Universidade Federal da Paraíba
Universidade Federal de Campina Grande
Programa Associado de Pós-Graduação em Matemática
Doutorado em Matemática

Existence results for some elliptic equations involving the fractional Laplacian operator and critical growth

por

Yane Lísley Ramos Araújo

João Pessoa - PB

Dezembro/2015

Existence results for some elliptic equations involving the fractional Laplacian operator and critical growth

por

Yane Lísley Ramos Araújo †

sob orientação do

Prof. Dr. Manassés Xavier de Souza

Tese apresentada ao Corpo Docente do Programa Associado de Pós-Graduação em Matemática - UFPB/UFCG, como requisito parcial para obtenção do título de Doutor em Matemática.

João Pessoa - PB

Dezembro/2015

†Este trabalho contou com apoio financeiro da Capes.

A663e Araújo, Yane Lísley Ramos.
Existence results for some elliptic equations involving the fractional Laplacian operator and critical growth / Yane Lísley Ramos Araújo.- João Pessoa, 2015.
111f.
Orientador: Manassés Xavier de Souza
Tese (Doutorado) - UFPB-UFCG
1. Matemática. 2. Laplaciano fracionário. 3. Métodos variacionais. 4. Desigualdade de Trudinger-Moser. 5. Expoente crítico de Sobolev.

UFPB/BC CDU: 51(043)

Universidade Federal da Paraíba

Universidade Federal de Campina Grande

Programa Associado de Pós-Graduação em Matemática

Doutorado em Matemática

Área de Concentração: Análise

Aprovada em:

Prof. Dr. Emerson Alves Mendonça de Abreu - UFMG

Prof. Dr. Everaldo Souto de Medeiros - UFPB

Prof. Dr. Olivâine Santana de Queiroz - UNICAMP

Prof. Dr. Wilberclay Gonçalves Melo - UFS

Prof. Dr. Manassés Xavier de Souza - UFPB

Orientador

Tese apresentada ao Corpo Docente do Programa Associado de Pós-Graduação em Matemática - UFPB/UFCG, como requisito parcial para obtenção do título de Doutor em Matemática.

Dezembro/2015

Resumo

Neste trabalho provamos alguns resultados de existência e multiplicidade de soluções para equações do tipo

$$(-\Delta)^\alpha u + V(x)u = f(x,u) \quad \text{em} \quad \mathbb{R}^N,$$

onde $0 < \alpha < 1$, $N \geq 2\alpha$, $(-\Delta)^\alpha$ denota o Laplaciano fracionário, $V : \mathbb{R}^N \to \mathbb{R}$ é uma função contínua que satisfaz adequadas condições e $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ é uma função contínua que pode ter crescimento crítico no sentido da desigualdade de Trudinger-Moser ou no sentido do expoente crítico de Sobolev. A fim de obter nossos resultados usamos métodos variacionais combinados com uma versão do Princípio de Concentração-Compacidade devido à Lions.

Palavras-chave: Laplaciano fracionário; métodos variacionais; desigualdade de Trudinger-Moser; expoente crítico de Sobolev.

Abstract

In this work we prove some results of existence and multiplicity of solutions for equations of the type

$$(-\Delta)^{\alpha}u + V(x)u = f(x,u) \quad \text{in} \quad \mathbb{R}^N,$$

where $0 < \alpha < 1$, $N \geq 2\alpha$, $(-\Delta)^{\alpha}$ denotes the fractional Laplacian, $V : \mathbb{R}^N \to \mathbb{R}$ is a continuous function that satisfy suitable conditions and $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a continuous function that may have critical growth in the sense of the Trudinger-Moser inequality or in the sense of the critical Sobolev exponent. In order to obtain our results we use variational methods combined with a version of the Concentration-Compactness Principle due to Lions.

Keywords: Fractional Laplacian; variational methods; Trudinger-Moser's inequality; critical Sobolev exponent.

Agradecimentos

A Deus pois só Ele é capaz de nos dar força para alcançarmos nossos objetivos.

Aos meus pais Ana Rita e Gildeval por todo amor, dedicação e apoio incondicional. Ao meu irmão Tarles Walker e a minha madrinha Ana por sempre acreditarem em mim.

Ao meu esposo Reginaldo Junior pelo amor, companheirismo, compreensão e toda ajuda indispensável à realização deste trabalho.

Ao Prof. Manassés Xavier de Souza por ter me concedido a oportunidade de estudar com o mesmo, por toda orientação, dedicação e empenho na realização deste trabalho.

Ao meu amigo-irmão Gilson por ser um porto seguro para mim ao longo de toda a jornada.

Ao amigo Nacib e a minha prima Sara pela disponibilidade e cuidado em me ajudar nas correções com o inglês.

Aos meus professores Éder Mateus de Souza, João de Azevedo Cardeal e Fabíola de Oliveira Pedreira Lima por serem inspirações para mim desde o ensino médio e a graduação. Aos professores de Pós-Graduação do DM por todos os ensinamentos transmitidos, em especial aos professores Everaldo Souto de Medeiros e Uberlandio Batista Severo.

A minha irmã de coração Pammella e aos demais colegas do DM/UFPB que por muitas vezes foram minha família dando apoio e diminuindo o fardo que é está longe de casa, em especial a Diego, Rodrigo, Ricardo Burity, Gustavo, Valdecir, Diana, Daniel.

Aos membros da banca pelas valiosas contribuições para a versão final deste trabalho.

A todos que direta ou indiretamente contribuíram para a conlusão deste trabalho.

À Capes pelo apoio financeiro.

Dedicatória

À minha família.

Contents

Introduction

In this work, we study the existence and multiplicity of solutions for elliptic equations of the type

$$(-\Delta)^\alpha u + V(x)u = f(x,u) \quad \text{in} \quad \mathbb{R}^N, \tag{0.1}$$

where $0 < \alpha < 1$, $N \geq 2\alpha$, $V : \mathbb{R}^N \to \mathbb{R}$ and $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ are continuous functions that satisfy suitable conditions and $(-\Delta)^\alpha$ denotes the fractional Laplacian which can be defined for a sufficiently regular function $u : \mathbb{R} \to \mathbb{R}$ by

$$(-\Delta)^\alpha u(x) = -\frac{1}{2}C(N,\alpha)\int\limits_{\mathbb{R}^N} \frac{u(x+y)+u(x-y)-2u(x)}{|y|^{N+2\alpha}}\mathrm{d}y, \quad \forall x \in \mathbb{R}^N.$$

For details about this operator see Appendix A.

Part of the interest on those equations arises in the search of standing waves solutions for the *fractional Schrödinger equation*

$$i\frac{\partial\psi}{\partial t} = (-\Delta)^\alpha\psi + (V(x)+\omega)\psi - f(x,\psi), \;\; (x,t) \in \mathbb{R}^N \times \mathbb{R}, \tag{0.2}$$

where $\omega \in \mathbb{R}$, $V : \mathbb{R}^N \to \mathbb{R}$ is an external potential function, $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a continuous function and $0 < \alpha < 1$ is a fixed parameter. Standing waves solutions to Equation (0.2) are solutions of the form

$$\psi(x,t) = u(x)\exp(-i\omega t),$$

where u solves elliptic Equation (0.1).

Equation (0.2) comes from an expansion of the Feynman path integral from Brownian-like to Lévy-like quantum mechanical paths (see [31] and [32]). It is known, but not completely trivial, that $(-\Delta)^\alpha$ reduces to the standard Laplacian $-\Delta$ if $\alpha \to 1^-$

(see [17]). Thus, when $\alpha = 1$, the Lévy dynamics becomes the Brownian dynamics, and Equation (0.2) reduces to the classical Schrödinger equation

$$i\frac{\partial \psi}{\partial t} = -\Delta\psi + (V(x) + \omega)\psi - f(x, \psi), \ \ (x, t) \in \mathbb{R}^N \times \mathbb{R}.$$

Motivated by Equation (0.2), several studies have been performed for elliptic equations involving the fractional Laplacian operator. In the sequel, we will list some papers related with the existence of solutions to Equation (0.1) that may be found in the literature.

Let us begin with the progress involving subcritical nonlinearities. Using the Nehari variational principle, in [11], Cheng proved the existence of a nontrivial solution of the fractional Schrödinger equation

$$(-\Delta)^\alpha u + V(x)u = |u|^{q-2}u, \ \ x \in \mathbb{R}^N, \tag{0.3}$$

with $2 < q < 2^*_\alpha$ if $N > 2\alpha$ or $2 < q < \infty$ if $N \leq 2\alpha$, where $2^*_\alpha := 2N/(N - 2\alpha)$ is the critical Sobolev exponent. Ground states are found by imposing a coercivity assumption on $V(x)$,

$$\lim_{|x|\to+\infty} V(x) = +\infty. \tag{0.4}$$

In [41], Secchi proved the existence of a nontrivial solution under less restrictive assumptions on $f(x, u)$. He obtained the existence of a ground state by the method used in [24]. It is worthwhile to notice that in [11] and [41] the hypothesis (0.4) is assumed on $V(x)$ in order to overcome the problem of lack of compactness, typical of elliptic problems defined in unbounded domains. In [18], Dipierro *et al.* considered the existence of radially symmetric solutions of (0.3) in the situation where $V(x)$ does not depend explicitly on the space variable x. For the first time, using rearrangement tools and following the ideas of Berestycki and Lions [5], the authors proved the existence of a nontrivial, radially symmetric solution to

$$(-\Delta)^\alpha u + u = |u|^{q-2}u, \ \ x \in \mathbb{R}^N,$$

where $2 < q < 2^*_\alpha$ if $N > 2\alpha$ or $2 < q < \infty$ if $N \leq 2\alpha$.

After the pioneering works by Brezis and Nirenberg in [8], elliptic problems with critical growth have had many progresses in several directions. For the fractional

Laplacian, we would like to mention [21, 28, 43] and the references therein. More specifically, Shang *et al.* in [43] considered the existence of solutions for the problem

$$(-\Delta)^{\alpha}u + V(x)u = |u|^{2^*_\alpha-2}u + \lambda|u|^{q-2}u, \;\; x \in \mathbb{R}^N,$$

where $\lambda > 0$ is a parameter, $2 < q < 2^*_\alpha$ and $N > 2\alpha$. The potential $V : \mathbb{R}^N \to \mathbb{R}$ is a continuous function satisfying $0 < \inf_{x\in\mathbb{R}^N} V(x) = \overline{V}_0 < \liminf_{|x|\to+\infty} V(x) = V_\infty$, where $V_\infty < \infty$. This kind of hypothesis was first introduced by Rabinowitz in [40].

J. M. do Ó *et al.* in [21] proved the existence of a solution of the fractional Schrödinger equation

$$(-\Delta)^{\alpha}u + V(x)u = |u|^{2^*_\alpha-2}u + K(x)f(u), \;\; x \in \mathbb{R}^N,$$

where V, K are continuous, $V, K > 0$ in $\mathbb{R}^N$ with $V(x) \to 0, K(x) \to 0$, as $|x| \to +\infty$, and $f(u)$ behaves like $|u|^{q-2}u$ at infinity, for some $2 < q < 2^*_\alpha$ and $N > 2\alpha$. Moreover, $f(u)$ satisfies the so-called Ambrosetti–Rabinowitz condition, namely,

$$\text{(AR) there exists}\, \mu \in (2, 2^*_\alpha) \;\text{ with}\, 0 < \mu F(s) \leq sf(s) \;\text{ for all}\, s \neq 0,\; F(s) = \int_0^s f(t)\mathrm{d}t.$$

With respect to the growth of the nonlinearity in problems of the type (0.1) in the limiting case $N = 2\alpha$, for $N = 1$ and $\alpha = 1/2$, there exists a special situation motivated by the Trudinger-Moser inequality. Precisely, it is known that the embedding $H^{1/2}(\mathbb{R}) \hookrightarrow L^q(\mathbb{R})$ is continuous for any $q \in [2, +\infty)$, but $H^{1/2}(\mathbb{R})$ is not continuously embedded in $L^\infty(\mathbb{R})$. However, T. Ozawa [39] and H. Kozono, T. Sato and H. Wadade [29] proved a version of the Trudinger-Moser inequality. More precisely, they proved that there exist positive constants ω and C such that, for all $u \in H^{1/2}(\mathbb{R})$ with $\|(-\Delta)^{1/4}u\|_2 \leq 1$,

$$\int_{\mathbb{R}} (e^{\beta u^2} - 1)\,\mathrm{d}x \leq C\|u\|_2^2, \quad \text{for all} \quad \beta \in (0, \omega]. \tag{0.5}$$

Consequently, the maximal growth which allows us to treat (0.1) variationally in $H^{1/2}(\mathbb{R})$ is of the type exponential.

Inequality (0.5) plays a crucial role in the study of problems that involved nonlinearities with exponential growth. For works involving this type of nonlinearities, we would like to mention two papers, [23] and [25]. J. M. do Ó *et al.* in [23] proved the existence of a solution for the fractional Schrödinger equation

$$(-\Delta)^{1/2}u + u = K(x)g(u),$$

where K is a positive function which can vanish at infinity and g has exponential growth. Iannizzotto and Squassina, in [25], considered the existence of solutions for the problem

$$\begin{cases} (-\Delta)^{1/2}u = f(u) \quad \text{in} \quad (0,1), \\ u = 0 \quad \text{in} \quad \mathbb{R}\setminus(0,1), \end{cases}$$

where the nonlinearity has exponential growth.

Motivated by these studies and taking into consideration the behavior of the potential $V(x)$ and the types of nonlinearity $f(x,s)$, in this work we obtain some results of existence and multiplicity of solutions to Equation (0.1). Precisely:

In Chapter 1, we treat the limiting case $N = 1$, $\alpha = 1/2$, more specifically, we study the equation

$$(-\Delta)^{1/2}u + V(x)u = f(x,u) + h \quad \text{in} \quad \mathbb{R}, \tag{0.6}$$

where $(-\Delta)^{1/2}$ is defined in the Section 1, $V : \mathbb{R} \to \mathbb{R}$ is a continuous potential, h belongs to the dual of an appropriate functional space, see Section 1, and $f : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is a continuous function that has *critical exponential growth*, that is, there exists $\beta_0 > 0$ such that

$$\lim_{|s|\to+\infty} f(x,s)e^{-\beta|s|^2} = \begin{cases} 0, & \text{for all } \beta > \beta_0, \\ +\infty, & \text{for all } \beta < \beta_0, \end{cases}$$

uniformly in $x \in \mathbb{R}$.

We consider the following hypotheses under $V(x)$:

(V_1) there exists a positive constant B such that $V(x) \geq -B$, for all $x \in \mathbb{R}$;

(V_2) the infimum

$$\lambda_1 := \inf_{\substack{u\in X \\ \|u\|_2=1}} \left(\frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{(u(x)-u(y))^2}{|x-y|^2}\,\mathrm{d}x\,\mathrm{d}y + \int_{\mathbb{R}} V(x)u^2\,\mathrm{d}x \right)$$

is positive;

(V_3) $\lim\limits_{R\to\infty} \nu(\mathbb{R}\setminus \overline{B}_R) = +\infty$, where

$$\nu(G) = \begin{cases} \inf\limits_{\substack{u\in X_0(G)\\ \|u\|_2=1}} \dfrac{1}{2\pi}\displaystyle\int\limits_{\mathbb{R}^2} \dfrac{(u(x)-u(y))^2}{|x-y|^2}\,\mathrm{d}x\,\mathrm{d}y + \int\limits_G V(x)u^2\,\mathrm{d}x & \text{if} \quad G \neq \varnothing; \\ \infty & \text{if} \quad G = \varnothing. \end{cases}$$

Here G is an open set in $\mathbb{R}$ and $X_0(G) = \{u \in X \,:\, u = 0 \quad \text{in} \quad \mathbb{R}\setminus G\}$, where X is defined in (1.3).

It is important to observe that the assumptions $(V_1)-(V_3)$ allow that the potentials may change sign.

In order to use a variational approach, we consider the following assumptions about $f(x,s)$:

(f_1) $0 \leq \lim\limits_{s\to 0} \dfrac{f(x,s)}{s} < \lambda_1$, uniformly in x;

(f_2) f is locally bounded in s, that is, for any bounded interval $J\subset\mathbb{R}$, there exists $C>0$ such that $|f(x,s)|\leq C$, for every $(x,s)\in\mathbb{R}\times J$;

(f_3) there exists $\theta > 2$ such that

$$0 < \theta F(x,s) := \theta \int\limits_0^s f(x,t)\,\mathrm{d}t \leq s f(x,s), \quad \text{for all} \quad (x,s)\in\mathbb{R}\times\mathbb{R}\setminus\{0\};$$

(f_4) there exist constants $s_0, M_0 > 0$ such that

$$0 < F(x,s) \leq M_0|f(x,s)|, \quad \text{for all} \quad |s|\geq s_0 \quad \text{and} \quad x\in\mathbb{R};$$

(f_5) there exist constants $p>2$ and C_p such that, for all $s\geq 0$ and $x\in\mathbb{R}$,

$$f(x,s) \geq C_p s^{p-1},$$

with $C_p > \left[\dfrac{\alpha_0(p-2)}{2\pi\kappa\omega p}\right]^{(p-2)/2} S_p^p$, where

$$S_p := \inf_{\substack{u\in X\\ \|u\|_p=1}} \left(\frac{1}{2\pi}\int\limits_{\mathbb{R}^2} \frac{(u(x)-u(y))^2}{|x-y|^2}\,\mathrm{d}x\,\mathrm{d}y + \int\limits_{\mathbb{R}} V(x)u^2\,\mathrm{d}x \right)^{1/2},$$

and κ is given in (1.7).

With this we obtain the main results of this chapter:

Theorem 0.1. *Suppose that* $(V_1)-(V_3)$ *and* $(f_1)-(f_5)$ *hold. Then there exists* $\delta_1>0$ *such that for each* $0<\|h\|_*<\delta_1$*, problem* (0.6) *has at least two weak solutions. One of them with positive energy, and the other one with negative energy.*

Theorem 0.2. *Suppose that* $(V_1)-(V_3)$ *and* $(f_1)-(f_5)$ *hold. If* $h\equiv 0$ *(i.e., there is no perturbation in* (0.6)*) then problem* (0.6) *has a weak solution with positive energy.*

In order to prove Theorems 0.1 and 0.2, we need to check some conditions concerning the mountain pass geometry and the compactness of the associated functional. More specifically, we show that the functional associated with the problem satisfies the Palais-Smale condition. Then we use minimization to find the first solution with negative energy and the Mountain Pass Theorem to obtain the existence of the second solution with positive energy. The main difficulties lie in the nonlocal operator involved and in the critical exponential growth of the nonlinearity.

Our results complement the work in [25] since we considered that the domain is all $\mathbb{R}$. It also complement [11, 23, 41, 42], once we work with nonlinearities more general than those treated by them and potentials that may change sign, vanish and be unbounded.

In Chapter 2, we deal with the problem of existence of weak solutions to a class of equations similar to those that we studied in Chapter 1, where V is a bounded potential that belongs to a different class of those treated therein. We study two class of problems:

The first one (a periodic problem) is the following,

$$\begin{cases} (-\Delta)^{1/2}u+V_0(x)u=f_0(x,u) \quad \text{in} \quad \mathbb{R}, \\ u\in H^{1/2}(\mathbb{R}) \quad \text{and} \quad u\geq 0. \end{cases} \tag{P_0}$$

We consider that the function $V_0:\mathbb{R}\to(0,+\infty)$ is a continuous $1-$periodic function and $f_0:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is a continuous $1-$periodic function in x, which has critical exponential growth in u. Since we are interested in the existence of nonnegative solutions, we set $f_0(x,s)=0$ for all $(x,s)\in\mathbb{R}\times(-\infty,0]$. We also assume that the nonlinearity $f_0(x,u)$ satisfies the conditions

$(f_{0,1})$ $\displaystyle\lim_{s\to 0}\frac{f_0(x,s)}{s}=0$ uniformly in $x\in\mathbb{R}$;

$(f_{0,2})$ there exists a constant $\theta > 2$ such that

$$0 < \theta F_0(x,s) := \theta \int_0^s f_0(x,t)\,\mathrm{d}t \leq s f_0(x,s), \quad \text{for all} \quad (x,s) \in \mathbb{R} \times (0,+\infty);$$

$(f_{0,3})$ for each fixed $x \in \mathbb{R}$, the function $f_0(x,s)/s$ is increasing with respect to $s \in \mathbb{R}$;

$(f_{0,4})$ there are constants $p > 2$ and $C_p > 0$ such that

$$f_0(x,s) \geq C_p s^{p-1}, \quad \text{for all} \quad (x,s) \in \mathbb{R} \times [0,+\infty),$$

where

$$C_p > \left[\frac{(p-2)\theta\alpha_0}{(\theta-2)p\omega}\right]^{(p-2)/2} S_p^p$$

and

$$S_p := \inf_{\substack{u \in H^{1/2}(\mathbb{R}) \\ \|u\|_p = 1}} \left(\frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{|u(x)-u(y)|^2}{|x-y|^2}\,\mathrm{d}x\mathrm{d}y + \|V\|_\infty \int_{\mathbb{R}} u^2\,\mathrm{d}x \right)^{1/2}.$$

Under these assumptions we have the first result of Chapter 2:

Theorem 0.3. *Assume that $(f_{0,1})$–$(f_{0,4})$ hold. Then (P_0) has a nonnegative nontrivial weak solution.*

The second problem (asymptotically periodic) that we study in this chapter is

$$\begin{cases} (-\Delta)^{1/2}u + V(x)u = f(x,u) & \text{in} \quad \mathbb{R}, \\ u \in H^{1/2}(\mathbb{R}) \quad \text{and} \quad u \geq 0, \end{cases} \tag{P}$$

on which we consider the following conditions on the function $V(x)$:

(V_1) $V : \mathbb{R} \to [0,+\infty)$ is a continuous function satisfying the conditions: $V(x) \leq V_0(x)$ for any $x \in \mathbb{R}$ and $V_0(x) - V(x) \to 0$ as $|x| \to \infty$;

We assume that the nonlinearity $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function that has critical exponential growth in s, $f(x,s) = 0$ for all $(x,s) \in \mathbb{R} \times (-\infty, 0]$ and also satisfies:

(f_1) $f(x,s) \geq f_0(x,s)$ for all $(x,s) \in \mathbb{R} \times [0,+\infty)$, and for all $\varepsilon > 0$, there exists $\eta > 0$ such that for $s \geq 0$ and $|x| \geq \eta$,

$$|f(x,s) - f_0(x,s)| \leq \varepsilon e^{\alpha_0 s^2};$$

(f_2) $\lim_{s\to 0} \dfrac{f(x,s)}{s} = 0$ uniformly in $x \in \mathbb{R}$;

(f_3) there exists a constant $\mu > 2$ such that

$$0 < \mu F(x,s) := \mu \int_0^s f(x,t)\,\mathrm{d}t \leq s f_0(x,s), \quad \text{for all} \quad (x,s) \in \mathbb{R} \times (0,+\infty);$$

(f_4) for each fixed $x \in \mathbb{R}$, the function $f(x,s)/s$ is increasing with respect to $s \in \mathbb{R}$;

(f_5) at least one of the nonnegative continuous functions $V_0(x) - V(x)$ and $f(x,s) - f_0(x,s)$ is positive on a set of positive measure.

The second result of Chapter 2 is the following:

Theorem 0.4. *Assume that* (V_1) *and* $(f_1) - (f_5)$ *hold. Then* (P) *has a nonnegative nontrivial weak solution.*

In order to prove our results, we show that the weak limit of an appropriate sequence of Palais Smale is a weak solution of the problem and we use a version of the Concentration-Compactness Priniple due to Lions to show that this limit is nontrivial.

We point out that our results complete the study presented in [22, 23], since we work with a general class of functions which are asymptotic to a nonautonomous periodic function at infinity. It also complements [10, 11, 18, 41], once we consider the limiting case for $N = 1$ and $\alpha = 1/2$ when the nonlinearity has exponential growth in the sense of the Trudinger-Moser inequality. Moreover, it also complements [14], the study of Chapter 1, once we consider that the potential $V(x)$ belongs to a different class from those treated there.

In Chapter 3, our main goal is to establish, under an asymptotic periodicity condition at infinity, the existence of a weak solution for the critical problem

$$(-\Delta)^{\alpha} u + V(x)u = |u|^{2^*_{\alpha}-2}u + g(x,u), \;\; x \in \mathbb{R}^N, \tag{0.7}$$

where $0 < \alpha < 1$, $N > 2\alpha$, $V : \mathbb{R}^N \to \mathbb{R}$ and $g : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ are continuous functions.

Considering $\mathcal{F}$ the class of functions $h \in C(\mathbb{R}^N) \cap L^{\infty}(\mathbb{R}^N)$ such that, for every $\varepsilon > 0$, the set $\{x \in \mathbb{R}^N : |h(x)| \geq \varepsilon\}$ has finite Lebesgue measure, we assume that V satisfies:

(V) there exist a constant $a_0 > 0$ and a function $V_0 \in C(\mathbb{R}^N)$, 1−periodic in x_i, $1 \leq i \leq N$, such that $V_0 - V \in \mathcal{F}$ and

$$V_0(x) \geq V(x) \geq a_0 > 0, \text{ for all } x \in \mathbb{R}^N.$$

Considering $G(x,s) = \int_0^s g(x,t)\,dt$, the primitive of g, we also suppose the following hypotheses:

(g_1) $g(x,s) = o(|s|)$, as $s \to 0^+$, uniformly in $\mathbb{R}^N$;

(g_2) there exist constants $a_1, a_2 > 0$ and $2 < q_1 < 2^*_\alpha$ such that

$$|g(x,s)| \leq a_1 + a_2|s|^{q_1-1}, \text{ for all } (x,s) \in \mathbb{R}^N \times [0,+\infty);$$

(g_3) there exist a constant $2 \leq q_2 < 2^*_\alpha$ and functions $h_1 \in L^1(\mathbb{R}^N)$, $h_2 \in \mathcal{F}$ such that

$$\frac{1}{2}g(x,s)s - G(x,s) \geq -h_1(x) - h_2(x)s^{q_2}, \text{ for all } (x,s) \in \mathbb{R}^N \times [0,+\infty).$$

The asymptotic periodicity of g at infinity is given by the following condition:

(g_4) there exist a constant $2 \leq q_3 \leq 2^*_\alpha - 1$ and functions $h_3 \in \mathcal{F}$, $g_0 \in C(\mathbb{R}^N \times \mathbb{R}, (0,+\infty))$, 1-periodic in x_i, $1 \leq i \leq N$, such that:

   (i) $G(x,s) \geq G_0(x,s) = \int_0^s g_0(x,t)\,dt$, for all $(x,s) \in \mathbb{R}^N \times [0,+\infty)$;

   (ii) $|g(x,s) - g_0(x,s)| \leq h_3(x)|s|^{q_3-1}$, for all $(x,s) \in \mathbb{R}^N \times [0,+\infty)$;

   (iii) the function $g_0(x,s)/s$ is nondecreasing in the variable $s > 0$, for each $x \in \mathbb{R}^N$.

Finally, we also suppose that g satisfies:

(g_5) there exist an open bounded set $\Omega \subset \mathbb{R}^N$, $2 < p < 2^*_\alpha$ and $C_0 > 0$ such that

   (i) $\dfrac{G(x,s)}{s^p} \to +\infty$, as $s \to +\infty$, uniformly in Ω, if $N \geq 4\alpha$;

   (ii) $\dfrac{G(x,s)}{s^p} \to +\infty$, as $s \to +\infty$, uniformly in Ω, if $2\alpha < N < 4\alpha$ and $\frac{4\alpha}{N-2\alpha} < p < 2^*_\alpha$;

   (iii) $G(x,s) \geq C_0 s^p$ almost everywhere in $\mathbb{R}^N$, if $2\alpha < N < 4\alpha$ and $2 < p < \frac{4\alpha}{N-2\alpha}$.

In Chapter 3 we prove the following result.

Theorem 0.5. *Assume* (V), $(g_1)-(g_5)$ *and that one of the following statements holds:*

(1) $N \geq 4\alpha$

(2) $2\alpha < N < 4\alpha$ *and* $\frac{4\alpha}{N-2\alpha} < p < 2^*_\alpha$

(3) $2\alpha < N < 4\alpha$ *and* $2 < p < \frac{4\alpha}{N-2\alpha}$, *with* C_0 *large enough in* (g_5).

Then, problem (0.7) *has a nonnegative nontrivial weak solution.*

Moreover, in the particular case: $V = V_0$, $g = g_0$, considering the problem

$$(-\Delta)^\alpha u + V_0(x)u = |u|^{2^*_\alpha - 2}u + g_0(x,u), \quad x \in \mathbb{R}^N, \tag{0.8}$$

under the hypothesis:

(V_0) the function $V_0 \in C(\mathbb{R}^N)$ is 1-periodic in x_i, $1 \leq i \leq N$, and there exists a constant $a_0 > 0$ such that

$$V_0(x) \geq a_0 > 0, \text{ for all } \in \mathbb{R}^N;$$

and the function g_0 satisfies $(g_1) - (g_3)$ and (g_5), we state:

Theorem 0.6. *Assume* (V_0), $(g_1)-(g_3)$, (g_5) *and that one of the following statements holds:*

(1) $N \geq 4\alpha$

(2) $2\alpha < N < 4\alpha$ *and* $\frac{4\alpha}{N-2\alpha} < p < 2^*_\alpha$

(3) $2\alpha < N < 4\alpha$ *and* $2 < p < \frac{4\alpha}{N-2\alpha}$, *with* C_0 *large enough in* (g_5).

Then, problem (0.8) *has a nonnegative nontrivial weak solution.*

Due to the loss of compactness, the study of critical problems have some additional difficulties. In order to overcome such difficulties, we follow the ideas of Brezis-Nirenberg (see [8]). Among the difficulties found, we can mention the estimating of the minimax level and the fact that the associated functional with problem (0.7) does not satisfy the compactness condition of Palais-Smale type. Moreover, we assume that the subcritical pertubation $g(x,u)$ does not satisfy the (AR) condition, this creates an extra difficulty in the proof of the limitation of Cerami sequence. Lastly, we prove Theorems 0.5 and 0.6 by combining two versions of the Mountain Pass Theorem and a version of the Concentration-Compactness Principle due to Lions.

Our results complement the study made in [10, 21, 43] in the sense that the nonlinearity behaves like $u^{2^*_\alpha - 1} + g(x, u)$, where the subcritical perturbation $g(x, u)$ does not satisfy (AR) condition. Moreover, we also complement [10, 11, 18, 41] in the sense that the potential $V(x)$ belongs to a different class from those treated by them.

In order to do not get resorting to Introduction, and, for the sake of independence of the chapters, we will present again, in each chapter, the main results and the hypotheses about the functions $V(x)$ and $f(x, u)$.

Notation and terminology

In this work we will use the following symbology:

- C, C_0, C_1, C_2, ... denote positive constants (possibly different);
- $\mathrm{supp}(f)$ denotes the support of the function f;
- $B_R(x)$ denotes an open ball of radius R and center x; B_R denotes an open ball of radius R and center at origin and $\overline{B}_R$ is the closed ball with center at origin and radius R;
- B_R^c denotes the complement of B_R;
- $\rightharpoonup, \to$ denote weak and strong convergence, respectively, in a normed space;
- $u^+ = \max\{u, 0\}$ and $u^- = \max\{-u, 0\}$;
- $\mathcal{X}_\Omega$ denotes the characteristic function of the set Ω;
- $\|\cdot\|_{1/2}$ denotes the norm in the space $H^{1/2}(\mathbb{R})$;
- $\|\cdot\|_*$ denotes the norm in the topologic dual space X^*;
- $\|\cdot\|_p$ denotes the standard $L^p(\mathbb{R}^N)$-norm;
- $\|\cdot\|_\infty$ denotes the standard $L^\infty(\mathbb{R}^N)$-norm;

Chapter 1

Semilinear elliptic equations for the fractional Laplacian operator involving critical exponential growth

This chapter is devoted to the paper [14], here we establish the existence and multiplicity of weak solutions for a class of equations involving the fractional Laplacian operator, potentials that may change sign and nonlinearities with critical exponential growth. The proofs of our existence results rely on minimization methods and the Mountain Pass Theorem.

Motivation and main results

The starting point of this chapter is to investigate the existence and multiplicity of weak solutions for the following class of equations

$$(-\Delta)^{1/2}u + V(x)u = f(x,u) + h \quad \text{in} \quad \mathbb{R}, \tag{1.1}$$

where $V : \mathbb{R} \to \mathbb{R}$ is a continuous potential which may change sign, the nonlinearity $f(x,s)$ behaves like $\exp(\alpha_0 s^2)$ when $|s| \to +\infty$ for some $\alpha_0 > 0$, h belongs to the dual of an appropriate functional space and $(-\Delta)^{1/2}$ is the fractional Laplacian operator which, for a sufficiently regular function $u : \mathbb{R} \to \mathbb{R}$, is defined by

$$(-\Delta)^{1/2}u(x) = -\frac{1}{2\pi}\int\limits_{\mathbb{R}} \frac{u(x+y)+u(x-y)-2u(x)}{|y|^2}\,\mathrm{d}y. \tag{1.2}$$

In order to study variationally (1.1), we consider a suitable subspace of the fractional Sobolev space $H^{1/2}(\mathbb{R})$. Recall that $H^{1/2}(\mathbb{R})$ is defined as the space

$$H^{1/2}(\mathbb{R}) := \left\{ u \in L^2(\mathbb{R}) \, : \, \frac{|u(x) - u(y)|}{|x-y|} \in L^2(\mathbb{R} \times \mathbb{R}) \right\};$$

endowed with the norm

$$\|u\|_{1/2} := \left([u]_{1/2}^2 + \int_{\mathbb{R}} |u|^2 \, \mathrm{d}x \right)^{1/2},$$

where

$$[u]_{1/2} := \left(\int_{\mathbb{R}^2} \frac{|u(x) - u(y)|^2}{|x-y|^2} \, \mathrm{d}x \, \mathrm{d}y \right)^{1/2}$$

is the so-called *Gagliardo semi-norm* of u. For more details see Appendix (A).

Some suitable conditions on the potential V are assumed in order to apply a variational framework considering the subspace of $H^{1/2}(\mathbb{R})$ given by

$$X = \left\{ u \in H^{1/2}(\mathbb{R}) \, : \, \int_{\mathbb{R}} V(x) u^2 \, \mathrm{d}x < \infty \right\}. \tag{1.3}$$

More precisely, we suppose the following assumptions on $V(x)$:

(V_1) there exists a positive constant B such that $V(x) \geq -B$, for all $x \in \mathbb{R}$;

(V_2) the infimum

$$\lambda_1 := \inf_{\substack{u \in X \\ \|u\|_2 = 1}} \left(\frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{(u(x) - u(y))^2}{|x-y|^2} \, \mathrm{d}x \, \mathrm{d}y + \int_{\mathbb{R}} V(x) u^2 \, \mathrm{d}x \right)$$

is positive;

(V_3) $\lim_{R \to \infty} \nu(\mathbb{R} \setminus \overline{B}_R) = +\infty$, where

$$\nu(G) = \begin{cases} \inf_{\substack{u \in X_0(G) \\ \|u\|_2 = 1}} \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{(u(x) - u(y))^2}{|x-y|^2} \, \mathrm{d}x \, \mathrm{d}y + \int_G V(x) u^2 \, \mathrm{d}x & \text{if} \quad G \neq \varnothing; \\ \infty & \text{if} \quad G = \varnothing. \end{cases}$$

Here G is an open set in $\mathbb{R}$, $X_0(G) = \{u \in X \, : \, u = 0 \quad \text{in} \quad \mathbb{R} \setminus G\}$.

The hypotheses (V_1) and (V_2) ensure that X is a Hilbert space when endowed with the inner product

$$\langle u,v\rangle = \frac{1}{2\pi}\int\limits_{\mathbb{R}^2} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^2}\,\mathrm{d}x\,\mathrm{d}y + \int\limits_{\mathbb{R}} V(x)uv\,\mathrm{d}x, \quad u,v\in X,$$

which induces the norm $\|u\| := \langle u,u\rangle^{1/2}$ (see Section 1).

In this context, we assume that $h\in X^*$ (dual space of X) and we say that $u\in X$ is a weak solution for (1.1) if for all $v\in X$,

$$\frac{1}{2\pi}\int\limits_{\mathbb{R}^2} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^2}\,\mathrm{d}x\,\mathrm{d}y + \int\limits_{\mathbb{R}} V(x)uv\,\mathrm{d}x = \int\limits_{\mathbb{R}} f(x,u)v\,\mathrm{d}x + (h,v), \quad (1.4)$$

where $(\cdot,\cdot)$ denotes the duality pairing between X and X^*.

We are interested in the case that the nonlinearity $f(x,s)$ has the maximal growth which allows us to study (1.1) by using a variational framework considering the space X. More specifically, we assume sufficient conditions such that the weak solutions of (1.1) become critical points of the Euler functional $I: X\to\mathbb{R}$ defined by

$$I(u) = \frac{1}{2}\|u\|^2 - \int\limits_{\mathbb{R}} F(x,u)\,\mathrm{d}x - (h,u),$$

where $F(x,s) = \int\limits_0^s f(x,t)\mathrm{d}t$.

In order to improve the presentation of the hypotheses on $f(x,s)$, we recall some well known facts involving the limiting Sobolev embedding Theorem in 1-dimension. The Sobolev embedding assures that $H^{1/2}(\mathbb{R}) \hookrightarrow L^q(\mathbb{R})$ for any $q\in[2,+\infty)$; but $H^{1/2}(\mathbb{R})$ is not continuously embedded in $L^\infty(\mathbb{R})$ (for more details, see [17], [39]). In this case the maximal growth of $f(x,s)$, which allows us to study (1.1) by applying a variational framework involving the space $H^{1/2}(\mathbb{R})$, is motivated by the Trudinger-Moser inequality that was proved by H. Kozono, T. Sato and H. Wadade [29] and T. Ozawa [39]. More precisely, they proved that there exist positive constants ω and C such that for all $u\in H^{1/2}(\mathbb{R})$ with $\|(-\Delta)^{1/4}u\|_2\le 1$,

$$\int\limits_{\mathbb{R}} (e^{\alpha u^2}-1)\,\mathrm{d}x \le C\|u\|_2^2, \quad \text{for all } \alpha\in(0,\omega]. \quad (1.5)$$

(See also some pioneering works such as [38], [45]).

Motivated by (1.5) we say that $f(x,s)$ has *critical exponential growth* if there exists $\alpha_0 > 0$ such that

$$\lim_{|s|\to+\infty} f(x,s)e^{-\alpha|s|^2} = \begin{cases} 0, & \text{for all } \alpha > \alpha_0, \\ +\infty, & \text{for all } \alpha < \alpha_0, \end{cases}$$

uniformly in $x \in \mathbb{R}$.

Now, we are able to establish our main assumptions on the nonlinearity $f(x,s)$. In order to find weak solutions to (1.1), by using variational methods, we assume the following conditions:

(f_1) $0 \leq \lim_{s\to 0} \dfrac{f(x,s)}{s} < \lambda_1$, uniformly in x;

(f_2) $f : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is continuous, it has critical exponential growth and it is locally bounded in s, that is, for any bounded interval $J \subset \mathbb{R}$, there exists $C > 0$ such that $|f(x,s)| \leq C$, for every $(x,s) \in \mathbb{R}\times J$;

(f_3) there exists $\theta > 2$ such that

$$0 < \theta F(x,s) := \theta \int_0^s f(x,t)\,\mathrm{d}t \leq s f(x,s), \quad \text{for all} \quad (x,s) \in \mathbb{R}\times\mathbb{R}\setminus\{0\};$$

(f_4) there exist constants $s_0, M_0 > 0$ such that

$$0 < F(x,s) \leq M_0|f(x,s)|, \quad \text{for all} \quad |s| \geq s_0 \quad \text{and} \quad x \in \mathbb{R};$$

(f_5) there exist constants $p > 2$ and C_p such that, for all $s \geq 0$ and $x \in \mathbb{R}$,

$$f(x,s) \geq C_p s^{p-1},$$

with $C_p > \left[\dfrac{\alpha_0(p-2)}{2\pi\kappa p\omega}\right]^{(p-2)/2} S_p^p$, where

$$S_p := \inf_{\substack{u\in X \\ \|u\|_p=1}} \left(\frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{(u(x)-u(y))^2}{|x-y|^2}\,\mathrm{d}x\,\mathrm{d}y + \int_{\mathbb{R}} V(x)u^2\,\mathrm{d}x \right)^{1/2},$$

and κ is given in (1.7).

We highlight that the hypotheses $(f_1)-(f_5)$ have been used in many papers to find a solution using variational framework (see for instance [2], [19], [20], [23], [25]). A simple example of a function that verifies our assumptions is $f(x,s) = C_p|s|^{p-2}s + 2s(e^{s^2}-1)$ for $(x,s) \in \mathbb{R}\times\mathbb{R}$.

Under these assumptions we presents the main results of this chapter.

Theorem 1.1. *Suppose that $(V_1)-(V_3)$ and $(f_1)-(f_5)$ hold. Then there exists $\delta_1 > 0$ such that for each $0 < \|h\|_* < \delta_1$, problem (1.1) has at least two weak solutions. One of them with positive energy, and the other one with negative energy.*

Theorem 1.2. *Suppose that $(V_1)-(V_3)$ and $(f_1)-(f_5)$ hold. If $h \equiv 0$ (i.e., there is no perturbation in (1.1)) then problem (1.1) has a weak solution with positive energy.*

Remark 1.3. Our work was mainly motivated by Iannizzotto and Squassina [25], and also by some recently published papers that discuss (1.1) by using a purely variational approach (see, for instance, [11, 23, 41, 42] and references therein). The goal is to extend and to improve the results obtained in [11, 25, 41, 42] since we work with nonlinearities with critical exponential growth and potentials that may change sign, vanish and be unbounded.

Remark 1.4. It is important to notice that many authors, in different ways, have studied problems involving the standard Laplacian instead of fractional Laplacian. One of these problems is to investigate the existence of solutions for the following class of equations:

$$-\Delta u + V(x)u = g(x,u), \quad x \in \mathbb{R}^N, \tag{1.6}$$

see e.g. [2], [4] for the case where $g(x,s)$ has subcritical growth in the Sobolev sense, and [19, 20, 30, 46] for the case where $g(x,s)$ has critical growth in the Trudinger-Moser sense. In these papers, the existence of solutions has been discussed under different conditions on the potential $V(x)$. The main reason of the hypotheses used is to overcome the problem of "lack of compactness", which usually appear in elliptic problems in unbounded domains. More specifically, the papers [4, 40] assume that the potential is continuous and positive and, furthermore, that one of the following assumptions holds:

(a) $V(x) \nearrow +\infty$ as $|x| \to +\infty$;

(b) for any $A > 0$, the sublevel set $\{x \in \mathbb{R}^N : V(x) \leq A\}$ has finite Lebesgue measure.

One of this conditions implies that the space

$$E := \left\{ u \in W^{1,2}(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x)u^2 \,\mathrm{d}x < \infty \right\}$$

is compactly embedded in the Lebesgue space $L^q(\mathbb{R}^N)$ for all $q \geq 2$.

We point out that (V_3) generalizes these two conditions above. It is also important to observe that the conditions $(V_1) - (V_3)$ were already considered by B. Sirakov [44] in order to study (1.6) by considering that $g(x,u)$ has subcritical growth in the Sobolev sense.

Remark 1.5. A usual example of function satisfying the assumptions $(V_1) - (V_3)$ it is a continuous function $V(x) = V^+(x) - V^-(x)$, where V^+ and V^- are the positive and negative parts of V, with V^+ and V^- satisfying:

(H_1) $\lim_{|x| \to +\infty} V^+(x) = +\infty$;

(H_2) $\|V^-\|_\infty < \nu_1 := \inf_{\substack{u \in X \\ \|u\|_2 = 1}} \left(\frac{1}{2\pi}[u]_{1/2}^2 + \int_{\mathbb{R}} V^+(x)u^2 \,\mathrm{d}x \right).$

By (H_1), it is not difficult to see that ν_1 is positive and, thus, for any $u \in X$ such that $\|u\|_2 = 1$, we have

$$\begin{aligned} \frac{1}{2\pi}[u]_{1/2}^2 + \int_{\mathbb{R}} V(x)u^2 &\geq \frac{1}{2\pi}[u]_{1/2}^2 + \int_{\mathbb{R}} V^+(x)u^2 - \|V^-\|_\infty \\ &\geq \nu_1 - \|V^-\|_\infty > 0. \end{aligned}$$

Consequently, we reach $\lambda_1 > 0$.

Remark 1.6. Similarly to [13, 19, 20, 25] we will use minimization to find the first weak solution with negative energy, and the Mountain Pass Theorem to obtain the existence of the second weak solution with positive energy. First of all, we need to check some conditions concerning the mountain pass geometry and the compactness of the associated functional. Trudinger-Moser's inequality to the space X and a version of a Concentration-Compactness Principle due to P. -L. Lions [34] to the space X have a crucial role in our proof (see Section 1). The main difficulties lie in the nonlocal operator involved and critical exponential growth of the nonlinearity.

Remark 1.7. In the papers [29, 39] Trudinger-Moser's inequality (1.5) was proved for the fractional Sobolev space $W^{N/p,p}(\mathbb{R}^N)$ with $1 < p < \infty$ and $N \geq 1$. However, for the class of operators considered in this work was fundamental equality (1.13) which is valid only if $p = 2$. Since we are interested in the case $0 < N/p < 1$, our approach is restricted to the case $N = 1$.

Remark 1.8. If a weak solution u is sufficiently regular, then, it is possible to get a pointwise expression of the fractional Laplacian as it is described in (1.2) (see, for example, [47]). In this case we may ensure that $u > 0$ if $u \neq 0$, (see Remark 3.4).

The outline of this chapter is as follows: Section 1.2 contains some preliminary results. Section 1.3 contains the variational framework and we also check the geometric conditions of the associated functional. Section 1.4 deals with Palais-Smale condition and Section 1.5 discusses the minimax level. Finally in Section 1.6, we complete the proofs of our main results.

Some preliminary results

Our first lemma enables us to settle the variational setting.

Lemma 1.9. *Suppose that* (V_1) *and* (V_2) *are satisfied. Then there exists* $\kappa > 0$ *satisfying*

$$\frac{1}{2\pi}\left(\int\limits_{\mathbb{R}^2} \frac{(u(x)-u(y))^2}{|x-y|^2}\,\mathrm{d}x\,\mathrm{d}y\right) + \int\limits_{\mathbb{R}} V(x)u^2\,\mathrm{d}x \geq \kappa\|u\|_{1/2}^2\,, \quad \textit{for any} \quad u \in X. \tag{1.7}$$

Proof. Suppose, by contradiction, that (1.7) does not hold. Then for each $n \in \mathbb{N}$ there exists $u_n \in X$ such that

$$\|u_n\|_{1/2}^2 = 1 \quad \text{and} \quad \frac{1}{2\pi}\left(\int\limits_{\mathbb{R}^2} \frac{(u_n(x)-u_n(y))^2}{|x-y|^2}\,\mathrm{d}x\,\mathrm{d}y\right) + \int\limits_{\mathbb{R}} V(x)u_n^2\,\mathrm{d}x < \frac{1}{n}. \tag{1.8}$$

It follows from (1.8) and (V_2) that

$$\lambda_1 \quad \leq \quad \frac{1}{\|u_n\|_2^2}\left(\frac{1}{2\pi}\left(\int\limits_{\mathbb{R}^2} \frac{(u_n(x)-u_n(y))^2}{|x-y|^2}\,\mathrm{d}x\,\mathrm{d}y\right) + \int\limits_{\mathbb{R}} V(x)u_n^2\,\mathrm{d}x\right) < \frac{1}{n\|u_n\|_2^2},$$

for all $n \in \mathbb{N}$. The last inequality, together with $\lambda_1 > 0$ and $\|u_n\|_{1/2}^2 = 1$, implies that $\|u_n\|_2 \to 0$ and $[u_n]_{1/2} \to 1$. Consequently, by using (V_1), we obtain the contradiction

$$o_n(1) = -B\|u_n\|_2^2 \leq \int\limits_{\mathbb{R}} V(x)\,u_n^2\,\mathrm{d}x < \frac{1}{n} - \frac{1}{2\pi}[u_n]_{1/2}^2 \to -\frac{1}{2\pi}.$$

Thus, the proof is complete. □

Using (1.7), we have that

$$\langle u, v\rangle := \frac{1}{2\pi}\left(\int_{\mathbb{R}^2} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^2}\,\mathrm{d}x\,\mathrm{d}y\right) + \int_{\mathbb{R}} V(x)uv\,\mathrm{d}x$$

defines an inner product in X which corresponds the norm

$$\|u\| = \left\{\frac{1}{2\pi}\left(\int_{\mathbb{R}^2} \frac{|u(x)-u(y)|^2}{|x-y|^2}\,\mathrm{d}x\,\mathrm{d}y\right) + \int_{\mathbb{R}} V(x)u^2\,\mathrm{d}x\right\}^{1/2}.$$

Moreover, X is a Hilbert space and the embedding $X \hookrightarrow H^{1/2}(\mathbb{R})$ is continuous. Therefore the embedding

$$X \hookrightarrow L^q(\mathbb{R}) \quad \text{for all} \quad q \in [2, \infty),$$

is continuous and the constant

$$S_p := \inf_{\substack{u\in X\\ \|u\|_p=1}} \left(\frac{1}{2\pi}\int_{\mathbb{R}^2} \frac{(u(x)-u(y))^2}{|x-y|^2}\,\mathrm{d}x\,\mathrm{d}y + \int_{\mathbb{R}} V(x)u^2\,\mathrm{d}x\right)^{1/2} \tag{1.9}$$

is positive.

Next, similar to Sirakov [44], we prove the following compactness result.

Lemma 1.10. *Suppose that* $(V_1)-(V_3)$ *hold. Then the embedding* $X \hookrightarrow L^q(\mathbb{R})$ *is compact for any* $q \in [2, \infty)$.

Proof. Let $(u_n) \subset X$ be a bounded sequence, up to a subsequence, we may assume that $u_n \rightharpoonup 0$ in X. We must prove that, up to a subsequence,

$$u_n \to 0 \quad \text{in} \quad L^2(\mathbb{R}), \text{ as } n \to \infty.$$

We take a function $\varphi \in C^\infty(\mathbb{R}, [0, 1])$ such that $\varphi \equiv 0$ in $\overline{B}_R$ and $\varphi \equiv 1$ in $\mathbb{R}\backslash\overline{B}_{R+1}$, where the constant $R > 0$ will be chosen later. Thus,

$$\begin{aligned}\|u_n\|_2 &= \|(1-\varphi)u_n + \varphi u_n\|_2\\ &\le \|(1-\varphi)u_n\|_2 + \|\varphi u_n\|_2\\ &= \|(1-\varphi)u_n\|_{L^2(B_{R+1})} + \|\varphi u_n\|_{L^2(\mathbb{R}\backslash\overline{B}_R)}.\end{aligned} \tag{1.10}$$

Since $H^{1/2}(B_{R+1})$ is compactly embedded into $L^2(B_{R+1})$, up to a subsequence, given $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$\|(1-\varphi)u_n\|_{L^2(B_{R+1})} < \frac{\varepsilon}{2}, \quad \text{for all} \quad n \ge n_0. \tag{1.11}$$

By the definition of $\nu(\mathbb{R}\setminus\overline{B}_R)$, it follows that

$$\|\varphi u_n\|^2_{L^2(\mathbb{R}\setminus\overline{B}_R)} \leq \frac{\|\varphi u_n\|^2}{\nu(\mathbb{R}\setminus\overline{B}_R)} \leq \frac{C}{\nu(\mathbb{R}\setminus\overline{B}_R)}, \quad \text{for all} \quad n\in\mathbb{N}.$$

Hence, by using (V_3), there exists $R=R(\varepsilon)>0$ sufficiently large, such that

$$\|\varphi u_n\|_{L^2(\mathbb{R}\setminus\overline{B}_R)} < \frac{\varepsilon}{2}, \quad \text{for all} \quad n\in\mathbb{N}. \tag{1.12}$$

Combining (1.10), (1.11) and (1.12), we conclude that

$$\|u_n\|_2 < \varepsilon, \quad \text{for all} \quad n\geq n_0,$$

which proves the lemma. □

In the sequel we will prove a version of (1.5) for the space X. This result is our main tool to prove Theorems 1.1 and 1.2. The ideas used in the proof are inspired in [19], [20], [25] and we present here for sake of completeness. We will need the following relation

$$\|(-\Delta)^{1/4}u\|_2 = (2\pi)^{-1/2}[u]_{1/2}, \quad \text{for all} \quad u\in H^{1/2}(\mathbb{R}), \tag{1.13}$$

which was proved in [17, Proposition 3.6].

Lemma 1.11. *If* $0<\alpha\leq 2\pi\kappa\omega$ *and* $u\in X$ *with* $\|u\|\leq 1$*, then there exists* $C>0$ *such that*

$$\int_{\mathbb{R}} (e^{\alpha u^2}-1)\,\mathrm{d}x \leq C. \tag{1.14}$$

Moreover, for any $\alpha>0$ *and* $u\in X$*, we have*

$$\int_{\mathbb{R}} (e^{\alpha u^2}-1)\,\mathrm{d}x < \infty. \tag{1.15}$$

Proof. First we observe that if a function $u\in X$ satisfies $\|u\|\leq 1$, setting $v=(2\pi\kappa)^{1/2}u$, then $v\in H^{1/2}(\mathbb{R})$ and by (1.7) and (1.13) we get

$$\|(-\Delta)^{1/4}v\|_2 = (2\pi)^{-1/2}[v]_{1/2} \leq \kappa^{1/2}\|u\|_{1/2} \leq \|u\| \leq 1.$$

Consequently, using (1.5),

$$\int_{\mathbb{R}} (e^{\alpha u^2}-1)\,\mathrm{d}x = \int_{\mathbb{R}} (e^{(\alpha/2\pi\kappa)v^2}-1)\,\mathrm{d}x \leq C_1\|v\|_2^2 \leq C.$$

Thus, we obtain (1.14).

Now we prove the second part of the lemma. Given $u \in X$ and $\varepsilon > 0$, there exists $\varphi \in C_0^\infty(\mathbb{R})$ such that $\|u - \varphi\| < \varepsilon$. Since

$$e^{\alpha u^2} - 1 \leq e^{\alpha(2(u-\varphi)^2 + 2\varphi^2)} - 1 \leq \frac{1}{2}\left(e^{4\alpha(u-\varphi)^2} - 1\right) + \frac{1}{2}\left(e^{4\alpha\varphi^2} - 1\right),$$

it follows that

$$\int_{\mathbb{R}} (e^{\alpha u^2} - 1)\,\mathrm{d}x \leq \frac{1}{2}\int_{\mathbb{R}} (e^{4\alpha\|u-\varphi\|^2\left(\frac{u-\varphi}{\|u-\varphi\|}\right)^2} - 1)\,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}} (e^{4\alpha\varphi^2} - 1)\,\mathrm{d}x. \tag{1.16}$$

Choosing $\varepsilon > 0$ such that $4\alpha\varepsilon^2 < 2\pi\kappa\omega$, we have $4\alpha\|u - \varphi\|^2 < 2\pi\kappa\omega$. Then, from (1.14) and (1.16), we obtain

$$\int_{\mathbb{R}} (e^{\alpha u^2} - 1)\,\mathrm{d}x \leq \frac{C}{2} + \frac{1}{2}\int_{\mathrm{supp}(\varphi)} (e^{4\alpha\varphi^2} - 1)\,\mathrm{d}x < \infty.$$

Thus, the proof is complete. □

The next lemma will be used to guarantee the geometry of the functional I.

Lemma 1.12. *If $v \in X$, $\alpha > 0$, $q > 2$ and $\|v\| \leq M$ with $\alpha M^2 < 2\pi\kappa\omega$, then there exists $C = C(\alpha, M, q) > 0$ such that*

$$\int_{\mathbb{R}} (e^{\alpha v^2} - 1)|v|^q\mathrm{d}x \leq C\|v\|^q.$$

Proof. Taking $r > 1$ close to 1 such that $\alpha r M^2 < 2\pi\kappa\omega$. By Hölder's inequality with $r' = r/(r-1)$, we have

$$\int_{\mathbb{R}} (e^{\alpha v^2} - 1)|v|^q\mathrm{d}x \leq \left(\int_{\mathbb{R}} (e^{\alpha v^2} - 1)^r\mathrm{d}x\right)^{1/r} \|v\|_{r'q}^q. \tag{1.17}$$

Notice that for $r > 1$, we have

$$(e^{\alpha s^2} - 1)^r \leq (e^{\alpha r s^2} - 1), \quad \text{for all} \quad s \in \mathbb{R}. \tag{1.18}$$

Hence, from (1.17) and (1.18), we get

$$\begin{aligned}\int_{\mathbb{R}} (e^{\alpha v^2} - 1)|v|^q\mathrm{d}x &\leq \left(\int_{\mathbb{R}} (e^{\alpha r v^2} - 1)\mathrm{d}x\right)^{1/r} \|v\|_{r'q}^q \\ &\leq \left(\int_{\mathbb{R}} (e^{\alpha r M^2\left(\frac{v}{\|v\|}\right)^2} - 1)\,\mathrm{d}x\right)^{1/r} \|v\|_{r'q}^q.\end{aligned}$$

Since $\alpha r M^2 < 2\pi\kappa\omega$, it follows by (1.14) and the continuous embedding $X \hookrightarrow L^{r'q}(\mathbb{R})$ that

$$\int_{\mathbb{R}} (e^{\alpha v^2} - 1)|v|^q\,\mathrm{d}x \leq C\|v\|^q.$$

Therefore, the proof is complete. □

In line with the Concentration-Compactness Principle due to P. -L. Lions [34], we will show a refinement of (1.14). This result will be crucial to show that the functional I satisfies the Palais-Smale condition.

Lemma 1.13. *If (v_n) is a sequence in X with $\|v_n\| = 1$ for all $n \in \mathbb{N}$ and $v_n \rightharpoonup v$ in X, $0 < \|v\| < 1$, then for all $0 < t < 2\pi\kappa\omega(1 - \|v\|^2)^{-1}$, we have*

$$\sup_n \int_{\mathbb{R}} (e^{tv_n^2} - 1)\,\mathrm{d}x < \infty.$$

Proof. Since $v_n \rightharpoonup v$ in X and $\|v_n\| = 1$, we conclude that

$$\|v_n - v\|^2 = 1 - 2\langle v_n, v\rangle + \|v\|^2 \to 1 - \|v\|^2 < \frac{2\pi\kappa\omega}{t}.$$

Then, for $n \in \mathbb{N}$ large enough, we have $t\|v_n - v\|^2 < 2\pi\kappa\omega$. Thus, we may choose $q > 1$ close to 1 and $\varepsilon > 0$ satisfying

$$qt(1 + \varepsilon^2)\|v_n - v\|^2 < 2\pi\kappa\omega, \tag{1.19}$$

for $n \in \mathbb{N}$ enough large. By (1.14) and (1.19), there exists $C > 0$ such that

$$\int_{\mathbb{R}} (e^{qt(1+\varepsilon^2)(v_n - v)^2} - 1)\,\mathrm{d}x = \int_{\mathbb{R}} \left(e^{qt(1+\varepsilon)^2\|v_n - v\|^2\left(\frac{v_n - v}{\|v_n - v\|}\right)^2} - 1 \right) \mathrm{d}x \le C. \tag{1.20}$$

Moreover, since

$$tv_n^2 \le t(1 + \varepsilon^2)(v_n - v)^2 + t\left(1 + \frac{1}{\varepsilon^2}\right)v^2,$$

it follows by the convexity of the exponential function, with $q^{-1} + r^{-1} = 1$, that

$$e^{tv_n^2} - 1 \le \frac{1}{q}(e^{qt(1+\varepsilon^2)(v_n - v)^2} - 1) + \frac{1}{r}(e^{rt(1+1/\varepsilon^2)v^2} - 1).$$

Therefore, by (1.15) and (1.20), we get

$$\int_{\mathbb{R}} (e^{tv_n^2} - 1)\,\mathrm{d}x \le \frac{1}{q}\int_{\mathbb{R}} (e^{qt(1+\varepsilon^2)(v_n - v)^2} - 1)\,\mathrm{d}x + \frac{1}{r}\int_{\mathbb{R}} (e^{rt(1+1/\varepsilon^2)v^2} - 1)\,\mathrm{d}x \le C,$$

and the result is proved. □

The variational framework

In order to apply the variational approach, we define the functional $I : X \to \mathbb{R}$, by

$$I(u) = \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}} F(x, u)\,\mathrm{d}x - (h, u).$$

Notice that, from (f_1) and (f_2), for each $\alpha > \alpha_0$ and $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$|F(x,s)| \leq \frac{(\lambda_1 - \varepsilon)}{2} s^2 + C_\varepsilon (e^{\alpha s^2} - 1), \quad \text{for all} \quad s \in \mathbb{R},$$

which combined with the continuous embedding $X \hookrightarrow L^2(\mathbb{R})$ and (1.15) assures that $F(x,u) \in L^1(\mathbb{R})$ for all $u \in X$. Consequently, I is well-defined and, by standard arguments, $I \in C^1(X,\mathbb{R})$, (see details in Appendix A), with

$$I'(u)v = \langle u, v \rangle - \int_{\mathbb{R}} f(x,u)v \,\mathrm{d}x - (h, v),$$

for all $u, v \in X$. Hence, a critical point of I is a weak solution of (1.1) and reciprocally.

The geometric conditions of the Mountain Pass Theorem for the functional I are established by the next lemmas.

Lemma 1.14. *Suppose that $(V_1)-(V_2)$ and $(f_1)-(f_2)$ hold. Then there exists $\delta_1 > 0$ such that for each $h \in X^*$ with $\|h\|_* < \delta_1$, there exists $\rho_h > 0$ such that*

$$I(u) > 0 \quad \text{if} \quad \|u\| = \rho_h.$$

Proof. From (f_1) and (f_2), given $\varepsilon > 0$, there exists $C > 0$ such that, for all $\alpha > \alpha_0$ and $q > 2$,

$$|F(x,s)| \leq \frac{(\lambda_1 - \varepsilon)}{2} s^2 + C(e^{\alpha s^2} - 1)|s|^q, \quad \text{for all} \quad s \in \mathbb{R}. \tag{1.21}$$

By using (1.21) and (V_2), we reach

$$\begin{aligned} I(u) &\geq \frac{1}{2}\|u\|^2 - \frac{(\lambda_1 - \varepsilon)}{2} \int_{\mathbb{R}} u^2 \,\mathrm{d}x - C \int_{\mathbb{R}} (e^{\alpha u^2} - 1)|u|^q \,\mathrm{d}x - \|h\|_* \|u\| \\ &\geq \frac{1}{2}\|u\|^2 - \frac{(\lambda_1 - \varepsilon)}{2\lambda_1} \|u\|^2 - C \int_{\mathbb{R}} (e^{\alpha u^2} - 1)|u|^q \,\mathrm{d}x - \|h\|_* \|u\|. \end{aligned}$$

Then, for $u \in X$ such that $\alpha\|u\|^2 < 2\pi\kappa\omega$, using Lemma 1.12, we obtain

$$I(u) \geq \left(\frac{1}{2} - \frac{(\lambda_1 - \varepsilon)}{2\lambda_1} \right) \|u\|^2 - C\|u\|^q - \|h\|_* \|u\|.$$

Consequently,

$$I(u) \geq \|u\| \left[\left(\frac{1}{2} - \frac{(\lambda_1 - \varepsilon)}{2\lambda_1} \right) \|u\| - C\|u\|^{q-1} - \|h\|_* \right].$$

Since $\frac{1}{2} - \frac{(\lambda_1 - \varepsilon)}{2\lambda_1} > 0$, we may choose $\rho_h > 0$ such that

$$\left(\frac{1}{2} - \frac{(\lambda_1 - \varepsilon)}{2\lambda_1} \right) \rho_h - C\rho_h^{q-1} > 0.$$

Thus, for $\|h\|_*$ sufficiently small, there exists ρ_h such that $I(u) > 0$ if $\|u\| = \rho_h$. Therefore, the proof is complete. □

Lemma 1.15. *Assume that* $(V_1) - (V_2)$ *and* $(f_1) - (f_3)$ *hold. Then there exists* $e \in X$ *with* $\|e\| > \rho_h$ *such that*

$$I(e) < \inf_{\|u\|=\rho_h} I(u).$$

Proof. Let $u \in C_0^\infty(\mathbb{R}) \setminus \{0\}$, $u \geq 0$ with compact support $K = \operatorname{supp}(u)$. By using (f_2) and (f_3), there exist positive constants C_1 and C_2 such that

$$F(x,s) \geq C_1 s^\theta - C_2, \quad \text{for all} \quad (x,s) \in K \times [0,\infty) \quad \text{and} \quad \theta > 2.$$

Then, for $t > 0$, we get

$$I(tu) \;\; \leq \;\; \frac{t^2}{2}\|u\|^2 - C_1 t^\theta \int\limits_K u^\theta \,\mathrm{d}x + C_2 \int\limits_K \mathrm{d}x + t|(h,u)|.$$

Since $\theta > 2$, we have $I(tu) \to -\infty$ as $t \to \infty$. Setting $e = tu$ with t large enough, we conclude the proof. □

In order to find an appropriate ball to use minimization argument, we prove the following result.

Lemma 1.16. *Suppose that* $(V_1) - (V_2)$ *and* $(f_1) - (f_2)$ *hold. If* $h \neq 0$, *there exist* $\eta > 0$ *and* $v \in X \setminus \{0\}$ *such that* $I(tv) < 0$ *for all* $0 < t < \eta$. *In particular,*

$$-\infty < c_0 \equiv \inf_{\|u\| \leq \eta} I(u) < 0.$$

Proof. For each $h \in X^*$, by applying the Riesz Representation Theorem in the space X, the problem

$$(-\Delta)^{1/2} u + V(x) u = h, \quad x \;\text{ in }\; \mathbb{R},$$

has a unique weak solution $v \in X$ such that

$$(h,v) = \|v\|^2 > 0.$$

Consequently, from (f_1) and (f_2), there exists $\eta > 0$ such that

$$\frac{d}{dt} I(tv) = t\|v\|^2 - \int\limits_{\mathbb{R}} f(x,tv) v \,\mathrm{d}x - (h,v) < 0,$$

for all $0 < t < \eta$. Using that $I(0) = 0$, it must occur $I(tv) < 0$ for all $0 < t < \eta$, this concludes the proof. □

Palais-Smale compactness condition

In this section we will show that the functional I satisfies the Palais-Smale condition for certain energy levels. Recall that the functional I satisfies the Palais-Smale condition at the level c, denoted by $(PS)_c$, if any sequence $(u_n) \subset X$ such that

$$I(u_n) \to c \quad \text{and} \quad I'(u_n) \to 0 \ \text{ as } \ n \to \infty, \tag{1.22}$$

has a strongly convergent subsequence in X.

Lemma 1.17. *Suppose that $(V_1)-(V_3)$ and $(f_1)-(f_4)$ are satisfied. Let $(u_n) \subset X$ be an arbitrary Palais-Smale sequence of I at level c. Then there exists a subsequence of (u_n) (also denoted by (u_n)) and $u \in X$ such that*

$$\begin{cases} u_n \rightharpoonup u & \text{in } X, \\ f(x,u_n) \to f(x,u) & \text{in } L^1_{loc}(\mathbb{R}), \\ F(x,u_n) \to F(x,u) & \text{in } L^1(\mathbb{R}). \end{cases}$$

Proof. By (f_3), for $\theta > 2$ we get

$$\begin{aligned} I(u_n) - \frac{1}{\theta}I'(u_n)u_n &= \left(\frac{1}{2}-\frac{1}{\theta}\right)\|u_n\|^2 + \int_{\mathbb{R}} \left(\frac{1}{\theta}f(x,u_n)u_n - F(x,u_n)\right) \mathrm{d}x + \left(\frac{1}{\theta}-1\right)(h,u_n) \\ &\geq \left(\frac{1}{2}-\frac{1}{\theta}\right)\|u_n\|^2 + \left(\frac{1}{\theta}-1\right)(h,u_n). \end{aligned} \tag{1.23}$$

Using (1.22), we obtain that for n sufficiently large

$$I(u_n) - \frac{1}{\theta}I'(u_n)u_n \leq C + \|u_n\|.$$

Combining this with (1.23), we have $\|u_n\| \leq C$. Since X is a Hilbert space, up to a subsequence, we may assume that there exists $u \in X$ such that

$$\begin{cases} u_n \rightharpoonup u & \text{in } X, \\ u_n \to u & \text{in } L^q(\mathbb{R}), \text{ for all } q \in [2,\infty), \\ u_n(x) \to u(x) & \text{almost everywhere in } \mathbb{R}. \end{cases}$$

From (1.22) and since $\|u_n\| \leq C$, there exists $C_1 > 0$ such that

$$\int_{\mathbb{R}} |f(x,u_n)u_n| \leq C_1.$$

Consequently, by [13, Lemma 2.1], we get

$$f(x,u_n) \to f(x,u) \ \text{ in } \ L^1_{loc}(\mathbb{R}), \ \text{ as } n \to \infty. \tag{1.24}$$

Next, similar to N. Lam and G. Lu [30], we will prove the last convergence of the lemma. Firstly, note that by using (f_3) and (f_4), for each $R > 0$, there exists $C_0 > 0$ such that

$$F(x, u_n) \leq C_0 |f(x, u_n)|.$$

This combined with (1.24) and the Generalized Lebesgue's Dominated Convergence Theorem, imply

$$F(x, u_n) \to F(x, u) \text{ in } L^1(B_R), \text{ for all } R > 0.$$

In order to conclude the last convergence of the lemma, it is sufficient to prove that given $\delta > 0$, there exists $R > 0$ such that

$$\int\limits_{B_R^c} F(x, u_n)\, \mathrm{d}x \leq \delta \text{ and } \int\limits_{B_R^c} F(x, u)\, \mathrm{d}x \leq \delta.$$

First, we note that by using (f_1), (f_3) and (f_4), there exist $C_1, C_2 > 0$ such that

$$|F(x, s)| \leq C_1 |s|^2 + C_2 |f(x, s)|, \quad \text{for all} \quad (x, s) \in \mathbb{R} \times \mathbb{R}.$$

Thus, for each $A > 0$, we obtain

$$\begin{aligned} \int\limits_{\substack{|x|>R \\ |u_n|>A}} F(x, u_n)\, \mathrm{d}x &\leq C_1 \int\limits_{\substack{|x|>R \\ |u_n|>A}} |u_n|^2\, \mathrm{d}x + C_2 \int\limits_{\substack{|x|>R \\ |u_n|>A}} |f(x, u_n)|\, \mathrm{d}x \\ &\leq \frac{C_1}{A} \int\limits_{\substack{|x|>R \\ |u_n|>A}} |u_n|^3\, \mathrm{d}x + \frac{C_2}{A} \int\limits_{\mathbb{R}} |f(x, u_n) u_n|\, \mathrm{d}x \\ &\leq \frac{C_1}{A} \|u_n\|^3 + \frac{C_2}{A} \int\limits_{\mathbb{R}} |f(x, u_n) u_n|\, \mathrm{d}x. \end{aligned}$$

Since $\|u_n\| \leq C$ and $\int\limits_{\mathbb{R}} |f(x, u_n) u_n|\, \mathrm{d}x \leq C_1$, given $\delta > 0$, we may choose $A > 0$ such that

$$\frac{C_1}{A} \|u_n\|^3 < \delta/3 \quad \text{and} \quad \frac{C_2}{A} \int\limits_{\mathbb{R}} |f(x, u_n) u_n|\, \mathrm{d}x < \delta/3.$$

Thus,

$$\int\limits_{\substack{|x|>R \\ |u_n|>A}} F(x, u_n)\, \mathrm{d}x \leq 2\delta/3. \tag{1.25}$$

Now, note that with such A, by (f_1) and (f_2), we have

$$F(x, s) \leq C(\alpha_0, A) |s|^2, \quad \text{for all} \quad (x, s) \in \mathbb{R} \times [-A, A].$$

Then, we get

$$\begin{aligned}\int\limits_{\substack{|x|>R\\|u_n|\leq A}} F(x,u_n)\,\mathrm{d}x &\leq C(\alpha_0,A)\int\limits_{\substack{|x|>R\\|u_n|\leq A}} |u_n|^2\,\mathrm{d}x\\ &\leq 2C(\alpha_0,A)\int\limits_{\substack{|x|>R\\|u_n|\leq A}} |u_n-u|^2\,\mathrm{d}x + 2C(\alpha_0,A)\int\limits_{\substack{|x|>R\\|u_n|\leq A}} |u|^2\,\mathrm{d}x.\end{aligned}$$

Hence, by Lemma 1.10, given $\delta > 0$, we may choose $R > 0$ such that

$$\int\limits_{\substack{|x|>R\\|u_n|\leq A}} F(x,u_n)\,\mathrm{d}x \leq \delta/3. \tag{1.26}$$

From (1.25) and (1.26), we have that given $\delta > 0$, there exists $R > 0$ such that

$$\int\limits_{|x|>R} F(x,u_n)\,\mathrm{d}x \leq \delta.$$

Similarly,

$$\int\limits_{|x|>R} F(x,u)\,\mathrm{d}x \leq \delta.$$

Combining all the above estimates and since $\delta > 0$ is arbitrary, we have

$$\int\limits_{\mathbb{R}} F(x,u_n)\,\mathrm{d}x \to \int\limits_{\mathbb{R}} F(x,u)\,\mathrm{d}x, \text{ as } n\to\infty,$$

which completes the proof. □

Finally, let us prove the main result of this section.

Proposition 1.18. *Under the hypotheses* $(V_1)-(V_3)$ *and* $(f_1)-(f_4)$*, if* $\|h\|_*$ *is sufficiently small then the functional* I *satisfies* $(PS)_c$ *for any* $0 \leq c < \pi\kappa\omega/\alpha_0$.

Proof. Let $(u_n) \subset X$ be an arbitrary Palais-Smale sequence of I at the level c. By Lemma 1.17, up to a subsequence, $u_n \rightharpoonup u$ in X. We will show that, up to a subsequence, $u_n \to u$ in X. In order to do this, we have two cases to consider:

<u>*Case 1:*</u> $u = 0$. In this case, Lemma 1.17, guarantees that

$$\int\limits_{\mathbb{R}} F(x,u_n) \to 0 \quad \text{and} \quad (h,u_n)\to 0 \quad \text{as} \quad n\to\infty.$$

Since

$$c + o_n(1) = I(u_n) = \frac{1}{2}\|u_n\|^2 - \int\limits_{\mathbb{R}} F(x,u_n) - (h,u_n),$$

we get

$$\lim_{n\to\infty} \|u_n\|^2 = 2c.$$

Hence, we can infer that for n large there exist $r_1 > 1$ sufficiently close to 1 and $\alpha > \alpha_0$ close to α_0 such that $r_1\alpha\|u_n\|^2 < 2\pi\kappa\omega$. Thus, by (1.18) and (1.14),

$$\int_{\mathbb{R}} (e^{\alpha u_n^2} - 1)^{r_1}\,\mathrm{d}x \le \int_{\mathbb{R}} (e^{r_1\alpha\|u_n\|^2\left(\frac{u_n}{\|u_n\|}\right)^2} - 1)\,\mathrm{d}x \le C. \tag{1.27}$$

Consequently,

$$\int_{\mathbb{R}} f(x,u_n)u_n\,\mathrm{d}x \to 0 \quad \text{as} \quad n\to\infty.$$

In fact, since $f(x,s)$ satisfies (f_1) and (f_2), for $\alpha > \alpha_0$ and $\varepsilon > 0$, there exists $C_1 > 0$ such that

$$|f(x,s)| \le (\lambda_1 - \varepsilon)|s| + C_1(e^{\alpha s^2} - 1), \quad \text{for all} \quad s \in \mathbb{R}.$$

Letting $r_1 > 1$ close to 1 such that $r_2 \ge 2$, where $1/r_1 + 1/r_2 = 1$, we obtain by Hölder's inequality that

$$\left|\int_{\mathbb{R}} f(x,u_n)u_n\,\mathrm{d}x\right| \le C\int_{\mathbb{R}} |u_n|^2\,\mathrm{d}x + C\left(\int_{\mathbb{R}} (e^{\alpha u_n^2} - 1)^{r_1}\,\mathrm{d}x\right)^{1/r_1}\left(\int_{\mathbb{R}} |u_n|^{r_2}\,\mathrm{d}x\right)^{1/r_2} \to 0,$$

where we have used (1.27) and Lemma 1.10. Therefore, since $I'(u_n)u_n = o_n(1)$, we conclude that, up to a subsequence, $u_n \to 0$ in X.

<u>*Case 2:*</u> $u \ne 0$. In this case, since (u_n) is a Palais-Smale sequence of I at the level c, we may define

$$v_n = \frac{u_n}{\|u_n\|} \quad \text{and} \quad v = \frac{u}{\lim\|u_n\|}.$$

It follows that $v_n \rightharpoonup v$ in X, $\|v_n\| = 1$ and $\|v\| \le 1$. If $\|v\| = 1$, we conclude the proof. If $\|v\| < 1$, we claim that there exist $r_1 > 1$ sufficiently close to 1, $\alpha > \alpha_0$ close to α_0 and $\beta > 0$ such that

$$r_1\alpha\|u_n\|^2 \le \beta < 2\pi\kappa\omega(1 - \|v\|^2)^{-1} \tag{1.28}$$

for $n \in \mathbb{N}$ large. In fact, since $I(u_n) = c + o_n(1)$, it follows that

$$\frac{1}{2}\lim_{n\to\infty}\|u_n\|^2 = c + \int_{\mathbb{R}} F(x,u)\,\mathrm{d}x + (h,u). \tag{1.29}$$

Setting

$$A = \left(c + \int_{\mathbb{R}} F(x,u)\,\mathrm{d}x + (h,u)\right)(1 - \|v\|^2),$$

from (1.29) and by the definition of v, we obtain

$$A = c - I(u),$$

which together with (1.29) imply

$$\frac{1}{2}\lim_{n\to\infty}\|u_n\|^2 = \frac{A}{1-\|v\|^2} = \frac{c-I(u)}{1-\|v\|^2}. \tag{1.30}$$

Note that

$$c - I(u) < c + \frac{1}{2}(h,u). \tag{1.31}$$

Indeed, since the norm is lower semicontinuous, from (1.29) it follows that $I(u) \leq c$. Moreover, for all $\varphi \in C_0^\infty(\mathbb{R})$,

$$I'(u_n)\varphi = \langle u_n, \varphi\rangle - \int_{\mathbb{R}} f(x,u_n)\varphi\,\mathrm{d}x - (h,\varphi).$$

Since $u_n \rightharpoonup u$ in X, passing to the limit in the above equality, by Lemma 1.17, we have

$$I'(u)\varphi = \langle u, \varphi\rangle - \int_{\mathbb{R}} f(x,u)\varphi\,\mathrm{d}x - (h,\varphi) = 0,$$

for all $\varphi \in C_0^\infty(\mathbb{R})$. By density, we conclude that $I'(u)v = 0$ for all $v \in X$. In particular, $I'(u)u = 0$. Thus, from (f_3) we obtain

$$\begin{aligned} 0 &= I'(u)u = \|u\|^2 - \int_{\mathbb{R}} f(x,u)u\,\mathrm{d}x - (h,u) \\ &< 2\left(\frac{1}{2}\|u\|^2 - \int_{\mathbb{R}} F(x,u)\,\mathrm{d}x - (h,u)\right) + (h,u) \\ &= 2I(u) + (h,u), \end{aligned}$$

which implies (1.31).

Now, note that

$$\|u\| \leq \frac{(\theta-1)\|h\|_* + \sqrt{(1-\theta)^2\|h\|_*^2 + 2\theta c(\theta-2)}}{(\theta-2)}. \tag{1.32}$$

Indeed, since $I(u) \leq c$ and $I'(u)u = 0$, we have

$$\theta I(u) - I'(u)u = \left(\frac{\theta}{2}-1\right)\|u\|^2 + \int_{\mathbb{R}}[f(x,u)u - \theta F(x,u)]\,\mathrm{d}x + (1-\theta)(h,u) \leq \theta c.$$

Thus, from (f_3), we have

$$\left(\frac{\theta-2}{2}\right)\|u\|^2 + (1-\theta)\|h\|_*\|u\| - \theta c \leq 0.$$

Consequently, (1.32) holds. Therefore, from (1.30), (1.31) and (1.32) for $\|h\|_*$ sufficiently small, we conclude

$$\frac{1}{2}\lim_{n\to\infty}\|u_n\|^2 = \frac{c - I(u)}{1-\|v\|^2} < \frac{\pi\kappa\omega}{\alpha_0(1-\|v\|^2)} \tag{1.33}$$

Consequently, (1.28) holds. By (1.18) and Lemma 1.13, we get

$$\int_{\mathbb{R}} (e^{\alpha u_n^2} - 1)^{r_1}\,\mathrm{d}x \le C.$$

By Hölder's inequality and similar computations done above we obtain

$$\int_{\mathbb{R}} f(x,u_n)(u_n - u)\,\mathrm{d}x \to 0 \quad \text{as} \quad n\to\infty.$$

This convergence and the fact that $I'(u_n)(u_n-u) = o_n(1)$, imply that

$$\|u_n\|^2 = (u_n, u) + o_n(1).$$

Since $u_n \rightharpoonup u$ in X, we obtain $u_n \to u$ in X and the proof is finished. □

Proposition 1.19. *Under the hypotheses $(V_1)-(V_3)$ and $(f_1)-(f_4)$, if $\|h\|_*$ is sufficiently small then the functional I satisfies $(PS)_{c_0}$.*

Proof. Let $(u_n) \subset \overline{B_{\rho_h}}$ be an arbitrary Palais-Smale sequence of I at the level c_0. By the Lemma 1.17, up to a subsequence, $u_n \rightharpoonup u$ in X. We will show that, up to a subsequence, $u_n \to u$ in X. Note that

$$\int_{\mathbb{R}} f(x,u_n)(u_n - u)\mathrm{d}x \to 0 \quad \text{as} \quad n\to\infty. \tag{1.34}$$

Firstly, since $\|u_n\| \le \rho_h$ making ρ_h sufficiently small, taking $r_1 > 1$ sufficiently close to 1 and α sufficiently close to α_0 we may infer that $r_1\alpha\|u_n\|^2 < 2\pi\kappa\omega$. Thus, by (1.18) and (1.14),

$$\int_{\mathbb{R}} (e^{\alpha u_n^2} - 1)^{r_1}\,\mathrm{d}x \le \int_{\mathbb{R}} (e^{r_1\alpha\|u_n\|^2\left(\frac{u_n}{\|u_n\|}\right)^2} - 1)\,\mathrm{d}x \le C. \tag{1.35}$$

Moreover, since $f(x,s)$ satisfies (f_1) and (f_2), for $\alpha > \alpha_0$ and $\varepsilon > 0$, there exists $C_1 > 0$ such that

$$|f(x,s)| \le (\lambda_1 - \varepsilon)|s| + C_1(e^{\alpha s^2} - 1), \quad \text{for all} \quad s\in\mathbb{R}.$$

Letting $r_1 > 1$ close to 1 such that $r_2 \ge 2$, where $1/r_1 + 1/r_2 = 1$, we obtain by Hölder's inequality that

$$\begin{aligned}\left|\int_{\mathbb{R}} f(x,u_n)(u_n-u)\,\mathrm{d}x\right| &\le C\left(\int_{\mathbb{R}} |u_n|^2\,\mathrm{d}x\right)^{1/2}\left(\int_{\mathbb{R}} (u_n-u)^2\,\mathrm{d}x\right)^{1/2} \\ &+ C\left(\int_{\mathbb{R}} (e^{\alpha u_n^2}-1)^{r_1}\,\mathrm{d}x\right)^{1/r_1}\left(\int_{\mathbb{R}} (u_n-u)^{r_2}\,\mathrm{d}x\right)^{1/r_2} \to 0,\end{aligned}$$

where we have used (1.35) and Lemma 1.10. Therefore, this convergence and the fact that $I'(u_n)(u_n - u) = o_n(1)$, imply that

$$\|u_n\|^2 = (u_n, u) + o_n(1).$$

Since $u_n \rightharpoonup u$ in X, we conclude that, up to a subsequence, $u_n \to u$ in X. □

Estimate of the minimax level

In this section we will prove an estimate for the minimax level. First, we will need the following lemma.

Lemma 1.20. *Suppose that* $(V_1) - (V_3)$ *hold. Then* S_p *given in* (1.9) *is attained by a nonnegative function* $u_p \in X$.

Proof. Let (u_n) be a minimizing sequence of nonnegative functions (if necessary, replace u_n by $|u_n|$, which is possible since by using the triangle inequality we have $|u_n(x) - u_n(y)| \geq ||u_n(x)| - |u_n(y)||$) for S_p in X, that is,

$$\|u_n\|_p = 1 \quad \text{and} \quad \left(\frac{1}{2\pi} \int\limits_{\mathbb{R}^2} \frac{(u_n(x) - u_n(y))^2}{|x-y|^2} \,\mathrm{d}x\,\mathrm{d}y + \int\limits_{\mathbb{R}} V(x) u_n^2 \,\mathrm{d}x \right)^{1/2} \to S_p.$$

Then, (u_n) is bounded in X. Since X is a Hilbert space and X is compactly embedded into $L^p(\mathbb{R})$, up to a subsequence, we may assume

$$\begin{aligned} &u_n \rightharpoonup u_p \quad \text{in } X,\\ &u_n \to u_p \quad \text{in } L^p(\mathbb{R}),\\ &u_n(x) \to u_p(x) \quad \text{almost everywhere in } \mathbb{R}. \end{aligned}$$

Consequently,

$$\begin{cases} \|u_p\|_p = 1,\\ \|u_p\| \leq \liminf\limits_{n\to+\infty} \|u_n\| = S_p,\\ u_p(x) \geq 0 \text{ almost everywhere in } \mathbb{R}. \end{cases}$$

Thus, $S_p = \|u_p\|$. This completes the proof. □

Now we prove the main result of this section.

Lemma 1.21. *Suppose that* $(V_1) - (V_3)$ *and* (f_5) *are satisfied, if* $\|h\|_*$ *is sufficiently small then*

$$\max_{t\geq 0} I(tu_p) < \frac{\pi\kappa\omega}{\alpha_0}.$$

Proof. Let $\Psi : [0, +\infty) \to \mathbb{R}$, given by

$$\Psi(t) = \frac{t^2}{2}\left(\frac{1}{2\pi}\int\limits_{\mathbb{R}^2}\frac{(u_p(x)-u_p(y))^2}{|x-y|^2}\,\mathrm{d}x\mathrm{d}y + \int\limits_{\mathbb{R}} V(x)u_p^2\,\mathrm{d}x\right) - \int\limits_{\mathbb{R}} F(x, tu_p)\,\mathrm{d}x.$$

By Lemma 1.20 and (f_5), we have

$$\Psi(t) \le \frac{t^2}{2}S_p^2 - \frac{C_p}{p}t^p \le \max_{t\ge 0}\left[\frac{t^2}{2}S_p^2 - \frac{C_p}{p}t^p\right] = \frac{(p-2)}{2p}\frac{S_p^{2p/(p-2)}}{C_p^{2/(p-2)}} < \frac{\pi\kappa\omega}{\alpha_0}. \tag{1.36}$$

To conclude, notice that $t|(h, u_p)| \le t\|h\|_*\|u_p\|$ with t in a compact interval. Therefore, taking $\|h\|_*$ sufficiently small and using (1.36) the result follows. □

Proofs of Theorem 1.1 and 1.2

Initially, it follows from Lemma 1.14 and Lemma 1.15 that the functional I satisfies the geometric conditions of the Mountain Pass Theorem. Consequently, the minimax level

$$c_m = \inf_{g\in\Gamma}\max_{t\in[0,1]} I(g(t))$$

is positive, where $\Gamma = \{g \in C([0,1], X) : g(0) = 0, g(1) = e\}$.

On the other hand, by Lemma 1.21 and Proposition 1.18, the functional I satisfies the $(PS)_{c_m}$ condition. Thus, by the Mountain Pass Theorem the functional I has a critical point u_m at the minimax level c_m.

Moreover, when $h \in X^*$ with $h \not\equiv 0$, we may find a second solution. In order to do this, we consider ρ_h like in Lemma 1.14 and we observe that $\overline{B}_{\rho_h}$ is a convex complete metric space with the metric induced by the norm of X, and the functional I is C^1 and bounded below on $\overline{B}_{\rho_h}$. Hence, by the Ekeland variational principle there exists a sequence (u_n) in $\overline{B}_{\rho_h}$ such that

$$I(u_n) \to c_0 < 0 \quad \text{and} \quad \|I'(u_n)\|_* \to 0.$$

By the Proposition 1.19, the functional I satisfies the $(PS)_{c_0}$ condition. Consequently, the functional I has a critical point u at the level c_0. Therefore, the proof of the results is complete.

Chapter 2

On nonlinear perturbations of a periodic fractional Schrödinger equation with critical exponential growth

In this chapter we present the results of the paper [15], more specifically, we study the existence of solutions for fractional Schrödinger equations of the form

$$(-\Delta)^{1/2}u + V(x)u = f(x,u) \quad \text{in} \quad \mathbb{R},$$

where V is a bounded potential, which belongs to a different class of those treated in Chapter 1, and the nonlinear term $f(x,u)$ is considered with critical exponential growth. We prove the existence of a nontrivial weak solution by combining the Mountain Pass Theorem, Trudinger-Moser's inequality and a version of Concentration-Compactness Principle due to Lions.

Motivation and main results

As mentioned in the introduction, some results have appeared, recently, in the literature concerning the equation

$$(-\Delta)^{1/2}u + V(x)u = f(x,u) \quad \text{in} \quad \mathbb{R}, \tag{2.1}$$

with interesting conditions on $V(x)$ and $f(x,u)$. The main purpose of this chapter is to study (2.1) considering the nonlinearity with exponential growth. As we have seen in Chapter 1, the Sobolev embedding states that $H^{1/2}(\mathbb{R}) \hookrightarrow L^q(\mathbb{R})$ for any $q \in [2,+\infty)$, but $H^{1/2}(\mathbb{R})$ is not continuous embedded in $L^\infty(\mathbb{R})$ (for details see [17, 39]). Thus the maximal growth, which allows us to treat (2.1) variationally in $H^{1/2}(\mathbb{R})$, is motivated by Trudinger-Moser's inequality proved by T. Ozawa [39] and H. Kozono, T. Sato and H. Wadade [29]. More precisely, they proved that there exist positive constants ω and $C = C(\omega)$ such that, for all $u \in H^{1/2}(\mathbb{R})$ with $\|(-\Delta)^{1/4}u\|_2 \leq 1$,

$$\int_{\mathbb{R}} (e^{\alpha u^2} - 1)\,\mathrm{d}x \leq C\|u\|_2^2, \quad \text{for all} \quad \alpha \in (0,\omega]. \tag{2.2}$$

Therefore, the maximal growth on the nonlinearity $f(x,u)$, that allows us to treat (2.1) variationally in $H^{1/2}(\mathbb{R})$, is given by $e^{\alpha_0 u^2}$ when $|u| \to +\infty$ for some $\alpha_0 > 0$ (see also the pioneers works [38, 45]).

Motivated by Trudinger-Moser inequality (2.2) and by the works [1, 14, 23, 25], we deal with two problems. First, we investigate (2.1) when $V(x)$ and $f(x,u)$ are periodic functions with respect to x, and $f(x,u)$ behaves like $e^{\alpha_0 u^2}$ when $|u| \to +\infty$ for some $\alpha_0 > 0$. Second, with the aid of the previous case, we study a more general problem assuming that $V(x)$ and $f(x,u)$ are just asymptotically periodic at infinity. Next, for easy reference, we recall the problems and assumptions.

A periodic problem

The first problem that we will study in this chapter is the following,

$$\begin{cases} (-\Delta)^{1/2}u + V_0(x)u = f_0(x,u) \quad \text{in} \quad \mathbb{R}, \\ u \in H^{1/2}(\mathbb{R}) \quad \text{and} \quad u \geq 0, \end{cases} \tag{P_0}$$

where $(-\Delta)^{1/2}$ is defined, for a sufficiently regular function, by

$$(-\Delta)^{1/2}u(x) = -\frac{1}{2\pi}\int_{\mathbb{R}} \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^2}\,\mathrm{d}y. \tag{2.3}$$

The assumptions on the functions $V_0(x)$ and $f_0(x,u)$ are the following: $V_0 : \mathbb{R} \to (0,+\infty)$ is a continuous $1-$periodic function and $f_0 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous $1-$periodic function in x, which has critical exponential growth in u, that is, there exists $\alpha_0 > 0$ such that

$$\lim_{|s|\to+\infty} f_0(x,s)e^{-\alpha s^2} = \begin{cases} 0, & \text{for all } \alpha > \alpha_0, \\ +\infty, & \text{for all } \alpha < \alpha_0, \end{cases} \tag{2.4}$$

uniformly in $x \in \mathbb{R}$.

Recall that this notion of criticality is directed by (2.2) and it has been used in several papers involving exponential growth, see for instance [13], [20] and [25]. Since we are interested in the existence of nonnegative solutions, we set $f_0(x,s) = 0$ for all $(x,s) \in \mathbb{R} \times (-\infty, 0]$. We also assume that the nonlinearity $f_0(x,u)$ satisfies the conditions

$(f_{0,1})$ $\displaystyle\lim_{s\to 0} \frac{f_0(x,s)}{s} = 0$ uniformly in $x \in \mathbb{R}$;

$(f_{0,2})$ there exists a constant $\theta > 2$ such that

$$0 < \theta F_0(x,s) := \theta \int_0^s f_0(x,t)\,\mathrm{d}t \le s f_0(x,s) \quad \text{for all} \quad (x,s) \in \mathbb{R} \times (0,+\infty);$$

$(f_{0,3})$ for each fixed $x \in \mathbb{R}$, the function $f_0(x,s)/s$ is increasing with respect to $s \in \mathbb{R}$;

$(f_{0,4})$ there are constants $p > 2$ and $C_p > 0$ such that

$$f_0(x,s) \ge C_p s^{p-1}, \quad \text{for all} \quad (x,s) \in \mathbb{R} \times [0,+\infty),$$

where

$$C_p > \left[\frac{(p-2)\theta\alpha_0}{(\theta-2)p\omega}\right]^{(p-2)/2} S_p^p \tag{2.5}$$

and

$$S_p := \inf_{\substack{u\in H^{1/2}(\mathbb{R}) \\ \|u\|_p=1}} \left(\frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{|u(x)-u(y)|^2}{|x-y|^2}\,\mathrm{d}x\mathrm{d}y + \|V\|_\infty \int_{\mathbb{R}} u^2\,\mathrm{d}x \right)^{1/2}.$$

Throughout this chapter, we say that $u \in H^{1/2}(\mathbb{R})$ is a weak solution for (P_0) if the following equality holds:

$$\frac{1}{2\pi}\int_{\mathbb{R}^2} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^2}\,\mathrm{d}x\,\mathrm{d}y + \int_{\mathbb{R}} V_0(x)uv\,\mathrm{d}x = \int_{\mathbb{R}} f_0(x,u)v\,\mathrm{d}x, \text{ for all } v \in H^{1/2}(\mathbb{R}).$$

Under these conditions we have the first result of this chapter:

Theorem 2.1. *Assume that* $(f_{0,1})-(f_{0,4})$ *hold. Then* (P_0) *has a nonnegative nontrivial weak solution.*

As a consequence of this theorem we find a nonnegative nontrivial weak solution for the following model problem:

$$(-\Delta)^{1/2}u + u = G_0'(u) \quad \text{in} \quad \mathbb{R},$$

with $G_0(u) = \frac{C_p}{p}\, u^p\, e^{\alpha_0 u^2}$, $u \geq 0$, $p > 2$ and C_p as defined in (2.5).

An asymptotically periodic problem

The second problem that we will study in this chapter is the following,

$$\begin{cases} (-\Delta)^{1/2}u + V(x)u = f(x,u) \quad \text{in} \quad \mathbb{R}, \\ u \in H^{1/2}(\mathbb{R}) \quad \text{and} \quad u \geq 0. \end{cases} \tag{P}$$

Next we will describe the conditions on the functions $V(x)$ and $f(x,s)$ in a more precise way.

(V_1) $V : \mathbb{R} \to [0, +\infty)$ is a continuous function satisfying the conditions: $V(x) \leq V_0(x)$ for any $x \in \mathbb{R}$ and $V_0(x) - V(x) \to 0$ as $|x| \to \infty$;

We assume that the nonlinearity $f : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is a continuous function satisfying (2.4), $f(x,s) = 0$ for all $(x,s) \in \mathbb{R} \times (-\infty, 0]$ and also the following conditions:

(f_1) $f(x,s) \geq f_0(x,s)$ for all $(x,s) \in \mathbb{R} \times [0,+\infty)$, and for all $\varepsilon > 0$, there exists $\eta > 0$ such that for $s \geq 0$ and $|x| \geq \eta$,

$$|f(x,s) - f_0(x,s)| \leq \varepsilon e^{\alpha_0 s^2};$$

(f_2) $\displaystyle\lim_{s\to 0} \frac{f(x,s)}{s} = 0$ uniformly in $x \in \mathbb{R}$;

(f_3) there exists a constant $\mu > 2$ such that

$$0 < \mu F(x,s) := \mu \int_0^s f(x,t)\,\mathrm{d}t \leq s f_0(x,s), \quad \text{for all} \quad (x,s) \in \mathbb{R} \times (0,+\infty);$$

(f_4) for each fixed $x \in \mathbb{R}$, the function $f(x,s)/s$ is increasing with respect to $s \in \mathbb{R}$;

(f_5) at least one of the nonnegative continuous functions $V_0(x) - V(x)$ and $f(x,s) - f_0(x,s)$ is positive on a set of positive measure.

We say that $u \in H^{1/2}(\mathbb{R})$ is a weak solution for (P) if the following equality holds:

$$\frac{1}{2\pi}\int\limits_{\mathbb{R}^2} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^2}\,\mathrm{d}x\,\mathrm{d}y + \int\limits_{\mathbb{R}} V(x)uv\,\mathrm{d}x = \int\limits_{\mathbb{R}} f(x,u)v\,\mathrm{d}x, \text{ for all } v \in H^{1/2}(\mathbb{R}).$$

The second result of this chapter is the following:

Theorem 2.2. *Assume that* (V_1) *and* $(f_1) - (f_5)$ *hold. Then* (P) *has a nonnegative nontrivial weak solution.*

As mentioned earlier, the results of this chapter were motivated by the works [1, 14, 23, 25]. Particularly, J. M. do Ó et al. in [23] have proved the existence of a nontrivial solution for the fractional Schrödinger equation

$$(-\Delta)^{1/2}u + u = K(x)g(u) \quad \text{in} \quad \mathbb{R},$$

where $g(u)$ behaves like $e^{\alpha_0 u^2}$ when $|u| \to +\infty$ for some $\alpha_0 > 0$ and $K : \mathbb{R} \to \mathbb{R}$ is a positive function such that $K \in L^\infty(\mathbb{R}) \cap C(\mathbb{R})$. Furthermore, if $\{A_n\}$ is a sequence of Borel sets of $\mathbb{R}$ with $|A_n| \leq R$ for some $R > 0$,

$$\lim_{r\to\infty} \int\limits_{A_n \cap B^c(0,R)} K(x)\mathrm{d}x = 0, \text{ uniformly with respect to } n \in \mathbb{N}.$$

We were inspired by Alves et al. [1], thus we studied (P) assuming that the potential $V(x)$ and the nonlinearity $f(x,u)$ are asymptotically periodic at infinity. Here we work with a general class of functions which are asymptotic to a nonautonomous periodic function at infinity. In this sense our work completes the study presented in [22, 23]. It complements also [10, 11, 18, 41] since we consider the limiting case for $N = 1$ and $s = 1/2$ when the nonlinearity has exponential growth in the sense of the Trudinger-Moser inequality. Moreover, also complements the study of Chapter 1 since we consider that the potential $V(x)$ belongs to a different class from those treated there.

Remark 2.3. The assumptions on the nonlinearity and the potential are standard, since we use a variational approach. Notice that our assumptions assure the mountain pass geometry of the functionals I_0 and I (which are defined in Sections 2 and 2). Furthermore, by (V_1) the potential $V(x)$ may be zero on bounded sets. For more details see Lemma 2.6.

Remark 2.4. As examples of functions that verifies our assumptions we may consider $V(x) = 2 + \dfrac{|x|^2}{|x|^2+1}|\sin(x)|$ with $V_0(x) = 2 + |\sin(x)|$, or

$$V(x) = \begin{cases} 0, & \text{if} \quad x^2 \leq 1 \\ x^2 - 1, & \text{if} \quad 1 \leq x^2 \leq 4 \\ 3, & \text{if} \quad x^2 \geq 4 \end{cases}$$

with $V_0(x) \equiv 3$.

Remark 2.5. We highlight that when u has sufficient regularity, it is possible to have a pointwise expression of the fractional Laplacian as (2.3) (see [47], for example). Again in this case we have $u > 0$ if $u \neq 0$, (see Remark 3.4).

To prove our main theorems we have used variational methods. An important point is a version of a Concentration-Compactness Principle. This one is crucial to show that S_p is attained and that the weak limit of an appropriate Palais-Smale sequence is nontrivial.

The outline of this chapter is as follows: Section 2.2 contains some preliminary results. Section 2.3 and Section 2.4 deal with the proof of Theorems 2.1 and 2.2, respectively.

Some preliminary results

In this section, we prove some technical results and we establish the appropriate setting to prove Theorems 2.1 and 2.2.

The functional setting

In order to study variationally (P) we consider a suitable subspace of the fractional Sobolev space $H^{1/2}(\mathbb{R})$, which is defined by

$$H^{1/2}(\mathbb{R}) := \left\{ u \in L^2(\mathbb{R}) \, : \, \frac{|u(x) - u(y)|}{|x-y|} \in L^2(\mathbb{R} \times \mathbb{R}) \right\},$$

endowed with the natural norm

$$\|u\|_{1/2} := ([u]_{1/2}^2 + \|u\|_2^2)^{1/2},$$

where the term

$$[u]_{1/2} := \left(\int_{\mathbb{R}^2} \frac{|u(x) - u(y)|^2}{|x-y|^2} \, \mathrm{d}x \, \mathrm{d}y \right)^{1/2}$$

is the so-called *Gagliardo semi-norm* of function u.

Recall that $H^{1/2}(\mathbb{R})$ is a Hilbert space and by [17, Proposition 3.6]

$$\|(-\Delta)^{1/4}u\|_2 = (2\pi)^{-1/2}[u]_{1/2} \quad \text{for all} \quad u \in H^{1/2}(\mathbb{R}). \tag{2.6}$$

The next lemma provides an inequality that we will use in some proofs.

Lemma 2.6. *Suppose that (V_1) holds. Then there exists a constant $\kappa > 0$ satisfying*

$$\frac{1}{2\pi}\int\limits_{\mathbb{R}^2} \frac{|u(x)-u(y)|^2}{|x-y|^2}\,\mathrm{d}x\mathrm{d}y + \int\limits_{\mathbb{R}} V(x)u^2\,\mathrm{d}x \geq \kappa\|u\|_2^2 \quad \textit{for all} \quad u \in H^{1/2}(\mathbb{R}). \tag{2.7}$$

Proof. Suppose that (2.7) does not hold. Then for each $n \in \mathbb{N}$ there exists $u_n \in H^{1/2}(\mathbb{R})$ such that

$$\|u_n\|_2 = 1 \quad \text{and} \quad \frac{1}{2\pi}\left(\int\limits_{\mathbb{R}^2} \frac{(u_n(x)-u_n(y))^2}{|x-y|^2}\,\mathrm{d}x\,\mathrm{d}y\right) + \int\limits_{\mathbb{R}} V(x)u_n^2\,\mathrm{d}x < \frac{1}{n}. \tag{2.8}$$

Since $V(x) \geq 0$, we get that $[u_n]_{1/2}^2 \to 0$, (u_n) is bounded in $H^{1/2}(\mathbb{R})$ and, up to a subsequence,

$$u_n \rightharpoonup u_0 \quad \text{in} \quad H^{1/2}(\mathbb{R}), \text{ as } n \to \infty. \tag{2.9}$$

We also have that

$$\int\limits_{\mathbb{R}} V(x)u_n^2\,\mathrm{d}x \to 0, \text{ as } n \to \infty. \tag{2.10}$$

Now, we use the following inequality, proved in [39, p.261], given by

$$\|u_n\|_r \leq C\|(-\Delta)^{1/4}u_n\|_2^{1-\theta}\|u_n\|_2^{\theta},$$

where $r > 2$, $C > 0$ and $\theta \in (0,1)$. This inequality together with (2.6) implies

$$\|u_n\|_r \leq C[u_n]_{1/2}^{(1-\theta)}\|u_n\|_2^{\theta},$$

from which it follows that $\|u_n\|_r \to 0$. On the other hand, by (2.9) $u_n \to u_0$ in $L^r_{loc}(\mathbb{R})$. Consequently, $u_0 \equiv 0$.

From (2.8), for each $R > 0$ we can write

$$1 = \int\limits_{\mathbb{R}} u_n^2\,\mathrm{d}x = \int\limits_{B_R} u_n^2\,\mathrm{d}x + \int\limits_{B_R^c} u_n^2\,\mathrm{d}x. \tag{2.11}$$

Next, in order to reach a contradiction we will use (V_1). More precisely, given $\varepsilon > 0$, there exists $R > 0$, such that

$$V_0(x) - V(x) \leq \varepsilon \quad \text{for all} \quad x \in B_R^c.$$

Thus, for all $x \in B_R^c$ and $\varepsilon > 0$ sufficiently small,

$$V(x) \geq V_0(x) - \varepsilon \geq C_0 - \varepsilon = b_0 > 0. \tag{2.12}$$

Combining (2.11), (2.12), (2.10) and the fact that $u_n \to 0$ in $L^2(B_R)$, we obtain the contradiction

$$1 \leq \int_{B_R} u_n^2 \,\mathrm{d}x + \frac{1}{b_0} \int_{B_R^c} V(x) u_n^2 \,\mathrm{d}x \to 0.$$

Therefore, (2.7) holds and the lemma is proved. □

We will use the following notations: X_0 will denote $H^{1/2}(\mathbb{R})$ endowed with the equivalent norm

$$\|u\|_{X_0} = \left\{ \frac{1}{2\pi} \left(\int_{\mathbb{R}^2} \frac{|u(x) - u(y)|^2}{|x-y|^2} \,\mathrm{d}x \,\mathrm{d}y \right) + \int_{\mathbb{R}} V_0(x) u^2 \,\mathrm{d}x \right\}^{1/2}$$

and X_1 will denote $H^{1/2}(\mathbb{R})$ endowed with the norm

$$\|u\|_{X_1} = \left\{ \frac{1}{2\pi} \left(\int_{\mathbb{R}^2} \frac{|u(x) - u(y)|^2}{|x-y|^2} \,\mathrm{d}x \,\mathrm{d}y \right) + \int_{\mathbb{R}} V(x) u^2 \,\mathrm{d}x \right\}^{1/2}.$$

As consequence of inequality (2.7) we have that $\|\cdot\|_{X_1}$ is a norm and also that the embedding $X_1 \hookrightarrow L^q(\mathbb{R})$ is continuous for all $2 \leq q < \infty$.

Trudinger-Moser type inequalities

In this subsection we prove a version of (2.2) to the space $H^{1/2}(\mathbb{R})$ with the norms $\|\cdot\|_{X_0}$ and $\|\cdot\|_{X_1}$. This will be our principal tool to prove our main results. The ideas used in the proof are inspired in [19, 20, 25] and we present them here for completeness of our work.

Lemma 2.7. *If $0 < \alpha < \omega$, then there exists a constant $C = C(\omega) > 0$, such that*

$$\sup_{\{u \in H^{1/2}(\mathbb{R}) \,:\, \|u\|_{X_i} \leq 1\}} \int_{\mathbb{R}} (e^{\alpha u^2} - 1) \,\mathrm{d}x \leq C \quad \text{for} \quad i = 0, 1. \tag{2.13}$$

Moreover, for any $\alpha > 0$ and $u \in H^{1/2}(\mathbb{R})$ we have

$$\int_{\mathbb{R}} (e^{\alpha u^2} - 1) \,\mathrm{d}x < \infty. \tag{2.14}$$

Proof. First, we observe that if a function $u \in H^{1/2}(\mathbb{R})$ satisfies $\|u\|_{X_i} \leq 1$, then by using (2.6), we get

$$\|(-\Delta)^{1/4}u\|_2 = (2\pi)^{-1/2}[u]_{1/2} \leq \|u\|_{X_i} \leq 1.$$

Consequently,

$$\int_{\mathbb{R}} (e^{\alpha u^2} - 1)\,\mathrm{d}x \leq C_1\|u\|_2^2 \leq C,$$

where we have used (2.2) and (2.7). Thus, we obtain (2.13).

Now we prove the second part of the lemma. Given $u \in H^{1/2}(\mathbb{R})$ and $\varepsilon > 0$ there exists $\varphi \in C_0^\infty(\mathbb{R})$ such that $\|u - \varphi\|_{X_i} < \varepsilon$. Since

$$e^{\alpha u^2} - 1 \leq e^{\alpha(2(u-\varphi)^2 + 2\varphi^2)} - 1 \leq \frac{1}{2}\left(e^{4\alpha(u-\varphi)^2} - 1\right) + \frac{1}{2}\left(e^{4\alpha\varphi^2} - 1\right),$$

it follows that

$$\int_{\mathbb{R}} (e^{\alpha u^2} - 1)\,\mathrm{d}x \leq \frac{1}{2}\int_{\mathbb{R}} (e^{4\alpha\|u-\varphi\|_{X_i}^2\left(\frac{u-\varphi}{\|u-\varphi\|_{X_i}}\right)^2} - 1)\,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}} (e^{4\alpha\varphi^2} - 1)\,\mathrm{d}x. \tag{2.15}$$

Choosing $\varepsilon > 0$ such that $4\alpha\varepsilon^2 < \omega$, we have $4\alpha\|u - \varphi\|_{X_i}^2 < \omega$. Then, from (2.13) and (2.15), it follows that

$$\int_{\mathbb{R}} (e^{\alpha u^2} - 1)\,\mathrm{d}x \leq \frac{C}{2} + \frac{1}{2}\int_{\mathrm{supp}(\varphi)} (e^{4\alpha\varphi^2} - 1)\,\mathrm{d}x < \infty.$$

This completes the proof of the lemma. □

Lemma 2.8. *If $\alpha > 0$, $q > 2$, $v \in X_i$ and $\|v\|_{X_i} \leq M$ with $\alpha M^2 < \omega$, then there exists $C = C(\alpha, M, q) > 0$, such that*

$$\int_{\mathbb{R}} (e^{\alpha v^2} - 1)|v|^q \mathrm{d}x \leq C\|v\|_{X_i}^q \quad \textit{for} \quad i = 0, 1.$$

Proof. Consider $r > 1$ close to 1 such that $\alpha r M^2 < \omega$. Using Hölder's inequality with $r' = r/(r-1)$, we have

$$\int_{\mathbb{R}} (e^{\alpha v^2} - 1)|v|^q \mathrm{d}x \leq \left(\int_{\mathbb{R}} (e^{\alpha v^2} - 1)^r \mathrm{d}x\right)^{1/r} \|v\|_{r'q}^q. \tag{2.16}$$

Notice that given $r > 1$ for all $s \in \mathbb{R}$,

$$(e^{\alpha s^2} - 1)^r \leq (e^{\alpha r s^2} - 1). \tag{2.17}$$

Hence, from (2.16) and (2.17) we get

$$\begin{aligned}\int_{\mathbb{R}} (e^{\alpha v^2}-1)|v|^q \mathrm{d}x &\le \left(\int_{\mathbb{R}} (e^{\alpha r v^2}-1)\mathrm{d}x\right)^{1/r} \|v\|_{r'q}^q \\ &\le \left(\int_{\mathbb{R}} (e^{\alpha r M^2\left(\frac{v}{\|v\|_{X_i}}\right)^2}-1)\,\mathrm{d}x\right)^{1/r} \|v\|_{r'q}^q.\end{aligned}$$

Thus, since $\alpha r M^2 < \omega$, it follows from (2.13) and the continuous embedding $X_i \hookrightarrow L^{r'q}(\mathbb{R})$ that

$$\int_{\mathbb{R}} (e^{\alpha v^2}-1)|v|^q \,\mathrm{d}x \le C\|v\|_{X_i}^q,$$

which proves the lemma. □

A Concentration-Compactness Principle

The next lemma is a version of a Lions's result (see P. L. Lions [35]).

Lemma 2.9. *If (u_k) is bounded in $H^{1/2}(\mathbb{R})$ and*

$$\lim_{k\to\infty} \sup_{y\in\mathbb{R}} \int_{B_R(y)} |u_k(x)|^2 \,\mathrm{d}x = 0, \tag{2.18}$$

for some $R>0$, then $u_k \to 0$ in $L^q(\mathbb{R})$ for $2<q<\infty$.

Proof. For each $r>q$, by standard interpolation, we obtain

$$\|u_k\|_{L^q(B_R(y))} \le \|u_k\|_{L^2(B_R(y))}^{1-\lambda} \|u_k\|_{L^r(B_R(y))}^{\lambda},$$

where $(1-\lambda)/2+\lambda/r = 1/q$ with $0<\lambda<1$. Covering $\mathbb{R}$ by balls of radius R, in such way that each point of $\mathbb{R}$ is contained in at most 2 balls, we find $C>0$ such that

$$\|u_k\|_{L^q(\mathbb{R})}^q \le C \sup_{y\in\mathbb{R}} \left(\int_{B_R(y)} |u_k|^2 \,\mathrm{d}x\right)^{(1-\lambda)q/2} \|u_k\|_{L^r(\mathbb{R})}^{\lambda q}.$$

By the continuous embedding $H^{1/2}(\mathbb{R}) \hookrightarrow L^r(\mathbb{R})$ and $\|u_k\|_{1/2} \le C_1$, we get

$$\int_{\mathbb{R}} |u_k|^q \,\mathrm{d}x \le C \sup_{y\in\mathbb{R}} \left(\int_{B_R(y)} |u_k|^2 \,\mathrm{d}x\right)^{(1-\lambda)q/2}$$

and so by (2.18) we conclude the proof. □

Using the previous lemma, we obtain the following result.

Lemma 2.10. *The constant S_p is attained by a nonnegative function $u_p \in H^{1/2}(\mathbb{R})$.*

Proof. Let (ϑ_k) be a minimizing sequence of nonnegative functions for S_p in $H^{1/2}(\mathbb{R})$ (if necessary, replace ϑ_k by $|\vartheta_k|$, which is possible since by using the triangle inequality we have $|\vartheta_k(x) - \vartheta_k(y)| \geq ||\vartheta_k(x)| - |\vartheta_k(y)||$), that is,

$$\int_{\mathbb{R}} |\vartheta_k|^p \, \mathrm{d}x = 1 \quad \text{and} \quad \left(\frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{(\vartheta_k(x) - \vartheta_k(y))^2}{|x-y|^2} \, \mathrm{d}x \, \mathrm{d}y + \|V\|_\infty \int_{\mathbb{R}} \vartheta_k^2 \, \mathrm{d}x \right)^{1/2} \to S_p.$$

Here, we consider $H^{1/2}(\mathbb{R})$ endowed with the norm

$$\|u\| = \left(\frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{|u(x) - u(y)|^2}{|x-y|^2} \, \mathrm{d}x\mathrm{d}y + \|V\|_\infty \int_{\mathbb{R}} u^2 \, \mathrm{d}x \right)^{1/2}.$$

It is clear that $\|\vartheta_k\| \leq C$ for some $C > 0$. Then, up to a subsequence, we may assume that $\vartheta_k \rightharpoonup \vartheta$ in $H^{1/2}(\mathbb{R})$ and

$$\|\vartheta\| \leq \liminf_{k \to +\infty} \|\vartheta_k\| = S_p.$$

Then ϑ is a minimizer provided that $\|\vartheta\|_p = 1$. But we know only that $\|\vartheta\|_p \leq 1$. Notice that, since $\|\vartheta_k\|_p = 1$, Lemma 2.9 implies

$$\delta = \lim_{k \to \infty} \sup_{y \in \mathbb{R}} \int_{B_1(y)} |\vartheta_k(x)|^2 \mathrm{d}x > 0.$$

Thus, up to a subsequence, we may assume the existence of $(y_k) \subset \mathbb{R}$ such that

$$\int_{B_1(y_k)} |\vartheta_k(x)|^2 \mathrm{d}x > \delta/2.$$

Let us define $u_k(x) := \vartheta_k(x + y_k)$. Hence, $\|u_k\|_p = \|\vartheta_k\|_p = 1$, $\|u_k\| = \|\vartheta_k\| \to S_p$ and

$$\int_{B_1(0)} |u_k(x)|^2 \mathrm{d}x > \delta/2. \tag{2.19}$$

Since (u_k) is bounded in $H^{1/2}(\mathbb{R})$, we may assume, up to a subsequence,

$$\begin{cases} u_k \rightharpoonup u_p \text{ in } H^{1/2}(\mathbb{R}), \\ u_k \to u_p \text{ in } L^2_{loc}(\mathbb{R}), \\ u_k \to u_p \text{ almost everywhere in } \mathbb{R}. \end{cases}$$

By the Brézis-Lieb Lemma (see [7]), we have that

$$1 = \|u_p\|_p^p + \lim_{k \to \infty} \|u_k - u_p\|_p^p \quad \text{and} \quad \lim_{k \to \infty} \|u_k\|^2 = \|u_p\|^2 + \lim_{k \to \infty} \|u_k - u_p\|^2.$$

Thus, we have

$$
\begin{aligned}
S_p^2 = \lim_{k\to\infty} \|u_k\|^2 &= \|u_p\|^2 + \lim_{k\to\infty} \|u_k - u_p\|^2 \\
&\geq S_p^2[(\|u_p\|_p^p)^{2/p} + (1 - \|u_p\|_p^p)^{2/p}].
\end{aligned} \tag{2.20}
$$

By (2.19), we have $u_p \neq 0$. If we suppose by contradiction that $\|u_p\|_p < 1$, by using (2.20) follows that

$$S_p^2 > S_p^2(\|u_p\|_p^p + 1 - \|u_p\|_p^p)^{2/p},$$

what is a contradiction. Therefore, $\|u_p\|_p = 1$ and u_p is a minimizer for S_p, and this completes the proof. □

Existence of a solution for the periodic problem

In order to apply the Mountain Pass Theorem without the Palais-Smale condition (see [37, Theorem 4.3]) to find a nontrivial solution for the problem (P_0), we will consider the functional $I_0 : X_0 \to \mathbb{R}$ given by

$$I_0(u) = \frac{1}{2}\|u\|_{X_0}^2 - \int_{\mathbb{R}} F_0(x,u)\,\mathrm{d}x.$$

Notice that I_0 is well defined. Indeed, combining the condition $(f_{0,1})$ and the fact that $f_0(x,s)$ has critical exponential growth, for each $\alpha > \alpha_0$ and $\varepsilon > 0$ there exists a positive constant C_ε such that

$$F_0(x,s) \leq \frac{\varepsilon}{2}s^2 + C_\varepsilon(e^{\alpha s^2} - 1) \quad \text{for all} \quad (x,s) \in \mathbb{R}\times\mathbb{R}.$$

Combining this estimate together with the continuous embedding $X_0 \hookrightarrow L^2(\mathbb{R})$ and (2.14), we obtain that $F_0(x,u) \in L^1(\mathbb{R})$ for all $u \in X_0$. Hence, I_0 is well defined.

By using standard arguments we can see that $I_0 \in C^1(X_0, \mathbb{R})$, see Appendix A, with

$$I_0'(u)\phi = \frac{1}{2\pi}\int_{\mathbb{R}^2} \frac{(u(x)-u(y))(\phi(x)-\phi(y))}{|x-y|^2}\,\mathrm{d}x\,\mathrm{d}y + \int_{\mathbb{R}} V_0(x)u\phi\,\mathrm{d}x - \int_{\mathbb{R}} f_0(x,u)\phi\,\mathrm{d}x,$$

for all $\phi \in X_0$. Therefore, a critical point of I_0 is a weak solution of (P_0) and reciprocally.

Now we prove some facts about the geometric structure of I_0 required by the minimax procedure.

Lemma 2.11. *Suppose that* $(f_{0,1})$ *is satisfied. Then there exist* $\rho, \sigma > 0$ *such that*

$$I_0(u) \geq \sigma \quad \textit{if} \quad \|u\|_{X_0} = \rho.$$

Proof. Combining $(f_{0,1})$ and the fact that $f_0(x,s)$ has critical exponential growth, for each $\alpha > \alpha_0$, $q > 2$ and $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$F_0(x,s) \leq \frac{\varepsilon}{2}s^2 + C_\varepsilon |s|^q (e^{\alpha s^2} - 1) \quad \text{for all} \quad (x,s) \in \mathbb{R} \times \mathbb{R}. \tag{2.21}$$

From (2.21) and the continuous embedding $X_0 \hookrightarrow L^2(\mathbb{R})$, we obtain

$$\begin{aligned} I_0(u) &\geq \frac{1}{2}\|u\|_{X_0}^2 - \frac{\varepsilon}{2}\|u\|_2^2 - C_\varepsilon \int_{\mathbb{R}} (e^{\alpha u^2} - 1)|u|^q \, \mathrm{d}x \\ &\geq \left(\frac{1}{2} - \frac{\varepsilon C_1}{2}\right)\|u\|_{X_0}^2 - C_\varepsilon \int_{\mathbb{R}} (e^{\alpha u^2} - 1)|u|^q \, \mathrm{d}x. \end{aligned}$$

Then, for each $u \in X_0$ with $\alpha\|u\|_{X_0}^2 < \omega$, by applying Lemma 2.8, we get

$$\begin{aligned} I_0(u) &\geq \left(\frac{1}{2} - \frac{\varepsilon C_1}{2}\right)\|u\|_{X_0}^2 - C_2\|u\|_{X_0}^q \\ &= \|u\|_{X_0}^2 \left[\left(\frac{1}{2} - \frac{\varepsilon C_1}{2}\right) - C_2\|u\|_{X_0}^{q-2}\right]. \end{aligned}$$

Since $q > 2$, we may choose $\varepsilon > 0$ and $\rho > 0$ sufficiently small such that

$$\left(\frac{1}{2} - \frac{\varepsilon C_1}{2}\right) - C_2\rho^{q-2} > 0.$$

Thus, there exist $\rho, \sigma > 0$ such that $I_0(u) \geq \sigma$ if $\|u\|_{X_0} = \rho$, which is the desired conclusion. □

Lemma 2.12. *Suppose that* $(f_{0,2})$ *is satisfied. Then there exists* $e \in X_0$ *with* $\|e\|_{X_0} > \rho$ *such that*

$$I_0(e) < \inf_{\|u\|_{X_0} = \rho} I_0(u).$$

Proof. Let $u \in C_0^\infty(\mathbb{R}) \backslash \{0\}$ with support K. By $(f_{0,2})$ there exist $C_1, C_2 > 0$ such that

$$F_0(x,u) \geq C_1|u|^\theta - C_2 \quad \text{for all} \quad x \in K.$$

Consequently, we have the following estimate for $t > 0$,

$$I_0(tu) \leq \frac{t^2}{2}\|u\|_{X_0}^2 - C_1 t^\theta \int_K |u|^\theta \, \mathrm{d}x + C_2 \int_K \mathrm{d}x.$$

Since $\theta > 2$, we obtain $I_0(tu) \to -\infty$ as $t \to \infty$. Setting $e = tu$ with t large enough, we conclude the proof. □

Minimax level

As a consequence of Lemmas 2.11 and 2.12, the minimax level

$$c_0 = \inf_{g\in\Gamma} \max_{t\in[0,1]} I_0(g(t))$$

is positive, where $\Gamma = \{g \in C([0,1], X_0) : g(0) = 0 \text{ and } g(1) = e\}$.

Next, we provide an estimate to the minimax level.

Lemma 2.13. *Suppose that* $(f_{0,4})$ *holds. Then*

$$\sigma \le c_0 < \frac{(\theta-2)\omega}{2\theta\alpha_0}.$$

Proof. Applying Lemma 2.11 we have that $c_0 \ge \sigma$. In order to get an upper estimate, we consider the function u_q given in Lemma 2.10. Thus, it follows that

$$\begin{aligned} c_0 &\le \max_{t\ge 0} I_0(tu_q) \\ &\le \max_{t\ge 0}\left[\frac{t^2}{2}\|u_q\|^2 - \frac{C_q}{q}t^q\|u_q\|_q^q\right] \\ &= \max_{t\ge 0}\left[\frac{t^2}{2}S_q^2 - \frac{C_q}{q}t^q\right] \\ &= \frac{(q-2)}{2q}\frac{S_q^{2q/(q-2)}}{C_q^{2/(q-2)}} \\ &< \frac{(\theta-2)\omega}{2\theta\alpha_0}, \end{aligned}$$

where we have used $(f_{0,4})$. This completes the proof of the lemma. □

On Palais-Smale sequences

By the Mountain Pass Theorem without the (PS) condition (see [37, Theorem 4.3]), there exists a sequence (u_k) in X_0 satisfying

$$I_0(u_k) \to c_0 \quad \text{and} \quad I_0'(u_k) \to 0. \tag{2.22}$$

Lemma 2.14. *Suppose that* $(f_{0,1})$ *and* $(f_{0,2})$ *hold. Then the sequence* (u_k) *is bounded in* X_0 *and its weak limit denoted by* u_0 *is a weak solution of* (P_0).

Proof. Using well-known arguments it is not difficult to check that (u_k) is a bounded sequence in X_0. Indeed, by $(f_{0,2})$ we have

$$\begin{aligned} I_0(u_k) - \frac{1}{\theta}I_0'(u_k)u_k &= \left(\frac{1}{2}-\frac{1}{\theta}\right)\|u_k\|_{X_0}^2 + \int_{\mathbb{R}}\left[\frac{1}{\theta}f_0(x,u_k)u_k - F_0(x,u_k)\right]\mathrm{d}x \\ &\ge \left(\frac{1}{2}-\frac{1}{\theta}\right)\|u_k\|_{X_0}^2. \end{aligned} \tag{2.23}$$

By (2.22), there exists $k_0 \in \mathbb{N}$ such that for all $k \in \mathbb{N}$ with $k \geq k_0$, it holds

$$I_0(u_k) - \frac{1}{\theta} I_0'(u_k) u_k \leq C + \|u_k\|_{X_0}.$$

This together with (2.23) imply that $\|u_k\|_{X_0} \leq C_1$.

Since X_0 is a Hilbert space, up to a subsequence, we can assume that there exists $u_0 \in X_0$ such that

$$\begin{cases} u_k \rightharpoonup u_0 \text{ in } X_0, \\ u_k \to u_0 \text{ in } L^q_{loc}(\mathbb{R}) \text{ for all } q \geq 1, \\ u_k(x) \to u_0(x) \text{ almost everywhere in } \mathbb{R}. \end{cases}$$

In order to complete the proof of the lemma, it is sufficient to prove that

$$\int\limits_{\mathbb{R}} f_0(x, u_k) v \, \mathrm{d}x \to \int\limits_{\mathbb{R}} f_0(x, u_0) v \, \mathrm{d}x, \text{ as } k \to \infty, \text{ for all } v \in C_0^\infty(\mathbb{R}). \tag{2.24}$$

Note that combining (2.22) and (2.23), we reach

$$c_0 \ \geq \ \frac{\theta - 2}{2\theta} \limsup_{k \to +\infty} \|u_k\|^2_{X_0}.$$

Thus, by Lemma 2.13 we obtain

$$\limsup_{k \to +\infty} \|u_k\|^2_{X_0} \leq \frac{2\theta c_0}{\theta - 2} < \frac{\omega}{\alpha_0}.$$

This implies $\alpha_0 \|u_k\|^2_{X_0} < \omega$ for k sufficiently large. Hence, we can choose $q > 1$ sufficiently close to 1 and $\delta > 0$ sufficiently small such that $q(\alpha_0 + \delta)\|u_k\|^2_{X_0} < \omega$ for k sufficiently large. Consequently, by (2.13) there exists $C > 0$ such that

$$\int\limits_{\mathbb{R}} (e^{q(\alpha_0+\delta)\|u_k\|^2_{X_0} (\frac{u_k}{\|u_k\|_{X_0}})^2} - 1) \, \mathrm{d}x \leq C. \tag{2.25}$$

Since $f_0(x, s)$ has critical exponential growth, combining condition $(f_{0,1})$ and Hölder's inequality for $q' = q/(q-1) > 2$, we get

$$\begin{aligned} \int\limits_{\mathbb{R}} f_0(x, u_k) u_k \, \mathrm{d}x &\leq \varepsilon \int\limits_{\mathbb{R}} u_k^2 \, \mathrm{d}x + C_\varepsilon \int\limits_{\mathbb{R}} (e^{(\alpha_0+\delta)u_k^2} - 1) u_k \, \mathrm{d}x \\ &\leq \varepsilon C + C_\varepsilon \|u_k\|_{q'} \left(\int\limits_{\mathbb{R}} (e^{q(\alpha_0+\delta)\|u_k\|^2_{X_0} (\frac{u_k}{\|u_k\|_{X_0}})^2} - 1) \, \mathrm{d}x \right)^{1/q}. \end{aligned} \tag{2.26}$$

Hence, by (2.25) we have

$$\int\limits_{\mathbb{R}} f_0(x, u_k) u_k \, \mathrm{d}x \leq C.$$

Consequently, thanks to Lemma 2.1 in [13], we reach

$$f_0(x, u_k) \to f_0(x, u_0) \text{ in } L^1_{loc}(\mathbb{R}), \text{ as } k \to \infty,$$

which implies (2.24). This completes the proof of the lemma. □

Proof of Theorem 2.1

Using Lemma 2.14, we have that u_0 is a weak solution of (P_0). Thus if u_0 is nontrivial the theorem is proved. If $u_0 = 0$, we have the following claim: there exist $(y_k) \subset \mathbb{R}$ and $R, a > 0$ such that

$$\liminf_{k\to\infty} \sup_{y_k\in\mathbb{R}} \int\limits_{B_R(y_k)} |u_k|^2 \,\mathrm{d}x > a. \tag{2.27}$$

Indeed, let us assume that (2.27) does not hold. Then for all sequences $(y_k) \subset \mathbb{R}$ and $R > 0$, we have

$$\liminf_{k\to\infty} \sup_{y_k\in\mathbb{R}} \int\limits_{B_R(y_k)} |u_k|^2 \,\mathrm{d}x = 0. \tag{2.28}$$

By combining (2.28) and Lemma 2.9, we obtain that $u_k \to 0$ in $L^t(\mathbb{R})$ for $2 < t < \infty$. Thus, by applying (2.25) and (2.26) we reach

$$\int\limits_{\mathbb{R}} f_0(x, u_k)u_k \,\mathrm{d}x \to 0, \text{ as } k \to \infty.$$

This estimate and (2.22) imply that $\|u_k\|_{X_0} \to 0$. Furthermore, in view of assumption $(f_{0,2})$ we conclude that

$$\int\limits_{\mathbb{R}} F_0(x, u_k) \,\mathrm{d}x \to 0, \text{ as } k \to \infty. \tag{2.29}$$

By combining the convergence $\|u_k\|_{X_0} \to 0$, (2.29) and (2.22), we get that $c_0 = 0$, which is contradiction. Thus, (2.27) holds.

We may assume, without loss of generality, that $(y_k) \subset \mathbb{Z}$. Letting $w_k(x) = u_k(x - y_k)$, since $V_0(\cdot)$, $f_0(\cdot, s)$ and $F_0(\cdot, s)$ are 1-periodic functions, by a careful calculation we obtain

$$\|u_k\|_{X_0} = \|w_k\|_{X_0},\ I_0(u_k) = I_0(w_k) \to c_0 \quad \text{and} \quad I_0'(w_k) \to 0.$$

Consequently, by similar arguments done in the previous sections, we obtain that (w_k) is bounded in X_0 and there exists $w_0 \in X_0$ such that $w_k \rightharpoonup w_0$ in X_0 and w_0 is a weak solution of the problem (P_0). Moreover, by (2.27), taking a subsequence and R sufficiently large, we get

$$a^{1/2} \leq \|w_k\|_{L^2(B_R(0))} \leq \|w_k - w_0\|_{L^2(B_R(0))} + \|w_0\|_{L^2(B_R(0))}. \tag{2.30}$$

Thus, from the Rellich-Kondrachov Embedding Theorem, we conclude that w_0 is nontrivial.

To finalize, notice that if u is a weak solution of (P_0), since $f_0(x,s) = 0$ for all $s \leq 0$ and $I_0'(u)v = 0$ for all $v \in X_0$, choosing the test function $v = -u^-$, by using the following inequality $|u^-(x) - u^-(y)|^2 \leq (u(x) - u(y))(u^-(y) - u^-(x))$ we get that $\|u^-\|_{X_0} \leq 0$. Thus, u is a nonnegative function. This completes the proof of Theorem 2.1.

Existence of a solution for the asymptotically periodic problem

In order to find a nontrivial solution for (P), we will consider the functional $I : X_1 \to \mathbb{R}$ given by

$$I(u) = \frac{1}{2}\|u\|_{X_1}^2 - \int\limits_{\mathbb{R}} F(x,u)\,\mathrm{d}x.$$

Similarly to Section 2 we can see that I is well defined and by using standard arguments $I \in C^1(X_1, \mathbb{R})$, see Appendix A, with

$$I'(u)\phi = \frac{1}{2\pi}\int\limits_{\mathbb{R}^2} \frac{(u(x) - u(y))(\phi(x) - \phi(y))}{|x-y|^2}\,\mathrm{d}x\,\mathrm{d}y + \int\limits_{\mathbb{R}} V(x)u\phi\,\mathrm{d}x - \int\limits_{\mathbb{R}} f(x,u)\phi\,\mathrm{d}x,$$

for all $\phi \in X_1$. Thus, a critical point of I is a weak solution of (P) and reciprocally. Moreover, the functional I has the geometry of the Mountain Pass Theorem, that is,

Lemma 2.15. *If $(f_2) - (f_3)$ hold, then*

(i) *there exist σ_1, $\rho_1 > 0$ such that $I(u) \geq \sigma_1$ if $\|u\|_{X_1} = \rho_1$;*

(ii) *there exists $e_1 \in X_1$, with $\|e_1\|_{X_1} > \rho_1$, such that $I(e_1) < 0$.*

As a consequence of Lemma 2.15, the minimax level

$$c_1 := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t))$$

is positive, where $\Gamma = \{\gamma \in C([0,1], X_1) : \gamma(0) = 0 \text{ and } \gamma(1) = e_1\}$.

Moreover, by applying the Mountain Pass Theorem without the (PS) condition (see [37, Theorem 4.3]), there exists a sequence $(v_k) \subset X_1$ such that $I(v_k) \to c_1$ and $I'(v_k) \to 0$. Using the arguments as in Section 2, we get the following result:

Lemma 2.16. *If* $(f_2)-(f_3)$ *hold, then*

(i) (v_k) *is a bounded sequence in* X_1*;*

(ii) $\int_{\mathbb{R}} (e^{\alpha_0 \beta v_k^2}-1)\,\mathrm{d}x \leq C$ *for* $\beta>1$ *sufficiently close to* 1*;*

(iii) $v_k \rightharpoonup v_0$ *in* X_1 *and* v_0 *is a critical point of functional* I.

Proof of Theorem 2.2

We will work in order to prove that v_0 is nontrivial. Assume, by contradiction, that v_0 is trivial. Then, we have the following convergence result:

Lemma 2.17. *If* (V_1), $(f_{0,1})-(f_{0,2})$ *and* $(f_1)-(f_3)$ *hold, then as* $k\to\infty$

(i) $\int_{\mathbb{R}} [f_0(x,v_k)-f(x,v_k)]v_k\,\mathrm{d}x \to 0$*;*

(ii) $\int_{\mathbb{R}} [F_0(x,v_k)-F(x,v_k)]\,\mathrm{d}x \to 0$*;*

(iii) $\int_{\mathbb{R}} [V_0(x)-V(x)]v_k^2\,\mathrm{d}x \to 0.$

Proof. By condition (f_1), given $\varepsilon>0$ there exists $\eta>0$ such that

$$\int_{|x|\geq\eta} |f(x,v_k)-f_0(x,v_k)||v_k|\,\mathrm{d}x \leq \varepsilon \int_{|x|\geq\eta} (e^{\alpha_0 v_k^2}-1)|v_k|\,\mathrm{d}x.$$

Hence, from (2.17) and Hölder's inequality with $1/\tau+1/\tau'=1$ such that $\tau>1$ and $\tau'>2$, we get

$$\begin{aligned}\int_{|x|\geq\eta} |f(x,v_k)-f_0(x,v_k)||v_k|\,\mathrm{d}x &\leq \varepsilon \left(\int_{\mathbb{R}} (e^{\alpha_0 v_k^2}-1)^{\tau}\,\mathrm{d}x\right)^{1/\tau} \left(\int_{\mathbb{R}} |v_k|^{\tau'}\,\mathrm{d}x\right)^{1/\tau'}\\ &\leq \varepsilon \left(\int_{\mathbb{R}} (e^{\alpha_0 \tau v_k^2}-1)\,\mathrm{d}x\right)^{1/\tau} \left(\int_{\mathbb{R}} |v_k|^{\tau'}\,\mathrm{d}x\right)^{1/\tau'}.\end{aligned}$$

By Lemma 2.16 (i),(ii), we obtain

$$\int_{|x|\geq\eta} |f(x,v_k)-f_0(x,v_k)||v_k|\,\mathrm{d}x \leq C\varepsilon. \tag{2.31}$$

On the other hand, using conditions $(f_{0,1})$, (f_2) and Hölder's inequality we reach

$$\int\limits_{|x|\leq\eta} |f(x,v_k)-f_0(x,v_k)||v_k|\,\mathrm{d}x \leq 2\varepsilon\|v_k\|_2^2+2C_\varepsilon\left(\int\limits_{\mathbb{R}}(e^{\alpha_0\tau v_k^2}-1)\,\mathrm{d}x\right)^{1/\tau}\left(\int\limits_{|x|\leq\eta}|v_k|^{\tau'}\,\mathrm{d}x\right)^{1/\tau'}.$$

From the Rellich-Kondrachov Embedding Theorem we have that, up to a subsequence, $\|v_k\|_{L^{\tau'}(B_\eta)} \to 0$. Moreover, since $\|v_k\|_2 \leq C$, we obtain

$$\int\limits_{|x|\leq\eta} |f(x,v_k)-f_0(x,v_k)||v_k|\,\mathrm{d}x \to 0, \text{ as } k\to\infty.$$

Combining this with (2.31), we obtain

$$\int\limits_{\mathbb{R}} |[f_0(x,v_k)-f(x,v_k)]v_k|\,\mathrm{d}x \to 0, \text{ as } k\to\infty.$$

Using the assumptions $(f_1)-(f_3)$, we have that

$$\int\limits_{\mathbb{R}} |F_0(x,v_k)-F(x,v_k)|\,\mathrm{d}x \leq C\int\limits_{\mathbb{R}} |[f_0(x,v_k)-f(x,v_k)]||v_k|\,\mathrm{d}x \to 0.$$

For last convergence, note that

$$0\leq \int\limits_{\mathbb{R}}[V_0(x)-V(x)]v_k^2\,\mathrm{d}x \quad\leq\quad C\int\limits_{|x|\leq R} v_k^2\,\mathrm{d}x + \int\limits_{|x|>R}[V_0(x)-V(x)]v_k^2\,\mathrm{d}x.$$

By (V_1), given $\varepsilon>0$, there exists $R>0$ sufficiently large such that $[V_0(x)-V(x)]<\varepsilon$ for $|x|>R$, then

$$\int\limits_{\mathbb{R}}[V_0(x)-V(x)]v_k^2\,\mathrm{d}x \leq C\int\limits_{|x|\leq R} v_k^2\,\mathrm{d}x + \varepsilon\int\limits_{|x|>R} v_k^2\,\mathrm{d}x.$$

From the Rellich-Kondrachov Embedding Theorem we have that, up to a subsequence, $\|v_k\|_{L^2(B_R)} \to 0$. Moreover, since $\|v_k\|_2 \leq C$, we conclude that

$$\int\limits_{\mathbb{R}}[V_0(x)-V(x)]v_k^2\,\mathrm{d}x \to 0, \text{ as } k\to\infty.$$

This completes the proof of the lemma. □

As a consequence of Lemma 2.17, it follows that

$$|I_0(v_k)-I(v_k)|\to 0 \quad\text{and}\quad \|I_0'(v_k)-I'(v_k)\|_* \to 0, \text{ as } k\to\infty.$$

Hence,

$$I_0(v_k) \to c_1 \quad \text{and} \quad I_0'(v_k) \to 0, \text{ as } k \to \infty.$$

Similar to the proof of Theorem 2.1, there exist $(y_k) \subset \mathbb{Z}$ and $R, a > 0$ such that

$$\liminf_{k\to+\infty} \sup_{y_k\in\mathbb{R}} \int\limits_{B_R(y_k)} |v_k|^2 \, \mathrm{d}x > a.$$

Consider $w_k(x) = v_k(x - y_k)$, since $V_0(x)$, $f_0(x,s)$ and $F_0(x,s)$ are $1-$periodic functions in x, we get

$$\|v_k\|_{X_0} = \|w_k\|_{X_0}, \;\; I_0(v_k) = I_0(w_k) \;\text{ and }\; I_0'(w_k) \to 0.$$

Then, there exists $w_0 \in X_0$ such that $w_k \rightharpoonup w_0$ in X_0 and $I_0'(w_0) = 0$. Moreover, $I_0(w_0) \le c_1$, indeed using Fatou's lemma we have

$$\begin{aligned} I_0(w_0) &= I_0(w_0) - \frac{1}{2} I_0'(w_0)w_0 \\ &= \frac{1}{2} \int\limits_{\mathbb{R}} [f_0(x,w_0)w_0 - 2F_0(x,w_0)] \, \mathrm{d}x \\ &\le \liminf_{k\to+\infty} \frac{1}{2} \int\limits_{\mathbb{R}} [f_0(x,w_k)w_k - 2F_0(x,w_k)] \, \mathrm{d}x \\ &= \lim_{k\to+\infty} [I_0(w_k) - \frac{1}{2} I_0'(w_k)w_k] = c_1. \end{aligned}$$

Arguing as in (2.30) we conclude that w_0 is nontrivial. Now, by $(f_{0,3})$, we have that $\max\{I_0(tw_0) \, : \, t \ge 0\}$ is unique and then

$$c_0 \le \max_{t\ge0} I_0(tw_0) = I_0(w_0) \le c_1. \tag{2.32}$$

On the other hand, considering u_0 the solution obtained in Theorem 2.1, from (V_1), (f_1), (f_5), (f_4) and $(f_{0,3})$, we have

$$c_1 \le \max_{t\ge0} I(tu_0) = I(t_1u_0) < I_0(t_1u_0) \le \max_{t\ge0} I_0(tu_0) = I_0(u_0) = c_0,$$

that is, $c_1 < c_0$, which contradicts (2.32). Therefore, v_0 is nontrivial.

To finalize, notice that if u is a weak solution of (P), since $f(x,s) = 0$ for all $s \le 0$ and $I'(u)v = 0$ for all $v \in X_1$, choosing the test function $v = -u^-$ and by using the following inequality $|u^-(x) - u^-(y)|^2 \le (u(x) - u(y))(u^-(y) - u^-(x))$ we get that $\|u^-\|_{X_1} \le 0$. Thus, u is a nonnegative function. This completes the proof of Theorem 2.2.

Chapter 3

A class of asymptotically periodic fractional Schrödinger equations with Sobolev critical growth

In this chapter we present the results of the paper [16], here we study a class of fractional Schrödinger equation of the form

$$(-\Delta)^{\alpha}u + V(x)u = |u|^{2^*_\alpha - 2}u + g(x,u), \quad \text{in} \quad \mathbb{R}^N,$$

where $0 < \alpha < 1$, $2\alpha < N$, $2^*_\alpha = 2N/(N-2\alpha)$ is the critical Sobolev exponent, $V : \mathbb{R}^N \to \mathbb{R}$ is a positive potential bounded away from zero, and the nonlinearity $g : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ behaves like $|u|^{q-1}$ at infinity for some $2 < q < 2^*_\alpha$, and does not satisfy the usual Ambrosetti–Rabinowitz condition. We also assume that the potential $V(x)$ and the nonlinearity $g(x,u)$ are asymptotically periodic at infinity. We prove the existence of at least one nonnegative weak solution $u \in H^{\alpha}(\mathbb{R}^N)$ by combining a version of the Mountain Pass Theorem and a version of Concentration-Compactness Principle due to Lions.

Motivation and main results

Our main goal is to establish, under an asymptotic periodicity condition at infinity, the existence of a weak solution for the critical problem

$$(-\Delta)^{\alpha}u + V(x)u = |u|^{2^*_\alpha - 2}u + g(x,u), \;\; x \in \mathbb{R}^N, \tag{3.1}$$

where $0 < \alpha < 1$, $N > 2\alpha$, $V : \mathbb{R}^N \to \mathbb{R}$ and $g : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ are continuous functions.

Considering $\mathcal{F} := \{h \in C(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N); \forall \varepsilon > 0, |\{x \in \mathbb{R}^N : |h(x)| \geq \varepsilon\}| < \infty\}$, we assume that V satisfies

(V) there exist a constant $a_0 > 0$ and a function $V_0 \in C(\mathbb{R}^N)$, 1−periodic in x_i, $1 \leq i \leq N$, such that $V_0 - V \in \mathcal{F}$ and

$$V_0(x) \geq V(x) \geq a_0 > 0, \text{ for all } x \in \mathbb{R}^N.$$

Considering $G(x,s) = \int\limits_0^s g(x,t)\,\mathrm{d}t$, the primitive of g, we also suppose the following hypotheses:

(g_1) $g(x,s) = o(|s|)$, as $s \to 0^+$, uniformly in $\mathbb{R}^N$;

(g_2) there exist constants $a_1, a_2 > 0$ and $2 < q_1 < 2^*_\alpha$ such that

$$|g(x,s)| \leq a_1 + a_2|s|^{q_1-1}, \text{ for all } (x,s) \in \mathbb{R}^N \times [0,+\infty);$$

(g_3) there exist a constant $2 \leq q_2 < 2^*_\alpha$ and functions $h_1 \in L^1(\mathbb{R}^N)$, $h_2 \in \mathcal{F}$ such that

$$\frac{1}{2}g(x,s)s - G(x,s) \geq -h_1(x) - h_2(x)s^{q_2}, \text{ for all } (x,s) \in \mathbb{R}^N \times [0,+\infty).$$

We observe that the conditions (g_1) and (g_2) allow us to employ variational methods to study problem (3.1) and to verify that the associated functional has a local minimum at the origin. Moreover, note that the condition (g_2) imposes a subcritical growth on g. Under the above hypotheses, the associated functional does not satisfy a compactness condition of Palais-Smale type since the term $|u|^{2^*_\alpha-2}u$ is critical and the domain is all $\mathbb{R}^N$.

The asymptotic periodicity of g at infinity is given by the following condition:

(g_4) there exist a constant $2 \leq q_3 \leq 2^*_\alpha - 1$ and functions $h_3 \in \mathcal{F}$, $g_0 \in C(\mathbb{R}^N \times \mathbb{R}, [0,+\infty))$, 1-periodic in x_i, $1 \leq i \leq N$, such that:

  (i) $G(x,s) \geq G_0(x,s) = \int\limits_0^s g_0(x,t)\,\mathrm{d}t$, for all $(x,s) \in \mathbb{R}^N \times [0,+\infty)$;

  (ii) $|g(x,s) - g_0(x,s)| \leq h_3(x)|s|^{q_3-1}$, for all $(x,s) \in \mathbb{R}^N \times [0,+\infty)$;

  (iii) the function $g_0(x,s)/s$ is nondecreasing in the variable $s > 0$, for each $x \in \mathbb{R}^N$.

Finally, we also suppose that g satisfies:

(g_5) there exist an open bounded set $\Omega \subset \mathbb{R}^N$, $2 < p < 2^*_\alpha$ and $C_0 > 0$ such that

(i) $\dfrac{G(x,s)}{s^p} \to +\infty$, as $s \to +\infty$, uniformly in Ω, if $N \geq 4\alpha$;

(ii) $\dfrac{G(x,s)}{s^p} \to +\infty$, as $s \to +\infty$, uniformly in Ω, if $2\alpha < N < 4\alpha$ and $\frac{4\alpha}{N-2\alpha} < p < 2^*_\alpha$;

(iii) $G(x,s) \geq C_0 s^p$ almost everywhere in $\mathbb{R}^N$, if $2\alpha < N < 4\alpha$ and $2 < p < \frac{4\alpha}{N-2\alpha}$.

Now, we may state our main result.

Theorem 3.1. *Assume* (V), $(g_1) - (g_5)$ *and that one of the following statements holds:*

(1) $N \geq 4\alpha$ *and* $2 < p < 2^*_\alpha$

(2) $2\alpha < N < 4\alpha$ *and* $\frac{4\alpha}{N-2\alpha} < p < 2^*_\alpha$

(3) $2\alpha < N < 4\alpha$ *and* $2 < p < \frac{4\alpha}{N-2\alpha}$*, with* C_0 *large enough.*

Then, problem (3.1) *has a nonnegative nontrivial weak solution.*

We observe that in the particular case: $V = V_0$, $g = g_0$, Theorem 3.1, clearly, gives us a solution for the periodic problem. Actually, the condition $(g_4)(iii)$ is not necessary when we look for the existence of a solution for the periodic problem. More specifically, considering the problem

$$(-\Delta)^\alpha u + V_0(x)u = |u|^{2^*_\alpha - 2}u + g_0(x,u), \ \ x \in \mathbb{R}^N, \tag{3.2}$$

under the hypothesis:

(V_0) the function $V_0 \in C(\mathbb{R}^N)$ is 1-periodic in x_i, $1 \leq i \leq N$, and there exists a constant $a_0 > 0$ such that

$$V_0(x) \geq a_0 > 0, \ \text{ for all } \ \in \mathbb{R}^N;$$

and the function g_0 satisfies $(g_1) - (g_3)$ and (g_5), we may state:

Theorem 3.2. *Assume* (V_0), $(g_1) - (g_3)$, (g_5) *and that one of the following statements holds:*

(1) $N \geq 4\alpha$ *and* $2 < p < 2^*_\alpha$

(2) $2\alpha < N < 4\alpha$ *and* $\frac{4\alpha}{N-2\alpha} < p < 2^*_\alpha$

(3) $2\alpha < N < 4\alpha$ *and* $2 < p < \frac{4\alpha}{N-2\alpha}$, *with* C_0 *large enough.*

Then, problem (3.2) *has a nonnegative nontrivial weak solution.*

The (AR) condition has appeared in most of the studies for superlinear problems and plays an important role in studying the existence of nontrivial solutions of many nonlinear elliptic problems. Since Ambrosetti and Rabinowitz proposed the Mountain Pass Theorem in their celebrated paper [2], the critical point theory has become one of the main tools for finding solutions to elliptic equations of variational type. In the subcritical case, the (AR) condition ensures that the Euler–Lagrange functional associated with a (3.1)- type problem has a mountain pass geometry and also guarantees the boundedness of the Palais–Smale sequence, so we can get the nontrivial solution by using suitable versions of the Mountain Pass Theorem.

On the other hand, there are many cases where the nonlinear term does not satisfy the (AR) condition (see Remark 3.3). Thus it becomes interesting to know if a nontrivial solution exists in such situations. When $\alpha = 1$, conditions weaker than (AR) were used first in [12, 27, 36]. In the case $0 < \alpha < 1$, we would like to mention two works, one by Chang and Wang [10] and a paper by J. M. do Ó *et al.* [21].

Motivated by the above mentioned papers and by Lins and Silva [33], we study the existence of a nontrivial solution to problem (3.1) where the subcritical perturbation $g(x, u)$ does not satisfy the (AR) condition. Moreover, we assume that the potential $V(x)$ and $g(x, u)$ are asymptotically periodic at infinity in x. In this sense our results complement the study made in [10,21,43]. Moreover, we also complement [10,11,18,41] in the sense that the potential $V(x)$ belongs to a different class from those treated by them.

Remark 3.3. An example of potential $V(x)$ satisfying the hypothesis (V) is given by $V(x) = e^{-1/(|x|+1)}$, where $a_0 = e^{-1}$ and $V_0 \equiv 1$, and if $\alpha = 1/2$ and $N = 2$, an example of nonlinearity $g(x, s)$, satisfying the hypotheses $(g_1) - (g_5)$, is given by

$$g(x,s) = \begin{cases} \left(1 + \dfrac{1}{|x|+1}\right) g_0(x,s), & \text{if } s > 0, \\ 0, & \text{if } s \leq 0, \end{cases}$$

where $g_0 : \mathbb{R}^2 \times \mathbb{R} \to [0, +\infty)$ is defined by

$$g_0(x,s) = \begin{cases} \varrho_1(x) s \ln(s+1) + \varrho_2(x) s^2, & \text{if } (x,s) \in \mathbb{R}^2 \times [0, +\infty), \\ 0, & \text{if } (x,s) \in \mathbb{R}^2 \times (-\infty, 0), \end{cases}$$

where $\varrho_i : \mathbb{R}^2 \to [0,2]$ is a continuous function, $\varrho_i \neq 0$, 1-periodic in x_i, with $i \in \{1,2\}$. Moreover, we consider that $\operatorname{supp}(\varrho_1) \cap \operatorname{supp}(\varrho_2) = \varnothing$ and that $B_1 \subset \operatorname{supp}(\varrho_2)$. Note that g does not satisfy the (AR) condition (see details in Appendix A).

The outline of this chapter is as follows: In Section 2, we present some notations and definitions about the fractional Laplacian operator, and we introduce the variational framework associated to Problems (3.1) and (3.2). In Section 3, we present some auxiliary results which are used in the proofs of our results, we verify the geometric conditions of the Mountain Pass Theorem and we present some results concerning the behavior of the Cerami sequences. In Section 4 we study the minimax level. In Section 5 we prove some convergence results and, finally, in Section 6 and Section 7, we prove Theorems 3.2 and 3.1.

Notations, definitions and variational setting

As seen in the introduction the operator $(-\Delta)^\alpha$ can be represented [17, Lemma 3.2] as

$$(-\Delta)^\alpha u(x) = -\frac{1}{2} C(N,\alpha) \int_{\mathbb{R}^N} \frac{u(x+z) + u(x-z) - 2u(x)}{|z|^{N+2\alpha}} \, \mathrm{d}z, \tag{3.3}$$

where

$$C(N,\alpha) = \left(\int_{\mathbb{R}^N} \frac{1 - \cos(\zeta_1)}{|\zeta|^{N+2\alpha}} \, \mathrm{d}\zeta \right)^{-1}, \quad \zeta = (\zeta_1, \zeta_2, \cdots, \zeta_N).$$

However, there is another way to define this operator. In fact, when $\alpha = 1/2$ there is an explicit form of calculating the half-Laplacian acting on a function u in the whole space $\mathbb{R}^N$, as the normal derivative on the boundary of its harmonic extension to the upper half-space $\mathbb{R}^{N+1}_+ = \{(x,y) \in \mathbb{R}^{N+1} : y > 0\}$, the so-called Dirichlet to Neumann operator. The α derivative $(-\Delta)^\alpha$ can be characterized in a similar way, defining the $\alpha-$harmonic extension to the upper half-space, see [9] and (3.5)-(3.6) for details. This extension is commonly used in the recent literature since it allows to write nonlocal problems in a local way and this permits to use the variational techniques for these kind of problems. In particular, for nonlocal case in bounded domain, we would like to mention two works, one by Barrios *et al.* [3], and a paper by Brandle *et al.* [6].

In order to prove our results, we consider the spaces $H^{\alpha}(\mathbb{R}^N)$ and $X^{2\alpha}(\mathbb{R}^{N+1}_+)$ defined as the completion of $C_0^{\infty}(\mathbb{R}^N)$ and $C_0^{\infty}(\overline{\mathbb{R}^{N+1}_+})$, respectively, under the norms

$$\|u\|^2_{H^{\alpha}} := \int\limits_{\mathbb{R}^N} |2\pi\xi|^{2\alpha}|\widehat{u}(\xi)|^2 \mathrm{d}\xi = \int\limits_{\mathbb{R}^N} |(-\Delta)^{\alpha/2}u|^2 \mathrm{d}x,$$

$$\|w\|^2_{X^{2\alpha}} := \int\limits_{\mathbb{R}^{N+1}_+} \kappa_{\alpha}\, y^{1-2\alpha}|\nabla w|^2 \mathrm{d}x\mathrm{d}y,$$

where $\kappa_{\alpha} = 2^{1-2\alpha}\Gamma(1-\alpha)/\Gamma(\alpha)$.

The extension operator $E_{2\alpha} : H^{\alpha}(\mathbb{R}^N) \to X^{2\alpha}(\mathbb{R}^{N+1}_+)$ is well defined (see [6, Lemmas 2.2 and 2.3]). For $\phi \in X^{2\alpha}(\mathbb{R}^{N+1}_+)$, let us denote its trace on $\mathbb{R}^N \times \{y = 0\}$ as $\phi(x, 0)$. This trace operator is also well defined and it was proved in [6, Lemmas 2.2 and 2.3] that

$$\|\phi(\cdot, 0)\|_{H^{\alpha}(\mathbb{R}^N)} \leq \kappa_{\alpha}^{-1/2}\|\phi\|_{X^{2\alpha}(\mathbb{R}^{N+1}_+)}. \tag{3.4}$$

For $u \in H^{\alpha}(\mathbb{R}^N)$, we say that $w = E_{2\alpha}(u)$ is the $\alpha-$harmonic extension of u to the upper half-space, $\mathbb{R}^{N+1}_+$, if w is a solution to the problem

$$\begin{cases} -div(y^{1-2\alpha}\nabla w) = 0 \quad \text{in} \quad \mathbb{R}^{N+1}_+, \\ w = u \quad \text{in} \quad \mathbb{R}^N \times \{0\}. \end{cases} \tag{3.5}$$

In [9] it is proved that

$$\lim_{y \to 0^+} y^{1-2\alpha} w_y(x, y) = -\frac{1}{\kappa_{\alpha}}(-\Delta)^{\alpha}u(x). \tag{3.6}$$

As we pointed out at the beginning of this section, identity (3.6) allows to formulate nonlocal problems involving the fractional powers of the Laplacian in $\mathbb{R}^N$ as local problems in divergence form in the half-space $\mathbb{R}^{N+1}_+$. Motivated by (3.5) and (3.6), we will consider the problem

$$\begin{cases} -div(y^{1-2\alpha}\nabla w) = 0 \quad \text{in} \quad \mathbb{R}^{N+1}_+, \\ -\kappa_{\alpha}\dfrac{\partial w}{\partial \nu} = -V(x)u + |u|^{2^*_{\alpha}-2}u + g(x, u) \quad \text{in} \quad \mathbb{R}^N \times \{0\}, \end{cases} \tag{3.7}$$

where

$$\frac{\partial w}{\partial \nu} = \lim_{y \to 0^+} y^{1-2\alpha} w_y(x, y).$$

In order to find a solution to problem (3.7), by using variational methods, we will consider the Hilbert space X,

$$X := \left\{ w \in X^{2\alpha}(\mathbb{R}^{N+1}_+) \, : \, \int\limits_{\mathbb{R}^N} V(x) w(x,0)^2 \mathrm{d}x < \infty \right\},$$

endowed with the inner product given by

$$\langle w, v \rangle = \int\limits_{\mathbb{R}^{N+1}_+} \kappa_\alpha \, y^{1-2\alpha} \nabla w \nabla v \, \mathrm{d}x\mathrm{d}y + \int\limits_{\mathbb{R}^N} V(x) w(x,0) v(x,0) \, \mathrm{d}x$$

and the induced norm

$$\|w\| = \langle w, w \rangle^{1/2}.$$

By condition (V), X is continuously embedded in $X^{2\alpha}(\mathbb{R}^{N+1}_+)$. Consequently, from (3.4), we find $C > 0$ such that

$$\|w(x,0)\|_q \leq C\|w\|, \text{ for all } 2 \leq q \leq 2^*_s. \tag{3.8}$$

Throughout this chapter, we say that $w \in X$ is a weak solution to problem (3.7), if

$$\langle w, \varphi \rangle - \int\limits_{\mathbb{R}^N} |w|^{2^*_\alpha - 2} w(x,0)\varphi(x,0)\mathrm{d}x + \int\limits_{\mathbb{R}^N} g(x, w(x,0))\varphi(x,0)\mathrm{d}x = 0, \text{ for all } \varphi \in X,$$

to which a weak solution $u = w(x,0) \in H^\alpha(\mathbb{R}^N)$ to problem (3.1) corresponds.

The Euler–Lagrange functional associated to problem (3.7) is given by

$$J(w) = \frac{1}{2}\|w\|^2 - \frac{1}{2^*_\alpha} \int\limits_{\mathbb{R}^N} |w(x,0)|^{2^*_\alpha} \mathrm{d}x - \int\limits_{\mathbb{R}^N} G(x, w(x,0))\mathrm{d}x, \tag{3.9}$$

which under the hypotheses (V), (g_1) and (g_2) is well defined in X and belongs to $C^1(X, \mathbb{R})$, with Gâteaux derivative given by

$$J'(w)v = \langle w, v \rangle - \int\limits_{\mathbb{R}^N} |w|^{2^*_\alpha - 2} w(x,0) v(x,0)\mathrm{d}x - \int\limits_{\mathbb{R}^N} g(x, w(x,0)) v(x,0)\mathrm{d}x.$$

Thus, a critical point of J is a weak solution to problem (3.7) and reciprocally.

By a similar approach, associated with the periodic problem, we have that the functional J_0 defined by

$$J_0(w) = \frac{1}{2}\|w\|_0 - \frac{1}{2^*_\alpha} \int\limits_{\mathbb{R}^N} |w(x,0)|^{2^*_\alpha} \mathrm{d}x - \int\limits_{\mathbb{R}^N} G_0(x, w(x,0))\mathrm{d}x,$$

belongs to $C^1(X_0, \mathbb{R})$, where

$$X_0 = \left\{ w \in X^{2\alpha}(\mathbb{R}^{N+1}_+) \,:\, \int\limits_{\mathbb{R}^N} V_0(x) w(x,0)^2 \mathrm{d}x < \infty \right\}.$$

Since we are interested in nonnegative solutions, we consider

$$f(x,s) = \begin{cases} |s|^{2^*_\alpha - 2} s + g(x,s) & \text{if } s \geq 0, \\ 0 & \text{if } s < 0, \end{cases}$$

throughout the remainder of this chapter.

Remark 3.4. Let w be a nonnegative weak solution to problem (3.7), to which a weak solution $u \in H^\alpha(\mathbb{R}^N)$ to problem (3.1) corresponds. Then $w = E_\alpha(u)$ and $w(x,0) = u$. If $w \not\equiv 0$, we have $u \neq 0$. Moreover, if u is sufficiently regular, we may ensure that $u > 0$. In fact, if $u(x_0) = 0$ for some $x_0 \in \mathbb{R}^N$, then $(-\Delta)^\alpha u(x_0) = 0$ and by the representation formula (3.3), one obtains, at x_0, that

$$\int\limits_{\mathbb{R}^N} \frac{u(x_0+z) + u(x_0-z)}{|z|^{N+2\alpha}} \mathrm{d}z = 0,$$

yielding $u = 0$, a contradiction.

Preliminary results

In this section, we present two versions of the Mountain Pass Theorem which are used in the proofs of Theorems 3.2 and 3.1. Furthermore, we verify the geometric conditions of the Mountain Pass Theorem and we also present some results concerning the behavior of the Cerami sequences of the associated functional: we show the boundedness for the Cerami sequences and a proposition which will be essential to guarantee that the solutions that we provide in our proofs of Theorems 3.2 and 3.1 are not trivial.

Versions of the Mountain Pass Theorem

As we observed in the introduction, the functional associated to problem (3.7) does not satisfy the condition Palais-Smale. To overcome this difficulty, we will use two versions of the Mountain Pass Theorem which we will present below.

Let E be a real Banach space and $I \in C^1(E, \mathbb{R})$. We recall that I satisfies the Cerami condition on level c, denoted by $(Ce)_c$, if any sequence $(u_n) \subset E$ for which (*i*) $I(u_n) \to c$ and (*ii*) $\|I'(u_n)\|_{E'}(\|u_n\|_E + 1) \to 0$, as $n \to \infty$, possesses a convergent subsequence. We say that $(u_n) \subset E$ is a $(Ce)_c$ sequence if it satisfies $(i) - (ii)$.

Theorem 3.5. *Let E be a real Banach space. Suppose that $I \in C^1(E, \mathbb{R})$, $I(0) = 0$ and*

(I_1) *there exist $\beta, \rho > 0$ such that $I\,|_{\partial B_\rho(0)} \geq \beta > 0$,*

(I_2) *there exists $e \in E$ with $\|e\| > \rho$ such that $I(e) \leq 0$.*

Then I possesses a $(Ce)_c$ sequence with $c \geq \beta > 0$ given by

$$c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t)),$$

where

$$\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = 0 \ \text{ and } \ \gamma(1) = e\}. \tag{3.10}$$

We will also need to establish a local version of Theorem 3.5, which has been proved in [33] (or [26, Theorem 7.10]). For this, we consider K the set of critical points of I and given $b \in \mathbb{R}$, we define $K_b = \{u \in X \,:\, u \in K \ \text{ and } \ I(u) = b\}$.

Theorem 3.6. *Let E be a real Banach space. Suppose that $I \in C^1(E, \mathbb{R})$ satisfies $I(0) = 0$, (I_1) and (I_2). If there exists $\gamma_0 \in \Gamma$, Γ defined by (3.10), such that*

$$c = \max_{t \in [0,1]} I(\gamma_0(t)) > 0,$$

then I possesses a nontrivial critical point $u \in K_c \cap \gamma_0([0,1])$.

Mountain pass geometry

The next lemma shows that the functional associated to problem (3.7) satisfies the geometric properties of the Mountain Pass Theorem.

Lemma 3.7. *Suppose that (V), (g_1) and (g_2) are satisfied. Then the functional J, defined by (3.9), satisfies the conditions of Theorem 3.5.*

Proof. Since $G(x, 0) = 0$ for all $x \in \mathbb{R}^N$, it follows that $J(0) = 0$. Thus we must show that J satisfies the conditions (I_1) and (I_2). To verify (I_1), note that by (g_1) and (g_2), given any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$|g(x,s)| \leq \varepsilon|s| + C_\varepsilon |s|^{q_1 - 1} \ \text{ for all } \ (x,s) \in \mathbb{R}^N \times \mathbb{R}, \tag{3.11}$$

$$|G(x,s)| \leq \frac{\varepsilon}{2}|s|^2 + \frac{C_\varepsilon}{q_1}|s|^{q_1} \ \text{ for all } \ (x,s) \in \mathbb{R}^N \times \mathbb{R}. \tag{3.12}$$

From (3.12),

$$\int_{\mathbb{R}^N} |G(x, w(x,0))|\,\mathrm{d}x \leq \frac{\varepsilon}{2}\int_{\mathbb{R}^N} w(x,0)^2\mathrm{d}x + \frac{C_\varepsilon}{q_1}\int_{\mathbb{R}^N} |w(x,0)|^{q_1}\mathrm{d}x, \text{ for all } w \in X. \quad (3.13)$$

By using the condition (V) in (3.13), we obtain

$$\int_{\mathbb{R}^N} G(x, w(x,0))\mathrm{d}x \leq \frac{\varepsilon}{2a_0}\int_{\mathbb{R}^N} V(x)w(x,0)^2\mathrm{d}x + \frac{C_\varepsilon}{q_1}\int_{\mathbb{R}^N} |w(x,0)|^{q_1}\mathrm{d}x. \quad (3.14)$$

Combining (3.8) and (3.14), we can find two positive constants, C_1 and C_2, such that

$$J(w) \geq \left(\frac{1}{2} - \frac{\varepsilon}{2a_0}\right)\rho^2 - C_1\rho^{2^*_\alpha} - C_2\rho^{q_1} \quad \text{if} \quad \|w\| = \rho.$$

Since $q_1, 2^*_\alpha > 2$, choosing $0 < \varepsilon < a_0$, we conclude, for ρ sufficiently small, that

$$\beta := \inf_{\|w\|=\rho} J > 0.$$

Hence (I_1) holds. In order to verify the condition (I_2), consider $\varphi \in C_0^\infty(\mathbb{R}^{N+1}_+, \mathbb{R}_+)$ with $\varphi(x,0) \not\equiv 0$. From $(g_4)(i)$, $G(x, t\varphi) \geq 0$ for every $t > 0$. Thus,

$$J(t\varphi) \leq \frac{t^2}{2}\|\varphi\|^2 - \frac{t^{2^*_\alpha}}{2^*_\alpha}\int_{\mathbb{R}^N} |\varphi(x,0)|^{2^*_\alpha}\mathrm{d}x \to -\infty, \quad \text{as} \quad t \to +\infty.$$

Setting $e_0 = tu$ with t large enough, the condition (I_2) is satisfied. This completes the proof. □

By Lemma 3.7 and Theorem 3.5, we have

Corollary 3.8. *Suppose that (V), (g_1) and (g_2) are satisfied. Then*

$$c_M = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} J(\gamma(t)) \geq \beta > 0,$$

where $\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = 0$ and $\gamma(1) = e_0\}$, and the functional J possesses a $(Ce)_{c_M}$ sequence.

Behaviour of the Cerami sequences

Here we verify the boundedness of the $(Ce)_c$ sequences associated with the functional J. Before stating the next lemma, we establish a simple result that will be employed several times throughout this chapter. In the following lemma, given $h \in \mathcal{F}$, we set $D_\varepsilon = \{x \in \mathbb{R}^N : |h(x)| \geq \varepsilon\}$ and $D_\varepsilon(R) = \{x \in \mathbb{R}^N : |h(x)| \geq \varepsilon \text{ and } |x| \geq R\}$.

Lemma 3.9. *Suppose that $h \in \mathcal{F}$. Then $|D_\varepsilon(R)| \to 0$ as $R \to \infty$.*

Proof. Since $h \in \mathcal{F}$, $|D_\varepsilon| < \infty$ for all $\varepsilon > 0$. Consequently, this Lemma is equivalent to the following claim:

$$\lim_{n\to\infty} |D_\varepsilon \cap (\mathbb{R}^N \setminus B_{R_n})| = 0,$$

for every sequence $(R_n) \subset \mathbb{R}$ such that $R_n \to \infty$. Consider the real function $\xi : \mathbb{R}^N \to \mathbb{R}$ given by

$$\xi(x) = \begin{cases} 1 & \text{for} \quad x \in D_\varepsilon, \\ 0 & \text{for} \quad x \in \mathbb{R}^N \setminus D_\varepsilon. \end{cases}$$

Note that $\|\xi\|_1 = |D_\varepsilon|$, then $\xi \in L^1(\mathbb{R}^N)$. Defining the sequence of functions $\xi_n : \mathbb{R}^N \to \mathbb{R}$ by

$$\xi_n(x) = \begin{cases} 1 & \text{for} \quad x \in D_\varepsilon \cap (\mathbb{R}^N \setminus B_{R_n}), \\ 0 & \text{for} \quad x \in \mathbb{R}^N \setminus D_\varepsilon \cap (\mathbb{R}^N \setminus B_{R_n}), \end{cases}$$

it follows that $|\xi_n(x)| \leq |\xi(x)|$ and $\xi_n(x) \to 0$ almost everywhere in $\mathbb{R}^N$ as $n \to \infty$. Consequently, by the dominated convergence theorem,

$$|D_\varepsilon \cap (\mathbb{R}^N \setminus B_{R_n})| = \|\xi_n\|_1 \to 0 \quad \text{as} \quad n \to +\infty.$$

This completes the proof of Lemma 3.9. □

Lemma 3.10. *Suppose that (V), $(g_1) - (g_3)$ are satisfied and let $(v_n) \subset X$ be an arbitrary Cerami sequence of J on level c, that is,*

$$J(v_n) = c + o_n(1) \quad \textit{and} \quad \|J'(v_n)\|_*(1 + \|v_n\|) = o_n(1). \tag{3.15}$$

Then (v_n) is bounded in X.

Proof. We must show that there exists $M > 0$ such that

$$\|v_n\|^2 = \int_{\mathbb{R}^{N+1}_+} \kappa_\alpha \, y^{1-2\alpha} |\nabla v_n|^2 \mathrm{d}x\mathrm{d}y + \int_{\mathbb{R}^N} V(x) v_n(x,0)^2 \mathrm{d}x \leq M.$$

By the first condition in (3.15), (3.12) and (V), we have

$$\frac{1}{2}\|v_n\|^2 \leq \frac{1}{2^*_\alpha} \int_{\mathbb{R}^N} |v_n(x,0)|^{2^*_\alpha} \mathrm{d}x + \frac{\varepsilon}{2a_0} \int_{\mathbb{R}^N} V(x)|v_n(x,0)|^2 \mathrm{d}x + \frac{C_\varepsilon}{q_1} \int_{\mathbb{R}^N} |v_n(x,0)|^{q_1} \mathrm{d}x + c + o_n(1).$$

Given $0 < \delta \leq 1$ to be chosen later, there exists $0 < \delta_1 < 1$ such that $|s|^{q_1} \leq \delta |s|^2$ for all $|s| \leq \delta_1$. Then, by using (V), it follows that

$$\begin{aligned} &\frac{1}{2} \int_{\mathbb{R}^{N+1}_+} \kappa_\alpha \, y^{1-2\alpha} |\nabla v_n|^2 \mathrm{d}x\mathrm{d}y + \left(\frac{1}{2} - \frac{\varepsilon}{2a_0}\right) \int_{\mathbb{R}^N} V(x)|v_n(x,0)|^2 \mathrm{d}x \\ &\quad \leq \frac{1}{2^*_\alpha} \int_{\mathbb{R}^N} |v_n(x,0)|^{2^*_\alpha} \mathrm{d}x + \frac{C_\varepsilon}{q_1} \int_{\{|v_n(x,0)| \leq \delta_1\}} |v_n(x,0)|^{q_1} \mathrm{d}x + \frac{C_\varepsilon}{q_1} \int_{\{|v_n(x,0)| > \delta_1\}} |v_n(x,0)|^{q_1} \mathrm{d}x + c + o_n(1) \\ &\quad \leq \frac{1}{2^*_\alpha} \int_{\mathbb{R}^N} |v_n(x,0)|^{2^*_\alpha} \mathrm{d}x + \frac{C_\varepsilon \delta}{q_1 a_0} \int_{\mathbb{R}^N} V(x)|v_n(x,0)|^2 \mathrm{d}x + \frac{C_\varepsilon}{q_1} \int_{\{|v_n(x,0)| > \delta_1\}} |v_n(x,0)|^{q_1} \mathrm{d}x + c + o_n(1), \end{aligned}$$

which yields

$$\frac{1}{2}\int\limits_{\mathbb{R}^{N+1}_+}\kappa_\alpha\, y^{1-2\alpha}|\nabla v_n|^2\mathrm{d}x\mathrm{d}y+\left(\frac{1}{2}-\frac{\varepsilon}{2a_0}-\frac{C_\varepsilon\delta}{q_1a_0}\right)\int\limits_{\mathbb{R}^N}V(x)|v_n(x,0)|^2\mathrm{d}x$$
$$\leq\frac{1}{2^*_\alpha}\int\limits_{\mathbb{R}^N}|v_n(x,0)|^{2^*_\alpha}\mathrm{d}x+\frac{C_\varepsilon}{q_1}\int\limits_{\{|v_n(x,0)|>\delta_1\}}|v_n(x,0)|^{q_1}\mathrm{d}x+c+o_n(1).$$

Note that if $|s|>\delta_1$, there exists $C_1>0$ such that $|s|^{q_1}\leq C_1|s|^{2^*_\alpha}$. Thus,

$$\begin{aligned}\frac{1}{2}\int\limits_{\mathbb{R}^{N+1}_+}\kappa_\alpha\, y^{1-2\alpha}|\nabla v_n|^2\mathrm{d}x\mathrm{d}y+\left(\frac{1}{2}-\frac{\varepsilon}{2a_0}-\frac{C_\varepsilon\delta}{q_1a_0}\right)\int\limits_{\mathbb{R}^N}V(x)|v_n(x,0)|^2\mathrm{d}x\\ \leq\left(\frac{1}{2^*_\alpha}+\frac{C_\varepsilon C_1}{q_1}\right)\int\limits_{\mathbb{R}^N}|v_n(x,0)|^{2^*_\alpha}\mathrm{d}x+c+o_n(1).\end{aligned}\tag{3.16}$$

Taking δ and ε sufficiently small such that $\frac{1}{2}-\frac{\varepsilon}{2a_0}-\frac{\delta C_\varepsilon}{q_1a_0}>0$. Hence, in order to conclude the proof of the lemma, it suffices to show that the right hand side in (3.16) is bounded. Note that without loss of generality we may assume that

$$\int\limits_{\mathbb{R}^N}|v_n(x,0)|^{2^*_\alpha}\mathrm{d}x\geq c.\tag{3.17}$$

To prove that $\|v_n(x,0)\|_{2^*_\alpha}$ is bounded, we begin with the estimate

$$J(v_n)-\frac{1}{2}J'(v_n)v_n=\left(\frac{1}{2}-\frac{1}{2^*_\alpha}\right)\int\limits_{\mathbb{R}^N}|v_n(x,0)|^{2^*_\alpha}\mathrm{d}x+\int\limits_{\mathbb{R}^N}\left[\frac{1}{2}g(x,v_n(x,0))v_n(x,0)-G(x,v_n(x,0)\right]\mathrm{d}x,$$

which by (g_3), implies

$$J(v_n)-\frac{1}{2}J'(v_n)v_n\geq\frac{\alpha}{N}\int\limits_{\mathbb{R}^N}|v_n(x,0)|^{2^*_\alpha}\mathrm{d}x-\int\limits_{\mathbb{R}^N}h_1(x)\mathrm{d}x-\int\limits_{\mathbb{R}^N}h_2(x)|v_n(x,0)|^{q_2}\mathrm{d}x.\tag{3.18}$$

Combining (3.15), (3.18) and the fact that $h_1\in L^1(\mathbb{R}^N)$, we can find a constant $C>0$ such that

$$\frac{\alpha}{N}\int\limits_{\mathbb{R}^N}|v_n(x,0)|^{2^*_\alpha}\mathrm{d}x\leq\int\limits_{\mathbb{R}^N}h_2(x)|v_n(x,0)|^{q_2}\mathrm{d}x+C.\tag{3.19}$$

Given $\epsilon>0$, we set $D_\epsilon(R)=\{x\in\mathbb{R}^N:|h_2(x)|\geq\epsilon\text{ and }|x|\geq R\}$ for all $R>0$. Then, since $h_2\in\mathcal{F}$, applying Lemma 3.9, there exists $R=R_\epsilon>0$ such that $|D_\epsilon(R)|<\epsilon$. By Hölder's inequality,

$$\begin{aligned}\int\limits_{D_\epsilon(R)}h_2(x)|v_n(x,0)|^{q_2}\mathrm{d}x&\leq\|h_2\|_\infty\left(\int\limits_{D_\epsilon(R)}1^{\frac{2^*_\alpha}{2^*_\alpha-q_2}}\mathrm{d}x\right)^{\frac{2^*_\alpha-q_2}{2^*_\alpha}}\left(\int\limits_{D_\epsilon(R)}|v_n(x,0)|^{2^*_\alpha}\mathrm{d}x\right)^{\frac{q_2}{2^*_\alpha}}\\&\leq\|h_2\|_\infty\epsilon^{\frac{2^*_\alpha-q_2}{2^*_\alpha}}\left(\int\limits_{D_\epsilon(R)}|v_n(x,0)|^{2^*_\alpha}\mathrm{d}x\right)^{\frac{q_2}{2^*_\alpha}}.\end{aligned}\tag{3.20}$$

On the other hand,

$$\int\limits_{\mathbb{R}^N\setminus D_\epsilon(R)} h_2(x)|v_n(x,0)|^{q_2}\mathrm{d}x \le \|h_2\|_\infty \left(\frac{\omega_N R^N}{N}\right)^{\frac{2^*_\alpha-q_2}{2^*_\alpha}} \left(\int\limits_{\mathbb{R}^N}|v_n(x,0)|^{2^*_\alpha}\mathrm{d}x\right)^{\frac{q_2}{2^*_\alpha}} + \epsilon\int\limits_{\mathbb{R}^N}|v_n(x,0)|^{q_2}\mathrm{d}x. \tag{3.21}$$

Furthermore, considering $0 < r \le 1$ such that $q_2 = 2r + (1-r)2^*_\alpha$, we apply Hölder's inequality, condition (V), (3.16) and (3.17) to find $C_2 > 0$ such that

$$\begin{aligned}\int\limits_{\mathbb{R}^N}|v_n(x,0)|^{q_2}\mathrm{d}x &\le \left(\int\limits_{\mathbb{R}^N}|v_n(x,0)|^{2}\mathrm{d}x\right)^{r}\left(\int\limits_{\mathbb{R}^N}|v_n(x,0)|^{2^*_\alpha}\mathrm{d}x\right)^{1-r}\\ &\le \left(\frac{1}{a_0}\int\limits_{\mathbb{R}^N}V(x)|v_n(x,0)|^{2}\mathrm{d}x\right)^{r}\left(\int\limits_{\mathbb{R}^N}|v_n(x,0)|^{2^*_\alpha}\mathrm{d}x\right)^{1-r}\\ &\le C_2\left(\int\limits_{\mathbb{R}^N}|v_n(x,0)|^{2^*_\alpha}\mathrm{d}x\right)^{r}\left(\int\limits_{\mathbb{R}^N}|v_n(x,0)|^{2^*_\alpha}\mathrm{d}x\right)^{1-r}.\end{aligned}$$

Consequently,

$$\int\limits_{\mathbb{R}^N}|v_n(x,0)|^{q_2}\mathrm{d}x \le C_2\int\limits_{\mathbb{R}^N}|v_n(x,0)|^{2^*_\alpha}\mathrm{d}x. \tag{3.22}$$

Thus, from (3.20)-(3.22), considering $\epsilon > 0$ sufficiently small we obtain $C_3 > 0$ such that

$$\int\limits_{\mathbb{R}^N} h_2(x)|v_n(x,0)|^{q_2}\mathrm{d}x \le \epsilon C_2\int\limits_{\mathbb{R}^N}|v_n(x,0)|^{2^*_\alpha}\mathrm{d}x + C_3\left(\int\limits_{\mathbb{R}^N}|v_n(x,0)|^{2^*_\alpha}\mathrm{d}x\right)^{\frac{q_2}{2^*_\alpha}}. \tag{3.23}$$

Combining (3.19) and (3.23), we get

$$\left(\frac{\alpha}{N} - \epsilon C_2\right)\int\limits_{\mathbb{R}^N}|v_n(x,0)|^{2^*_\alpha}\mathrm{d}x \le C_3\left(\int\limits_{\mathbb{R}^N}|v_n(x,0)|^{2^*_\alpha}\mathrm{d}x\right)^{\frac{q_2}{2^*_\alpha}} + C.$$

Since $q_2 < 2^*_\alpha$, taking $\epsilon > 0$ sufficiently small such that $\dfrac{\alpha}{N} - \epsilon C_2 > 0$, we obtain the desired result. □

In order to show the next result, we recall the following Sobolev inequality proved in [6, Theorem 2.1],

$$\left(\int\limits_{\mathbb{R}^N}|w(x,0)|^{2^*_\alpha}\mathrm{d}x\right)^{\frac{2}{2^*_\alpha}} \le \frac{1}{S(\alpha,N)}\int\limits_{\mathbb{R}^{N+1}_+} y^{1-2\alpha}|\nabla w|^2\mathrm{d}x\mathrm{d}y, \text{ for all } w \in X^{2\alpha}(\mathbb{R}^{N+1}_+), \tag{3.24}$$

where

$$S(\alpha,N)=\frac{\Gamma(\alpha)\Gamma(\frac{N-2\alpha}{2})(\Gamma(N))^{2\alpha/N}}{(2\pi)^{\alpha}\Gamma(1-\alpha)\Gamma(\frac{N+2\alpha}{2})(\Gamma(\frac{N}{2}))^{2\alpha/N}}.$$

From (3.24),

$$\|w(x,0)\|_{2^*_\alpha}^2\le\frac{1}{\kappa_\alpha S(\alpha,N)}\|w\|^2,\ \text{ for all }\ w\in X. \tag{3.25}$$

Proposition 3.11. *Suppose that (V), (g_1) and (g_2) are satisfied. Let $(v_n)\subset X$ be a $(Ce)_c$ sequence with $0<c<\frac{\alpha}{N}(S(\alpha,N)\kappa_\alpha)^{N/2\alpha}$ and $v_n\rightharpoonup 0$ in X. Then there exist a sequence $(y_n)\subset\mathbb{R}^N$ and $R,\eta>0$ such that $|y_n|\to\infty$ and*

$$\limsup_{n\to\infty}\int_{B_R(y_n)}|v_n(x,0)|^2\,\mathrm{d}x\ge\eta>0. \tag{3.26}$$

Proof. Supposing that the result does not hold. Then, arguing similarly as in [48, Lemma 1.21], we can assume that

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}|v_n(x,0)|^{\sigma}\mathrm{d}x=0,\ \text{ for every }\ \sigma\in(2,2^*_\alpha).$$

Since g is subcritical and (v_n) is bounded in X, by using (3.11) and (3.12), we obtain

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}g(x,v_n(x,0))v_n(x,0)\mathrm{d}x=\lim_{n\to\infty}\int_{\mathbb{R}^N}G(x,v_n(x,0))\mathrm{d}x=0. \tag{3.27}$$

Moreover,

$$\begin{aligned}c+o_n(1)&=J(v_n)-\frac{1}{2}J'(v_n)v_n\\&=\frac{\alpha}{N}\int_{\mathbb{R}^N}|v_n(x,0)|^{2^*_\alpha}\mathrm{d}x+\int_{\mathbb{R}^N}\left[\frac{1}{2}g(x,v_n(x,0))v_n(x,0)-G(x,v_n(x,0))\right]\mathrm{d}x.\end{aligned}$$

Taking the limit in the above equality and using (3.27), we obtain

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}|v_n(x,0)|^{2^*_\alpha}\mathrm{d}x=\frac{cN}{\alpha}. \tag{3.28}$$

Combining

$$o_n(1)=J'(v_n)v_n=\|v_n\|^2-\int_{\mathbb{R}^N}|v_n(x,0)|^{2^*_\alpha}\mathrm{d}x-\int_{\mathbb{R}^N}g(x,v_n(x,0))v_n(x,0)\mathrm{d}x$$

with (3.27) and (3.28), we reach

$$\lim_{n\to\infty}\|v_n\|^2=\frac{cN}{\alpha}. \tag{3.29}$$

By (3.25), it follows that

$$\lim_{n\to\infty}\left(\int\limits_{\mathbb{R}^N}|v_n(x,0)|^{2^*_\alpha}\mathrm{d}x\right)^{\frac{2}{2^*_\alpha}}\leq\frac{1}{\kappa_\alpha S(\alpha,N)}\lim_{n\to\infty}\|v_n\|^2.$$

The last inequality, together with (3.28) and (3.29), implies

$$c\geq\frac{\alpha}{N}(\kappa_\alpha S(\alpha,N))^{N/2\alpha},$$

which is a contradiction and this completes the proof of Proposition 3.11. □

Estimate of the minimax level

In this section we will verify that the minimax level associated with the Mountain Pass Theorem is in the interval where Proposition 3.11 may be applied. To show this result, we use appropriate test functions as the ones employed by Brézis and Nirenberg [8].

Test functions

Theorem 2.1 of [6] states that $S(\alpha,N)$ is achieved on the family of functions $w_\varepsilon=E_{2\alpha}(u_\varepsilon)$, where

$$u_\varepsilon(x)=\frac{\varepsilon^{\frac{N-2\alpha}{2}}}{(|x|^2+\varepsilon^2)^{\frac{N-2\alpha}{2}}},\quad \varepsilon>0.$$

This family of functions will be crucial to estimate the minimax level. First, we define $\phi:\mathbb{R}^{N+1}_+\to\mathbb{R}$ by $\phi(x,y)=\phi_0(|(x,y)|)$, where $\phi_0\in C^\infty([0,+\infty))$ is a non-increasing cut-off such that $\phi_0(t)=1$ if $0\leq t\leq 1/2$ and $\phi_0(t)=0$ if $t\geq 1$. It is obvious that $\phi w_\varepsilon\in X^{2\alpha}(\mathbb{R}^{N+1}_+)$ and by Lemma 3.8 of [3] and Lemma 2.4 of [21] we have the following result:

Lemma 3.12. *The family $\{\phi w_\varepsilon\}$ and its trace on $\{y=0\}$, namely, ϕu_ε, satisfy*

$$\|\phi w_\varepsilon\|^2_{X^{2\alpha}}\leq\|w_\varepsilon\|^2_{X^{2\alpha}}+\mathcal{O}(\varepsilon^{N-2\alpha}),\tag{3.30}$$

$$\|\phi u_\varepsilon\|_2^2=\begin{cases}\mathcal{O}(\varepsilon^{2\alpha}), & \textit{if } N>4\alpha,\\ \mathcal{O}(\varepsilon^{2\alpha}\log(1/\varepsilon)), & \textit{if } N=4\alpha,\\ \mathcal{O}(\varepsilon^{N-2\alpha}), & \textit{if } N<4\alpha,\end{cases}\tag{3.31}$$

for $\varepsilon > 0$ sufficiently small. Define

$$\eta_\varepsilon = \frac{\phi w_\varepsilon}{\|\phi u_\varepsilon\|_{2^*_\alpha}},$$

then

$$\|\eta_\varepsilon\|^2_{X^{2\alpha}} \leq \kappa_\alpha S(\alpha, N) + \mathcal{O}(\varepsilon^{N-2\alpha}), \tag{3.32}$$

$$\|\eta_\varepsilon(x,0)\|_2^2 = \begin{cases} \mathcal{O}(\varepsilon^{2\alpha}), & \text{if } N > 4\alpha, \\ \mathcal{O}(\varepsilon^{2\alpha}\log(1/\varepsilon)), & \text{if } N = 4\alpha, \\ \mathcal{O}(\varepsilon^{N-2\alpha}), & \text{if } N < 4\alpha, \end{cases} \tag{3.33}$$

and

$$\|\eta_\varepsilon(x,0)\|_q^q = \begin{cases} \mathcal{O}(\varepsilon^{\frac{2N-(N-2\alpha)q}{2}}), & \text{if } q \geq \frac{N}{N-2\alpha}, \\ \mathcal{O}(\varepsilon^{\frac{(N-2\alpha)q}{2}}), & \text{if } q \leq \frac{N}{N-2\alpha}. \end{cases} \tag{3.34}$$

As a first consequence of Lemma 3.12, we obtain the following result.

Lemma 3.13. *Suppose that (V), (g_1), (g_2) and $(g_4)(i)$ are satisfied. Consider $t_\varepsilon > 0$ such that*

$$J(t_\varepsilon \eta_\varepsilon) = \max\{J(t\eta_\varepsilon) : t > 0\}.$$

Then, there exist $\varepsilon_0 > 0$ and positive constants T and $\overline{T}$ such that $T \leq t_\varepsilon \leq \overline{T}$ for every $0 < \varepsilon < \varepsilon_0$.

Proof. In view of Lemma 3.7, $(g_4)(i)$ and the definition η_ε, there exists a positive constant β such that

$$\beta \leq J(t_\varepsilon \eta_\varepsilon) \leq \frac{t_\varepsilon^2}{2}\|\eta_\varepsilon\|^2 - \frac{t_\varepsilon^{2^*_\alpha}}{2^*_\alpha}\int_{\mathbb{R}^N} \eta_\varepsilon^{2^*_\alpha}(x,0)\mathrm{d}x = \frac{t_\varepsilon^2}{2}\|\eta_\varepsilon\|^2 - \frac{t_\varepsilon^{2^*_\alpha}}{2^*_\alpha},$$

which implies that

$$\beta \leq \frac{t_\varepsilon^2}{2}\|\eta_\varepsilon\|^2 \quad \text{and} \quad \frac{t_\varepsilon^{2^*_\alpha}}{2^*_\alpha} \leq \frac{t_\varepsilon^2}{2}\|\eta_\varepsilon\|^2 - \beta. \tag{3.35}$$

From (3.32) and (3.33), we obtain a positive constant C such that $\|\eta_\varepsilon\| \leq C$ for $\varepsilon > 0$ sufficiently small. By using (3.35), there exist positive constants T and $\overline{T}$ such that $T \leq t_\varepsilon \leq \overline{T}$. This completes the proof. □

Now we are ready to prove the main result of this section.

Proposition 3.14. *Suppose that (V), (g_1), (g_2), (g_4) are satisfied. Furthermore, we assume one of the following conditions*

$(g_5)(i)$ *if $N \geq 4\alpha$.*

$(g_5)(ii)$ *if $2\alpha < N < 4\alpha$ and $\dfrac{4\alpha}{N-2\alpha} < p < 2^*_\alpha$.*

$(g_5)(iii)$ *if* $2\alpha < N < 4\alpha$ *and* $2 < p < \dfrac{4\alpha}{N-2\alpha}$.

Then there exists $v \in X \setminus \{0\}$ *such that*

$$\max_{t\geq 0} J(tv) < \frac{\alpha}{N}(S(\alpha,N)\kappa_\alpha)^{N/2\alpha}. \tag{3.36}$$

Proof. Consider t_ε as defined by Lemma 3.13, it is clear that

$$J(t_\varepsilon\eta_\varepsilon) = \frac{t_\varepsilon^2}{2}\|\eta_\varepsilon\|^2 - \frac{t_\varepsilon^{2^*_\alpha}}{2^*_\alpha} - \int_{\mathbb{R}^N} G(x, t_\varepsilon\eta_\varepsilon(x,0))\mathrm{d}x.$$

Considering the function $h_\varepsilon : [0,\infty) \to \mathbb{R}$ given by $h_\varepsilon(t) = \frac{1}{2}\|\eta_\varepsilon\|^2 t^2 - \frac{1}{2^*_\alpha}t^{2^*_\alpha}$, we have that $t_{\varepsilon,0} = \|\eta_\varepsilon\|^{2/(2^*_\alpha-2)}$ is a maximum point of h_ε and $h_\varepsilon(t_{\varepsilon,0}) = \frac{\alpha}{N}\|\eta_\varepsilon\|^{N/\alpha}$. This, together with (3.32), implies

$$\begin{aligned} J(t_\varepsilon\eta_\varepsilon) &\leq \frac{\alpha}{N}(\|\eta_\varepsilon\|^2)^{N/2\alpha} - \int_{\mathbb{R}^N} G(x, t_\varepsilon\eta_\varepsilon(x,0))\mathrm{d}x \\ &\leq \frac{\alpha}{N}\left(\kappa_\alpha S(\alpha,N) + \mathcal{O}(\varepsilon^{N-2\alpha}) + \|V\|_\infty\|\eta_\varepsilon\|_2^2\right)^{N/2\alpha} - \int_{\mathbb{R}^N} G(x, t_\varepsilon\eta_\varepsilon(x,0))\mathrm{d}x. \end{aligned}$$

Consequently, by using (3.33) we reach

$$J(t_\varepsilon\eta_\varepsilon) \leq \begin{cases} \dfrac{\alpha}{N}\left(\kappa_\alpha S(\alpha,N)\right)^{N/2\alpha} + \mathcal{O}\left(\varepsilon^{2\alpha}\right) - \int\limits_{\mathbb{R}^N} G(x, t_\varepsilon\eta_\varepsilon(x,0))\mathrm{d}x, & \text{if } N > 4\alpha, \\ \dfrac{\alpha}{N}\left(\kappa_\alpha S(\alpha,N)\right)^{N/2\alpha} + \mathcal{O}\left(\varepsilon^{2\alpha}\log(1/\varepsilon)\right) - \int\limits_{\mathbb{R}^N} G(x, t_\varepsilon\eta_\varepsilon(x,0))\mathrm{d}x, & \text{if } N = 4\alpha, \\ \dfrac{\alpha}{N}\left(\kappa_\alpha S(\alpha,N)\right)^{N/2\alpha} + \mathcal{O}(\varepsilon^{N-2\alpha}) - \int\limits_{\mathbb{R}^N} G(x, t_\varepsilon\eta_\varepsilon(x,0))\mathrm{d}x, & \text{if } N < 4\alpha. \end{cases} \tag{3.37}$$

Indeed, if $N > 4\alpha$,

$$\begin{aligned} J(t_\varepsilon\eta_\varepsilon) &\leq \frac{\alpha}{N}\left(\kappa_\alpha S(\alpha,N) + \mathcal{O}(\varepsilon^{N-2\alpha}) + \mathcal{O}(\varepsilon^{2\alpha})\right)^{N/2\alpha} - \int_{\mathbb{R}^N} G(x, t_\varepsilon\eta_\varepsilon(x,0))\mathrm{d}x \\ &= \frac{\alpha}{N}\left(\kappa_\alpha S(\alpha,N) + \mathcal{O}(\varepsilon^{2\alpha})\right)^{N/2\alpha} - \int_{\mathbb{R}^N} G(x, t_\varepsilon\eta_\varepsilon(x,0))\mathrm{d}x. \end{aligned}$$

Applying the inequality

$$(b+c)^\sigma \leq b^\sigma + \sigma(b+c)^{\sigma-1}c, \ \text{ with } b,c \geq 0, \sigma \geq 1, \tag{3.38}$$

we get

$$J(t_\varepsilon\eta_\varepsilon) \leq \frac{\alpha}{N}\left(\kappa_\alpha S(\alpha,N)\right)^{N/2\alpha} + \mathcal{O}\left(\varepsilon^{2\alpha}\right) - \int_{\mathbb{R}^N} G(x, t_\varepsilon\eta_\varepsilon(x,0))\mathrm{d}x.$$

This proves the first estimate of (3.37). For $N = 4\alpha$, note that

$$\begin{aligned} J(t_\varepsilon\eta_\varepsilon) &\leq \frac{\alpha}{N}\left(\kappa_\alpha S(\alpha,N) + \mathcal{O}(\varepsilon^{N-2\alpha}) + \mathcal{O}\left(\varepsilon^{2\alpha}\log(1/\varepsilon)\right)\right)^{N/2\alpha} - \int_{\mathbb{R}^N} G(x, t_\varepsilon\eta_\varepsilon(x,0))\mathrm{d}x \\ &= \frac{\alpha}{N}\left(\kappa_\alpha S(\alpha,N) + \mathcal{O}\left(\varepsilon^{2\alpha}\log(1/\varepsilon)\right)\right)^{N/2\alpha} - \int_{\mathbb{R}^N} G(x, t_\varepsilon\eta_\varepsilon(x,0))\mathrm{d}x. \end{aligned}$$

Again by using (3.38), we get

$$J(t_\varepsilon\eta_\varepsilon) \leq \frac{\alpha}{N}\left(\kappa_\alpha S(\alpha,N)\right)^{N/2\alpha} + \mathcal{O}\left(\varepsilon^{2\alpha}\log(1/\varepsilon)\right) - \int\limits_{\mathbb{R}^N} G(x, t_\varepsilon\eta_\varepsilon(x,0))\mathrm{d}x.$$

This proves the second estimate of (3.37). For $N < 4\alpha$, combining (3.33) and (3.38), it is immediate that

$$J(t_\varepsilon\eta_\varepsilon) \leq \frac{\alpha}{N}\left(\kappa_\alpha S(\alpha,N)\right)^{N/2\alpha} + \mathcal{O}(\varepsilon^{N-2\alpha}) - \int\limits_{\mathbb{R}^N} G(x, t_\varepsilon\eta_\varepsilon(x,0))\mathrm{d}x.$$

This completes the proof of (3.37).

Now consider

$$\gamma(\varepsilon) = \begin{cases} \varepsilon^{2\alpha}, & \text{if } N > 4\alpha, \\ \varepsilon^{2\alpha}\log(1/\varepsilon), & \text{if } N = 4\alpha, \\ \varepsilon^{N-2\alpha}, & \text{if } N < 4\alpha. \end{cases}$$

By using (3.37) we find a positive constant Θ such that

$$J(t_\varepsilon\eta_\varepsilon) \leq \frac{\alpha}{N}\left(\kappa_\alpha S(\alpha,N)\right)^{N/2\alpha} + \gamma(\varepsilon)\left[\Theta - \frac{1}{\gamma(\varepsilon)}\int\limits_{\mathbb{R}^N} G(x, t_\varepsilon\eta_\varepsilon(x,0))\mathrm{d}x\right]. \tag{3.39}$$

In order to prove Proposition 3.14, we just need to verify that

$$\lim_{\varepsilon\to 0^+}\frac{1}{\gamma(\varepsilon)}\int\limits_{\mathbb{R}^N} G(x, t_\varepsilon\eta_\varepsilon(x,0))\mathrm{d}x > \Theta. \tag{3.40}$$

Initially, observe that without loss of generality we may assume that $B_1 \subset \Omega$. On the other hand, by (g_5), given $A > 0$ there exists $R_A > 0$ such that

$$G(x,s) \geq As^p, \text{ for all } (x,s) \in \Omega \times [R_A, +\infty). \tag{3.41}$$

Moreover, by using Lemma 3.13 it is easy to see that there exists positive constant ν such that

$$t_\varepsilon\eta_\varepsilon(x,0) \geq t_\varepsilon C_1 u_\varepsilon(x) \geq TC_1(2\varepsilon)^{-(N-2\alpha)/2} \geq \nu\varepsilon^{-(N-2\alpha)/2}, \text{ for } |x| < \varepsilon, \tag{3.42}$$

with $\varepsilon > 0$ sufficiently small. It is clear that we may choose $\varepsilon_1 > 0$ such that

$$\nu\varepsilon^{-(N-2\alpha)/2} \geq R_A, \text{ for } 0 < \varepsilon < \varepsilon_1. \tag{3.43}$$

Combining (3.41), (3.42) and (3.43), we have

$$G(x, t_\varepsilon\eta_\varepsilon(x,0)) \geq A\nu\varepsilon^{-(N-2\alpha)p/2}, \text{ for } |x| < \varepsilon.$$

Furthermore, by (g_4)

$$\int\limits_{\mathbb{R}^N} G(x, t_\varepsilon\eta_\varepsilon(x,0))\mathrm{d}x \;\geq\; \int\limits_{B_\varepsilon(0)} A\nu\varepsilon^{-(N-2\alpha)p/2}\mathrm{d}x - \int\limits_{\Omega\setminus B_\varepsilon(0)} (t_\varepsilon\eta_\varepsilon(x,0))^2\mathrm{d}x,$$

which, together with Lemma 3.13, gives

$$\int_{\mathbb{R}^N} G(x, t_\varepsilon \eta_\varepsilon(x,0))\mathrm{d}x \geq A\nu\omega_N \varepsilon^{-(N-2\alpha)p/2+N} - \overline{T}^2 \|\eta_\varepsilon\|_2^2. \tag{3.44}$$

Now, in order to verify (3.40), we should consider the following cases:

Case 1: $N > 4\alpha$.

From (3.44) and (3.33), we obtain

$$\frac{1}{\gamma(\varepsilon)} \int_{\mathbb{R}^N} G(x, t_\varepsilon \eta_\varepsilon(x,0))\mathrm{d}x \geq A\nu\omega_N \varepsilon^{-(N-2\alpha)p/2+(N-2\alpha)} - \mathcal{O}(1).$$

Since $-(N-2\alpha)p/2 + (N-2\alpha) < 0$, we have that for $\varepsilon > 0$ sufficiently small (3.40) is satisfied.

Case 2: $N = 4\alpha$.

From (3.44) and (3.33), we have

$$\frac{1}{\gamma(\varepsilon)} \int_{\mathbb{R}^N} G(x, t_\varepsilon \eta_\varepsilon(x,0))\mathrm{d}x \geq A\nu\omega_N \frac{\varepsilon^{-(N-2\alpha)p/2+N}}{\varepsilon^{2\alpha}\log(1/\varepsilon)} - \mathcal{O}(1).$$

But,

$$\lim_{\varepsilon \to 0^+} \frac{\varepsilon^{-(N-2\alpha)p/2+N}}{\varepsilon^{2\alpha}\log(1/\varepsilon)} = +\infty.$$

Consequently, we get that (3.40) holds, for $\varepsilon > 0$ sufficiently small.

Case 3: $2\alpha < N < 4\alpha$ and $\frac{4\alpha}{N-2\alpha} < p < 2^*_\alpha$.

Combining (3.44) and (3.33), we obtain

$$\frac{1}{\gamma(\varepsilon)} \int_{\mathbb{R}^N} G(x, t_\varepsilon \eta_\varepsilon)\mathrm{d}x \geq A\nu\omega_N \varepsilon^{-(N-2\alpha)p/2+2\alpha} - \mathcal{O}(1).$$

Since $-(N-2\alpha)p/2 + 2\alpha < 0$, we get that (3.40) holds, for $\varepsilon > 0$ sufficiently small.

Case 4: $2\alpha < N < 4\alpha$ and $2 < p < \dfrac{4\alpha}{N-2\alpha}$.

By $(g_5)(iii)$,

$$G(x,s) \geq C_0 s^p, \text{ for all } (x,s) \in \mathbb{R}^N \times [0, +\infty), \tag{3.45}$$

where $C_0 \simeq \varepsilon^\tau$ with τ to be chosen later. Here, we need to consider two cases:

If $2 < p \leq N/(N-2\alpha)$. By applying Lemma 3.13, (3.45) and (3.34), we get

$$\frac{1}{\gamma(\varepsilon)} \int\limits_{\mathbb{R}^N} G(x, t_\varepsilon \eta_\varepsilon) \mathrm{d}x \geq \mathcal{O}(\varepsilon^{(N-2\alpha)p/2-(N-2\alpha)+\tau}),$$

which implies that (3.40) holds, since $\tau < (N-2\alpha)(2-p)/2$.

If $N/(N-2\alpha) \leq p < 4\alpha/(N-2\alpha)$. By applying Lemma 3.13, (3.45) and (3.34), we get

$$\frac{1}{\gamma(\varepsilon)} \int\limits_{\mathbb{R}^N} G(x, t_\varepsilon \eta_\varepsilon) \mathrm{d}x \geq \mathcal{O}(\varepsilon^{N-(N-2\alpha)p/2-(N-2\alpha)+\tau}),$$

which implies that (3.40) holds, since $\tau < (N-2\alpha)p/2 - 2\alpha$. This concludes the proof. □

Convergence results

In this section, we prove two lemmas which are required in later sections.

Lemma 3.15. *Suppose that (V) and (g_4) are satisfied. Let $(v_n) \subset X$ be a bounded sequence in X and $w_n(x,0) = w(x-y_n, 0)$, where $w \in X$ and $(y_n) \subset \mathbb{R}^N$. If $|y_n| \to \infty$, then*

$$[V_0(x) - V(x)] v_n(x,0) w_n(x,0) \to 0,$$

$$[g_0(x, v_n(x,0)) - g(x, v_n(x,0))] v_n(x,0) w_n(x,0) \to 0,$$

in $L^1(\mathbb{R}^N)$, as $n \to \infty$.

Proof. Given $\delta > 0$, since $w(x,0) \in L^q(\mathbb{R}^N)$ for all $2 \leq q \leq 2^*_\alpha$, we find $0 < \varepsilon < \delta$ such that, for each measurable set $A \subset \mathbb{R}^N$ satisfying $|A| < \varepsilon$,

$$\int\limits_A |w(x,0)|^2 \mathrm{d}x < \delta \quad \text{and} \quad \int\limits_A |w(x,0)|^{\frac{2^*_\alpha}{2^*_\alpha - q_3}} \mathrm{d}x < \delta. \tag{3.46}$$

The condition (V) implies $V_0 - V \in \mathcal{F}$ and by Lemma 3.9 there exists $R_1 > 0$ such that $|D_\varepsilon(R_1)| < \varepsilon$, where $D_\varepsilon(R_1) = \{x \in \mathbb{R}^N : |V_0(x) - V(x)| \geq \varepsilon \text{ and } |x| > R_1\}$.

Thus, applying Hölder's inequality and (V), we obtain

$$\begin{aligned}
\int_{\mathbb{R}^N\setminus B_{R_1}} & |V_0(x)-V(x)||v_n(x,0)||w_n(x,0)|\mathrm{d}x \\
&\le \|V_0\|_\infty \int_{D_\varepsilon(R_1)} |v_n(x,0)||w_n(x,0)|\mathrm{d}x + \varepsilon \int_{F_\varepsilon} |v_n(x,0)||w_n(x,0)|\mathrm{d}x \\
&\le \|V_0\|_\infty \left(\int_{D_\varepsilon(R_1)} |v_n(x,0)|^2\mathrm{d}x \right)^{\frac{1}{2}} \left(\int_{D_\varepsilon(R_1)} |w_n(x,0)|^2\mathrm{d}x \right)^{\frac{1}{2}} \\
&\quad + \varepsilon \left(\int_{F_\varepsilon} |v_n(x,0)|^2\mathrm{d}x \right)^{\frac{1}{2}} \left(\int_{F_\varepsilon} |w_n(x,0)|^2\mathrm{d}x \right)^{\frac{1}{2}},
\end{aligned}$$

where $F_\varepsilon := \mathbb{R}^N \setminus (B_{R_1} \cup D_\varepsilon(R_1))$. Consequently,

$$\begin{aligned}
\int_{\mathbb{R}^N\setminus B_{R_1}} & |V_0(x)-V(x)||v_n(x,0)||w_n(x,0)|\mathrm{d}x \\
&\le \|V_0\|_\infty \|v_n(x,0)\|_2 \|w_n(x,0)\|_{L^2(D_\varepsilon(R_1))} + \delta \|v_n(x,0)\|_2 \|w(x,0)\|_2.
\end{aligned}$$

Using (3.46) and the fact that (v_n) is bounded in X, we obtain a constant $C_1 > 0$ such that

$$\int_{\mathbb{R}^N\setminus B_{R_1}} |V_0(x)-V(x)||v_n(x,0)||w_n(x,0)|\mathrm{d}x \le C_1(\delta^{1/2}+\delta). \tag{3.47}$$

On the other hand, by Hölder's inequality, (V) and the boundedness of (v_n) in X, we find $C_2 > 0$ such that

$$\begin{aligned}
\int_{B_{R_1}} & |V_0(x)-V(x)||v_n(x,0)||w_n(x,0)|\mathrm{d}x \\
&\le \|V_0\|_\infty \left(\int_{B_{R_1}} |v_n(x,0)|^2\mathrm{d}x \right)^{\frac{1}{2}} \left(\int_{B_{R_1}} |w_n(x,0)|^2\mathrm{d}x \right)^{\frac{1}{2}} \\
&\le \|V_0\|_\infty \|v_n(x,0)\|_2 \left(\int_{B_{R_1}(-y_n)} |w(x,0)|^2\mathrm{d}x \right)^{\frac{1}{2}} \\
&\le C_2 \left(\int_{B_{R_1}(-y_n)} |w(x,0)|^2\mathrm{d}x \right)^{\frac{1}{2}}.
\end{aligned} \tag{3.48}$$

Then, since $w(x,0) \in L^2(\mathbb{R}^N)$ and $|y_n| \to \infty$, there exists $n_0 \in \mathbb{N}$ such that

$$\int_{B_{R_1}} |V_0(x)-V(x)||v_n(x,0)||w_n(x,0)|\mathrm{d}x \le C_2\delta, \ \text{ for all } \ n \ge n_0. \tag{3.49}$$

The inequalities (3.47), (3.49) and the fact that $\delta > 0$ can be chosen arbitrarily small imply that

$$[V_0(x) - V(x)]v_n(x,0)w_n(x,0) \to 0 \text{ in } L^1(\mathbb{R}^N), \text{ as } n \to \infty.$$

This proves the first convergence of the lemma. In order to verify the second limit, given $R_2 > 0$, we define $A_n = (\mathbb{R}^N \setminus B_{R_2}) \cap \{x \in \mathbb{R}^N : |v_n(x)| \leq 1\}$, $B_n = \mathbb{R}^N \setminus (A_n \cup B_{R_2})$, $D_\varepsilon(R_2) = \{x \in \mathbb{R}^N : |h_3(x)| \geq \varepsilon \text{ and } |x| \geq R_2\}$ and we split

$$\int_{\mathbb{R}^N} |g_0(x, v_n(x,0)) - g(x, v_n(x,0))||v_n(x,0)||w_n(x,0)|\mathrm{d}x = I_{1,n} + I_{2,n} + I_{3,n},$$

where

$$\begin{aligned} I_{1,n} &:= \int_{A_n} |g_0(x, v_n(x,0)) - g(x, v_n(x,0))||v_n(x,0)||w_n(x,0)|\mathrm{d}x, \\ I_{2,n} &:= \int_{B_n} |g_0(x, v_n(x,0)) - g(x, v_n(x,0))||v_n(x,0)||w_n(x,0)|\mathrm{d}x, \\ I_{3,n} &:= \int_{B_{R_2}} |g_0(x, v_n(x,0)) - g(x, v_n(x,0))||v_n(x,0)||w_n(x,0)|\mathrm{d}x. \end{aligned}$$

We will estimate each one of these integrals. For $I_{1,n}$, note that by the condition $(g_4)(ii)$,

$$I_{1,n} \leq \int_{A_n} |h_3(x)||v_n(x,0)|^{q_3}|w_n(x,0)|\mathrm{d}x =: \overline{I}_{1,n}.$$

By Lemma 3.9 and the fact that $h_3 \in \mathcal{F}$, we can find $R_2 > 0$ such that $|D_\varepsilon(R_2)| < \varepsilon$. Then, by Hölder's inequality

$$\begin{aligned} \overline{I}_{1,n} \leq \|h_3\|_\infty &\left(\int_{A_n \cap D_\varepsilon(R_2)} |v_n(x,0)|^{2^*_\alpha}\mathrm{d}x \right)^{\frac{q_3}{2^*_\alpha}} \left(\int_{A_n \cap D_\varepsilon(R_2)} |w_n(x,0)|^{\frac{2^*_\alpha}{2^*_\alpha - q_3}}\mathrm{d}x \right)^{\frac{2^*_\alpha - q_3}{2^*_\alpha}} \\ + \varepsilon &\left(\int_{A_n \setminus D_\varepsilon(R_2)} |v_n(x,0)|^{2^*_\alpha}\mathrm{d}x \right)^{\frac{q_3}{2^*_\alpha}} \left(\int_{A_n \setminus D_\varepsilon(R_2)} |w_n(x,0)|^{\frac{2^*_\alpha}{2^*_\alpha - q_3}}\mathrm{d}x \right)^{\frac{2^*_\alpha - q_3}{2^*_\alpha}}. \end{aligned}$$

Consequently,

$$I_{1,n} \leq \|h_3\|_\infty \|v_n(x,0)\|_{2^*_\alpha}^{q_3} \|w_n(x,0)\|_{L^{\frac{2^*_\alpha}{(2^*_\alpha - q_3)}}(D_\varepsilon(R_2))} + \delta \|v_n(x,0)\|_{2^*_\alpha}^{q_3} \|w(x,0)\|_{\frac{2^*_\alpha}{2^*_\alpha - q_3}}.$$

Again using (3.46) and the fact that (v_n) is bounded in X, we obtain a constant $C_3 > 0$ such that

$$I_{1,n} \leq C_3(\delta^{(2^*_\alpha - q_3)/2^*_\alpha} + \delta). \tag{3.50}$$

On the other hand, again using the condition $(g_4)(ii)$ and Hölder's inequality we have

$$
\begin{aligned}
I_{2,n} \leq \quad & \|h_3\|_\infty \left(\int\limits_{D_\varepsilon(R_2)} |v_n(x,0)|^{2_\alpha^*} \mathrm{d}x \right)^{\frac{q_3}{2_\alpha^*}} \left(\int\limits_{D_\varepsilon(R_2)} |w_n(x,0)|^{\frac{2_\alpha^*}{2_\alpha^*-q_3}} \mathrm{d}x \right)^{\frac{2_\alpha^*-q_3}{2_\alpha^*}} \\
& + \varepsilon \left(\int\limits_{B_n \setminus D_\varepsilon(R_2)} |v_n(x,0)|^{2_\alpha^*} \mathrm{d}x \right)^{\frac{q_3}{2_\alpha^*}} \left(\int\limits_{B_n \setminus D_\varepsilon(R_2)} |w_n(x,0)|^{\frac{2_\alpha^*}{2_\alpha^*-q_3}} \mathrm{d}x \right)^{\frac{2_\alpha^*-q_3}{2_\alpha^*}} .
\end{aligned}
$$

Thus, by the boundedness of (v_n) in X and (3.46), there exists $C_4 > 0$ such that

$$
I_{2,n} \leq C_4 \delta^{\frac{2_\alpha^*-q_3}{2_\alpha^*}}. \tag{3.51}
$$

Similarly to (3.48), we obtain that

$$
I_{3,n} \leq C_5 \left(\int\limits_{B_{R_2}(-y_n)} |w(x,0)|^{\frac{2_\alpha^*}{2_\alpha^*-q_3}} \mathrm{d}x \right)^{\frac{2_\alpha^*-q_3}{2_\alpha^*}},
$$

for some constant $C_5 > 0$. Therefore, since $w(x,0) \in L^{2_\alpha^*/(2_\alpha^*-q_3)}(\mathbb{R}^N)$ and $|y_n| \to +\infty$, it follows that there exists $n_0 \in \mathbb{N}$ such that

$$
I_{3,n} \leq C_5 \delta, \ \text{ for all } \ n \geq n_0. \tag{3.52}
$$

Finally, the inequalities $(3.50)-(3.52)$ and the fact that $\delta > 0$ can be chosen arbitrarily small imply that

$$
[g_0(x, v_n(x,0)) - g(x, v_n(x,0))] v_n(x,0) w_n(x,0) \to 0 \ \text{ in } \ L^1(\mathbb{R}^N), \ \text{ as } \ n \to \infty.
$$

This concludes the proof. □

Lemma 3.16. *Suppose that $2 \leq q < 2_\alpha^*$ and $h \in \mathcal{F}$. Let $(v_n) \subset X$ be a sequence such that $v_n \rightharpoonup v$ in X. Then*

$$
h(x)|v_n(x,0)|^q \to h(x)|v(x,0)|^q \ \text{ in } \ L^1(\mathbb{R}^N), \ \text{ as } \ n \to \infty.
$$

Proof. Arguing by contradiction, we suppose that there is a subsequence, still denoted by (v_n) and $\varepsilon > 0$ such that

$$
\int\limits_{\mathbb{R}^N} |h(x)| \, \big| |v_n(x,0)|^q - |v(x,0)|^q \big| \mathrm{d}x \geq \varepsilon \ \text{ for all } \ n \in \mathbb{N}. \tag{3.53}
$$

Now we define $D_\delta(R) = \{x \in \mathbb{R}^N : |h(x)| \geq \delta \ \text{and} \ |x| > R\}$. Since $h \in \mathcal{F}$, by Lemma 3.9 there exists $R = R_\delta > 0$ such that $|D_\delta(R)| < \delta$. Consequently, since the sequence (v_n) is bounded in X, by using Hölder's inequality and (3.8), we get

$$
\int\limits_{D_\delta(R)} |v_n(x,0)|^q \mathrm{d}x \leq \left(\int\limits_{D_\delta(R)} 1^{\frac{2_\alpha^*}{2_\alpha^*-q}} \mathrm{d}x \right)^{\frac{2_\alpha^*-q}{2_\alpha^*}} \left(\int\limits_{D_\delta(R)} |v_n(x,0)|^{2_\alpha^*} \mathrm{d}x \right)^{\frac{q}{2_\alpha^*}} \leq C \delta^{(2_\alpha^*-q)/2_\alpha^*},
$$

for some constant $C > 0$, which implies that

$$\int_{D_\delta(R)} |h(x)|\,||v_n(x,0)|^q - |v(x,0)|^q|\mathrm{d}x \leq \|h\|_\infty \int_{D_\delta(R)} (|v_n(x,0)|^q + |v(x,0)|^q)\mathrm{d}x \leq C_1\delta^{(2^*_\alpha - q)/2^*_\alpha}, \quad (3.54)$$

for some constant $C_1 > 0$. Still by the definition of $D_\delta(R)$ and boundedness of (v_n) in X, there exists $C_2 > 0$ such that

$$\int_{\mathbb{R}^N\setminus(B_R\cup D_\delta(R))} |h(x)|\,||v_n(x,0)|^q - |v(x,0)|^q|\mathrm{d}x \leq \delta \int_{\mathbb{R}^N} (|v_n(x,0)|^q + |v(x,0)|^q)\mathrm{d}x \leq C_2\delta. \quad (3.55)$$

Using that $v_n \rightharpoonup v$ in X, (3.4) and (V), up to a subsequence, we have $v_n(x,0) \to v(x,0)$ in $L^q_{loc}(\mathbb{R}^N)$ for all $1 \leq q < 2^*_\alpha$. Consequently, up to a subsequence, we obtain that $v_n(x,0) \to v(x,0)$ almost everywhere in B_R and also that there exists $w_q \in L^q(B_R)$ such that $|v_n(x,0)| \leq w_q$. Thus, it follows by the Dominated Convergence Theorem that

$$\int_{B_R} ||v_n(x,0)|^q - |v(x,0)|^q|\mathrm{d}x \to 0, \text{ as } n \to \infty.$$

Since $h \in L^\infty(\mathbb{R}^N)$ there exists $n_0 \in \mathbb{N}$ such that

$$\int_{B_R} |h(x)|||v_n(x,0)|^q - |v(x,0)|^q|\mathrm{d}x < \delta, \text{ for all } n \geq n_0. \quad (3.56)$$

Combining (3.54) – (3.56), for n sufficiently large, we get

$$\int_{\mathbb{R}^N} |h(x)|||v_n(x,0)|^q - |v(x,0)|^q|\mathrm{d}x \leq C_1\delta^{(2^*_\alpha - q)/2^*_\alpha} + C_2\delta + \delta.$$

But δ can be chosen arbitrarily small and so the inequality above contradicts (3.53), which completes the proof. □

Proof of Theorem 3.1

With the aid of the results of the previous sections the goal of this section is to find a nontrivial critical point of J. First, recall that by Corollary 3.8 there exists $(v_n) \subset X$ such that

$$J(v_n) \to c_M \quad \text{and} \quad \|J'(v_n)\|_*(1 + \|v_n\|) \to 0, \quad \text{as} \quad n \to \infty. \quad (3.57)$$

By Lemma 3.10 we may assume that $v_n \rightharpoonup v$ in X. Note that v is a critical point of J. Indeed, since $v_n \rightharpoonup v$ in X, from (3.4) and (V), up to a subsequence, we have that

$v_n(x,0) \to v(x,0)$ in $L^q_{loc}(\mathbb{R}^N)$ for all $1 \le q < 2^*_\alpha$. Consequently, up to a subsequence, combining the Dominated Convergence Theorem and (3.11), we obtain

$$\int_{\mathbb{R}^N} g(x, v_n(x,0))\varphi(x,0)\mathrm{d}x \to \int_{\mathbb{R}^N} g(x, v(x,0))\varphi(x,0)\mathrm{d}x, \text{ as } n \to +\infty$$

and

$$\int_{\mathbb{R}^N} |v_n(x,0)|^{2^*_\alpha - 1}\varphi(x,0)\mathrm{d}x \to \int_{\mathbb{R}^N} |v(x,0)|^{2^*_\alpha - 1}\varphi(x,0)\mathrm{d}x, \text{ as } n \to +\infty,$$

for each $\varphi \in C_0^\infty(\overline{\mathbb{R}^{N+1}_+})$. These convergences, together with the fact that $v_n \rightharpoonup v$ in X and (3.57), imply

$$J'(v)\varphi = \lim_{n\to+\infty} J'(v_n)\varphi = 0, \text{ for all } \varphi \in C_0^\infty(\overline{\mathbb{R}^{N+1}_+}).$$

By density this holds for every $\varphi \in X$. This concludes the assertion.

If $v \not\equiv 0$, Theorem 3.1 is proved. Assuming that $v \equiv 0$, the existence of a nontrivial point critical of J is more delicate and it will involve several steps. Initially, when $v \equiv 0$ in view of Proposition 3.14 we have $c_M < \frac{\alpha}{N}(\kappa_\alpha S(\alpha, N))^{N/2\alpha}$. Furthermore, by Proposition 3.11 there are a sequence $(y_n) \subset \mathbb{R}^N$ and $R, \eta > 0$ such that $|y_n| \to \infty$, as $n \to \infty$, and

$$\limsup_{n\to\infty} \int_{B_R(y_n)} |v_n(x,0)|^2 \mathrm{d}x \ge \eta > 0. \tag{3.58}$$

Without loss of generality we may assume that $(y_n) \subset \mathbb{Z}^N$. Then, defining $u_n(x,y) = v_n(x+y_n, y)$, by the periodicity of V_0 we have that $\|u_n(\cdot,\cdot)\|_0 = \|v_n(\cdot + y_n, \cdot)\|_0$. Since (v_n) is also bounded in X_0, it follows that (u_n) is bounded in X_0. Thus, passing to a subsequence if necessary, there is $u \in X_0$ such that $u_n \rightharpoonup u$ in X_0.

Step 1. u is nonzero.

Indeed, by (3.58), up to a subsequence,

$$\eta^{1/2} \le \|u_n(\cdot,0)\|_{L^2(B_R)} \le \|(u_n - u)(\cdot,0)\|_{L^2(B_R)} + \|u(\cdot,0)\|_{L^2(B_R)}.$$

Thus, from the Rellich-Kondrachov Embedding Theorem, we conclude that u is nonzero.

Step 2. $J_0'(u_n)\varphi \to J_0'(u)\varphi$, as $n \to \infty$, for all $\varphi \in C_0^\infty(\overline{\mathbb{R}_+^{N+1}})$.

By the Dominated Convergence Theorem, it is easy to see that for $\varphi \in C_0^\infty(\overline{\mathbb{R}_+^{N+1}})$, up to a subsequence,

$$\int_{\mathbb{R}^N} |u_n(x,0)|^{2_\alpha^*-2}u_n(x,0)\varphi(x,0)\mathrm{d}x \to \int_{\mathbb{R}^N} |u(x,0)|^{2_\alpha^*-2}u(x,0)\varphi(x,0)\mathrm{d}x, \text{ as } n \to +\infty. \tag{3.59}$$

We also claim that

$$\int_{\mathbb{R}^N} g_0(x,u_n(x,0))\varphi(x,0)\mathrm{d}x \to \int_{\mathbb{R}^N} g_0(x,u(x,0))\varphi(x,0)\mathrm{d}x, \text{ as } n \to +\infty. \tag{3.60}$$

In order to prove (3.60), observe that

$$\begin{aligned}[g_0(x,u(x,0)) - g_0(x,u_n(x,0))]\,\varphi(x,0) = {} & [g_0(x,u(x,0)) - g(x,u_n(x,0))]\varphi(x,0) \\ & + [g(x,u_n(x,0)) - g_0(x,u_n(x,0))]\varphi(x,0).\end{aligned} \tag{3.61}$$

Now, using that $u_n \rightharpoonup u$ in X_0, (3.4) and (V), up to a subsequence, we have $u_n(x,0) \to u(x,0)$ in $L^q_{loc}(\mathbb{R}^N)$ for all $1 \leq q < 2_\alpha^*$. Consequently, from (g_1), (g_2) and the Dominated Convergence Theorem, up to a subsequence,

$$\int_{\mathbb{R}^N} g(x,u_n(x,0))\varphi(x,0)\mathrm{d}x \to \int_{\mathbb{R}^N} g(x,u(x,0))\varphi(x,0)\mathrm{d}x, \text{ as } n \to +\infty, \tag{3.62}$$

and by $(g_4)(ii)$ and the Dominated Convergence Theorem, up to a subsequence,

$$\int_{\mathbb{R}^N} [g(x,u_n(x,0)) - g_0(x,u_n(x,0))]\varphi(x,0)\mathrm{d}x \to \int_{\mathbb{R}^N} [g(x,u(x,0)) - g_0(x,u(x,0))]\varphi(x,0)\mathrm{d}x, \tag{3.63}$$

as $n \to +\infty$. Thus, combining (3.61)-(3.63), we obtain (3.60).

Since $u_n \rightharpoonup u$ in X_0, from (3.59) and (3.60) we conclude Step 2.

Step 3. Given $\varphi \in C_0^\infty(\overline{\mathbb{R}_+^{N+1}})$, we have that $J_0'(u_n)\varphi = J_0'(v_n)\varphi_n$, where $\varphi_n(x,y) = \varphi(x-y_n,y)$ and $n \in \mathbb{N}$.

First, observe that

$$
\begin{aligned}
J_0'(u_n)\varphi = & \int_{\mathbb{R}^{N+1}_+} \kappa_\alpha\, y^{1-2\alpha} \nabla v_n(x+y_n,y)\nabla\varphi(x,y)\mathrm{d}x\mathrm{d}y \\
& + \int_{\mathbb{R}^N} V_0(x)v_n(x+y_n,0)\varphi(x,0)\mathrm{d}x - \int_{\mathbb{R}^N} g_0(x,v_n(x+y_n,0))\varphi(x,0)\mathrm{d}x \\
& - \int_{\mathbb{R}^N} |v_n(x+y_n,0)|^{2^*_\alpha-1}v_n(x+y_n,0)\varphi(x,0)\mathrm{d}x.
\end{aligned}
$$

By applying the definition of $\varphi_n(x,y)$ and the periodicity of V_0 and g_0, we obtain

$$
\begin{aligned}
J_0'(u_n)\varphi = & \int_{\mathbb{R}^{N+1}_+} \kappa_\alpha\, y^{1-2\alpha} \nabla v_n(z,y)\nabla\varphi_n(z,y)\mathrm{d}z\mathrm{d}y \\
& + \int_{\mathbb{R}^N} V_0(z)v_n(z,0)\varphi_n(z,0)\mathrm{d}z - \int_{\mathbb{R}^N} g_0(z,v_n(z,0))\varphi_n(z,0)\mathrm{d}z \\
& - \int_{\mathbb{R}^N} |v_n(z,0)|^{2^*_\alpha-1}v_n(z,0)\varphi_n(z,0)\mathrm{d}z \\
= & \ J_0'(v_n)\varphi_n.
\end{aligned}
$$

This concludes Step 3.

Step 4. u is a critical point of J_0.

Initially, note that

$$
\begin{aligned}
|J_0'(v_n)\varphi_n - J'(v_n)\varphi_n| \leq & \int_{\mathbb{R}^N} |V_0(z)-V(z)||v_n(z,0)||\varphi_n(z,0)|\mathrm{d}z \\
& + \int_{\mathbb{R}^N} |g(z,v_n(z,0))\varphi_n(z,0) - g_0(z,v_n(z,0))\varphi_n(z,0)|\mathrm{d}z.
\end{aligned}
$$

Then, applying Lemma 3.15, we get that

$$
|J_0'(v_n)\varphi_n - J'(v_n)\varphi_n| \to 0, \quad \text{as} \quad n\to\infty. \tag{3.64}
$$

By (3.57) and using that $\|\varphi_n\|_0 = \|\varphi\|_0$, for all $n\in\mathbb{N}$, we have $J'(v_n)\varphi_n \to 0$ as $n\to+\infty$. Hence, by (3.64), we obtain

$$
J_0'(v_n)\varphi_n \to 0, \quad \text{as} \quad n\to\infty.
$$

From Step 2 and Step 3, we obtain that u is a critical point of J_0 as claimed.

Step 5. We claim that

$$\liminf_{n\to\infty}\int\limits_{\mathbb{R}^N}\left[\frac{1}{2}g(x,v_n(x,0))v_n(x,0)-G(x,v_n(x,0))\right]\mathrm{d}x$$
$$\geq\int\limits_{\mathbb{R}^N}\left[\frac{1}{2}g_0(x,u(x,0))u(x,0)-G_0(x,u(x,0))\right]\mathrm{d}x. \tag{3.65}$$

Since $v_n \rightharpoonup 0$ in X, in view of Lemma 3.16,

$$\int\limits_{\mathbb{R}^N} h_i(x)|v_n(x,0)|^{q_i}\mathrm{d}x\to 0,\quad \text{as}\quad n\to\infty, \tag{3.66}$$

for $i\in\{1,2,3\}$, $h_i\in\mathcal{F}$ and $2\le q_i<2^*_\alpha$. Recall that by $(g_4)(ii)$

$$|g(x,s)s-g_0(x,s)s|\le h_3(x)|s|^{q_3}.$$

This together with (3.66) imply

$$\int\limits_{\mathbb{R}^N}\left[g(x,v_n(x,0))v_n(x,0)-g_0(x,v_n(x,0))v_n(x,0)\right]\mathrm{d}x\to 0,\quad \text{as}\quad n\to\infty. \tag{3.67}$$

Similarly,

$$\int\limits_{\mathbb{R}^N}\left[G(x,v_n(x,0))-G_0(x,v_n(x,0))\right]\mathrm{d}x\to 0,\quad \text{as}\quad n\to\infty. \tag{3.68}$$

From (3.67) and (3.68), we reach

$$\liminf_{n\to\infty}\int\limits_{\mathbb{R}^N}\left[\frac{1}{2}g(x,v_n(x,0))v_n(x,0)-G(x,v_n(x,0))\right]\mathrm{d}x$$
$$=\liminf_{n\to\infty}\int\limits_{\mathbb{R}^N}\left[\frac{1}{2}g_0(x,v_n(x,0))v_n(x,0)-G_0(x,v_n(x,0))\right]\mathrm{d}x.$$

Consequently, by the periodicity of g_0 and the definition of (v_n), we obtain

$$\liminf_{n\to\infty}\int\limits_{\mathbb{R}^N}\left[\frac{1}{2}g(x,v_n(x,0))v_n(x,0)-G(x,v_n(x,0))\right]\mathrm{d}x$$
$$=\liminf_{n\to\infty}\int\limits_{\mathbb{R}^N}\left[\frac{1}{2}g_0(x,u_n(x,0))u_n(x,0)-G_0(x,u_n(x,0))\right]\mathrm{d}x. \tag{3.69}$$

Now, from (g_3) and (g_4), we have that for $s \geq 0$,

$$\begin{aligned}\frac{1}{2}g_0(x,s)s - G_0(x,s) &\geq \frac{1}{2}[g_0(x,s)-g(x,s)]s + \frac{1}{2}g(x,s)s - G(x,s) + [G(x,s)-G_0(x,s)]\\ &\geq -h_1(x) - h_2(x)s^{q_2} - \frac{1}{2}h_3(x)s^{q_3}.\end{aligned}$$

Hence, since $g_0(x,s)=0$ if $s<0$, by Fatou's Lemma, it follows that

$$\begin{aligned}&\liminf_{n\to\infty}\int_{\mathbb{R}^N}\left[\frac{1}{2}g_0(x,u_n(x,0))u_n(x,0) - G_0(x,u_n(x,0)) + h_1(x) + h_2(x)|u_n(x,0)|^{q_2} + \frac{1}{2}h_3(x)|u_n(x,0)|^{q_3}\right]\mathrm{d}x\\ &\quad\geq \int_{\mathbb{R}^N}\left[\frac{1}{2}g_0(x,u(x,0))u(x,0) - G_0(x,u(x,0)) + h_1(x) + h_2(x)|u(x,0)|^{q_2} + \frac{1}{2}h_3(x)|u(x,0)|^{q_3}\right]\mathrm{d}x.\end{aligned}$$

By using (3.66) and (3.69), we obtain (3.65).

Step 6. $J_0(u) \leq c_M$.

In order to show this fact, note that

$$J(v_n) - \frac{1}{2}J'(v_n)v_n = \frac{\alpha}{N}\int_{\mathbb{R}^N}|v_n(x,0)|^{2^*_\alpha}\mathrm{d}x + \int_{\mathbb{R}^N}\left[\frac{1}{2}g(x,v_n(x,0))v_n(x,0) - G(x,v_n(x,0))\right]\mathrm{d}x. \quad (3.70)$$

By using the definition of (u_n) and the Fatou's Lemma we have

$$\liminf_{n\to\infty}\int_{\mathbb{R}^N}|v_n(x,0)|^{2^*_\alpha}\mathrm{d}x \geq \int_{\mathbb{R}^N}|u(x,0)|^{2^*_\alpha}\mathrm{d}x. \quad (3.71)$$

Combining (3.57), (3.65), (3.70), (3.71) and Step 3, we reach

$$\begin{aligned}c_M &\geq \frac{\alpha}{N}\int_{\mathbb{R}^N}|u(x,0)|^{2^*_\alpha}\mathrm{d}x + \int_{\mathbb{R}^N}\left[\frac{1}{2}g_0(x,u(x,0))u(x,0) - G_0(x,u(x,0))\right]\mathrm{d}x\\ &= J_0(u) - \frac{1}{2}J_0'(u)u = J_0(u),\end{aligned} \quad (3.72)$$

where we have used the fact that u is a critical point of J_0 (see Step 4). This completes Step 6.

Step 7. There exists $\gamma_0 \in \Gamma$ such that

$$c_M = \max_{t\in[0,1]} J(\gamma_0(t)). \quad (3.73)$$

Note that $\max\{J_0(tu); t \geq 0\}$ is unique. Indeed, we suppose that exist t_1 and t_2 such that t_1u and t_2u are maximum points of J_0 with $t_2 > t_1$, without loss of generality. By using the fact that t_2u is a critical point of J_0, we have

$$t_2^2\|u\|_0^2 = \int_{\mathbb{R}^N} g_0(x,t_2u)t_2u\mathrm{d}x + t_2^{2^*_s}\int_{\mathbb{R}^N}|u|^{2^*_s}\mathrm{d}x.$$

By using $(g_4)(iii)$, we have that

$$0 = t_2^2 \left(\|u\|_0^2 - \int\limits_{\mathbb{R}^N} g_0(x, t_2 u) t_2^{-1} u \mathrm{d}x - t_2^{2_s^*-2} \|u\|_{2_s^*}^{2_s^*} \right)$$
$$< t_2^2 \left(\|u\|_0^2 - \int\limits_{\mathbb{R}^N} g_0(x, t_1 u) t_1^{-1} u \mathrm{d}x - t_1^{2_s^*-2} \|u\|_{2_s^*}^{2_s^*} \right).$$

Multiplying the above inequality by t_1^2 and using the fact that $t_1 u$ is also a critical point of J_0 we obtain

$$0 < t_2^2 \left(t_1^2 \|u\|_0^2 - \int\limits_{\mathbb{R}^N} g_0(x, t_1 u) t_1 u \mathrm{d}x - t_1^{2_s^*} \|u\|_{2_s^*}^{2_s^*} \right)$$
$$= t_2^2 \left(t_1^{2_s^*} \|u\|_{2_s^*}^{2_s^*} - t_1^{2_s^*} \|u\|_{2_s^*}^{2_s^*} \right) = 0,$$

but this is a contradiction, thus the $\max\{J_0(tu); t \geq 0\}$ is unique.

Using this fact combined with (V), $(g_4)(i)$ and Step 6, we have

$$c_M \leq \max_{t \geq 0} J(tu) \leq \max_{t \geq 0} J_0(tu) = J_0(u) \leq c_M.$$

This implies that there exists $\gamma_0 \in \Gamma$ such that (3.73) holds.

In view of Theorem 3.6, J possesses a nontrivial critical point $\overline{v}$ on level c_M. This concludes the proof of Theorem 3.1. □

Proof of Theorem 3.2

The proof of Theorem 3.2 follows the same arguments of Theorem 3.1. In fact, since g_0 satisfies $(g_1) - (g_3)$, applying Corollary 3.8, we find a sequence $(v_n) \subset X_0$ such that

$$J_0(v_n) \to c_M \quad \text{and} \quad \|J_0'(v_n)\|_*(1 + \|v_n\|_0) \to 0, \quad \text{as} \quad n \to \infty.$$

By Lemma 3.10, we may suppose, without loss of generality, that $v_n \rightharpoonup v_0$ in X_0. From (3.11) for g_0, by using the same arguments as in the proof of Theorem 3.1, we have that v_0 is a critical point of J_0. Hence, in order to prove Theorem 3.2 it suffices to assume that $v_0 \equiv 0$.

In view of Proposition 3.14, it follows that $c_M < \frac{\alpha}{N}(\kappa_\alpha S(\alpha, N))^{N/2\alpha}$. Furthermore, by Proposition 3.11 there are a sequence $(y_n) \subset \mathbb{R}^N$ and $R, \eta > 0$ such that $|y_n| \to \infty$, as $n \to \infty$, and

$$\limsup_{n\to\infty} \int_{B_R(y_n)} |v_n(x,0)|^2 \mathrm{d}x \geq \eta > 0. \tag{3.74}$$

As in the proof of Theorem 3.1, we may assume that $(y_n) \subset \mathbb{Z}^N$. Then, defining $u_n(\cdot,*) = v_n(\cdot + y_n,*)$, $n \in \mathbb{N}$, by the periodicity of V_0 we have that $\|u_n(\cdot,*)\|_0 = \|v_n(\cdot + y_n,*)\|_0$ for all $n \in \mathbb{N}$. Consequently, passing to a subsequence, if necessary, there exists $u_0 \in X_0$ such that $u_n \rightharpoonup u_0$ in X_0. Similar to Step 4 we have that u_0 is a critical point of J_0. Furthermore, (3.74) implies that $u_0 \not\equiv 0$. This completes the proof of Theorem 3.2. □

Appendix A

Auxiliary results

Fractional Sobolev spaces and the fractional Laplacian operator

In this section we present the suitable environment for the study of the nonlocal equations: Fractional Sobolev spaces.

Considering $0 < \alpha < 1$ we may define the fractional Sobolev space $W^{\alpha,p}(\mathbb{R}^N)$ for any $p \in [1, +\infty)$ as follows

$$W^{\alpha,p}(\mathbb{R}^N) := \left\{ u \in L^p(\mathbb{R}^N); \frac{|u(x) - u(y)|}{|x - y|^{\frac{N}{p} + \alpha}} \in L^p(\mathbb{R}^N \times \mathbb{R}^N) \right\},$$

endowed with the natural norm

$$\|u\|_{\alpha,p} := \left(\int_{\mathbb{R}^N} |u|^p \mathrm{d}x + \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x - y|^{N + \alpha p}} \mathrm{d}x\mathrm{d}y \right)^{1/p},$$

where the term

$$[u]_{\alpha,p} := \left(\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x - y|^{N + \alpha p}} \mathrm{d}x\mathrm{d}y \right)^{1/p}$$

is the so-called *Gagliardo (semi)norm* of u, see details in [17].

As in the classic case when α is an integer, any function in the fractional Sobolev space $W^{\alpha,p}(\mathbb{R}^N)$ can be approximated by a sequence of smooth functions with compact support, it is what guarantees us the following result:

Theorem A.1. [17, Theorem 2.4] *For any $\alpha > 0$, the space $C_0^\infty(\mathbb{R}^N)$ of smooth functions with compact support is dense in $W^{\alpha,p}(\mathbb{R}^N)$.*

When $p = 2$ we have an important case since $W^{\alpha,2}(\mathbb{R}^N)$ turn out to be Hilbert space usually denoted by $H^\alpha(\mathbb{R}^N)$, which is strictly related to the fractional Laplacian operator $(-\Delta)^\alpha$.

This nonlocal operator $(-\Delta)^\alpha$ in $\mathbb{R}^N$ is defined on the Schwartz class through the Fourier transform,

$$(\widehat{(-\Delta)^\alpha} u)(\xi) = |2\pi\xi|^{2\alpha}\widehat{u}(\xi),$$

where $\widehat{u}$ denotes the Fourier Transform of u, that is,

$$\widehat{u}(\xi) = \int_{\mathbb{R}^N} e^{-2\pi i x\cdot\xi} u(x)\,\mathrm{d}x.$$

The operator $(-\Delta)^\alpha$ can be equivalently represented as

$$(-\Delta)^\alpha u(x) = C(N,\alpha) \lim_{\epsilon\to 0^+} \int_{B_\epsilon^c(x)} \frac{u(x)-u(y)}{|x-y|^{N+2\alpha}}\mathrm{d}y, \tag{A.1}$$

where

$$C(N,\alpha) = \left(\int_{\mathbb{R}^N} \frac{1-\cos(\zeta_1)}{|\zeta|^{N+2\alpha}}\,\mathrm{d}\zeta \right)^{-1}, \quad \zeta = (\zeta_1, \zeta_2, \cdots, \zeta_N).$$

E. Di Nezza *et al.* in [17] proved that we may write the singular integral in (A.1) as a weighted second order differential quotient.

Lemma A.2. [17, Lemma 3.2] *Let $0 < \alpha < 1$ for any u on the Schwartz class,*

$$(-\Delta)^\alpha u(x) = -\frac{1}{2}C(N,\alpha) \int_{\mathbb{R}^N} \frac{u(x+y)+u(x-y)-2u(x)}{|y|^{N+2\alpha}}\mathrm{d}y, \quad \forall x \in \mathbb{R}^N.$$

Moreover in [17] it has proved a relation between the fractional Laplacian operator $(-\Delta)^\alpha$ and the fractional Sobolev space $H^\alpha(\mathbb{R}^N)$ given by following result:

Lemma A.3. [17, Proposition 3.6] *Let $0 < \alpha < 1$ and let $u \in H^\alpha(\mathbb{R}^N)$. Then,*

$$[u]_\alpha^2 = 2C(N,\alpha)^{-1}\|(-\Delta)^{\frac{\alpha}{2}}u\|_2^2.$$

Among some properties of these spaces we may mention the fractional Sobolev inequalities. In this sense it is also important note that for $N > \alpha p$ the space $W^{\alpha,p}(\mathbb{R}^N)$ is continuously embedded in $L^q(\mathbb{R}^N)$ for any $q \in [p, p_\alpha^*]$ and in the limiting case $N = \alpha p$

we have that $W^{\alpha,p}(\mathbb{R}^N)$ is continuously embedded in $L^q(\mathbb{R}^N)$ for any $q \in [p, \infty)$. For a proof of these results see [17, Theorems 6.5 and 6.9].

We conclude this section presenting an other approach for the fractional Laplacian operator proposed by Caffarelli and Silvestre in [9] which reduces the nonlocal operator $(-\Delta)^\alpha$ on functions defined in $\mathbb{R}^N$ to a local operator on functions sitting in the higher dimensional half-space $\mathbb{R}^{N+1}_+$, thus we have that

$$(-\Delta)^\alpha u(x) = \kappa_\alpha \lim_{y \to 0^+} \left(y^{1-2\alpha} w_y(x,y) \right),$$

where the function $w : \mathbb{R}^{N+1}_+ \to \mathbb{R}$ solves

$$\begin{cases} -div(y^{1-2\alpha} \nabla w) = 0 & \text{in} \quad \mathbb{R}^{N+1}_+, \\ w = u & \text{in} \quad \mathbb{R}^N \times \{0\}. \end{cases}$$

For more details see [9].

Variational formulation

In this section we present the variational formulation for the class of the problems studied in Chapters 1 and 2. In order to do this, we consider the following class of problems:

$$\begin{cases} (-\Delta)^{1/2} u + V(x) u = f(x,u) & \text{in} \quad \mathbb{R}, \\ u \in H^{1/2}(\mathbb{R}) \quad \text{and} \quad u \geq 0, \end{cases} \tag{P}$$

and we denote X_1 the space $H^{1/2}(\mathbb{R})$ endowed with the norm

$$\|u\|_{X_1} = \left\{ \frac{1}{2\pi} \left(\int_{\mathbb{R}^2} \frac{|u(x) - u(y)|^2}{|x-y|^2} \, \mathrm{d}x \, \mathrm{d}y \right) + \int_{\mathbb{R}} V(x) u^2 \, \mathrm{d}x \right\}^{1/2}.$$

Recall that $u \in H^{1/2}(\mathbb{R})$ is a weak solution to problem (P) if

$$\langle u, v \rangle - \int_{\mathbb{R}} f(x,u) v \mathrm{d}x = 0, \text{ for all } v \in H^{1/2}(\mathbb{R}). \tag{A.2}$$

The purpose of this section is to prove that if $u \in C^\infty(\mathbb{R})$ is sufficiently regular such that $(-\Delta)^{1/2} u + V(x) u = f(x,u)$ in $\mathbb{R}$, then u satisfies (A.2) for every function $v \in C_0^\infty(\mathbb{R})$ such that $\int_{\mathbb{R}} V(x) uv \mathrm{d}x < \infty$ and $\int_{\mathbb{R}} f(x,u) v \mathrm{d}x < \infty$. Indeed, since

$$(-\Delta)^{1/2} u(x) + V(x) u(x) = f(x,u) \quad \text{in} \quad \mathbb{R}.$$

Multiplying both members of this equality by v and integrating, we obtain

$$\int\limits_{\mathbb{R}} (-\Delta)^{1/2} u(x)v(x)\mathrm{d}x + \int\limits_{\mathbb{R}} V(x)u(x)v(x)\mathrm{d}x - \int\limits_{\mathbb{R}} f(x,u)v(x)\mathrm{d}x = 0.$$

Now note that to obtain (A.2) it is sufficient to show that

$$\int\limits_{\mathbb{R}} (-\Delta)^{1/2} u(x)v(x)\mathrm{d}x + \int\limits_{\mathbb{R}} V(x)u(x)v(x)\mathrm{d}x = \langle u, v\rangle\,. \tag{A.3}$$

First, remember that

$$(-\Delta)^{1/2}u(x) = -\frac{1}{2\pi}\int_{\mathbb{R}} \frac{u(x+y)+u(x-y)-2u(x)}{|y|^2}\mathrm{d}y, \text{ for all } x \in \mathbb{R}.$$

Thus,

$$\int\limits_{\mathbb{R}} (-\Delta)^{1/2} u(x)v(x)\mathrm{d}x = \frac{1}{2\pi}\int_{\mathbb{R}^2} \frac{-v(x)(u(x+y)+u(x-y)-2u(x))}{|y|^2}\mathrm{d}y\mathrm{d}x.$$

Then,

$$\begin{aligned}\int\limits_{\mathbb{R}} (-\Delta)^{1/2} u(x)v(x)\mathrm{d}x &= \frac{1}{2\pi}\int_{\mathbb{R}^2} \frac{v(x)(u(x)-u(x+y))}{|y|^2}\mathrm{d}y\mathrm{d}x \\ &+ \frac{1}{2\pi}\int_{\mathbb{R}^2} \frac{v(x)(u(x)-u(x-y))}{|y|^2}\mathrm{d}y\mathrm{d}x.\end{aligned}$$

By the change of variables formula we obtain

$$\frac{1}{2\pi}\int_{\mathbb{R}^2} \frac{v(x)(u(x)-u(x+y))}{|y|^2}\mathrm{d}y\mathrm{d}x = \frac{1}{2\pi}\int_{\mathbb{R}^2} \frac{v(\theta)(u(\theta)-u(\xi))}{|\xi-\theta|^2}\mathrm{d}\theta\mathrm{d}\xi, \tag{A.4}$$

and

$$\frac{1}{2\pi}\int_{\mathbb{R}^2} \frac{v(x)(u(x)-u(x-y))}{|y|^2}\mathrm{d}y\mathrm{d}x = \frac{1}{2\pi}\int_{\mathbb{R}^2} \frac{v(\xi)(u(\xi)-u(\theta))}{|\xi-\theta|^2}\mathrm{d}\theta\mathrm{d}\xi. \tag{A.5}$$

Combining (A.4) and (A.5), we get

$$\int\limits_{\mathbb{R}} (-\Delta)^{1/2} u(x)v(x)\mathrm{d}x = \frac{1}{2\pi}\int_{\mathbb{R}^2} \frac{(u(\xi)-u(\theta)(v(\xi)-v(\theta)}{|\xi-\theta|^2}\mathrm{d}\theta\mathrm{d}\xi.$$

Thus, the equality (A.3) is valid and, therefore, (A.2) holds.

Differentiability of the functional I

In this section we will study the properties of the functional $I : X_1 \to \mathbb{R}$ defined by

$$I(u) = \frac{1}{2}\|u\|_{X_1}^2 - \int_{\mathbb{R}} F(x,u)\,\mathrm{d}x,$$

where $F(x,s) = \int_0^s f(x,t)\mathrm{d}t$. In order to show that the functional $I \in C^1(X_1,\mathbb{R})$, we need the following result.

Proposition A.4. *Let (u_n) a sequence that converges strongly in $H^{1/2}(\mathbb{R})$. Then there are a subsequence (u_{n_k}) of (u_n) and v in $H^{1/2}(\mathbb{R})$ satisfying*

$$|u_{n_k}(x)| \le v(x), \text{ almost everywhere in } \mathbb{R}.$$

Proof. Let (u_n) be a strongly convergent sequence in $H^{1/2}(\mathbb{R})$, let us say, $u_n \to u$ in $H^{1/2}(\mathbb{R})$. Hence (u_n) is Cauchy in $H^{1/2}(\mathbb{R})$. Thus, there is a subsequence (u_{n_k}) of (u_n), which, for simplicity, we denote by (u_k) satisfying

$$\|u_{k+1} - u_k\|_{1/2} \le \frac{1}{2^k}, \text{ for all } k \in \mathbb{N}. \tag{A.6}$$

We define

$$g_n(x) = \sum_{k=1}^{n} |u_{k+1}(x) - u_k(x)|.$$

By the triangle inequality and by (A.6), it follows that

$$\|g_n\|_{1/2} \le \sum_{k=1}^{n} \||u_{k+1} - u_k|\|_{1/2} = \sum_{k=1}^{n} \|u_{k+1} - u_k\|_{1/2} \le \sum_{k=1}^{n} \frac{1}{2^k} \le 1.$$

Therefore, $\|g_n\|_{1/2} \le 1$. In particular,

$$[g_n]_{1/2} \le 1 \text{ and } \|g_n\|_2 \le 1. \tag{A.7}$$

Note that (g_n) is a sequence of measurable functions that converges, let us say, to a function $g(x) = \sum_{k=1}^{\infty} |u_{k+1}(x) - u_k(x)|$, almost everywhere in $\mathbb{R}$. Then g is a measurable function. From this we have that $(g_n^2(x))$ converges for $g^2(x)$ almost everywhere in $\mathbb{R}$. Moreover, we have that (g_n) is a sequence of increasing functions and nonnegative, then (g_n^2) is also increasing and nonnegative. Thus, by the monotone convergence theorem

$$\lim_{n\to\infty} \int_{\mathbb{R}} g_n^2(x)\mathrm{d}x = \int_{\mathbb{R}} g^2(x)\mathrm{d}x. \tag{A.8}$$

Let us show that $g \in H^{1/2}(\mathbb{R})$. First, note that $g \in L^2(\mathbb{R})$. Indeed, by (A.7) and (A.8) we have that

$$\int\limits_{\mathbb{R}} g^2(x)\mathrm{d}x = \lim_{n\to\infty} \int\limits_{\mathbb{R}} g_n^2(x)\mathrm{d}x \leq 1.$$

Now, for $x \neq y$, we denote

$$h_n(x,y) = \frac{(g_n(x) - g_n(y))^2}{|x-y|^2}.$$

Note that (h_n) is a nonnegative functions sequence such that

$$h_n(x,y) \to \frac{(g(x) - g(y))^2}{|x-y|^2}, \text{ almost everywhere in } \mathbb{R}^2,$$

then, using Fatou's lemma and (A.7),

$$\begin{aligned} \int\limits_{\mathbb{R}^2} \frac{(g(x)-g(y))^2}{|x-y|^2}\mathrm{d}x\mathrm{d}y &= \int\limits_{\mathbb{R}^2} \liminf_{n\to\infty} \frac{(g_n(x)-g_n(y))^2}{|x-y|^2}\mathrm{d}x\mathrm{d}y \\ &\leq \liminf_{n\to\infty} \int\limits_{\mathbb{R}^2} \frac{(g_n(x)-g_n(y))^2}{|x-y|^2}\mathrm{d}x\mathrm{d}y \leq 1. \end{aligned}$$

Therefore, $g \in L^2(\mathbb{R})$ and $[g]_{1/2} \leq 1$, this is, $g \in H^{1/2}(\mathbb{R})$. Now, notice that for every $k \in \mathbb{N}$, we may write

$$\begin{aligned} g(x) &= \sum_{i=1}^{k-1} |u_{i+1}(x) - u_i(x)| + \sum_{i=k}^{k+j-1} |u_{i+1}(x) - u_i(x)| + \sum_{i=k+j}^{\infty} |u_{i+1}(x) - u_i(x)| \\ &= g_{k-1}(x) + \sum_{i=k}^{k+j-1} |u_{i+1}(x) - u_i(x)| + \sum_{i=k+j}^{\infty} |u_{i+1}(x) - u_i(x)|, \end{aligned}$$

then,

$$\begin{aligned} |u_{k+j}(x) - u_k(x)| &\leq |u_{k+j}(x) - u_{k+j-1}(x)| + \cdots + |u_{k+1}(x) - u_k(x)| \\ &= \sum_{i=k}^{k+j-1} |u_{i+1}(x) - u_i(x)| \leq g(x) - g_{k-1}(x) \leq g(x). \end{aligned}$$

Since $u_k \to u$ in $H^{1/2}(\mathbb{R})$, it follows that $u_k \to u$ in $L^2(\mathbb{R})$, then, up to a subsequence, $u_k(x) \to u(x)$ almost everywhere in $\mathbb{R}$. Making $j \to \infty$ in the previous estimate, we obtain that almost everywhere in $\mathbb{R}$

$$|u(x) - u_k(x)| \leq g(x), \text{ for all } k \in \mathbb{N}.$$

Thus,

$$|u_k(x)| = |u_k(x) - u(x) + u(x)| \leq g(x) + |u(x)| \text{ almost everywhere in } \mathbb{R}.$$

In order to conclude the proof, it is sufficient to choose $v = g + |u| \in H^{1/2}(\mathbb{R})$. □

Proposition A.5. *The functional $I \in C^1(X_1, \mathbb{R})$ and the Fréchet derivative of I is given by*

$$I'(u)v = \langle u, v\rangle - \int_{\mathbb{R}} f(x,u)v\mathrm{d}x.$$

Proof. We denote the two terms of the functional I, by

$$I_1(u) = \|u\|^2_{X_1} \text{ and } I_2(u) = \int_{\mathbb{R}} F(x,u)\mathrm{d}x.$$

Since I_1 is coming from an inner product, is immediate that $I_1 \in C^1(X_1, \mathbb{R})$ with Fréchet derivative given by $I_1'(u)v = \langle u, v\rangle$. Let us see now that $I_2 : X_1 \to \mathbb{R}$ is Fréchet differentiable and its derivative is given by

$$I_2'(u)v = \int_{\mathbb{R}} f(x,u)v\mathrm{d}x.$$

Let $u \in X_1$ be a fixed function. For every $v \in X_1$, we denote

$$r(v) = I_2(u+v) - I_2(v) - \int_{\mathbb{R}} f(x,u)v\mathrm{d}x. \tag{A.9}$$

We claim that

$$\lim_{\|v\|_{X_1}\to 0} \frac{r(v)}{\|v\|_{X_1}} = 0,$$

or equivalently, that

$$\lim_{\|v_n\|_{X_1}\to 0} \frac{r(v_n)}{\|v_n\|_{X_1}} = 0.$$

Indeed, we define $h : [0,1] \to \mathbb{R}$, given by $h(t) = F(x, u + tv_n)$. Note that h is differentiable and

$$h'(t) = f(x, u + tv_n)v_n.$$

By the Fundamental Theorem of Calculus, it follows that

$$F(x, u+v_n) - F(x,u) = h(1) - h(0) = \int_0^1 h'(t)\mathrm{d}t = \int_0^1 f(x, u+tv_n)v_n\mathrm{d}t.$$

Thus,

$$|r(v_n)| = \left|\int_{\mathbb{R}}\int_0^1 [f(x,u+tv_n) - f(x,u)]v_n\mathrm{d}t\mathrm{d}x\right|. \tag{A.10}$$

Now for each $n \in \mathbb{N}$, we define $g_n : \mathbb{R}\times[0,1] \to \mathbb{R}$ given by

$$g_n(x,t) = [f(x, u(x) + tv_n(x)) - f(x, u(x))]v_n(x),$$

and, for convenience, it will be denoted by

$$g_n = [f(x, u + tv_n) - f(x, u)]v_n.$$

Let us see that $g_n \in L^1(\mathbb{R} \times [0, 1])$. Indeed, by Young's inequality

$$|g_n| = |f(x, u + tv_n) - f(x, u)|\, |v_n| \leq \frac{|f(x, u + tv_n) - f(x, u)|^2}{2} + \frac{|v_n|^2}{2}. \tag{A.11}$$

Note that,

$$|f(x, u + tv_n) - f(x, u)|^2 \leq (|f(x, u + tv_n)| + |f(x, u)|)^2, \tag{A.12}$$

and since $(a + b)^2 \leq 2(a^2 + b^2)$, for all $a, b \geq 0$, we have that

$$(|f(x, u + tv_n)| + |f(x, u)|)^2 \leq 2\left[|f(x, u + tv_n)|^2 + |f(x, u)|^2\right]. \tag{A.13}$$

From (A.12) and (A.13) we obtain

$$|f(x, u + tv_n) - f(x, u)|^2 \leq 2\left(|f(x, u + tv_n)|^2 + |f(x, u)|^2\right). \tag{A.14}$$

Moreover, by (2.4), we can find positive constants $C_1, C_2 > 0$ such that for all $\alpha > \alpha_0$

$$|f(x, u + tv_n)| \leq C_1\left(e^{\alpha(|u|+t|v_n|)^2} - 1\right) \text{ and } |f(x, u)| \leq C_2\left(e^{\alpha u^2} - 1\right).$$

By using that for all $t \in [0, 1]$ we have $(|u| + t|v_n|)^2 \leq (|u| + |v_n|)^2$. It follows that

$$|f(x, u + tv_n)| \leq C_1\left(e^{\alpha(|u|+|v_n|)^2} - 1\right) \text{ and } |f(x, u)| \leq C_2\left(e^{\alpha u^2} - 1\right). \tag{A.15}$$

Thus, from (A.14) and (A.15) we obtain

$$|f(x, u + tv_n) - f(x, u)|^2 \leq 2\left[C_1'\left(e^{2\alpha(|u|+|v_n|)^2} - 1\right) + C_2'\left(e^{2\alpha u^2} - 1\right)\right], \tag{A.16}$$

and, consequently, from (A.11) and (A.16) we get

$$|g_n| \leq C_1'\left(e^{2\alpha(|u|+|v_n|)^2} - 1\right) + C_2'\left(e^{2\alpha u^2} - 1\right) + \frac{|v_n|^2}{2}.$$

From (2.14) we have that $(e^{2\alpha(|u|+|v_n|)^2} - 1)$, $(e^{2\alpha u^2} - 1) \in L^1(\mathbb{R})$, and since $v_n \in L^2(\mathbb{R})$, we obtain that

$$\int_{\mathbb{R}\times[0,1]} |g_n(x, t)|\, \mathrm{d}t\mathrm{d}x < \infty,$$

then, for each $n \in \mathbb{N}$, g_n is integrable. Thus we can use Fubini's Theorem in (A.10) to obtain

$$|r(v_n)| = \left|\int_0^1 \int_{\mathbb{R}} [f(x, u + tv_n) - f(x, u)]v_n \mathrm{d}x\mathrm{d}t\right|.$$

By applying Hölder's inequality, we obtain

$$|r(v_n)| \leq \int_0^1 \|f(u+tv_n)-f(u)\|_2 \|v_n\|_2 \, \mathrm{d}t.$$

Since the embedding $X_1 \hookrightarrow L^2(\mathbb{R})$ is continuous, there exists $C>0$ such that

$$|r(v_n)| \leq C\|v_n\|_{X_1} \int_0^1 \|f(x,u+tv_n)-f(x,u)\|_2 \, \mathrm{d}t. \tag{A.17}$$

On the other hand, since (v_n) converges strongly in X_1, by Proposition A.4, up to a subsequence, there exists $w \in H^{1/2}(\mathbb{R})$, satisfying

$$|v_n(x)| \leq w(x), \text{ almost everywhere in } \mathbb{R},$$

this, combined with (A.16), implies that

$$|f(x,u+tv_n)-f(x,u)|^2 \leq 2\left[C_1'\left(e^{2\alpha(|u|+|w|)^2}-1\right)+C_2'\left(e^{2\alpha u^2}-1\right)\right] =: l(x), \tag{A.18}$$

almost everywhere in $\mathbb{R}$. By (2.14), we have that $l \in L^1(\mathbb{R})$. Thus, since the embedding $X_1 \hookrightarrow L^2(\mathbb{R})$ is continuous and $\|v_n\|_{X_1} \to 0$, we have that for some constant $C>0$

$$\|u+tv_n-u\|_2 \leq \|v_n\|_2 \leq C\|v_n\|_{X_1} \to 0,$$

that is, $u+tv_n \to u$ in $L^2(\mathbb{R})$. Then, up to a subsequence, $u(x)+tv_n(x) \to u(x)$ almost everywhere in $\mathbb{R}$. By using the continuity of f, it follows that

$$f(x,u(x)+tv_n(x)) \to f(x,u(x)), \text{ almost everywhere in } \mathbb{R},$$

this implies that,

$$|f(x,u(x)+tv_n(x))-f(x,u(x))|^2 \to 0, \text{ almost everywhere in } \mathbb{R}. \tag{A.19}$$

Since (A.18) and (A.19) hold, by using the dominated convergence theorem, we conclude that

$$\|f(x,u+tv_n)-f(x,u)\|_2 \to 0, \text{ as } n \to \infty. \tag{A.20}$$

From (A.17) we have

$$\frac{r(v_n)}{\|v_n\|_{X_1}} \leq C\int_0^1 \|f(x,u+tv_n)-f(x,u)\|_2 \, \mathrm{d}t,$$

then, by (A.20), we obtain

$$\lim_{\|v_n\|_{X_1}\to 0} \frac{r(v_n)}{\|v_n\|_{X_1}} = 0.$$

The linearity of $I_2'(u) : X_1 \to \mathbb{R}$ follows by the integral properties. Let us show its boundedness. By Hölder's inequality, we have

$$\left|I_2'(u)v\right| = \left|\int\limits_{\mathbb{R}} f(x,u)v\mathrm{d}x\right| \leq \|f(x,u)\|_2 \|v\|_2 ,$$

From (2.4) and the fact that the embedding $X_1 \hookrightarrow L^2(\mathbb{R})$ is continuous, there exists constant $C > 0$ such that

$$\left|I_2'(u)v\right| = C\left(\int\limits_{\mathbb{R}} (e^{2\alpha u^2} - 1)\mathrm{d}x\right)^{1/2} \|v\|_{X_1} .$$

Again, by inequality (2.14), we have that $(e^{2\alpha u^2} - 1) \in L^1(\mathbb{R})$. Then we find a positive constant C_1, which depends on u such that

$$\left|I_2'(u)v\right| \leq C_1 \|v\| , \text{for all } v \in X_1;$$

then, $I_2'(u)$ is a bounded functional. Therefore, I_2 is Fréchet differentiable and its derivative is given by $I_2'(u)v = \int\limits_{\mathbb{R}} f(x,u)v\mathrm{d}x$. It remains to show that I_2' is continuous. In order to prove this, we consider $(u_n) \subset X_1$ such that $u_n \to u$ in X_1. Note that,

$$\left\|I_2'(u_n) - I_2'(u)\right\|_* \leq \sup_{\substack{v\in X_1 \\ \|v\|_{X_1}=1}} \int\limits_{\mathbb{R}} |f(x,u_n) - f(x,u)|\,|v|\,\mathrm{d}x.$$

By Hölder's inequality and of the fact that the embedding $X_1 \hookrightarrow L^2(\mathbb{R})$ is continuous, we have that for some constant $C > 0$,

$$\left\|I_2'(u_n) - I_2'(u)\right\|_* \leq \sup_{\substack{v\in X_1 \\ \|v\|_{X_1}=1}} \|f(x,u_n) - f(x,u)\|_2 \|v\|_2 \leq C\|f(x,u_n) - f(x,u)\|_2 .$$

Since $u_n \to u$ in X_1, similarly to (A.20), we conclude that

$$\|f(x,u_n) - f(x,u)\|_2 \to 0, \text{ as } n \to \infty,$$

which shows the continuity of I_2'. This implies that $I_2 \in C^1(X_1,\mathbb{R})$. Finally, joining all these statements, we obtain that the functional $I \in C^1(X_1,\mathbb{R})$. □

Remark A.6. Note that the functionals associated with problems (1.1) and (P_0) are also of class $C^1(\cdot,\mathbb{R})$. This fact follows by the proof presented herein similarly.

A example without the (AR) condition

We will present an example of nonlinearity $g(x,s)$ satisfying the hypotheses (g_1)–$(g_5)(i)$ but that does not satisfy the (AR) condition. In order to prove this, we consider

$\alpha = 1/2$ and $N = 2$ with $2^*_\alpha = 4$. Let $g(x,s) : \mathbb{R}^2 \times \mathbb{R} \to [0,+\infty)$ given by

$$g(x,s) = \begin{cases} \left(1 + \dfrac{1}{|x|+1}\right) g_0(x,s), & \text{if } (x,s) \in \mathbb{R}^2 \times [0,+\infty), \\ 0, & \text{if } (x,s) \in \mathbb{R}^2 \times (-\infty, 0), \end{cases}$$

where $g_0 : \mathbb{R}^2 \times \mathbb{R} \to [0,+\infty)$ is defined by

$$g_0(x,s) = \begin{cases} \varrho_1(x) s \ln(s+1) + \varrho_2(x) s^2, & \text{if } (x,s) \in \mathbb{R}^2 \times [0,+\infty), \\ 0, & \text{if } (x,s) \in \mathbb{R}^2 \times (-\infty, 0), \end{cases}$$

where $\varrho_i : \mathbb{R}^2 \to [0,2]$ is a continuous function, $\varrho_i \neq 0$, 1-periodic in x_i, with $i \in \{1,2\}$. Moreover, we consider that $\mathrm{supp}(\varrho_1) \cap \mathrm{supp}(\varrho_2) = \varnothing$ and that $B_1 \subset \mathrm{supp}(\varrho_2)$.

It is clear that $g_0 \in C(\mathbb{R}^2 \times \mathbb{R}, \mathbb{R}_+)$ and g_0 is also 1-periodic in x_i, with $i \in \{1,2\}$. Moreover, it is easy to see that $g_0(x,s) \geq 0$ since $\ln(s+1)$ is an increasing function and then $\ln(s+1) \geq \ln 1 = 0$.

By the continuity of the function $(1 + 1/(|x|+1))$ and of the $g_0(x,s)$ it follows that $g(x,s)$ is a continuous function. Note that g is asymptotically periodic at infinity since as $|x| \to +\infty$ we have that $g(x,s) \to g_0(x,s)$ which is periodic.

Let us show that g satisfies $(g_1) - (g_5)(i)$, note that for $(x,s) \in \mathbb{R}^2 \setminus [\mathrm{supp}(\varrho_1) \cup \mathrm{supp}(\varrho_2)] \times [0,+\infty)$ we have clearly that all the hypotheses are satisfied since $g(x,s) = 0$.

(g_1) Note that uniformly in x, we have the following limit

$$\begin{aligned} \lim_{s \to 0^+} \frac{g(x,s)}{s} &= \lim_{s \to 0^+} \left(1 + \frac{1}{|x|+1}\right) \left[\frac{\varrho_1(x) s \ln(s+1) + \varrho_2(x) s^2}{s}\right] \\ &= \left(1 + \frac{1}{|x|+1}\right) \lim_{s \to 0^+} [\varrho_1(x) \ln(s+1) + \varrho_2(x) s] = 0. \end{aligned}$$

(g_2) We must show that there exist $a_1, a_2 > 0$ and $2 < q_1 < 4$ such that

$$|g(x,s)| \leq a_1 + a_2 |s|^{q_1 - 1} \text{ for all } (x,s) \in \mathbb{R}^2 \times [0,+\infty).$$

For $(x,s) \in \mathrm{supp}(\varrho_1) \times [0,+\infty)$ note that

$$\begin{aligned} &\lim_{s \to \infty} \frac{\left(1 + \dfrac{1}{|x|+1}\right) \varrho_1(x)\, s \ln(s+1)}{s^{q_1 - 1}} \\ &= \left(1 + \frac{1}{|x|+1}\right) \varrho_1(x) \lim_{s \to \infty} \frac{1}{(q_1 - 2) s^{q_1 - 2} \left(1 + \dfrac{1}{s}\right)} = 0. \end{aligned}$$

Then, we have that there exist a_1, $a_2 > 0$ and $2 < q_1 < 4$ such that

$$\left(1 + \frac{1}{|x|+1}\right) \varrho_1(x)\, s \ln(s+1) \leq a_1 + a_2 s^{q_1 - 1}.$$

In this case, we have that (g_2) is satisfied. On the other hand, for $(x, s) \in \mathrm{supp}(\varrho_2) \times [0, +\infty)$, we obtain

$$|g(x,s)| = \left(1 + \frac{1}{|x|+1}\right) \varrho_2(x)\, s^2 \leq a_2 s^2 \leq a_1 + a_2 s^2.$$

Taking $q_1 = 3$, we have that (g_2) is also satisfied in this case. Therefore, we conclude that (g_2) is satisfied for all $(x, s) \in \mathbb{R}^2 \times [0, +\infty)$.

(g_3) We must show that there exist $2 \leq q_2 \leq 4$ and fuctions $h_1 \in L^1(\mathbb{R}^2)$, $h_2 \in \mathcal{F}$ such that

$$\frac{1}{2} g(x,s)\, s - G(x,s) \geq -h_1(x) - h_2(x)\, s^{q_2} \ \text{ for all}\,(x,s) \in \mathbb{R}^2 \times [0, \infty).$$

For $(x, s) \in \mathrm{supp}(\varrho_1) \times [0, +\infty)$, we have that

$$G(x,s) = \left(1 + \frac{1}{|x|+1}\right) \varrho_1(x) \int_0^s t \ln(t+1)\, \mathrm{d}t$$

We consider the change of variable $w = t + 1$, with $\mathrm{d}w = \mathrm{d}t$, then we use integration by parts and obtain

$$G(x,s) = \left(1 + \frac{1}{|x|+1}\right) \varrho_1(x) \left[\frac{s^2}{2} \ln(s+1) - \frac{1}{2} \ln(s+1) + \frac{1}{2} s - \frac{1}{4} s^2\right].$$

Note that by standard calculus,

$$\frac{1}{2} g(x,s) s - G(x,s) = \left(1 + \frac{1}{|x|+1}\right) \varrho_1(x) \left[\frac{1}{2} \ln(s+1) + \frac{1}{4} s^2 - \frac{1}{2} s\right].$$

Thus, considering $h(s) = \frac{1}{2}\ln(s+1) + \frac{1}{4}s^2 - \frac{1}{2}s$ we have

$$h'(s) = \frac{1}{2(s+1)} + \frac{1}{2} s - \frac{1}{2} \geq 0,$$

which implies that h is not decreasing, then $h(s) \geq h(0) = 0$. Therefore, $\frac{1}{2} g(x,s)s - G(x,s) \geq 0$, this implies that the hypothesis is satisfied in this case.

On the other hand, for $(x, s) \in \mathrm{supp}(\varrho_2) \times [0, +\infty)$, we have that

$$G(x,s) = \left(1 + \frac{1}{|x|+1}\right) \varrho_1(x) \frac{s^3}{3}.$$

Then

$$\frac{1}{2}g(x,s)s - G(x,s) = \left(\frac{1}{2} - \frac{1}{3}\right)\left(1 + \frac{1}{|x|+1}\right)\varrho_2(x)s^3 \geq 0.$$

This concludes the proof and assures that hypothesis (g_3) is satisfied for $(x,s) \in \mathbb{R}^2 \times [0,+\infty)$.

(g_4) (i) Since $\left(1 + \frac{1}{|x|+1}\right) \geq 1$ it follows that $g \geq g_0$. Thus, $G \geq G_0$.

(ii) We must show that there exist $2 \leq q_3 \leq 3$ and function $h_3 \in \mathcal{F}$ such that

$$|g(x,s) - g_0(x,s)| \leq h_3(x)s^{q_3-1} \text{ for all } (x,s) \in \mathbb{R}^2 \times [0,+\infty).$$

Note that for $(x,s) \in \operatorname{supp}(\varrho_1) \times [0,+\infty)$ and $(x,s) \in \operatorname{supp}(\varrho_2) \times [0,+\infty)$, we obtain

$$|g(x,s) - g_0(x,s)| \leq \left(\frac{1}{|x|+1}\right)\varrho_i(x)s^2,$$

with $i \in \{1,2\}$. Taking $q_3 = 3$ and observing that $\left(\frac{1}{|x|+1}\right)\varrho_i(x) \in \mathcal{F}$, we conclude that $(g_4)(ii)$ holds.

(iii) Let us now see that $g_0(x,s)/s$ is not decreasing in $s > 0$.

Note that for $(x,s) \in \operatorname{supp}(\varrho_1) \times [0,+\infty)$. By considering

$$h(s) = \frac{g_0(x,s)}{s} = \varrho_1(x)\ln(s+1),$$

we have that

$$h'(s) = \varrho_1(x)\frac{1}{s+1} \geq 0,$$

which implies that h is not decreasing in s. Analogously, for $(x,s) \in \operatorname{supp}(\varrho_2) \times [0,+\infty)$, we have that

$$h(s) = \frac{g_0(x,s)}{s} = \varrho_2(x)s.$$

Then $h'(s) = \varrho_2(x) \geq 0$, which implies that h is not decreasing in s. Therefore, $(g_4)(iii)$ is satisfied for all $(x,s) \in \mathbb{R}^2 \times [0,+\infty)$.

(g_5) (i) Consider $\Omega := B_1 \subset \operatorname{supp}(\varrho_2)$ and note that

$$\lim_{s\to\infty}\frac{G(x,s)}{s^p} = \left(1 + \frac{1}{|x|+1}\right)\varrho_2(x)\lim_{s\to\infty}\frac{s^{3-p}}{3} = +\infty,$$

uniformly in Ω, for $2 < p < 3$. This implies that (g_5) (i) is satisfied.

Let us now that $g(x,s)$ does not satisfy the (AR) condition for $(x,s) \in \text{supp}(\varrho_1) \times [R, +\infty)$. In fact,

$$\frac{G(x,s)}{sg(x,s)} = \frac{1}{2} - \frac{1}{2s^2} - \frac{1}{4\ln(s+1)} + \frac{1}{2s\ln(s+1)}$$

Therefore,

$$\lim_{s \to +\infty} \frac{G(x,s)}{sg(x,s)} = \frac{1}{2}.$$

References

[1] C. O. Alves, J. M. do Ó, and O.H. Miyagaki, *On nonlinear perturbations of a periodic elliptic problem in $\mathbb{R}^2$ involving critical growth*, Nonlinear Analysis **56** (2004), 781–791.

[2] A. Ambrosetti and P. H. Rabinowitz, *Dual variational methods in critical point theory and applications*, J. Funct. Anal. **14** (1973), 349–381.

[3] B. Barrios, E. Colorado, A. de Pablo, and U. Sánchez, *On some critical problems for the fractional Laplacian operator*, J. Differential Equations **252** (2012), 6133–6162.

[4] T. Bartsch and Z.-Q. Wang, *Existence and multiplicity results for some superlinear elliptic problems on $\mathbb{R}^N$*, Comm. Part. Diff. Eq. **20** (1995), 1725–1741.

[5] H. Berestycki and P. -L. Lions, *Nonlinear scalar field equations I. Existence of a ground state*, Arch. Rational Mech. Anal. **82** (1983), 313–345.

[6] C. Brandle, E. Colorado, A. de Pablo, and U. Sánchez, *A concave-convex elliptic problem involving the fractional laplacian*, Proc. Roy. Soc. Edinburgh Sect A. **143** (2013), 39–71.

[7] H. Brézis and E. Lieb, *A relation between pointwise convergence of functions and convergence of functionals*, Proc. Amer. Math. Soc. **88** (1983), 486–490.

[8] H. Brézis and L. Nirenberg, *Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents*, Comm. Pure Appl. Math. **36** (1983), 437–477.

[9] L. A. Caffarelli and L. Silvestre, *An extension problems related to the fractional Laplacian*, Comm. PDE. **36** (2007), 1245–1260.

[10] X. Chang and Z. -Q. Wang, *Ground state of scalar field equations involving a fractional Laplacian with general nonlinearity*, Nonlinearity **26** (2013), 479–494.

[11] M. Cheng, *Bound state for the fractional Schrödinger equation with unbounded potential*, J. Math. Phys. **53** (2012), 043507.

[12] D. G. Costa and C. A. Magalhaes, *Variational elliptic problems which are nonquadratic at infinity*, Nonlinear Anal. **23** (1994), 1401–1412.

[13] D. G. de Figueiredo, O. H. Miyagaki, and B. Ruf, *Elliptic equations in $\mathbb{R}^2$ with nonlinearities in the critical growth range*, Calc. Var. Partial Differential Equations **3** (1995), 139–153.

[14] M. de Souza and Y. L. Araújo, *Semilinear elliptic equations for the fractional Laplacian involving critical exponential growth*, Submitted.

[15] ______, *On nonlinear perturbations of a periodic fractional Schrödinger equation with critical exponential growth*, Mathematische Nachrichten, DOI:10.1002/mana.201500120 (2015).

[16] ______, *A class of asymptotically periodic fractional Schrödinger equations with critical growth*, Preprint.

[17] E. Di Nezza, G. Palatucci, and E. Valdinoci, *Hitchhiker's guide to the fractional Sobolev spaces*, Bull. Sci. Math. **136** (2012), 521–573.

[18] S. Dipierro, G. Palatucci, and E. Valdinoci, *Existence and symmetry results for a Schrödinger type problem involving the fractional Laplacian*, Matematiche **68** (2013), 201–216.

[19] J. M. do Ó and M. de Souza, *On a class of singular Trudinger-Moser inequalities*, Mathematishe Nachrichten **284** (2011), 1754–1776.

[20] J. M. do Ó, E. Medeiros, and U. B. Severo, *A nonhomogeneous elliptic problem involving critical growth in dimension two*, J. Math. Anal. Appl. **345** (2008), 286–304.

[21] J. M. do Ó, O. H. Miyagaki, and M. Squassina, *Critical and subcritical fractional problems with vanishing potentials*, Communications in Contemporary Mathematics, DOI:10.1142/s0219199715500637 (2015).

[22] ______, *Ground states of nonlocal scalar field equations with Trudinger-Moser critical nonlinearity*, arXiv:1504.07291v1.

[23] ______, *Nonautonomous fractional problems with exponential growth*, Nonlinear Differ. Equ. Appl. DOI:10.1007/s00030-015-0327-0 (2015).

[24] P. Felmer, A. Quaas, and J. Tan, *Positive solutions of nonlinear Schrödinger equation with the fractional Laplacian*, Proceedings of the Royal Society of Edinburgh: Section A Mathematics. **142** (2012), 1237–1262.

[25] A. Iannizzotto and M. Squassina, *1/2-Laplacian problems with exponential nonlinearity*, J. Math. Anal. Appl. **414** (2014), 372–385.

[26] Y. Jabri, *The Mountain Pass Theorem: variants, generalizations and some applications*, Encyclopedia of Mathematics and its Applications 95. Cambridge University Press., 2003.

[27] L. Jeanjean, *On the existence of bounded Palais–Smale sequences and application to a Landesman Lazer type problem set on* $\mathbb{R}^N$, Proc. Roy. Soc. Edinburgh Sect. A **129** (1999), 787–809.

[28] T. Jin, Y. Li, and J. Xiong, *On a fractional Nirenberg problem, part I: blow up analysis and compactness of solutions*, J. Eur. Math. Soc. **16** (2014), 1111–1171.

[29] H. Kozono, T. Sato, and H. Wadade, *Upper bound of the best constant of a Trudinger-Moser inequality and its application to a Gagliardo-Nirenberg inequality*, Indiana Univ. Math. J. **55** (2006), 1951–1974.

[30] N. Lam and G. Lu, *Existence and multiplicity of solution to equations of N-Laplacian type with critical exponential growth in* $\mathbb{R}^N$, J. Funct. Anal. **262** (2012), 1132–1165.

[31] N. Laskin, *Fractional quantum mechanics and Lévy path integrals*, Phys. Lett. A **268** (2000), 298–305.

[32] ______, *Fractional Schrödinger equation*, Phys. Rev. E **66** (2002), 056108.

[33] H. F. Lins and E. A. B. Silva, *Quasilinear asymptotically periodic elliptic equations with critical growth*, Nonlinear Anal. **71** (2009), 2890–2905.

[34] P. -L. Lions, *The concentration-compactness principle in the calculus of variations, Part I*, Rev. Mat. Iberoamericana **1** (1985), 145–201.

[35] ______, *Symétrie et compacité dans les espaces de Sobolev*, J. Funct. Anal. **49** (1982), 315–334.

[36] Z. Liu and Z.-Q. Wang, *On the Ambrosetti Rabinowitz superlinear condition*, Nonlinear Stud. **4** (2004), 563–574.

[37] J. Mawhin and M. Willem, *Critical Point Theory and Hamiltonian Systems*, Springer, Berlim, 1989.

[38] J. Moser, *A sharp form of an inequality by N. Trudinger*, Indiana Univ. Math. J. **20** (1971), 1077–1092.

[39] T. Ozawa, *On critical cases of Sobolev's inequalities*, J. Funct. Anal. **127** (1995), 259–269.

[40] P. H. Rabinowitz, *On a class of nonlinear Schrödinger equations*, Z. Angew. Math. Phys. **43** (1992), 270–291.

[41] S. Secchi, *Ground state solutions for nonlinear fractional Schrödinger equations in* $\mathbb{R}^N$, J. of Math. Phys. **54** (2013), 031501.

[42] R. Servadei and E. Valdinoci, *Mountain pass solution for non-local elliptic operators*, J. Math. Anal. Appl. **389** (2012), 887–898.

[43] X. Shang, J. Zhang, and Y. Yang, *On fractional Schrödinger equation in* $\mathbb{R}^N$ *with critical growth*, J. Math. Phys. **54** (2013), ID: 121502, 19 pp.

[44] B. Sirakov, *Existence and multiplicity of solutions of semi-linear elliptic equations in* $\mathbb{R}^N$, Cal. Var. Partial Differential Equations **11** (2000), 119–142.

[45] N. S. Trudinger, *On the imbedding into Orlicz spaces and some applications*, J. Math. Mech. **17** (1967), 473–484.

[46] Y. Yang, *Existence of positive solutions to quasi-linear elliptic equations with exponential growth in the whole Euclidean space*, J. Funct. Anal. **262** (2012), 1679–1704.

[47] M. Weinstein, *Solitary waves of nonlinear dispersive evolution equations with critical power nonlinearities*, J. Diff. Equa. **69** (1987), 192–203.

[48] M. Willem, *Minimax Theorems*, Birkhauser, Boston, 1996.