon^{-2}p^1(\mathbf{x},\varepsilon^{-1}(T-t)), & (\mathbf{x},t) \in \overline{\Omega} \times [T-\varepsilon, T]. \\ \theta(\mathbf{x},t) = \varepsilon^{-2}\theta^1(\mathbf{x},\varepsilon^{-1}(T-t)), \end{cases}$$

Then, $(\mathbf{y},\theta) \in C^0([0,T];\mathbf{C}^{1,\alpha}(\overline{\Omega};\mathbb{R}^{N+1})$ and the triplet $(\mathbf{y},p,\theta)$ satisfies (3.76) and (3.75).

3.4 Proof of Proposition 9. The 2D case

Let $\mu \in C^\infty([0,1];\mathbb{R})$ a function such that $\mu \equiv 1$ in $[0,1/4]$, $\mu \equiv 0$ in $[1/2,1]$ and $0 < \mu < 1$.
The proof of Proposition 9 is a consequence of the following result:

Proposition 10. *There exists $\delta > 0$ such that if* $\max\{\|\mathbf{y}_0\|_{2,\alpha}, \|\theta_0\|_{2,\alpha}\} \le \delta$, *then the system*

$$\begin{cases} \zeta_t + \mathbf{y}\cdot\nabla\zeta = -\vec{\mathbf{k}}\times\nabla\theta & \textit{in} \quad \Omega\times(0,1), \\ \theta_t + \mathbf{y}\cdot\nabla\theta = 0 & \textit{in} \quad \Omega\times(0,1), \\ \mathbf{y}\cdot\mathbf{n} = (\overline{\mathbf{y}} + \mu\,\mathbf{y}_0)\cdot\mathbf{n} & \textit{on} \quad \Gamma\times(0,1), \\ \nabla\cdot\mathbf{y} = 0, \ \nabla\times\mathbf{y} = \zeta & \textit{in} \quad \Omega\times(0,1), \\ \zeta(0) = \nabla\times\mathbf{y}_0, \ \theta(0) = \theta_0 & \textit{in} \quad \Omega, \end{cases} \tag{3.22}$$

possesses at least one solution

$$(\zeta,\theta,\mathbf{y}) \in C^0([0,1];C^{0,\alpha}(\overline{\Omega};\mathbb{R})) \times C^0([0,1];C^{1,\alpha}(\overline{\Omega};\mathbb{R})) \times C^0([0,1];\mathbf{C}^{1,\alpha}(\overline{\Omega};\mathbb{R}^2)) \tag{3.23}$$

such that $\theta(\mathbf{x},t) = 0$ *in* $\Omega\times[1/2,1]$ *and* $\zeta(\mathbf{x},1) = 0$ *in* Ω.

The reminder of this section is devoted to prove Proposition 10. We are going to adapt some ideas from Bardos and Frisch [4] and Kato [60] already used in [15] and [43].

Let us give a sketch.

We will start from an arbitrary flow $\mathbf{z} := \mathbf{z}(\mathbf{x}, t)$ in a suitable class $\mathbf{S}$ of continuous functions. Together with this $\mathbf{z}$, we will construct a scalar function θ verifying

$$\begin{cases} \theta_t + \mathbf{z} \cdot \nabla\theta = 0 & \text{in} \quad \overline{\Omega} \times [0,1], \\ \theta(0) = \theta_0 & \text{in} \quad \overline{\Omega}. \end{cases} \tag{3.24}$$

and

$$\theta \equiv 0 \text{ in } \overline{\Omega} \times [1/2, 1].$$

Then, with this θ we construct a function ζ satisfying

$$\begin{cases} \zeta_t + \mathbf{z} \cdot \nabla\zeta = -\vec{\mathbf{k}} \times \nabla\theta & \text{in} \quad \overline{\Omega} \times [0,1], \\ \zeta(0) = \nabla \times \mathbf{y}_0 & \text{in} \quad \overline{\Omega}. \end{cases} \tag{3.25}$$

and

$$\zeta(\cdot, 1) \equiv 0 \text{ in } \overline{\Omega}.$$

In this way, we shall have assigned ζ to each $\mathbf{z} \in \mathbf{S}$. Now, we can construct a flow $\mathbf{y}$ such that $\nabla \times \mathbf{y} = \zeta$ and, therefore, we will have defined a mapping F with $F(\mathbf{z}) = \mathbf{y}$. We will choose the class $\mathbf{S}$ in such a way that F maps $\mathbf{S}$ into itself and satisfies the conditions of Banach's Fixed Point Theorem.

Let $\mathbf{y}$ be the unique fixed point of F in $\mathbf{S}$ and let θ and ζ be the associated *temperature* and *vorticity*. Then, the triplet $(\zeta, \theta, \mathbf{y})$ should solve (3.54) and satisfy (3.55).

Let us now give the details. Consider $\mathbf{y}_0 \in \mathbf{C}^{3,\alpha}(\overline{\Omega})$ and $\theta \in C^{3,\alpha}(\overline{\Omega})$ and let us introduce

$$\mathbf{S}' = \left\{ \mathbf{z} \in C^0([0,1]; \mathbf{C}^{2,\alpha}(\overline{\Omega}; \mathbb{R}^2)) : \nabla \cdot \mathbf{z} = 0 \text{ in } \overline{\Omega} \times [0,1],\, \mathbf{z} \cdot \mathbf{n} = (\overline{\mathbf{y}} + \mu \mathbf{y}_0) \cdot \mathbf{n} \text{ on } \Gamma \times [0,1] \right\}.$$

For $\nu > 0$, we will denote by $\mathbf{S}_\nu$ the set

$$\mathbf{S}_\nu = \left\{ \mathbf{z} \in \mathbf{S}' : \|\mathbf{z}(\cdot, t) - \overline{\mathbf{y}}(\cdot, t)\|_{2,\alpha} \le \nu, \ \forall t \in [0,1] \right\}.$$

Let $\nu > 0$, be the constant furnished by Lemma 11 and let us carry out the previous process for $\mathbf{z} \in \mathbf{S}_\nu$ ($\mathbf{S} = \mathbf{S}_\nu$). Notice that, if $\|\mathbf{y}_0\|_{2,\alpha} \le \nu$ then $\overline{\mathbf{y}} + \mu \mathbf{y}_0 \in \mathbf{S}_\nu$.

First, let us set $\mathbf{z}^* = \mathbf{y}^* + \pi_2(\mathbf{z} - \overline{\mathbf{y}})$. Then, we have the estimate

$$\|\mathbf{z}^*(\cdot, t)\|_{2,\alpha} \le \|\mathbf{y}^*(\cdot, t)\|_{2,\alpha} + C\|(\mathbf{z} - \overline{\mathbf{y}})(\cdot, t)\|_{2,\alpha}, \quad \forall t \in [0,1] \tag{3.26}$$

and the following result:

Lemma 17. *There exists a unique global solution* $\mathbf{Z}^* \in C^1([0,1] \times [0,1]; \mathbf{C}^{3,\alpha}(\overline{\Omega}_3; \mathbb{R}^2))$

$$\begin{cases} \mathbf{Z}^*_t(\mathbf{x}, t, s) = \mathbf{z}^*(\mathbf{Z}^*(\mathbf{x}, t, s), t), \\ \mathbf{Z}^*(\mathbf{x}, s, s) = \mathbf{x}, \end{cases} \tag{3.27}$$

with

$$\mathbf{Z}^*(\mathbf{x}, t, s) \in \overline{\Omega}_3 \quad \forall (\mathbf{x}, t, s) \in \overline{\Omega}_3 \times [0,1] \times [0,1].$$

For the proof, it is suffices to apply directly the well known (classical) existence, uniqueness and regularity theory of ODEs.

Since $\mathbf{Z}^* \in C^1([0,1]\times[0,1];\mathbf{C}^{3,\alpha}(\overline{\Omega}_3;\mathbb{R}^2))$ and $\theta_0 \in C^{3,\alpha}(\overline{\Omega};\mathbb{R})$, from the properties of π_1, we can obtain a unique solution $\theta^* \in C^0([0,1/2];C^{3,\alpha}(\overline{\Omega}_3;\mathbb{R}))$ to the problem

$$\begin{cases} \theta^*_t + \mathbf{z}^*\cdot\nabla\theta^* = 0 & \text{in} \quad \overline{\Omega}_3\times[0,1/2], \\ \theta^*(\mathbf{x},0) = \pi_1(\theta_0)(\mathbf{x}) & \text{in} \quad \overline{\Omega}_3. \end{cases} \tag{3.28}$$

Note that, in (3.28), boundary condition on θ does not appear. Obviously, this is because $\mathbf{z}^*$ has support contained in Ω_3.

The solution to (3.28) verifies

$$\theta^*(\mathbf{Z}^*(\mathbf{x},t,0),t) = \pi_1(\theta_0)(\mathbf{x}) \quad \forall(\mathbf{x},t)\in\overline{\Omega}_3\times[0,1/2] \tag{3.29}$$

and, consequently,

$$\textit{supp}\ \theta^*(\cdot,t) \subset \mathbf{Z}^*(\Omega_2,t,0) \quad \text{in } [0,1/2].$$

In particular, in view of the choice of ν, we get:

$$\textit{supp}\ \theta^*(\cdot,1/2) \subset \mathbf{Z}^*(\Omega_2,1/2,0) \subset \Omega_3\setminus\overline{\Omega}_2.$$

Therefore, $\theta^*(\cdot,1/2)\equiv 0$ in $\overline{\Omega}_2$.

Now, applying Lemma 6 to the equation verified for θ^*, we have:

$$\frac{d}{dt^+}\|\theta^*(\cdot,t)\|_{2,\alpha} \le K\|\mathbf{z}^*(\cdot,t)\|_{2,\alpha}\|\theta^*(\cdot,t)\|_{2,\alpha}. \tag{3.30}$$

Then, from Gronwall's Lemma, we obtain:

$$\|\theta^*(\cdot,t)\|_{2,\alpha} \le \|\pi_1(\theta_0)\|_{2,\alpha}\exp\left(K\int_0^t \|\mathbf{z}^*(\cdot,\tau)\|_{2,\alpha}\,d\tau\right). \tag{3.31}$$

Let θ be the function

$$\theta(\mathbf{x},t) = \begin{cases} \theta^*(\mathbf{x},t), & (\mathbf{x},t)\in\overline{\Omega}\times[0,1/2], \\ 0, & (\mathbf{x},t)\in\overline{\Omega}\times[1/2,1]. \end{cases} \tag{3.32}$$

Then $\theta\in C^0([0,1];C^{3,\alpha}(\overline{\Omega};\mathbb{R}))$ and

$$\begin{cases} \theta_t + \mathbf{z}\cdot\nabla\theta = 0 & \text{in} \quad \Omega\times(0,1), \\ \theta(\mathbf{x},0) = \theta_0(\mathbf{x}), & \text{in} \quad \Omega. \end{cases} \tag{3.33}$$

Now, we set $\zeta^*_0 := \nabla\times(\pi_2(\mathbf{y}_0))$ and $\zeta^*\in C^0([0,1/2];C^{2,\alpha}(\overline{\Omega}_3;\mathbb{R}))$ unique solution to the problem

$$\begin{cases} \zeta^*_t + \mathbf{z}^*\cdot\nabla\zeta^* = -\vec{\mathbf{k}}\times\nabla\theta^* & \text{in} \quad \overline{\Omega}_3\times[0,1/2], \\ \zeta^*(\mathbf{x},0) = \zeta^*_0(\mathbf{x}) & \text{in} \quad \overline{\Omega}_3. \end{cases} \tag{3.34}$$

Applying again Lemma 6, we get:

$$\frac{d}{dt^+}\|\zeta^*(\cdot,t)\|_{1,\alpha} \le C\|\theta^*(\cdot,t)\|_{2,\alpha} + K\|\mathbf{z}^*(\cdot,t)\|_{2,\alpha}\|\zeta^*(\cdot,t)\|_{1,\alpha}. \tag{3.35}$$

From Gronwall's Lemma and (3.31), we obtain:

$$\|\zeta^*(\cdot,t)\|_{1,\alpha} \leq C(\|\pi_2(\mathbf{y}_0)\|_{2,\alpha} + \|\pi_1(\theta_0)\|_{2,\alpha}) \exp\left(K\int_0^t \|\mathbf{z}^*(\cdot,\tau)\|_{2,\alpha}\,d\tau\right). \tag{3.36}$$

With this ζ^*, we define $\zeta^*_{1/2} \in C^{2,\alpha}(\overline{\Omega};\mathbb{R})$ given by $\zeta^*_{1/2}(\mathbf{x}) = \zeta^*(\mathbf{x},1/2)$ for all $\mathbf{x} \in \overline{\Omega}$. Then, we can obtain a unique solution $\zeta^{**} \in C^0([1/2,1];C^{2,\alpha}(\overline{\Omega}_3;\mathbb{R}))$ of the problem

$$\begin{cases} \zeta^{**}_t + \mathbf{z}^*\cdot\nabla\zeta^{**} = 0, & \text{in } \overline{\Omega}_3 \times [1/2,1], \\ \zeta^{**}(\mathbf{x},1/2) = \pi_1(\zeta^*_{1/2})(\mathbf{x}), & \text{in } \overline{\Omega}_3. \end{cases} \tag{3.37}$$

Also, we have that the solution of the problem (3.37) verifies

$$\zeta^{**}(\mathbf{Z}^*(\mathbf{x},t,1/2),t) = \pi_1(\zeta^*_{1/2})(\mathbf{x}), \quad \forall(\mathbf{x},t) \in \overline{\Omega}_3 \times [1/2,1]$$

and then

$$\mathit{supp}\ \zeta^{**}(\cdot,t) \subset \mathbf{Z}^*(\Omega_2,t,1/2) \quad \text{in } [1/2,1].$$

Again, by the choose of ν and the identity above, we have

$$\mathit{supp}\ \zeta^{**}(\cdot,t) \subset \mathbf{Z}^*(\Omega_2,1,1/2) \subset \Omega_3 \setminus \overline{\Omega}_2.$$

Therefore, $\zeta^{**}(\cdot,1) \equiv 0$ in $\overline{\Omega}_2$.

With similar arguments to get (3.31) and combining with (3.36), we have:

$$\|\zeta^{**}(\cdot,t)\|_{1,\alpha} \leq C(\|\pi_2(\mathbf{y}_0)\|_{2,\alpha} + \|\pi_1(\theta_0)\|_{2,\alpha}) \exp\left(K\int_0^t \|\mathbf{z}^*(\cdot,\tau)\|_{2,\alpha}(\tau)d\tau\right), \tag{3.38}$$

for all $t \in [1/2,1]$.

So, we can define $\zeta \in C^0([0,1];C^{2,\alpha}(\overline{\Omega};\mathbb{R}))$ given by

$$\zeta(\mathbf{x},t) = \begin{cases} \zeta^*(\mathbf{x},t), & (\mathbf{x},t) \in \overline{\Omega} \times [0,1/2], \\ \zeta^{**}(\mathbf{x},t), & (\mathbf{x},t) \in \overline{\Omega} \times [1/2,1] \end{cases} \tag{3.39}$$

and we have that ζ is solution of

$$\begin{cases} \zeta_t + \mathbf{z}\cdot\nabla\zeta = -\vec{\mathbf{k}} \times \nabla\theta, & \text{in } \overline{\Omega} \times [0,1], \\ \zeta(0) = \nabla \times \mathbf{y}_0, & \text{in } \overline{\Omega}. \end{cases} \tag{3.40}$$

Therefore, thanks to (3.36) and (3.38), we obtain the for ζ:

$$\|\zeta(\cdot,t)\|_{1,\alpha} \leq C(\|\mathbf{y}_0\|_{2,\alpha} + \|\theta_0\|_{2,\alpha}) \exp\left(K\int_0^t \|\mathbf{z}^*\|_{2,\alpha}(\tau)d\tau\right). \tag{3.41}$$

Whence, with this ζ we get a unique $\mathbf{y} \in C^0([0,1];\mathbf{C}^{3,\alpha}(\overline{\Omega};\mathbb{R}^2))$ such that $\nabla \times \mathbf{y} = \zeta$ and $\mathbf{y}\cdot\mathbf{n} = (\overline{\mathbf{y}} + \mu\mathbf{y}_0)\cdot\mathbf{n}$ on $\Gamma \times [0,1]$. In fact, let $\psi \in C^0([0,1];C^{4,\alpha}(\overline{\Omega};\mathbb{R}))$ a unique solution of the elliptic equation

$$\left| \begin{array}{ll} -\Delta\psi = \zeta - \mu\nabla \times \mathbf{y}_0, & \text{in } \Omega \times [0,1], \\ \psi = 0, & \text{on } \Gamma \times [0,1]. \end{array} \right. \tag{3.42}$$

Then, we define $\mathbf{y} := \nabla \times \psi + \overline{\mathbf{y}} + \mu \mathbf{y}_0$. Obviously, we have that $\mathbf{y}$ is a flow, $\mathbf{y} \in C^0([0,1]; \mathbf{C}^{3,\alpha}(\overline{\Omega}; \mathbb{R}^2))$, $\mathbf{y} \cdot \mathbf{n} = (\overline{\mathbf{y}} + \mu \mathbf{y}_0) \cdot \mathbf{n}$ on $\Gamma \times [0,1]$ and $\nabla \times \mathbf{y} = \zeta$. Also, easily we can see that $\mathbf{y}$ is unique.

Therefore, since $\mathbf{y}$ is determined by $\mathbf{z}$, we write

$$\begin{array}{rcl} F : \mathbf{S}_\nu & \to & \mathbf{S}' \\ \mathbf{z} & \mapsto & F(\mathbf{z}) := \mathbf{y}. \end{array} \tag{3.43}$$

The following result holds:

Lemma 18. *There exists $\delta > 0$ such that if*

$$\max\{\|\mathbf{y}_0\|_{1,\alpha}, \|\theta_0\|_{1,\alpha}\} < \delta$$

then $F(\mathbf{S}_\nu) \subset \mathbf{S}_\nu$.

Demonstração. Let $\mathbf{z} \in \mathbf{S}_\nu$, then $F(\mathbf{z}) - \overline{\mathbf{y}} = \nabla \times \psi + \mu \mathbf{y}_0$ and we have:

$$\|F(\mathbf{z})(\cdot,t) - \overline{\mathbf{y}}(\cdot,t)\|_{2,\alpha} \leq C(\|\zeta(\cdot,t)\|_{1,\alpha} + \|\mathbf{y}_0\|_{2,\alpha}).$$

Therefore, using (3.41), (3.26) and the definition of $\mathbf{S}_\nu$, we obtain

$$\begin{aligned} \|F(\mathbf{z})(\cdot,t) - \overline{\mathbf{y}}(\cdot,t)\|_{2,\alpha} &\leq C_1(\|\mathbf{y}_0\|_{2,\alpha} + \|\theta_0\|_{2,\alpha}) \exp\left(C_2 \int_0^t \|\mathbf{z}(\cdot,\tau) - \overline{\mathbf{y}}(\cdot,\tau)\|_{2,\alpha} \, d\tau\right) \\ &\leq C_1(\|\mathbf{y}_0\|_{2,\alpha} + \|\theta_0\|_{2,\alpha}) \exp(C_2 \nu). \end{aligned}$$

So, we take $\delta > 0$ such that

$$C_1(\|\mathbf{y}_0\|_{2,\alpha} + \|\theta_0\|_{2,\alpha}) \exp(C_2 \nu) \leq \nu.$$

In this way, if $\mathbf{z} \in \mathbf{S}_\nu$ then we have that

$$\|F(\mathbf{z})(\cdot,t) - \overline{\mathbf{y}}(\cdot,t)\|_{2,\alpha} \leq \nu, \quad \forall t \in [0,1].$$

Then, $F : \mathbf{S}_\nu \to \mathbf{S}_\nu$ is a well-defined application. □

Now, to apply the Banach's fixed point Theorem to F, we have the result:

Lemma 19. *For $\mathbf{z}$, $\mathbf{z}' \in \mathbf{S}_\nu$ we have the inequalities*

$$\|(\zeta - \zeta')(\cdot,t)\|_{0,\alpha} \leq C_\nu(\|\mathbf{y}_0\|_{2,\alpha} + \|\theta_0\|_{2,\alpha}) \int_0^t \|(\mathbf{z} - \mathbf{z}')(\cdot,\tau)\|_{1,\alpha} \, ds.$$

Demonstração. The equation verified by $\Theta^* = \theta^* - \theta'^*$ is

$$\Theta_t^* + \mathbf{z}^* \cdot \nabla \Theta^* = -\mathbf{Z}^* \cdot \nabla \theta'^*, \tag{3.44}$$

where $\mathbf{Z} := \mathbf{z} - \mathbf{z}'$.

Applying the lemma 6 to the equation above, we have

$$\frac{\partial}{\partial t^+}\|\Theta^*(\cdot,t)\|_{1,\alpha} \leq C\|\mathbf{Z}^*(\cdot,t)\|_{1,\alpha}\|\theta'^*(\cdot,t)\|_{2,\alpha} + K\|\mathbf{z}^*(\cdot,t)\|_{1,\alpha}\|\Theta^*(\cdot,t)\|_{1,\alpha}. \tag{3.45}$$

Since,

$$\frac{\partial}{\partial t^+}\|\theta'^*(\cdot,t)\|_{2,\alpha} \leq K\|\mathbf{z}'^*(\cdot,t)\|_{2,\alpha}\|\theta'^*(\cdot,t)\|_{2,\alpha}$$

we have

$$\|\theta'^*(\cdot,t)\|_{2,\alpha} \leq C\|\theta_0\|_{2,\alpha}\exp\left(K\int_0^t \|\mathbf{z}'^*(\cdot,\tau)\|_{2,\alpha}\, d\tau\right). \tag{3.46}$$

The equation verified by $\Lambda^* = \zeta^* - \zeta'^*$ and $\Lambda^{**} = \zeta^{**} - \zeta'^{**}$ are, respectively,

$$\Lambda^*_t + \mathbf{z}^*\cdot\nabla\Lambda^* = -\mathbf{Z}^*\cdot\nabla\zeta'^* - \vec{k}\times\nabla\Theta^* \tag{3.47}$$

and

$$\Lambda^{**}_t + \mathbf{z}^*\cdot\nabla\Lambda^{**} = -\mathbf{Z}^*\cdot\nabla\zeta'^{**}. \tag{3.48}$$

Applying the Lemma 6 to the equations above, we have

$$\begin{aligned}\frac{\partial}{\partial t^+}\|\Lambda^*(\cdot,t)\|_{0,\alpha} \leq{}& \|\mathbf{Z}^*(\cdot,t)\|_{0,\alpha}\|\zeta'^*(\cdot,t)\|_{1,\alpha} + \|\Theta^*(\cdot,t)\|_{1,\alpha} + \|\mathbf{z}^*(\cdot,t)\|_{1,\alpha}\|\Lambda^*(\cdot,t)\|_{0,\alpha}\\ \leq{}& C\|\mathbf{Z}(\cdot,t)\|_{0,\alpha}\|\zeta'^*(\cdot,t)\|_{1,\alpha} + \|\Theta^*(\cdot,t)\|_{1,\alpha} + C(1+\nu)\|\Lambda^*(\cdot,t)\|_{0,\alpha}\end{aligned} \tag{3.49}$$

and

$$\begin{aligned}\frac{\partial}{\partial t^+}\|\Lambda^{**}(\cdot,t)\|_{0,\alpha} \leq{}& \|\mathbf{Z}^*(\cdot,t)\|_{0,\alpha}\|\zeta'^{**}(\cdot,t)\|_{1,\alpha} + \|\mathbf{z}^*(\cdot,t)\|_{1,\alpha}\|\Lambda^{**}(\cdot,t)\|_{0,\alpha}\\ \leq{}& C\|\mathbf{Z}(\cdot,t)\|_{0,\alpha}\|\zeta'^{**}(\cdot,t)\|_{1,\alpha} + C(1+\nu)\|\Lambda^{**}(\cdot,t)\|_{0,\alpha}\end{aligned} \tag{3.50}$$

We sum the inequalities (3.45) and (3.49) to obtain

$$\begin{aligned}\frac{\partial}{\partial t^+}\left(\|\Lambda^*(\cdot,t)\|_{0,\alpha} + \|\Theta^*(\cdot,t)\|_{1,\alpha}\right) \leq{}& C\|\mathbf{Z}\|_{1,\alpha}(\|\zeta'^*(\cdot,t)\|_{1,\alpha} + \|\theta'^*(\cdot,t)\|_{2,\alpha})\\ &+ C(1+\nu)(\|\Lambda^*(\cdot,t)\|_{0,\alpha} + \|\Theta^*(\cdot,t)\|_{1,\alpha}).\end{aligned}$$

Applying the Gronwall's Lemma, we deduce

$$\|\Lambda^*(\cdot,t)\|_{0,\alpha} + \|\Theta(\cdot,t)\|_{1,\alpha} \leq C_\nu(\|\mathbf{y}_0\|_{2+\alpha} + \|\theta_0\|_{2+\alpha})\int_0^t \|\mathbf{Z}(\cdot,\tau)\|_{1,\alpha}\, d\tau.$$

Also, applying the Gronwall's Lemma to (3.50) and using the inequality above, we deduce

$$\begin{aligned}\|\Lambda^{**}(\cdot,t)\|_{0,\alpha} &\leq C_\nu(\|\mathbf{y}_0\|_{2,\alpha} + \|\theta_0\|_{2,\alpha})\int_{1/2}^t \|\mathbf{Z}(\cdot,\tau)\|_{1,\alpha}\, d\tau + C_\nu\|\Lambda^*(\cdot,1/2)\|_{0,\alpha}\\ &\leq C_\nu(\|\mathbf{y}_0\|_{2+\alpha} + \|\theta_0\|_{2+\alpha})\int_0^t \|\mathbf{Z}(\cdot,\tau)\|_{1,\alpha}\, d\tau.\end{aligned}$$

Therefore, we can define $\Lambda = \zeta - \zeta'$. We can see that

$$\Lambda(\mathbf{x},t) = \begin{cases}\Lambda^*(\mathbf{x},t), & (\mathbf{x},t)\in\overline{\Omega}\times[0,1/2],\\ \Lambda^{**}(\mathbf{x},t), & (\mathbf{x},t)\in\overline{\Omega}\times[1/2,1]\end{cases} \tag{3.51}$$

and then

$$\|\Lambda(\cdot,t)\|_{0,\alpha} \leq C_\nu(\|\mathbf{y}_0\|_{2,\alpha} + \|\theta_0\|_{2,\alpha}) \int_0^t \|\mathbf{Z}(\cdot,\tau)\|_{1,\alpha}\, d\tau.$$

□

Note that $\mathbf{y} - \mathbf{y}' = \nabla \times (\psi - \psi')$. We can note that

$$\nabla \cdot (\nabla \times (\psi - \psi')) = 0,\ \nabla \times (\nabla \times (\psi - \psi')) = \zeta - \zeta',\ \nabla \times (\psi - \psi') \cdot \mathbf{n}|_{\Gamma \times [0,1]} = 0.$$

Then we have that

$$|||(\mathbf{y} - \mathbf{y}')(\cdot,t)|||_{1,\alpha} = \|(\zeta - \zeta')(\cdot,t)\|_{0,\alpha}$$

is a norm equivalent to the norm $\|\cdot\|_{1,\alpha}$.

Lemma 20. *For each* $\mathbf{z},\ \mathbf{z}' \in \mathbf{S}_\nu$*, if* $\widetilde{C} = C_\nu(\|\mathbf{y}_0\|_{2,\alpha} + \|\theta_0\|_{2,\alpha})$ *then we have that*

$$|||(F^m(\mathbf{z}) - F^m(\mathbf{z}'))(\cdot,t)|||_{1,\alpha} \leq \frac{\widetilde{C}t^m}{m!}\|\mathbf{z} - \mathbf{z}'\|_{C^0(\mathbf{C}^{1,\alpha})} \tag{3.52}$$

for all $m \in \mathbb{N}$.

Demonstração. We are going to prove by induction.
► $m = 1$:
Thanks to the lemma 19;
► We suppose that it is true for $m = k$ and we want to prove for $m = k + 1$:
Applying the lemma 19 to $\mathbf{y} = F^k(\mathbf{z})$ and $\mathbf{y}' = F^k(\mathbf{z}')$ we have

$$|||(F(\mathbf{y}) - F(\mathbf{y}'))(\cdot,t)|||_{1,\alpha}(t) \leq \widetilde{C} \int_0^t \|(\mathbf{y} - \mathbf{z}')(\cdot,s)\|_{1,\alpha} ds.$$

so, using the induction hypothesis, we obtain

$$|||F^{k+1}(\mathbf{z}) - F^{k+1}(\mathbf{z}')|||_{1,\alpha} \leq \widetilde{C}\|\mathbf{z} - \mathbf{z}'\|_{C^0(\mathbf{C}^{1,\alpha})} \int_0^t \frac{\widetilde{C}s^k}{k!} ds.$$

Therefore, (3.52) holds for all $m = 1, 2, \ldots$. □

So, how $\widetilde{C} \leq K := K(\nu,\delta)$ we can take m large enough such that $\frac{K1^m}{m!} < 1$.

We fix this m, so we have that $F^m : \mathbf{S}_\nu \to \mathbf{S}_\nu$ is a contraction in $C^0([0,1];\mathbf{C}^{1,\alpha}(\overline{\Omega}))$.

Then, defining $B := \mathbf{S}_\nu$ and applying the theorem 14 to $G := F^m$ we obtain that F^m has a unique fixed point $\mathbf{y} \in \overline{\mathbf{S}_\nu}^{\|\cdot\|_{C^0(\mathbf{C}^{1,\alpha})}}$. This ends the proof.

Therefore, $(\mathbf{y},\zeta,\theta)$ verifies

$$\begin{cases} \zeta_t + \mathbf{y}\cdot\nabla\zeta = -\vec{k} \times \nabla\theta, & \text{in} \quad \overline{\Omega} \times [0,1], \\ \theta_t + \mathbf{y}\cdot\nabla\theta = 0, & \text{in} \quad \overline{\Omega} \times [0,1], \\ \mathbf{y}(\mathbf{x},t)\cdot\mathbf{n}(\mathbf{x}) = (\overline{\mathbf{y}}(\mathbf{x},t) + \mu(t)\mathbf{y}_0(\mathbf{x}))\cdot\mathbf{n}(\mathbf{x}), & \text{on} \quad \Gamma \times [0,1], \\ \nabla\cdot\mathbf{y} = 0,\ \nabla\times\mathbf{y} = \zeta & \text{in} \quad \overline{\Omega} \times [0,1], \\ \zeta(0) = \nabla\times\mathbf{y}_0,\ \theta(0) = \theta_0, & \text{in} \quad \overline{\Omega}, \end{cases} \tag{3.53}$$

From the first and fourth equation of (3.53), we have

$$\nabla\times\left(\mathbf{y}_t-(\mathbf{y}\cdot\nabla)\mathbf{y}-\vec{\mathbf{k}}\theta\right)=0$$

and then there exists a pressure $p\in\mathcal{D}'(\overline{\Omega}\times[0,1])$ such that

$$\mathbf{y}_t-(\mathbf{y}\cdot\nabla)\mathbf{y}=-\nabla p+\vec{\mathbf{k}}\theta,\ \text{ in } \Omega\times[0,1].$$

From the fourth and fifth equation of (3.53), we have

$$\nabla\times(\mathbf{y}(0)-\mathbf{y}_0)=0$$

and then there exists a function $g\in\mathcal{D}'(\overline{\Omega})$ such that

$$\mathbf{y}(0)-\mathbf{y}_0=\nabla g.$$

How $\nabla\cdot\mathbf{y}=\nabla\cdot\mathbf{y}_0=0$ and $(\mathbf{y}(0)-\mathbf{y}_0)\cdot\mathbf{n}=0$ in Γ we have that g is solution of the elliptic problem

$$\begin{cases}\Delta g=0 & \text{in} \quad \Omega,\\ \dfrac{\partial g}{\partial\mathbf{n}}=0 & \text{on} \quad \Gamma,\end{cases}$$

Whence, g is constant and then $\mathbf{y}(0)=\mathbf{y}_0$.

From the fact that $\zeta(\cdot,1)=0$ in $\overline{\Omega}$ and the fourth equation of (3.53), we have

$$\nabla\times\mathbf{y}(\cdot,1)=0$$

and then there exists a function $h\in\mathcal{D}'(\overline{\Omega})$ such that

$$\mathbf{y}(\cdot,1)=\nabla h.$$

How $\nabla\cdot\mathbf{y}=0$ and $\mathbf{y}(\cdot,1)\cdot\mathbf{n}=0$ in Γ we have that h is solution of the elliptic problem

$$\begin{cases}\Delta h=0 & \text{in} \quad \Omega,\\ \dfrac{\partial h}{\partial\mathbf{n}}=0 & \text{on} \quad \Gamma,\end{cases}$$

Whence, h is constant and then $\mathbf{y}(\cdot,1)=0$ in $\overline{\Omega}$.

Remark 7. *The condition $\mathbf{y}_0\in\mathbf{C}^{3,\alpha}(\overline{\Omega})$ and $\theta\in C^{3,\alpha}(\overline{\Omega})$ can be relaxed. In fact, if $\mathbf{y}_0\in\mathbf{C}^{2,\alpha}(\overline{\Omega})$ and $\theta\in C^{2,\alpha}(\overline{\Omega})$, for the Lemma* 12*, we can construct a sequence $(\mathbf{y}_0^n)_n\in\mathbf{C}^{3,\alpha}(\overline{\Omega})$ and $(\theta_0^n)_n\in C^{3,\alpha}(\overline{\Omega})$ such that*

$$\mathbf{y}_0^n\to\mathbf{y}_0 \ \textit{ in } \ \mathbf{C}^{2,\alpha}(\overline{\Omega}) \qquad \textit{and} \qquad \theta_0^n\to\theta_0 \ \textit{ in } \ C^{2,\alpha}(\overline{\Omega}).$$

Then, from the previous result, we obtain $(\mathbf{y}^n,p^n,\theta^n)$ a fixed point for F that is a solution of (3.54)*. This solution is bounded by a constant which depends of the $C^{2,\alpha}$-norm of the data. So, we can pass the limit obtained the Proposition 11.*

3.5 Proof of theorem 2 - Tridimensional case

In this section, we will use the notations of Section 3.2.2. Let $\{\psi^i\}$ a partition of the unity associated to the balls B^i such that $\sum_{i=1}^{k} \psi^i = 1$.

The 3D version of proposition 9 is given by the following result:

Proposition 11. *There exists $\delta > 0$ such that if* $\max\{\|\mathbf{y}_0\|_{2,\alpha}, \|\theta_0\|_{2,\alpha}\} \leq \delta$*, then the system*

$$\begin{cases} \omega_t + \mathbf{y} \cdot \nabla\omega = \omega \cdot \nabla\mathbf{y} - \vec{\mathbf{k}} \times \nabla\theta & \textit{in} \quad \Omega \times (0,1), \\ \theta_t + \mathbf{y} \cdot \nabla\theta = 0 & \textit{in} \quad \Omega \times (0,1), \\ \mathbf{y} \cdot \mathbf{n} = (\overline{\mathbf{y}} + \mu\, \mathbf{y}_0) \cdot \mathbf{n} & \textit{on} \quad \Gamma \times (0,1), \\ \nabla \cdot \mathbf{y} = 0, \ \nabla \times \mathbf{y} = \omega & \textit{in} \quad \Omega \times (0,1), \\ \omega(0) = \nabla \times \mathbf{y}_0, \ \theta(0) = \theta_0 & \textit{in} \quad \Omega, \end{cases} \tag{3.54}$$

possesses at least one solution

$$(\omega, \theta, \mathbf{y}) \in C^0([0,1]; \mathbf{C}^{0,\alpha}(\overline{\Omega}; \mathbb{R}^3)) \times C^0([0,1]; C^{1,\alpha}(\overline{\Omega}; \mathbb{R})) \times C^0([0,1]; \mathbf{C}^{1,\alpha}(\overline{\Omega}; \mathbb{R}^3)) \tag{3.55}$$

such that $\theta(\mathbf{x}, t) = 0$ *in* $\Omega \times [1/2, 1]$ *and* $\omega(\mathbf{x}, 1) = 0$ *in* Ω*.*

Demonstração. In this existence result, we will follow the ideas from Bardos and Frisch (see [4]).

First, we consider the class

$$R' = \left\{ \mathbf{z} \in C^0([0,1]; \mathbf{C}^{2,\alpha}(\overline{\Omega})); \ \nabla \cdot \mathbf{z} = 0 \text{ in } \Omega \times [0,1] \text{ and } \mathbf{z} \cdot \mathbf{n} = (\overline{\mathbf{y}} + \mu\mathbf{y}_0) \cdot \mathbf{n} \text{ on } \Gamma \right\}.$$

For $\nu > 0$ we are going to denote by

$$R_\nu = \{\mathbf{z} \in R'; \|\mathbf{z}(\cdot\,, t) - \overline{\mathbf{y}}(\cdot\,, t)\|_{2,\alpha} \leq \nu \quad \forall t \in [0,1]\}.$$

For now on, R_ν is fixed for the value of $\nu > 0$ guaranteed by the lemma 16. As in the 2D case, the idea is to define a application $F : R_\nu \to R_\nu$ and show that it has a fixed point. We start from a arbitrary vector function $\mathbf{z} = \mathbf{z}(\mathbf{x}, t) \in R_\nu$. With this $\mathbf{z}$ we define $\tilde{\mathbf{z}}$ as in the Lemma 16 and then find θ^* verifying

$$\begin{cases} \theta^*_t + \widetilde{\mathbf{z}} \cdot \nabla\theta^* = 0 & \text{in} \quad G \times [0, 1/2], \\ \theta^*(\mathbf{x}, 0) = \sum_{i=1}^{k} \psi^i \pi_1(\theta_0)(\mathbf{x}) & \text{in} \quad G. \end{cases}$$

For $\mathbf{x} \in \Omega$, let $\theta^*_0(\mathbf{x}) = \theta^*(\mathbf{x}\,, 1/4)$ and consider, for $i = 1, ..., k$, the functions $\theta^i \in C^0([0,1]; C^{2,\alpha}(G; \mathbb{R}))$ solutions to the following problem

$$\begin{cases} \theta^i_t + \widetilde{\mathbf{z}} \cdot \nabla\theta^i = 0 & \text{in} \quad G \times [1/4, 1/2], \\ \theta^i(\mathbf{x}, 1/4) = \psi^i \pi_1(\theta^*_0)(\mathbf{x}) & \text{in} \quad G. \end{cases} \tag{3.56}$$

The function θ of Proposition 11 will be constructed as a restriction of a function defined in G. This extension will still be denoted in the same way as θ. We start defining:

$$\begin{cases} \theta = \theta^* & \text{in} \quad G \times [0, 1/4] \\ \theta_t + \widetilde{\mathbf{z}} \cdot \nabla\theta = 0 & \text{in} \quad G \times \left([1/4, 1/2] \setminus \bigcup_{i=1}^{k} \left\{ t_{i-\frac{1}{2}} \right\} \right). \end{cases} \tag{3.57}$$

To define θ properly, we have yet to define it on the values $t_{i-\frac{1}{2}}$. This follows by an induction argument. Indeed, in the first interval we can write

$$\theta(\mathbf{x}, t) = \sum_{i=1}^{k} \theta^i(\mathbf{x}, t), \quad (\mathbf{x}, t) \in G \times [1/4, t_{1/2}]. \tag{3.58}$$

The equality

$$\theta^i(\mathbf{X}_{\widetilde{\mathbf{z}}}(\mathbf{x}, t, 0), t) = \psi^i(\mathbf{x})\pi_1(\theta_0)(\mathbf{x}) \tag{3.59}$$

implies that

$$supp\ \theta^i(\cdot, t^-_{i-\frac{1}{2}}) \subset \mathbf{X}_{\widetilde{\mathbf{z}}}(B^i, t_{i-\frac{1}{2}}, 0). \tag{3.60}$$

Using Lemma 16 and the inclusion (3.60) we have that

$$\theta^i(\mathbf{x}, t^-_{i-\frac{1}{2}}) = 0, \quad \mathbf{x} \in \Omega. \tag{3.61}$$

This shows that it is natural to define

$$\theta(\mathbf{x}, t^+_{1/2}) = \sum_{l=2}^{k} \theta^l(\mathbf{x}, t^+_{1/2}), \quad \mathbf{x} \in G. \tag{3.62}$$

More generally, in a analogous way, we can ask that

$$\theta(\mathbf{x}, t^+_{i-1/2}) = \sum_{l=i+1}^{k} \theta^l(\mathbf{x}, t^+_{i-1/2}), \quad \mathbf{x} \in G. \tag{3.63}$$

with the convention

$$\theta(\mathbf{x}, t_{k-\frac{1}{2}}) := 0 \quad \text{in} \quad G.$$

We remark that θ defined in this way is not necessarily continuous at the points $\{t_{i-1/2}\}$, but the restriction to Ω is indeed a function in $C^0([0, T]; C^{2,\alpha}(\Omega))$. In this way, we clearly have

$$\theta(\mathbf{x}, t) = \sum_{l=i+1}^{k} \theta^l(\mathbf{x}, t) \quad \text{in} \quad \Omega \times [t_{i-\frac{1}{2}}, t_{i+\frac{1}{2}}], \tag{3.64}$$

in particular

$$\begin{cases} \theta_t + \mathbf{z} \cdot \nabla\theta = 0, & \text{in} \quad \Omega \times [0, 1/2], \\ \theta(0) = \theta_0, & \text{in} \quad \Omega. \end{cases} \tag{3.65}$$

with $\theta(\mathbf{x}, t) := 0$ in $\overline{\Omega} \times [t_{k-\frac{1}{2}}, 1]$.

With this θ, we obtain ω^* given by

$$\begin{cases} \omega^*_t + (\widetilde{\mathbf{z}} \cdot \nabla)\omega^* = (\omega^* \cdot \nabla)\widetilde{\mathbf{z}} - (\nabla \cdot \widetilde{\mathbf{z}})\omega^* - \overrightarrow{\mathbf{k}} \times \nabla \pi_1(\theta) & \text{in} \quad G \times [0, 1/2], \\ \omega^*(\mathbf{x}, 0) = \nabla \times \pi_3(\mathbf{y}_0)(\mathbf{x}) & \text{in} \quad G. \end{cases}$$

By Lemma 7, there exist $\boldsymbol{v} \in \mathbf{C}^{2,\alpha}(\Omega)$ such that $\omega^*(\cdot, 1/2) = \nabla \times \boldsymbol{v}$. In this way, consider the solution of the problem

$$\begin{cases} \omega^{**}_t + (\widetilde{\mathbf{z}} \cdot \nabla)\omega^{**} = (\omega^{**} \cdot \nabla)\widetilde{\mathbf{z}} - (\nabla \cdot \widetilde{\mathbf{z}})\omega^{**} & \text{in} \quad G \times [1/2, 1], \\ \omega^{**}(\mathbf{x}, 1/2) = \sum_{i=1}^{k} \nabla \times (\psi^i \pi_3(\boldsymbol{v}))(\mathbf{x}) & \text{in} \quad G. \end{cases} \tag{3.66}$$

For $i = 1, .., k$, let

$$\begin{cases} \omega^i_t + (\widetilde{\mathbf{z}} \cdot \nabla)\omega^i = (\omega^i \cdot \nabla)\widetilde{\mathbf{z}} - (\nabla \cdot \widetilde{\mathbf{z}})\omega^i & \text{in} \quad G \times [1/2, 1], \\ \omega^i(\mathbf{x}, 1/2) = \nabla \times (\psi^i \pi_3(\boldsymbol{v}))(\mathbf{x}) & \text{in} \quad G. \end{cases} \tag{3.67}$$

The solutions ω^i are divergence free and moreover has the property

$$\omega^i(\mathbf{X}_{\widetilde{\mathbf{z}}}(\mathbf{x}, t, 1/2), t) = \omega^i(\mathbf{x}, 1/2) + \int_{1/2}^{t} [(\omega^i \cdot \nabla)\widetilde{\mathbf{z}} - (\nabla \cdot \widetilde{\mathbf{z}})\omega^i](\mathbf{X}_{\widetilde{\mathbf{z}}}(\mathbf{x}, \sigma, 1/2), \sigma) d\sigma.$$

Then,

$$|\omega^i(\mathbf{X}_{\widetilde{\mathbf{z}}}(\mathbf{x}, t, 1/2), t)| \leq |\omega^i(\mathbf{x}, 1/2)| + C\|\widetilde{\mathbf{z}}\|_{C^0([0,1];\mathbf{C}^1(\Omega))} \int_{1/2}^{t} |\omega^i(\mathbf{X}_{\widetilde{\mathbf{z}}}(\mathbf{x}, \sigma, 1/2), \sigma)| d\sigma.$$

Notice that if $\mathbf{x} \notin B^i$ we have

$$|\omega^i(\mathbf{X}_{\widetilde{\mathbf{z}}}(\mathbf{x}, t, 1/2), t)| \leq C\|\widetilde{\mathbf{z}}\|_{C^0([0,1];\mathbf{C}^1(\Omega))} \int_{1/2}^{t} |\omega^i(\mathbf{X}_{\widetilde{\mathbf{z}}}(\mathbf{x}, \sigma, 1/2), \sigma)| d\sigma.$$

From the Gronwall's Lemma, we see that

$$\omega^i(\mathbf{X}_{\widetilde{\mathbf{z}}}(\mathbf{x}, t, 1/2), t) = 0, \quad (\mathbf{x}, t) \in (G \setminus B^i) \times (1/2, 1)$$

whence

$$supp\ \omega^i(\cdot, t) \subset \mathbf{X}_{\widetilde{\mathbf{z}}}(B^i, t, 1/2).$$

Using Lemma 14 we get

$$\omega^i(\mathbf{x}, t_{k+i-\frac{1}{2}}) = 0, \ \forall \mathbf{x} \in \overline{\Omega}.$$

Then, we can define ω by

$$\omega(\mathbf{x}, t) := \omega^*(\mathbf{x}, t) \ \text{ in } \ G \times [0, 1/2]. \tag{3.68}$$

To define properly ω in $[1/2, 1]$, we proceed as for the temperature. Precisely we have:

$$\begin{cases} \omega_t + (\widetilde{\mathbf{z}} \cdot \nabla)\omega = (\omega \cdot \nabla)\widetilde{\mathbf{z}} - (\nabla \cdot \widetilde{\mathbf{z}})\omega & \text{in} \quad G \times \left([1/2, 1] \setminus \bigcup_{i=1}^{k} \left\{ t_{k+i-\frac{1}{2}} \right\} \right), \\ \omega(\mathbf{x}, t^+_{k+i-\frac{1}{2}}) = \sum_{l=i+1}^{k} \omega^l(\mathbf{x}, t^+_{k+i-\frac{1}{2}}) & \text{in} \quad G. \end{cases} \tag{3.69}$$

In particular, we have:

$$\omega(\mathbf{x},t) = \sum_{l=i+1}^{k} \omega^l(\mathbf{x},t) \quad \text{in} \quad \Omega \times [t_{k+i-\frac{1}{2}}, t_{k+i+\frac{1}{2}}] \tag{3.70}$$

and

$$\omega(\mathbf{x}, t_{2k-\frac{1}{2}}) = 0. \tag{3.71}$$

In this way, we get that the restriction of ω to $\overline{\Omega} \times [0,1]$ is $C^0([0,1]; C^{1,\alpha}(\overline{\Omega}))$ and the pair (ω, θ) satisfies:

$$\begin{cases} \omega_t + (\mathbf{z}\cdot\nabla)\omega = (\omega\cdot\nabla)\mathbf{z} - (\nabla\cdot\mathbf{z})\omega - \overrightarrow{\mathbf{k}} \times \nabla\theta & \text{in} \quad \Omega \times [0,1], \\ \theta_t + \mathbf{z}\cdot\nabla\theta = 0 & \text{in} \quad \Omega \times [0,1], \\ \theta(0) = \theta_0, \ \omega(0) = \nabla \times \mathbf{y}_0 & \text{in} \quad \Omega. \end{cases}$$

Moreover $\omega(\mathbf{x},t) := 0$ in $\Omega \times [t_{2k-\frac{1}{2}}, 1]$ and $\theta(\mathbf{x},t) := 0$ in $\Omega \times [t_{k-\frac{1}{2}}, 1]$.

The task now is to recover the velocity field in order to well define the fixed point application.

Firstly, thanks to (3.68),(3.69) and Lemma 7, ω stays divergence-free in $G \times [0,1]$.

Consider Ψ^*, the unique solution of the problem

$$\begin{cases} -\Delta\Psi^* = \omega - \mu\nabla\times\mathbf{y}_0, & \text{in} \quad \overline{\Omega}\times[0,1], \\ \nabla\cdot\Psi^* = 0 & \text{in} \quad \overline{\Omega}\times[0,1], \\ (\nabla\times\Psi^*)\cdot\mathbf{n} = 0, & \text{on} \quad \partial\Omega\times[0,1]. \end{cases}$$

So, with this Ψ^*, there exists a unique $\mathbf{y}$ in $C([0,1]; C^{2+\alpha}(\overline{\Omega}))$ such that

$$\begin{cases} \nabla\times\mathbf{y} = \omega & \text{in} \quad \overline{\Omega}\times[0,1], \\ \nabla\cdot\mathbf{y} = 0 & \text{in} \quad \Omega\times[0,1], \\ \mathbf{y}\cdot\mathbf{n} = (\mu\mathbf{y}_0 + \overline{\mathbf{y}})\cdot\mathbf{n} & \text{on} \quad \partial\Omega\times[0,1]. \end{cases} \tag{3.72}$$

In fact, $\mathbf{y} = \nabla\times\Psi^* + \overline{\mathbf{y}} + \mu\mathbf{y}_0$.

Therefore, since $\mathbf{y}$ is uniquely determined by $\mathbf{z}$, we write

$$\begin{aligned} F : R_\nu &\to R' \\ \mathbf{z} &\mapsto F(\mathbf{z}) = \mathbf{y} \end{aligned} \tag{3.73}$$

and we have the following Lemma:

Lemma 21. *There exists $\delta > 0$ such that if $\{\|\mathbf{y}_0\|_{2,\alpha}, \|\theta_0\|_{2,\alpha}\} \leq \delta$ $F : R_\nu \to R_\nu$ is well defined.*

In order to prove that F has a fixed point, we need to prove that, in some sense, F is a contraction. For this purpose, we will apply the Theorem 14, as was done before.

Using the same ideas of Lemma 20 and recalling remark 7, we can prove the result:

Lemma 22. *For each* $\mathbf{z},\ \mathbf{z}' \in R_\nu$*, if* $\widetilde{C} = C_\nu(\|\mathbf{y}_0\|_{2,\alpha} + \|\theta_0\|_{2,\alpha})$ *then we have that*

$$|||(F^m(\mathbf{z}) - F^m(\mathbf{z}'))(\cdot,t)|||_{1,\alpha} \leq \frac{\widetilde{C}t^m}{m!}\|\mathbf{z} - \mathbf{z}\|_{C^0(\mathbf{C}^{1,\alpha})} \tag{3.74}$$

for all $m \in \mathbb{N}$.

Then, if m is great enough, we have that $F^m : R_\nu \to R_\nu$ is a contraction map. Therefore, applying the theorem 14 to F^m we obtain that F^m has a unique fixed point $\mathbf{y} \in \overline{R_\nu}^{\|\cdot\|_{0,1,\alpha}}$. This ends the proof. □

3.6 Additional Comments

- An interesting question which arise if it is possible to obtain for the solution the same regularity of the data. More precisely, given $\mathbf{y}_0, \mathbf{y}_1 \in \mathbf{C}(2,\alpha,\Gamma_0)$ and $\theta_0, \theta_1 \in C^{2,\alpha}(\overline{\Omega};\mathbb{R})$, find $\mathbf{y} \in C^0([0,T];\mathbf{C}(2,\alpha,\Gamma_0))$, $\theta \in C^0([0,T];C^{2,\alpha}(\overline{\Omega};\mathbb{R}))$ and $p \in \mathcal{D}'(\Omega \times (0,T))$ such that (3.76) holds and

$$\mathbf{y}(\mathbf{x},T) = \mathbf{y}_1(\mathbf{x}), \quad \theta(\mathbf{x},T) = \theta_1(\mathbf{x}) \ \text{ in } \ \Omega. \tag{3.75}$$

- Another interesting question is about controllability of the *incompressible, heat conductive, inviscid Boussinesq* system given by:

$$\begin{cases} \mathbf{y}_t + (\mathbf{y}\cdot\nabla)\mathbf{y} = -\nabla p + \vec{\mathbf{k}}\theta & \text{in} \quad \Omega \times (0,T), \\ \nabla \cdot \mathbf{y} = 0 & \text{in} \quad \Omega \times (0,T), \\ \theta_t + \mathbf{y}\cdot\nabla\theta = \kappa\,\Delta\theta & \text{in} \quad \Omega \times (0,T), \\ \mathbf{y}(\mathbf{x},0) = \mathbf{y}_0(\mathbf{x}),\ \theta(\mathbf{x},0) = \theta_0(\mathbf{x}) & \text{in} \quad \Omega, \end{cases} \tag{3.76}$$

  where $\kappa > 0$ can be viewed as the heat diffusion coefficient.

Both question will be subject of a forthcoming paper.

Capítulo 4

On the control of some coupled systems of the Boussinesq kind with few controls

On the control of some coupled systems of the Boussinesq kind with few controls

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Abstract. This paper is devoted to prove the local exact controllability to the trajectories for a coupled system, of the Boussinesq kind, with a reduced number of controls. In the state system, the unknowns are the velocity field and pressure of the fluid $(\mathbf{y}, p)$, the temperature θ and an additional variable c that can be viewed as the concentration of a contaminant solute. We prove several results, that essentially show that it is sufficient to act locally in space on the equations satisfied by θ and c.

4.1 Introduction

Let $\Omega \subset \mathbb{R}^N$ be a bounded connected open set whose boundary $\partial\Omega$ is regular enough (for instance of class C^2; $N = 2$ or $N = 3$). Let $\mathcal{O} \subset \Omega$ be a (small) nonempty open subset and assume that $T > 0$. We will use the notation $Q = \Omega \times (0, T)$ and $\Sigma = \partial\Omega \times (0, T)$ and we will denote by $\mathbf{n} = \mathbf{n}(x)$ the outward unit normal to Ω at any point $x \in \partial\Omega$.

In the sequel, we will denote by $C,\ C_1,\ C_2, \dots$ various positive constants (usually depending on Ω, $\mathcal{O}$ and T).

We will be concerned with the following controlled system

$$\begin{cases} \mathbf{y}_t - \Delta\mathbf{y} + (\mathbf{y}\cdot\nabla)\mathbf{y} + \nabla p = \mathbf{v}1_{\mathcal{O}} + \mathbf{F}(\theta, c) & \text{in} \quad Q, \\ \nabla\cdot\mathbf{y} = 0 & \text{in} \quad Q, \\ \theta_t - \Delta\theta + \mathbf{y}\cdot\nabla\theta = w_1 1_{\mathcal{O}} + f_1(\theta, c) & \text{in} \quad Q, \\ c_t - \Delta c + \mathbf{y}\cdot\nabla c = w_2 1_{\mathcal{O}} + f_2(\theta, c) & \text{in} \quad Q, \\ \mathbf{y} = \mathbf{0},\ \theta = c = 0 & \text{on} \quad \Sigma, \\ \mathbf{y}(0) = \mathbf{y}_0,\ \theta(0) = \theta_0,\ c(0) = c_0 & \text{in} \quad \Omega, \end{cases} \tag{4.1}$$

where $\mathbf{v} = \mathbf{v}(x, t),\ w_1 = w_1(x, t)$ and $w_2 = w_2(x, t)$ stand for the control functions. They are assumed to act on the (small) set $\mathcal{O}$ during the whole time interval $(0, T)$. The symbol $1_{\mathcal{O}}$ stands for the characteristic function of $\mathcal{O}$. It will be assumed that the functions $\mathbf{F} = (F_1, \dots, F_N)$, f_1 and f_2 satisfy:

$$\begin{cases} F_i, f_1, f_2 \in C^1(\mathbb{R}^2),\ \text{with}\ \nabla F_i,\ \nabla f_1,\ \nabla f_2 \in \mathbf{L}^\infty(\mathbb{R}^2)\ \text{and} \\ F_i(0,0) = f_1(0,0) = f_2(0,0) = 0\ (1 \le i \le N). \end{cases} \tag{4.2}$$

In (4.1), $\mathbf{y}$ and p can be respectively interpreted as the velocity field and the pressure of a fluid. The function θ (resp. c) can be viewed as the temperature of the fluid (resp. the concentration of a contaminant solute). On the other hand, $\mathbf{v}$, w_1 and w_2 must be regarded as source terms, locally supported in space, respectively for the PDEs satisfied by $(\mathbf{y},p)$, θ and c.

From the viewpoint of control theory, $(\mathbf{v},w_1,w_2)$ is the control and $(\mathbf{y},p,\theta,c)$ is the state. In the problems considered in this paper, the main goal will always be related to choose $(\mathbf{v},w_1,w_2)$ such that $(\mathbf{y},p,\theta,c)$ satisfies a desired property at $t=T$.

More precisely, we will present some results that show that the system (4.1) can be controlled, at least locally, with only N scalar controls in $L^2(\mathcal{O}\times(0,T))$. We will also see that, when $N=3$, (4.1) can be controlled, at least under some geometrical assumptions, with only 2 (i.e. $N-1$) scalar controls.

Thus, let us introduce the spaces $\mathbf{H}$, $\mathbf{E}$ and $\mathbf{V}$, with

$$\begin{aligned}&\mathbf{V}=\{\ \varphi\in\mathbf{H}^1_0(\Omega):\nabla\cdot\varphi=0 \text{ in } \Omega\ \},\\ &\mathbf{H}=\{\ \varphi\in\mathbf{L}^2(\Omega):\nabla\cdot\varphi=0 \text{ in } \Omega \text{ and } \varphi\cdot\mathbf{n}=0 \text{ on } \partial\Omega\ \}\end{aligned}$$

and

$$\mathbf{E}=\begin{cases}\mathbf{H}, & \text{if } \ N=2,\\ \mathbf{L}^4(\Omega)\cap\mathbf{H}, & \text{if } \ N=3\end{cases}\tag{4.3}$$

and let us fix a *trajectory* $(\overline{\mathbf{y}},\overline{p},\overline{\theta},\overline{c})$, that is, a sufficiently regular solution to the related noncontrolled system:

$$\begin{cases}\overline{\mathbf{y}}_t-\Delta\overline{\mathbf{y}}+(\overline{\mathbf{y}}\cdot\nabla)\overline{\mathbf{y}}+\nabla\overline{p}=\mathbf{F}(\overline{\theta},\overline{c}) & \text{in } \ Q,\\ \nabla\cdot\overline{\mathbf{y}}=0 & \text{in } \ Q,\\ \overline{\theta}_t-\Delta\overline{\theta}+\overline{\mathbf{y}}\cdot\nabla\overline{\theta}=f_1(\overline{\theta},\overline{c}) & \text{in } \ Q,\\ \overline{c}_t-\Delta\overline{c}+\overline{\mathbf{y}}\cdot\nabla\overline{c}=f_2(\overline{\theta},\overline{c}) & \text{in } \ Q,\\ \overline{\mathbf{y}}=\mathbf{0},\ \overline{\theta}=\overline{c}=0 & \text{on } \ \Sigma,\\ \overline{\mathbf{y}}(0)=\overline{\mathbf{y}}_0,\ \overline{\theta}(0)=\overline{\theta}_0,\ \overline{c}(0)=\overline{c}_0 & \text{in } \ \Omega.\end{cases}\tag{4.4}$$

It will be assumed that

$$\overline{\mathbf{y}}_i,\ \overline{\theta},\ \overline{c}\in L^\infty(Q) \text{ and } (\overline{\mathbf{y}}_i)_t,\overline{\theta}_t,\overline{c}_t\in L^2\big(0,T;L^\kappa(\Omega)\big),\ 1\le i\le N,\tag{4.5}$$

with

$$\kappa>\begin{cases}1, & \text{if } \ N=2,\\ 6/5, & \text{if } \ N=3.\end{cases}\tag{4.6}$$

Notice that, if the initial data in (4.4) satisfy appropriate regularity conditions and $(\overline{\mathbf{y}},\overline{p},\overline{\theta},\overline{c})$ solves (4.4) (for instance in the usual weak sense) and $\overline{\mathbf{y}}_i,\overline{\theta},\overline{c}\in L^\infty(Q)$, then we have (4.5). For example, if $\overline{\mathbf{y}}_0\in\mathbf{V}$ and $\overline{\theta}_0,\ \overline{c}_0\in H^1_0(\Omega)$, we actually have from the parabolic regularity theory that $(\overline{\mathbf{y}}_i)_t,\ \overline{\theta}_t,\ \overline{c}_t\in L^2(Q)$.

In our first main result, we will assume the following:

$f_1\equiv f_2\equiv 0$ and $\mathbf{F}(a_1,a_2)=a_1\mathbf{e}_N+a_2\overrightarrow{\mathbf{h}}$, where:

- $\mathbf{e}_N$ is the N-th vector of the canonical basis of $\mathbb{R}^N$ and (4.7)
- $\mathbf{e}_N$ and $\overrightarrow{\mathbf{h}}$ are linearly independent.

Then, we have the following result:

Theorem 15. *Assume that $T > 0$ is given and the assumptions* (4.4)–(4.7) *are satisfied. Then there exists $\delta > 0$ such that, whenever $(\mathbf{y}_0,\ \theta_0,\ c_0) \in \mathbf{E} \times L^2(\Omega) \times L^2(\Omega)$ and*

$$\|(\mathbf{y}_0,\ \theta_0,\ c_0) - (\overline{\mathbf{y}}_0,\ \overline{\theta}_0,\ \overline{c}_0)\|_{\mathbf{L}^2(\Omega)} \leq \delta,$$

we can find L^2 controls $\mathbf{v}$, w_1 and w_2 with $\mathbf{v}_i \equiv \mathbf{v}_N \equiv 0$ for some $i < N$ and associated states $(\mathbf{y}, p, \theta, c)$ satisfying

$$\mathbf{y}(T) = \overline{\mathbf{y}}(T),\ \theta(T) = \overline{\theta}(T) \quad \text{and} \quad c(T) = \overline{c}(T). \tag{4.8}$$

In our second result, we will consider more general (and maybe nonlinear) functions $\mathbf{F}$. We will denote by $\mathbf{G}$ and $\mathbf{L}$ the partial derivatives of $\mathbf{F}$ with respect to θ and c:

$$\mathbf{G} = \frac{\partial \mathbf{F}}{\partial \theta}, \quad \mathbf{L} = \frac{\partial \mathbf{F}}{\partial c}.$$

The folowing will be assumed:

$$\begin{aligned}&\text{There exists a non-empty open set } \mathcal{O}_* \subset \mathcal{O} \text{ such that}\\ &\mathbf{G}(\overline{\theta}, \overline{c}) \text{ and } \mathbf{L}(\overline{\theta}, \overline{c}) \text{ are continuous and linearly independent in } \mathcal{O}_* \times (0,T).\end{aligned} \tag{4.9}$$

Then, we get a generalization of theorem 15:

Theorem 16. *Assume that $T > 0$ is given and the assumptions* (4.4)–(4.6) *and* (4.9) *are satisfied. Then there exists $\delta > 0$ such that, whenever $(\mathbf{y}_0,\ \theta_0,\ c_0) \in \mathbf{E} \times L^2(\Omega) \times L^2(\Omega)$ and*

$$\|(\mathbf{y}_0,\ \theta_0,\ c_0) - (\overline{\mathbf{y}}_0,\ \overline{\theta}_0,\ \overline{c}_0)\|_{\mathbf{L}^2(\Omega)} \leq \delta,$$

we can find L^2 controls $\mathbf{v}$, w_1 and w_2 with $\mathbf{v}_i \equiv \mathbf{v}_j \equiv 0$ for some $i \neq j$ and associated states $(\mathbf{y}, p, \theta, c)$ satisfying (4.8).

In the three-dimensional case, we can improve theorem 15 if we add to the hypotheses an appropriate geometrical assumption on $\mathcal{O}$. More precisely, let us assume that

$$\exists x^0 \in \partial\Omega,\ \exists a > 0 \ \text{ such that } \ \overline{\mathcal{O}} \cap \partial\Omega \supset B_a(x^0) \cap \partial\Omega \tag{4.10}$$

(here, $B_a(x^0)$ is the ball centered at x^0 of radius a).
Then the following holds:

Theorem 17. *Assume that $N = 3$, $T > 0$ is given, the assumptions in theorem* (15) *are satisfied,* (4.10) *holds and*

$$\mathbf{h}_1\mathbf{n}_2(x^0) - \mathbf{h}_2\mathbf{n}_1(x^0) \neq 0. \tag{4.11}$$

Then, the conclusion of theorem (15) *holds good with L^2 controls $\mathbf{v}$, w_1 and w_2 such that $\mathbf{v} \equiv \mathbf{0}$.*

The rest of this paper is organized as follows. In Section 4.2, we recall a previous result, needed for the proofs of theorems 15 to 17. In Section 4.3, we give the proof of theorem 15. We will adapt the arguments in [29] and [30], that lead to the local exact controllability to the trajectories for Navier-Stokes and Boussinesq systems; see also [38, 49, 56]. It will be seen that the main ingredients of this proof are appropriate global Carleman estimates for the solutions to linear systems similar to (4.1) and an inverse mapping theorem of the Liusternik kind. Sections 4.4 and 4.5 respectively deal with the proofs of theorems 16 and 17. In Section 4.6, we present some additional questions and comments. Finally, for completeness, we recall the main ideas of the proof of the Carleman estimates that serve as a starting point in an Appendix (see Section 4.7).

4.2 A preliminary result

A considerable part of this paper follows from the arguments and results in [29] and [30] adapted to the present context. Thus, let us set $\mathbf{y} = \overline{\mathbf{y}} + \mathbf{u},\ p = \overline{p} + q,\ \theta = \overline{\theta} + \phi,\ c = \overline{c} + z$ and let us use these identities in (4.1). Taking into account that $(\overline{\mathbf{y}}, \overline{p}, \overline{\theta}, \overline{c})$ solves (4.4), we find:

$$\begin{cases} \mathbf{u}_t - \Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\overline{\mathbf{y}} + (\overline{\mathbf{y}}\cdot\nabla)\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla q = \mathbf{v}1_{\mathcal{O}} + \mathbf{F}(\theta, c) - \overline{\mathbf{F}} & \text{in} \quad Q, \\ \nabla\cdot\mathbf{u} = 0 & \text{in} \quad Q, \\ \phi_t - \Delta\phi + \mathbf{u}\cdot\nabla\overline{\theta} + \overline{\mathbf{y}}\cdot\nabla\phi + \mathbf{u}\cdot\nabla\phi = w_1 1_{\mathcal{O}} + f_1(\theta, c) - \overline{f}_1 & \text{in} \quad Q, \\ z_t - \Delta z + \mathbf{u}\cdot\nabla\overline{c} + \overline{\mathbf{y}}\cdot\nabla z + \mathbf{u}\cdot\nabla z = w_2 1_{\mathcal{O}} + f_2(\theta, c) - \overline{f}_2 & \text{in} \quad Q, \\ \mathbf{u} = \mathbf{0},\ \phi = z = 0 & \text{on} \quad \Sigma, \\ \mathbf{u}(0) = \mathbf{y}_0 - \overline{\mathbf{y}}_0,\ \phi(0) = \theta_0 - \overline{\theta}_0,\ z(0) = c_0 - \overline{c}_0 & \text{in} \quad \Omega, \end{cases} \tag{4.12}$$

where we have introduced $\overline{\mathbf{F}} := \mathbf{F}(\overline{\theta}, \overline{c}),\ \overline{f}_1 := f_1(\overline{\theta}, \overline{c})$ and $\overline{f}_2 := f_2(\overline{\theta}, \overline{c})$.

This way, the local exact controllability to the trajectories for the system (4.1) is reduced to a local null controllability problem for the solution $(\mathbf{u}, q, \phi, z)$ to the nonlinear problem (4.12).

In order to solve the latter, following a standard approach, we will first deduce the (global) null controllability of a suitable linearized version, namely:

$$\begin{cases} \mathbf{u}_t - \Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\overline{\mathbf{y}} + (\overline{\mathbf{y}}\cdot\nabla)\mathbf{u} + \nabla q = \mathbf{S} + \mathbf{v}1_{\mathcal{O}} + \overline{\mathbf{G}}\phi + \overline{\mathbf{L}}z & \text{in} \quad Q, \\ \nabla\cdot\mathbf{u} = 0 & \text{in} \quad Q, \\ \phi_t - \Delta\phi + \mathbf{u}\cdot\nabla\overline{\theta} + \overline{\mathbf{y}}\cdot\nabla\phi = r_1 + w_1 1_{\mathcal{O}} + \overline{g_1}\phi + \overline{l_1}z & \text{in} \quad Q, \\ z_t - \Delta z + \mathbf{u}\cdot\nabla\overline{c} + \overline{\mathbf{y}}\cdot\nabla z = r_2 + w_2 1_{\mathcal{O}} + \overline{g_2}\phi + \overline{l_2}z & \text{in} \quad Q, \\ \mathbf{u} = \mathbf{0},\ \phi = z = 0 & \text{on} \quad \Sigma, \\ \mathbf{u}(0) = \mathbf{u}_0,\ \phi(0) = \phi_0,\ z(0) = z_0 & \text{in} \quad \Omega, \end{cases} \tag{4.13}$$

where

$$\overline{\mathbf{G}} = \mathbf{G}(\overline{\theta}, \overline{c}),\quad \overline{\mathbf{L}} = \mathbf{L}(\overline{\theta}, \overline{c}),\quad \overline{g_i} = g_i(\overline{\theta}, \overline{c}),\quad \overline{l_i} = l_i(\overline{\theta}, \overline{c}),$$

g_i and l_i denote the partial derivatives of f_i with respect to θ and c, $\mathbf{u}_0$, ϕ_0 and z_0 are the initial data and $\mathbf{S}$, r_1 and r_2 are appropriate functions that decay exponentially as $t \to T^-$. Then, appropriate and rather classical arguments will be used to deduce the local null controllability of the nonlinear system (4.12).

In this Section, we will present a suitable Carleman inequality for the so called adjoint of (4.13). This will lead easily to the null controllability result.

Thus, let us first introduce some weight functions:

$$
\begin{array}{rcl}
\alpha(x,t) & = & \frac{e^{5/4\lambda m\|\eta^0\|_\infty}-e^{\lambda(m\|\eta^0\|_\infty+\eta^0(x))}}{t^4(T-t)^4}, \\
\xi(x,t) & = & \frac{e^{\lambda(m\|\eta^0\|_\infty+\eta^0(x))}}{t^4(T-t)^4}, \\
\widehat{\alpha}(t) & = & \min_{x\in\overline{\Omega}} \alpha(x,t) = \dfrac{e^{5/4\lambda m\|\eta^0\|_\infty}-e^{\lambda(m+1)\|\eta^0\|_\infty}}{t^4(T-t)^4}, \\
\alpha^*(t) & = & \max_{x\in\overline{\Omega}} \alpha(x,t) = \dfrac{e^{5/4\lambda m\|\eta^0\|_\infty}-e^{\lambda m\|\eta^0\|_\infty}}{t^4(T-t)^4}, \\
\hat{\xi}(t) & = & \min_{x\in\overline{\Omega}} \xi(x,t) = \dfrac{e^{\lambda m\|\eta^0\|_\infty}}{t^4(T-t)^4}, \\
\xi^*(t) & = & \max_{x\in\overline{\Omega}} \xi(x,t) = \dfrac{e^{\lambda(m+1)\|\eta^0\|_\infty}}{t^4(T-t)^4}, \\
\hat{\mu}(t) & = & s\lambda e^{-s\hat{\alpha}}\xi^*,\ \mu(t) = s^{15/4}e^{-2s\hat{\alpha}+s\alpha^*}\xi^{*15/4},
\end{array}
\tag{4.14}
$$

where $m > 4$ is a fixed real number, $\eta^0 \in C^2(\overline{\Omega})$ is a function that verifies

$$
\eta^0 > 0 \text{ in } \Omega,\ \eta^0 \equiv 0 \text{ on } \partial\Omega \text{ and } |\nabla\eta^0| > 0 \text{ in } \overline{\Omega\setminus\mathcal{O}_0}
$$

and $\mathcal{O}_0$ is a non-empty open subset of $\mathcal{O}$ such that $\overline{\mathcal{O}_0} \subset \mathcal{O}$.

The adjoint system of (4.13) is:

$$
\begin{cases}
-\varphi_t - \Delta\varphi - D\varphi\overline{\mathbf{y}} + \nabla\pi = \widetilde{\mathbf{G}} + \overline{\theta}\nabla\psi + \overline{c}\nabla\zeta & \text{in} \quad Q, \\
\nabla\cdot\varphi = 0 & \text{in} \quad Q, \\
-\psi_t - \Delta\psi - \overline{\mathbf{y}}\cdot\nabla\psi = \widetilde{g}_1 + \overline{\mathbf{G}}\cdot\varphi + \overline{g_1}\psi + \overline{g_2}\zeta & \text{in} \quad Q, \\
-\zeta_t - \Delta\zeta - \overline{\mathbf{y}}\cdot\nabla\zeta = \widetilde{g}_2 + \overline{\mathbf{L}}\cdot\varphi + \overline{l_1}\psi + \overline{l_2}\zeta & \text{in} \quad Q, \\
\varphi = \mathbf{0},\ \psi = \zeta = 0 & \text{on} \quad \Sigma, \\
\varphi(T) = \varphi_0,\ \psi(T) = \psi_0,\ \zeta(T) = \zeta_0 & \text{in} \quad \Omega,
\end{cases}
\tag{4.15}
$$

where $D\varphi = \nabla\varphi + (\nabla\varphi)^T$ denotes the symmetric part of the gradient of φ. Here, the final and right hand side data are assumed to satisfy:

$$
\varphi_0 \in \mathbf{H},\ \psi_0,\ \zeta_0 \in L^2(\Omega),\ (\widetilde{\mathbf{G}})_i, \widetilde{g}_1, \widetilde{g}_2 \in L^2(Q) \quad (1 \le i \le N).
$$

Let us introduce the following notation:

$$
\begin{aligned}
I(s,\lambda;g) = \ & s^{-1}\iint_Q e^{-2s\alpha}\xi^{-1}|g_t|^2 + s^{-1}\iint_Q e^{-2s\alpha}\xi^{-1}|\Delta g|^2 \\
& + s\lambda^2\iint_Q e^{-2s\alpha}\xi|\nabla g|^2 + s^3\lambda^4\iint_Q e^{-2s\alpha}\xi^3|g|^2
\end{aligned}
$$

for any $s, \lambda > 0$ and for any function $g = g(x,t)$ such that these integrals of g make sense. Let us also set

$$
K(\varphi,\psi,\zeta) = I(s,\lambda;\varphi) + I(s,\lambda;\psi) + I(s,\lambda;\zeta). \tag{4.16}
$$

For the moment, we will accept the following proposition, whose proof is sketched in the Appendix:

Proposition 12. *Assume that* $(\overline{\mathbf{y}},\, \overline{p},\, \overline{\theta},\, \overline{c})$ *satisfies* (4.4)–(4.6). *There exist positive constants* $\hat{s}$, $\hat{\lambda}$ *and* $\widehat{C}$, *only depending on* Ω *and* $\mathcal{O}$ *such that, for any* $(\varphi_0,\, \psi_0,\, \zeta_0) \in \mathbf{H} \times L^2(\Omega) \times L^2(\Omega)$ *and any* $(\widetilde{\mathbf{G}}, \widetilde{g}_1, \widetilde{g}_2) \in \mathbf{L}^2(Q) \times L^2(Q) \times L^2(Q)$, *the solution to the adjoint system* (4.15) *satisfies:*

$$\begin{aligned} K(\varphi,\psi,\zeta) \leq\ & \widehat{C}(1+T^2)\Bigg(s^{\frac{15}{2}}\lambda^{24} \iint_Q e^{-4s\hat{\alpha}+2s\alpha^*} \xi^{*\frac{15}{2}} (|\mathbf{G}|^2 + |\widetilde{g}_1|^2 + |\widetilde{g}_2|^2) \\ & + s^{16}\lambda^{48} \iint_{\mathcal{O}\times(0,T)} e^{-8s\hat{\alpha}+6s\alpha^*} \xi^{*16} (|\varphi|^2 + |\psi|^2 + |\zeta|^2) \Bigg) \end{aligned} \tag{4.17}$$

for any $s \geq \hat{s}(T^4 + T^8)$ *and any*

$$\begin{aligned} \lambda \ \geq\ & \hat{\lambda}\Big(1 + \|\overline{\mathbf{y}}\|_\infty + \|\overline{\theta}\|_\infty + \|\overline{c}\|_\infty + \|\overline{\mathbf{G}}\|_\infty^{1/2} + \|\overline{\mathbf{L}}\|_\infty^{1/2} + \|\overline{g}_1\|_\infty^{1/2} + \|\overline{g}_2\|_\infty^{1/2} \\ & + \|\overline{l}_1\|_\infty^{1/2} + \|\overline{l}_2\|_\infty^{1/2} + \|\overline{\mathbf{y}}_t\|^2_{L^2(0,T;\mathbf{L}^\kappa(\Omega))} + \|\overline{\theta}_t\|^2_{L^2(0,T;L^\kappa(\Omega))} \\ & + \|\overline{c}_t\|^2_{L^2(0,T;L^\kappa(\Omega))} + \exp\left\{\hat{\lambda}T(1 + \|\overline{\mathbf{y}}\|^2_\infty + \|\overline{\theta}\|^2_\infty + \|\overline{c}\|^2_\infty)\right\}\Big). \end{aligned}$$

4.3 Proof of theorem 15

Without any lack of generality, we can assume that $N = 3$ and $\mathbf{h}_2 \neq 0$. In order to prove the result, we have to establish some new Carleman estimates. The first one is given in the following result:

Lemma 23. *Assume that* $(\overline{\mathbf{y}},\ \overline{p},\ \overline{\theta},\ \overline{c})$ *satisfies* (4.4)–(4.6). *Under the assumptions of theorem* 15, *there exist positive constants* C, $\overline{\alpha}$ *and* $\tilde{\alpha}$ *only depending on* Ω, $\mathcal{O}$, T, $\overline{\mathbf{y}}$, $\overline{\theta}$ *and* $\overline{c}$ *and satisfying* $0 < \tilde{\alpha} < \overline{\alpha}$ *and* $16\tilde{\alpha} - 15\overline{\alpha} > 0$ *such that, for any* $(\varphi_0, \psi_0, \zeta_0) \in \mathbf{H} \times L^2(\Omega) \times L^2(\Omega)$ *and any* $(\widetilde{\mathbf{G}}, \widetilde{g}_1, \widetilde{g}_2) \in \mathbf{L}^2(Q) \times L^2(Q) \times L^2(Q)$, *the solution to the adjoint system* (4.15) *satisfies:*

$$\begin{aligned} K(\varphi,\psi,\zeta) \ \leq\ & C\Bigg(\iint_Q e^{\frac{-4\tilde{\alpha}+2\overline{\alpha}}{t^4(T-t)^4}} t^{-30}(T-t)^{-30}|\widetilde{\mathbf{G}}|^2) \\ & + \iint_Q e^{\frac{-32\tilde{\alpha}+30\overline{\alpha}}{t^4(T-t)^4}} t^{-116}(T-t)^{-116}(|\widetilde{g}_1|^2 + |\widetilde{g}_2|^2) \\ & + \iint_{\mathcal{O}\times(0,T)} e^{\frac{-8\tilde{\alpha}+6\overline{\alpha}}{t^4(T-t)^4}} t^{-64}(T-t)^{-64}|\varphi_1|^2 \\ & + \iint_{\mathcal{O}\times(0,T)} e^{\frac{-32\tilde{\alpha}+30\overline{\alpha}}{t^4(T-t)^4}} t^{-132}(T-t)^{-132}(|\psi|^2 + |\zeta|^2). \end{aligned} \tag{4.18}$$

Demonstração. By choosing

$$\overline{\alpha} = s_0(e^{5/4\lambda_1 m\|\eta^0\|_\infty} - e^{\lambda_1 m\|\eta^0\|_\infty}), \quad \widetilde{\alpha} = s_0(e^{5/4\lambda_1 m\|\eta^0\|_\infty} - e^{\lambda_1(m+1)\|\eta^0\|_\infty}),$$

$$C_1 = \widehat{C}(1+T^2)s_1^{17}\lambda_1^{48}e^{17\lambda_1(m+1)\|\eta^0\|_\infty}$$

and $\omega \subset\subset \mathcal{O}$, we see from (4.17) that

$$\begin{aligned} &\iint_Q e^{\frac{-2\overline{\alpha}}{t^4(T-t)^4}} t^4(T-t)^4(|\varphi_t|^2+|\psi_t|^2+|\zeta_t|^2+|\Delta\varphi|^2+|\Delta\psi|^2+|\Delta\zeta|^2) \\ &+\iint_Q e^{\frac{-2\overline{\alpha}}{t^4(T-t)^4}} t^{-4}(T-t)^{-4}(|\nabla\varphi|^2+|\nabla\psi|^2+|\nabla\zeta|^2) \\ &+\iint_Q e^{\frac{-2\overline{\alpha}}{t^4(T-t)^4}} t^{-12}(T-t)^{-12}(|\varphi|^2+|\psi|^2+|\zeta|^2) \\ &\leq C_1\Bigg(\iint_Q e^{\frac{-4\widetilde{\alpha}+2\overline{\alpha}}{t^4(T-t)^4}} t^{-30}(T-t)^{-30}(|\widetilde{\mathbf{G}}|^2+|\widetilde{g}_1|^2+|\widetilde{g}_2|^2) \\ &+\iint_{\omega\times(0,T)} e^{\frac{-8\widetilde{\alpha}+6\overline{\alpha}}{t^4(T-t)^4}} t^{-64}(T-t)^{-64}(|\varphi|^2+|\psi|^2+|\zeta|^2)\Bigg). \end{aligned} \tag{4.19}$$

Notice that $0<\widetilde{\alpha}<\overline{\alpha}$. Moreover, by taking λ_1 large enough, it can be assumed that $16\widetilde{\alpha}-15\overline{\alpha}>0$.

Since $\mathbf{h}_2 \neq 0$, we have

$$|\varphi_1|^2+|\varphi_2|^2+|\varphi_3|^2 \leq C_1(|\varphi_1|^2+|\varphi\cdot\overrightarrow{\mathbf{h}}|^2+|\varphi_3|^2). \tag{4.20}$$

Thus, by combining (4.20) with (4.19), the task is reduced to prove an estimate of the integrals

$$I_3 := \iint_{\omega\times(0,T)} e^{\frac{-8\widetilde{\alpha}+6\overline{\alpha}}{t^4(T-t)^4}} t^{-64}(T-t)^{-64}|\varphi_3|^2 \tag{4.21}$$

and

$$I_{\mathbf{h}} := \iint_{\omega\times(0,T)} e^{\frac{-8\widetilde{\alpha}+6\overline{\alpha}}{t^4(T-t)^4}} t^{-64}(T-t)^{-64}|\varphi\cdot\overrightarrow{\mathbf{h}}|^2 \tag{4.22}$$

of the form

$$I_3+I_{\mathbf{h}} \leq \varepsilon K(\varphi,\psi,\zeta)+C\varepsilon(\dots),$$

where the dots contain local integrals of ψ, ζ, $\widetilde{g}_1$ and $\widetilde{g}_2$.

To do this, let us set

$$\beta(t) = e^{\frac{-8\widetilde{\alpha}+6\overline{\alpha}}{t^4(T-t)^4}} t^{-64}(T-t)^{-64} \quad \forall t\in(0,T)$$

and let us introduce a cut-off function $\vartheta \in C_0^2(\mathcal{O})$ such that

$$\vartheta \equiv 1 \text{ in } \omega,\ 0\leq\vartheta\leq 1.$$

For instance, from the differential equation satisfied by ζ, see (4.15), we have:

$$\begin{aligned} \iint_{\omega\times(0,T)} \beta|\varphi\cdot\overrightarrow{\mathbf{h}}|^2 &\leq \iint_{\mathcal{O}\times(0,T)} \beta\vartheta(\varphi\cdot\overrightarrow{\mathbf{h}})(-\zeta_t-\Delta\zeta-\overline{\mathbf{y}}\cdot\nabla\zeta-\widetilde{g}_2) \\ &= C\iint_{\mathcal{O}\times(0,T)} \overrightarrow{\mathbf{h}}\cdot[\beta\vartheta\varphi(-\zeta_t-\Delta\zeta-\overline{\mathbf{y}}\cdot\nabla\zeta-\widetilde{g}_2)]. \end{aligned} \tag{4.23}$$

To get the estimate of $I_{\mathbf{h}}$, we will now perform integrations by parts in the last integral and we will “pass” all the derivatives from ζ to φ:

• First, we integrate by parts with respect to t, taking into account that $\beta(0) = \beta(T) = 0$:

$$\begin{aligned} -C \iint_{\mathcal{O}\times(0,T)} \overrightarrow{\mathbf{h}} \cdot (\beta\vartheta\varphi\zeta_t) &= C \iint_{\mathcal{O}\times(0,T)} \overrightarrow{\mathbf{h}} \cdot (\beta_t\vartheta\varphi\zeta + \beta\vartheta\varphi_t\zeta) \\ &\leq \varepsilon K(\varphi,\psi,\zeta) \\ &\quad + C(\varepsilon) \iint_{\mathcal{O}\times(0,T)} e^{\frac{-32\widetilde{\alpha}+30\overline{\alpha}}{t^4(T-t)^4}} t^{-132}(T-t)^{-132}|\zeta|^2. \end{aligned} \tag{4.24}$$

• Next, we integrate by parts twice with respect to x. Here, we use the properties of the cut-off function ϑ:

$$\begin{aligned} & -C \iint_{\mathcal{O}\times(0,T)} \overrightarrow{\mathbf{h}} \cdot (\beta\vartheta\varphi\Delta\zeta) = -C \iint_{\mathcal{O}\times(0,T)} \overrightarrow{\mathbf{h}} \cdot [\beta\Delta(\vartheta\varphi)\zeta] \\ & \quad + C \iint_{\partial\mathcal{O}\times(0,T)} \overrightarrow{\mathbf{h}} \cdot [\beta\zeta\partial_{\mathbf{n}}(\vartheta\varphi) - \beta\vartheta\varphi\partial_{\mathbf{n}}\zeta]\, dS\, dt \\ & = C \iint_{\mathcal{O}\times(0,T)} \overrightarrow{\mathbf{h}} \cdot \{\beta[-\Delta\vartheta\varphi - 2(\nabla\vartheta\cdot\nabla\varphi_1, \nabla\vartheta\cdot\nabla\varphi_2, \nabla\vartheta\cdot\nabla\varphi_3) - \vartheta\Delta\varphi]\zeta\} \\ & \leq \varepsilon K(\varphi,\psi,\zeta) + C(\varepsilon) \iint_{\mathcal{O}\times(0,T)} e^{\frac{-32\widetilde{\alpha}+30\overline{\alpha}}{t^4(T-t)^4}} t^{-132}(T-t)^{-132}|\zeta|^2. \end{aligned} \tag{4.25}$$

• We also integrate by parts in the third term with respect to x and we use the incompressibility condition on $\overline{\mathbf{y}}$:

$$-C \iint_{\mathcal{O}\times(0,T)} \overrightarrow{\mathbf{h}} \cdot \ [\beta\vartheta\varphi(\overline{\mathbf{y}}\cdot\nabla\zeta)] \tag{4.26}$$

$$\begin{aligned} &= -C \iint_{\mathcal{O}\times(0,T)} \overrightarrow{\mathbf{h}} \cdot \ \{\beta \sum_{i=1}^{N} [\nabla\cdot(\vartheta\varphi_i\overline{\mathbf{y}}\zeta) - [\nabla(\vartheta\varphi_i)\cdot\overline{\mathbf{y}}]\zeta - \vartheta\varphi(\nabla\cdot\overline{\mathbf{y}})\zeta]\,\mathbf{e}_i\} \\ &\leq \varepsilon K(\varphi,\psi,\zeta) + C(\varepsilon) \iint_{\mathcal{O}\times(0,T)} e^{\frac{-32\widetilde{\alpha}+30\overline{\alpha}}{t^4(T-t)^4}} t^{-132}(T-t)^{-132}|\zeta|^2. \end{aligned}$$

• We finally apply Young's inequality in the last term and we get:

$$\begin{aligned} -\iint_{\mathcal{O}\times(0,T)} \overrightarrow{\mathbf{h}} \cdot (\beta\vartheta\varphi\widetilde{g}_2) \ &\leq \ \varepsilon K(\varphi,\psi,\zeta) \\ &\quad + C(\varepsilon) \iint_{\mathcal{O}\times(0,T)} e^{\frac{-32\widetilde{\alpha}+30\overline{\alpha}}{t^4(T-t)^4}} t^{-116}(T-t)^{-116}|\widetilde{g}_2|^2. \end{aligned} \tag{4.27}$$

From (4.19) and (4.23)-(4.27), by choosing $\varepsilon > 0$ sufficiently small, it is easy to deduce the desired estimate of (4.22). We can argue in the same way starting from (4.21) and the equation satisfied by ψ, which leads to a similar estimate.

Finally, putting all these inequalities together, we find (4.18). $\square$

We will now deduce a second Carleman inequality with weights that do not vanish at $t = 0$. More precisely, let us consider the function

$$l(t) = \begin{cases} T^2/4 & \text{for} \quad 0 \leq t \leq T/2, \\ t(T-t) & \text{for} \quad T/2 \leq t \leq T \end{cases}$$

and the following associated weight functions:

$$\begin{aligned} &\beta_1 = e^{\frac{\overline{\alpha}}{[l(t)]^4}}[l(t)]^2, \quad \beta_2(t) = e^{\frac{\overline{\alpha}}{[l(t)]^4}}[l(t)]^6, \quad \beta_3(t) = e^{\frac{2\tilde{\alpha}-\overline{\alpha}}{[l(t)]^4}}[l(t)]^{15}, \\ &\beta_4(t) = e^{\frac{16\tilde{\alpha}-15\overline{\alpha}}{[l(t)]^4}}[l(t)]^{58}, \quad \beta_5(t) = e^{\frac{4\tilde{\alpha}-3\overline{\alpha}}{[l(t)]^4}}[l(t)]^{32} \quad \text{and} \quad \beta_6(t) = e^{\frac{16\tilde{\alpha}-15\overline{\alpha}}{l(t)[l(t)]^4}}[l(t)]^{66}. \end{aligned}$$

By combining lemma 23 and the classical energy estimates satisfied by φ, ψ and ζ, we easily deduce the following:

Lemma 24. *Assume that* $(\overline{\mathbf{y}},\ \overline{p},\ \overline{\theta},\ \overline{c})$ *satisfies* (4.4)–(4.6). *Under the assumptions of theorem* 15, *there exist positive constants* C, $\overline{\alpha}$ *and* $\tilde{\alpha}$ *depending on* Ω, $\mathcal{O}$, T, $\overline{\mathbf{y}}$, $\overline{\theta}$ *and* $\overline{c}$ *and satisfying* $0 < \tilde{\alpha} < \overline{\alpha}$ *and* $16\tilde{\alpha} - 15\overline{\alpha} > 0$ *such that, for any* $(\varphi_0, \psi_0, \zeta_0) \in \mathbf{H} \times L^2(\Omega) \times L^2(\Omega)$ *and any* $(\widetilde{\mathbf{G}}, \widetilde{g}_1, \widetilde{g}_2) \in \mathbf{L}^2(Q) \times L^2(Q) \times L^2(Q)$, *the solution to the adjoint system* (4.15) *satisfies:*

$$\begin{aligned} &\iint_Q \left[\beta_1^{-2}(|\nabla\varphi|^2 + |\nabla\psi|^2 + |\nabla\zeta|^2) + \beta_2^{-2}(|\varphi|^2 + |\psi|^2 + |\zeta|^2)\right] \\ &\quad + \|\varphi(0)\|^2_{\mathbf{L}^2(\Omega)} + \|\psi(0)\|^2_{L^2(\Omega)} + \|\zeta(0)\|^2_{L^2(\Omega)} \\ &\quad \leq C\Bigg(\iint_Q \left[\beta_3^{-2}|\widetilde{\mathbf{G}}|^2 + \beta_4^{-2}(|\widetilde{g}_1|^2 + |\widetilde{g}_2|^2)\right] \\ &\qquad + \iint_{\mathcal{O}\times(0,T)} \left[\beta_5^{-2}|\varphi_1|^2 + \beta_6^{-2}(|\psi|^2 + |\zeta|^2)\right]\Bigg). \end{aligned} \tag{4.28}$$

The next step is to prove the null controllability of the linear system (4.13). Of course, we will need some specific conditions on the data $\mathbf{S}$, r_1 and r_2. Thus, let us introduce the linear operators M_i, with

$$M_1(\mathbf{u}) = \mathbf{u}_t - \Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\overline{\mathbf{y}} + (\overline{\mathbf{y}}\cdot\nabla)\mathbf{u}, \quad M_2(\phi) = \phi_t - \Delta\phi + \mathbf{u}\cdot\nabla\overline{\theta} \tag{4.29}$$

and

$$M_3(z) = z_t - \Delta z + \mathbf{u}\cdot\nabla\overline{c}$$

and the spaces

$$\begin{aligned} E_0 \ = \ &\{\, (\mathbf{u}, \phi, z, \mathbf{v}, w_1, w_2) : \beta_3\mathbf{u} \in \mathbf{L}^2(Q),\ \beta_4\phi, \beta_4 z \in L^2(Q), \\ &\beta_5\mathbf{v}1_{\mathcal{O}} \in \mathbf{L}^2(Q),\ \beta_6 w_1 1_{\mathcal{O}}, \beta_6 w_2 1_{\mathcal{O}} \in L^2(Q), \\ &\mathbf{v}_2 \equiv \mathbf{v}_3 \equiv 0,\ \beta_1^{1/2}\mathbf{u} \in L^2(0,T;\mathbf{V}) \cap L^\infty(0,T;\mathbf{H}), \\ &\beta_1^{1/2}\phi,\ \beta_1^{1/2}z \in L^2(0,T;H_0^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega)) \,\} \end{aligned}$$

and

$$
\begin{aligned}
E_3 \;=\; \{\, & (\mathbf{u}, q, \phi, z, \mathbf{v}, w_1, w_2) : (\mathbf{u}, \phi, z, \mathbf{v}, w_1, w_2) \in E_0, \\
& \beta_1^{1/2}\mathbf{u} \in L^4(0,T;\mathbf{L}^{12}(\Omega)),\ q \in L^2_{loc}(Q), \\
& \beta_1(M_1\mathbf{u} + \nabla q - \phi \mathbf{e}_N - z\overrightarrow{\mathbf{h}} - \mathbf{v}1_{\mathcal{O}}) \in L^2(0,T;\mathbf{W}^{-1,6}(\Omega)), \\
& \beta_1(M_2\phi + \overline{\mathbf{y}}\cdot\nabla\phi - w_1 1_{\mathcal{O}}) \in L^2(0,T;H^{-1}(\Omega)), \\
& \beta_1(M_3 z + \overline{\mathbf{y}}\cdot\nabla z - w_2 1_{\mathcal{O}}) \in L^2(0,T;H^{-1}(\Omega)) \,\}.
\end{aligned}
$$

It is clear that E_0 and E_3 are Banach spaces for the norms $\|\cdot\|_{E_0}$ and $\|\cdot\|_{E_3}$, where

$$
\begin{aligned}
\|(\mathbf{u}, \phi, z, \mathbf{v}, w_1, w_2)\|_{E_0} = \big(& \|\beta_3\mathbf{u}\|^2_{\mathbf{L}^2(Q)} + \|\beta_4\phi\|^2_{L^2(Q)} + \|\beta_4 z\|^2_{L^2(Q)} \\
& + \|\beta_5\mathbf{v}1_{\mathcal{O}}\|^2_{\mathbf{L}^2(Q)} + \|\beta_6 w_1 1_{\mathcal{O}}\|^2_{L^2(Q)} + \|\beta_6 w_2 1_{\mathcal{O}}\|^2_{L^2(Q)} \\
& + \|\beta_1^{1/2}\mathbf{u}\|^2_{L^2(0,T;\mathbf{V})} + \|\beta_1^{1/2}\mathbf{u}\|^2_{L^\infty(0,T;\mathbf{H})} \\
& + \|\beta_1^{1/2}\phi\|^2_{L^2(0,T;H^1_0(\Omega))} + \|\beta_1^{1/2}\phi\|^2_{L^\infty(0,T;L^2(\Omega))} \\
& + \|\beta_1^{1/2}z\|^2_{L^2(0,T;H^1_0(\Omega))} + \|\beta_1^{1/2}z\|^2_{L^\infty(0,T;L^2(\Omega))}\big)^{1/2}
\end{aligned}
$$

and

$$
\begin{aligned}
\|(\mathbf{u}, q, \phi, z, \mathbf{v}, w_1, w_2)\|_{E_3} = \big(& \|(\mathbf{u}, \phi, z, \mathbf{v}, w_1, w_2)\|^2_{E_0} + \|\beta_1^{1/2}\mathbf{u}\|^2_{L^4(0,T;\mathbf{L}^{12}(\Omega))} \\
& + \|\beta_1(M_1\mathbf{u} + \nabla q - \phi\mathbf{e}_N - z\overrightarrow{\mathbf{h}} - \mathbf{v}1_{\mathcal{O}})\|^2_{L^2(0,T;\mathbf{W}^{-1,6}(\Omega))} \\
& + \|\beta_1(M_2\phi + \overline{\mathbf{y}}\cdot\nabla\phi - w_1 1_{\mathcal{O}})\|^2_{L^2(0,T;H^{-1}(\Omega))} \\
& + \|\beta_1(M_3 z + \overline{\mathbf{y}}\cdot\nabla z - w_2 1_{\mathcal{O}})\|^2_{L^2(0,T;H^{-1}(\Omega))}\big)^{1/2}.
\end{aligned}
$$

Proposition 13. *Assume that* $(\overline{\mathbf{y}}, \overline{p}, \overline{\theta}, \overline{c})$ *satisfies* (4.4)–(4.6). *Also, assume that* $\mathbf{u}_0 \in \mathbf{E}$, $\phi_0, z_0 \in L^2(\Omega)$ *and*

$$\beta_1(\mathbf{S}, r_1, r_2) \in L^2(0,T;\mathbf{W}^{-1,6}(\Omega)) \times L^2(0,T;H^{-1}(\Omega)) \times L^2(0,T;H^{-1}(\Omega)).$$

Then, there exist controls $\mathbf{v}$*,* w_1 *and* w_2 *such that the associated solution to* (4.13) *belongs to* E_3*. In particular,* $\mathbf{v}_2 \equiv \mathbf{v}_3 \equiv 0$*,* $\mathbf{u}(T)) = \mathbf{0}$ *and* $\phi(T) = z(T) = 0$*.*

Demonstração. We will follow the general method introduced and used in [40] for linear parabolic scalar problems.

Thus, let us introduce the auxiliary extremal problem

$$
\left\{
\begin{array}{l}
\text{Minimize } \ J(\mathbf{u}, q, \phi, z, \mathbf{v}, w_1, w_2) \\
\text{Subject to } \ \mathbf{v} \in \mathbf{L}^2(Q),\ w_1,\ w_2 \in L^2(Q), \\
\qquad\qquad \text{supp}(\mathbf{v}), \text{supp}(w_1), \text{supp}(w_2) \subset \mathcal{O}\times(0,T),\ \mathbf{v}_2 \equiv \mathbf{v}_3 \equiv 0 \ \text{ and} \\
\left\{
\begin{array}{lll}
M_1(\mathbf{u}) + \nabla q = \mathbf{S} + \mathbf{v}1_{\mathcal{O}} + \phi\mathbf{e}_N + z\overrightarrow{\mathbf{h}} & \text{in} & Q, \\
\nabla\cdot\mathbf{u} = 0 & \text{in} & Q, \\
M_2(\phi) + \overline{\mathbf{y}}\cdot\nabla\phi = r_1 + w_1 1_{\mathcal{O}} & \text{in} & Q, \\
M_3(z) + \overline{\mathbf{y}}\cdot\nabla z = r_2 + w_2 1_{\mathcal{O}} & \text{in} & Q, \\
\mathbf{u} = \mathbf{0}, \phi = z = 0 & \text{on} & \Sigma, \\
\mathbf{u}(0) = \mathbf{u}_0, \phi(0) = \phi_0, z(0) = z_0 & \text{in} & \Omega.
\end{array}
\right.
\end{array}
\right.
\tag{4.30}
$$

Here, we have used the notation

$$J(\mathbf{u}, q, \phi, z, \mathbf{v}, w_1, w_2) = \frac{1}{2}\left\{\iint_Q \left[\beta_3^{-2}|\mathbf{u}|^2 + \beta_4^{-2}(|\phi|^2 + |z|^2)\right]\right.$$
$$\left. + \iint_{\mathcal{O}\times(0,T)} \left[\beta_5^{-2}|\mathbf{v}|^2 + \beta_6^{-2}(|w_1|^2 + |w_2|^2)\right]\right\}.$$

Observe that a solution $(\widehat{\mathbf{u}}, \widehat{q}, \widehat{\phi}, \widehat{z}, \widehat{\mathbf{v}}, \widehat{w_1}, \widehat{w_2})$ to (4.30) is a good candidate to satisfy $(\widehat{\mathbf{u}}, \widehat{q}, \widehat{\phi}, \widehat{z}, \widehat{\mathbf{v}}, \widehat{w_1}, \widehat{w_2}) \in E_3$.

Let us suppose for the moment that $(\widehat{\mathbf{u}}, \widehat{q}, \widehat{\phi}, \widehat{z}, \widehat{\mathbf{v}}, \widehat{w_1}, \widehat{w_2})$ solves (4.30). Then, in view of *Lagrange's multipliers theorem,* there must exist dual variables $\widehat{\varphi}$, $\widehat{\pi}$, $\widehat{\psi}$ and $\widehat{\zeta}$ such that

$$\begin{cases} \widehat{\mathbf{u}} = \beta_3^{-2}(M_1^*\widehat{\varphi} + \nabla\widehat{\pi} - \overline{\theta}\nabla\widehat{\psi} - \overline{c}\nabla\widehat{\zeta}), \quad \nabla\cdot\widehat{\varphi} = 0 & \text{in } Q, \\ \widehat{\phi} = \beta_4^{-2}(M_2^*\widehat{\psi} - \widehat{\varphi}\cdot\mathbf{e}_N) & \text{in } Q, \\ \widehat{z} = \beta_4^{-2}(M_3^*\widehat{\zeta} - \widehat{\varphi}\cdot\overrightarrow{\mathbf{h}}) & \text{in } Q, \\ \widehat{\mathbf{v}} = -\beta_5^{-2}\widehat{\varphi},\ \widehat{w}_1 = -\beta_6^{-2}\widehat{\psi},\ \widehat{w}_2 = -\beta_6^{-2}\widehat{\zeta}, & \text{in } \mathcal{O}\times(0,T), \\ \widehat{\varphi} = \mathbf{0},\ \widehat{\psi} = \widehat{\zeta} = 0 & \text{on } \Sigma, \end{cases} \tag{4.31}$$

where M_i^* is the adjoint operator of M_i $(i = 1, 2, 3)$, i.e.,

$$M_1^*\varphi = -\varphi_t - \Delta\varphi - D\varphi\overline{\mathbf{y}}, \quad M_2^*\psi = -\psi_t - \Delta\psi - \overline{\mathbf{y}}\cdot\nabla\psi$$

and

$$M_3^*\zeta = -\zeta_t - \Delta\zeta - \overline{\mathbf{y}}\cdot\nabla\zeta.$$

Let us introduce the linear space

$$P_0 = \left\{ (\varphi, \pi, \psi, \zeta) \in \mathbf{C}^2(\overline{Q}) : \nabla\cdot\varphi = 0 \text{ in } Q,\ \varphi|_\Sigma = \mathbf{0}, \right.$$
$$\left. \psi|_\Sigma = \zeta|_\Sigma = 0,\ \int_{\mathcal{O}} \pi(x,t) = 0 \right\},$$

the bilinear form

$$b((\widehat{\varphi}, \widehat{\pi}, \widehat{\psi}, \widehat{\zeta}), (\varphi, \pi, \psi, \zeta))$$
$$= \iint_Q \beta_3^{-2}(M_1^*\widehat{\varphi} + \nabla\widehat{\pi} - \overline{\theta}\nabla\widehat{\psi} - \overline{c}\nabla\widehat{\zeta})\cdot(M_1^*\varphi + \nabla\pi - \overline{\theta}\nabla\psi - \overline{c}\nabla\zeta)$$
$$+ \iint_Q \beta_4^{-2}(M_2^*\widehat{\psi} - \widehat{\varphi}\cdot\mathbf{e}_N)(M_2^*\psi - \varphi\cdot\mathbf{e}_N)$$
$$+ \iint_Q \beta_4^{-2}(M_3^*\widehat{\zeta} - \widehat{\varphi}\cdot\overrightarrow{\mathbf{h}})(M_3^*\zeta - \varphi\cdot\overrightarrow{\mathbf{h}})$$
$$+ \iint_{\mathcal{O}\times(0,T)} (\beta_5^{-2}\widehat{\varphi}_2\cdot\varphi_2 + \beta_6^{-2}\widehat{\psi}\,\psi + \beta_6^{-2}\widehat{\zeta}\,\zeta)$$

and the linear form

$$\langle l_0,(\varphi,\pi,\psi,\zeta)\rangle = \int_0^T \langle \mathbf{S},\varphi\rangle_{\mathbf{H}^{-1},\mathbf{H}_0^1}\,dt + \int_0^T \langle r_1,\psi\rangle_{H^{-1},H_0^1}\,dt$$
$$+\int_0^T \langle r_2,\zeta\rangle_{H^{-1},H_0^1}\,dt + \int_\Omega \mathbf{u}_0\cdot\varphi(0)\,dx$$
$$+\int_\Omega \phi_0\,\psi(0)\,dx + \int_\Omega z_0\,\zeta(0)\,dx.$$

Then we must have

$$b((\widehat{\varphi},\widehat{\pi},\widehat{\psi},\widehat{\zeta}),(\varphi,\pi,\psi,\zeta)) = \langle l_0,(\varphi,\pi,\psi,\zeta)\rangle \quad \forall(\varphi,\pi,\psi,\zeta)\in P_0, \tag{4.32}$$

i.e., the solution to (4.30) satisfies (4.32).

Conversely, if we are able to solve (4.32) in some sense and then use (4.31) to define $(\widehat{\mathbf{u}},\widehat{q},\widehat{\phi},\widehat{z},\widehat{\mathbf{v}},\widehat{w_1},\widehat{w_2})$, we will have probably found a solution to (4.30).

It is clear that $b(\cdot\,,\cdot) : P_0\times P_0\to\mathbb{R}$ is a symmetric, definite positive and bilinear form on P_0, i.e., a scalar product in this linear space. We will denote by P the completion of P_0 for the norm induced by $b(\cdot\,,\cdot)$. Then P is a Hilbert space for $b(\cdot\,,\cdot)$. On the other hand, in view of the Carleman estimate (4.28), the linear form $(\varphi,\pi,\psi,\zeta)\mapsto\langle l_0,(\varphi,\pi,\psi,\zeta)\rangle$ is well-defined and continuous on P. Hence, from *Lax-Milgram's lemma,* we deduce that the variational problem

$$\begin{cases} b((\widehat{\varphi},\widehat{\pi},\widehat{\psi},\widehat{\zeta}),(\varphi,\pi,\psi,\zeta)) = \langle l_0,(\varphi,\pi,\psi,\zeta)\rangle, \\ \forall(\varphi,\pi,\psi,\zeta)\in P, \quad (\widehat{\varphi},\widehat{\pi},\widehat{\psi},\widehat{\zeta})\in P \end{cases} \tag{4.33}$$

possesses exactly one solution.

Let $(\widehat{\varphi},\widehat{\pi},\widehat{\psi},\widehat{\zeta})$ be the unique solution to (4.33) and let $\widehat{\mathbf{u}},\widehat{\phi},\widehat{z},\widehat{\mathbf{v}},\widehat{w_1}$ and $\widehat{w_2}$ be given by (4.31). Then, it is readily seen that

$$\iint_Q [\beta_3^2|\widehat{\mathbf{u}}|^2 + \beta_4^2(|\widehat{\phi}|^2+|\widehat{z}|^2)] + \iint_{\mathcal{O}\times(0,T)} [\beta_5|\widehat{\mathbf{v}}|^2 + \beta_6^2(|\widehat{w}_1|^2+|\widehat{w}_2|^2)] < +\infty \tag{4.34}$$

and, also, that $\widehat{\mathbf{u}},\widehat{\phi},\widehat{z}$ is, together with some $\widehat{q}$, the unique weak solution (belonging to $L^2(0,T;\mathbf{V})\cap L^\infty(0,T;\mathbf{H})\times[L^2(0,T;H_0^1(\Omega))\cap L^\infty(0,T;L^2(\Omega))]^2$) to the system in (4.30) for $\mathbf{v}=\widehat{\mathbf{v}}$, $w_1=\widehat{w_1}$ and $w_2=\widehat{w_2}$.

From the arguments in [40], we also see that $(\widehat{\mathbf{u}},\widehat{q},\widehat{\phi},\widehat{z},\widehat{\mathbf{v}},\widehat{w_1},\widehat{w_2})\in E_3$ and, consequently, the proof is achieved. □

We can now end the proof of theorem 15.

We will use the following inverse mapping theorem (see [1]):

Theorem 18. *Let B and G be two Banach spaces and let $\mathcal{A}: B\mapsto G$ satisfy $\mathcal{A}\in C^1(B;G)$. Assume that $e_0\in B$, $\mathcal{A}(e_0)=w_0$ and $\mathcal{A}'(e_0): B\mapsto G$ is surjective. Then there exists $\delta>0$ such that, for every $w\in G$ satisfying $||w-w_0||_G<\delta$, there exists a solution to the equation*

$$\mathcal{A}(e) = w, \quad e\in B.$$

We will apply this result with $B = E_3$, $G = G_1 \times G_2$ and

$$\mathcal{A}(e) = (\mathcal{A}_1(\mathbf{u}, q, \phi, z, \mathbf{v}), \mathcal{A}_2(\mathbf{u}, \phi, w_1), \mathcal{A}_3(\mathbf{u}, z, w_2), \mathbf{u}(0), \phi(0), z(0))$$

for any $e = (\mathbf{u}, q, \phi, z, \mathbf{v}, w_1, w_2) \in E_3$, where

$$\begin{aligned} &G_1 = L^2(\beta_1(0,T); \mathbf{W}^{-1,6}(\Omega)) \times L^2(\beta_1(0,T); H^{-1}(\Omega)) \times L^2(\beta_1(0,T); H^{-1}(\Omega)), \\ &G_2 = (\mathbf{L}^4(\Omega) \cap \mathbf{H}) \times L^2(\Omega) \times L^2(\Omega) \end{aligned} \tag{4.35}$$

and

$$\begin{aligned} \mathcal{A}_1(\mathbf{u}, q, \phi, z, \mathbf{v}) &= M_1\mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla q - \mathbf{v}1_{\mathcal{O}} - \phi\mathbf{e}_N - z\mathbf{h}, \\ \mathcal{A}_2(\mathbf{u}, \phi, w_1) &= M_2\phi + \overline{\mathbf{y}} \cdot \nabla\phi + \mathbf{u} \cdot \nabla\phi - w_1 1_{\mathcal{O}} \\ \mathcal{A}_3(\mathbf{u}, z, w_2) &= M_3 z + \overline{\mathbf{y}} \cdot \nabla z + \mathbf{u} \cdot \nabla z - w_2 1_{\mathcal{O}}, \end{aligned} \tag{4.36}$$

for all $(\mathbf{u}, q, \phi, z, \mathbf{v}, w_1, w_2) \in E_3$.

It is not difficult to check that $\mathcal{A}$ is bilinear and satisfies $\mathcal{A} \in C^1(B, G)$. Let e_0 be the origin of B. Notice that $\mathcal{A}'(\mathbf{0}, 0, 0, 0, \mathbf{0}, 0, 0) : B \mapsto G$ is the mapping that, to each $(\mathbf{u}, q, \phi, z, \mathbf{v}, w_1, w_2) \in B$, associates the function in G whose components are

$$\begin{aligned} &M_1\mathbf{u} + \nabla q - \mathbf{v}1_{\mathcal{O}} - \phi\mathbf{e}_N - z\mathbf{h}, \\ &M_2\phi + \overline{\mathbf{y}} \cdot \nabla\phi - w_1 1_{\mathcal{O}}, \\ &M_3 z + \overline{\mathbf{y}} \cdot \nabla z - w_2 1_{\mathcal{O}} \end{aligned}$$

and the initial values $\mathbf{u}(0)$, $\phi(0)$ and $z(0)$. In view of the null controllability result for (4.13) given in proposition 13, $\mathcal{A}'(\mathbf{0}, 0, 0, 0, \mathbf{0}, 0, 0)$ is surjective.

Consequently, we can indeed apply theorem 18 with these data and, in particular, there exists $\delta > 0$ such that, if

$$\left\|(\mathbf{0}, 0, 0, \mathbf{y}_0 - \overline{\mathbf{y}}_0, \theta_0 - \overline{\theta}_0, c_0 - \overline{c}_0)\right\|_G = \left\|(\mathbf{y}_0 - \overline{\mathbf{y}}_0, \theta_0 - \overline{\theta}_0, c_0 - \overline{c}_0)\right\|_{\mathbf{E} \times L^2(\Omega) \times L^2(\Omega)} \le \delta,$$

we can find controls $\mathbf{v}$, w_1 and w_2 with $\mathbf{v}_2 \equiv \mathbf{v}_3 \equiv 0$ and associated solutions to (4.12) that satisfy $\mathbf{u}(T) = \mathbf{0}$, $\phi(T) = 0$ and $z(T) = 0$.

This ends the proof of theorem 15.

4.4 Proof of Theorem 16

Again, it is not restrictive to assume that $N = 3$. We will provisionally impose something stronger than (4.9):

$$\exists \mathbf{k} \in \mathbb{R}^3, \ \exists a_0 > 0 \ \text{ such that } \ \det\left[\,\overline{\mathbf{G}}\,|\,\overline{\mathbf{L}}\,|\,\mathbf{k}\,\right] \ge a_0 \ \text{ in } \ \mathcal{O} \times (0,T). \tag{4.37}$$

We will need a new different Carleman estimate, which is given in the following lemma:

Lemma 25. *Assume that* $(\overline{\mathbf{y}}, \overline{p}, \overline{\theta}, \overline{c})$ *satisfies* (4.4)–(4.6) *and* (4.37) *holds. There exist three positive constants* C, $\overline{\alpha}$ *and* $\tilde{\alpha}$, *depending on* Ω, $\mathcal{O}$, T, $\overline{\mathbf{y}}$, $\overline{\theta}$ *and* $\overline{c}$ *with* $0 < \tilde{\alpha} < \overline{\alpha}$ *and*

$16\widetilde{\alpha}-15\overline{\alpha}>0$ *such that, for any* $(\varphi_0,\psi_0,\zeta_0)\in\mathbf{H}\times L^2(\Omega)\times L^2(\Omega)$ *and any* $(\widetilde{\mathbf{G}},\widetilde{g}_1,\widetilde{g}_2)\in\mathbf{L}^2(Q)\times L^2(Q)\times L^2(Q)$, *the solution to the adjoint system* (4.15) *satisfies:*

$$\begin{aligned}K(\varphi,\psi,\zeta) \;\le\; & C\Big(\iint_Q e^{\frac{-4\widetilde{\alpha}+2\overline{\alpha}}{t^4(T-t)^4}}t^{-30}(T-t)^{-30}|\widetilde{\mathbf{G}}|^2)\\ &+\iint_Q e^{\frac{-32\widetilde{\alpha}+30\overline{\alpha}}{t^4(T-t)^4}}t^{-116}(T-t)^{-116}(|\widetilde{g}_1|^2+|\widetilde{g}_2|^2)\\ &+\iint_{\mathcal{O}\times(0,T)} e^{\frac{-8\widetilde{\alpha}+6\overline{\alpha}}{t^4(T-t)^4}}t^{-64}(T-t)^{-64}|\varphi_1|^2\\ &+\iint_{\mathcal{O}\times(0,T)} e^{\frac{-32\widetilde{\alpha}+30\overline{\alpha}}{t^4(T-t)^4}}t^{-132}(T-t)^{-132}(|\psi|^2+|\zeta|^2)\Big).\end{aligned}\tag{4.38}$$

Demonstração. As in the proof of lemma 23, by choosing

$$\overline{\alpha}=s_0(e^{5/4\lambda_1 m\|\eta^0\|_\infty}-e^{\lambda_1 m\|\eta^0\|_\infty}),\quad \widetilde{\alpha}=s_0(e^{5/4\lambda_1 m\|\eta^0\|_\infty}-e^{\lambda_1(m+1)\|\eta^0\|_\infty}),$$

$$C_1=\widehat{C}(1+T^2)s_1^{17}\lambda_1^{48}e^{17\lambda_1(m+1)\|\eta^0\|_\infty}$$

and $\omega\subset\subset\mathcal{O}$, we see from (4.17) that

$$\begin{aligned}&\iint_Q e^{\frac{-2\overline{\alpha}}{t^4(T-t)^4}}t^4(T-t)^4(|\varphi_t|^2+|\psi_t|^2+|\zeta_t|^2+|\Delta\varphi|^2+|\Delta\psi|^2+|\Delta\zeta|^2)\\ &+\iint_Q e^{\frac{-2\overline{\alpha}}{t^4(T-t)^4}}t^{-4}(T-t)^{-4}(|\nabla\varphi|^2+|\nabla\psi|^2+|\nabla\zeta|^2)\\ &+\iint_Q e^{\frac{-2\overline{\alpha}}{t^4(T-t)^4}}t^{-12}(T-t)^{-12}(|\varphi|^2+|\psi|^2+|\zeta|^2)\\ &\le C_1\Big(\iint_Q e^{\frac{-4\widetilde{\alpha}+2\overline{\alpha}}{t^4(T-t)^4}}t^{-30}(T-t)^{-30}(|\widetilde{\mathbf{G}}|^2+|\widetilde{g}_1|^2+|\widetilde{g}_2|^2)\\ &+\iint_{\omega\times(0,T)} e^{\frac{-8\widetilde{\alpha}+6\overline{\alpha}}{t^4(T-t)^4}}t^{-64}(T-t)^{-64}(|\varphi|^2+|\psi|^2+|\zeta|^2)\Big).\end{aligned}\tag{4.39}$$

Notice that $0<\widetilde{\alpha}<\overline{\alpha}$. Moreover, taking λ_1 large enough, it can be assumed that $16\widetilde{\alpha}-15\overline{\alpha}>0$.

Recall that $\mathbf{F}$ satisfies (4.37). Let us suppose that, for instance, $\mathbf{k}=\mathbf{e}_1$. Then we have:

$$|\varphi_1|^2+|\varphi_2|^2+|\varphi_3|^2\le C_2(|\overline{\mathbf{G}}\cdot\varphi|^2+|\overline{\mathbf{L}}\cdot\varphi|^2+|\varphi_1|^2)\ \text{ in }\ \mathcal{O}\times(0,T)\tag{4.40}$$

for some $C_2>0$. Combining (4.39) and (4.40), we thus see that the task is reduced to estimate the integrals

$$\iint_{\omega\times(0,T)} e^{\frac{-8\widetilde{\alpha}+6\overline{\alpha}}{t^4(T-t)^4}}t^{-64}(T-t)^{-64}|\overline{\mathbf{G}}\cdot\varphi|^2\tag{4.41}$$

and

$$\iint_{\omega\times(0,T)} e^{\frac{-8\widetilde{\alpha}+6\overline{\alpha}}{t^4(T-t)^4}}t^{-64}(T-t)^{-64}|\overline{\mathbf{L}}\cdot\varphi|^2\tag{4.42}$$

in terms of $\varepsilon K(\varphi,\psi,\zeta)$ and a constant C_ε local integrals of ψ, ζ, $\widetilde{g}_1$ and $\widetilde{g}_2$.

These estimates can be obtained by following the final steps of lemma 23; as a result, we obtain the inequality (4.38). □

Let us now give the proof of theorem 16.

First, it is not restrictive to assume that we have (4.37) instead of (4.9). Indeed, if (4.9) holds, since $\overline{\mathbf{G}}$ and $\overline{\mathbf{L}}$ are continuous, there exist $\tau, a_0 > 0$, a non-empty open set $\omega \subset\subset \mathcal{O}_*$ and a vector $\mathbf{k} \in \mathbb{R}^3$ such that

$$\det \left[\overline{\mathbf{G}} \,|\, \overline{\mathbf{L}} \,|\, \mathbf{k} \right] \geq a_0 \ \text{ in } \ \overline{\omega} \times [\tau, T-\tau].$$

We can first take $\mathbf{v} \equiv \mathbf{0}$ and $w_1 \equiv w_2 \equiv 0$ for $t \in [0, \tau]$; then, we can try to get local exact controllability to $(\overline{\mathbf{y}}, \overline{p}, \overline{\theta}, \overline{c})$ at time $T - \tau$. If appropriate controls are found, they serve to prove theorem 16.

Hence, we can assume that (4.37) is satisfied. Arguing as in Section 4.3, we can deduce from lemma 25 the null controllability of the linearized system (4.13) with controls like in theorem 16 (that is, an analog of proposition 13); then, using again the inverse mapping theorem, we can easily achieve the proof of the desired result.

4.5 Proof of Theorem 17

The proof of our third main result, theorem 17, relies on a different and stronger Carleman estimate:

Lemma 26. *Assume that* $N = 3$ *and* $(\overline{\mathbf{y}}, \overline{p}, \overline{\theta}, \overline{c})$ *satisfies* (4.4)–(4.6)*. Under the assumptions of theorem* 17*, there exist three positive constants* C*,* $\overline{\alpha}$ *and* $\tilde{\alpha}$ *depending on* Ω*,* $\mathcal{O}$*,* T*,* $\overline{\mathbf{y}}$*,* $\overline{\theta}$ *and* $\overline{c}$ *satisfying* $0 < \tilde{\alpha} < \overline{\alpha}$ *and* $16\tilde{\alpha} - 15\overline{\alpha} > 0$ *such that, for any* $(\varphi_0, \psi_0, \zeta_0) \in \mathbf{H} \times L^2(\Omega) \times L^2(\Omega)$ *and any* $(\widetilde{\mathbf{G}}, \widetilde{g}_1, \widetilde{g}_2) \in \mathbf{L}^2(Q) \times L^2(Q) \times L^2(Q)$*, the solution to the adjoint system* (4.15) *satisfies:*

$$\begin{aligned} K(\varphi, \psi, \zeta) \;\leq\; & C \left(\iint_Q e^{\frac{-4\tilde{\alpha}+2\overline{\alpha}}{t^4(T-t)^4}} t^{-30}(T-t)^{-30} |\widetilde{\mathbf{G}}|^2 \right. \\ & + \iint_Q e^{\frac{-32\tilde{\alpha}+30\overline{\alpha}}{t^4(T-t)^4}} t^{-252}(T-t)^{-252}(|\widetilde{g}_1|^2 + |\widetilde{g}_2|^2) \\ & \left. + \iint_{\mathcal{O}\times(0,T)} e^{\frac{-32\tilde{\alpha}+30\overline{\alpha}}{t^4(T-t)^4}} t^{-268}(T-t)^{-268}(|\psi|^2 + |\zeta|^2) \right). \end{aligned} \tag{4.43}$$

Demonstração. Again, by choosing

$$\overline{\alpha} = s_0(e^{5/4\lambda_1 m\|\eta^0\|_\infty} - e^{\lambda_1 m\|\eta^0\|_\infty}), \quad \widetilde{\alpha} = s_0(e^{5/4\lambda_1 m\|\eta^0\|_\infty} - e^{\lambda_1 (m+1)\|\eta^0\|_\infty}),$$

$$C_1 = \widehat{C}(1+T^2) s_1^{17} \lambda_1^{48} e^{17\lambda_1 (m+1)\|\eta^0\|_\infty}$$

and $\omega \subset \mathcal{O}$, we obtain:

$$\begin{aligned}
&\iint_Q e^{\frac{-2\overline{\alpha}}{t^4(T-t)^4}} t^4(T-t)^4(|\varphi_t|^2 + |\psi_t|^2 + |\zeta_t|^2 + |\Delta\varphi|^2 + |\Delta\psi|^2 + |\Delta\zeta|^2) \\
&\quad + \iint_Q e^{\frac{-2\overline{\alpha}}{t^4(T-t)^4}} t^{-4}(T-t)^{-4}(|\nabla\varphi|^2 + |\nabla\psi|^2 + |\nabla\zeta|^2) \\
&\quad + \iint_Q e^{\frac{-2\overline{\alpha}}{t^4(T-t)^4}} t^{-12}(T-t)^{-12}(|\varphi|^2 + |\psi|^2 + |\zeta|^2) \\
&\le C_1 \Bigg(\iint_Q e^{\frac{-4\widetilde{\alpha}+2\overline{\alpha}}{t^4(T-t)^4}} t^{-30}(T-t)^{-30}(|\widetilde{\mathbf{G}}|^2 + |\widetilde{g}_1|^2 + |\widetilde{g}_2|^2) \\
&\quad + \iint_{\omega\times(0,T)} e^{\frac{-8\widetilde{\alpha}+6\overline{\alpha}}{t^4(T-t)^4}} t^{-64}(T-t)^{-64}(|\varphi|^2 + |\psi|^2 + |\zeta|^2) \Bigg).
\end{aligned} \tag{4.44}$$

We notice that $0 < \widetilde{\alpha} < \overline{\alpha}$ and, by taking λ_1 large enough, it can be assumed that $16\widetilde{\alpha} - 15\overline{\alpha} > 0$.

Using the incompressibility condition (we can assume that $\mathbf{h}_1 \neq 0$), we get

$$(-\mathbf{h}_2, \mathbf{h}_1, 0) \cdot \nabla\varphi_2 = -\partial_1(\mathbf{h}\cdot\varphi) + (\mathbf{h}_3, 0, -\mathbf{h}_1) \cdot \nabla\varphi_3. \tag{4.45}$$

We will apply (4.44) for the open set ω defined as follows. By assumption, $\mathbf{n}_1(x^0) \neq 0$ or $\mathbf{n}_2(x^0) \neq 0$; for example, let us assume that the former holds. First, we choose $\nu > 0$ such that

$$\mathbf{n}_1(x) \neq 0 \quad \forall x \in \Gamma_\nu := B_\nu(x^0) \cap \partial\mathcal{O} \cap \partial\Omega.$$

Then, we introduce

$$\omega := \{\, \overline{x} \in \Omega : \ \overline{x} = x + \tau(-\mathbf{h}_2, \mathbf{h}_1, 0), \ x \in \Gamma_\nu, \ |\tau| < \tau^0 \,\},$$

with $\nu, \tau^0 > 0$ small enough, so that we still have

$$\omega \subset \mathcal{O} \text{ and } d := \mathrm{dist}(\overline{\omega}, \partial\mathcal{O} \cap \Omega) > 0.$$

Observe that, with this choice, each point $x_* \in \omega$ has the property that one of the point at which the straight line $\{\, x_* + r(-\mathbf{h}_2, \mathbf{h}_1, 0) : r \in \mathbb{R} \,\}$ intersects $\partial\Omega$ belongs to $\partial\omega$.

Once ω is defined, we apply the inequality (4.44) in this open set and we try to bound the term

$$\iint_{\omega\times(0,T)} e^{\frac{-8\widetilde{\alpha}+6\overline{\alpha}}{t^4(T-t)^4}} t^{-64}(T-t)^{-64}|\varphi_2|^2$$

in terms of $\varepsilon K(\varphi, \psi, \zeta)$ and local integrals of $\mathbf{h}\cdot\varphi$ and φ_3.

To this end, for each $(x,t) \in \omega \times (0,T)$ we denote by $l(x,t)$ (resp. $\tilde{l}(x,t)$) the segment that starts from (x,t) with direction $(-\mathbf{h}_2, \mathbf{h}_1, 0)$ in the positive (resp. negative) sense and ends at $\partial\omega \times \{t\}$. Then, since φ verifies (4.45) and $\varphi = \mathbf{0}$ on Σ, it is not difficult to see that

$$\varphi_2(x,t) = \int_{l(x,t)} [\partial_1(\mathbf{h}\cdot\varphi) + (\mathbf{h}_3, 0, -\mathbf{h}_1)\cdot\nabla\varphi_3]\,(\overline{x},t)\, d\overline{x} \quad \forall (x,t) \in \omega \times (0,T).$$

Applying at this point Hölder's inequality and Fubini's formula, we obtain:

$$
\begin{aligned}
&\iint_{\omega\times(0,T)} |\beta\varphi_2|^2 \\
&\le C \iint_{\omega\times(0,T)} \beta \left(\int_{l(x,t)} \left[|\partial_1(\mathbf{h}\cdot\varphi)|^2 + |(\mathbf{h}_3, 0, -\mathbf{h}_1)\cdot\nabla\varphi_3|^2 \right] d\overline{x} \right) \\
&= C \iint_{\omega\times(0,T)} \left[|\partial_1(\mathbf{h}\cdot\varphi)|^2 + |(\mathbf{h}_3, 0, -\mathbf{h}_1)\cdot\nabla\varphi_3|^2 \right] \left(\int_{\tilde{l}(\overline{x},t)} \beta \, dx \right) d\overline{x}\, dt \\
&\le C \iint_{\omega\times(0,T)} \beta \left[|\partial_1(\mathbf{h}\cdot\varphi)|^2 + |(\mathbf{h}_3, 0, -\mathbf{h}_1)\cdot\nabla\varphi_3|^2 \right].
\end{aligned}
\tag{4.46}
$$

Then, let us introduce an appropriate non-empty open set ω_0 verifying $\omega \subset \omega_0 \subset \mathcal{O}$, $d_1 := \text{dist}(\overline{\omega_0}, \partial\mathcal{O}\cap\Omega) > 0$ and $d_2 := \text{dist}(\overline{\omega}, \partial\omega_0\cap\Omega) > 0$ and a cut-off function $\vartheta_0 \in C^2(\overline{\omega_0})$ such that

$$
\begin{gathered}
\vartheta_0 \equiv 1 \text{ in } \omega,\ 0 \le \vartheta_0 \le 1 \text{ and} \\
\vartheta_0(x) = 0 \text{ whenever } x \in \omega_0 \text{ and } \text{dist}(x, \partial\omega_0\cap\Omega) \le d_2/2.
\end{gathered}
$$

In particular, ϑ_0 and its derivatives vanish on $\partial\omega_0\cap\Omega$. This and the fact that $\varphi = \mathbf{0}$ on Σ imply:

$$
\begin{aligned}
&\iint_{\omega\times(0,T)} \beta|\partial_1(\mathbf{h}\cdot\varphi)|^2 \le \iint_{\omega_0\times(0,T)} \vartheta_0\beta|\partial_1(\mathbf{h}\cdot\varphi)|^2 \\
&= \iint_{\omega_0\times(0,T)} \beta \left[\frac{1}{2}\partial_1(\vartheta_0\partial_1|(\mathbf{h}\cdot\varphi)|^2) - \vartheta_0\partial_{11}(\mathbf{h}\cdot\varphi)(\mathbf{h}\cdot\varphi) \right. \\
&\qquad \left. -\frac{1}{2}\partial_1(\partial_1\vartheta_0|(\mathbf{h}\cdot\varphi)|^2) + \frac{1}{2}\partial_{11}\vartheta_0|(\mathbf{h}\cdot\varphi)|^2 \right] \\
&\le \frac{C}{2} \iint_{\omega_0\times(0,T)} \beta|(\mathbf{h}\cdot\varphi)|^2 + C \iint_{\omega_0\times(0,T)} \beta|\partial_{11}(\mathbf{h}\cdot\varphi)(\mathbf{h}\cdot\varphi)|
\end{aligned}
\tag{4.47}
$$

and

$$
\begin{aligned}
&\iint_{\omega\times(0,T)} \beta|(\mathbf{h}_3, 0, -\mathbf{h}_1)\cdot\nabla\varphi_3|^2 \le C \iint_{\omega_0\times(0,T)} \vartheta_0\beta|\nabla\varphi_3|^2 \\
&= C\sum_{j=1}^{N} \iint_{\omega_0\times(0,T)} \beta \left[\frac{1}{2}\partial_j(\vartheta_0\partial_j|\varphi_3|^2) - \vartheta_0\partial_{jj}(\varphi_3)\varphi_3 \right. \\
&\qquad \left. - \frac{1}{2}\partial_j(\partial_j\vartheta_0|\varphi_3|^2) + \frac{1}{2}\partial_{jj}\vartheta_0|\varphi_3|^2 \right] \mathbf{e}_j \\
&\le C\sum_{j=1}^{N} \iint_{\omega_0\times(0,T)} \left[\frac{1}{2}\beta|\varphi_3|^2 - \beta\partial_{jj}(\varphi_3)\varphi_3 \right].
\end{aligned}
\tag{4.48}
$$

Finally, in view of Young's inequality and classical Sobolev estimates, we see that

$$\begin{aligned}
&\iint_{\omega\times(0,T)} \beta|\partial_1(\mathbf{h}\cdot\varphi)|^2 \le \frac{C}{2}\iint_{\omega_0\times(0,T)} \beta|(\mathbf{h}\cdot\varphi)|^2 \\
&\quad +\frac{1}{2C}\iint_{\omega_0\times(0,T)} e^{\frac{-2\overline{\alpha}}{t^4(T-t)^4}} t^4(T-t)^4|\partial_{11}\varphi|^2 \\
&\quad +2C\iint_{\omega_0\times(0,T)} e^{\frac{-16\widetilde{\alpha}+14\overline{\alpha}}{t^4(T-t)^4}} t^{-132}(T-t)^{-132}|(\mathbf{h}\cdot\varphi)|^2 \\
&\le C\iint_{\omega_0\times(0,T)} e^{\frac{-16\widetilde{\alpha}+14\overline{\alpha}}{t^4(T-t)^4}} t^{-132}(T-t)^{-132}|(\mathbf{h}\cdot\varphi)|^2 \\
&\quad +\frac{1}{2C}\iint_{Q} e^{\frac{-2\overline{\alpha}}{t^4(T-t)^4}} t^4(T-t)^4|\Delta\varphi|^2
\end{aligned} \tag{4.49}$$

and

$$\begin{aligned}
&\iint_{\omega\times(0,T)} \beta|(\mathbf{h}_3,0,-\mathbf{h}_1)\cdot\nabla\varphi_3|^2 \\
&\le \sum_{j=1}^{N}\Bigg[\frac{1}{6C}\iint_{\omega_0\times(0,T)} e^{\frac{-2\overline{\alpha}}{t^4(T-t)^4}} t^4(T-t)^4|\partial_{jj}\varphi_3|^2 \\
&\quad +6C\iint_{\omega_0\times(0,T)} e^{\frac{-16\widetilde{\alpha}+14\overline{\alpha}}{t^4(T-t)^4}} t^{-132}(T-t)^{-132}|\varphi_3|^2 \\
&\quad +\frac{C}{2}\iint_{\omega_0\times(0,T)} \beta|\varphi_3|^2\Bigg] \\
&\le C\iint_{\omega_0\times(0,T)} e^{\frac{-16\widetilde{\alpha}+14\overline{\alpha}}{t^4(T-t)^4}} t^{-132}(T-t)^{-132}|\varphi_3|^2 \\
&\quad +\frac{1}{2C}\iint_{Q} e^{\frac{-2\overline{\alpha}}{t^4(T-t)^4}} t^4(T-t)^4|\Delta\varphi|^2
\end{aligned} \tag{4.50}$$

Therefore, combining (4.44), (4.46), (4.49) and (4.50), we obtain

$$\begin{aligned}
&\iint_{Q} e^{\frac{-2\overline{\alpha}}{t^4(T-t)^4}} t^4(T-t)^4(|\varphi_t|^2+|\psi_t|^2+|\zeta_t|^2+|\Delta\varphi|^2+|\Delta\psi|^2+|\Delta\zeta|^2) \\
&\quad +\iint_{Q} e^{\frac{-2\overline{\alpha}}{t^4(T-t)^4}} t^{-4}(T-t)^{-4}(|\nabla\varphi|^2+|\nabla\psi|^2+|\nabla\zeta|^2) \\
&\quad +\iint_{Q} e^{\frac{-2\overline{\alpha}}{t^4(T-t)^4}} t^{-12}(T-t)^{-12}(|\varphi|^2+|\psi|^2+|\zeta|^2) \\
&\le C\Bigg(\iint_{Q} e^{\frac{-4\widetilde{\alpha}+2\overline{\alpha}}{t^4(T-t)^4}} t^{-30}(T-t)^{-30}(|\widetilde{\mathbf{G}}|^2+|\widetilde{g}_1|^2+|\widetilde{g}_2|^2) \\
&\quad +\iint_{\omega_0\times(0,T)} e^{\frac{-16\widetilde{\alpha}+14\overline{\alpha}}{t^4(T-t)^4}} t^{-132}(T-t)^{-132}\left[|(\mathbf{h}\cdot\varphi)|^2+|\varphi_3|^2\right] \\
&\quad +\iint_{\omega\times(0,T)} e^{\frac{-8\widetilde{\alpha}+6\overline{\alpha}}{t^4(T-t)^4}} t^{-64}(T-t)^{-64}(|\psi|^2+|\zeta|^2)\Bigg).
\end{aligned} \tag{4.51}$$

Once more, our task is reduced to estimate the integrals

$$\iint_{\omega_0\times(0,T)} e^{\frac{-16\tilde{\alpha}+14\overline{\alpha}}{t^4(T-t)^4}} t^{-132}(T-t)^{-132}|(\mathbf{h}\cdot\varphi)|^2 \tag{4.52}$$

and

$$\iint_{\omega_0\times(0,T)} e^{\frac{-16\tilde{\alpha}+14\overline{\alpha}}{t^4(T-t)^4}} t^{-132}(T-t)^{-132}|\varphi_3|^2 \tag{4.53}$$

in terms of $\varepsilon K(\varphi,\psi,\zeta)$ and local integrals of ψ, ζ, $\widetilde{g}_1$ and $\widetilde{g}_2$.

To this end, we can again follow the steps of lemma 23; after some work, we are led to (4.43). □

4.6 Final comments and questions

4.6.1 The case $N = 2$

We see from theorems 15 and 16 that, for $N = 2$, even without imposing geometrical hypotheses to $\mathcal{O}$ like (4.10) the local exact controllability to the trajectories holds with two scalar controls w_1 and w_2. In other words, in this case, we only have to act on the PDEs satisfied by θ and c (no purely mechanical action is needed).

A natural question is thus whether theorem 16 can be improved (in the sense that the whole system can be controlled with just one scalar control by imposing (4.10) or any other condition.

4.6.2 Nonlinear F and geometrical conditions on $\mathcal{O}$

In theorem 17, we have assumed that $\mathbf{F}$ depends linearly on θ and c. This allowed to use the incompressibility condition (written in the form (4.45)) and, after several integrations by parts and estimates, led to (4.51).

It is thus reasonable to ask whether a similar result holds for more general functions $\mathbf{F}$ satisfying (4.9) and maybe other conditions. But this is to our knowledge an open question.

4.6.3 Generalizations to coupled systems with more unknowns

The results in this paper admit several straightforward generalizations. For instance, let us assume that $N = 3$. With suitable hypotheses, we can obtain a result similar theorem 16 for the following system in Q

$$\begin{cases} \mathbf{y}_t - \Delta\mathbf{y} + (\mathbf{y}\cdot\nabla)\mathbf{y} + \nabla p = \mathbf{v}1_{\mathcal{O}} + \mathbf{F}(\theta, c^1, c^2), \\ \nabla\cdot\mathbf{y} = 0, \\ \begin{pmatrix} \theta \\ c^1 \\ c^2 \end{pmatrix}_t - \begin{pmatrix} \widetilde{a}\Delta\theta \\ \widetilde{a}^1\Delta c^1 \\ \widetilde{a}^2\Delta c^2 \end{pmatrix} + \mathbf{y}\cdot\nabla \begin{pmatrix} \theta \\ c^1 \\ c^2 \end{pmatrix} = \begin{pmatrix} f(\theta, c^1, c^2) \\ f^1(\theta, c^1, c^2) \\ f^2(\theta, c^1, c^2) \end{pmatrix} + \begin{pmatrix} w1_{\mathcal{O}} \\ w^1 1_{\mathcal{O}} \\ w^2 1_{\mathcal{O}} \end{pmatrix}, \end{cases}$$

completed with homogeneous Dirichlet boundary conditions and initial conditions at $t = 0$.

This means that the whole system can be controlled, at least locally, by acting on the PDEs satisfied by θ, c^1 and c^2, but not on the motion equation.

Nevertheless, it is unknown whether this can be improved and local controllability can also hold, under some specific assumptions, with at most two scalar controls.

4.6.4 Local null controllability without geometrical hypotheses

Let us come back to theorem 15. Suppose that $(\overline{\mathbf{y}}, \overline{p}, \overline{\theta}, \overline{c}) \equiv \mathbf{0}$ and let us try to prove a local null controllability result with L^2 controls $\mathbf{v} \equiv \mathbf{0}$, w_1 and w_2, without any assumption on $\mathcal{O}$.

Arguing as in Section 4.3, we readily see that the task is reduced to the proof of a Carleman inequality for the solutions to (4.15) with only local integrals of ψ and ζ in the right hand side.

But this inequality is true. Indeed, with a self-explained notation, the following holds:

a) $\widetilde{I}(s,\lambda;\varphi) \leq C \displaystyle\iint_{\mathcal{O}\times(0,T)} \rho_1^{-2}(|\varphi_2|^2 + |\varphi_3|^2) + \dots$

(from the results in [18]; here and below, the dots contain weighted integrals of $|\widetilde{\mathbf{G}}|^2$ and $|\widetilde{g}_1|^2 + |\widetilde{g}_2|^2$).

b) $K(\psi,\zeta) \leq \varepsilon \widetilde{I}(s,\lambda;\varphi) + C \displaystyle\iint_{\mathcal{O}\times(0,T)} \rho_2^{-2}(|\psi|^2 + |\zeta|^2) + \dots$

(from the usual Carleman estimates for the heat equation).

Using in a) the arguments in the final part of the proof of lemma 23, we obtain an estimate of the form

$$\widetilde{I}(s,\lambda;\varphi) \leq \varepsilon \widetilde{I}(s,\lambda;\varphi) + C_\varepsilon \iint_{\mathcal{O}\times(0,T)} (\rho_3^{-2}|\psi|^2 + \rho_4^{-2}|\zeta|^2) + \dots$$

Then, after addition, we find:

$$\widetilde{I}(s,\lambda;\varphi) + K(\psi,\zeta) \leq C \iint_{\mathcal{O}\times(0,T)} \rho^{-2}(|\psi|^2 + |\zeta|^2) + \dots,$$

which easily leads to the desired estimates.

4.7 Appendix

Let us now present a sketch of the proof of proposition 12.

• FIRST ESTIMATES:

In view of the usual Carleman estimates for the heat equations, we easily obtain

$$\begin{aligned} K(\varphi,\psi,\zeta) \;\; &\leq \;\; C\left(\iint_Q e^{-2s\alpha}|\nabla\pi|^2 + \iint_Q e^{-2s\alpha}(|\widetilde{\mathbf{G}}|^2 + |\widetilde{g}_1|^2 + |\widetilde{g}_2|^2) \right. \\ &\left. + \, s^3\lambda^4 \iint_{\mathcal{O}\times(0,T)} e^{-2s\alpha}\xi^3(|\varphi|^2 + |\psi|^2 + |\zeta|^2)\, dx\, dt \right) \end{aligned} \tag{4.54}$$

for all $s \geq s_0(T^7 + T^8)$ and

$$\begin{aligned}\lambda \ &\geq \ \hat{\lambda}\big(1 + ||\overline{\mathbf{y}}||_\infty + ||\overline{\theta}||_\infty + ||\overline{c}||_\infty + ||\overline{\mathbf{G}}||_\infty^{1/2} + ||\overline{\mathbf{L}}||_\infty^{1/2} \\ &+||\overline{g}_1||_\infty^{1/2} + ||\overline{g}_2||_\infty^{1/2} + ||\overline{l}_1||_\infty^{1/2} + ||\overline{l}_2||_\infty^{1/2}\big).\end{aligned}$$

• ELIMINATING THE GLOBAL INTEGRAL OF $\nabla\pi$:

Let us look at the (weak) equation satisfied by the pressure, which can be found by applying the divergence operator to the motion equation of (4.15):

$$\Delta\pi(t) = \nabla \cdot \Big[D\varphi(t)\overline{\mathbf{y}}(t) + \widetilde{\mathbf{G}}(t) + \overline{\theta}(t)\nabla\psi(t) + \overline{c}(t)\nabla\zeta(t)\Big] \quad \text{in } \Omega, \quad t \in (0,T) \text{ a.e.} \tag{4.55}$$

Regarding the right hand side of (4.55) like a H^{-1} term, we can apply the main result in [57] and deduce that

$$\begin{aligned}K(\varphi,\psi,\zeta) \ &\leq \ C\Bigg(s^3\lambda^4 \iint_{\mathcal{O}\times(0,T)} e^{-2s\alpha}\xi^3|(\varphi|^2 + |\psi|^2 + |\zeta|^2) \\ &+s\iint_Q e^{-2s\alpha}\xi\left|\widetilde{\mathbf{G}}\right|^2 + \iint_Q e^{-2s\alpha}(|\widetilde{g}_1|^2 + |\widetilde{g}_2|^2) \\ &+\iint_{\mathcal{O}_1\times(0,T)} |\hat{\mu}|^2\,|\nabla\pi|^2\Bigg)\end{aligned} \tag{4.56}$$

for all $s \geq s_0(T^7 + T^8)$ and all

$$\begin{aligned}\lambda \ &\geq \ \hat{\lambda}\big(1 + ||\overline{\mathbf{y}}||_\infty + ||\overline{\theta}||_\infty + ||\overline{c}||_\infty + ||\overline{\mathbf{G}}||_\infty^{1/2} + ||\overline{\mathbf{L}}||_\infty^{1/2} \\ &+||\overline{g}_1||_\infty^{1/2} + ||\overline{g}_2||_\infty^{1/2} + ||\overline{l}_1||_\infty^{1/2} + ||\overline{l}_2||_\infty^{1/2}\big).\end{aligned}$$

Taking into account the motion equation in (4.15), we have:

$$\begin{aligned}\iint_{\mathcal{O}_1\times(0,T)} |\hat{\mu}|^2\,|\nabla\pi|^2 \leq C\Bigg(&\iint_{\mathcal{O}_1\times(0,T)} |\hat{\mu}|^2\left|\widetilde{\mathbf{G}}\right|^2 \\ &+\iint_{\mathcal{O}_1\times(0,T)} |\hat{\mu}|^2\,|\varphi_t|^2 + \|\overline{\mathbf{y}}\|_\infty^2 \iint_{\mathcal{O}_1\times(0,T)} |\hat{\mu}|^2\,|\nabla\varphi|^2 \\ &+\|\overline{\theta}\|_\infty^2 \iint_{\mathcal{O}_1\times(0,T)} |\hat{\mu}|^2\,|\nabla\psi|^2 + \|\overline{c}\|_\infty^2 \iint_{\mathcal{O}_1\times(0,T)} |\hat{\mu}|^2\,|\nabla\zeta|^2 \\ &+\iint_{\mathcal{O}_1\times(0,T)} |\hat{\mu}|^2\,|\Delta\varphi|^2\Bigg).\end{aligned} \tag{4.57}$$

for all $s \geq s_0(T^7 + T^8)$ and all

$$\begin{aligned}\lambda \ &\geq \ \hat{\lambda}\big(1 + ||\overline{\mathbf{y}}||_\infty + ||\overline{\theta}||_\infty + ||\overline{c}||_\infty + ||\overline{\mathbf{G}}||_\infty^{1/2} + ||\overline{\mathbf{L}}||_\infty^{1/2} \\ &+||\overline{g}_1||_\infty^{1/2} + ||\overline{g}_2||_\infty^{1/2} + ||\overline{l}_1||_\infty^{1/2} + ||\overline{l}_2||_\infty^{1/2}\big).\end{aligned}$$

• ESTIMATES OF THE LOCAL TERMS ON $\Delta\varphi$ AND φ_t

The remainder of the proof is devoted to estimate the local terms on $\Delta\varphi$ and φ_t. To do this, we can follow the ideas in [49], which gives

a) An estimate of $|\Delta\varphi|^2$:

$$\iint_{\mathcal{O}_1\times(0,T)} |\hat{\mu}|^2|\Delta\varphi|^2 \;\leq\; C(1+T)\Bigg(\iint_{\mathcal{O}_2\times(0,T)}|\hat{\mu}|^2(|D\varphi\overline{\mathbf{y}}|^2+|\overline{\theta}\nabla\psi|^2+|\overline{c}\nabla\zeta|^2) + \iint_{\mathcal{O}_2\times(0,T)}(|\hat{\mu}'\varphi|^2+|\hat{\mu}\varphi|^2+|\hat{\mu}\widetilde{\mathbf{G}}|^2)\Bigg)\,; \tag{4.58}$$

b) An estimate of $|\varphi_t|^2$:

$$\begin{aligned}\iint_{\mathcal{O}_1\times(0,T)} |\hat{\mu}|^2|\varphi_t|^2 \;\leq\; & C_\varepsilon\lambda^{24}(1+T)\Big(\big\|\mu\widetilde{\mathbf{G}}\big\|^2_{\mathbf{L}^2(Q)} + \|\mu\widetilde{g}_1\|^2_{L^2(Q)} + \|\mu\widetilde{g}_2\|^2_{L^2(Q)} \\ & +\|\mu\varphi\|^2_{L^2(0,T;\mathbf{L}^2(\mathcal{O}_3))} + |\mu'\varphi|^2_{L^2(0,T;\mathbf{L}^2(\mathcal{O}_3))} \\ & +\|\mu\nabla\varphi\|^2_{L^2(0,T;\mathbf{L}^2(\mathcal{O}_3))} + \|\mu\nabla\psi\|^2_{L^2(0,T;\mathbf{L}^2(\mathcal{O}_3))} \\ & +\|\mu\nabla\zeta\|^2_{L^2(0,T;\mathbf{L}^2(\mathcal{O}_3))}\Big) + \varepsilon K(\varphi,\psi,\zeta),\end{aligned} \tag{4.59}$$

with $\mathcal{O}_1 \subset\subset \mathcal{O}_2 \subset\subset \mathcal{O}_3 \subset\subset \mathcal{O}$.

Combining (4.54) and (4.56)–(4.59), after several additional computations, we find (4.17).

This ends the proof.

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