Universidade Federal da Paraíba
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Programa Associado de Pós-Graduação em Matemática
Doutorado em Matemática

On Local Cohomology and Local Homology Based on an Arbitrary Support

por

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Tese apresentada ao Corpo Docente do Programa Associado de Pós-Graduação em Matemática - UFPB/UFCG, como requisito parcial para obtenção do título de Doutor em Matemática.

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Resumo

Este trabalho desenvolve as teorias de cohomologia e homologia locais com respeito a um conjunto arbitrário de ideais e generaliza vários dos resultados importantes das teorias clássicas. Também, introduz a categoria dos $\mathscr{D}$-módulos quase-holonômicos e prova alguns resultados de finitude de cohomologia local que generalizam, em algum sentido, os resultados de G. Lyubeznik.

Palavras-chave: Cohomologia local; Família boa; Homologia local; Topologia linear; Dualidade de Matlis; $\mathscr{D}$-módulos quase-holonômicos.

Abstract

This work develops the theories of local cohomology and local homology with respect to an arbitrary set of ideals and generalises most of the important results from the classical theories. It also introduces the category of quasi-holonomic $\mathscr{D}$-modules and proves some finiteness properties of local cohomology modules which generalise Lyubeznik's results in some sense.

Keywords: Local cohomology; Good family; Local homology; Linear topology; Matlis' duality; Quasi-holonomic $\mathscr{D}$-modules.

Contents

Introduction

The study of local cohomology has its roots in Algebraic Geometry and it serves to general purposes in calculations of invariants in Commutative Algebra. Its starting point can be determined in the work of J. P. Serre [Ser55] as an approach to the study of projective varieties in terms of graded rings or complete local rings. It was then presented in a seminar given by A. Grothendieck in the context of abelian sheaves on an affine scheme, see [Har67]. Local cohomology was then thought as the right derived functors of the assignation to sections of sheaves with support on a locally closed subspace of said affine scheme. It is thus a specialisation of sheaf cohomology defined by A. Grothendieck himself in [Gro57]. This theory of local cohomology defined on a locally closed support would readily find a generalisation to an arbitrary family of supports as it can be read in [Har66, p. 218] and it constitutes the main object of study of this work by means of Commutative Algebra.

R. Hartshorne and R. Speiser posed in [HS77] the following question: When are the local cohomology modules $H^i_I(R)$ Artinian or zero for large values of i? Regarding the latter situation, A. Grothendieck proved for every d-dimensional R-module that $H^i_I(M) = 0$ when $i > d$ and that $H^d_{\mathfrak{m}}(M) \neq 0$ when M is finitely generated over a local ring (see [Har66, Propositions 1.12 and 6.4, 4)] and [BS98, Theorem 6.1.2]). These results are nowadays known as Grothendieck's Vanishing and Non-Vanishing theorems. In this direction, another important result is the Lichtenbaum-Hartshorne Vanishing Theorem which establishes a characterisation for the vanishing of $H^d_I(M)$ in terms of the $\mathfrak{m}$-adic completion.

For the former condition, Artinianness property for $H^i_I(M)$ was extensively studied by A. Grothendieck when $\dim(R/I) = 0$ (see [Gro68]). For an arbitrary ideal I, this property is only assured in the top dimension by studies of L. Melkersson (see [Mel95]).

Recall that for an R-module T, a prime ideal $\mathfrak{p} \in R$ is said to be an attached prime ideal of T if $\mathfrak{p} = \mathrm{Ann_R}(T/N)$ for a proper submodule N of T. The theory of attached primes and secondary representations of modules has been developed by I. G. Macdonald in [Mac73], which is in a certain sense dual to the theory of associated prime ideals and primary decompositions. It is well known that every Artinian module has a secondary representation. The theory of attached primes and secondary representation was successfully applied to the theory of local cohomology by I. G. Macdonald and R. Y. Sharp in [MS72]. Thus there exists a secondary representation for the Artinian module $H^d_I(M)$ and it makes sense to study its attached primes as done by R. Y. Sharp in [Sha81]. M. Dibaei and S. Yassemi also studied this set of prime ideals in [DY05] and deduced that the set of attached prime ideals of the top local cohomology module $H^d_I(M)$ is actually a subset of the minimal prime ideals of M.

Many results on derived categories have motivated studies for the behaviour of the local cohomology of a dualising sheaf of the n-dimensional affine scheme X, this is, a sheaf ω_X such that the Serre's duality $H^{n-i}(X, F^\vee \otimes \omega_X) \cong H^i(X, F)^*$ holds for every coherent sheaf F on X. Matlis' duality $(-)^\vee = \mathrm{Hom}_R(-, E_R(R/\mathfrak{m}))$ gives a translation of this to homomorphic images of Gorenstein local rings in the context of Commutative Algebra. Namely, the local duality isomorphisms $H^i_\mathfrak{m}(M) \cong \mathrm{Ext}^{n-i}_R(M, R)^\vee$ and $H^i_\mathfrak{m}(M)^\vee \cong \mathrm{Ext}^{n-i}_R(M, R)^\wedge$, where $(-)^\wedge$ denotes the $\mathfrak{m}$-adic completion, hold for every finitely generated module M over the n-dimensional Gorenstein local ring $(R, \mathfrak{m})$, giving the Artinian R-module $H^n_\mathfrak{m}(R) = E_R(R/\mathfrak{m})$ as a dualising module for R. Ideas for generalising this concept of dualising module for arbitrary modules were thought by P. Schenzel in [Sch93]. We use these ideas in order to generalise some results from [ES12] related to endomorphism rings of top local cohomology modules with respect to an arbitrary support.

E. Matlis investigated in [Mat58] some characterisations for Artinianness. Among them, it is included, by one side, the antiequivalence relation between Artinian modules and Noetherian ones. On the other hand, $\mathrm{Ext}^i_R(R/\mathfrak{m}, M)$ is always finitely generated for every Artinian R-module M. This situation motivated R. Hartshorne to define I-cofinite complexes in [Har70], where I is an ideal of the local ring R. Later, C. Huneke and J. Koh used this definition of cofiniteness in [HK91] in order to prove the finiteness of Bass numbers of local cohomology modules on ideals of dimension 1.

E. Matlis studied in [Mat74] the left derived functors of the I-adic completion functor $\Lambda_I(-) = \varprojlim_{n\in\mathbb{N}}(-)\otimes_R R/I^n$, where the ideal I was generated by a regular sequence in a local Noetherian ring R. This functor is the Matlis dual of the local cohomology functor. More precisely, the functor $\Gamma_I(-)^\vee$ is isomorphic to the functor $\Lambda_I((-)^\vee)$. On the other hand, since the local cohomology functors can be defined via Ext as $H^i_I(-) = \varinjlim_{n\in\mathbb{N}} \operatorname{Ext}^i_R(R/I^n, -)$, N. Cuong and T. Nam considered in [CN01] a definition for local homology via Tor, this is, $H^I_i(-) = \varprojlim_{n\in\mathbb{N}} \operatorname{Tor}^R_i(R/I^n, -)$. In the same work it was proved that these local homology functors are indeed the left derived functors of the I-adic completion functor $\Lambda_I(-)$ in the category of Artinian R-modules.

Now consider a regular k-algebra R, where k is a field of characteristic zero, and I be an ideal of R. In his seminal paper [Lyu93], G. Lyubeznik uses the theory of $\mathscr{D}$-modules over the ring of power series with coefficients in a field of characteristic zero to study some finiteness properties of local cohomology modules, more specifically, he proves the following statements:

(i) $\operatorname{inj.dim}_R(H^j_I(R)) \leq \dim_R(H^j_I(R))$.

(ii) The set of associated primes of $H^j_I(R)$ contained in every maximal ideal is finite.

(iii) All the Bass numbers of $H^j_I(R)$ are finite.

Recall that local cohomology modules with respect to I can be computed using Čech complex of M which is described by localising M at generators of I. A key point in the work of G. Lyubeznik relating the theory of $\mathscr{D}$-modules to local cohomology modules is a non-trivial result due to J.-E. Björk which establishes that holonomicity of $\mathscr{D}$-modules over the ring of differential operators of the ring of power series is preserved via localisation at elements of this ring. In the same direction, Z. Mebkhout and L. Narváez-Macarro prove in [MNM91], using the theory of Bernstein-Sato polynomials, that localisation of holonomic modules over rings of differential operators of certain more general rings are also holonomic. The rings considered in [MNM91] are those commutative Noetherian regular algebras over a field k of characteristic zero having the following properties:

(i) R is equidimensional of dimension n, that is, the height of any maximal ideal is equal to n.

(ii) Every residual field with respect to a maximal ideal is an algebraic extension of k.

(iii) There are k-linear derivations $D_1, \ldots, D_n \in Der_k(R)$ and $a_1, \ldots, a_n \in R$ such that $D_i(a_j) = 1$ if $i = j$ and 0 otherwise.

This class of algebras led L. Núñez-Betancourt in [NB13] to define a more general class of algebras substituting condition (iii) by the following one:

(iii)' $Der_k(R)$ is a finitely generated projective R-module of rank n and the canonical map $R_{\mathfrak{m}} \otimes_R Der_k(R) \to Der_k(R_{\mathfrak{m}})$ is an isomorphism for any maximal ideal $\mathfrak{m} \subset R$.

For this kind of algebras, L. Núñez-Betancourt proves that localisations at an element of R of holonomic $\mathscr{D}$-modules are holonomic. Here $\mathscr{D}$ is the ring of differential operators of R. L. Núñez-Betancourt also uses this result to prove that the set of associated primes of a holonomic $\mathscr{D}$-module is finite.

This work presents a generalisation of local cohomology functors for an arbitrary support and proves some of their fundamental properties, including those of vanishing and non-vanishing theorems as well as local duality theorems. In order to develop this theory we define good families of a ring, that is, a set α of ideals which is stable under multiplication and under inclusion, and for any R-module M, the α-torsion module $\Gamma_\alpha(M)$ as the set $\{x \in M : \mathrm{Supp}(Rx) \subseteq \alpha\}$. The i-th right derived functors are denoted by $H^i_\alpha(-)$ and called the i-th local cohomology module with respect to α.

The first chapter defines the basic notions and notations to establish the functors and proves some basic properties, including an improved version of the fundamental Independence Theorem. Later, it studies the vanishing and non-vanishing of the local cohomology modules with respect to a family and gives important generalisations of Grothendieck's Vanishing and Non-Vanishing and Lichtenbaum-Hartshorne Vanishing Theorems. Closing this chapter, the third section describes some duality properties of the local cohomology modules in the local case.

The second chapter is concerned with the properties of Artinianness of the top local cohomology functors with respect to an arbitrary support and gives a suitable definition of cohomological dimension of a module. It also studies the set of attached primes of the top local cohomology module and gives another generalisation of the Lichtenbaum-Hartshorne Vanishing Theorem which covers the one observed by K. Divaani-Aazar, R. Naghipour and M. Tousi in [DANT02].

The third chapter investigates the first non-vanishing local cohomology modules with respect to a family α and suggests a definition of α-cofiniteness. This number is called the α-depth of a module. The notion of α-cofiniteness is introduced as an extension of the notion of I-cofiniteness of [Har70], precisely, the R-module N is said to be α-cofinite if $\mathrm{Supp}(N) \subseteq \alpha$ and $\mathrm{Ext}^i_R(R/I, N)$ is finitely generated for every $I \in \alpha$ and every i. The main result establishes the α-cofiniteness of the top local cohomology modules. We also study the associated primes of $H^c_\alpha(M)$, where c is the α-depth of M.

Next, the fourth chapter deals with the modules of endomorphisms of local cohomology modules and investigates them in two phases. The first one studies the endomorphisms in the α-depth level, along with the special case when the α-depth equals the cohomological dimension. This part basically extends some ideas from [Mah13]. The second one takes care of the top local cohomology modules. It exploits the Lichtenbaum-Hartshorne Vanishing Theorem conditions and the Artinian nature of said modules to obtain informations on the ring structure of their modules of endomorphisms in an analogous way as done in [ES12].

The fifth chapter extends the notion of local homology with respect to an ideal to a good family α as $H^\alpha_i(-) = \varprojlim_{I \in \alpha} \mathrm{Tor}^R_i(R/I, -)$. Since $H^\alpha_0(-) = \varprojlim_{I \in \alpha}(R/I \otimes_R -)$, it also explores the linear topology induced by the family α which we call the α-adic topology. The study includes dual versions of classical results from local cohomology as the Independence Theorem, Vanishing and Non-Vanishing, acyclicity and Artinianness criteria and Matlis' duality with local cohomology.

The last chapter introduces the class of quasi-holonomic $\mathscr{D}$-modules which is a full subcategory of $\mathscr{D}$-modules stable under submodules, quotients, extensions and direct limits. This category extends the category of holonomic $\mathscr{D}$-modules. In fact, holonomic $\mathscr{D}$-modules are exactly the finitely generated quasi-holonomic $\mathscr{D}$-modules. We also prove that quasi-holonomicity is preserved via localisation on any multiplicative set of R. We also prove that local cohomology modules, with respect to any family of supports, of quasi-holonomic $\mathscr{D}$-modules are quasi-holonomic.

The main result of this work extends most of G. Lyubeznik's finiteness properties for $\mathscr{D}$-modules over rings of differential operators of the class of rings introduced by L. Núñez-Betancourt. Precisely, we prove the following statements:

(a) If $\dim_R(H_I^j(M)) = 0$, then $H_I^j(M)$ is an injective R-module. In particular, $H^j_{\mathfrak{m}_1\cdots\mathfrak{m}_s}(H_I^i(M))$ is an injective R-module for every finite family $\{\mathfrak{m}_1, \ldots, \mathfrak{m}_s\}$ of maximal ideals of R and every pair of integers i and j.

(b) If M is an quasi-holonomic $\mathscr{D}$-module and N is a finitely generated $\mathscr{D}$-submodule of $H_I^j(M)$, then the set of associated primes of N is finite.

(c) If M is quasi-holonomic, then every finitely generated $\mathscr{D}$-submodule of $H_I^j(M)$ has finite Bass numbers with respect to the maximal ideals.

Chapter 1

Foundations on local cohomology

1.1 Basic properties of local cohomology modules

In this section we define the local cohomology modules with respect to a pair of families of ideals and prove some of their basic properties, including an improved version of the fundamental Independence Theorem. Unless stated explicitly, all the rings through this work are commutative with identity.

Definition 1.1. Any set of ideals will be called a *family*. A family α of R will be called *good* when the following three conditions are satisfied:

(i) (*Non-emptiness*) $R \in \alpha$.

(ii) (*Stability under inclusion*) If $I \in \alpha$ and J is an ideal of R containing I, then $J \in \alpha$.

(iii) (*Stability under multiplication*) If $\{I, J\} \subseteq \alpha$, then $IJ \in \alpha$.

We will say that a family α of R is *trivial* when $\alpha \subseteq \{R\}$.

Definition 1.2. For any pair of families, φ and ψ, of R, we define the family

$$\tilde{W}(\varphi, \psi) := \{I \trianglelefteq R : I + J \in \varphi \text{ for all } J \in \psi\}.$$

Example 1.3. The most important examples of families we will consider in this work are the following: for any ideal I of R, we set

$$\mathcal{I} := \{J \trianglelefteq R : J \supseteq I^n \text{ for some integer } n \geq 1\}.$$

It can be seen that $\mathcal{I}$ is a good family. If $I_1, \ldots, I_s$ are ideals of R, we define $\tilde{W}(I_s, \ldots, I_1)$ inductively as follows: for $s = 1$, we set $\tilde{W}(I_1) = \mathcal{I}_1$; for $s \geq 2$, set

$$\tilde{W}(I_s, \ldots, I_1) = \tilde{W}(\tilde{W}(I_s), \tilde{W}(I_{s-1}, \ldots, I_1)).$$

If $s = 2$, the family $\tilde{W}(I_2, I_1)$ coincides with the family defined in [TYY09, Definition 3.1] for a pair of ideals.

Remark 1.4. Let α be a non-empty family of R. We define the family

$$\langle \alpha \rangle := \{K \trianglelefteq R : K \supseteq I_1 \cdots I_m \text{ for some } I_i \in \alpha\}.$$

When $\alpha = \emptyset$, $\langle \alpha \rangle$ is defined as the trivial family $\{R\}$. Any family $\langle \alpha \rangle$ defines a subspace of $\operatorname{Spec} R$ which is stable under specialisation, more exactly,

$$\langle \alpha \rangle \cap \operatorname{Spec} R = \bigcup_{I \in \alpha} V(I). \tag{1.1}$$

Conversely, if Z is a subspace of $\operatorname{Spec} R$ which is stable under specialisation, then $Z = \bigcup_{\mathfrak{p} \in Z} V(\mathfrak{p})$ and the family $\mathcal{Z} = \langle Z \rangle = \{K \trianglelefteq R : K \supseteq \mathfrak{p}_1 \cdots \mathfrak{p}_r \text{ for some } \mathfrak{p}_i \in Z\}$ is a good family of R such that $\mathcal{Z} \cap \operatorname{Spec} R = Z$.

It can also be seen that $\langle \alpha \rangle$ is the smallest good family containing α. In fact, α is a good family if and only if $\langle \alpha \rangle = \alpha$.

If I is an ideal of R and ψ is any family of R, we denote $\tilde{W}(\tilde{W}(I), \psi)$ by $\tilde{W}(I, \psi)$. In the same form, we denote $\tilde{W}(\varphi, \tilde{W}(J))$ by $\tilde{W}(\varphi, J)$ for any ideal J of R and any family φ. Next we state without proof some basic properties of the family $\tilde{W}(\varphi, \psi)$.

Lemma 1.5. *Let I and J be ideals of R and φ, φ', ψ and ψ' be families of R.*

(i) $\tilde{W}(\varphi, \psi) + \psi \subseteq \varphi$, *where* $\alpha + \beta := \{I + J : I \in \alpha, J \in \beta\}$. *Thus* $\tilde{W}(\varphi, \psi) \cap \psi \subseteq \varphi$.

(ii) If $\varphi \subseteq \varphi'$, *then* $\tilde{W}(\varphi, \psi) \subseteq \tilde{W}(\varphi', \psi)$. *In particular,* $\tilde{W}(I, \psi) \subseteq \tilde{W}(J, \psi)$ *every time* $I \supseteq J$.

(iii) If $\psi \subseteq \psi'$, *then* $\tilde{W}(\varphi, \psi) \supseteq \tilde{W}(\varphi, \psi')$. *In particular,* $\tilde{W}(\varphi, I) \supseteq \tilde{W}(\varphi, J)$ *every time* $I \supseteq J$.

(iv) $\tilde{W}(\varphi, \psi) \cap \tilde{W}(\varphi', \psi) = \tilde{W}(\varphi \cap \varphi', \psi)$. *In particular,*

$$\tilde{W}(I, \psi) \cap \tilde{W}(J, \psi) = \tilde{W}(I + J, \psi).$$

(v) $\tilde{W}(\varphi, \psi) \cap \tilde{W}(\varphi, \psi') = \tilde{W}(\varphi, \psi \cup \psi')$.

(vi) $\varphi = \tilde{W}(\varphi, \{(0)\})$.

(vii) If φ is a good family, then $\tilde{W}(\varphi,\psi)$ is also a good family and $\varphi \subseteq \tilde{W}(\varphi,\psi)$. In particular, $\tilde{W}(I,\psi)$ is a good family and $\tilde{W}(I) \subseteq \tilde{W}(I,\psi)$.

(viii) If φ is a good family, then $\tilde{W}(\varphi,\psi) = \tilde{W}(\varphi,\langle\psi\rangle)$. In particular,

$$\tilde{W}(\varphi,I) \cap \tilde{W}(\varphi,J) = \tilde{W}(\varphi,IJ)$$

when φ is a good family.

We now define the main object of this work.

Definition 1.6. Let M be an R-module and φ and ψ be families of R. The (φ,ψ)-*torsion* subset of M is the set $\Gamma_{\varphi,\psi}(M) = \left\{x \in M : \mathrm{Supp}(Rx) \subseteq \tilde{W}(\langle\varphi\rangle,\psi)\right\}$.

We will also use the notations $\Gamma_{I,\psi}(M)$, $\Gamma_{\varphi,J}(M)$ and $\Gamma_{I_s,\dots,I_1}(M)$ for $\Gamma_{\tilde{W}(I),\psi}(M)$, $\Gamma_{\varphi,\tilde{W}(J)}(M)$ and $\Gamma_{\tilde{W}(I_s),\tilde{W}(I_{s-1},\dots,I_1)}(M)$ respectively.

Let φ and ψ be families of R. For every morphism $f : M \to N$ between objects in R-mod, we define $\Gamma_{\varphi,\psi}(f) = f|_{\Gamma_{\varphi,\psi}(M)}$.

Proposition 1.7. *The assignation $\Gamma_{\varphi,\psi}(-) : R\text{-mod} \to R\text{-mod}$ is a left-exact R-linear functor.*

Proof. If φ and ψ are families of R and M is an R-module, then $\Gamma_{\varphi,\psi}(M)$ is an R-submodule of M because $\tilde{W}(\langle\varphi\rangle,\psi)$ is a good family by Lemma 1.5, (vii), and the relations $\mathrm{Ann}(x-y) \supseteq \mathrm{Ann}(x)\,\mathrm{Ann}(y)$ and $\mathrm{Ann}(ax) \supseteq \mathrm{Ann}(x)$ hold for every $a \in R$, $x, y \in M$. For every homomorphism of R-modules $f : M \to N$, the application

$$\Gamma_{\varphi,\psi}(f) : \Gamma_{\varphi,\psi}(M) \to \Gamma_{\varphi,\psi}(N)$$

is well defined as $\Gamma_{\varphi,\psi}(f)(x) = f(x)$ because $\mathrm{Ann}(f(x)) \supseteq \mathrm{Ann}(x)$ for every $x \in M$. More than this, every time f is injective, we have that $\mathrm{Ann}(f(x)) = \mathrm{Ann}(x)$ for every x and this equality suggests the left-exactness of $\Gamma_{\varphi,\psi}(-)$. $\square$

For any pair of families, φ and ψ, we can define the right derived functors of $\Gamma_{\varphi,\psi}(-)$ and we shall denote them by $H^i_{\varphi,\psi}(-)$ for every $i \geq 0$. For any R-module M, the module $H^i_{\varphi,\psi}(M)$ will be called *the i-th local cohomology module of M with respect to (φ,ψ)*. Due to the left-exactness of $\Gamma_{\varphi,\psi}(-)$, we have that $H^0_{\varphi,\psi}(-) \cong \Gamma_{\varphi,\psi}(-)$. Moreover, every short exact sequence of R-modules $0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$ induces a long exact sequence of R-modules $0 \longrightarrow \Gamma_{\varphi,\psi}(L) \longrightarrow \Gamma_{\varphi,\psi}(M) \longrightarrow \Gamma_{\varphi,\psi}(N) \longrightarrow H^1_{\varphi,\psi}(L) \longrightarrow H^1_{\varphi,\psi}(M) \longrightarrow H^1_{\varphi,\psi}(N) \longrightarrow \cdots$.

Remark 1.8. It is worth to notice that whenever φ is a system of ideals as defined in [BZ79, p. 403] and $(0) \in \langle\psi\rangle$, then the functor $H^i_{\varphi,\psi}(-)$ coincides with the functor $H^i_\varphi(-)$ defined in [BZ79, p. 405] for every i in the Noetherian case. So we will use the notation $H^i_\alpha(M)$ instead of $H^i_{\alpha,\{(0)\}}(M)$ for any family α and every i.

From now, the order relation considered in any family α of R will be the reverse inclusion. Recall that $\tilde{W}(\varphi, I) \cap \tilde{W}(\varphi, J) = \tilde{W}(\varphi, IJ)$ for every pair of ideals, I and J, of R when φ is a good family by Lemma 1.5, (viii). This implies that, whenever ψ is stable under multiplication, the projective system $\{\Gamma_{\varphi,J}(M), \iota_{JJ'}\}_{J\in\psi}$ is stable under finite intersections. Thus we have the following statement.

Lemma 1.9. *Let φ and ψ be families of R and M be an R-module. Then*

$$\Gamma_{\varphi,\psi}(M) = \bigcap_{J\in\psi} \Gamma_{\varphi,J}(M) \cong \varprojlim_{J\in\langle\psi\rangle} \Gamma_{\varphi,J}(M).$$

Proof. If $x \in \Gamma_{\varphi,\psi}(M)$, then $J + \mathfrak{p} \in \langle\varphi\rangle$ for every $J \in \psi$ and every $\mathfrak{p} \in \operatorname{Supp}(Rx)$. Let I be an element of $\tilde{W}(J)$. Then $I \supseteq J^n$ and $I + \mathfrak{p} \supseteq J^n + \mathfrak{p} \supseteq (J + \mathfrak{p})^n \in \langle\varphi\rangle$. Hence every $\mathfrak{p} \in \operatorname{Supp}(Rx)$ satisfies that $\mathfrak{p} \in \tilde{W}(\langle\varphi\rangle, J)$ and we deduce that $x \in \Gamma_{\varphi,J}(M)$ for every $J \in \psi$. The converse is clear because $J \in \tilde{W}(J)$ for every ideal J of R. As $\tilde{W}(\langle\varphi\rangle, \psi) = \tilde{W}(\langle\varphi\rangle, \langle\psi\rangle)$ by Lemma 1.5, (viii), we have that

$$\Gamma_{\varphi,\psi}(M) = \Gamma_{\varphi,\langle\psi\rangle}(M) = \bigcap_{J\in\langle\psi\rangle} \Gamma_{\varphi,J}(M) \cong \varprojlim_{J\in\langle\psi\rangle} \Gamma_{\varphi,J}(M).$$

The last isomorphism follows because the projective system $\{\Gamma_{\varphi,J}(M), \iota_{JJ'}\}_{J\in\langle\psi\rangle}$ is stable under finite intersections. □

We can characterise good families in the Noetherian case by the following property.

Lemma 1.10. *Let φ be a family satisfying the following property for every ideal I of R:*

$$I \in \varphi \text{ if and only if } V(I) \subseteq \varphi.$$

Then φ is a good family. The converse holds when R is Noetherian.

Proof. Let us suppose that φ is a family such that $I \in \varphi$ if and only if $V(I) \subseteq \varphi$. Since $V(IJ) = V(I) \cup V(J)$ for any ideals, I and J, of R, it follows that φ is closed under multiplication. Moreover, $V(J) \subseteq V(I)$ every time $J \supseteq I$; thus this family is also good.

Now let us suppose that R is Noetherian and φ is a good family and let I be an ideal of R such that $V(I) \subseteq \varphi$. Naming $\mathfrak{p}_1, \dots, \mathfrak{p}_s$ the minimal elements of $V(I)$, we have that $I \supseteq \left(\sqrt{I}\right)^r = (\mathfrak{p}_1 \cap \dots \cap \mathfrak{p}_s)^r \supseteq \mathfrak{p}_1^r \cdots \mathfrak{p}_s^r$ for some big enough r. Then $I \in \varphi$. The converse is straightforward. □

The previous lemma concludes also that if α and β are families of the Noetherian ring R such that $\langle\alpha\rangle \cap \operatorname{Spec} R \subseteq \langle\beta\rangle \cap \operatorname{Spec} R$, then $\langle\alpha\rangle \subseteq \langle\beta\rangle$. Thus there exists a bijective correspondence between good families of a Noetherian ring and stable under specialisation (abbr. s. u. s.) subspaces of its spectrum, namely,

$$\begin{array}{rcl} \{W \subseteq \operatorname{Spec} R : W \text{ is s. u. s.}\} & \leftrightarrow & \{\alpha : \alpha \text{ is a good family of } R\} \\ W & \mapsto & \langle W\rangle \\ \alpha \cap \operatorname{Spec} R & \leftarrow\!\shortmid & \alpha \end{array}$$

When α is stable under multiplication, reverse inclusion ordering defines an inductive system $\{H_I^i(M), \iota_{I'I}^i\}_{I\in\alpha}$ where $\iota_{I'I}^i : H_I^i(M) \to H_{I'}^i(M)$ is induced by the inclusion $\Gamma_I(E^i(M)) \to \Gamma_{I'}(E^i(M))$ whenever $I \supseteq I'$ for every non-negative integer i and any injective resolution $(E^i(M), \partial^i)$ of M.

Theorem 1.11. *For any pair of families, φ and ψ, of a Noetherian ring R, every R-module M and every $i \geq 0$, $H_{\varphi,\psi}^i(M) \cong \varinjlim_{I\in\tilde{W}(\langle\varphi\rangle,\psi)} H_I^i(M)$. In particular,*

$$H_\alpha^i(M) \cong \varinjlim_{I\in\langle\alpha\rangle} H_I^i(M)$$

for every family α.

Proof. Allow us to call $W = \tilde{W}(\langle\varphi\rangle, \psi)$. For any ring R,

$$\varinjlim_{I\in W} \Gamma_I(M) = \bigcup_{I\in W} \Gamma_I(M) \subseteq \Gamma_{\varphi,\psi}(M).$$

If R is Noetherian, Lemma 1.10 says that $I \in W$ if and only if $V(I) \subseteq W$. Thus $\Gamma_{\varphi,\psi}(M) \subseteq \bigcup_{I\in W} \Gamma_I(M)$. For $i > 0$ and $I \in W$, each short exact sequence of R-modules $0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$ leads to a long exact sequence of R-modules $0 \longrightarrow H_I^0(L) \longrightarrow H_I^0(M) \longrightarrow H_I^0(N) \longrightarrow H_I^1(L) \longrightarrow \cdots$. Since W is a filtered small category, $0 \longrightarrow \varinjlim_{I\in W} H_I^0(L) \longrightarrow \varinjlim_{I\in W} H_I^0(M) \longrightarrow \varinjlim_{I\in W} H_I^0(N) \longrightarrow \varinjlim_{I\in W} H_I^1(L) \longrightarrow \cdots$ is an exact sequence and we conclude that $\left(\varinjlim_{I\in W} H_I^i(-)\right)$ is a family of ∂-functors. If E is an injective R-module, then $H_I^i(E) = 0$ for every $i > 0$ and every $I \in W$; hence $\varinjlim_{I\in W} H_I^i(E) = 0$ for $i > 0$. By [Rot09, Theorem 6.51] we have that there exists a unique isomorphism of functors $\tau : \left(\varinjlim_{I\in W} H_I^i(-)\right) \longrightarrow (H_{\varphi,\psi}^i(-))$, whence

$$H_{\varphi,\psi}^i(M) \cong \varinjlim_{I\in W} H_I^i(M)$$

for every R-module M. $\square$

Remark 1.12. We say that the family α *is cofinal to* a family β (or simply that α and β *are cofinal*) when, for every $I \in \alpha$, there exists $J \in \beta$ such that $I \supseteq J$ and, for every $J \in \beta$, there exists $K \in \alpha$ such that $J \supseteq K$. When $\alpha \subseteq \beta$ and they are cofinal, we will also say that α *is a cofinal subfamily of* β. For any two cofinal families, α and β, of R, we have that $\langle\alpha\rangle = \langle\beta\rangle$ and thus $H^i_{\alpha,\psi}(M) = H^i_{\beta,\psi}(M)$ for every i, every R-module M and every family ψ of R. We also must observe that any family α cofinal to a good family φ (e.g., when α is stable under multiplication or α is a system of ideals as defined in [BS98, Definition 2.1.10] among other examples) is necessarily a subfamily of this and thus $H^i_\alpha(-) = H^i_\varphi(-) = \varinjlim_{I\in\alpha} H^i_I(-)$ for every i when R is Noetherian. It is worth to notice that this statement is a refinement of the particular case considered in Theorem 1.11.

Remark 1.13. Let $F : \mathcal{A} \to \mathcal{B}$ be a left-exact additive functor from an abelian category $\mathcal{A}$ with enough injectives to another abelian category $\mathcal{B}$. Recall that an object E in $\mathcal{A}$ is called *right F-acyclic* (shortly, F-acyclic) when $(R^iF)(E) = 0$ for every $i > 0$. It is well known that, for every object M of $\mathcal{A}$, the object $(R^iF)(M)$ can be calculated via F-acyclic resolutions of M, see [Har77, Proposition 1.2A, p. 205]. In this way, if R is a Noetherian ring, S is a commutative Noetherian R-algebra and E is an injective S-module, then E is a Γ_α-acyclic R-module for every family α of R by [BS98, Theorem 4.1.6] and Theorem 1.11.

Let $f : R \to R'$ be a ring homomorphism and let us denote $\alpha R' := \{JR' : J \in \alpha\}$ for any family α of R. Viewing the R'-module N' as an R-module via f, we may observe that, for any pair of families, φ and ψ, of R, the R-module $\Gamma_{\varphi,\psi}(N')$ is also an R'-module. We can get more with some additional conditions and the outcome is a generalisation for [TYY09, Theorem 2.7].

Theorem 1.14. *Consider two Noetherian rings, R and R', two families, φ and ψ, of R, a ring homomorphism $f : R \to R'$ and an R'-module M'. Suppose that $f(J) = JR'$ for every $J \in \psi$. Then the R'-modules $H^i_{\varphi,\psi}(M')$ and $H^i_{\varphi R',\psi R'}(M')$ are isomorphic for every i.*

Proof. By Remark 1.13 it is enough to show that $\Gamma_{\varphi,\psi}(N') = \Gamma_{\varphi R',\psi R'}(N')$ for every R'-module N'. If $x \in N'$ is such that $V(\mathrm{Ann_R}(x)) \subseteq \tilde{W}(\langle\varphi\rangle, \psi)$, consider a prime $\mathfrak{p} \in V(\mathrm{Ann_{R'}}(x))$. Then $f^{-1}(\mathfrak{p}) \in V(\mathrm{Ann_R}(x))$, whence $J + f^{-1}(\mathfrak{p}) \in \langle\varphi\rangle$ for every $J \in \psi$. It follows that $JR' + f^{-1}(\mathfrak{p})R' \in \langle\varphi\rangle R' \subseteq \langle\varphi R'\rangle$, concluding for every ideal $J \in \psi$ that $JR' + \mathfrak{p} \in \langle\varphi R'\rangle$ and $\mathfrak{p} \in \tilde{W}(\langle\varphi R'\rangle, \psi R')$. Now consider $x \in N'$ such that $\mathrm{Ann_{R'}}(x) \in \tilde{W}(\langle\varphi R'\rangle, \psi R')$. For each $J \in \psi$, there exist $K_1, \ldots, K_s \in \varphi$ such that $JR' + \mathrm{Ann_{R'}}(x) \supseteq K_1R' \cdots K_sR'$. Choose any $(k_1, \ldots, k_s) \in K_1 \times \cdots \times K_s$. Then $f(k_1 \cdots k_s) = \hat{\jmath} + r'$ for some $\hat{\jmath} \in JR'$ and $r' \in \mathrm{Ann_{R'}}(x)$. Since $f(J) = JR'$, we have that $f(k_1 \cdots k_s - j) = r'$ for some $j \in J$. Thus $k_1 \cdots k_s - j \in \mathrm{Ann_R}(x)$, leading to $K_1 \cdots K_s \subseteq J + \mathrm{Ann_R}(x)$, whence $\mathrm{Ann_R}(x) \in \tilde{W}(\langle\varphi\rangle, \psi)$ and we conclude that $\Gamma_{\varphi,\psi}(N') = \Gamma_{\varphi R',\psi R'}(N')$. $\square$

The following consequence appears in [DANT02, Remark 2.5, (ii)] for a special kind of families.

Corollary 1.15 (Independence Theorem). *Let R and R' be Noetherian rings, α be a family of R, $f : R \to R'$ be a ring homomorphism and M' be an R'-module. For every i we have the isomorphism of R'-modules $H^i_\alpha(M') \cong H^i_{\alpha R'}(M')$.*

Remark 1.16. For any ideal I of R and any ring homomorphism $f : R \to R'$, we have that $\langle \tilde{W}(I)R' \rangle = \{K \trianglelefteq R' : K \supseteq (IR')^n \text{ for some integer } n \geq 1\}$. The previous result gives also the isomorphism of R'-modules $H^i_{I,\psi}(M') \cong H^i_{IR',\psi R'}(M')$ for every i. Furthermore:

Corollary 1.17. *Let R and R' be Noetherian rings, $I_1, \ldots, I_s$ be ideals of R and β be a family of R such that $\langle \beta \rangle = \left\langle \bigcup_{j=2}^{s-1} \tilde{W}(I_j, I_{j-1}) \right\rangle$. If a ring homomorphism $f : R \to R'$ satisfies $f(I_1) = I_1R'$ and $f(K) = KR'$ for every $K \in \beta$, then, for every R'-module M' and every non-negative i, the R'-modules $H^i_{I_s,\ldots,I_1}(M')$ and $H^i_{I_sR',\ldots,I_1R'}(M')$ are isomorphic.*

Proof. We will show inductively at a first stage that the family $\tilde{W}(I_{s-1}, \ldots, I_1)R'$ is cofinal to $\tilde{W}(I_{s-1}R', \ldots, I_1R')$. We will also assume without loss of generality that $\beta = \bigcup_{j=2}^{s-1} \tilde{W}(I_j, I_{j-1})$. When $s = 2$, we have that the family $\tilde{W}(I_1)R'$ is cofinal to $\tilde{W}(I_1R')$ by Remark 1.16. Consider now $s > 2$ and an ideal $\mathfrak{a} \in \tilde{W}(I_{s-1}, \ldots, I_1)$. Then, for every ideal $J \in \tilde{W}(I_{s-2}, \ldots, I_1)$ there exists a positive integer n_J such that $\mathfrak{a} + J \supseteq I_{s-1}^{n_J}$, whence $\mathfrak{a}R' \in \tilde{W}(I_{s-1}R', \tilde{W}(I_{s-2}, \ldots, I_1)R')$. By induction hypothesis we have that $\tilde{W}(I_{s-2}, \ldots, I_1)R'$ is cofinal to the good family $\tilde{W}(I_{s-2}R', \ldots, I_1R')$ because $\bigcup_{j=2}^{s-2} \tilde{W}(I_j, I_{j-1}) \subseteq \beta$. Thus

$$\begin{aligned} \tilde{W}(I_{s-1}R', \tilde{W}(I_{s-2}, \ldots, I_1)R') &= \tilde{W}(I_{s-1}R', \tilde{W}(I_{s-2}R', \ldots, I_1R')) \\ &= \tilde{W}(I_{s-1}R', \ldots, I_1R') \end{aligned}$$

and $\mathfrak{a}R' \in \tilde{W}(I_{s-1}R', \ldots, I_1R')$, giving us that

$$\tilde{W}(I_{s-1}, \ldots, I_1)R' \subseteq \tilde{W}(I_{s-1}R', \ldots, I_1R').$$

Consider now $\mathfrak{b} \in \tilde{W}(I_{s-1}R', \ldots, I_1R')$. By similar arguments used in order to prove Theorem 1.14, we get that $\mathfrak{a} = f^{-1}(\mathfrak{b})$ is an ideal in $\tilde{W}(I_{s-1}, \ldots, I_1)$ such that $\mathfrak{b} \supseteq \mathfrak{a}R'$. It has been stated now that $\langle \tilde{W}(I_{s-1}, \ldots, I_1)R' \rangle = \tilde{W}(I_{s-1}R', \ldots, I_1R')$.

Finally, we can apply the Theorem 1.14 to $\varphi = \tilde{W}(I_s)$ and $\psi = \tilde{W}(I_{s-1}, \dots, I_1)$ because $\tilde{W}(I_{s-1}, \dots, I_1) \subseteq \tilde{W}(I_{s-1}, I_{s-2}) \subseteq \beta$. $\square$

It is also obtained a generalised version of the Flat Base Change Theorem (cf. [Lyu93, Lemma 3.1]) which we shall now state.

Lemma 1.18 (Flat Base Change). *Let R' be a commutative Noetherian algebra over a Noetherian ring R, α be a family of R and M' be an R'-module which is flat over R. Then there exists an isomorphism of functors $(H^i_{\alpha R'}(- \otimes_R M')) \cong (H^i_\alpha(-) \otimes_R M')$.*

Proof. At a first stage, allow us to recall that if F is an exact functor between two abelian categories, $\mathcal{A}$ and $\mathcal{B}$, and (C_*, ∂_*) is a complex in $\mathcal{A}$, then the isomorphism $H_i(F(C_*)) \cong F(H_i(C_*))$ holds for every i. Indeed, for every i we obtain the isomorphisms $F(\ker \partial_i) \cong \ker F(\partial_i)$ and $F(\operatorname{im} \partial_{i+1}) \cong \operatorname{im} F(\partial_{i+1})$. Thus the claimed isomorphism follows.

Next, recall that $\Gamma_\alpha(N) = \varinjlim_{I \in \langle\alpha\rangle} \operatorname{Hom}_R(R/I, N)$ for every R-module N by Theorem 1.11. Since M' is a flat R-module, we have the natural isomorphism

$$\operatorname{Hom}_R(R/I, N) \otimes_R M' \cong \operatorname{Hom}_R(R/I, N \otimes_R M')$$

for every $I \in \langle\alpha\rangle$, see [AK12, Proposition 9.14]. Hence we have a natural isomorphism of R-modules $\Gamma_\alpha(N) \otimes_R M' \cong \Gamma_\alpha(N \otimes_R M')$ for every R-module N. Observe now that if E is an injective R-module, then $E \otimes_R M'$ is Γ_α-acyclic: in fact, Theorem 1.11 gives that $H^i_\alpha(N) = \varinjlim_{I \in \langle\alpha\rangle} \operatorname{Ext}^i_R(R/I, N)$ for every R-module N and every i. On the other hand, denoting by $P_*(L)$ a projective resolution of an R-module L, we have that

$$\begin{aligned}
\operatorname{Ext}^i_R(R/I, E \otimes_R M') &= H^i(\operatorname{Hom}_R(P_*(R/I), E \otimes_R M')) \\
&\cong H^i(\operatorname{Hom}_R(P_*(R/I), E) \otimes_R M') \\
&\cong H^i(\operatorname{Hom}_R(P_*(R/I), E)) \otimes_R M' \\
&= 0
\end{aligned}$$

for every $I \in \langle\alpha\rangle$ and every $i > 0$. Hence $H^i_\alpha(E \otimes_R M') = 0$ for every $i > 0$. We have thus the following isomorphisms of R-modules for every R-module M, every family α of R and every i:

$$\begin{aligned}
H^i_\alpha(M) \otimes_R M' &= H^i(\Gamma_\alpha(E^*(M))) \otimes_R M' \\
&\cong H^i(\Gamma_\alpha(E^*(M)) \otimes_R M') \\
&\cong H^i(\Gamma_\alpha(E^*(M) \otimes_R M')) \\
&\cong H^i_\alpha(M \otimes_R M').
\end{aligned}$$

Here $E^*(N)$ is an injective resolution of an R-module N. Finally, Corollary 1.15 says that the R'-modules $H^i_\alpha(M \otimes_R M')$ and $H^i_{\alpha R'}(M \otimes_R M')$ are isomorphic. Hence the isomorphism of R'-modules $H^i_{\alpha R'}(M \otimes_R M') \cong H^i_\alpha(M) \otimes_R M'$ holds for every R-module M, every family α of R and every i. □

As in the usual terminology of local cohomology theory and its current extensions, an R-module M will be called (φ, ψ)-*torsion* when $\Gamma_{\varphi,\psi}(M) = M$ (equivalently, when $\operatorname{Supp}(M) \subseteq \tilde{W}(\langle\varphi\rangle, \psi)$). From definition, we observe that $\Gamma_{\varphi,\psi}(M)$ is (φ, ψ)-torsion for every R-module M and every pair of families, φ and ψ, of R. On the other hand, we say that M is (φ, ψ)-*torsion-free* when $\Gamma_{\varphi,\psi}(M) = 0$. If $I_1, \dots, I_s$ are ideals of R, we say that M is $(I_1, \dots, I_s)$-*torsion* when it is $(\tilde{W}(I_1), \tilde{W}(I_2, \dots, I_s))$-torsion. Similarly, M is $(I_1, \dots, I_s)$-*torsion-free* when it is $(\tilde{W}(I_1), \tilde{W}(I_2, \dots, I_s))$-torsion-free.

Remark 1.19. It is straightforward to see that $(0) \in \tilde{W}(\varphi, \psi)$ if and only if $\psi \subseteq \varphi$. Thus every R-module will be (φ, ψ)-torsion if and only if $\psi \subseteq \langle\varphi\rangle$ provided R is Noetherian.

Example 1.20. We list now two important examples for the development of the work:

(i) If $\mathfrak{p} \in \operatorname{Spec} R$, then $R/\mathfrak{p}$ is (φ, ψ)-torsion if and only if $\mathfrak{p} \in \tilde{W}(\langle\varphi\rangle, \psi)$. On the other hand, $R/\mathfrak{p}$ is (φ, ψ)-torsion-free if and only if $\mathfrak{p} \notin \tilde{W}(\langle\varphi\rangle, \psi)$.

(ii) Let N be an essential extension of the R-module M. Then $\Gamma_{\varphi,\psi}(N)$ is an essential extension of $\Gamma_{\varphi,\psi}(M)$. In particular, an R-module M is (φ, ψ)-torsion-free if and only if its injective hull $E(M)$ is (φ, ψ)-torsion-free.

Proposition 1.21. *Let φ and ψ be families of R.*

(i) Let $0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$ be an exact sequence of R-modules. Then M is (φ, ψ)-torsion if and only if L and N are (φ, ψ)-torsion.

(ii) Let s be a positive integer and $I_1, \dots, I_s$ be ideals of R. If M is an $(I_1, \dots, I_s)$-torsion R-module, then M is $(I_1, \dots, I_j)$-torsion for every even integer $2 \leq j \leq s$. If M is an $(I_1, \dots, I_s)$-torsion-free R-module, then M is $(I_1, \dots, I_j)$-torsion-free for every odd integer $1 \leq j \leq s$.

(iii) The R-module $H^i_{\varphi,\psi}(M)$ is (φ, ψ)-torsion for every $i \geq 0$.

(iv) $\operatorname{Ass}(\Gamma_{\varphi,\psi}(M)) = \operatorname{Ass}(M) \cap \tilde{W}(\langle\varphi\rangle, \psi)$ *for every R-module M.*

Proof. Item (i) follows because $\operatorname{Supp}(M) = \operatorname{Supp}(N) \cup \operatorname{Supp}(L)$.

Item (ii) follows from the inclusions

$$\tilde{W}(I_1) \subseteq \tilde{W}(I_1, I_2, I_3) \subseteq \cdots \subseteq \tilde{W}(I_1, I_2, I_3, I_4) \subseteq \tilde{W}(I_1, I_2).$$

In order to prove (iii), we observe that $H^i_{\varphi,\psi}(M)$ is a sub-quotient of a (φ, ψ)-torsion module for every i, thus we have by (i) that $H^i_{\varphi,\psi}(M)$ is (φ, ψ)-torsion for every i.

We now prove (iv): consider $\mathfrak{p} \in \operatorname{Ass}(\Gamma_{\varphi,\psi}(M))$. Then $\mathfrak{p} = \operatorname{Ann}(x)$ for some $x \in \Gamma_{\varphi,\psi}(M)$ and $V(\mathfrak{p}) = \operatorname{Supp}(Rx) \subseteq \tilde{W}(\langle\varphi\rangle, \psi)$, whence $\mathfrak{p} \in \operatorname{Ass}(M) \cap \tilde{W}(\langle\varphi\rangle, \psi)$. For the converse, consider $\mathfrak{p} \in \operatorname{Ass}(M) \cap \tilde{W}(\langle\varphi\rangle, \psi)$. Then $\mathfrak{p} = \operatorname{Ann}(x)$ for some $x \in M$ and $\mathfrak{p} \in \tilde{W}(\langle\varphi\rangle, \psi)$. Hence $\operatorname{Supp}(Rx) \subseteq \tilde{W}(\langle\varphi\rangle, \psi)$ and we conclude that $x \in \Gamma_{\varphi,\psi}(M)$. □

Proposition 1.22. *If φ and ψ are families of a Noetherian ring R and M is a (φ, ψ)-torsion finitely generated R-module, then M is I-torsion for some ideal $I \in \tilde{W}(\langle\varphi\rangle, \psi)$.*

Proof. If $\{x_1, \ldots, x_n\}$ generates M, then $\operatorname{Ann}(M) = \operatorname{Ann}(x_1) \cap \cdots \cap \operatorname{Ann}(x_n)$. Since M is (φ, ψ)-torsion, we have that $\operatorname{Ann}(x_i) = I_i \in \tilde{W}(\langle\varphi\rangle, \psi)$ for every i. Set

$$I = I_1 \cdots I_n \in \tilde{W}(\langle\varphi\rangle, \psi).$$

Then $\operatorname{Ann}(x) \supseteq \operatorname{Ann}(M) \supseteq I$ for every $x \in M$, whence $x \in \Gamma_I(M)$. □

Corollary 1.23. *If α is a family of the Noetherian ring R and M is a finitely generated R-module, then there exists $I \in \langle\alpha\rangle$ such that $\Gamma_\alpha(M) = \Gamma_I(M)$. If α is cofinal to a good family, we can take $I \in \alpha$.*

Proof. Since $\Gamma_\alpha(M)$ is a finitely generated α-torsion R-module, by Proposition 1.22 there exists $I \in \tilde{W}(\langle\alpha\rangle, \{(0)\}) = \langle\alpha\rangle$ such that $\Gamma_I(\Gamma_\alpha(M)) = \Gamma_\alpha(M)$. By Definition 1.6 and by Lemma 1.5, (iv), $\Gamma_I(\Gamma_\alpha(M)) = \Gamma_{\tilde{W}(I)\cap\langle\alpha\rangle}(M)$ and, since $\tilde{W}(I) \subseteq \langle\alpha\rangle$, we conclude the statement. Now if α is cofinal to $\langle\alpha\rangle$, then there exists $J \in \alpha$ such that $J \subseteq I$. Hence $\Gamma_I(M) \subseteq \Gamma_J(M) \subseteq \Gamma_\alpha(M) = \Gamma_I(M)$. □

It is straightforward to check that torsion functors $\Gamma_{\varphi,\psi}(-)$ commute with formation of arbitrary direct sums. Hence local cohomology functors with respect to any pair of families commute with formation of arbitrary direct sums. Moreover, we shall observe that local cohomology functors with respect to a pair of families commute with inductive limits in the Noetherian case.

Proposition 1.24. *If R is a Noetherian ring and $\{M_\lambda, f_{\mu\lambda}\}_{\lambda\in\Lambda}$ is an inductive system of R-modules, then $H^i_{\varphi,\psi}\left(\varinjlim_{\lambda\in\Lambda} M_\lambda\right) \cong \varinjlim_{\lambda\in\Lambda} H^i_{\varphi,\psi}(M_\lambda)$.*

Proof. We have the following isomorphisms:

$$\begin{aligned}
H^i_{\varphi,\psi}\left(\varinjlim_{\lambda\in\Lambda} M_\lambda\right) &\cong \varinjlim_{I\in\tilde{W}(\langle\varphi\rangle,\psi)} H^i_I\left(\varinjlim_{\lambda\in\Lambda} M_\lambda\right) && \text{(Theorem 1.11)}\\
&\cong \varinjlim_{I\in\tilde{W}(\langle\varphi\rangle,\psi)} \left(\varinjlim_{\lambda\in\Lambda} H^i_I(M_\lambda)\right) && \text{([BS98, Theorem 3.4.10])}\\
&\cong \varinjlim_{\lambda\in\Lambda}\left(\varinjlim_{I\in\tilde{W}(\langle\varphi\rangle,\psi)} H^i_I(M_\lambda)\right)\\
&\cong \varinjlim_{\lambda\in\Lambda} H^i_{\varphi,\psi}(M_\lambda) && \text{(Theorem 1.11).}
\end{aligned}$$

□

This section ends with a discussion about torsion and torsion-free modules and establishes the main results of local cohomology theory related to these features.

Proposition 1.25. *Let φ and ψ be families of a Noetherian ring R and M be an R-module. The following statements are equivalent:*

(i) *M is a (φ,ψ)-torsion R-module.*

(ii) $\mathrm{Ass}(M) \subseteq \tilde{W}(\langle\varphi\rangle,\psi)$.

Proof. Since $\mathrm{Ass}(M) \subseteq \mathrm{Supp}(M)$, the implication (i)⇒(ii) is clear. Now R is Noetherian, whence $\mathrm{Supp}(M) = \bigcup_{\mathfrak{p}\in\mathrm{Ass}(M)} V(\mathfrak{p})$. On the other hand, $\tilde{W}(\langle\varphi\rangle,\psi)$ is a good family. Hence (ii)⇒(i). □

As direct consequences, we have the following statements.

Corollary 1.26. *Let φ and ψ be families of a Noetherian ring R and M be an R-module.*

(i) *M is (φ,ψ)-torsion-free if and only if* $\mathrm{Ass}(M)\cap\tilde{W}(\langle\varphi\rangle,\psi)=\emptyset$.

(ii) *M is (φ,ψ)-torsion if and only if its injective hull $E(M)$ is (φ,ψ)-torsion.*

(iii) *If M is (φ,ψ)-torsion, then every term of any minimal injective resolution of M is (φ,ψ)-torsion.*

(iv) *The R-module $E_R(R/\mathfrak{p})$ is (φ,ψ)-torsion when $\mathfrak{p}\in\tilde{W}(\langle\varphi\rangle,\psi)$. On the other hand, it is (φ,ψ)-torsion-free when $\mathfrak{p}\notin\tilde{W}(\langle\varphi\rangle,\psi)$.*

Proof. Note that (i) follows from Proposition 1.21, (iv). Item (ii) follows because $\mathrm{Ass}(E(M)) = \mathrm{Ass}(M)$ and (iv) follows because $\mathrm{Ass}_R(E_R(R/\mathfrak{p})) = \mathrm{Ass}(R/\mathfrak{p}) = \{\mathfrak{p}\}$ for every $\mathfrak{p} \in \mathrm{Spec}\, R$.

We now prove (iii): if $(E^*(M), d^*)$ is a minimal injective resolution of M, we get that $E^0(M) = E(M)$ and $E^i(M) = E(\mathrm{im}\, d^{i-1})$ for $i \geq 1$. Since homomorphic images of (φ, ψ)-torsion modules are (φ, ψ)-torsion by Proposition 1.21, (i), the statement follows in an inductive way. □

Corollary 1.27. *Let α be a family of a Noetherian ring R and M be a finitely generated R-module. Then*

(i) M is α-torsion-free if and only if every $I \in \alpha$ contains an M-regular element.

(ii) The R-modules $R/\mathfrak{p}$ and $E_R(R/\mathfrak{p})$ are α-torsion when $I \subseteq \mathfrak{p}$ for some $I \in \alpha$. On the other hand, they are α-torsion-free when $I \nsubseteq \mathfrak{p}$ for every $I \in \alpha$.

From Corollary 1.26, we have the following result.

Proposition 1.28. *Let φ and ψ be families of a Noetherian ring R and M be an R-module.*

(i) If M is a (φ, ψ)-torsion module, then $H^i_{\varphi,\psi}(M) = 0$ for every $i > 0$ (i.e., every (φ, ψ)-torsion R-module is $\Gamma_{\varphi,\psi}$-acyclic).

(ii) The R-module $M/\Gamma_{\varphi,\psi}(M)$ is (φ, ψ)-torsion-free and

$$H^i_{\varphi,\psi}(M) \cong H^i_{\varphi,\psi}\left(M/\Gamma_{\varphi,\psi}(M)\right)$$

for every $i > 0$.

Proof. Note that (i) can be obtained from Corollary 1.26, (iii). For (ii), consider the short exact sequence $0 \to \Gamma_{\varphi,\psi}(M) \to M \to M/\Gamma_{\varphi,\psi}(M) \to 0$. This leads to the long exact sequence $0 \to \Gamma_{\varphi,\psi}(M) \to \Gamma_{\varphi,\psi}(M) \to \Gamma_{\varphi,\psi}(M/\Gamma_{\varphi,\psi}(M)) \to H^1_{\varphi,\psi}(\Gamma_{\varphi,\psi}(M)) \to H^1_{\varphi,\psi}(M) \to H^1_{\varphi,\psi}(M/\Gamma_{\varphi,\psi}(M)) \to \cdots$, being the first non-trivial arrow an isomorphism. Also, as $H^i_{\varphi,\psi}(\Gamma_{\varphi,\psi}(M)) = 0$ for every $i > 0$ by the previous item, the result follows. □

We will show later that the class of finitely generated Γ_α-acyclic modules actually coincides with the class of finitely generated α-torsion modules.

Corollary 1.29. *Let φ and ψ be families of R and M be an R-module. If M is (φ, ψ)-torsion, then M/JM is φ-torsion (i.e., M/JM is $(\varphi, \{(0)\})$-torsion) for every $J \in \psi$. The converse holds when R is Noetherian and M is finitely generated.*

In particular, if M is $(I_1, \ldots, I_s)$-torsion, then M/JM is I_1-torsion for every ideal $J \in \tilde{W}(I_2, \ldots, I_j)$, where $2 \leq j \leq s$ is any even integer or $j = s$. Conversely, *if R is Noetherian, M is finitely generated and M/JM is I_1-torsion for every ideal $J \in \tilde{W}(I_2, \ldots, I_s)$, then M is $(I_1, \ldots, I_j)$-torsion for every even integer $2 \leq j \leq s$ and for $j = s$.*

Proof. If M is (φ, ψ)-torsion, then $\mathrm{Supp}(M) \subseteq \tilde{W}(\langle\varphi\rangle, \psi)$. For every $J \in \psi$ it also holds that $V(J) \subseteq \langle\psi\rangle$, thus

$$\mathrm{Supp}\,(M/JM) \subseteq \tilde{W}(\langle\varphi\rangle, \psi) \cap \langle\psi\rangle = \tilde{W}(\langle\varphi\rangle, \langle\psi\rangle) \cap \langle\psi\rangle \subseteq \langle\varphi\rangle = \tilde{W}(\langle\varphi\rangle, \{(0)\}).$$

Hence M/JM is φ-torsion for every $J \in \psi$.

For the converse, let us suppose that R is Noetherian and M is a finitely generated R-module such that M/JM is φ-torsion for every $J \in \psi$. We have that $V(J + \mathrm{Ann}(M)) \subseteq \langle\varphi\rangle$ when $J \in \psi$ and $J + \mathrm{Ann}(x) \in \langle\varphi\rangle$ for every $x \in M$, whence $\mathrm{Ann}(x) \in \tilde{W}(\langle\varphi\rangle, \psi)$ and $M = \Gamma_{\varphi,\psi}(M)$. □

Proposition 1.30. *Let α be a family of a Noetherian ring R and M be an R-module. Then $\mathrm{Ass}(M) = \mathrm{Ass}(\Gamma_\alpha(M)) \cup \mathrm{Ass}(M/\Gamma_\alpha(M))$, being the right term a disjoint union.*

Proof. From Corollary 1.26, (i), and Proposition 1.28, (ii), we get that

$$\mathrm{Ass}(\Gamma_\alpha(M)) \cap \mathrm{Ass}(M/\Gamma_\alpha(M)) = \emptyset.$$

Consider now the exact sequence $0 \to \Gamma_\alpha(E(M)) \to E(M) \to E(M)/\Gamma_\alpha(E(M)) \to 0$. By Theorem 1.11 we have that $\Gamma_\alpha(E(M))$ is an injective R-module, thus the sequence splits and $\mathrm{Ass}(E(M)) = \mathrm{Ass}(\Gamma_\alpha(E(M))) \cup \mathrm{Ass}(E(M)/\Gamma_\alpha(E(M)))$. We have also a natural monomorphism $M/\Gamma_\alpha(M) \to E(M)/\Gamma_\alpha(E(M))$ and this leads to

$$\mathrm{Ass}(M/\Gamma_\alpha(M)) \subseteq \mathrm{Ass}(E(M)/\Gamma_\alpha(E(M))) \subseteq \mathrm{Ass}(E(M)) = \mathrm{Ass}(M)$$

which concludes the statement. □

Proposition 1.31. *Let φ and ψ be families of R and M be an R-module which is J-torsion for some $J \in \langle\psi\rangle$. Then $\Gamma_\varphi(M) = \Gamma_{\varphi,\psi}(M)$. If in addition R is Noetherian, then $H^i_\varphi(M) = H^i_{\varphi,\psi}(M)$ for every i.*

Proof. Note that $\Gamma_\varphi(M) \subseteq \Gamma_{\varphi,\psi}(M)$ because $\tilde{W}(\langle\varphi\rangle, \{(0)\}) = \langle\varphi\rangle \subseteq \tilde{W}(\langle\varphi\rangle, \psi)$ by Lemma 1.5, (vi) and (vii). Since M is J-torsion, we have for every $x \in M$ that $V(\mathrm{Ann}(x)) \subseteq V(J) \subseteq \langle\psi\rangle$. Consider now an element $x \in \Gamma_{\varphi,\psi}(M)$. It follows that $V(\mathrm{Ann}(x)) \subseteq \tilde{W}(\langle\varphi\rangle, \psi) \cap \langle\psi\rangle \subseteq \langle\varphi\rangle$ and $x \in \Gamma_\varphi(M)$.

Suppose now that R is Noetherian. If M is J-torsion, then $E^i(M)$ is also J-torsion for every injective $E^i(M)$ in a minimal injective resolution of M by Corollary 1.26, (iii). We conclude that $\Gamma_\varphi(E^i(M)) = \Gamma_{\varphi,\psi}(E^i(M))$ and $H^i_\varphi(M) = H^i_{\varphi,\psi}(M)$ for every i. □

Corollary 1.32. *Let s be a positive integer, M be an R-module and $I_1, \ldots, I_s$ be ideals of R. If M is J-torsion for some $J \in \tilde{W}(I_2, \ldots, I_j)$, where $2 \leq j \leq s$ is an even integer or $j = s$, then $\Gamma_{I_1,\ldots,I_s}(M) = \Gamma_{I_1}(M)$. If in addition R is Noetherian, then $H^i_{I_1,\ldots,I_s}(M) = H^i_{I_1}(M)$ for every i.*

If $\mathfrak{a}$ and $\mathfrak{b}$ are ideals of the Noetherian ring R and E is an injective R-module, then as seen in [BS98, p. 53], the sequence

$$0 \longrightarrow \Gamma_{\mathfrak{a}+\mathfrak{b}}(E) \longrightarrow \Gamma_{\mathfrak{a}}(E) \oplus \Gamma_{\mathfrak{b}}(E) \longrightarrow \Gamma_{\mathfrak{a}\cap\mathfrak{b}}(E) \longrightarrow 0 \tag{1.2}$$

is exact. This result can be easily extended.

Proposition 1.33 (Mayer-Vietoris Sequence). *For every pair of families, α and β, of a Noetherian ring R and every R-module M, there exists an exact sequence $0 \to \Gamma_{\langle\alpha\rangle\cap\langle\beta\rangle}(M) \to \Gamma_{\alpha}(M) \oplus \Gamma_{\beta}(M) \to \Gamma_{\alpha\cup\beta}(M) \to H^1_{\langle\alpha\rangle\cap\langle\beta\rangle}(M) \to H^1_{\alpha}(M) \oplus H^1_{\beta}(M) \to H^1_{\alpha\cup\beta}(M) \to H^2_{\langle\alpha\rangle\cap\langle\beta\rangle}(M) \to \cdots$.*

Proof. It suffices to show that if E is an injective R-module, then the sequence of R-modules $0 \longrightarrow \Gamma_{\langle\alpha\rangle\cap\langle\beta\rangle}(E) \longrightarrow \Gamma_{\alpha}(E) \oplus \Gamma_{\beta}(E) \longrightarrow \Gamma_{\alpha\cup\beta}(E) \longrightarrow 0$ is exact. Exactness at $\Gamma_{\langle\alpha\rangle\cap\langle\beta\rangle}(E)$ and $\Gamma_{\alpha}(E) \oplus \Gamma_{\beta}(E)$ is clear. Now if $x \in \Gamma_{\alpha\cup\beta}(E)$, then we have that $\mathrm{Ann}(x) \supseteq I_1 \cdots I_r J_1 \cdots J_s$ for some $I_i \in \alpha$ and $J_j \in \beta$. Then $x \in \Gamma_{IJ}(E) = \Gamma_{I\cap J}(E)$, where $I = I_1 \cdots I_r \in \langle\alpha\rangle$ and $J = J_1 \cdots J_s \in \langle\beta\rangle$ and exist $x_1 \in \Gamma_I(E) \subseteq \Gamma_{\alpha}(E)$ and $x_2 \in \Gamma_J(E) \subseteq \Gamma_{\beta}(E)$ such that $x_1 - x_2 = x$ by Equation (1.2). □

1.2 Vanishing and non-vanishing

In this section we establish generalised versions of the classic vanishing and non-vanishing theorems from usual local cohomology theory.

From now on, we will assume that R is Noetherian. Every time $E = \bigoplus_{\mathfrak{p}\in\Lambda} E\left(R/\mathfrak{p}\right)$, where Λ is a family of prime ideals, we get that $\Gamma_{\varphi,\psi}(E) = \bigoplus_{\mathfrak{p}\in\Lambda\cap\tilde{W}(\langle\varphi\rangle,\psi)} E\left(R/\mathfrak{p}\right)$ for every pair of families, φ and ψ, of R by Corollary 1.26, (iv).

For every R-module M, every prime ideal $\mathfrak{p}$ and every non-negative integer i, we recall the definition of the i-th Bass number $\mu^i(\mathfrak{p}, M)$ of M with respect to $\mathfrak{p}$ as the cardinality of the set $\{\lambda \in \Lambda : \mathfrak{p}_\lambda = \mathfrak{p}\}$ of indices of prime ideals of the decomposition $E^i(M) = \bigoplus_{\lambda\in\Lambda} E\left(R/\mathfrak{p}_\lambda\right)$ of the i-th term in a minimal injective resolution of M. It is well known that the number $\mu^i(\mathfrak{p}, M)$ can be calculated also by the formula

$$\mu^i(\mathfrak{p}, M) = \dim_{\kappa(\mathfrak{p})} \mathrm{Ext}^i_{R_\mathfrak{p}}(\kappa(\mathfrak{p}), M_\mathfrak{p}),$$

where $\kappa(\mathfrak{p}) = R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$ is the residue field of the local ring $R_\mathfrak{p}$.

Next we write extensions of some results of [TYY09]. The following characterises the depth in terms of the non-vanishing of local cohomology modules.

Proposition 1.34. *If φ and ψ are families of R and M is a finitely generated R-module, then* $\inf\left\{i : H^i_{\varphi,\psi}(M) \neq 0\right\} = \inf\left\{\operatorname{depth}_{R_\mathfrak{p}}(M_\mathfrak{p}) : \mathfrak{p} \in \tilde{W}(\langle\varphi\rangle, \psi) \cap \operatorname{Spec} R\right\}$.

Proof. We set $n = \inf\left\{\operatorname{depth}_{R_\mathfrak{p}}(M_\mathfrak{p}) : \mathfrak{p} \in \tilde{W}(\langle\varphi\rangle, \psi) \cap \operatorname{Spec} R\right\}$ and a minimal injective resolution $(E^*(M), d^*)$ of M. If $\mathfrak{p} \in \tilde{W}(\langle\varphi\rangle, \psi) \cap \operatorname{Spec} R$, then

$$n \leq \operatorname{depth}_{R_\mathfrak{p}}(M_\mathfrak{p}) = \inf\left\{i : \mu^i(\mathfrak{p}, M) \neq 0\right\}.$$

Hence we have that $\Gamma_{\varphi,\psi}(E^i(M)) = \bigoplus_{\mathfrak{p} \in \tilde{W}(\langle\varphi\rangle,\psi)\cap \operatorname{Spec} R} E\left(R/\mathfrak{p}\right)^{\mu^i(\mathfrak{p},M)} = 0$ for $i < n$ and $\Gamma_{\varphi,\psi}(E^n(M)) \neq 0$. It follows that $H^i_{\varphi,\psi}(M) = 0$ for $i < n$ and

$$H^n_{\varphi,\psi}(M) = \ker \Gamma_{\varphi,\psi}(d^n) = \Gamma_{\varphi,\psi}(E^n(M)) \cap \ker d^n \neq 0$$

because $E^i(M)$ is an essential extension of $\ker d^i$ for each i. □

Corollary 1.35. *If α is a family of R and M is a finitely generated R-module, then* $\inf\left\{i : H^i_\alpha(M) \neq 0\right\} = \inf_{I\in\alpha} \operatorname{grade}(I, M)$.

We also write a converse for Proposition 1.28, (i).

Corollary 1.36. *Let M be a finitely generated R-module, φ be a non-trivial family and ψ be any family of R. Then, every time $H^i_{\varphi,\psi}(M) = 0$ for every $i > 0$, we have that M is a (φ, ψ)-torsion R-module.*

Proof. Let us assume first that R is a local ring with maximal ideal $\mathfrak{m}$. Setting $N = M/\Gamma_{\varphi,\psi}(M)$, we have by Proposition 1.28, (ii), that N is (φ, ψ)-torsion-free and $H^i_{\varphi,\psi}(N) \cong H^i_{\varphi,\psi}(M) = 0$ for $i > 0$. Since φ is non-trivial, we have that $\tilde{W}(\langle\varphi\rangle, \psi)$ is also non-trivial by Lemma 1.5, (vii), and $\mathfrak{m} \in \tilde{W}(\langle\varphi\rangle, \psi)$, whence

$$\inf\left\{\operatorname{depth}_{R_\mathfrak{p}}(N_\mathfrak{p}) : \mathfrak{p} \in \tilde{W}(\langle\varphi\rangle, \psi) \cap \operatorname{Spec} R\right\} \leq \operatorname{depth}(N).$$

If $N \neq 0$, then $H^i_{\varphi,\psi}(N) \neq 0$ for some $0 \leq i \leq \operatorname{depth}(N)$, which is absurd. Hence $\Gamma_{\varphi,\psi}(M) = M$.

If R is any ring, then $H^i_{\tilde{W}(\langle\varphi\rangle,\psi)R_\mathfrak{p}}(M_\mathfrak{p}) = 0$ for every $i \geq 1$ and every $\mathfrak{p} \in \operatorname{Spec} R$ by Lemma 1.18. Thus $\Gamma_{\tilde{W}(\langle\varphi\rangle,\psi)R_\mathfrak{p}}(M_\mathfrak{p}) = M_\mathfrak{p}$ for every $\mathfrak{p} \in \operatorname{Spec} R$ by the previous arguments and $\Gamma_{\varphi,\psi}(M) = M$ again by Lemma 1.18. □

Hence the class of finitely generated (φ, ψ)-torsion R-modules coincides with the class of finitely generated $\Gamma_{\varphi,\psi}$-acyclic R-modules. Furthermore:

Corollary 1.37. *Let M be a finitely generated R-module, φ be a non-trivial family and ψ be any family of R. The following conditions are equivalent:*

(i) M is $\Gamma_{\varphi,\psi}$-acyclic.

(ii) M is (φ, ψ)-torsion.

(iii) M is I-torsion for some $I \in \tilde{W}(\langle\varphi\rangle, \psi)$.

(iv) M is Γ_I-acyclic for some $I \in \tilde{W}(\langle\varphi\rangle, \psi)$.

We inherit an upper bound from the usual local cohomology for the non-vanishing of the cohomology modules.

Lemma 1.38. *Consider two families, φ and ψ, of R and an R-module M. Then $H^i_{\varphi,\psi}(M) = 0$ for every $i > \dim M$. In particular, $H^i_\alpha(M) = 0$ and $H^i_{I_s,\ldots,I_1}(M) = 0$ for every $i > \dim M$, every family α of R and ideals $I_1, \ldots, I_s$.*

Proof. For every $I \in \tilde{W}(\langle\varphi\rangle, \psi)$, Grothendieck's Vanishing Theorem (see [BS98, Theorem 6.1.2]) states that $H^i_I(M) = 0$ if $i > \dim M$. By Theorem 1.11, we have that $H^i_\alpha(M) \cong \varinjlim_{I \in \tilde{W}(\langle\varphi\rangle,\psi)} H^i_I(M) = 0$ if $i > \dim M$. □

It is readily observed that the class of zero-dimensional R-modules is contained in the class of Γ_α-acyclic R-modules for any family α of R.

Lemma 1.39. *Let n be a non-negative integer. If $H^i_{\varphi,\psi}(R) = 0$ for every $i > n$, then $H^i_{\varphi,\psi}(M) \cong H^i_{\varphi,\psi}(R) \otimes_R M$ for every $i \geq n$ and every R-module M.*

Proof. If $\tilde{W}(\langle\varphi\rangle, \psi)$ is trivial, the statement holds in an obvious way. Then we may assume that $\tilde{W}(\langle\varphi\rangle, \psi)$ is non-trivial and let us suppose initially that M is finitely generated. Then there exists a short exact sequence $0 \longrightarrow N \longrightarrow R^m \longrightarrow M \longrightarrow 0$ where m is a positive integer and N is a finitely generated R-module. For each i, this sequence induces the exact sequence $H^i_{\varphi,\psi}(R^m) \longrightarrow H^i_{\varphi,\psi}(M) \longrightarrow H^{i+1}_{\varphi,\psi}(N)$. We already observed that $H^i_{\varphi,\psi}(M) = 0$ when $i > \dim M$ in Lemma 1.38. So, we may assume the induction hypothesis: *if $H^i_{\varphi,\psi}(R) = 0$ for every $i > n+1$, then $H^i_{\varphi,\psi}(M) = 0$ for every $i > n+1$.* Thus $H^{i+1}_{\varphi,\psi}(N) = 0$ and $H^i_{\varphi,\psi}(M) = 0$ if $i > n$. If M is any R-module, then it is the inductive limit of its finitely generated submodules; hence we can conclude that $H^i_{\varphi,\psi}(M) = 0$ for every $i > n$.

Now the functor $H^n_{\varphi,\psi}(-)$ is R-linear, right-exact and preserves direct sums. Then the R-modules $H^n_{\varphi,\psi}(M)$ and $H^n_{\varphi,\psi}(R) \otimes_R M$ are isomorphic for every R-module M by Watts' Theorem (see [Rot09, Theorem 5.45]). □

The next result is a generalised version of the fundamental Grothendieck's Vanishing Theorem.

Theorem 1.40. *Let M be a finitely generated module over a local ring $(R, \mathfrak{m})$, φ be any family and ψ be a non-trivial family. Then $H^i_{\varphi,\psi}(M) = 0$ for every integer number $i > \sup_{J \in \psi} \dim(M/JM)$.*

Proof. Again, if φ is trivial, the result is obviously satisfied. The statement will be proved by induction on $r = \sup_{J \in \psi} \dim(M/JM)$. If $r = -1$, then $M = 0$ and $H^i_{\varphi,\psi}(M) = 0$ for any $i \geq 0$.

Now assume $r \geq 0$. Let us suppose also that $M = R$ is an integral domain and $H^l_{\varphi,\psi}(R) \neq 0$ for some $l > r$. Then there exists $\mathfrak{q} \in \operatorname{Ass}(H^l_{\varphi,\psi}(R))$. If $\mathfrak{q} \neq (0)$, choose a non-zero $x \in \mathfrak{q}$. From the exact sequence

$$0 \longrightarrow R \xrightarrow{\mu_x} R \longrightarrow R/Rx \longrightarrow 0$$

we get the exact sequence

$$H^{l-1}_{\varphi,\psi}(R/Rx) \longrightarrow H^l_{\varphi,\psi}(R) \xrightarrow{\mu_x} H^l_{\varphi,\psi}(R) .$$

As $\dim(R/(J + Rx)) \leq r - 1 < l - 1$ for every $J \in \psi$, we have that $H^{l-1}_{\varphi,\psi}(R/Rx) = 0$. Thus x is $H^l_{\varphi,\psi}(R)$-regular, which is absurd because $x \in \mathfrak{q} \in \operatorname{Ass}(H^l_{\varphi,\psi}(R))$. Then $\operatorname{Ass}(H^l_{\varphi,\psi}(R)) = \{(0)\}$. Since $H^l_{\varphi,\psi}(R)$ is a (φ, ψ)-torsion R-module, we may conclude that $(0) \in \tilde{W}(\langle\varphi\rangle, \psi)$ and any R-module is (φ, ψ)-torsion by Remark 1.19. This implies that $H^i_{\varphi,\psi}(R) = 0$ for every $i > 0$ and this leads to a contradiction.

Now if $R = M$ is not an integral domain, then the projection $\pi : R \to R/\mathfrak{p}$ leads to $H^i_{\varphi,\psi}(R/\mathfrak{p}) \cong H^i_{\varphi(R/\mathfrak{p}),\psi(R/\mathfrak{p})}(R/\mathfrak{p})$ for every i and every $\mathfrak{p} \in \operatorname{Spec} R$ by Theorem 1.14. Finally, if M is any finitely generated R-module, then we have a filtration of R-modules $0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_{s-1} \subsetneq M_s = M$ such that $M_j/M_{j-1} \cong R/\mathfrak{p}_j$ for some $\mathfrak{p}_j \in \operatorname{Supp}(M)$ and $j = 1, \ldots, s$. For every i and every j, we obtain exact sequences $0 \longrightarrow M_{j-1} \longrightarrow M_j \longrightarrow R/\mathfrak{p}_j \longrightarrow 0$ and $H^i_{\varphi,\psi}(M_{j-1}) \longrightarrow H^i_{\varphi,\psi}(M_j) \longrightarrow H^i_{\varphi,\psi}(R/\mathfrak{p}_j)$. Since $\dim(R/(J + \mathfrak{p}_j)) \leq \dim(R/(J + \operatorname{Ann}(M))) = \dim(M/JM) \leq r$ for every $J \in \psi$, we have that $H^i_{\varphi,\psi}(R/\mathfrak{p}_j) = 0$ for $i > r$ and every j, making the first arrow surjective. Hence we will have that $H^i_{\varphi,\psi}(M_j) = 0$ for every j. $\square$

Corollary 1.41. *Let $(R, \mathfrak{m})$ be a local ring, M be an R-module, φ be any family and ψ be a non-trivial family. Then $H^i_{\varphi,\psi}(M) = 0$ for every $i > \sup_{J \in \psi} \dim(R/J)$.*

Proof. M is the inductive limit of all its finitely generated submodules $\{M_\lambda\}_{\lambda \in \Lambda}$. Since $\dim(M_\lambda/JM_\lambda) \leq \dim(R/J)$ for every $\lambda \in \Lambda$ and every $J \in \psi$, we have that

$$H^i_{\varphi,\psi}(M) = \varinjlim_{\lambda \in \Lambda} H^i_{\varphi,\psi}(M_\lambda) = 0$$

when $i > \sup_{J \in \psi} \dim(R/J)$. $\square$

The upper bound considered in the Lemma 1.38 is slightly improved.

Proposition 1.42. *Let M be a finitely generated R-module and φ and ψ be families of R. Then $H^i_{\varphi,\psi}(M) = 0$ for every $i > 1 + \sup_{J \in \psi} \dim(M/JM)$.*

Proof. We prove the statement by induction over $r = \sup_{J\in\psi} \dim(M/JM)$. Let us suppose that $r = -1$. Then, for every $J \in \psi$, there exists $a_J \in J$ such that $(1+a_J)M = 0$. Thus $Jx = Rx$ for every $J \in \psi$ and every $x \in M$. From this, we have that $J + \operatorname{Ann}(x) \supseteq I$ for every $I \in \varphi$, every $J \in \psi$ and every $x \in M$, whence M is (φ, ψ)-torsion and $H^i_{\varphi,\psi}(M) = 0$ for $i > 0$ by Proposition 1.28, (i). When $r \geq 0$, the arguments are the same as those used in the proof of Theorem 1.40. □

Next we state a generalisation of the classic Grothendieck's Non-Vanishing Theorem. We recall the notation $\tilde{W}(\mathfrak{m})$ for the good family of all the ideals containing a power of $\mathfrak{m}$.

Theorem 1.43. *Let M be a finitely generated module over $(R, \mathfrak{m})$ and φ and ψ be non-trivial families of R such that $\varphi + \psi \subseteq \tilde{W}(\mathfrak{m})$. Then*

$$\sup\left\{i : H^i_{\varphi,\psi}(M) \neq 0\right\} = \sup_{J\in\psi} \dim(M/JM).$$

Proof. It suffices to show that $H^r_{\varphi,\psi}(M) \neq 0$ for $r = \sup_{J\in\psi} \dim(M/JM)$ by Theorem 1.40. Since $I + J \in \tilde{W}(\mathfrak{m})$ for every $I \in \varphi$ and every $J \in \psi$, it is straightforward to see that $\tilde{W}(\langle\varphi\rangle, \psi) = \tilde{W}(\mathfrak{m}, \psi)$ and $H^i_{\varphi,\psi}(-) = H^i_{\mathfrak{m},\psi}(-)$ for every i. Hence we may suppose $\varphi = \tilde{W}(\mathfrak{m})$. The exact sequence $0 \longrightarrow JM \longrightarrow M \longrightarrow M/JM \longrightarrow 0$ induces the exact sequence $H^r_{\mathfrak{m},\psi}(M) \longrightarrow H^r_{\mathfrak{m},\psi}(M/JM) \longrightarrow H^{r+1}_{\mathfrak{m},\psi}(JM)$ for each $J \in \psi$. If $J' \in \psi$, then $\dim(JM/J'JM) \leq \dim(M/J'JM) = \max\{\dim(M/JM), \dim(M/J'M)\} \leq r$. Thus $H^{r+1}_{\mathfrak{m},\psi}(JM) = 0$ by Theorem 1.40. Since M/JM is J-torsion, we have by Proposition 1.31 that $H^r_{\mathfrak{m},\psi}(M/JM) = H^r_{\mathfrak{m}}(M/JM)$. Now if $\dim(M/JM) = r$, then $H^r_{\mathfrak{m}}(M/JM) \neq 0$ by Grothendieck's Non-Vanishing Theorem (see [BS98, Theorem 6.1.4]). We conclude that $H^r_{\mathfrak{m},\psi}(M) \neq 0$, whence $H^r_{\varphi,\psi}(M) \neq 0$. □

Corollary 1.44. *Let M be a finitely generated module over $(R, \mathfrak{m})$ and $I_1, \ldots, I_s$ be ideals of R with I_1 and I_2 proper and $I_1+\mathfrak{p}$ is $\mathfrak{m}$-primary for every prime $\mathfrak{p} \in \tilde{W}(I_2, \ldots, I_s)$. Then $\sup\left\{i : H^i_{I_1,\ldots,I_j}(M) \neq 0\right\} = \sup_{J\in\tilde{W}(I_2,\ldots,I_j)} \dim(M/JM)$ for every even $2 \leq j \leq s$ and for $j = s$.*

Now we present a generalisation of the classic Lichtenbaum-Hartshorne Vanishing Theorem.

Theorem 1.45. *Let $(R, \mathfrak{m})$ be a local ring of dimension d and φ and ψ be non-trivial families of R. The following conditions are equivalent:*

(i) $H^d_{\varphi,\psi}(R) = 0$.

(ii) *For each prime ideal $\mathfrak{p}$ of $\hat{R}$ such that $\dim(\hat{R}/\mathfrak{p}) = d$ and $J\hat{R} \subseteq \mathfrak{p}$ for some $J \in \psi$, we have that $\dim(\hat{R}/(I\hat{R} + \mathfrak{p})) > 0$ for some $I \in \varphi$.*

Proof. Let us suppose that $H^d_{\varphi,\psi}(R) = 0$ and that there exists a prime ideal $\mathfrak{p}$ of $\hat{R}$ such that $\dim(\hat{R}/\mathfrak{p}) = d$, $J\hat{R} \subseteq \mathfrak{p}$ for some $J \in \psi$ and $\dim(\hat{R}/(I\hat{R}+\mathfrak{p})) \leq 0$ for every $I \in \varphi$. The first assumption gives $H^d_{\varphi,\psi}(\hat{R}/\mathfrak{p}) = 0$ because $H^i_{\varphi,\psi}(R) = 0$ for $i > d-1$ (see Corollary 1.41 and Lemma 1.39). On the other hand, the R-module $\hat{R}/\mathfrak{p}$ is J-torsion, whence $H^d_{\varphi,\psi}(\hat{R}/\mathfrak{p}) = H^d_{\varphi}(\hat{R}/\mathfrak{p}) \cong H^d_{\varphi(\hat{R}/\mathfrak{p})}(\hat{R}/\mathfrak{p})$ by Theorem 1.31 and Theorem 1.14. Since $(I\hat{R}+\mathfrak{p})/\mathfrak{p}$ is an $\mathfrak{m}\hat{R}/\mathfrak{p}$-primary ideal of the d-dimensional local ring $(\hat{R}/\mathfrak{p}, \mathfrak{m}\hat{R}/\mathfrak{p})$ for every proper ideal $I \in \varphi$, it follows from Theorem 1.43 that $H^d_{\varphi,\psi}(\hat{R}/\mathfrak{p}) \cong H^d_{\mathfrak{m}\hat{R}/\mathfrak{p}}(\hat{R}/\mathfrak{p}) \neq 0$ and this is a contradiction.

For the converse, let us suppose that $H^d_{\varphi,\psi}(R) \neq 0$ and the second condition. From Lemma 1.39 and Corollary 1.41 we have that $H^d_{\varphi,\psi}(\hat{R}) \cong H^d_{\varphi,\psi}(R) \otimes_R \hat{R}$, hence $H^d_{\varphi,\psi}(\hat{R}) \neq 0$ because $\hat{R}$ is a faithfully flat R-module. Consider a filtration

$$0 = K_0 \subsetneq K_1 \subsetneq \cdots \subsetneq K_{s-1} \subsetneq K_s = \hat{R}$$

of ideals of $\hat{R}$ such that $K_j/K_{j-1} \cong \hat{R}/\mathfrak{p}_j$ for some prime ideals $\mathfrak{p}_j$ of $\hat{R}$. Thus we have exact sequences $H^d_{\varphi,\psi}(K_{j-1}) \longrightarrow H^d_{\varphi,\psi}(K_j) \longrightarrow H^d_{\varphi,\psi}(\hat{R}/\mathfrak{p}_j)$. If every $\mathfrak{p}_j$ is such that $H^d_{\varphi,\psi}(\hat{R}/\mathfrak{p}_j) = 0$, then $H^d_{\varphi,\psi}(\hat{R}) = 0$; hence there must be a prime ideal $\mathfrak{p}$ of $\hat{R}$ such that $H^d_{\varphi,\psi}(\hat{R}/\mathfrak{p}) \neq 0$. Now we shall consider two possibilities:

(i) There exists $J \in \psi$ such that $J\hat{R} \subseteq \mathfrak{p}$: as $\hat{R}/\mathfrak{p}$ is a J-torsion R-module, we have that $H^d_{\varphi,\psi}(\hat{R}/\mathfrak{p}) = H^d_{\varphi}(\hat{R}/\mathfrak{p}) \cong H^d_{\varphi(\hat{R}/\mathfrak{p})}(\hat{R}/\mathfrak{p})$ by Theorem 1.31 and Theorem 1.14. If $\dim(\hat{R}/\mathfrak{p}) < d$, then $H^d_{\varphi(\hat{R}/\mathfrak{p})}(\hat{R}/\mathfrak{p}) = 0$, which is a contradiction. So $\dim(\hat{R}/\mathfrak{p}) = d$ and $\dim(\hat{R}/(I\hat{R}+\mathfrak{p})) > 0$ for some $I \in \varphi$ by our assumption. Consider the family $\varphi' = \{I^s J_1 \cdots J_s \text{ for some } J_i \in \varphi, s \geq 1\}$. Observe that φ' is stable under multiplication, whence $\varphi'(\hat{R}/\mathfrak{p})$ is also stable under multiplication. By Lichtenbaum-Hartshorne Vanishing Theorem (see [BS98, Theorem 8.2.1]), we get that $H^d_{\mathfrak{a}(\hat{R}/\mathfrak{p})}(\hat{R}/\mathfrak{p}) = 0$ for every $\mathfrak{a} \in \varphi'$ because $\dim(\hat{R}/(\mathfrak{a}\hat{R}+\mathfrak{p})) > 0$. Now $\langle\varphi'\rangle = \langle\varphi\rangle$, whence $H^i_{\varphi,\beta}(-) = H^i_{\varphi',\beta}(-)$ for every family β and every i. Thus we get from Theorem 1.14 and Remark 1.12 that

$$H^d_{\varphi(\hat{R}/\mathfrak{p})}(\hat{R}/\mathfrak{p}) \cong H^d_{\varphi'(\hat{R}/\mathfrak{p})}(\hat{R}/\mathfrak{p}) \cong \varinjlim_{\mathfrak{a}\in\varphi'} H^d_{\mathfrak{a}(\hat{R}/\mathfrak{p})}(\hat{R}/\mathfrak{p}) = 0$$

and this is a contradiction.

(ii) For all $J \in \psi$, we have that $J\hat{R} \nsubseteq \mathfrak{p}$: setting $\bar{R} = R/(\mathfrak{p} \cap R)$, we have that $\hat{R}/\mathfrak{p}$ is an $\bar{R}$-module and that $H^d_{\varphi,\psi}(\hat{R}/\mathfrak{p}) \cong H^d_{\varphi\bar{R},\psi\bar{R}}(\hat{R}/\mathfrak{p})$ by Theorem 1.14. If $J \in \psi$, we have that $J \nsubseteq \mathfrak{p} \cap R$ and thus $\dim(\bar{R}/J\bar{R}) < \dim \bar{R} \leq d$. We conclude from Corollary 1.41 that $H^d_{\varphi\bar{R},\psi\bar{R}}(\hat{R}/\mathfrak{p}) = 0$ and this is another contradiction.

□

Corollary 1.46. *Let α be a non-trivial family of the d-dimensional local ring $(R, \mathfrak{m})$. The following statements are equivalent:*

(i) $H^d_\alpha(R) = 0$.

(ii) *For each prime ideal* $\mathfrak{p}$ *of* $\hat{R}$ *such that* $\dim(\hat{R}/\mathfrak{p}) = \dim \hat{R}$*, there exists* $I \in \alpha$ *such that* $\dim(\hat{R}/(I\hat{R} + \mathfrak{p})) > 0$.

(iii) $H^d_I(R) = 0$ *for some proper* $I \in \alpha$.

Proof. Since $\langle\alpha\rangle = \tilde{W}(\langle\alpha\rangle, \{(0)\})$ for any family α by Lemma 1.5, (vi), we apply the previous theorem to the families $\varphi = \alpha$ and $\psi = \{(0)\}$ in order to obtain the equivalence (i)⇔(ii). The equivalence (ii)⇔(iii) is just the classic Lichtenbaum-Hartshorne Vanishing Theorem. □

Corollary 1.47. *Let* $(R, \mathfrak{m})$ *be a local ring of dimension* d *and* α *and* β *be non-trivial families of* R *such that* $\alpha \subseteq \langle\beta\rangle$*. Then* $H^d_\alpha(R) = 0$ *implies* $H^d_\beta(R) = 0$.

Corollary 1.48. *Let* $(R, \mathfrak{m})$ *be a local ring of dimension* d *and* $I_1, \ldots, I_s$ *be ideals of* R *with* I_1 *and* I_2 *proper. The following conditions are equivalent:*

(i) $H^d_{I_1,\ldots,I_s}(R) = 0$.

(ii) *For each prime ideal* $\mathfrak{p}$ *of* $\hat{R}$ *such that* $\dim(\hat{R}/\mathfrak{p}) = \dim \hat{R}$ *and* $J\hat{R} \subseteq \mathfrak{p}$ *for some* $J \in \tilde{W}(I_2, \ldots, I_j)$*, being* $2 \leq j \leq s$ *an even integer or* $j = s$*, we have that* $\dim(\hat{R}/(I_1\hat{R} + \mathfrak{p})) > 0$.

Corollary 1.49. *Let* $(R, \mathfrak{m})$ *be a local ring of dimension* d *and* $I_1, \ldots, I_s$ *be ideals of* R *with* I_1 *and* I_2 *proper. Consider the following statements:*

(i) $H^d_{I_1,\ldots,I_j}(R) = 0$ *for some odd integer* $1 \leq j \leq s$.

(ii) $H^d_{I_1,\ldots,I_s}(R) = 0$.

(iii) $H^d_{I_1,\ldots,I_j}(R) = 0$ *for every even integer* $2 \leq j \leq s$.

Then (i)⇒(ii)⇒(iii).

1.3 Local duality

In this section we prove some results related to local duality for the local cohomology modules with respect to a pair of families.

Lemma 1.50. *Let* $(R, \mathfrak{m})$ *be a Cohen-Macaulay local ring of dimension* d *and* ψ *be a family of* R*. If there exists* $J \in \psi$ *contained in a perfect ideal* I *of grade* t *(this is,* $\mathrm{gr}(I, R) = \mathrm{proj.dim}(R/I) = t$*), then* $\mathrm{ht}\,\mathfrak{p} \geq d - t$ *for every prime ideal* $\mathfrak{p} \in \tilde{W}(\mathfrak{m}, \psi)$.

Proof. Since $\tilde{W}(\mathfrak{m},\psi) = \tilde{W}(\mathfrak{m},\langle\psi\rangle)$, we may assume that J is a perfect ideal of grade t. Observe also that $\tilde{W}(\mathfrak{m},\psi) = \bigcap_{K\in\langle\psi\rangle} \tilde{W}(\mathfrak{m},K) \subseteq \tilde{W}(\mathfrak{m},J)$ by Lemma 1.9. Thus $\operatorname{ht}\mathfrak{p} \geq d-t$ for every prime ideal $\mathfrak{p} \in \tilde{W}(\mathfrak{m},\psi)$ by [TYY09, Lemma 5.2]. □

For any module M over a local ring $(R,\mathfrak{m})$, the Matlis dual of M, denoted by $M^\vee$, is the module $\operatorname{Hom}_R(M, E_R(R/\mathfrak{m}))$. It can be seen that $(-)^\vee : R\text{-mod} \to R\text{-mod}$ is a contravariant exact R-linear functor.

We recall the definition of canonical module of a ring which is an important object in classical local duality theory.

Definition 1.51. Let $(R,\mathfrak{m})$ be a local ring of dimension n. A finitely generated R-module K_R is said to be a *canonical module* of R when $K_R \cong H^n_{\mathfrak{m}}(R)^\vee$.

The following result gives a characterisation of the associated prime ideals of the top local cohomology module of the canonical module.

Proposition 1.52. *Let $(R,\mathfrak{m})$ be a Cohen-Macaulay local ring of dimension d with canonical module K_R and ψ be a family of R. Let us suppose that there exists $J \in \psi$ contained in a perfect ideal of grade t. Then*

$$\operatorname{Ass}(H^{d-t}_{\mathfrak{m},\psi}(K_R)) = \left\{\mathfrak{p} \in \tilde{W}(\mathfrak{m},\psi) \cap \operatorname{Spec} R : \operatorname{ht}\mathfrak{p} = d-t\right\}.$$

Proof. As $\tilde{W}(\mathfrak{m},\psi) = \tilde{W}(\mathfrak{m},\langle\psi\rangle)$ by Lemma 1.5, (viii), we may suppose that J is a perfect ideal of grade t. Let $(E^*(K_R),\partial^*)$ be a minimal injective resolution of K_R. Then for each i, $E^i(K_R) = \bigoplus_{\substack{\mathfrak{p}\in\operatorname{Spec} R\\ \operatorname{ht}\mathfrak{p}=i}} E(R/\mathfrak{p})$ by [BH98, Theorem 3.3.10], whence $\Gamma_{\mathfrak{m},\psi}(E^i(K_R)) = \bigoplus_{\substack{\mathfrak{p}\in\tilde{W}(\mathfrak{m},\psi)\cap\operatorname{Spec} R\\ \operatorname{ht}\mathfrak{p}=i}} E(R/\mathfrak{p})$. Since $\operatorname{ht}\mathfrak{p} \geq d-t$ for every $\mathfrak{p} \in \tilde{W}(\mathfrak{m},\psi)$ by Lemma 1.50, we have that $H^{d-t}_{\mathfrak{m},\psi}(K_R) = \ker\partial^{d-t} \cap \Gamma_{\mathfrak{m},\psi}(E^{d-t}(K_R))$ and there is an exact sequence $0 \longrightarrow H^{d-t}_{\mathfrak{m},\psi}(K_R) \longrightarrow \bigoplus_{\substack{\mathfrak{p}\in\tilde{W}(\mathfrak{m},\psi)\cap\operatorname{Spec} R\\ \operatorname{ht}\mathfrak{p}=d-t}} E(R/\mathfrak{p}) \longrightarrow \bigoplus_{\substack{\mathfrak{p}\in\tilde{W}(\mathfrak{m},\psi)\cap\operatorname{Spec} R\\ \operatorname{ht}\mathfrak{p}=d-t+1}} E(R/\mathfrak{p})$. This implies that $\operatorname{Ass}(H^{d-t}_{\mathfrak{m},\psi}(K_R)) \subseteq \left\{\mathfrak{p} \in \tilde{W}(\mathfrak{m},\psi) \cap \operatorname{Spec} R : \operatorname{ht}\mathfrak{p} = d-t\right\}$.

Conversely, if $\mathfrak{p} \in \tilde{W}(\mathfrak{m},\psi)$ is a prime ideal such that $\operatorname{ht}\mathfrak{p} = d-t$, then we have that $(H^{d-t}_{\mathfrak{m},\psi}(K_R))_{\mathfrak{p}} = E_{R_{\mathfrak{p}}}(\kappa(\mathfrak{p})) \supseteq \kappa(\mathfrak{p})$. Hence $\mathfrak{p} \in \operatorname{Min}(H^{d-t}_{\mathfrak{m},\psi}(K_R)) \subseteq \operatorname{Ass}(H^{d-t}_{\mathfrak{m},\psi}(K_R))$ and the statement is proved. □

Now we prove the main result of this section.

Theorem 1.53. *Let $(R,\mathfrak{m})$ be a Cohen-Macaulay complete local ring of dimension d and ψ be a family of R. Consider $t = d - \sup_{J\in\psi}\dim(R/J)$ and assume that there exists* *a perfect ideal $J \in \psi$ such that $\dim(R/J) = d - t$. Then for any finitely generated R-module M, there is a functorial isomorphism $H^{d-i}_{\mathfrak{m},\psi}(M)^\vee \cong \operatorname{Ext}^{i-t}_R(M, K)$ where $K = H^{d-t}_{\mathfrak{m},\psi}(R)^\vee$.*

Proof. Setting $T^j(-) = H^{d-t-j}_{\mathfrak{m},\psi}(-)^\vee$, we shall show the isomorphism of functors

$$T^j(-) \cong \operatorname{Ext}^j_R(-, K).$$

There exists a perfect ideal $J \in \psi$ such that $\dim(R/J) = d - t$. We have an isomorphism $H^{d-t}_{\mathfrak{m},\psi}(M) \cong H^{d-t}_{\mathfrak{m},\psi}(R) \otimes_R M$ for any R-module M by Corollary 1.41 and Lemma 1.39, hence $T^0(M) \cong (M \otimes_R H^{d-t}_{\mathfrak{m},\psi}(R))^\vee \cong \operatorname{Hom}(M, K)$. Every time we have an exact sequence $0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$, we obtain the long exact sequences $\cdots \longrightarrow H^{d-t-1}_{\mathfrak{m},\psi}(N) \longrightarrow H^{d-t}_{\mathfrak{m},\psi}(L) \longrightarrow H^{d-t}_{\mathfrak{m},\psi}(M) \longrightarrow H^{d-t}_{\mathfrak{m},\psi}(N) \longrightarrow 0$ and $0 \longrightarrow T^0(N) \longrightarrow T^0(M) \longrightarrow T^0(L) \longrightarrow T^1(N) \longrightarrow \cdots$. For any free R-module R^n we have that $H^{d-t-j}_{\mathfrak{m},\psi}(R^n)^\vee \cong (H^{d-t-j}_{\mathfrak{m},\psi}(R)^\vee)^n$ for $j > 0$ by Theorem 1.24 and [Rot09, Theorem 2.31]. Thus $H^{d-t-j}_{\mathfrak{m},\psi}(R^n)^\vee = 0$ because $\operatorname{depth}_{R_\mathfrak{p}}(R_\mathfrak{p}) = \operatorname{ht}\mathfrak{p} \geq d - t$ for every prime ideal $\mathfrak{p} \in \tilde{W}(\mathfrak{m}, \psi)$, see Proposition 1.34 and Lemma 1.50. Then there exists a unique isomorphism $T^j(-) \cong \operatorname{Ext}^j_R(-, K)$ for each j and the isomorphism $H^{d-i}_{\mathfrak{m},\psi}(M)^\vee \cong \operatorname{Ext}^{i-t}_R(M, H^{d-t}_{\mathfrak{m},\psi}(R)^\vee)$ holds for every finitely generated R-module M. □

For any ring R, any ideal J of R and any R-module M, we denote the completion for M with respect to the J-adic topology as $M^\wedge_J$.

Theorem 1.54. *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension d with canonical module K_R and ψ be a family of R. For every $J \in \psi$, there is a natural isomorphism $H^{d-t}_{\mathfrak{m},\psi}(R)^\wedge_J \cong H^t_J(K_R)^\vee$, where $t = d - \sup\limits_{J \in \psi} \dim(R/J)$.*

Proof. For every $J \in \psi$ and every $n \in \mathbb{N}$, we have the isomorphisms

$$\begin{aligned} H^{d-t}_{\mathfrak{m},\psi}(R)/J^n H^{d-t}_{\mathfrak{m},\psi}(R) &\cong H^{d-t}_{\mathfrak{m},\psi}(R) \otimes R/J^n \\ &\cong H^{d-t}_{\mathfrak{m},\psi}(R/J^n) && \text{(by Lemma 1.39)} \\ &\cong H^{d-t}_{\mathfrak{m}}(R/J^n) && \text{(by Proposition 1.31)} \\ &\cong \operatorname{Ext}^t_R(R/J^n, K_R)^\vee && \text{(by [BS98, Theorem 12.1.20, (ii)]).} \end{aligned}$$

Thus

$$\begin{aligned} H^{d-t}_{\mathfrak{m},\psi}(R)^\wedge_J &\cong \varprojlim_{n \in \mathbb{N}} \operatorname{Ext}^t_R(R/J^n, K_R)^\vee \\ &\cong \left(\varinjlim_{n \in \mathbb{N}} \operatorname{Ext}^t_R(R/J^n, K_R)\right)^\vee && \text{(by [Rot09, Proposition 5.26])} \\ &\cong H^t_J(K_R)^\vee && \text{(by [BS98, Theorem 1.3.8]).} \end{aligned}$$

□

For any pair of R-modules, M and N, and any family α of R, we can define the generalised local cohomology as

$$H^i_\alpha(M,N) := \varinjlim_{I\in\langle\alpha\rangle} H^i_I(M,N) = \varinjlim_{\substack{n\in\mathbb{N}\\ I\in\langle\alpha\rangle}} \operatorname{Ext}^i_R(M/I^nM,N) = \varinjlim_{I\in\langle\alpha\rangle} \operatorname{Ext}^i_R(M/IM,N).$$

We generalise a result of [TYY09].

Proposition 1.55. *Let $(R,\mathfrak{m})$ be a Gorenstein local ring of dimension d and ψ be a family such that R is J-adically complete for some $J\in\psi$. Then there is an isomorphism $\Gamma_{\mathfrak{m},\psi}(M)\cong H^d_\psi(M,R)^\vee$ for every finitely generated R-module M.*

Proof. The family $\psi' = \{J^sI_1\cdots I_s \text{ for some } I_i\in\psi, s\geq 1\}$ is cofinal to $\langle\psi\rangle$ and every $I\in\psi'$ is such that R is I-adically complete. Thus $\Gamma_{\mathfrak{m},I}(M) = H^d_I(M,R)^\vee$ for every $I\in\psi'$ by [TYY09, Theorem 5.7] and

$$\begin{aligned}
\Gamma_{\mathfrak{m},\psi}(M) &\cong \varprojlim_{I\in\langle\psi\rangle}\Gamma_{\mathfrak{m},I}(M) && \text{(Lemma 1.9)}\\
&\cong \varprojlim_{I\in\psi'}\Gamma_{\mathfrak{m},I}(M)\\
&\cong \varprojlim_{I\in\psi'} H^d_I(M,R)^\vee\\
&\cong \left(\varinjlim_{I\in\psi'} H^d_I(M,R)\right)^\vee && \text{([Rot09, Proposition 5.26])}\\
&\cong \left(\varinjlim_{I\in\langle\psi\rangle} H^d_I(M,R)\right)^\vee\\
&\cong H^d_\psi(M,R)^\vee.
\end{aligned}$$

□

If the d-dimensional ring R admits D_R as a dualising complex, we denote by K_M the canonical module of the r-dimensional R-module M, which is defined as

$$K_M = H^{d-r}(\mathbf{R}\operatorname{Hom}_R(M,D_R)).$$

We generalise another result of [TYY09].

Proposition 1.56. *Let $(R,\mathfrak{m})$ be a complete local ring, α be a family of R and M be a finitely generated R-module of dimension r. Then we have an isomorphism*

$$H^r_\alpha(M)^\vee \cong \Gamma_{\mathfrak{m},\alpha}(K_M).$$

Proof.

$$\begin{aligned} H^r_\alpha(M)^\vee &\cong \left(\varinjlim_{I\in\langle\alpha\rangle} H^r_I(M)\right)^\vee \\ &\cong \varprojlim_{I\in\langle\alpha\rangle} H^r_I(M)^\vee \\ &\cong \varprojlim_{I\in\langle\alpha\rangle} \Gamma_{\mathfrak{m},I}(K_M) \\ &\cong \Gamma_{\mathfrak{m},\alpha}(K_M). \end{aligned}$$

□

Chapter 2

Top local cohomology modules

2.1 Artinianness and cohomological dimension

As seen in [BS98, Chapter 7], it is well known that $H^n_I(M)$ is an Artinian R-module for any n-dimensional R-module M and any ideal I of R. The purpose of this section is exactly to show the same property for any family of R. Unless stated explicitly, all the rings in the following sections are local Noetherian.

Lemma 2.1. *Let M be a finitely generated R-module, φ and ψ be families of R and set $l = \sup_{J\in\psi} \dim(M/JM)$. Then $H^i_{\varphi,\psi}(M) \cong H^i_{\varphi,\psi}(R/\operatorname{Ann}(M)) \otimes_R M$ for every $i \geq l$.*

Proof. Set $\bar{R} = R/\operatorname{Ann}(M)$. Thus $H^i_{\varphi\bar{R},\psi\bar{R}}(\bar{R}) = 0$ for $i > \sup_{J\in\psi} \dim(\bar{R}/J\bar{R}) = l$ by Theorem 1.40 and both sides of the claimed isomorphism are equal to zero, so it suffices to prove it for $i = l$. By Lemma 1.39 we have the isomorphism of $\bar{R}$-modules $H^l_{\varphi\bar{R},\psi\bar{R}}(M) \cong H^l_{\varphi\bar{R},\psi\bar{R}}(\bar{R}) \otimes_{\bar{R}} M$ and since $M = M \otimes_R \bar{R}$, we have that $H^l_{\varphi\bar{R},\psi\bar{R}}(\bar{R}) \otimes_{\bar{R}} M \cong H^l_{\varphi\bar{R},\psi\bar{R}}(\bar{R}) \otimes_R M$. From Theorem 1.14 we get the isomorphisms of $\bar{R}$-modules $H^l_{\varphi\bar{R},\psi\bar{R}}(M) \cong H^l_{\varphi,\psi}(M)$ and $H^l_{\varphi\bar{R},\psi\bar{R}}(\bar{R}) \cong H^l_{\varphi,\psi}(\bar{R})$. We conclude that the $\bar{R}$-modules $H^l_{\varphi,\psi}(M)$ and $H^l_{\varphi,\psi}(\bar{R}) \otimes_R M$ are isomorphic. By reducing scalars we obtain the statement. □

If $\tilde{W}(\langle\varphi\rangle, \psi) \subseteq \tilde{W}(\langle\varphi'\rangle, \psi')$, then we have natural maps $H^i_{\varphi,\psi}(-) \to H^i_{\varphi',\psi'}(-)$ for all i. Moreover, the top cohomology functor $H^d_{\varphi,\psi}(-)$ displays a dual behaviour with respect to $H^0_{\varphi,\psi}(-)$ in the following sense.

Theorem 2.2. *Let M be a finitely generated R-module of dimension d and let α and β be families of R such that $\{R\} \subsetneq \langle\alpha\rangle \subseteq \langle\beta\rangle$. Then the natural map $H^d_\alpha(M) \to H^d_\beta(M)$ is surjective. In particular $H^d_\alpha(M)$ is Artinian, more precisely, it is a quotient of $H^d_{\mathfrak{m}}(M)$.*

Proof. The proof will be done in several steps.

Step 1: Suppose that R is complete Gorenstein of dimension d and $M = R$. Then every element of a minimal injective resolution (E^i, ∂^i) of R is of the form $E^i = \bigoplus_{\operatorname{ht}\mathfrak{p}=i} E(R/\mathfrak{p})$, where $E(R/\mathfrak{p})$ is the injective hull of the R-module $R/\mathfrak{p}$. Since $\Gamma_\alpha(E^d) = E^d$ when α is a non-trivial family and $\Gamma_\alpha(E^{d-1}) \subseteq \Gamma_\beta(E^{d-1})$, we have that $\operatorname{im}\Gamma_\alpha(\partial^{d-1}) \subseteq \operatorname{im}\Gamma_\beta(\partial^{d-1})$ and the homomorphism

$$H^d_\alpha(R) = E^d/\operatorname{im}\Gamma_\alpha(\partial^{d-1}) \to E^d/\operatorname{im}\Gamma_\beta(\partial^{d-1}) = H^d_\beta(R)$$

is surjective.

Step 2: Suppose now that R is complete of dimension d and $M = R$. By Cohen Structure Theorem there exists a complete regular (hence Gorenstein) local ring $(S, \mathfrak{n})$ of dimension d and a surjective ring homomorphism $\phi : S \to R$. Then Theorem 1.14 says that $H^d_\alpha(R) \cong H^d_{\phi^{-1}(\alpha)}(R)$, where $\phi^{-1}(\gamma)$ is the family $\{\phi^{-1}(I) : I \in \gamma\}$ of ideals of S. Observe that $H^d_{\phi^{-1}(\alpha)}(S) \to H^d_{\phi^{-1}(\beta)}(S)$ is surjective by the previous step. Thus $H^d_{\phi^{-1}(\alpha)}(S) \otimes_S S/K \to H^d_{\phi^{-1}(\beta)}(S) \otimes_S S/K$ is also surjective, where $K = \ker\phi$. Now $H^d_{\phi^{-1}(\alpha)}(R) \cong H^d_{\phi^{-1}(\alpha)}(S) \otimes_S S/K$ and $H^d_{\phi^{-1}(\beta)}(R) \cong H^d_{\phi^{-1}(\beta)}(S) \otimes_S S/K$ by Lemma 1.39. Hence $H^d_\alpha(R) \to H^d_\beta(R)$ is surjective.

Step 3: Suppose that R is any ring of dimension d and $M = R$. Lemma 1.18 gives that $H^d_{\alpha\hat{R}}(\hat{R}) \cong H^d_\alpha(R) \otimes_R \hat{R}$, where $\hat{R}$ is the $\mathfrak{m}$-adic completion of R, and the surjectivity of $H^d_\alpha(R) \to H^d_\beta(R)$ comes from the surjectivity of the natural map of $\hat{R}$-modules $H^d_{\alpha\hat{R}}(\hat{R}) \to H^d_{\beta\hat{R}}(\hat{R})$ by step 2.

Step 4: Suppose that $\dim M = \dim R = d$. Then, by step 3, the natural map $H^d_\alpha(R) \to H^d_\beta(R)$ is surjective. It follows that $H^d_\alpha(R) \otimes M \to H^d_\beta(R) \otimes M$ is also surjective. Thus $H^d_\alpha(M) \to H^d_\beta(M)$ is surjective by Lemma 1.14.

Step 5: In general, we have that $H^d_\alpha(M) \cong H^d_{\alpha\bar{R}}(M)$ for $\bar{R} = R/\operatorname{Ann}(M)$ and every family α of ideals of R by Theorem 1.14. Hence we obtain that the natural map $H^d_\alpha(M) \to H^d_\beta(M)$ is also surjective by step 4.

Finally, since α is a non-trivial family, we have that $\tilde{W}(\mathfrak{m}) \subseteq \langle\alpha\rangle$. Then the natural map $H^d_\mathfrak{m}(M) \to H^d_\alpha(M)$ is surjective and we conclude that $H^d_\alpha(M)$ is Artinian because $H^d_\mathfrak{m}(M)$ is Artinian by [BS98, Theorem 7.1.3]. $\square$

We shall observe that the Artinianness of the top local cohomology $H^d_\alpha(M)$ was proved in [DANT02, Theorem 2.6].

Corollary 2.3. *Let M be a finitely generated R-module of dimension d and let φ, φ', ψ and ψ' be families of R such that $\{R\} \subsetneq \tilde{W}(\langle\varphi\rangle, \psi) \subseteq \tilde{W}(\langle\varphi'\rangle, \psi')$. Then the natural map $H^d_{\varphi,\psi}(M) \to H^d_{\varphi',\psi'}(M)$ is surjective. In particular $H^d_{\varphi,\psi}(M)$ is Artinian, more precisely, it is a quotient of $H^d_\mathfrak{m}(M)$.*

The inclusion $\tilde{W}(\langle\varphi\rangle, \psi) \subseteq \tilde{W}(\langle\varphi'\rangle, \psi')$ holds whenever $\varphi \subseteq \langle\varphi'\rangle$ and $\langle\psi\rangle \supseteq \psi'$. Thus we can state the following.

Corollary 2.4. *Let M be a finitely generated R-module of dimension d, $s \leq t$ be positive integers and $\{I_1, \ldots, I_s\}$ and $\{J_1, \ldots, J_t\}$ be two sets of ideals such that I_1 and J_1 are proper.*

- *If s is odd, $\sqrt{I_i} \supseteq \sqrt{J_i}$ for every odd $1 \leq i \leq s$ and $\sqrt{I_j} \subseteq \sqrt{J_j}$ for every even $2 \leq j \leq s-1$, then there exists a natural map $H^d_{I_1,\ldots,I_s}(M) \to H^d_{J_1,\ldots,J_t}(M)$ which is surjective and $H^d_{I_1,\ldots,I_s}(M)$ is a quotient of $H^d_{\mathfrak{m}}(M)$, hence Artinian.*
- *If s is even, $\sqrt{J_j} \subseteq \sqrt{I_j}$ for every even $2 \leq j \leq s$ and $\sqrt{J_i} \supseteq \sqrt{I_i}$ for every odd $1 \leq i \leq s-1$, then there exists a natural map $H^d_{J_1,\ldots,J_t}(M) \to H^d_{I_1,\ldots,I_s}(M)$ which is surjective and $H^d_{J_1,\ldots,J_t}(M)$ is a quotient of $H^d_{\mathfrak{m}}(M)$, hence Artinian.*

Proof. The first situation gives that $\tilde{W}(\mathfrak{m}) \subseteq \tilde{W}(I_1, \ldots, I_s) \subseteq \tilde{W}(J_1, \ldots, J_t)$, while the second one gives that $\tilde{W}(\mathfrak{m}) \subseteq \tilde{W}(J_1, \ldots, J_t) \subseteq \tilde{W}(I_1, \ldots, I_s)$. □

For any R-module M, set $l = \sup_{J\in\psi} \dim(M/JM)$. The R-module $H^l_{\varphi,\psi}(M)$ is not always Artinian (see for example Proposition 1.52), but the following property holds.

Theorem 2.5. *Let M be a finitely generated R-module, φ and ψ be families of R and consider $l = \sup_{J\in\psi} \dim(M/JM)$. Then $H^l_{\varphi,\psi}(M)/JH^l_{\varphi,\psi}(M)$ is Artinian for every $J \in \psi$.*

Proof. We shall prove the statement by induction on $d = \dim M$. If $d = 0$, then $l \leq 0$ and $H^l_{\varphi,\psi}(M)$ is Artinian, whence $H^l_{\varphi,\psi}(M)/JH^l_{\varphi,\psi}(M)$ is Artinian for every $J \in \psi$.

Suppose now that $d > 0$. Let us suppose first that M is ψ-torsion-free, this is, $\Gamma_\psi(M) = 0$. Then, for every $J \in \psi$ there exists an M-regular element $x \in J$ by Corollary 1.27, (i). The exact sequence

$$0 \longrightarrow M \xrightarrow{\mu_x} M \longrightarrow M/xM \longrightarrow 0$$

leads to the exact sequence

$$H^l_{\varphi,\psi}(M) \xrightarrow{\mu_x} H^l_{\varphi,\psi}(M) \longrightarrow H^l_{\varphi,\psi}(M/xM) \longrightarrow 0$$

by Theorem 1.40. Set $N = M/xM$ and $r = \sup_{K\in\psi} \dim(N/KN)$. Then $r \leq l$ and $H^i_{\varphi,\psi}(N)/KH^i_{\varphi,\psi}(N)$ is Artinian for every $i \geq r$ and every $K \in \psi$ by inductive hypothesis. We also obtain the exact sequence

$$\frac{H^l_{\varphi,\psi}(M)}{JH^l_{\varphi,\psi}(M)} \xrightarrow{\mu_x} \frac{H^l_{\varphi,\psi}(M)}{JH^l_{\varphi,\psi}(M)} \longrightarrow \frac{H^l_{\varphi,\psi}(N)}{JH^l_{\varphi,\psi}(N)} \longrightarrow 0$$

and since $x \in J$ we conclude that $H^l_{\varphi,\psi}(M)/JH^l_{\varphi,\psi}(M) \cong H^l_{\varphi,\psi}(N)/JH^l_{\varphi,\psi}(N)$.

Now if M is not ψ-torsion-free, consider the following short exact sequence of R-modules $0 \longrightarrow \Gamma_\psi(M) \longrightarrow M \longrightarrow M/\Gamma_\psi(M) \longrightarrow 0$. Consider also the integer numbers $r = \sup_{K\in\psi} \dim((M/\Gamma_\psi(M))/K(M/\Gamma_\psi(M)))$ and $k = \sup_{K\in\psi} \dim(\Gamma_\psi(M)/K\Gamma_\psi(M))$. We observe that $\max\{r,k\} \leq l$, thus we obtain the exact sequence of R-modules $H^l_{\varphi,\psi}(\Gamma_\psi(M)) \longrightarrow H^l_{\varphi,\psi}(M) \longrightarrow H^l_{\varphi,\psi}(M/\Gamma_\psi(M)) \longrightarrow 0$ which induces the exact sequence

$$\frac{H^l_{\varphi,\psi}(\Gamma_\psi(M))}{JH^l_{\varphi,\psi}(\Gamma_\psi(M))} \longrightarrow \frac{H^l_{\varphi,\psi}(M)}{JH^l_{\varphi,\psi}(M)} \longrightarrow \frac{H^l_{\varphi,\psi}(M/\Gamma_\psi(M))}{JH^l_{\varphi,\psi}(M/\Gamma_\psi(M))} \longrightarrow 0 \tag{2.1}$$

for every $J \in \psi$. We also observe that $\dim \Gamma_\psi(M) \leq l$ by Proposition 1.21, (iv), whence $H^l_{\varphi,\psi}(\Gamma_\psi(M))$ is Artinian by Theorem 2.2. Furthermore, the right-hand side of equation (2.1) is Artinian by the previous case. Then the statement follows. □

Corollary 2.6. *Let M be a finitely generated R-module and $I_1,\ldots,I_s$ be ideals of R. Then $H^l_{I_s,\ldots,I_1}(M)/JH^l_{I_s,\ldots,I_1}(M)$ is Artinian for every $J \in \tilde{W}(I_{s-1},\ldots,I_1)$, where $l = \sup\left\{\dim(M/JM) : J \in \tilde{W}(I_{s-1},\ldots,I_1)\right\}$.*

When $s = 2$, the above corollary was stated in [CW09, Theorem 2.3].

Theorem 2.7. *Let M be a finitely generated R-module and α be a non-trivial family of R. Then*

$$\inf\left\{i : H^i_\alpha(M) \text{ is not Artinian}\right\} = \inf\left\{\operatorname{depth}_{R_\mathfrak{p}}(M_\mathfrak{p}) : \mathfrak{p} \in \langle\alpha\rangle \cap \operatorname{Spec} R - \{\mathfrak{m}\}\right\}.$$

Proof. We set $n = \inf\left\{\operatorname{depth}_{R_\mathfrak{p}}(M_\mathfrak{p}) : \mathfrak{p} \in \langle\alpha\rangle \cap \operatorname{Spec} R - \{\mathfrak{m}\}\right\}$ and a minimal injective resolution $(E^*(M), d^*)$ of M. Thus $\Gamma_\alpha(E^i(M)) = E(R/\mathfrak{m})^{\mu^i(\mathfrak{m},M)}$ for every $i < n$ by Corollary 1.26, (iv). Since $E(R/\mathfrak{m})$ is Artinian and $\mu^i(\mathfrak{m}, M)$ is finite, we have that $\Gamma_\alpha(E^i(M))$ is Artinian too for $i < n$ and so is $H^i_\alpha(M)$. This implies that $\inf\{i : H^i_\alpha(M) \text{ is not Artinian}\} \geq n$.

For the other inequality we observe that there exists a prime ideal $\mathfrak{q} \neq \mathfrak{m}$ in $\langle\alpha\rangle$ such that $\mu^n(\mathfrak{q}, M) > 0$. Thus $\mathfrak{q} \in \operatorname{Ass}_R(\Gamma_\alpha(E^n(M)))$ by Corollary 1.26, (iv). Then $\Gamma_\alpha(E^n(M))$ is not Artinian. Now $\Gamma_\alpha(E^n(M))$ is an essential extension of $\ker\Gamma_\alpha(d^n)$, leading to $\ker\Gamma_\alpha(d^n)$ not being Artinian. On the other hand, $\operatorname{im}\Gamma_\alpha(d^{n-1})$ is Artinian. Thus the exact sequence $0 \longrightarrow \operatorname{im}\Gamma_\alpha(d^{n-1}) \longrightarrow \ker\Gamma_\alpha(d^n) \longrightarrow H^n_\alpha(M) \longrightarrow 0$ implies that $H^n_\alpha(M)$ is not Artinian. □

Corollary 2.8. *Let M be a finitely generated R-module and φ and ψ be families of R. Then*

$$\inf\left\{i : H^i_{\varphi,\psi}(M) \text{ is not Artinian}\right\} = \inf\left\{\operatorname{depth}_{R_\mathfrak{p}}(M_\mathfrak{p}) : \mathfrak{p} \in \tilde{W}(\langle\varphi\rangle,\psi) \cap \operatorname{Spec} R - \{\mathfrak{m}\}\right\}.$$

Corollary 2.9. *Let M be a finitely generated R-module and α and β be families of R such that $\alpha \subseteq \langle\beta\rangle$. Then*

$$\inf\left\{i : H^i_\beta(M) \text{ is not Artinian}\right\} \leq \inf\left\{i : H^i_\alpha(M) \text{ is not Artinian}\right\}.$$

In particular,

$$\inf\left\{i : H^i_{I_1,\ldots,I_j}(M) \textit{ is not Artinian}\right\} \leq \inf\left\{i : H^i_{I_1,\ldots,I_s}(M) \textit{ is not Artinian}\right\}$$

for every even integer $2 \leq j \leq s$ *and*

$$\inf\left\{i : H^i_{I_1,\ldots,I_s}(M) \textit{ is not Artinian}\right\} \leq \inf\left\{i : H^i_{I_1,\ldots,I_i}(M) \textit{ is not Artinian}\right\}$$

for every odd integer $1 \leq i \leq s$.

Corollary 2.10. *Let* M *be a finitely generated* R*-module and* α *be a non-trivial family of* R*. Then*

$$\inf\left\{i : H^i_\alpha(M) \textit{ is not Artinian}\right\} = \inf\left\{i : H^i_\alpha(M) \ncong H^i_{\mathfrak{m}}(M)\right\}.$$

Proof. Set $n = \inf\{i : H^i_\alpha(M) \text{ is not Artinian}\}$. Then $\Gamma_\alpha(E^i(M)) = \Gamma_{\mathfrak{m}}(E^i(M))$ for every $i < n$ by Theorem 2.7 and Corollary 1.26, (iv). Conversely, $H^n_\alpha(M)$ is not Artinian while $H^n_{\mathfrak{m}}(M)$ is and we conclude that $H^n_\alpha(M) \ncong H^n_{\mathfrak{m}}(M)$. $\square$

The next result generalises [CW09, Theorem 2.4 and Proposition 2.5].

Corollary 2.11. *Let* M *be a finitely generated* R*-module and* $I_1, \ldots, I_s$ *be ideals of* R*. Then*

$$\begin{aligned}\inf\left\{i : H^i_{I_s,\ldots,I_1}(M) \textit{ is not Artinian}\right\} &= \inf\left\{\operatorname{depth}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) : \mathfrak{p} \in W(I_s,\ldots,I_1) - \{\mathfrak{m}\}\right\} \\ &= \inf\left\{i : H^i_{I_s,\ldots,I_1}(M) \ncong H^i_{\mathfrak{m}}(M)\right\}\end{aligned}$$

where $W(I_s,\ldots,I_1) = \tilde{W}(I_s,\ldots,I_1) \cap \operatorname{Spec} R$.

In the same fashion of local cohomology theory and its current extensions, we define, for each R-module M and every family α of R, the *cohomological dimension of* M *with respect to* α as $\operatorname{cd}(\alpha, M) = \sup\{r : H^r_\alpha(M) \neq 0\}$.

When $\langle\alpha\rangle = \tilde{W}(\langle\varphi\rangle, \psi)$ for some families, φ and ψ, of R, we denote $\operatorname{cd}(\alpha, M)$ as $\operatorname{cd}(\varphi, \psi, M)$ and we call it the *cohomological dimension of* M *with respect to the pair* (φ, ψ). Also, when $\tilde{W}(\langle\varphi\rangle, \psi) = \tilde{W}(I_s, \ldots, I_1)$ for some ideals $I_1, \ldots, I_s$ of R, we shall write $\operatorname{cd}(\varphi, \psi, M)$ as $\operatorname{cd}(I_s, \ldots, I_1, M)$ and call it the *cohomological dimension of* M *with respect to the* s*-tuple* $(I_s, \ldots, I_1)$.

Let M be a finitely generated R-module. If $\langle\psi\rangle \supseteq \psi'$, Theorem 1.43 states that $\operatorname{cd}(\mathfrak{m}, \psi, M) \geq \operatorname{cd}(\mathfrak{m}, \psi', M)$. Thus $\operatorname{cd}(\mathfrak{m}, I_2, \ldots, I_s, M) \leq \operatorname{cd}(\mathfrak{m}, J_2, \ldots, J_t, M)$ every time s is even, $s \leq t$, $\sqrt{I_j} \supseteq \sqrt{J_j}$ for every even integer $2 \leq j \leq s$ and $\sqrt{I_i} \subseteq \sqrt{J_i}$ for every odd integer $3 \leq i \leq s-1$. Similarly, $\operatorname{cd}(\mathfrak{m}, I_2, \ldots, I_s, M) \geq \operatorname{cd}(\mathfrak{m}, J_2, \ldots, J_t, M)$

every time s is odd, $s \leq t$, $\sqrt{J_j} \supseteq \sqrt{I_j}$ for every even integer $2 \leq j \leq s-1$ and $\sqrt{J_i} \subseteq \sqrt{I_i}$ for every odd integer $3 \leq i \leq s$ (compare with Corollary 2.4).

We now establish some properties of this invariant by seeing first that the cohomological dimension of a finitely generated module only depends on its support.

Theorem 2.12. *Let α be a family of R, M and N be finitely generated R-modules with $\operatorname{Supp}(N) \subseteq \operatorname{Supp}(M)$ and r be a non-negative integer number such that $H^r_\alpha(R/\mathfrak{p}) = 0$ for every prime ideal $\mathfrak{p} \in \operatorname{Supp}(M)$. Then*

(i) $H^r_\alpha(N) = 0$.

(ii) $\operatorname{cd}(\alpha, R/\mathfrak{p}) < r$ *for all* $\mathfrak{p} \in \operatorname{Supp}(M)$.

Proof. We now prove (i). If $H^r_\alpha(R/\mathfrak{p}) = 0$ for every $\mathfrak{p} \in \operatorname{Supp}(M)$, then, for any finitely generated R-module N such that $\operatorname{Supp}(N) \subseteq \operatorname{Supp}(M)$ and any filtration $0 = N_0 \subsetneq N_1 \subsetneq \cdots \subsetneq N_{t-1} \subsetneq N_t = N$ of submodules of N such that the isomorphism $N_j/N_{j-1} \cong R/\mathfrak{p}_j$ holds for some $\mathfrak{p}_j \in \operatorname{Supp}(N)$ and every $j = 1, \ldots, t$, we obtain, from the exact sequence $H^r_\alpha(N_{j-1}) \longrightarrow H^r_\alpha(N_j) \longrightarrow H^r_\alpha(R/\mathfrak{p}_j)$, that $H^r_\alpha(N_j) = 0$ for every $j = 1, \ldots, t$.

In order to prove (ii), it suffices to show that $H^{r+1}_\alpha(R/\mathfrak{p}) = 0$ for every prime ideal $\mathfrak{p} \in \operatorname{Supp}(M)$. Otherwise there would exist a prime $\mathfrak{q} \in \operatorname{Ass}_R(H^{r+1}_\alpha(R/\mathfrak{p}_0))$ for some $\mathfrak{p}_0 \in \operatorname{Supp}(M)$. If $\mathfrak{q} \neq \mathfrak{p}_0$, we can choose $x \in \mathfrak{q} - \mathfrak{p}_0$ in order to obtain the exact sequence

$$0 \longrightarrow R/\mathfrak{p}_0 \xrightarrow{\mu_x} R/\mathfrak{p}_0 \longrightarrow R/(\mathfrak{p}_0 + Rx) \longrightarrow 0\ .$$

This leads to the exact sequence

$$H^r_\alpha(R/(\mathfrak{p}_0 + Rx)) \longrightarrow H^{r+1}_\alpha(R/\mathfrak{p}_0) \xrightarrow{\mu_x} H^{r+1}_\alpha(R/\mathfrak{p}_0)$$

and, since $\operatorname{Supp}(R/(\mathfrak{p}_0 + Rx)) \subseteq \operatorname{Supp}(R/\mathfrak{p}_0) \subseteq \operatorname{Supp}(M)$, we conclude by (i) that $H^r_\alpha(R/(\mathfrak{p}_0 + Rx)) = 0$ and x is $H^{r+1}_\alpha(R/\mathfrak{p}_0)$-regular, which is absurd. Thus $\mathfrak{q} = \mathfrak{p}_0$.

Now, $\operatorname{Ass}_R(H^{r+1}_\alpha(R/\mathfrak{p}_0)) \subseteq \langle\alpha\rangle$ and $\mathfrak{p}_0 \in \langle\alpha\rangle$. Then $R/\mathfrak{p}_0$, being a $\mathfrak{p}_0$-torsion module, is also α-torsion. Thus $H^i_\alpha(R/\mathfrak{p}_0) = 0$ for every $i > 0$, which is a contradiction and $H^{r+1}_\alpha(R/\mathfrak{p}) = 0$ for every $\mathfrak{p} \in \operatorname{Supp}(M)$. □

Proposition 2.13. *Let α be a non-trivial family of R and M and N be finitely generated R-modules such that $\operatorname{Ann}(M) \subseteq \operatorname{Ann}(N)$. Then $\operatorname{cd}(\alpha, N) \leq \operatorname{cd}(\alpha, M)$ and $\operatorname{cd}(I_s, \ldots, I_1, N) \leq \operatorname{cd}(I_s, \ldots, I_1, M)$ for every s-tuple of ideals $(I_1, \ldots, I_s)$.*

Proof. We will show that every time $H^i_\alpha(M) = 0$ we get that $H^i_\alpha(N) = 0$. Since $\dim N \leq \dim M$, we will prove the statement only for $\operatorname{cd}(\alpha, M) < i \leq \dim M$. Since N is an $(R/\operatorname{Ann}(M))$-module, there exists a filtration of R-modules

$$0 = N_0 \subseteq N_1 \subseteq \cdots \subseteq N_{t-1} \subseteq N_t = N$$

such that each N_j/N_{j-1} is an homomorphic image of a direct sum of finitely many copies of M (see [Vas74, Theorem 4.1]). Let us suppose that $t = 1$. Then there exists an exact sequence $0 \longrightarrow L \longrightarrow M^m \longrightarrow N \longrightarrow 0$ where L is a finitely generated R-module. This leads to the long exact sequence

$$\cdots \longrightarrow H^i_\alpha(L) \longrightarrow H^i_\alpha(M^m) \longrightarrow H^i_\alpha(N) \longrightarrow H^{i+1}_\alpha(L) \longrightarrow \cdots .$$

Since $\operatorname{Ann}(M) \subseteq \operatorname{Ann}(L)$, we have that $H^{i+1}_\alpha(L) = 0$ by descending induction. If $H^i_\alpha(M) = 0$, then $H^i_\alpha(N) = 0$. Now $H^i_\alpha(N_t/N_{t-1}) = 0$ and $H^i_\alpha(N_{t-1}) = 0$, provided $H^i_\alpha(M) = 0$ and $t > 1$. We conclude that $H^i_\alpha(N_t) = 0$ and $\operatorname{cd}(\alpha, N) \leq \operatorname{cd}(\alpha, M)$. $\square$

Corollary 2.14. *If M is a finitely generated R-module, then there exists a prime ideal $\mathfrak{p} \in \operatorname{Min}(M)$ such that $\operatorname{cd}(\alpha, M) = \operatorname{cd}(\alpha, R/\mathfrak{p})$. In particular, for every s-tuple $(I_1, \ldots, I_s)$ of ideals of R, there exists a prime ideal $\mathfrak{p} \in \operatorname{Min}(M)$ such that $\operatorname{cd}(I_s, \ldots, I_1, M) = \operatorname{cd}(I_s, \ldots, I_1, R/\mathfrak{p})$.*

Proof. Again, Theorem 2.12 gives us the inequality $\operatorname{cd}(\alpha, M) \leq \operatorname{cd}(\alpha, R/\mathfrak{p}')$ for some $\mathfrak{p}' \in \operatorname{Supp}(M)$. Since $\operatorname{Ann}(M) \subseteq \mathfrak{q}$ for every $\mathfrak{q} \in \operatorname{Supp}(M)$, we also have that $\operatorname{cd}(\alpha, M) \geq \operatorname{cd}(\alpha, R/\mathfrak{q})$ for every $\mathfrak{q} \in \operatorname{Supp}(M)$ by Proposition 2.13. Now there exists $\mathfrak{p} \in \operatorname{Min}(M)$ such that $\mathfrak{p} \subseteq \mathfrak{p}'$, whence

$$\operatorname{cd}(\alpha, M) \leq \operatorname{cd}(\alpha, R/\mathfrak{p}') \leq \operatorname{cd}(\alpha, R/\mathfrak{p}) \leq \operatorname{cd}(\alpha, M)$$

and we conclude the statement. $\square$

The following result generalises [CW09, Corollary 3.3].

Corollary 2.15. *For any finitely generated R-module M and any family α of R,*

$$\operatorname{cd}(\alpha, M) = \inf\left\{i \in \mathbb{N} : H^i_\alpha(R/\mathfrak{p}) = 0 \textit{ for all } \mathfrak{p} \in \operatorname{Supp}(M)\right\} - 1.$$

In particular, if $I_1, \ldots, I_s$ are ideals of R, then

$$\operatorname{cd}(I_s, \ldots, I_1, M) = \inf\left\{i \in \mathbb{N} : H^i_{I_s,\ldots,I_1}(R/\mathfrak{p}) = 0 \textit{ for all } \mathfrak{p} \in \operatorname{Supp}(M)\right\} - 1.$$

Proof. Theorem 2.12 gives that

$$\operatorname{cd}(\alpha, M) \leq \inf\left\{i \in \mathbb{N} : H^i_\alpha(R/\mathfrak{p}) = 0 \text{ for all } \mathfrak{p} \in \operatorname{Supp}(M)\right\} - 1.$$

For the converse, we will show that $H^i_\alpha(R/\mathfrak{p}) = 0$ for every $\mathfrak{p} \in \operatorname{Supp}(M)$ and every integer $i > \operatorname{cd}(\alpha, M)$. So consider $\mathfrak{p} \in \operatorname{Supp}(M)$ and $i > \operatorname{cd}(\alpha, M)$. It follows from Proposition 2.13 that $\operatorname{cd}(\alpha, R/\mathfrak{p}) \leq \operatorname{cd}(\alpha, M) < i$, whence $H^i_\alpha(R/\mathfrak{p}) = 0$. $\square$

Corollary 2.16. *For any family α of R and any exact sequence of finitely generated R-modules $0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$, $\operatorname{cd}(\alpha, M) = \max\{\operatorname{cd}(\alpha, L), \operatorname{cd}(\alpha, N)\}$. In particular, $\operatorname{cd}(I_s, \ldots, I_1, M) = \max\{\operatorname{cd}(I_s, \ldots, I_1, L), \operatorname{cd}(I_s, \ldots, I_1, N)\}$ for every s-tuple $(I_1, \ldots, I_s)$ of ideals of R.*

Proof. Since $\mathrm{Ann}(M) \subseteq \mathrm{Ann}(L) \cap \mathrm{Ann}(N)$, Proposition 2.13 gives that

$$\max\{\mathrm{cd}(\alpha, L), \mathrm{cd}(\alpha, N)\} \leq \mathrm{cd}(\alpha, M).$$

For the converse, there exists an exact sequence $H^i_\alpha(L) \longrightarrow H^i_\alpha(M) \longrightarrow H^i_\alpha(N)$. Then $H^i_\alpha(L) = 0$ and $H^i_\alpha(N) = 0$ imply $H^i_\alpha(M) = 0$. □

Corollary 2.17. *Suppose that M and N are finitely generated R-modules such that $\mathrm{Supp}(N) \subseteq \mathrm{Supp}(M)$ and consider a non-trivial family α of R. Then*

$$\mathrm{cd}(\alpha, N) \leq \mathrm{cd}(\alpha, M)$$

and $\mathrm{cd}(I_s, \ldots, I_1, N) \leq \mathrm{cd}(I_s, \ldots, I_1, M)$ for every s-tuple of ideals $(I_1, \ldots, I_s)$.

Proof. Let $0 = N_0 \subsetneq N_1 \subsetneq \cdots \subsetneq N_{t-1} \subsetneq N_t = N$ be a filtration of submodules of N such that $N_i/N_{i-1} \cong R/\mathfrak{p}_i$ for some $\mathfrak{p}_i \in \mathrm{Supp}(N)$ and for every $i = 1, \ldots, t$. If $t = 1$, the statement follows directly from Proposition 2.13. If $t > 1$, the exact sequence $0 \longrightarrow N_{t-1} \longrightarrow N_t \longrightarrow N_t/N_{t-1} \longrightarrow 0$ leads to

$$\mathrm{cd}(\alpha, N_t) = \max\{\mathrm{cd}(\alpha, N_{t-1}), \mathrm{cd}(\alpha, N_t/N_{t-1})\} \leq \mathrm{cd}(\alpha, M)$$

and the statement is proved. □

2.2 Attached primes of top local cohomology modules

Let M be an R-module. A prime ideal $\mathfrak{p}$ is said to be an *attached prime* of M when $\mathfrak{p} = \mathrm{Ann}(M/T)$ for some submodule T of M. The concept of attached prime ideal is closely related to a secondary representation of M (see [Mac73]).

There is a well-known property in usual local cohomology which says that the attached primes of the n-th cohomology module of the n-dimensional finitely generated R-module M is a subset of the minimal primes of M (see [DY05, Theorem A]). This is also valid in a more general context as we can see in the following results.

Theorem 2.18. *Let M be a finitely generated R-module of dimension d and α be a non-trivial family of R. Then*

$$\mathrm{Att}(H^d_\alpha(M)) = \{\mathfrak{p} \in \mathrm{Supp}(M) : \mathrm{cd}\,(\alpha, R/\mathfrak{p}) = d\}.$$

Proof. If $d = 0$, then M has finite length and

$$\mathrm{Att}(H^0_\alpha(M)) = \mathrm{Att}(M) = \{\mathfrak{m}\} = \mathrm{Supp}(M).$$

Let us suppose that $d > 0$. By Corollary 2.15, we obtain that $H_\alpha^d(M) = 0$ if and only if $H_\alpha^d(R/\mathfrak{p}) = 0$ for every $\mathfrak{p} \in \operatorname{Supp}(M)$. Thus $\operatorname{Att}(H_\alpha^d(M)) = \emptyset$ if and only if $\{\mathfrak{p} \in \operatorname{Supp}(M) : \operatorname{cd}(\alpha, R/\mathfrak{p}) = d\} = \emptyset$. Hence we may assume that $H_\alpha^d(M) \neq 0$.

Assume first that every non-trivial submodule of M has cohomological dimension with respect to α equal to d. We claim that $\operatorname{Ass}(M) = \{\mathfrak{p} \in \operatorname{Supp}(M) : \operatorname{cd}(\alpha, R/\mathfrak{p}) = d\}$. Indeed, if $\mathfrak{p} \in \operatorname{Ass}(M)$, then $R/\mathfrak{p}$ is isomorphic to a non-trivial submodule of M, thus $\operatorname{cd}(\alpha, R/\mathfrak{p}) = d$. For the converse, observe that if $\mathfrak{p} \in \operatorname{Supp}(M)$ is such that $\operatorname{cd}(\alpha, R/\mathfrak{p}) = d$, then $d \leq \dim(R/\mathfrak{p}) \leq \dim(M) = d$, whence $\mathfrak{p} \in \operatorname{Ass}(M)$ and the claim is proved. Thus we shall prove that $\operatorname{Att}(H_\alpha^d(M)) = \operatorname{Ass}(M)$.

Let $r \in R$ be an M-regular element. Then the exact sequence

$$0 \longrightarrow M \xrightarrow{\mu_r} M \longrightarrow M/rM \longrightarrow 0$$

induces the exact sequence

$$H_\alpha^d(M) \xrightarrow{\mu_r} H_\alpha^d(M) \longrightarrow H_\alpha^d(M/rM) .$$

Now $H_\alpha^d(M/rM) = 0$ by Lemma 1.38, thus $\mu_r : H_\alpha^d(M) \to H_\alpha^d(M)$ is surjective, hence $r \notin \bigcup_{\mathfrak{p} \in \operatorname{Att}(H_\alpha^d(M))} \mathfrak{p}$. Therefore, $\bigcup_{\mathfrak{p} \in \operatorname{Att}(H_\alpha^d(M))} \mathfrak{p} \subseteq \bigcup_{\mathfrak{p} \in \operatorname{Ass}(M)} \mathfrak{p}$ and, for every prime ideal $\mathfrak{p} \in \operatorname{Att}(H_\alpha^d(M))$, there exists $\mathfrak{q} \in \operatorname{Ass}(M)$ such that $\mathfrak{p} \subseteq \mathfrak{q}$. We also have that $\operatorname{Ann}(M) \subseteq \operatorname{Ann}(H_\alpha^d(M)) \subseteq \mathfrak{p}$ for every $\mathfrak{p} \in \operatorname{Att}(H_\alpha^d(M))$. Then we get the inequalities $d = \operatorname{cd}(\alpha, R/\mathfrak{q}) \leq \dim(R/\mathfrak{q}) \leq \dim(R/\mathfrak{p}) \leq d$ and $\mathfrak{p} = \mathfrak{q}$, implying the relation $\operatorname{Att}(H_\alpha^d(M)) \subseteq \operatorname{Ass}(M)$. For the converse consider a prime ideal $\mathfrak{p} \in \operatorname{Ass}(M)$. Then there exists a $\mathfrak{p}$-primary submodule T of M such that $\operatorname{Ass}(M/T) = \{\mathfrak{p}\}$. Hence Corollary 2.14 implies that $\operatorname{cd}(\alpha, M/T) = \operatorname{cd}(\alpha, R/\mathfrak{p}) = d$. Since $\operatorname{Ass}(L/T) = \{\mathfrak{p}\}$ for every submodule L of M such that $L \supsetneq T$, we have that $\operatorname{cd}(\alpha, L/T) = d$. Thus every non-trivial submodule of M/T has also cohomological dimension with respect to α equal to d. Hence, we obtain as before that $\operatorname{Att}\left(H_\alpha^d(M/T)\right) \subseteq \operatorname{Ass}(M/T)$ and this implies that $\operatorname{Att}\left(H_\alpha^d(M/T)\right) = \operatorname{Ass}(M/T)$. Finally the exact sequence of R-modules $H_\alpha^d(M) \longrightarrow H_\alpha^d(M/T) \longrightarrow 0$ leads to $\operatorname{Att}\left(H_\alpha^d(M/T)\right) \subseteq \operatorname{Att}(H_\alpha^d(M))$. By varying T over all the primary submodules of M we get that

$$\operatorname{Ass}(M) = \bigcup_T \operatorname{Ass}(M/T) = \bigcup_T \operatorname{Att}\left(H_\alpha^d(M/T)\right) \subseteq \operatorname{Att}(H_\alpha^d(M)) \subseteq \operatorname{Ass}(M).$$

Suppose now that M has a non-trivial submodule with cohomological dimension with respect to α lower than d. We claim that there is a unique maximal submodule N of M, with respect to inclusion, such that $\operatorname{cd}(\alpha, N) \leq d - 1$. In fact, existence of maximal submodules with this property is assured since M is finitely generated. Now uniqueness follows because if M_1 and M_2 are submodules of M such that $\max\{\operatorname{cd}(\alpha, M_1), \operatorname{cd}(\alpha, M_2)\} \leq d - 1$, then the short exact sequence of R-modules $0 \longrightarrow M_1 \longrightarrow M_1 + M_2 \longrightarrow (M_1 + M_2)/M_1 \longrightarrow 0$ and Corollary 2.16 give that

$\operatorname{cd}(\alpha, M_1 + M_2) = \max\{\operatorname{cd}(\alpha, M_1), \operatorname{cd}(\alpha, M_2/(M_1 \cap M_2))\} \leq d - 1$. Hence the claim is stated.

From the exact sequence $0 \longrightarrow N \longrightarrow M \longrightarrow M/N \longrightarrow 0$ we get by Corollary 2.16 that $\operatorname{cd}(\alpha, M) = \operatorname{cd}(\alpha, M/N)$. The exact sequence

$$H^d_\alpha(N) \longrightarrow H^d_\alpha(M) \longrightarrow H^d_\alpha(M/N) \longrightarrow H^{d+1}_\alpha(N)$$

gives the isomorphism $H^d_\alpha(M) \cong H^d_\alpha(M/N)$. Observe now that the R-module M/N has no non-trivial submodules with cohomological dimension with respect to α lower than d. Hence the result follows from the previous case. $\square$

We can get a refinement of the previous result when $\langle\alpha\rangle = \tilde{W}(\langle\varphi\rangle, \psi)$.

Theorem 2.19. *Let M be a finitely generated R-module of dimension d and φ and ψ be non-trivial families of R. Then*

$$\operatorname{Att}(H^d_{\varphi,\psi}(M)) = \{\mathfrak{p} \in \operatorname{Supp}(M) \cap \langle\psi\rangle : \operatorname{cd}(\varphi, R/\mathfrak{p}) = d\}.$$

Proof. Consider $\bar{R} = R/\operatorname{Ann}(M)$. By Theorem 1.14 we get that $H^i_{\varphi,\psi}(M) \cong H^i_{\varphi\bar{R},\psi\bar{R}}(M)$ and $H^i_{\varphi,\psi}(R/\mathfrak{p}) \cong H^i_{\varphi\bar{R},\psi\bar{R}}(R/\mathfrak{p})$ for every i and every $\mathfrak{p} \in \operatorname{Supp}_R(M)$. Thus we may assume that M is faithful, so that $\dim R = \dim M = d$.

Recall that $\langle\alpha\rangle \cap \operatorname{Spec} R = \bigcup_{I\in\alpha} V(I)$ for any family α of R (see equation (1.1)). Thus the prime ideal $\mathfrak{p} \notin \langle\psi\rangle$ if and only if $J \not\subseteq \mathfrak{p}$ for every $J \in \psi$. In this case $\dim(R/(\mathfrak{p} + J)) \leq d - 1$ for every $J \in \psi$ and $H^d_{\varphi,\psi}(R/\mathfrak{p}) = 0$ for every prime ideal $\mathfrak{p} \notin \langle\psi\rangle$ by Lemma 1.38. If $\mathfrak{p} \in \langle\psi\rangle$, we conclude from Proposition 1.31 that $H^i_{\varphi,\psi}(R/\mathfrak{p}) = H^i_\varphi(R/\mathfrak{p})$ for every i. We also have from Corollary 2.15 that $\operatorname{cd}(\varphi, \psi, M) = \inf\left\{i \in \mathbb{N} : H^i_{\varphi,\psi}(R/\mathfrak{p}) = 0 \text{ for all } \mathfrak{p} \in \operatorname{Supp}(M)\right\} - 1$. In this way we obtain that $H^d_{\varphi,\psi}(M) = 0$ if and only if $\{\mathfrak{p} \in \operatorname{Supp}(M) \cap \langle\psi\rangle : \operatorname{cd}(\varphi, R/\mathfrak{p}) = d\} = \emptyset$. Let us consider $\mathfrak{p} \in \operatorname{Att}(H^d_{\varphi,\psi}(M))$. Then $H^d_{\varphi,\psi}(M/\mathfrak{p}M) \cong H^d_{\varphi,\psi}(M)/\mathfrak{p}H^d_{\varphi,\psi}(M) \neq 0$ by Lemma 1.39. By Theorem 1.40, there exists $J \in \psi$ such that

$$d \leq \dim((M/\mathfrak{p}M)/J(M/\mathfrak{p}M)) = \dim(R/(\mathfrak{p} + J)) \leq d,$$

whence $J \subseteq \mathfrak{p}$ and $\dim(R/\mathfrak{p}) = d$. From this we have that $M/\mathfrak{p}M$ is J-torsion and $H^d_{\varphi,\psi}(M/\mathfrak{p}M) = H^d_\varphi(M/\mathfrak{p}M)$ by Proposition 1.31. Since M is faithful, we also conclude that $\sqrt{\operatorname{Ann}(M/\mathfrak{p}M)} = \mathfrak{p}$ and

$$\begin{aligned}
d &\leq \operatorname{cd}(\varphi, R/\mathfrak{p}) \\
&\leq \operatorname{cd}(\varphi, M/\mathfrak{p}M) \quad \text{(by Proposition 2.13)} \\
&\leq \dim(M/\mathfrak{p}M) \qquad \text{(by Lemma 1.38)} \\
&= \dim(R/\mathfrak{p}) \\
&= d.
\end{aligned}$$

In sum, we have that $\mathfrak{p} \in \operatorname{Supp}(M) \cap \langle \psi \rangle$ and $\operatorname{cd}(\varphi, R/\mathfrak{p}) = d$. Conversely let us consider a prime ideal $\mathfrak{p} \in \operatorname{Supp}(M) \cap \langle \psi \rangle$ such that $\operatorname{cd}(\varphi, R/\mathfrak{p}) = d$. Then Proposition 1.31 states that $H^i_{\varphi,\psi}(M/\mathfrak{p}M) = H^i_{\varphi}(M/\mathfrak{p}M)$ for every i. Observe that $\operatorname{Att}_R\left(H^d_{\varphi}(M/\mathfrak{p}M)\right) = \{\mathfrak{q} \in \operatorname{Supp}_R(M/\mathfrak{p}M) : \operatorname{cd}(\varphi, R/\mathfrak{q}) = d\}$ by Theorem 2.18. Then $\mathfrak{p} \in \operatorname{Att}_R\left(H^d_{\varphi}(M/\mathfrak{p}M)\right) = \operatorname{Att}_R\left(H^d_{\varphi,\psi}(M/\mathfrak{p}M)\right)$. Since the isomorphism of R-modules $H^d_{\varphi,\psi}(M/\mathfrak{p}M) \cong H^d_{\varphi,\psi}(M)/\mathfrak{p}H^d_{\varphi,\psi}(M)$ holds by Lemma 1.39, we get that

$$\operatorname{Att}_R\left(H^d_{\varphi,\psi}(M/\mathfrak{p}M)\right) \subseteq \operatorname{Att}(H^d_{\varphi,\psi}(M)).$$

Therefore, $\mathfrak{p} \in \operatorname{Att}(H^d_{\varphi,\psi}(M))$. $\square$

Corollary 2.20. *Let M be a d-dimensional finitely generated R-module and φ and ψ be non-trivial families of R such that $\varphi + \psi \subseteq \tilde{W}(\mathfrak{m})$. Then*

$$\operatorname{Att}(H^d_{\varphi,\psi}(M)) = \{\mathfrak{p} \in \operatorname{Supp}(M) \cap \langle \psi \rangle : \dim(R/\mathfrak{p}) = d\}.$$

Proof. It was proved in Theorem 1.43 that $\operatorname{cd}(\varphi, \psi, N) = \sup_{J \in \psi} \dim(N/JN)$ for any finitely generated R-module N. Now the prime ideal $\mathfrak{p} \in \langle \psi \rangle$ if and only if there exists $J \in \psi$ such that $J \subseteq \mathfrak{p}$ by equation (1.1). Thus

$$\sup_{J \in \psi} \dim((R/\mathfrak{p})/J(R/\mathfrak{p})) = \sup_{J \in \psi} \dim(R/(J + \mathfrak{p})) = \dim(R/\mathfrak{p}).$$

By the previous theorem we get the statement. $\square$

The next result generalises [Chu11, Theorem 2.2].

Corollary 2.21. *Let M be a d-dimensional finitely generated R-module and $I_1, \ldots, I_s$ be ideals of R. Suppose that $I_1 + \mathfrak{p}$ is $\mathfrak{m}$-primary for every prime $\mathfrak{p} \in \tilde{W}(I_2, \ldots, I_s)$. Then $\operatorname{Att}(H^d_{I_1,\ldots,I_j}(M)) = \left\{\mathfrak{p} \in \operatorname{Supp}(M) \cap \tilde{W}(I_2, \ldots, I_j) : \dim(R/\mathfrak{p}) = d\right\}$ for every even integer $2 \leq j \leq s$ and for $j = s$.*

Theorem 2.22. *Let M be a finitely generated R-module of dimension d and φ and ψ be families of R. Suppose that $H^d_{\varphi,\psi}(M) \neq 0$. Then there exists a quotient M/N such that $\operatorname{Supp}(M/N) \subseteq \langle \psi \rangle$, $\dim(M/N) = d$ and $H^d_{\varphi,\psi}(M) \cong H^d_{\varphi}(M/N)$.*

Proof. If $\mathfrak{p} \in \operatorname{Supp}(M)$ is such that $\operatorname{cd}(\varphi, R/\mathfrak{p}) = d$, then $\dim(R/\mathfrak{p}) = d$. Then $\operatorname{Att}(H^d_{\varphi,\psi}(M)) \subseteq \operatorname{Min}(M)$ by Theorem 2.19. There exists a submodule N of M such that $\operatorname{Ass}(N) = \operatorname{Ass}(M) - \operatorname{Att}(H^d_{\varphi,\psi}(M))$ and $\operatorname{Ass}(M/N) = \operatorname{Att}(H^d_{\varphi,\psi}(M))$ by [Bou89, Proposition 4, p. 263], see also Lemma A.3. Consider now the short exact sequence $H^d_{\varphi,\psi}(N) \longrightarrow H^d_{\varphi,\psi}(M) \longrightarrow H^d_{\varphi,\psi}(M/N) \longrightarrow 0$. If $H^d_{\varphi,\psi}(N) \neq 0$, then there exists $\mathfrak{p} \in \operatorname{Supp}(N) \cap \langle \psi \rangle$ such that $\operatorname{cd}(\varphi, R/\mathfrak{p}) = d$. Therefore $\mathfrak{p} \in \operatorname{Att}(H^d_{\varphi,\psi}(M))$ by Theorem 2.19 and $\mathfrak{p} \in \operatorname{Ass}(N) \cap \operatorname{Ass}(M/N) = \emptyset$. We conclude that $H^d_{\varphi,\psi}(N) = 0$ and hence $H^d_{\varphi,\psi}(M) \cong H^d_{\varphi,\psi}(M/N)$. Notice that $\operatorname{Ass}(M/N) \subseteq \langle \psi \rangle$ by Theorem 2.19. Then M/N is ψ-torsion and $\operatorname{Supp}(M/N) \subseteq \langle \psi \rangle$, whence M/N is J-torsion for some $J \in \langle \psi \rangle$ by Proposition 1.22. Thus we get that $H^d_{\varphi,\psi}(M/N) = H^d_{\varphi}(M/N)$ by Proposition 1.31. Since $H^d_{\varphi,\psi}(M) \neq 0$, we conclude that $H^d_{\varphi}(M/N) \neq 0$, whence $\dim(M/N) = d$. $\square$

The next result generalises [Chu11, Theorem 2.3].

Corollary 2.23. *Let M be a finitely generated R-module of dimension d and $I_1, \ldots, I_s$ be ideals of R. Suppose that $H^d_{I_s,\ldots,I_1}(M) \neq 0$. Then there exists an $(I_{s-1}, \ldots, I_1)$-torsion quotient M/N such that $\dim(M/N) = d$ and $H^d_{I_s,\ldots,I_1}(M) \cong H^d_{I_s}(M/N)$.*

Proposition 2.24. *Let M be a d-dimensional finitely generated R-module and φ and ψ be families of R. Then there exists $J \in \langle\psi\rangle$ such that*

$$\mathrm{Att}(H^d_{\varphi,\psi}(M)) = \mathrm{Att}_R\left(H^d_{\varphi}(M/JM)\right).$$

Proof. We have that $\mathrm{Att}(H^d_{\varphi,\psi}(M)) = \{\mathfrak{p} \in \mathrm{Supp}(M) \cap \langle\psi\rangle : \mathrm{cd}(\varphi, R/\mathfrak{p}) = d\}$ by Theorem 2.19 and $\mathrm{Att}_R\left(H^d_{\varphi}(M/KM)\right) = \{\mathfrak{p} \in \mathrm{Supp}(M) \cap V(K) : \mathrm{cd}(\varphi, R/\mathfrak{p}) = d\}$ for every ideal K such that $\dim(M/KM) = d$ by Theorem 2.18. If $\dim(M/KM) < d$, then $\mathrm{Att}_R\left(H^d_{\varphi}(M/KM)\right) = \emptyset$. From equation (1.1), we get that

$$\mathrm{Att}(H^d_{\varphi,\psi}(M)) = \bigcup_{K\in\psi} \mathrm{Att}_R\left(H^d_{\varphi}(M/KM)\right) \tag{2.2}$$

and this union actually runs on the ideals $K \in \psi$ such that $\dim(M/KM) = d$. Since $\mathrm{Att}(H^d_{\varphi,\psi}(M))$ is a finite subset $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_s\}$ of $\langle\psi\rangle$, there exists a subfamily $\{J_1, \ldots, J_s\}$ of ψ such that $\mathfrak{p}_i \in V(J_i)$ for each i by equation (1.1). Consider now the ideal $J = J_1 \cdots J_s \in \langle\psi\rangle$. Then $\dim(M/JM) = d$ by equation (2.2). By Theorem 2.18, we have that $\mathrm{Att}_R\left(H^d_{\varphi}(M/JM)\right) = \bigcup_{i=1}^{s} \mathrm{Att}_R\left(H^d_{\varphi}(M/J_iM)\right)$. Hence $\mathrm{Att}_R\left(H^d_{\varphi}(M/JM)\right) \subseteq \mathrm{Att}(H^d_{\varphi,\psi}(M))$ by equation (2.2). Conversely, for every i we have that $\mathfrak{p}_i \in \mathrm{Att}_R\left(H^d_{\varphi}(M/J_iM)\right)$. Then

$$\mathrm{Att}(H^d_{\varphi,\psi}(M)) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_s\} \subseteq \mathrm{Att}_R\left(H^d_{\varphi}(M/JM)\right)$$

and the proof is complete. □

We state now another generalisation of Lichtenbaum-Hartshorne Vanishing Theorem. For this goal, we shall first translate a result of [DANT02] to the present terms.

Proposition 2.25. *Let M be a finitely generated R-module of dimension d and let α be a non-trivial family of R. The following statements are equivalent:*

(i) $H^d_{\alpha}(M) = 0$.

(ii) *For each prime ideal $\mathfrak{p} \in \hat{R}$ such that $\dim(\hat{R}/\mathfrak{p}) = d$ and $\mathfrak{p} \in \mathrm{Supp}_{\hat{R}}(\hat{M})$, there exists $I \in \alpha$ such that $\dim(\hat{R}/(I\hat{R} + \mathfrak{p})) > 0$.*

Proof. If (ii) is satisfied, then condition (iii) of [DANT02, Theorem 2.8] is satisfied for $\Phi = \langle\alpha\rangle$. Hence $H^d_\alpha(M) = H^d_{\langle\alpha\rangle}(M) = 0$.

Now, if $H^d_\alpha(M) = 0$, then, for each prime ideal $\mathfrak{p} \in \hat{R}$ such that $\dim(\hat{R}/\mathfrak{p}) = d$ and $\mathfrak{p} \in \operatorname{Supp}_{\hat{R}}(\hat{M})$, there exists $I \in \langle\alpha\rangle$ such that $\dim(\hat{R}/(I\hat{R}+\mathfrak{p})) > 0$ again by [DANT02, Theorem 2.8]. Thus there exist $I_1, \ldots, I_s \in \alpha$ such that $I \supseteq I_1 \cdots I_s$. It follows that $0 < \dim(\hat{R}/(I\hat{R}+\mathfrak{p})) \leq \max\left\{\dim(\hat{R}/(I_1\hat{R}+\mathfrak{p})), \ldots, \dim(\hat{R}/(I_s\hat{R}+\mathfrak{p}))\right\}$ and condition (ii) of this statement is true. □

Theorem 2.26. *Let M be a finitely generated R-module of dimension d and let φ and ψ be non-trivial families of R. The following statements are equivalent:*

(i) $H^d_{\varphi,\psi}(M) = 0$.

(ii) For each prime ideal $\mathfrak{p} \in \hat{R}$ such that $\dim(\hat{R}/\mathfrak{p}) = d$ *and* $\mathfrak{p} \in \operatorname{Supp}_{\hat{R}}(\hat{M}/J\hat{M})$ *for some $J \in \psi$, we have that* $\dim(\hat{R}/(I\hat{R}+\mathfrak{p})) > 0$ *for some $I \in \varphi$.*

Proof. If $H^d_{\varphi,\psi}(M) = 0$, then $H^d_\varphi(M/JM) = 0$ for every $J \in \psi$ by equation (2.2). If $\mathfrak{p} \in \operatorname{Spec}\hat{R}$ is such that $\dim(\hat{R}/\mathfrak{p}) = d$ and $\mathfrak{p} \in \operatorname{Supp}_{\hat{R}}(\hat{M}/J\hat{M})$ for some $J \in \psi$, then $\dim(\hat{R}/(I\hat{R}+\mathfrak{p})) > 0$ for some $I \in \varphi$ by Proposition 2.25.

For the converse, by Theorem 2.12, it is enough to prove that $H^d_{\varphi,\psi}(R/\mathfrak{q}) = 0$ for every $\mathfrak{q} \in \operatorname{Supp}(M)$. If $\mathfrak{q} \notin \langle\psi\rangle$, then $H^d_{\varphi,\psi}(R/\mathfrak{q}) = 0$ because $\dim(R/(\mathfrak{q}+J)) \leq d-1$ for every $J \in \psi$. So let us assume that $\mathfrak{q} \in \operatorname{Supp}(M) \cap \langle\psi\rangle$. If $H^d_{\varphi,\psi}(R/\mathfrak{q}) \neq 0$, then $\mathfrak{q} \in \operatorname{Att}(H^d_{\varphi,\psi}(M))$ by Theorem 2.19, whence $\mathfrak{q} \in \operatorname{Att}\left(H^d_\varphi(M/JM)\right)$ for some $J \in \psi$ by equation (2.2). We also have that $\dim(R/\mathfrak{q}) = d$, whence $\dim(\hat{R}/\mathfrak{q}\hat{R}) = d$. Consider $\mathfrak{p} \in \operatorname{Supp}_{\hat{R}}(\hat{R}/\mathfrak{q}\hat{R})$ such that $\dim(\hat{R}/\mathfrak{p}) = d$. Since $J \subseteq \mathfrak{q} \in \operatorname{Supp}(M)$, we have that $\mathfrak{p} \in \operatorname{Supp}_{\hat{R}}(\hat{M}/J\hat{M})$. By assumption, there exists $I \in \varphi$ such that $\dim(\hat{R}/(I\hat{R}+\mathfrak{p})) > 0$. By Proposition 1.31 and Proposition 2.25 we conclude that $H^d_{\varphi,\psi}(R/\mathfrak{q}) = H^d_\varphi(R/\mathfrak{q}) = 0$. □

Corollary 2.27. *Let M be a finitely generated R-module of dimension d and α be a non-trivial family of R. Then $H^d_\alpha(M) = 0$ if and only if $H^d_I(M) = 0$ for some proper $I \in \alpha$.*

Chapter 3

The α-depth

3.1 Cofiniteness and local cohomology modules

Here we present a proposal to define α-cofiniteness. We recall the assumption that R is a Noetherian ring.

Definition 3.1. Let α be a family of R. The R-module N is said to be *α-cofinite* if $\operatorname{Supp}(N) \subseteq \langle \alpha \rangle$ and $\operatorname{Ext}^i_R(R/I, N)$ is finitely generated for every $I \in \alpha$ and every i.

For the sake of completeness, here we show some statements from [DM97].

Proposition 3.2. *Let M be a finitely generated R-module, N be an arbitrary R-module and p be a non-negative integer. Suppose that $\operatorname{Ext}^i_R(M, N)$ is finitely generated for every $i \leq p$. Then, for any finitely generated R-module L with $\operatorname{Ann}(M) \subseteq \operatorname{Ann}(L)$, $\operatorname{Ext}^i_R(L, N)$ is finitely generated for every $i \leq p$.*

Proof. Using induction on p, we may suppose that $\operatorname{Ext}^i_R(L, N)$ is finitely generated for every $i < p$ and every finitely generated R-module L such that $\operatorname{Ann}(M) \subseteq \operatorname{Ann}(L)$. Since L is an $(R/\operatorname{Ann}(M))$-module, we have by Gruson's Theorem that there exists a finite filtration $0 = L_0 \subseteq L_1 \subseteq \cdots \subseteq L_{n-1} \subseteq L_n = L$ of submodules of L such that L_j/L_{j-1} is an homomorphic image of a direct sum of finitely many copies of M for every j. Let us assume first that $n = 1$. Then we have an exact sequence $0 \to K \to M^k \to L \to 0$ for some non-negative integer k and some finitely generated module K. Thus we have an exact sequence

$$\cdots \to \operatorname{Ext}^{p-1}_R(K, N) \to \operatorname{Ext}^p_R(L, N) \to \operatorname{Ext}^p_R(M^k, N) \to \cdots .$$

Now $\operatorname{Ext}^p_R(M^k, N) \cong \operatorname{Ext}^p_R(M, N)^k$ is finitely generated and $\operatorname{Ann}(M) \subseteq \operatorname{Ann}(K)$, whence $\operatorname{Ext}^{p-1}_R(K, N)$ is also finitely generated. Hence $\operatorname{Ext}^p_R(L, N)$ is finitely generated.

Finally, for $n > 1$, the exact sequence $0 \to L_{n-1} \to L_n \to L_n/L_{n-1} \to 0$ induces the exact sequence $\cdots \to \mathrm{Ext}^p_R(L_n/L_{n-1}, N) \to \mathrm{Ext}^p_R(L_n, N) \to \mathrm{Ext}^p_R(L_{n-1}, N) \to \cdots$. Hence, the finiteness of $\mathrm{Ext}^p(L_{n-1}, N)$ and $\mathrm{Ext}^p_R(L_n/L_{n-1}, N)$ implies the finiteness of $\mathrm{Ext}^p_R(L_n, N)$. □

Corollary 3.3. *Let M be a finitely generated R-module, N be an arbitrary R-module and p be a non-negative integer. Suppose that $\mathrm{Ext}^i_R(M, N)$ is finitely generated for every $i \leq p$. Then, for any finitely generated R-module L such that $\mathrm{Supp}(L) \subseteq \mathrm{Supp}(M)$, $\mathrm{Ext}^i_R(L, N)$ is finitely generated for every $i \leq p$.*

Proof. As in the first part of the proof of the previous statement, consider a finite filtration $0 = L_0 \subseteq L_1 \subseteq \cdots \subseteq L_{n-1} \subseteq L_n = L$ of submodules of L such that L_j/L_{j-1} is isomorphic to $R/\mathfrak{p}_j$ for some $\mathfrak{p}_j \in \mathrm{Supp}(L) \subseteq \mathrm{Supp}(M)$ and every j. Now $\mathrm{Ann}(L_j/L_{j-1}) \supseteq \mathrm{Ann}(M)$, whence $\mathrm{Ext}^i_R(L_j/L_{j-1}, N)$ is finitely generated for every $i \leq p$ and every j. Thus $\mathrm{Ext}^i_R(L, N)$ is finitely generated for every $i \leq p$. □

Corollary 3.4. *Let I be an ideal of R and N be an R-module. The following conditions are equivalent:*

(i) $\mathrm{Ext}^i_R(R/I, N)$ *is finitely generated for every $i \leq p$.*

(ii) $\mathrm{Ext}^i_R(R/J, N)$ *is finitely generated for every $i \leq p$ and every ideal $J \supseteq I$.*

(iii) $\mathrm{Ext}^i_R(R/\mathfrak{p}, N)$ *is finitely generated for every $i \leq p$ and every prime $\mathfrak{p} \in \mathrm{Min}(R/I)$.*

Proof. It is enough to show that statement (iii) implies statement (i). Consider the set $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$ of minimal primes of R/I. Hence, $\mathrm{Supp}(R/I) = \mathrm{Supp}(R/\mathfrak{p}_1 \oplus \cdots \oplus R/\mathfrak{p}_n)$ and $\mathrm{Ext}^i_R(R/I, N)$ is finitely generated for every $i \leq p$ by Corollary 3.3. □

We obtain in these lines that the concept of α-cofiniteness only depends on the good family $\langle \alpha \rangle$ and not on a particular set of generators.

Lemma 3.5. *Let N be an R-module and p be a non-negative integer. Then the collection of all the ideals of R such that $\mathrm{Ext}^i_R(R/I, N)$ is finitely generated for every $i \leq p$ is a good family. Furthermore, the collection of all the ideals of R such that $\mathrm{Ext}^i_R(R/I, N)$ is finitely generated for every i is a good family.*

Proof. By Corollary 3.4, this collection is closed under inclusion. If I and J are ideals of R such that $\mathrm{Ext}^i_R(R/I, N)$ and $\mathrm{Ext}^i_R(R/J, N)$ are finitely generated for every $i \leq p$, then $\mathrm{Ext}^i_R(R/I \oplus R/J, N) \cong \mathrm{Ext}^i_R(R/I, N) \oplus \mathrm{Ext}^i_R(R/J, N)$ is finitely generated for every $i \leq p$. On the other hand,

$$\mathrm{Supp}(R/IJ) = \mathrm{Supp}(R/I) \cup \mathrm{Supp}(R/J) = \mathrm{Supp}(R/I \oplus R/J),$$

hence $\operatorname{Ext}^i_R(R/IJ, N)$ is finitely generated for every $i \leq p$ and the family is also closed under multiplication.

Calling $\psi(p) = \{I \trianglelefteq R : \operatorname{Ext}^i_R(R/I, N) \text{ is finitely generated for every } i \leq p\}$ and $\psi = \{I \trianglelefteq R : \operatorname{Ext}^i_R(R/I, N) \text{ is finitely generated for every } i\}$, we get that $\psi = \bigcap_{p\in\mathbb{N}} \psi(p)$. Thus the collection of all the ideals of R such that $\operatorname{Ext}^i_R(R/I, N)$ is finitely generated for every i is a good family. $\square$

Remark 3.6. If M is a finitely generated α-torsion R-module and N is any α-torsion-free R-module, then $\operatorname{Ass}(\operatorname{Hom}_R(M, N)) = \operatorname{Supp}(M) \cap \operatorname{Ass}(N) = \emptyset$. In general, for every α-torsion R-module M we have that $M = \varinjlim M_\lambda$ where each M_λ is a finitely generated submodule of M. Since M_λ is α-torsion, we get that $\operatorname{Hom}_R(M_\lambda, N) = 0$ for every λ. Hence $\operatorname{Hom}_R(M, N) = \varprojlim \operatorname{Hom}_R(M_\lambda, N) = 0$. For every R-module N the exact sequence $0 \to \Gamma_\alpha(N) \to N \to N/\Gamma_\alpha(N) \to 0$ implies that

$$\operatorname{Hom}_R(M, N) \cong \operatorname{Hom}_R(M, \Gamma_\alpha(N)) \tag{3.1}$$

for every α-torsion R-module M.

Consider now an R-module M and a family α of R. Setting the R-modules $E = E(M/\Gamma_\alpha(M))$ and $L = E/(M/\Gamma_\alpha(M))$, we have that E is α-torsion-free and $\operatorname{Hom}_R(R/I, E) = 0$ for every $I \in \alpha$. Furthermore, the exact sequence of R-modules $0 \longrightarrow M/\Gamma_\alpha(M) \longrightarrow E \longrightarrow L \longrightarrow 0$ leads to the isomorphisms of R-modules $\operatorname{Ext}^i_R(R/I, L) \cong \operatorname{Ext}^{i+1}_R(R/I, M/\Gamma_\alpha(M))$ and $H^i_\alpha(L) \cong H^{i+1}_\alpha(M)$ for every i because E is injective.

The number $t = \inf\{i : H^i_\alpha(M) \neq 0\}$ is called the *α-depth of the R-module M* and it is denoted as $\operatorname{depth}(\alpha, M)$. When $H^i_\alpha(M) = 0$ for every i, define $\operatorname{depth}(\alpha, M) = \infty$. Observe that $\operatorname{depth}(\alpha, M) = \inf_{I\in\alpha} \operatorname{grade}(I, M)$ when M is finitely generated as seen in Corollary 1.35.

Proposition 3.7. *Let α be a family of R, M be an R-module and set $t = \operatorname{depth}(\alpha, M)$. Then $\operatorname{Hom}_R(R/I, H^t_\alpha(M)) \cong \operatorname{Ext}^t_R(R/I, M)$ for every $I \in \alpha$.*

Proof. For every ideal $I \in \alpha$ and every R-module N there is an isomorphism of R-modules $\operatorname{Hom}_R(R/I, N) \cong \operatorname{Hom}_R(R/I, \Gamma_\alpha(N))$. Since $\Gamma_\alpha(E)$ is injective for every injective R-module E, it is true by [Rot09, Theorem 10.47] that

$$\operatorname{Ext}^p_R(R/I, H^q_\alpha(N)) \Rightarrow \operatorname{Ext}^{p+q}_R(R/I, N). \tag{3.2}$$

Now $H^j_\alpha(M) = 0$ for every $j < t$. Hence $\operatorname{Hom}_R(R/I, H^t_\alpha(M)) \cong \operatorname{Ext}^t_R(R/I, M)$ for every $I \in \alpha$. $\square$

Recall that a canonical module of the d-dimensional local ring $(R, \mathfrak{m})$ is a finitely generated R-module K_R such that $K_R^\vee = \operatorname{Hom}_R(K_R, E_R(R/\mathfrak{m})) = H^d_\mathfrak{m}(R)$.

Corollary 3.8. *Let R be a Gorenstein local ring of dimension d and α be a family of R. Consider $t = \operatorname{depth}(\alpha, R)$ and $I \in \alpha$ such that $\operatorname{ht} I = t$. Then $\operatorname{Hom}_R(R/I, H^t_\alpha(R))$ is isomorphic to the canonical module $K_{R/I}$ of R/I.*

Proof. Since $\bar{R} = R/I$ is the image of a Gorenstein local ring and $\dim \bar{R} = d - t$, we get the isomorphisms

$$\begin{array}{rcl} K_{R/I} & = & \operatorname{Hom}_{\bar{R}}(H^{d-t}_{\mathfrak{m}\bar{R}}(R/I), E_{\bar{R}}(\bar{R}/\mathfrak{m}\bar{R})) \\ & \cong & \operatorname{Hom}_R(H^{d-t}_{\mathfrak{m}}(R/I), E_R(R/\mathfrak{m})) \\ & \cong & \operatorname{Ext}^t_R(R/I, R) \end{array}$$

and the statement follows. □

Proposition 3.9. *Let R be a Gorenstein local ring of dimension d and α be a family of R. Consider an ideal $I \in \alpha$ such that $\operatorname{ht} I = t = \operatorname{depth}(\alpha, R)$ and R/I is Cohen-Macaulay. If $H^j_\alpha(R) = 0$ for every $j \notin \{t, t+1\}$, then $\operatorname{Ext}^1_R(R/I, H^t_\alpha(R)) = 0$ and $\operatorname{Ext}^i_R(R/I, H^{t+1}_\alpha(R)) \cong \operatorname{Ext}^{i+2}_R(R/I, H^t_\alpha(R))$ for every i.*

Proof. Observe that $\operatorname{Ext}^i_R(R/I, R) = 0$ for every integer $i \neq t$: since $\operatorname{grade}(I, R) = t$, we have that $\operatorname{Ext}^i_R(R/I, R) = 0$ for every $i < t$. On the other hand, since R/I is a Cohen-Macaulay ring of dimension $d - t$, $H^{d-t-i}_{\mathfrak{m}/I}(R/I) = 0$ for $i > 0$. Thus $\operatorname{Ext}^{t+i}_R(R/I, R) = 0$ for every $i > 0$ by Matlis' duality. Now $H^j_\alpha(R) = 0$ for every integer $j \notin \{t, t+1\}$ and we get from [Rot09, Proposition 10.28] the long exact sequence $\cdots \to \operatorname{Ext}^{j-t}_R(R/I, H^t_\alpha(R)) \to \operatorname{Ext}^j_R(R/I, R) \to \operatorname{Ext}^{j-t-1}_R(R/I, H^{t+1}_\alpha(R)) \to \operatorname{Ext}^{j-t+1}_R(R/I, H^t_\alpha(R)) \to \cdots$. Then the statement follows. □

Theorem 3.10. *Consider a non-negative integer t, a family α of R and an ideal $I \in \alpha$. Let M be an R-module such that $\operatorname{Ext}^t_R(R/I, M)$ is finitely generated and $H^i_\alpha(M)$ is α-cofinite for every $i < t$. If $N \subseteq H^t_\alpha(M)$ is such that $\operatorname{Ext}^1_R(R/I, N)$ is finitely generated, then $\operatorname{Hom}_R(R/I, H^t_\alpha(M)/N)$ is finitely generated.*

Proof. Observe that the exact sequence $0 \longrightarrow N \longrightarrow H^t_\alpha(M) \longrightarrow H^t_\alpha(M)/N \longrightarrow 0$ leads to an exact sequence

$$\operatorname{Hom}_R(R/I, H^t_\alpha(M)) \longrightarrow \operatorname{Hom}_R(R/I, H^t_\alpha(M)/N) \longrightarrow \operatorname{Ext}^1_R(R/I, N). \tag{3.3}$$

So if we proved the statement for $N = 0$, it also holds for the general case. Assume then that $N = 0$.

Consider first $t = 0$. Then $\operatorname{Hom}_R(R/I, \Gamma_\alpha(M)) = \operatorname{Hom}_R(R/I, M)$ by equation (3.1), thus the left member of sequence (3.3) will be finitely generated.

Suppose now that $t > 0$. Then $\Gamma_\alpha(M) = H^0_\alpha(M)$ is α-cofinite, whence the R-module $\operatorname{Ext}^i_R(R/I, \Gamma_\alpha(M))$ is finitely generated for every i. The exact sequence of R-modules $0 \longrightarrow \Gamma_\alpha(M) \longrightarrow M \longrightarrow M/\Gamma_\alpha(M) \longrightarrow 0$ gives that $\operatorname{Ext}^t_R(R/I, M/\Gamma_\alpha(M))$

is also finitely generated. Setting $E = E(M/\Gamma_\alpha(M))$ and $L = E/(M/\Gamma_\alpha(M))$ as in Remark 3.6, we also observe that $\mathrm{Ext}^i_R(R/I, L) \cong \mathrm{Ext}^{i+1}_R(R/I, M/\Gamma_\alpha(M))$ and $H^i_\alpha(L) \cong H^{i+1}_\alpha(M)$ for every i. Thus $\mathrm{Ext}^{t-1}_R(R/I, L)$ is also finitely generated and $H^i_\alpha(L)$ is α-cofinite for every $i < t-1$. Thus $\mathrm{Hom}_R(R/I, H^{t-1}_\alpha(L))$ is finitely generated which implies that $\mathrm{Hom}_R(R/I, H^t_\alpha(M))$ is also finitely generated. □

Theorem 3.11. *Let α be a family of R, t be a non-negative integer and M be an R-module such that $H^i_\alpha(M)$ is α-cofinite for every $i < t$. Then*

(i) if $\mathrm{Ext}^{t+1}_R(R/I, M)$ is finitely generated for some $I \in \alpha$, then $\mathrm{Ext}^1_R(R/I, H^t_\alpha(M))$ is finitely generated.

(ii) if $\mathrm{Ext}^i_R(R/I, M)$ is finitely generated for some $I \in \alpha$ and every i, then the R-module $\mathrm{Hom}_R(R/I, H^{t+1}_\alpha(M))$ is finitely generated if and only if the R-module $\mathrm{Ext}^2_R(R/I, H^t_\alpha(M))$ is finitely generated.

Proof. We prove (1) by induction on t. When $t = 0$, the exact sequence of R-modules $0 \longrightarrow \Gamma_\alpha(M) \longrightarrow M \longrightarrow M/\Gamma_\alpha(M) \longrightarrow 0$ gives that $\mathrm{Ext}^1_R(R/I, \Gamma_\alpha(M))$ is finitely generated.

Suppose now that $t > 0$. Since $\Gamma_\alpha(M) = H^0_\alpha(M)$ is α-cofinite, we have that $\mathrm{Ext}^i_R(R/I, \Gamma_\alpha(M))$ is finitely generated for every $I \in \alpha$ and every i. Thus the former exact sequence implies that $\mathrm{Ext}^{t+1}_R(R/I, M/\Gamma_\alpha(M))$ is finitely generated. Taking E and L as in Remark 3.6 we obtain that $H^i_\alpha(L)$ is α-cofinite for every $i < t-1$ and $\mathrm{Ext}^t_R(R/I, L)$ is also finitely generated when $\mathrm{Ext}^{t+1}_R(R/I, M)$ is. Hence $\mathrm{Ext}^1_R(R/I, H^{t-1}_\alpha(L))$ is finitely generated and this implies that $\mathrm{Ext}^1_R(R/I, H^t_\alpha(M))$ is also finitely generated.

In order to prove (2), consider an ideal $I \in \alpha$ such that $\mathrm{Ext}^i_R(R/I, M)$ is finitely generated for every i and $\mathrm{Hom}_R(R/I, H^{t+1}_\alpha(M))$ is finitely generated. Assume first that $t = 0$. We have thus an exact sequence

$$\mathrm{Ext}^1_R(R/I, M/\Gamma_\alpha(M)) \longrightarrow \mathrm{Ext}^2_R(R/I, \Gamma_\alpha(M)) \longrightarrow \mathrm{Ext}^2_R(R/I, M).$$

Taking $L = E(M/\Gamma_\alpha(M))/(M/\Gamma_\alpha(M))$ as in Remark 3.6 we get the isomorphisms

$$\begin{aligned}\mathrm{Ext}^1_R(R/I, M/\Gamma_\alpha(M)) &\cong \mathrm{Hom}_R(R/I, L)\\ &\cong \mathrm{Hom}_R(R/I, \Gamma_\alpha(L))\\ &\cong \mathrm{Hom}_R(R/I, H^1_\alpha(M))\end{aligned}$$

and this implies that $\mathrm{Ext}^2_R(R/I, \Gamma_\alpha(M))$ is finitely generated.

When $t > 0$ we have that $\Gamma_\alpha(M) = H^0_\alpha(M)$ is α-cofinite, hence the R-module $\mathrm{Ext}^i_R(R/I, \Gamma_\alpha(M))$ is finitely generated for every i. Thus $\mathrm{Ext}^i_R(R/I, M/\Gamma_\alpha(M))$ is finitely generated for every i. Hence $\mathrm{Ext}^i_R(R/I, L)$ is also finitely generated for every i, as well as $\mathrm{Hom}_R(R/I, H^t_\alpha(L)) \cong \mathrm{Hom}_R(R/I, H^{t+1}_\alpha(M))$. Then the R-module

$\mathrm{Ext}_R^2(R/I, H_\alpha^{t-1}(L))$ is finitely generated, implying that $\mathrm{Ext}_R^2(R/I, H_\alpha^t(M))$ is also finitely generated.

Conversely, suppose that the R-module $\mathrm{Ext}_R^2(R/I, H_\alpha^t(M))$ is finitely generated and assume again that $t = 0$. Then we have an exact sequence of R-modules

$$\mathrm{Ext}_R^1(R/I, M) \longrightarrow \mathrm{Ext}_R^1(R/I, M/\Gamma_\alpha(M)) \longrightarrow \mathrm{Ext}_R^2(R/I, \Gamma_\alpha(M))$$

and this immediately implies that $\mathrm{Ext}_R^1(R/I, M/\Gamma_\alpha(M))$ is finitely generated. Hence $\mathrm{Hom}_R(R/I, L) \cong \mathrm{Hom}_R(R/I, \Gamma_\alpha(L)) \cong \mathrm{Hom}_R(R/I, H_\alpha^1(M))$ is finitely generated.

When $t > 0$ we have that $\mathrm{Ext}_R^2(R/I, H_\alpha^{t-1}(L))$ and $\mathrm{Ext}_R^i(R/I, L)$ are finitely generated for each i. Thus $\mathrm{Hom}_R(R/I, H_\alpha^t(L)) \cong \mathrm{Hom}_R(R/I, H_\alpha^{t+1}(M))$ is also finitely generated. $\square$

Corollary 3.12. *Let M be a finitely generated R-module, α be a family of R and consider $t = \mathrm{depth}(\alpha, M)$. Then*

(i) the R-module $\mathrm{Ext}_R^1(R/I, H_\alpha^t(M))$ is finitely generated for every $I \in \alpha$.

(ii) for every $I \in \alpha$ the R-module $\mathrm{Ext}_R^2(R/I, H_\alpha^t(M))$ is finitely generated if and only if $\mathrm{Hom}_R(R/I, H_\alpha^{t+1}(M))$ is finitely generated.

Let us recall some basic facts.

Proposition 3.13. *Let α be a family of R and M be a finitely generated R-module such that $\mathrm{depth}(\alpha, M) = t = \mathrm{cd}(\alpha, M)$. If $(E^i(M), d^i)$ is a minimal injective resolution of M, then $(\Gamma_\alpha(E^{t+i}(M)), \Gamma_\alpha(d^{t+i}))$ is a minimal injective resolution of $H_\alpha^t(M)$. If $\mathrm{inj.dim}(M) = d$, then $\mathrm{inj.dim}(H_\alpha^t(M)) \leq d - t$. In particular, if R is local Gorenstein of dimension d, then $\mathrm{inj.dim}(H_\alpha^t(R)) = d - t$.*

Proof. Since $t = \inf\{i : \mu^i(\mathfrak{p}, M) \neq 0 \text{ for some } \mathfrak{p} \in \langle\alpha\rangle \cap \mathrm{Spec}\, R\}$ by Proposition 1.34, we have for every $j < t$ that $\Gamma_\alpha(E^j(M)) = 0$. Thus $H_\alpha^t(M) = \ker \Gamma_\alpha(d^t) \subseteq \Gamma_\alpha(E^t(M))$. Now $H_\alpha^j(M) = 0$ for every $j \neq t$. Thus we obtain an exact sequence of R-modules $0 \longrightarrow H_\alpha^t(M) \longrightarrow \Gamma_\alpha(E^t(M)) \longrightarrow \Gamma_\alpha(E^{t+1}(M)) \longrightarrow \cdots$ which happens to be a minimal injective resolution of $H_\alpha^t(M)$ because $\Gamma_\alpha(E^j(M))$ is the injective hull of $\Gamma_\alpha(\ker d^j) = \ker \Gamma_\alpha(d^j)$ for every j. The second statement is a direct consequence of the first one. The third statement follows because $E^i(R) = \bigoplus_{\substack{\mathfrak{p} \in \mathrm{Spec}\, R \\ \mathrm{ht}\, \mathfrak{p} = i}} E_R(R/\mathfrak{p})$ for every i, hence $\Gamma_\alpha(E^j(R)) \neq 0$ exactly when $t \leq j \leq d$. $\square$

Proposition 3.14. *Let R be a Gorenstein local ring of dimension d and α be a non-trivial family of R. If $H_\alpha^j(R) = 0$ for every $j \neq d - 1$, then, for every $I \in \alpha$, $\mathrm{Ext}_R^1(R/I, H_\alpha^{d-1}(R)) = \Gamma_\mathfrak{m}(R/I)^\vee$ and $\mathrm{Ext}_R^i(R/I, H_\alpha^{d-1}(R)) = 0$ for every $i > 1$.*

Proof. Observe that $H^{d-1}_\alpha(R) \neq 0$. We get that $\operatorname{inj.dim}(H^{d-1}_\alpha(R)) = 1$ from Proposition 3.13. Thus $\operatorname{Ext}^i_R(N, H^{d-1}_\alpha(R)) = 0$ for every R-module N and every $i > 1$. Since $H^q_\alpha(R) \neq 0$ only when $q = d-1$, we obtain the isomorphisms

$$\operatorname{Ext}^1_R(R/I, H^{d-1}_\alpha(R)) \cong \operatorname{Ext}^d_R(R/I, R) \cong H^0_{\mathfrak{m}}(R/I)^\vee$$

for every $I \in \alpha$ by [Rot09, Proposition 10.21] and Matlis' duality. □

Lemma 3.15. *Let $(R, \mathfrak{m})$ be a Gorenstein local ring of dimension d, α be a family of R such that $\sup\limits_{I\in\alpha} \dim(R/I) = 1$ and $H^d_\alpha(R) = 0$. If N is a submodule of a finitely generated free R-module F, then $\operatorname{Hom}_R(N, H^{d-1}_\alpha(R))$ is α-cofinite.*

Proof. By Lemma 3.5, it is enough to show that $\operatorname{Ext}^i_R(R/\mathfrak{p}, \operatorname{Hom}_R(N, H^{d-1}_\alpha(R)))$ is finitely generated for every prime ideal $\mathfrak{p} \in \langle\alpha\rangle$ and every i. By Proposition 3.14 we get that $\operatorname{Ext}^i_R(R/I, H^{d-1}_\alpha(R)) = 0$ for every $I \in \langle\alpha\rangle$ and every $i > 1$. If in addition I is a prime ideal such that $\dim(R/I) = 1$, we get that $\Gamma_{\mathfrak{m}}(R/I) = 0$, whence $\operatorname{Ext}^i_R(R/I, H^{d-1}_\alpha(R)) = 0$ for every $i > 0$ by Proposition 3.14. For such I, the contravariant functor $\operatorname{Hom}_R(-, H^{d-1}_\alpha(R))$ sends projective R-modules to $\operatorname{Hom}_R(R/I, -)$-acyclic modules: in fact, if P is a projective R-module and P_* is a projective resolution of R/I, then

$$\begin{aligned}
\operatorname{Ext}^i_R(R/I, \operatorname{Hom}_R(P, H^{d-1}_\alpha(R))) &= H^i(\operatorname{Hom}_R(P_*, \operatorname{Hom}_R(P, H^{d-1}_\alpha(R)))) \\
&\cong H^i(\operatorname{Hom}_R(P, \operatorname{Hom}_R(P_*, H^{d-1}_\alpha(R)))) \\
&\cong \operatorname{Hom}_R(P, H^i(\operatorname{Hom}_R(P_*, H^{d-1}_\alpha(R)))) \\
&\cong \operatorname{Hom}_R(P, \operatorname{Ext}^i_R(R/I, H^{d-1}_\alpha(R)))
\end{aligned}$$

for every i. Noting $\mathscr{F}(-) = \operatorname{Hom}_R(R/I, \operatorname{Hom}_R(-, H^{d-1}_\alpha(R)))$, we obtain the spectral sequence $\operatorname{Ext}^p_R(R/I, \operatorname{Ext}^q_R(N, H^{d-1}_\alpha(R))) \Rightarrow (R^{p+q}\mathscr{F})(N)$. We also have exact sequences $\operatorname{Ext}^q_R(F, H^{d-1}_\alpha(R)) \longrightarrow \operatorname{Ext}^q_R(N, H^{d-1}_\alpha(R)) \longrightarrow \operatorname{Ext}^{q+1}_R(F/N, H^{d-1}_\alpha(R))$. Consider $q > 0$. Since F is free, we have that $\operatorname{Ext}^q_R(F, H^{d-1}_\alpha(R)) = 0$. Now Proposition 3.13 gives that $\operatorname{inj.dim}(H^{d-1}_\alpha(R)) = 1$, whence $\operatorname{Ext}^{q+1}_R(F/N, H^{d-1}_\alpha(R)) = 0$. So $\operatorname{Ext}^q_R(N, H^{d-1}_\alpha(R)) = 0$ for every $q > 0$ and the spectral sequence collapses to produce isomorphisms $\operatorname{Ext}^p_R(R/I, \operatorname{Hom}_R(N, H^{d-1}_\alpha(R))) \cong (R^p\mathscr{F})(N)$ for every p.

We affirm that $(R^p\mathscr{F})(N)$ is finitely generated for every p. In fact, consider a projective resolution F_* of N. Then

$$(R^p\mathscr{F})(N) \cong H^p(\operatorname{Hom}_R(F_*, \operatorname{Hom}_R(R/I, H^{d-1}_\alpha(R)))).$$

By Corollary 3.8 we get that $\operatorname{Hom}_R(R/I, H^{d-1}_\alpha(R)) \cong K_{R/I}$, thus the isomorphism $(R^p\mathscr{F})(N) \cong \operatorname{Ext}^p_R(N, K_{R/I})$ holds. Hence the affirmation is true and the R-module $\operatorname{Ext}^i_R(R/I, \operatorname{Hom}_R(N, H^{d-1}_\alpha(R)))$ is finitely generated for every i and every prime ideal $I \in \langle\alpha\rangle$ such that $\dim(R/I) = 1$.

Finally, Lemma 3.5 guarantees that $\operatorname{Ext}^i_R(R/\mathfrak{m}, \operatorname{Hom}_R(N, H^{d-1}_\alpha(R)))$ is finitely generated for every i. Thus $\operatorname{Hom}_R(N, H^{d-1}_\alpha(R))$ is α-cofinite. □

We regard now some properties of change of rings.

Proposition 3.16. *Let $f : R \to S$ be a surjective homomorphism of rings, α be a family of R and M be an S-module. Then M is α-cofinite if and only if M is αS-cofinite.*

Proof. First, observe that $\operatorname{Supp}_R(M) \subseteq \langle\alpha\rangle$ if and only if $\operatorname{Supp}_S(M) \subseteq \langle\alpha S\rangle$. For every $I \in \alpha$, [Rot09, Theorem 10.62] gives a spectral sequence

$$E_2^{pq} = \operatorname{Ext}_S^p(\operatorname{Tor}_q^R(S, R/I), M) \Rightarrow \operatorname{Ext}_R^{p+q}(R/I, M).$$

If M is αS-cofinite, then $E_2^{p0} = \operatorname{Ext}_S^p(S/IS, M)$ is finitely generated for every p. Now $\operatorname{Supp}(\operatorname{Tor}_q^R(S, R/I)) \subseteq \operatorname{Supp}(S/IS)$ for every q. Hence E_2^{pq} is finitely generated for every p and every q. Since the spectral sequence is bounded, we obtain that $\operatorname{Ext}_R^p(R/I, M)$ is finitely generated for every p. Thus M is α-cofinite.

For the converse, observe that $E_2^{00} = \operatorname{Hom}_S(S/IS, M) = \operatorname{Hom}_R(R/I, M)$ is finitely generated. Suppose now that $n > 0$ and that E_2^{p0} is finitely generated for every $p < n$. Hence E_2^{pq} is finitely generated for every q and every $p < n$. Since $H^n = \operatorname{Ext}_R^n(R/I, M)$ is finitely generated, we get that E_2^{n0} is finitely generated by [DM97, Lemma 1]. Thus $\operatorname{Ext}_S^n(I/IS, M)$ is finitely generated for every n and every $I \in \alpha$, whence M is αS-cofinite. $\square$

Lemma 3.17. *Let α be a family of a local ring $(R, \mathfrak{m})$, M be an R-module and denote by $\hat{R}$ the $\mathfrak{m}$-adic completion of R. Then $H_\alpha^i(M)$ is α-cofinite if and only if $H_{\alpha\hat{R}}^i(M \otimes \hat{R})$ is $\alpha\hat{R}$-cofinite.*

Proof. Recall that there exists a natural homomorphism

$$\operatorname{Hom}_R(L, M) \otimes N \to \operatorname{Hom}_R(L, M \otimes N)$$

which is an isomorphism if N is flat and L is finitely presented. Then, for every j, $\operatorname{Ext}_{\hat{R}}^j(\hat{R}/I\hat{R}, H_{\alpha\hat{R}}^i(M \otimes \hat{R})) \cong \operatorname{Ext}_{\hat{R}}^j(R/I \otimes \hat{R}, H_\alpha^i(M) \otimes \hat{R}) \cong \operatorname{Ext}_R^j(R/I, H_\alpha^i(M)) \otimes \hat{R}$ and we conclude the statement. $\square$

Let $(R, \mathfrak{m})$ be a local ring and M be an R-module. Recall that a prime $\mathfrak{p}$ is called a *coassociated prime ideal of* M when $\mathfrak{p}$ is an associated prime ideal of its Matlis dual $M^\vee = \operatorname{Hom}_R(M, E_R(R/\mathfrak{m}))$. Observe that $\operatorname{Coass}(M \otimes_R N) = \operatorname{Supp}(M) \cap \operatorname{Coass}(N)$ for every finitely generated R-module M and every R-module N, see [DM97, Remark 1]. In particular $\operatorname{Coass}(H_\alpha^n(M)) = \operatorname{Supp}(M) \cap \operatorname{Coass}(H_\alpha^n(R))$ when $n \geq \operatorname{cd}(\alpha, R)$.

Proposition 3.18. *Let $(R, \mathfrak{m})$ be a complete local ring, α be a non-trivial family of R and M be a finitely generated R-module of dimension $n \geq 0$. Then*

$$\begin{aligned} \operatorname{Coass}(H_\alpha^n(M)) &= \left\{\mathfrak{p} \in \operatorname{Supp}(M) : \dim(R/\mathfrak{p}) = n \textit{ and } I + \mathfrak{p} \in \tilde{W}(\mathfrak{m}) \textit{ for every } I \in \alpha\right\} \\ &= \bigcap_{I \in \alpha - \{R\}} \operatorname{Coass}(H_I^n(M)). \end{aligned}$$

Proof. Set $\bar{R} = R/\operatorname{Ann}(M)$ and $E = E_R(R/\mathfrak{m})$. Then $E_{\bar{R}}(\bar{R}/\mathfrak{m}\bar{R}) = \operatorname{Hom}_R(\bar{R}, E)$ and

$$\operatorname{Hom}_{\bar{R}}(H^n_{\alpha\bar{R}}(M), \operatorname{Hom}_R(\bar{R}, E)) \cong \operatorname{Hom}_R(H^n_{\alpha\bar{R}}(M) \otimes_{\bar{R}} \bar{R}, E) \cong \operatorname{Hom}_R(H^n_\alpha(M), E).$$

Hence we can assume that M is faithful and $n = \dim(R)$. In this case, we also have that $\operatorname{Coass}(H^n_\alpha(M)) = \operatorname{Coass}(H^n_\alpha(R))$, so it is enough to suppose $M = R$. Suppose that $H^n_\alpha(R) \neq 0$ for both sets of the statement are empty when $H^n_\alpha(R) = 0$. Consider $\mathfrak{q} \in \operatorname{Coass}(H^n_\alpha(R))$. Then $H^n_\alpha(R/\mathfrak{q}) \neq 0$. Hence $\dim(R/\mathfrak{q}) = n$ and $I + \mathfrak{q} \in \tilde{W}(\mathfrak{m})$ for every $I \in \alpha$ by Corollary 1.46. The converse also holds: if $\dim(R/\mathfrak{q}) = n$ and $I + \mathfrak{q}$ is $\mathfrak{m}$-primary for every proper $I \in \alpha$, then $H^n_\alpha(R) \otimes_R R/\mathfrak{q} = H^n_\alpha(R/\mathfrak{q}) \neq 0$. If $\mathfrak{p} \in \operatorname{Coass}(H^n_\alpha(R/\mathfrak{q}))$, then $\mathfrak{p} \supseteq \mathfrak{q}$ and $\mathfrak{p} \in \operatorname{Coass}(H^n_\alpha(R))$. But $\dim(R/\mathfrak{p}) = n$. Hence $\mathfrak{q} \in \operatorname{Coass}(H^n_\alpha(R))$ when $\dim(R/\mathfrak{q}) = n$ and $I + \mathfrak{q} \in \tilde{W}(\mathfrak{m})$ for every $I \in \alpha$. □

When $\langle\alpha\rangle = \tilde{W}(\langle\varphi\rangle, \psi)$, we get a refinement of the previous result.

Corollary 3.19. *Let $(R, \mathfrak{m})$ be a complete local ring, φ and ψ be non-trivial families of R and M be a finitely generated R-module of dimension $n \geq 0$. Then*

$$\operatorname{Coass}(H^n_{\varphi,\psi}(M)) = \left\{\mathfrak{p} \in \operatorname{Supp}(M) \cap \langle\psi\rangle : \dim(R/\mathfrak{p}) = n \text{ and } I + \mathfrak{p} \in \tilde{W}(\mathfrak{m}) \text{ for every } I \in \varphi\right\}.$$

Proof. We will show that the right-hand term of the statement is equal to the right-hand term of Proposition 3.18. Indeed, if $\mathfrak{p} \in \operatorname{Supp}(M) \cap \langle\psi\rangle$ is such that $I + \mathfrak{p} \in \tilde{W}(\mathfrak{m})$ for every $I \in \varphi$, consider $K \in \tilde{W}(\langle\varphi\rangle, \psi)$. Thus $K + \mathfrak{p} \in \langle\varphi\rangle$ and there exist $I_1, \ldots, I_s \in \varphi$ such that $K + \mathfrak{p} \supseteq I_1 \cdots I_s$. Since $I_i + \mathfrak{p} \in \tilde{W}(\mathfrak{m})$ for every i, we get that $K + \mathfrak{p} \in \tilde{W}(\mathfrak{m})$ and thus $\mathfrak{p} \in \operatorname{Coass}(H^n_{\varphi,\psi}(M))$.

For the converse, consider a prime ideal $\mathfrak{p} \in \operatorname{Supp}(M)$ such that $I + \mathfrak{p} \in \tilde{W}(\mathfrak{m})$ for every ideal $I \in \tilde{W}(\langle\varphi\rangle, \psi)$. If $\mathfrak{p} \notin \langle\psi\rangle$, then $J \nsubseteq \mathfrak{p}$ for every $J \in \psi$. Hence $\dim(R/(J + \mathfrak{p})) < \dim(R/\mathfrak{p}) \leq n$ for every $J \in \psi$ and $H^n_{\varphi,\psi}(R/\mathfrak{p}) = 0$. Thus we obtain that $V(\mathfrak{p}) \cap \operatorname{Coass}(H^n_{\varphi,\psi}(R)) = \operatorname{Coass}(H^n_{\varphi,\psi}(R/\mathfrak{p})) = \emptyset$ and $\mathfrak{p} \notin \operatorname{Coass}(H^n_{\varphi,\psi}(R))$. So $\mathfrak{p} \in \langle\psi\rangle$. If $I \in \varphi$, then $I + \mathfrak{p} \in \tilde{W}(\mathfrak{m})$ because $\varphi \subseteq \tilde{W}(\langle\varphi\rangle, \psi)$ and the statement is proved. □

Theorem 3.20. *Let R be a local ring, α be a family of R and M be a finitely generated R-module of dimension n. Then $H^n_\alpha(M)$ is α-cofinite. Moreover, $\operatorname{Ext}^i_R(R/I, H^n_\alpha(M))$ has finite length for every $I \in \alpha$ and every i.*

Proof. By Lemma 3.17, we can suppose that R is complete. The R-module $H^n_\alpha(M)$ is Artinian, whence the R-module $H^n_\alpha(M)^\vee$ is finitely generated. Consider the finite set $\operatorname{Coass}(H^n_\alpha(M)) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_s\}$. Then $\operatorname{Supp}(H^n_\alpha(M)^\vee) = V(\mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_s)$. Now, for each $I \in \alpha$ and every i, the R-module $\operatorname{Tor}^R_i(R/I, H^n_\alpha(M)^\vee)$ is finitely generated. Furthermore,

$$\begin{aligned}
\operatorname{Supp}_R(\operatorname{Tor}^R_i(R/I, H^n_\alpha(M)^\vee)) &\subseteq V(I) \cap \operatorname{Supp}(H^n_\alpha(M)^\vee) \\
&= V(I) \cap V(\mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_s) \\
&\subseteq \{\mathfrak{m}\}.
\end{aligned}$$

Thus $\operatorname{Tor}_i^R(R/I, H_\alpha^n(M)^\vee)$ has finite length. Also,

$$\operatorname{Tor}_i^R(R/I, H_\alpha^n(M)^\vee)^\vee \cong \operatorname{Ext}_R^i(R/I, H_\alpha^n(M))$$

for every i (see [HK91, Remark 2.1]). Hence $\operatorname{Ext}_R^i(R/I, H_\alpha^n(M))$ has finite length for every $I \in \alpha$ and every i by Matlis' duality. $\square$

3.2 Associated primes of local cohomology modules

In this section we investigate the associated prime ideals of the first non-zero local cohomology modules. We improve some results appearing in [TT10].

Observe that $\operatorname{Ass}(H_\alpha^i(M)) \subseteq \bigcup_{I \in \langle\alpha\rangle} \operatorname{Ass}(H_I^i(M))$ for every family α of R, every R-module M and every i. In fact, if $\mathfrak{p} \in \operatorname{Ass}(H_\alpha^i(M))$, then $\mathfrak{p}R_\mathfrak{p} \in \operatorname{Ass}(H_\alpha^i(M)_\mathfrak{p})$ and $\operatorname{Hom}_{R_\mathfrak{p}}(R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}, H_\alpha^i(M)_\mathfrak{p}) \neq 0$. Hence $\varinjlim_{I \in \langle\alpha\rangle} \operatorname{Hom}_{R_\mathfrak{p}}(R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}, H_I^i(M)_\mathfrak{p}) \neq 0$ and this implies that $\mathfrak{p} \in \operatorname{Ass}(H_I^i(M))$ for some $I \in \langle\alpha\rangle$.

Moreover, $\operatorname{Ass}(H_\alpha^t(M)) \subseteq \bigcup_{\substack{I \in \langle\alpha\rangle \\ \operatorname{grade}(I,M)=t}} \operatorname{Ass}(H_I^t(M))$ when M is finitely generated and $t = \operatorname{depth}(\alpha, M)$. This follows because $t \leq \operatorname{grade}(I, M)$ for every $I \in \langle\alpha\rangle$, hence $H_I^t(M) = 0$ when $\operatorname{grade}(I, M) > t$. We have thus a relation between families.

Lemma 3.21. *Let M be a finitely generated R-module, α and β be families of R such that $\beta \subseteq \langle\alpha\rangle$ and set $t = \operatorname{depth}(\alpha, M)$. Then $\operatorname{Ass}(H_\beta^t(M)) \subseteq \operatorname{Ass}(H_\alpha^t(M))$. In particular, $\operatorname{Ass}(H_\alpha^t(M)) = \bigcup_{\substack{I \in \langle\alpha\rangle \\ \operatorname{grade}(I,M)=t}} \operatorname{Ass}(H_I^t(M))$.*

Proof. Observe that $\operatorname{depth}(\beta, M) \geq t$. First statement follows for if $(E^i(M), \partial^i)$ is a minimal injective resolution of M, then

$$H_\beta^t(M) = \ker \Gamma_\beta(\partial^t) \subseteq \ker \Gamma_\alpha(\partial^t) = H_\alpha^t(M).$$

Second statement follows because $\operatorname{grade}(I, M) = \operatorname{depth}(I, M) \geq \operatorname{depth}(\alpha, M)$ for every $I \in \langle\alpha\rangle$. $\square$

We can improve the second statement of the previous result.

Proposition 3.22. *Let M be a finitely generated R-module and α be a family of R. Consider $t = \operatorname{depth}(\alpha, M)$. Then*

$$\operatorname{Ass}(H_\alpha^t(M)) = \bigcup_{\substack{\mathfrak{p} \in \langle\alpha\rangle \cap \operatorname{Spec} R \\ \operatorname{grade}(\mathfrak{p},M)=t}} \operatorname{Ass}(H_\mathfrak{p}^t(M)) = \bigcup_{\substack{I \in \alpha \\ \operatorname{grade}(I,M)=t}} \operatorname{Ass}(H_I^t(M)).$$

Proof. In order to prove that $\operatorname{Ass}(H^t_\alpha(M)) \subseteq \bigcup_{\substack{\mathfrak{p}\in\langle\alpha\rangle\cap\operatorname{Spec} R\\ \operatorname{grade}(\mathfrak{p},M)=t}} \operatorname{Ass}(H^t_\mathfrak{p}(M))$ it is enough to show that $\operatorname{Ass}(H^t_\alpha(M)) \subseteq \bigcup_{\mathfrak{p}\in\langle\alpha\rangle\cap\operatorname{Spec} R} \operatorname{Ass}(H^t_\mathfrak{p}(M))$. Consider a minimal injective resolution $(E^*(M), \partial^*)$ of M. Then $H^t_\alpha(M) = \Gamma_\alpha(\ker\partial^t)$. If $\mathfrak{p} \in \operatorname{Ass}(H^t_\alpha(M))$, then $\mathfrak{p} = \operatorname{Ann}(x)$ for some element $x \in \Gamma_\alpha(\ker\partial^t)$. Thus $\mathfrak{p} \in \langle\alpha\rangle$ and $x \in H^t_\mathfrak{p}(M)$. Now if $\mathfrak{p} \in \operatorname{Ass}(H^t_\mathfrak{q}(M))$ for some prime ideal $\mathfrak{q} \in \langle\alpha\rangle$, then $\operatorname{grade}(\mathfrak{q}, M) = t$ and there exists an ideal $I \in \alpha$ such that $I \subseteq \mathfrak{q}$. Hence $t \leq \operatorname{grade}(I, M) \leq \operatorname{grade}(\mathfrak{q}, M) = t$. All of this imply that $H^t_\mathfrak{q}(M) = \Gamma_\mathfrak{q}(\ker\partial^t) \subseteq \Gamma_I(\ker\partial^t) = H^t_I(M)$ and $\mathfrak{p} \in \operatorname{Ass}(H^t_I(M))$. Finally, the relation $\bigcup_{\substack{I\in\alpha\\ \operatorname{grade}(I,M)=t}} \operatorname{Ass}(H^t_I(M)) \subseteq \operatorname{Ass}(H^t_\alpha(M))$ follows readily from Lemma 3.21. □

We remark an additional fact that was proved in the previous result.

Corollary 3.23. *Let M be a finitely generated R-module and α be a family of R. Consider $t = \operatorname{depth}(\alpha, M)$. Then $\operatorname{Ass}(H^t_\alpha(M)) \subseteq \{\mathfrak{p} \in \langle\alpha\rangle : \operatorname{grade}(\mathfrak{p}, M) = t\}$.*

Corollary 3.24. *Let α be a family of R, M be a finitely generated R-module and consider $t = \operatorname{depth}(\alpha, M)$. If $I \in \langle\alpha\rangle$ is such that $\operatorname{Hom}_R(R/I, H^t_\alpha(M)) \neq 0$, then $\operatorname{grade}(I, M) = t$.*

Proof. If $I \in \langle\alpha\rangle$, then $\operatorname{grade}(I, M) = \operatorname{depth}(I, M) \geq \operatorname{depth}(\alpha, M) = t$. On the other hand, if the prime $\mathfrak{p} \in \operatorname{Ass}(\operatorname{Hom}_R(R/I, H^t_\alpha(M)))$, then $I \subseteq \mathfrak{p}$ and

$$\operatorname{grade}(I, M) \leq \operatorname{grade}(\mathfrak{p}, M) = t.$$

Hence $\operatorname{grade}(I, M) = t$. □

In general, the first non-zero local cohomology module is not Artinian as we shall see next.

Corollary 3.25. *Let $(R, \mathfrak{m})$ be a local ring and M be a Cohen-Macaulay R-module. Consider a family α of R such that $\dim(M/IM) > 0$ for some ideal $I \in \langle\alpha\rangle$ and set $t = \operatorname{depth}(\alpha, M)$. Then $\mathfrak{m} \notin \operatorname{Ass}(H^t_\alpha(M))$.*

Proof. If $\mathfrak{m} \in \operatorname{Ass}(H^t_\alpha(M))$, then $\operatorname{grade}(\mathfrak{m}, M) = t$. On the other hand, there exists $I \in \langle\alpha\rangle$ such that $\operatorname{grade}(I, M) = \dim(M) - \dim(M/IM) < \operatorname{depth}(M)$ and this is a contradiction. □

Chapter 4

Endomorphism modules

Unless stated otherwise, all the rings in this chapter are Noetherian. Rings are also assumed local where the Matlis dual functor $(-)^\vee = \mathrm{Hom}_R(-, E)$ is used (here E always denotes the injective hull of the residual field of the local ring R).

In this chapter, we are concerned with the modules of endomorphisms of local cohomology modules and we investigate them in two phases. The first section studies the endomorphisms of first local cohomology modules, along with the special case when the cohomological depth equals the cohomological dimension. This part basically extends some ideas from [Mah13].

The second one takes care of the top local cohomology modules. It exploits the Lichtenbaum-Hartshorne Vanishing Theorem conditions and the Artinian nature of said modules to obtain informations on the ring structure of their modules of endomorphisms in an analogous way as done in [ES12].

4.1 On α-depth level

Recall that there exists a natural transformation

$$\mathrm{Hom}_R(N, P) \otimes_R M \to \mathrm{Hom}_R(\mathrm{Hom}_R(M, N), P) \tag{4.1}$$

which is an isomorphism when M is finitely generated and P is injective (see [Rot09, Lemma 9.71]). Hence $M \otimes_R N^\vee \cong \mathrm{Hom}_R(M, N)^\vee$ for every finitely generated R-module

M. Furthermore, if M is α-torsion, then

$$M \otimes_R \Gamma_\alpha(N)^\vee = M \otimes_R N^\vee \tag{4.2}$$

for every R-module N by equation (3.1). Observe that the natural isomorphism (4.1) suggests that $\operatorname{Hom}_R(I, J)$ is a flat R-module for every pair of injective R-modules I and J. In particular, the Matlis dual of any injective R-module is flat.

Notice that the composite functor $G = \Gamma_\alpha(-)^\vee$ is right exact contravariant. Hence, it makes sense to talk about its left derived functors $(L_i G)$ by taking injective resolutions of R-modules. Since $\Gamma_\alpha(-)$ is left exact and $(-)^\vee$ is contravariant exact, it is readily seen that $(L_i G)(M) = H^i_\alpha(M)^\vee$ for every R-module M and every i. Similar to right derived functors, we say that an R-module M is *left G-acyclic* exactly when $(L_i G)(M) = 0$ for every $i \geq 1$. In order to put in clearer evidence the dual relation between Tor and Ext, we use a modification of Grothendieck's spectral sequences (Theorem A.1) in order to prove the following statement.

Proposition 4.1. *Let α be a family of R, M be a finitely generated R-module and set $c = \operatorname{depth}(\alpha, M)$. Suppose that N is an α-torsion R-module. Then*

(i) There is an isomorphism $\operatorname{Ext}^c_R(N, M) \cong \operatorname{Hom}_R(N, H^c_\alpha(M))$ and $\operatorname{Ext}^i_R(N, M) = 0$ for every $i < c$.

(ii) There is an isomorphism $\operatorname{Tor}^R_c(N, M^\vee) \cong N \otimes_R H^c_\alpha(M)^\vee$ and $\operatorname{Tor}^R_i(N, M^\vee) = 0$ for every $i < c$.

Proof. The proof of the first part of (i) is similar to that of Proposition 3.7. For the second part, just observe that if $(E^*(M)) = (E^i(M))_{i \geq 0}$ is a minimal injective resolution of a finitely generated R-module M, then $E^i(M)$ is α-torsion-free for $i < c$, see Proposition 1.34 or Proposition 3.13. Thus $\operatorname{Hom}_R(N, E^i(M)) = 0$ for $i < c$ by equation (3.1).

Now we prove (ii): regarding N as an inductive limit of its finitely generated submodules, we obtain that $N \otimes_R \Gamma_\alpha(M)^\vee \cong N \otimes_R M^\vee$ for any R-module M by equation (4.2). Since the Matlis dual of every injective module is flat, we obtain a spectral sequence $\operatorname{Tor}^R_p(N, H^q_\alpha(M)^\vee) \Rightarrow \operatorname{Tor}^R_{p+q}(N, M^\vee)$ by Theorem A.1. Now $H^q_\alpha(M)^\vee = 0$ for every $q < c$. Thus $N \otimes_R H^c_\alpha(M)^\vee \cong \operatorname{Tor}^R_c(N, M^\vee)$. On the other hand,

$$N \otimes_R E^i(M)^\vee \cong N \otimes_R \Gamma_\alpha(E^i(M))^\vee = 0$$

for every $i < c$ when $(E^i(M))$ is a minimal injective resolution of a finitely generated R-module M. Thus $\operatorname{Tor}^R_i(N, M^\vee) = 0$ for every $i < c$ because $(E^i(M)^\vee)_{i \geq 0}$ is a flat resolution of $M^\vee$. □

In this way, we obtain a dual characterisation for depth in terms of Tor functors.

Corollary 4.2. *For every finitely generated module M over a local ring R,*

$$\operatorname{depth}(\alpha, M) = \inf\left\{i : \operatorname{Tor}_i^R(R/I, M^\vee) \neq 0 \text{ for some } I \in \alpha\right\}.$$

Proof. Set $c = \operatorname{depth}(\alpha, M) = \inf_{I\in\alpha} \operatorname{grade}(I, M)$ by Corollary 1.35. We already know that $\operatorname{Tor}_i^R(R/I, M^\vee) = 0$ for every $I \in \alpha$ and every $i < c$ by the previous proposition. Now $\operatorname{Tor}_c^R(R/I, M^\vee) \cong R/I \otimes_R H_\alpha^c(M)^\vee$ for every $I \in \alpha$. Lemma A.2, (ii), gives that $R/I \otimes_R H_\alpha^c(M)^\vee \cong \operatorname{Hom}_R(R/I, H_\alpha^c(M))^\vee$. Thus $\operatorname{Tor}_c^R(R/I, M^\vee) \cong \operatorname{Ext}_R^c(R/I, M)^\vee$ for every $I \in \alpha$. In this way, if $\operatorname{Tor}_c^R(R/I, M^\vee) = 0$ for every $I \in \alpha$, we have that $\varprojlim_{I\in\langle\alpha\rangle} \operatorname{Ext}_\alpha^c(R/I, M)^\vee = 0$, but this means that $H_\alpha^c(M)^\vee = 0$, which is an absurd. Hence $c \geq \inf\left\{i : \operatorname{Tor}_i^R(R/I, M^\vee) \neq 0 \text{ for some } I \in \alpha\right\}$. Conversely, apply the previous arguments to each family $\{I\} \subseteq \alpha$ in order to obtain

$$\operatorname{grade}(I, M) = \inf\left\{i : \operatorname{Tor}_i^R(R/I, M^\vee) \neq 0\right\},$$

as seen in [MZ14]. Hence $c \leq \inf\left\{i : \operatorname{Tor}_i^R(R/I, M^\vee) \neq 0 \text{ for some } I \in \alpha\right\}$. □

We state now an extension of the Local Duality Theorem which serves as a generalisation to [Mah13].

Theorem 4.3. *Let α be a family of a local ring R. Assume that α is* cohomologically complete intersection *(this is,* $\operatorname{depth}(\alpha, R) = n = \operatorname{cd}(\alpha, R)$*). Then, for every R-module M and every integer i,*

(i) $\operatorname{Tor}_{n-i}^R(M, H_\alpha^n(R)) \cong H_\alpha^i(M)$.

(ii) $H_\alpha^i(M)^\vee \cong \operatorname{Ext}_R^{n-i}(M, H_\alpha^n(R)^\vee)$.

Proof. It is enough to prove (i) because of Lemma A.2. Noting $T_j(-) = H_\alpha^{n-j}(-)$, we shall show the isomorphism of ∂-functors $\operatorname{Tor}_i^R(-, H_\alpha^n(R)) \cong T_i(-)$ for $i \geq 0$. Since $n = \operatorname{cd}(\alpha, R)$, we have that $H_\alpha^n(M) = M \otimes_R H_\alpha^n(R)$ for every R-module M by Lemma 1.39. Thus $\operatorname{Tor}_0^R(-, H_\alpha^n(R)) \cong T_0(-)$. Suppose now that M is free and $i > 0$. Hence $\operatorname{Tor}_i^R(M, H_\alpha^n(R)) = 0 = T_i(M)$. Thus the desired isomorphism follows by [Rot09, Theorem 6.36]. □

Lemma 4.4. *Let R be a complete local ring of dimension n and α be a family of R. Consider a finitely generated R-module M and set $c = \operatorname{depth}(\alpha, M)$. Then*

(i) There is an isomorphism $\operatorname{Hom}_R(H_\alpha^c(M), H_\alpha^c(M)) \cong \operatorname{Hom}_R(H_\alpha^c(M)^\vee, H_\alpha^c(M)^\vee)$.

(ii) The natural homomorphism $R \to \operatorname{Hom}_R(H_\alpha^c(M), H_\alpha^c(M))$ is an isomorphism if and only if the natural homomorphism $R \to \operatorname{Hom}_R(H_\alpha^c(M)^\vee, H_\alpha^c(M)^\vee)$ is an isomorphism if and only if the natural homomorphism $H_\alpha^c(M) \otimes_R H_\alpha^c(M)^\vee \to E$ is an isomorphism.

Proof. There is an isomorphism $\operatorname{Hom}_R(H^c_\alpha(M)^\vee, H^c_\alpha(M)^\vee) \cong (H^c_\alpha(M)^\vee \otimes_R H^c_\alpha(M))^\vee$ by Lemma A.2 and the module on the right side is isomorphic to $\operatorname{Tor}^R_c(H^c_\alpha(M), M^\vee)^\vee$ by Proposition 4.1 which, in turn, is isomorphic to $\operatorname{Ext}^c_R(H^c_\alpha(M), M^{\vee\vee})$ again by Lemma A.2. Since R is complete, we have that $M^{\vee\vee} = M$ and statement (i) follows.

We now prove the equivalences in (ii): observe that the natural homomorphism $R \to \operatorname{Hom}_R(H^c_\alpha(M), H^c_\alpha(M))$ is an isomorphism if and only if the natural homomorphism $R \to \operatorname{Hom}_R(H^c_\alpha(M)^\vee, H^c_\alpha(M)^\vee)$ is an isomorphism by (i). By Matlis' duality, a homomorphism is an isomorphism if and only if its induced homomorphism in the duals is an isomorphism. Hence the natural homomorphism $H^c_\alpha(M) \otimes_R H^c_\alpha(M)^\vee \to E$ is an isomorphism if and only if the natural homomorphism $E^\vee \to \operatorname{Hom}_R(H^c_\alpha(M), H^c_\alpha(M))$ is an isomorphism. Thus the statement follows by completeness of R. $\square$

Lemma 4.5. *Let α be a family of ideals of a complete local ring R. Consider a finitely generated R-module M such that* $\operatorname{depth}(\alpha, M) = c = \operatorname{cd}(\alpha, M)$. *Then, for every integer $i \neq c$, the following statements hold:*

(i) There exist isomorphisms:

1. $\operatorname{Ext}^{i-c}_R(H^c_\alpha(M), H^c_\alpha(M)) \cong \operatorname{Ext}^i_R(H^c_\alpha(M), M)$.
2. $\operatorname{Tor}^R_{i-c}(H^c_\alpha(M), H^c_\alpha(M)^\vee) \cong \operatorname{Tor}^R_i(H^c_\alpha(M), M^\vee)$.

(ii) These statements are equivalent:

1. $\operatorname{Ext}^{i-c}_R(H^c_\alpha(M), H^c_\alpha(M)) = 0$.
2. $\operatorname{Ext}^{i-c}_R(H^c_\alpha(M)^\vee, H^c_\alpha(M)^\vee) = 0$.
3. $\operatorname{Tor}^R_{i-c}(H^c_\alpha(M), H^c_\alpha(M)^\vee) = 0$.

Proof. Consider a minimal injective resolution $(E^*(M)) = (E^i(M))_{i\geq 0}$ of M. By Proposition 3.13, $(\Gamma_\alpha(E^{i+c}(M)))_{i\geq 0}$ is a minimal injective resolution of $H^c_\alpha(M)$. Since $H^c_\alpha(M)$ is α-torsion, we get an isomorphism

$$\operatorname{Ext}^{i-c}_R(H^c_\alpha(M), H^c_\alpha(M)) \cong \operatorname{Ext}^i_R(H^c_\alpha(M), M)$$

as follows: for every integer j there are isomorphisms

$$\begin{aligned} \operatorname{Ext}^j_R(H^c_\alpha(M), H^c_\alpha(M)) &= H^j(\operatorname{Hom}_R(H^c_\alpha(M), \Gamma_\alpha(E^{*+c}(M)))) \\ &= H^j(\operatorname{Hom}_R(H^c_\alpha(M), E^{*+c}(M))) \\ &= \operatorname{Ext}^{j+c}_R(H^c_\alpha(M), M). \end{aligned}$$

So we obtained the first isomorphism of (i). By applying the Matlis dual to the minimal injective resolutions of M and $H^c_\alpha(M)$ we obtain flat resolutions of $M^\vee$ and $H^c_\alpha(M)^\vee$. Now we have natural isomorphisms $H^c_\alpha(M) \otimes_R \Gamma_\alpha(E^i(M)) \cong H^c_\alpha(M) \otimes_R E^i(M)$ for every i. Hence $\operatorname{Tor}^R_{i-c}(H^c_\alpha(M), H^c_\alpha(M)^\vee) \cong \operatorname{Tor}^R_i(H^c_\alpha(M), M^\vee)$.

We now prove the equivalences in (ii): there exists an isomorphism

$$\operatorname{Ext}_R^{i-c}(H^c_\alpha(M)^\vee, H^c_\alpha(M)^\vee) \cong \operatorname{Tor}^R_{i-c}(H^c_\alpha(M)^\vee, H^c_\alpha(M))^\vee$$

by Lemma A.2. Together with Matlis' duality, we also get the isomorphism

$$\operatorname{Tor}^R_{i-c}(H^c_\alpha(M), M)^\vee \cong \operatorname{Ext}_R^{i-c}(H^c_\alpha(M), M).$$

Then the equivalences follow from these isomorphisms and those of (i). □

Proposition 4.6. *Let R be a local ring of dimension n and $\alpha \subseteq \langle\beta\rangle$ be two families of R. Set $c = \operatorname{depth}(\beta, M)$, where M is a finitely generated R-module. Then*

(i) There exists a natural homomorphism

$$\operatorname{Hom}_R(H^c_\beta(M), H^c_\beta(M)) \to \operatorname{Hom}_R(H^c_\alpha(M), H^c_\alpha(M)).$$

(ii) Suppose in addition that $\langle\alpha R_\mathfrak{p}\rangle = \langle\beta R_\mathfrak{p}\rangle$ for every $\mathfrak{p} \in \langle\beta\rangle \cap \operatorname{Supp}(M)$ such that $\operatorname{depth}_{R_\mathfrak{p}}(M_\mathfrak{p}) = c$. Then the homomorphism in (i) is an isomorphism.

(iii) Suppose now that R is complete. Then there exists a natural homomorphism $\operatorname{Hom}_R(H^c_\beta(M)^\vee, H^c_\beta(M)^\vee) \to \operatorname{Hom}_R(H^c_\alpha(M)^\vee, H^c_\alpha(M)^\vee)$ which turns out to be an isomorphism when the additional condition of (ii) is satisfied.

Proof. Notice that $H^c_\alpha(M) = 0$ when $\operatorname{depth}(\alpha, M) \neq c$. Similar to Lemma 3.21, we obtain a natural injection $H^c_\alpha(M) \to H^c_\beta(M)$, for instance, by taking a minimal injective resolution of M. Hence we have a natural homomorphism

$$\operatorname{Hom}_R(H^c_\beta(M), M) \to \operatorname{Hom}_R(H^c_\alpha(M), M).$$

By Proposition 4.1, (i), the natural homomorphism of the first part of the present statement follows.

In order to prove (ii), recall that $\Gamma_\gamma(E^i(M)) = \bigoplus_{\mathfrak{p}\in\langle\gamma\rangle\cap\operatorname{Spec} R} E(R/\mathfrak{p})^{\mu^i(\mathfrak{p},M)}$ for every family γ of R and every term $E^i(M)$ of a minimal injective resolution of M and that if $\mu^i(\mathfrak{p}, M) \neq 0$, then $\mathfrak{p} \in \operatorname{Supp}(M)$ and $\operatorname{depth}_{R_\mathfrak{p}}(M_\mathfrak{p}) \leq i$. We shall show that $\Gamma_\alpha(E^c(M)) = \Gamma_\beta(E^c(M))$ for a minimal injective resolution $(E^i(M))$ of M. This situation will imply that the natural injection $H^c_\alpha(M) \to H^c_\beta(M)$ is an isomorphism. So consider $x \in \Gamma_\beta(E^c(M))$. Then $x \in E(R/\mathfrak{p})$ for some $\mathfrak{p} \in \langle\beta\rangle \cap \operatorname{Supp}(M)$ such that $\mu^c(\mathfrak{p}, M) \neq 0$. Thus this $\mathfrak{p}$ also satisfies that $c \leq \operatorname{grade}(\mathfrak{p}, M) \leq \operatorname{depth}_{R_\mathfrak{p}}(M_\mathfrak{p}) \leq c$. Now $x \in E(R/\mathfrak{p}) \subseteq \Gamma_{\beta R_\mathfrak{p}}(E^c(M_\mathfrak{p})) = \Gamma_{\alpha R_\mathfrak{p}}(E^c(M_\mathfrak{p}))$ by hypothesis. We conclude in this way that $x \in \Gamma_\alpha(E^c(M))$ and the statement follows.

Finally, statement (iii) follows by Lemma 4.4. □

Observe that locality in the first two items can be dropped.

4.2 On top local cohomology modules

In this section R always denotes a local ring and $\hat{R}$ is its completion with respect to the linear topology induced by its maximal ideal, as usual.

Definition 4.7. Let α be a family of ideals of R and M be a d-dimensional finitely generated R-module. Consider the disjoint sets

$$U_\alpha(M) = \{\mathfrak{p} \in \operatorname{Ass}(M) : \dim(R/\mathfrak{p}) = d \text{ and } \dim(R/(I+\mathfrak{p})) \leq 0 \text{ for every } I \in \alpha\}$$

and $V_\alpha(M) = \operatorname{Ass}(M) - U_\alpha(M)$. Explicitly, $V_\alpha(M)$ is the set of associated primes $\mathfrak{p}$ of M such that $\dim(R/\mathfrak{p}) < d$ or $\dim(R/\mathfrak{p}) = d$ and $\dim(R/(I+\mathfrak{p})) > 0$ for some $I \in \alpha$. Of course, if $\langle\alpha\rangle = \langle\beta\rangle$, then $U_\alpha(M) = U_\beta(M)$ and $V_\alpha(M) = V_\beta(M)$, so we shall denote these sets shortly by U and V respectively if no confusion arise. For a minimal primary decomposition of $0 = \bigcap_{i=1}^{n} Q_i(M)$ in M, denote $Q_\alpha(M) = \bigcap_{\mathfrak{p}_i \in U} Q_i(M)$. If $U = \emptyset$, put $Q_\alpha(M) = M$.

Lemma 4.8. $\operatorname{Ass}(Q_\alpha(M)) = V$ *and* $\operatorname{Ass}(M/Q_\alpha(M)) = U$.

Proof. It is a straightforward consequence of [Sch07, Lemma 2.7], see Lemma A.3. $\square$

Theorem 4.9. *Let α be a non-trivial family of $(R, \mathfrak{m})$. Consider a finitely generated R-module M and set $d = \dim(M)$. Then there exists a natural isomorphism*

$$H^d_\alpha(M) \cong H^d_{\mathfrak{m}\hat{R}}(\hat{M}/Q_{\alpha\hat{R}}(\hat{M})).$$

Proof. Since $H^d_\alpha(M)$ is Artinian by Theorem 2.2, the natural homomorphism of $\hat{R}$-modules $H^d_{\alpha\hat{R}}(\hat{M}) \cong H^d_\alpha(M) \otimes_R \hat{R} \to H^d_\alpha(M)$ is an isomorphism. So we shall assume that R is complete. The exact sequence $0 \to Q_\alpha(M) \to M \to M/Q_\alpha(M) \to 0$ induces an isomorphism of R-modules $H^d_\alpha(M) \cong H^d_\alpha(M/Q_\alpha(M))$ by Proposition 2.25. If we denote $\bar{R} = R/\operatorname{Ann}_R(M/Q_\alpha(M))$, we obtain another isomorphism of $\bar{R}$-modules $H^d_\alpha(M/Q_\alpha(M)) \cong H^d_{\alpha\bar{R}}(M/Q_\alpha(M))$. Finally, if $I \in \alpha - \{R\}$, then

$$V(I + \operatorname{Ann}_R(M/Q_\alpha(M)) = V(I) \cap \bigcup_{\mathfrak{p} \in U} V(\mathfrak{p}) = \{\mathfrak{m}\}$$

by Lemma 4.8. Thus $\langle\alpha\bar{R}\rangle = \langle\mathfrak{m}\bar{R}\rangle$ and $H^d_\alpha(M/Q_\alpha(M)) \cong H^d_\mathfrak{m}(M/Q_\alpha(M))$. So the desired isomorphism follows. $\square$

Definition 4.10. Let M be a finitely generated R-module and α be a family of R. Set $P_\alpha(M)$ as the intersection of all the primary components $\mathfrak{p}$ of $\operatorname{Ann}(M)$ such that $\dim(R/\mathfrak{p}) = \dim(M)$ and $\dim(R/(I+\mathfrak{p})) \leq 0$ for every ideal $I \in \alpha$. Observe that $P_\alpha(M) = \pi^{-1}(Q_{\alpha\bar{R}}(\bar{R}))$ where $\pi : R \to \bar{R} = R/\operatorname{Ann}(M)$ is the natural projection.

Lemma 4.11. *Let M be a finitely generated R-module, α be a non-trivial family of R and set $d = \dim(M)$. Then $H^d_\alpha(M) \cong H^d_{\mathfrak{m}\hat{R}}(\hat{M}/P_\alpha(\hat{M})\hat{M})$.*

Proof. Again, we may assume that R is complete. Set $\bar{R} = R/\operatorname{Ann}(M)$. Then there is an isomorphism $H^d_\alpha(M) \cong H^d_{\alpha\bar{R}}(\bar{R}) \otimes_R M$. By Theorem 4.9, there is an isomorphism $H^d_{\alpha\bar{R}}(\bar{R}) \cong H^d_{\mathfrak{m}}(R/P_\alpha(M))$. Thus $H^d_{\alpha\bar{R}}(\bar{R}) \otimes_R M \cong H^d_{\mathfrak{m}}(M/P_\alpha(M)M)$ and the statement follows. $\square$

For a finitely generated R-module M of dimension d we recall the notation $K(M) = H^d_{\mathfrak{m}}(M)^\vee$ used by M. Eghbali and P. Schenzel in [ES12]. Observe that $K(M)$ is isomorphic to the $\hat{R}$-module $K_{\hat{M}} = \operatorname{Ext}^{n-d}_{\hat{S}}(\hat{M}, \hat{S})$ when $(R, \mathfrak{m})$ is the image of an n-dimensional Gorenstein ring $(S, \mathfrak{n})$. Indeed, by Local Duality (see [Har67, Theorem 6.3]), we have a natural equivalence $\operatorname{Ext}^{n-i}_S(-, S)^\wedge \to \operatorname{Hom}_S(H^i_{\mathfrak{n}}(-), E_S)$. Here E_S denotes the injective hull of the residual field of S. Left side of this equivalence gives $\operatorname{Ext}^{n-d}_S(M, S)^\wedge \cong \operatorname{Ext}^{n-d}_S(M, S) \otimes_S \hat{S} \cong \operatorname{Ext}^{n-d}_{\hat{S}}(M \otimes_S \hat{S}, \hat{S})$ and right side gives $\operatorname{Hom}_S(H^d_{\mathfrak{n}}(M), E_S) \cong \operatorname{Hom}_S(H^d_{\mathfrak{n}}(M) \otimes_R R, E_S) \cong \operatorname{Hom}_R(H^d_{\mathfrak{m}}(M), \operatorname{Hom}_S(R, E_S))$, proving the claim. We also point out that $\operatorname{Ann}_{\hat{R}}(K(M))$ equals the intersection of the primary components of dimension d of $\operatorname{Ann}_{\hat{R}}(\hat{M})$ for every d-dimensional finitely generated R-module M as seen in [Har67, Proposition 6.6, 7)].

Lemma 4.12. *Let α be a non-trivial family of $(R, \mathfrak{m})$ and M be a finitely generated R-module of dimension d.*

(i) $H^d_\alpha(M)^\vee$ *is a finitely generated $\hat{R}$-module.*

(ii) $\operatorname{Ass}_{\hat{R}}(H^d_\alpha(M)^\vee) = U_{\alpha\hat{R}}(\hat{M})$.

(iii) $K_{\hat{R}}(\hat{M}/Q_{\alpha\hat{R}}(\hat{M})) \cong H^d_\alpha(M)^\vee$.

Proof. In order to prove (i), we notice that the Matlis dual of an Artinian R-module is the same as its Matlis dual when regarded as an $\hat{R}$-module: in fact, if A is such R-module, then $A \cong A \otimes_R \hat{R}$, $E_{\hat{R}}(R/\mathfrak{m}) = E$ and we obtain natural isomorphisms

$$\begin{aligned} \operatorname{Hom}_{\hat{R}}(A, E) &\cong \operatorname{Hom}_{\hat{R}}(A \otimes_R \hat{R}, E) \\ &\cong \operatorname{Hom}_R(A, \operatorname{Hom}_{\hat{R}}(\hat{R}, E)) \\ &\cong \operatorname{Hom}_R(A, E). \end{aligned}$$

For (ii), we get that $\operatorname{Ass}_{\hat{R}}(H^d_\alpha(M)^\vee) = \operatorname{Coass}_{\hat{R}}(H^d_{\alpha\hat{R}}(\hat{M})) = U_{\alpha\hat{R}}(\hat{M})$, the last equality follows from Proposition 3.18.

Finally, $K_{\hat{R}}(\hat{M}/Q_{\alpha\hat{R}}(\hat{M}))^\vee = H^d_{\mathfrak{m}\hat{R}}(\hat{M}/Q_{\alpha\hat{R}}(\hat{M})) = H^d_\alpha(M)$ and the statement remains proved. $\square$

Recall that a finitely generated R-module M satisfies Serre's condition S_r, for an integer r, when $\operatorname{depth}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) \geq \min\left\{r, \dim_{R_{\mathfrak{p}}}(M_{\mathfrak{p}})\right\}$ for every $\mathfrak{p} \in \operatorname{Spec} R$.

Theorem 4.13. *Let α be a family of $(R, \mathfrak{m})$ and consider a finitely generated R-module M of dimension d. Consider the natural homomorphism*

$$\Phi_M : \hat{R} \to \operatorname{Hom}_{\hat{R}}(H^d_\alpha(M), H^d_\alpha(M)).$$

(i) $\ker \Phi_M = P_{\alpha\hat{R}}(\hat{M})$.

(ii) $\operatorname{Hom}_{\hat{R}}(H^d_\alpha(M), H^d_\alpha(M))$ *is a finitely generated $\hat{R}$-module.*

(iii) Φ_R *is surjective if and only if $\hat{R}/Q_{\alpha\hat{R}}(\hat{R})$ satisfies S_2.*

(iv) $\operatorname{Hom}_{\hat{R}}(H^d_\alpha(R), H^d_\alpha(R))$ *is a commutative semilocal Noetherian ring.*

Proof. As usual, we may assume that R is complete. Lemma 4.11 gives an isomorphism $H^d_\alpha(M) \cong H^d_{\mathfrak{m}}(M/P_\alpha(M)M)$. Hence $K(M/P_\alpha(M)M) \cong H^d_\alpha(M)^\vee$. In this way we obtain another isomorphism

$$\operatorname{Hom}_R(H^d_\alpha(M), H^d_\alpha(M)) \cong \operatorname{Hom}_R(K(M/P_\alpha(M)M), K(M/P_\alpha(M)M)). \tag{4.3}$$

Thus $\ker \Phi_M = \operatorname{Ann}(K(M/P_\alpha(M)M)) = P_\alpha(M)$, the last equality follows from [Har67, Proposition 6.6, 7)] and (i) remains proved. Item (ii) follows also by equation (4.3) because $K(M)$ is a finitely generated $\hat{R}$-module for every finitely generated R-module M. Items (iii) and (iv) follow for $M = R$ because in this case $K(M/P_\alpha(M)M)$ is isomorphic to the canonical module of the ring $\hat{R}/Q_{\alpha\hat{R}}(\hat{R})$. Hence, results of Y. Aoyama and S. Goto (see [AG85, Proposition 1.2]) and Y. Aoyama (see [Aoy83, Theorem 3.2]) apply. $\square$

Chapter 5

Linear topologies and local homology

Throughout this chapter, all the rings are assumed to be commutative Noetherian. We shall introduce linear topologies induced by a family α of a ring R. The local case serves as a motivation for this since the composite functor $\Gamma_\alpha(-)^\vee$ is naturally isomorphic to the composite functor $\varprojlim_{I\in\langle\alpha\rangle}(-)^\vee \otimes_R R/I$.

5.1 The α-adic topology

Consider a family α of R and an R-module M. For each $m \in M$ we define its fundamental open neighbourhoods as the cosets of the form $m+IM$ for some $I \in \langle\alpha\rangle$. By these means, M is a topological module and this topology is called the *α-adic topology on M*. Notice that M is Hausdorff if and only if $\bigcap_{I\in\langle\alpha\rangle} IM = 0$. When $M = R$, we also have that $\langle\alpha\rangle$ is exactly the set of open ideals of R. Furthermore, if β is another family, then the β-adic topology on R coincides with the α-adic topology on R if and only if $\langle\beta\rangle = \langle\alpha\rangle$. If $M = R/J$ for some $J \in \langle\alpha\rangle$, we see that the α-adic topology on M is just the discrete topology and the projective system $\{R/I, \pi_{IJ}\}_{I\in\langle\alpha\rangle}$, with natural projections $\pi_{IJ} : R/J \to R/I$, is a system of topological rings. Therefore it is defined the *α-adic completion* of R as the topological ring

$$\Lambda_\alpha(R) := \varprojlim_{I\in\langle\alpha\rangle} R/I$$

in such a way that the canonical homomorphism of rings $\varphi_R : R \to \Lambda_\alpha(R)$ is continuous. Namely, the topology considered in $\Lambda_\alpha(R)$ is the coarsest topology such that every projection $\pi_I : \Lambda_\alpha(R) \to R/I$ is continuous. As before, φ_R is injective if and only if R is Hausdorff in the α-adic topology.

Definition 5.1. The ring R is called α-adically complete if φ_R is an isomorphism.

Now consider an R-module M. It is seen as before that the α-adic topology on M/IM is just the discrete topology for every $I \in \langle\alpha\rangle$ and it is defined the *α-completion of* M as the $\Lambda_\alpha(R)$-module $\Lambda_\alpha(M) = \varprojlim_{I\in\langle\alpha\rangle} M/IM$. Recall that this structure is given component-wise by the structure of the R/I-module M/IM. If N is an open submodule of M, there exists a monomorphism of $\Lambda_\alpha(R)$-modules $\Lambda_\alpha(N) \to \Lambda_\alpha(M)$ which sends $(n_I + IN)_{I\in\langle\alpha\rangle}$ to $(n_I + IM)_{I\in\langle\alpha\rangle}$: in fact, if N is an open submodule of M, then $JM \subseteq N$ for some $J \in \langle\alpha\rangle$. Assuming that $n_I \in IM$ for every $I \in \langle\alpha\rangle$, we obtain that $n_{IJ} \in IJM \subseteq IN$. But $n_I - n_{IJ} \in IN$. Hence $n_I \in IN$ and the injection is now clear. We have even more.

Lemma 5.2. *Suppose that N is an open submodule of M with respect to the α-adic topology. Then there exists an exact sequence of $\Lambda_\alpha(R)$-modules*

$$0 \to \Lambda_\alpha(N) \to \Lambda_\alpha(M) \to \Lambda_\alpha(M/N) \to 0.$$

Proof. Since $JM \subseteq N$ for some $J \in \langle\alpha\rangle$, we have that the α-adic topology on M/N is just the discrete topology. Thus $\Lambda_\alpha(M/N) = M/N$ and the surjectivity of the homomorphism $\Lambda_\alpha(M) \to \Lambda_\alpha(M/N)$ follows. Consider now $(m_I + IM)_{I\in\langle\alpha\rangle} \in \Lambda_\alpha(M)$ such that $(m_I + IM + N)_{I\in\langle\alpha\rangle} = (IM + N)_{I\in\langle\alpha\rangle}$ in $\Lambda_\alpha(M/N)$. So, for every m_I there exists $n_I \in N$ such that $m_I + IM = n_I + IM$. Hence the statement follows. $\square$

It follows that $\Lambda_\alpha(N) = \pi_J^{-1}(N/JM)$ is an open submodule of $\Lambda_\alpha(M)$ such that $\varphi_R(N) \subseteq \Lambda_\alpha(N)$. Moreover, the submodules $\Lambda_\alpha(IM)$, with $I \in \langle\alpha\rangle$, form a fundamental system of open neighbourhoods of the zero element $0 \in \Lambda_\alpha(M)$. As before, there exists a continuous homomorphism of R-modules $\varphi_M : M \to \Lambda_\alpha(M)$ and M is Hausdorff if and only if φ_M is injective.

Proposition 5.3. *The following statements are true:*

(i) For every open submodule N of M, we have that $\Lambda_\alpha(N) = \overline{\varphi_M(N)}$.

(ii) There exists a bijective correspondence

$$\begin{array}{rcl} \{\textit{open submodules in } M\} & \leftrightarrow & \{\textit{open submodules in } \Lambda_\alpha(M)\} \\ N & \mapsto & \Lambda_\alpha(N) \\ \varphi_M^{-1}(\mathfrak{n}) & \leftarrow\!\shortmid & \mathfrak{n} \end{array}$$

Proof. Since $\Lambda_\alpha(N)$ is an open submodule in a linear topology, it is also closed and therefore $\overline{\varphi_M(N)} \subseteq \Lambda_\alpha(N)$. If $JM \subseteq N$ for some $J \in \langle\alpha\rangle$ and $(n_I + IM)_{I\in\langle\alpha\rangle} \in \Lambda_\alpha(N)$, then $n_I + JM + IM = n_J + JM + IM$ for every $I \in \langle\alpha\rangle$. Hence $n_J \in n_I + JM + IM$ for every $I \in \langle\alpha\rangle$. This implies finally that $\varphi_M(n_J) \in (n_I + IM)_{I\in\langle\alpha\rangle} + \Lambda_\alpha(JM)$ and thus (i) holds. For (iii), consider an open submodule $\mathfrak{n}$ of $\Lambda_\alpha(M)$. By the continuity of φ_M it follows that $N := \varphi_N^{-1}(\mathfrak{n})$ is an open submodule of M. Since $\mathfrak{n}$ is also closed, it follows from (i) that $\Lambda_\alpha(N) \subseteq \mathfrak{n}$. In the natural diagram

$$\begin{tikzcd} M/N \arrow[rr, "\cong"] \arrow[dr] & & \Lambda_\alpha(M)/\Lambda_\alpha(N) \arrow[dl] \\ & \Lambda_\alpha(M)/\mathfrak{n} & \end{tikzcd}$$

the left oblique arrow is injective and the right oblique arrow is surjective. According to Lemma 5.2, the horizontal arrow is bijective. This means $\Lambda_\alpha(N) = \mathfrak{n}$ and (iii) follows. □

The assignation $\Lambda_\alpha(-) : R\text{-mod} \to \Lambda_\alpha(R)\text{-mod}$ is functorial (furthermore, additive): indeed, every homomorphism of R-modules $f : M \to N$ is continuous with respect to *any* linear topology α. Hence it induces a continuous homomorphism of $\Lambda_\alpha(R)$-modules $\Lambda_\alpha(f) : \Lambda_\alpha(M) \to \Lambda_\alpha(N)$ in a natural way.

Definition 5.4. The module M is called *α-adically complete* if φ_M is bijective.

The α-torsion modules have coefficients in the α-adic completion of the base ring as we shall see next.

Theorem 5.5. *Consider an α-torsion R-module M. Then M has a natural structure of $\Lambda_\alpha(R)$-module and the canonical homomorphism of $\Lambda_\alpha(R)$-modules $M \otimes_R \Lambda_\alpha(R) \to M$ is an isomorphism.*

Proof. If $x \in M$, then there exists $J \in \langle\alpha\rangle$ such that $Jx = 0$ by Lemma 1.10. In this way, for every element $(r_I + I)_{I\in\langle\alpha\rangle} \in \Lambda_\alpha(R)$, the filter $(r_I x)_{I\in\langle\alpha\rangle}$ acquires a constant value $r_J x$: indeed, if K is another ideal in $\langle\alpha\rangle$ such that $Kx = 0$, then $r_J x = r_K x$. So let us define the $\Lambda_\alpha(R)$-module structure for M according to this: $(r_I + x)_{I\in\langle\alpha\rangle} := r_J x$. This structure is compatible with the usual R-module structure of M and the natural map $M \otimes_R \Lambda_\alpha(R) \to M$ which sends $x \otimes (r_I + I)$ to $r_J x$ has natural inverse $x \mapsto x \otimes (1 + I)$. □

Now that α-adic topologies are defined, it is time to translate to this language a partial result from Section 1.3.

Theorem 5.6. *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension d with canonical module K_R and ψ be a family of R. There is a natural isomorphism*

$$\Lambda_\psi(H^{d-t}_{\mathfrak{m},\psi}(R)) \cong H^t_\psi(K_R)^\vee,$$

where $t = d - \sup_{J\in\psi} \dim(R/J)$.

Proof. Exactly as in Theorem 1.54, we obtain a natural isomorphism

$$H^{d-t}_{\mathfrak{m},\psi}(R)/JH^{d-t}_{\mathfrak{m},\psi}(R) \cong \operatorname{Ext}^t_R(R/J, K_R)^\vee$$

for every $J \in \langle\alpha\rangle$. Thus

$$\Lambda_\psi(H^{d-t}_{\mathfrak{m},\psi}(R)) \cong \varprojlim_{J\in\langle\psi\rangle} \operatorname{Ext}^t_R(R/J, K_R)^\vee \cong \left(\varinjlim_{J\in\langle\psi\rangle} \operatorname{Ext}^t_R(R/J, K_R)\right)^\vee \cong H^t_\psi(K_R)^\vee$$

and we obtain the statement. $\square$

5.2 Local homology modules

Recall that Theorem 1.11 gives a functorial isomorphism

$$H^i_\alpha(M) \cong \varinjlim_{I\in\langle\alpha\rangle} \operatorname{Ext}^i_R(R/I, M) \tag{5.1}$$

which reproduces the classic one for the closed support case

$$H^i_I(M) \cong \varinjlim_{n\in\mathbb{N}} \operatorname{Ext}^i_R(R/I^n, M) \tag{5.2}$$

Equation (5.2) was used by N. T. Cuong and T. T. Nam in [CN01] to define the i-th local homology module $H^I_i(M)$ of M with respect to an ideal I of R as the projective limit

$$H^I_i(M) = \varprojlim_{n\in\mathbb{N}} \operatorname{Tor}^R_i(R/I^n, M).$$

This situation, together with equation (5.1) suggest the following definition.

Definition 5.7. Let α be a family and M an R-module. The i-th local homology module $H^\alpha_i(M)$ of M with respect to α is defined by

$$H^\alpha_i(M) := \varprojlim_{I\in\langle\alpha\rangle} \operatorname{Tor}^R_i(R/I, M).$$

Remark 5.8. Clearly, $H_0^\alpha(M) \cong \Lambda_\alpha(M)$. It is also clear that, when α is the family $\{I^n : n \in \mathbb{N}\}$, this definition agrees with N. T. Cuong and T. T. Nam (*loc. cit.*) definition of local homology modules.

Next we show a useful relation between the local homology modules $H_i^\alpha(M)$ and $H_i^I(M)$.

Proposition 5.9. *Let M be an R-module and α be a family of R. Then*

$$H_i^\alpha(M) \cong \varprojlim_{I\in\langle\alpha\rangle} H_i^I(M)$$

for every i.

Proof. The claimed isomorphism follows since

$$\varprojlim_{I\in\langle\alpha\rangle} H_i^I(M) = \varprojlim_{I\in\langle\alpha\rangle} \varprojlim_{n\in\mathbb{N}} \operatorname{Tor}_i^R(R/I^n, M) \cong \varprojlim_{n\in\mathbb{N}} \varprojlim_{I\in\langle\alpha\rangle} \operatorname{Tor}_i^R(R/I^n, M).$$

Now we observe that, for every fixed $n \in \mathbb{N}$, the set $\{I^n : I \in \langle\alpha\rangle\}$ is cofinal with $\langle\alpha\rangle$. Therefore,

$$\varprojlim_{I\in\langle\alpha\rangle} \operatorname{Tor}_i^R(R/I^n, M) = \varprojlim_{I\in\langle\alpha\rangle} \operatorname{Tor}_i^R(R/I, M).$$

Thus

$$\varprojlim_{I\in\langle\alpha\rangle} H_i^I(M) \cong \varprojlim_{n\in\mathbb{N}} \varprojlim_{I\in\langle\alpha\rangle} \operatorname{Tor}_i^R(R/I, M) = \varprojlim_{n\in\mathbb{N}} H_i^\alpha(M) = H_i^\alpha(M).$$

This proves the statement. □

Proposition 5.10. *Let M be an R-module and α be a family of R. Then $H_i^\alpha(M)$ is* α-separated, *i.e.,* $\bigcap_{I\in\langle\alpha\rangle} IH_i^\alpha(M) = 0$.

Proof. Note that

$$\bigcap_{I\in\langle\alpha\rangle} IH_i^\alpha(M) \cong \varprojlim_{I\in\langle\alpha\rangle} IH_i^\alpha(M) = \varprojlim_{I\in\langle\alpha\rangle} I \varprojlim_{J\in\langle\alpha\rangle} \operatorname{Tor}_i^R(R/J, M).$$

But

$$\varprojlim_{I\in\langle\alpha\rangle} I \varprojlim_{J\in\langle\alpha\rangle} \operatorname{Tor}_i^R(R/I, M) \cong \varprojlim_{I\in\langle\alpha\rangle} \varprojlim_{J\in\langle\alpha\rangle} I \operatorname{Tor}_i^R(R/J, M) \cong \varprojlim_{J\in\langle\alpha\rangle} \varprojlim_{I\in\langle\alpha\rangle} I \operatorname{Tor}_i^R(R/J, M).$$

Since for each $J \in \langle\alpha\rangle$ we have that

$$\varprojlim_{I\in\langle\alpha\rangle} I \operatorname{Tor}_i^R(R/J, M) = \varprojlim_{I\geq J} I \operatorname{Tor}_i^R(R/J, M)$$

and $I \operatorname{Tor}_i^R(R/J, M) = 0$ for every $I \geq J$, we conclude that

$$\bigcap_{I\in\langle\alpha\rangle} IH_i^\alpha(M) = 0$$

and the statement is proved. □

A. Ooishi introduced in [Ooi76] a generalised Matlis dual functor in the following way: let R be a semi-local Noetherian ring with maximal ideals $\mathfrak{m}_1, \ldots, \mathfrak{m}_n$ and set $E_R := \bigoplus_{i=1}^{n} E_R(R/\mathfrak{m}_i)$. For an R-module M it is defined the *Matlis dual* of M by $D(M) := \operatorname{Hom}_R(M, E_R)$. With this notion we get a duality relation between local homology and local cohomology for modules over semi-local Noetherian rings as follows.

Proposition 5.11. *Suppose that R is a semi-local ring and let α be a family. Then $H_i^{\alpha}(D(M)) \cong D(H_{\alpha}^i(M))$ for every $i \geq 0$.*

Proof. By [Ooi76, Corollary 1.5],

$$H_i^{\alpha}(D(M)) = \varprojlim_{I \in \langle \alpha \rangle} \operatorname{Tor}_i^R(R/I, D(M)) \cong \varprojlim_{I \in \langle \alpha \rangle} D(\operatorname{Ext}_R^i(R/I, M)).$$

But

$$\varprojlim_{I \in \langle \alpha \rangle} D(\operatorname{Ext}_R^i(R/I, M)) \cong D\left(\varinjlim_{I \in \langle \alpha \rangle} \operatorname{Ext}_R^i(R/I, M) \right) = D(H_{\alpha}^i(M)).$$

Hence the statement follows. □

Next we state dual versions of some classic results from local cohomology theory which serve as generalisations of some results from [CN01]. For this purpose, we now recall the notion of *Noetherian dimension* of an Artinian R-module M denoted by $\operatorname{Ndim}(M)$. This concept was introduced by R. N. Roberts in [Rob75] by the name *Krull dimension*. Later, D. Kirby changed in [Kir90] this terminology of R. N. Roberts and referred to Noetherian dimension to avoid confusion with the well-known Krull dimension of finitely generated modules. Let M be an Artinian R-module. When $M = 0$, set $\operatorname{Ndim}(M) = -1$. By induction, for any ordinal α, set $\operatorname{Ndim}(M) = \alpha$, when $\operatorname{Ndim}(M) < \alpha$ is false and for every ascending chain $M_0 \subseteq M_1 \subseteq \cdots$ of submodules of M there exists a positive integer m_0 such that $\operatorname{Ndim}(M_{m+1}/M_m) < \alpha$ for every $m \geq m_0$. Thus an Artinian module M is non-zero and finitely generated if and only if $\operatorname{Ndim}(M) = 0$.

If $0 \to M'' \to M \to M' \to 0$ is a short exact sequence of R-modules, then

$$\operatorname{Ndim}(M) = \max \{\operatorname{Ndim}(M''), \operatorname{Ndim}(M')\}. \tag{5.3}$$

Proposition 5.12. *Let M be an Artinian R-module with* $\operatorname{Ndim}(M) = d$. *Then*

$$H_i^{\alpha}(M) = 0$$

for every $i > d$.

Proof. Since $H_i^\alpha(M) \cong \varprojlim_{I\in\langle\alpha\rangle} H_i^I(M)$ by Proposition 5.9, the result follows from [CN01, Proposition 4.8]. □

Proposition 5.13 (Independence Theorem). *Let $f : R \to S$ be a homomorphism of Noetherian rings and M be an S-module. Then we get an isomorphism of $\Lambda_\alpha(R)$-modules $H_i^\alpha(M) \cong H_i^{\alpha S}(M)$ for every $i \geq 0$.*

Proof. By [CN01, Corollary 3.7], for each $I \in \langle\alpha\rangle$, we have an isomorphism of $\Lambda_I(R)$-modules $H_i^I(M) \cong H_i^{IS}(M)$ for each $i \geq 0$. Now, by taking projective limits, we get the desired isomorphism. □

Proposition 5.14. *Let R be a semi-local ring and let M be an Artinian R-module. Then the following conditions are equivalent:*

(i) $D(M)$ is α-torsion R-module.

(ii) $H_i^\alpha(M) = 0$ for every $i > 0$.

Proof. (i)⇒(ii): Since $D(M)$ is α-torsion, it follows by Proposition 1.28, (i), that $H_\alpha^i(D(M)) = 0$ for every $i > 0$. Therefore, since M is Artinian, we have by [Ooi76, Theorem 1.6, (5)] and Proposition 5.11, that $H_i^\alpha(M) \cong H_i^\alpha(DD(M)) \cong D(H_\alpha^i(D(M))) = 0$ for every $i > 0$.

(ii)⇒(i): By the Independence Theorem (Proposition 5.13) we can assume that R is J-adically complete, where J is the Jacobson radical of R. If $H_i^\alpha(M) = 0$ for every $i > 0$, then by [Ooi76, Theorem 1.6 (5)] and Proposition 5.11 we get that $D(H_\alpha^i(D(M))) = 0$. Therefore $H_\alpha^i(D(M)) = 0$ by [Ooi76, Theorem 1.6 (8)]. On the other hand, since M is Artinian, $D(M)$ is finitely generated by [Ooi76, Theorem 1.6, (3)]. Hence Corollary 1.37 gives that $D(M)$ is α-torsion. □

Proposition 5.15. *Let R be a semi-local ring and let M be a non-zero Artinian R-module with $\operatorname{Ndim}(M) = d$. Then $H_d^J(M) \neq 0$, where J is the Jacobson radical of R.*

Proof. By [Sha92, Lemma 2.2] M has a natural structure as a module over $\Lambda_J(R)$ in such a way that a subset of M is an R-submodule if and only if it is a $\Lambda_J(R)$-module. Thus, $\operatorname{Ndim}_R(M) = \operatorname{Ndim}_{\Lambda_J(R)}(M)$. Therefore, since $H_i^J(M) \cong H_i^{J\Lambda_J(R)}(M)$ as $\Lambda_J(R)$-module for every i by the Independence Theorem (Proposition 5.13), we may assume that R is J-adically complete. Then the Matlis dual $D(M)$ is a non-zero finitely generated R-module (see [Ooi76, Theorem 1.6, (3)]). It follows by [Ooi76, Theorema 1.6, (8)] that $\dim_R(D(M)) = \dim_R(R/\operatorname{Ann}_R(M))$. On the other hand, by [Sha89, Proposition 1.4], we can write $M = \Gamma_{\mathfrak{m}_1}(M) \oplus \cdots \oplus \Gamma_{\mathfrak{m}_k}(M)$ for some maximal ideals $\mathfrak{m}_1, \ldots, \mathfrak{m}_k$ of R. Set $M_i = \Gamma_{\mathfrak{m}_i}(M)$. Then, by [Sha89, Remark 1.7], each M_i has a natural structure as an $R_{\mathfrak{m}_i}$-module in such a way that a subset of

M is an R-submodule if and only if it is a $R_{\mathfrak{m}_i}$-module. Moreover, $M_i \cong (M_i)_{\mathfrak{m}_i}$ as $R_{\mathfrak{m}_i}$-modules. By equation (5.3), $\operatorname{Ndim}(M) = \max\{\operatorname{Ndim}(M_1), \dots, \operatorname{Ndim}(M_k)\}$. Hence $\operatorname{Ndim}_{R_{\mathfrak{m}_i}}(M_i) = \operatorname{Ndim}_{R_{\mathfrak{m}_i}}((M_i)_{\mathfrak{m}_i}) = \dim_{R_{\mathfrak{m}_i}}(R_{\mathfrak{m}_i}/\operatorname{Ann}_{R_{\mathfrak{m}_i}}((M_i)_{\mathfrak{m}_i}))$ for every $i = 1, \dots, k$, the second equality following from [Rob75, Theorem 6]. Therefore we get $d = \operatorname{Ndim}_R M = \dim_R(R/\operatorname{Ann}(M)) = \dim_R(D(M))$. Then $H_J^d(D(M)) \neq 0$ by Grothendieck's Non-Vanishing Theorem and Flat Base Change. Therefore, by Proposition 5.11, $H_d^J(M) \cong H_d^J(DD(M)) \cong D(H_J^d(D(M))) \neq 0$ and the statement is now proved. □

Remark 5.16. Observe that if M is an R-module and α is a family of R, then $H_i^\alpha(M)$ has a natural structure of $\Lambda_\alpha(R)$-modules.

The next result generalises [CN01, Proposition 4.6].

Proposition 5.17. *Let R be a semi-local ring and M be an Artinian R-module. Then $H_i^J(M)$ is a Noetherian $\Lambda_J(R)$-module for every i, where J is the Jacobson radical of R and $\Lambda_J(R)$ is the J-adic completion of R.*

Proof. By Proposition 5.13, we can assume that R is J-adically complete. By [Ooi76, Theorem 1.6, (3)], $D(M)$ is a Noetherian R-module. We shall prove first that the module $H_J^i(D(M))$ is Artinian. In fact, since R is semi-local, it is enough to prove that the localisation of $H_J^i(D(M))$ at every maximal ideal is Artinian, but this follows by [BS98, Theorem 4.3.2 and Theorem 7.1.3]. Moreover, $DD(M)$ is isomorphic to M, by [Ooi76, Theorem 1.6, (5)]. Finally, by using Proposition 5.11 and [Ooi76, Theorem 1.6, (3)] we get that $H_i^J(M) \cong H_i^J(DD(M)) \cong D(H_J^i(D(M)))$ is a Noetherian module. □

Proposition 5.18. *Let R be an Artinian ring and let M be a finitely generated R-module. Then $H_i^\alpha(M)$ has finite length. In particular, $H_i^\alpha(M)$ is an Artinian R-module.*

Proof. There exists an ideal J in $\langle\alpha\rangle$ such that $J = I$ for every $I \geq J$ in $\langle\alpha\rangle$. Then $H_i^\alpha(M) = \varprojlim_{I \geq J} \operatorname{Tor}_i^R(R/I, M) = \operatorname{Tor}_i^R(R/J, M)$ for every $i \geq 0$.

On the other hand, since M is a finitely generated R-module, there is a free resolution $\mathbf{F}_{\bullet}$ of M in which the free R-modules F_i are finitely generated. Therefore, $\operatorname{Tor}_i^R(R/J, M) \cong H_i(R/J \otimes_R \mathbf{F}_{\bullet})$ has finite length for every $i \geq 0$. □

Proposition 5.19. *Let M be an Artinian R-module. Then,*

$$H_i^\alpha\left(\bigcap_{I\in\langle\alpha\rangle} IM\right) = \begin{cases} 0 & \text{if } i = 0 \\ H_i^\alpha(M) & \text{if } i > 0 \end{cases}$$

Proof. Since M is an Artinian R-module, the family $\{IM : I \in \langle\alpha\rangle\}$ of submodules of M has an minimal element JM for some $J \in \langle\alpha\rangle$. On the other hand, we can write

$$\bigcap_{I\in\langle\alpha\rangle} IM = \varprojlim_{I\in\langle\alpha\rangle} IM = \varprojlim_{I\geq J} IM = JM,$$

because for any $I \geq J$, we have that $IM = JM$ by the minimality of JM.

Therefore, by Proposition 5.9,

$$H_i^{\alpha}\left(\bigcap_{I \in \langle\alpha\rangle} IM\right) = H_i^{\alpha}(JM) = \varprojlim_{I \in \langle\alpha\rangle} H_i^I(JM) = \varprojlim_{I \geq J} H_i^I(JM).$$

Furthermore, if $I \geq J$ and $n \geq 1$, we get $I^n M = JM$. Thus $\bigcap_{n \geq 1} I^n M = JM$. By [CN01, Corollary 4.5], we have that

$$H_i^I\left(\bigcap_{n \geq 1} I^n M\right) = \begin{cases} 0 & \text{if } \ i = 0 \\ H_i^I(M) & \text{if } \ i > 0 \end{cases}$$

Therefore the result follows from Proposition 5.9. $\square$

The following theorem provides a characterisation for α-adically complete Artinian modules.

Theorem 5.20. *Let M be an Artinian R-module. The following statements are equivalent:*

(i) M is Hausdorff with respect to the α-adic topology.

(ii) M is α-adically complete.

(iii) $H_0^{\alpha}(M) \cong M$ and $H_i^{\alpha}(M) = 0$ for all $i > 0$.

Proof. We consider the following short exact sequence of projective systems of Artinian R-modules

$$0 \to \{IM\}_{I \in \langle\alpha\rangle} \to \{M\}_{I \in \langle\alpha\rangle} \to \{M/IM\}_{I \in \langle\alpha\rangle} \to 0$$

By [Jen72, Corollary 7.2], the sequence of projective limits

$$0 \to \bigcap_{I \in \langle\alpha\rangle} IM \to M \to \Lambda_{\alpha}(M) \to 0$$

is exact.

The equivalence between (i) and (ii) follows from the exact sequence above.

Now let us suppose condition (ii). We have that $H_0^{\alpha}(M) \cong \Lambda_{\alpha}(M) \cong M$. On the other hand, by Proposition 5.19 and condition (i) we get that

$$H_i^{\alpha}(M) = H_i^{\alpha}\left(\bigcap_{I \in \langle\alpha\rangle} IM\right) = 0$$

for every $i > 0$.

Finally, condition (iii) trivially implies condition (ii). $\square$

Corollary 5.21. *Let R be an Artinian ring and let M be a finitely generated R-module. Then $H_i^\alpha(M)$ is α-adically complete.*

Proof. This is a direct consequence of the Proposition 5.18 and Theorem 5.20. $\square$

Proposition 5.22. *Let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence of Artinian modules. Then we have a long exact sequence*

$$\cdots \to H_1^\alpha(M') \to H_1^\alpha(M) \to H_1^\alpha(M'') \to H_0^\alpha(M') \to H_0^\alpha(M) \to H_0^\alpha(M'') \to 0.$$

Proof. For each $I \in \langle\alpha\rangle$, the short exact sequence $0 \to M' \to M \to M'' \to 0$ induces a long exact sequence $\cdots \to \mathrm{Tor}_1^R(R/I, M') \to \mathrm{Tor}_1^R(R/I, M) \to \mathrm{Tor}_1^R(R/I, M'') \to R/I \otimes M' \to R/I \otimes M \to R/I \otimes M'' \to 0$. Since M', M and M'' are Artinian, the modules in the latter long exact sequence are Artinian. By [Jen72, Corollary 7.2] the projective limit is exact on the category of Artinian R-modules. Thus we have a long exact sequence of local homology modules with respect to α. $\square$

Remark 5.23. Let $\varphi : M \to N$ be a homomorphism between Artinian modules and consider a family $\mathscr{F}$ of submodules of M such that, for every pair F and G of elements of $\mathscr{F}$, there exists an element $H \in \mathscr{F}$ contained in $F \cap G$. Then $\varphi\left(\bigcap_{F\in\mathscr{F}} F\right) = \bigcap_{F\in\mathscr{F}} \varphi(F)$.

Proposition 5.24. *Let M be an Artinian R-module and N a submodule of M. Then M is α-adically complete if and only if N and M/N are α-adically complete.*

Proof. Suppose that M is α-adically complete. By Theorem 5.20 it is sufficient to prove that N and M/N are Hausdorff with respect to the α-adic topology. It is clear that $\bigcap_{I\in\langle\alpha\rangle} IN \subseteq \bigcap_{I\in\langle\alpha\rangle} IM = 0$. On the other hand,

$$\bigcap_{I\in\langle\alpha\rangle} I(M/N) = \bigcap_{I\in\langle\alpha\rangle} (IM+N)/N = \left(\left(\bigcap_{I\in\langle\alpha\rangle} IM\right) + N\right)/N = 0$$

by Remark 5.23. Conversely, assume that N and M/N are α-adically complete. We have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & N \arrow[r] \arrow[d, "\varphi_N"] & M \arrow[r] \arrow[d, "\varphi_M"] & M/N \arrow[r] \arrow[d, "\varphi_{M/N}"] & 0 \\
0 \arrow[r] & \Lambda_\alpha(N) \arrow[r] & \Lambda_\alpha(M) \arrow[r] & \Lambda_\alpha(M/N) \arrow[r] & 0
\end{tikzcd}$$

The first row is obviously exact. The exactness of the second row is a consequence of Proposition 5.22 and Theorem 5.20, (iii). Since φ_N and $\varphi_{M/N}$ are isomorphisms, the homomorphism φ_M is an isomorphism. $\square$

Proposition 5.25. *Let M be an Artinian R-module and t a positive integer. The following assertions are equivalent:*

(i) $H_i^\alpha(M)$ is Artinian for every $i < t$.

(ii) There exists an ideal $J \in \langle \alpha \rangle$ such that $J \subseteq \sqrt{\mathrm{Ann}_R(H_i^\alpha(M))}$ for every $i < t$.

Proof. (i)⇒(ii): Consider $i < t$. Since $H_i^\alpha(M)$ is Artinian, there exists an ideal $J \in \langle \alpha \rangle$ such that $JH_i^\alpha(M) = IH_i^\alpha(M)$ for all $I \geq J$. By Proposition 5.10 we have that

$$0 = \bigcap_{I \in \langle \alpha \rangle} IH_i^\alpha(M) = JH_i^\alpha(M).$$

Therefore, $J \subseteq \mathrm{Ann}_R(H_i^\alpha(M))$.

(ii)⇒(i): The argument goes by induction on t. For $t = 1$, since M is Artinian, there exists an ideal $I_0 \in \langle \alpha \rangle$ such that $I_0 M = IM$ for all $I \geq I_0$. Therefore

$$H_0^\alpha(M) = \varprojlim_{I \geq I_0} M/IM = M/I_0M$$

is Artinian.

Let $t > 1$ and set $K = I_0 M$. By Proposition 5.19 we can replace M by K because $\bigcap_{I \in \langle \alpha \rangle} IM = I_0 M = K$. On the other hand, note that $JK = JI_0M = I_0M = K$ because $JI_0 \geq I_0$. Since K is Artinian, there exists $x \in J$ such that $xK = K$. Thus, by hypothesis, $xH_i^\alpha(K) = xH_i^\alpha(M) = 0$ for all $i < t$. Then the short exact sequence of Artinian modules

$$0 \longrightarrow (0 :_K x) \longrightarrow K \xrightarrow{x} K \longrightarrow 0$$

gives rise by Proposition 5.22 to an exact sequence

$$0 \longrightarrow H_{i+1}^\alpha(K) \longrightarrow H_i^\alpha((0 :_K x)) \longrightarrow H_i^\alpha(K) \longrightarrow 0$$

for every $i < t - 1$. It follows that $J \subseteq \sqrt{\mathrm{Ann}_R(H_i^\alpha((0 :_K x)))}$ and, by inductive hypothesis, that $H_i^\alpha((0 :_K x))$ is Artinian for every $i < t - 1$. Consequently, $H_i^\alpha(K)$ is Artinian for any $i < t$ and the statement is proved. □

5.3 Co-localisation and co-support

Let S be a multiplicative set of R. L. Melkersson and P. Schenzel introduced in [MS95] the *co-localisation* of an R-module M with respect to S as the $S^{-1}R$-module ${}_SM = Hom_R(S^{-1}R, M)$. This is a functor from the category of R-modules to the category of $S^{-1}R$-modules. Such functor is exact in the category of Artinian R-modules (see [MS95, Proposition 2.4]). If $\mathfrak{p}$ is a prime ideal of R and $S = R - \mathfrak{p}$, then instead of ${}_SM$, we write ${}_{\mathfrak{p}}M$. For an R-module M, the *co-support* of M to be the set

$$\mathrm{Cos}_R(M) := \{\mathfrak{p} \in \operatorname{Spec} R : \ {}_{\mathfrak{p}}M \neq 0\}.$$

Proposition 5.26. *Let S be a multiplicative set of R. Suppose that $S \cap I \neq \emptyset$ for every $I \in \alpha$. If M is an α-separated module, then ${}_SM = 0$.*

Proof. Set $f \in \ {}_SM$ and consider $\frac{r}{s} \in S^{-1}R$. For each $I \in \alpha$, there exists $t \in S \cap I$. Therefore $f\left(\frac{r}{s}\right) = f\left(\frac{tr}{ts}\right) = tf\left(\frac{r}{ts}\right) \in IM$. Since $\bigcap\limits_{I \in \langle\alpha\rangle} IM = 0$, we have the desired result. □

Corollary 5.27. *Let S be a multiplicative set of R and let M be an R-module. Suppose that $S \cap I \neq \emptyset$ for every $I \in \alpha$. Then ${}_SH_i^{\alpha}(M) = 0$ for all $i \geq 0$.*

Proof. This follows from Proposition 5.10 and Proposition 5.26. □

Lemma 5.28. *Let N be a finitely generated R-module and M an Artinian R-module. Then*

$${}_S \operatorname{Tor}_i^R(N, M) \cong \operatorname{Tor}_i^{S^{-1}R}(S^{-1}N, \ {}_SM).$$

Proof. Consider a resolution $F_* = (F_i)_{i \geq 0}$ of finitely generated free R-modules for N. Then $S^{-1}F_* = (S^{-1}F_i)_{i \geq 0}$ is a resolution of finitely generated flat $S^{-1}R$-modules for $S^{-1}N$. On the other hand $F_* \otimes M = (F_i \otimes M)_{i \geq 0}$ is a complex of Artinian R-modules. Since co-localisation is an additive functor and exact on the category of Artinian R-modules ([MS95, Proposition 2.4]), it commutes with homology functors. Moreover, by [MS95, Lemma 5.1], we get that

$$\begin{aligned}
{}_S \operatorname{Tor}_i^R(N, M)) &= {}_SH_i(F_* \otimes M) \\
&\cong H_i({}_S(F_* \otimes M)) \\
&\cong H_i(S^{-1}F_* \otimes \ {}_SM) \\
&= \operatorname{Tor}_i^{S^{-1}R}(S^{-1}N, \ {}_SM)
\end{aligned}$$

as required. □

Proposition 5.29. *Let M be an Artinian R-module. Then*

$${}_SH_i^{\alpha}(M) \cong H_i^{\alpha S^{-1}R}({}_SM)$$

for every $i \geq 0$.

Proof. By [Rot09, Proposition 5.21], the co-localisation functor preserves projective limits. Therefore

$${}_SH_i^{\alpha}(M) = \ {}_S \varprojlim_{I \in \langle\alpha\rangle} \operatorname{Tor}_i^R(R/I, M) \cong \varprojlim_{I \in \langle\alpha\rangle} \ {}_S \operatorname{Tor}_i^R(R/I, M).$$

Using Lemma 5.28 we get that

$$_SH_i^{\alpha}(M) \cong \varprojlim_{I \in \langle\alpha\rangle} \operatorname{Tor}_i^{S^{-1}R}(S^{-1}R/IS^{-1}R,\ _SM) = H_i^{\alpha S^{-1}R}(_SM)$$

as desired. $\square$

Corollary 5.30. *Let M be an Artinian R-module. Then*

$$\operatorname{Cos}_R(H_i^{\alpha}(M)) \subseteq \operatorname{Cos}_R(M) \cap \langle\alpha\rangle$$

for every $i \geq 0$.

Proof. Let $\mathfrak{p} \in \operatorname{Cos}_R(H_i^{\alpha}(M))$. By Proposition 5.29 we have an isomorphism

$$_{\mathfrak{p}}H_i^{\alpha}(M) \cong H_i^{\alpha R_{\mathfrak{p}}}(_{\mathfrak{p}}M)$$

for every $i \geq 0$. Since $_{\mathfrak{p}}H_i^{\alpha}(M) \neq 0$, it follows that $_{\mathfrak{p}}M \neq 0$. Thus $\mathfrak{p} \in \operatorname{Cos}_R(M)$. On the other hand, by Corollary 5.27, there exists an ideal $I \in \alpha$ such that $I \subseteq \mathfrak{p}$. Thus $\mathfrak{p} \in \langle\alpha\rangle$ and the statement is now proved. $\square$

Chapter 6

Local cohomology and $\mathscr{D}$-modules

6.1 Rings of differential operators over differentiable admissible algebras

In this section k denotes a field of characteristic zero and R is a commutative k-algebra. We begin recalling the definition of the ring of k-linear differential operators on R denoted by $\mathscr{D}_{R/k}$. First we define $\mathscr{D}^i_{R/k} \subseteq End_k(R)$ for every $i \geq 0$ in an inductive way: set $\mathscr{D}^0_{R/k} = R$ and define $\mathscr{D}^i_{R/k}$ for $i > 0$ as

$$\mathscr{D}^i_{R/k} = \left\{ P \in End_k(R) : [P, r] = Pr - rP \in \mathscr{D}^{i-1}_{R/k} \text{ for each } r \in R \right\}.$$

We note that each $\mathscr{D}^i_{R/k}$ is a sub-(R-R)-bimodule of $End_k(R)$. Moreover, we have that $\mathscr{D}^1_{R/k} = R \oplus Der_k(R)$, where $Der_k(R)$ denotes the R-module of all k-derivations of R, and $\mathscr{D}^i_{R/k}\mathscr{D}^j_{R/k} \subset \mathscr{D}^{i+j}_{R/k}$. Furthermore, if $P \in \mathscr{D}^i_{R/k}$ and $Q \in \mathscr{D}^j_{R/k}$, then $[P, Q] \in \mathscr{D}^{i+j-1}_{R/k}$.

The ring of *k-linear differential operators on* R is the ring $\mathscr{D}_{R/k} := \bigcup_{i \geq 0} \mathscr{D}^i_{R/k}$. Now recall that an associative ring A with identity is *filtered* if there exists an ascending filtration of additive subgroups $\Gamma = \{\Gamma_i, i \in \mathbb{Z}\}$ such that $\Gamma_i = 0$ for every $i < 0$, $1 \in \Gamma_0$, $\bigcup_{i \in \mathbb{Z}} \Gamma_i = A$ and $\Gamma_i \Gamma_j \subset \Gamma_{i+j}$ for every $i, j \geq 0$. It is clear from definition that Γ_0 is a subring of A.

We denote by $\mathrm{gr}^\Gamma(A)$ the *associated graded ring* $\mathrm{gr}^\Gamma(A) := \bigoplus_{i \in \mathbb{Z}} \Gamma_i/\Gamma_{i-1}$. Now $\Gamma_i = \mathscr{D}^i_{R/k}$ defines naturally a filtration of $\mathscr{D}_{R/k}$. Notice that the associated graded ring $\mathrm{gr}^\Gamma(\mathscr{D}_{R/k})$ is commutative because $[\mathscr{D}^i_{R/k}, \mathscr{D}^j_{R/k}] \subset \mathscr{D}^{i+j-1}_{R/k}$, for all $i, j \geq 0$.

In [NB13], L. Núñez-Betancourt introduced an important class of algebras that are essential in our work. The rings of differential operators of this kind of algebras behave like those of polynomials or power series rings.

Definition 6.1. A commutative k-algebra R is called *differentiable admissible* if it is Noetherian and regular and satisfies the following properties:

(A-1) R is equidimensional of dimension n, that is, the height of any maximal ideal is equal to n.

(A-2) Every residual field with respect to a maximal ideal is an algebraic extension of k.

(A-3) $Der_k(R)$ is a finitely generated projective R-module of rank n and the canonical map $R_{\mathfrak{m}} \otimes_R Der_k(R) \to Der_k(R_{\mathfrak{m}})$ is an isomorphism for any maximal ideal $\mathfrak{m} \subset R$.

Consider now the condition:

(A-3)' There are k-linear derivations $D_1, \ldots, D_n \in Der_k(R)$ and $a_1, \ldots, a_n \in R$ such that $D_i(a_j) = 1$ if $i = j$ and 0 otherwise.

The properties (A-1), (A-2) and (A-3)' appear in [MNM91] as conditions (i), (ii) and (iii) and any commutative Noetherian regular k-algebra satisfying these conditions will be called *strong differentiable admissible.* These conditions inspired L. Núñez-Betancourt to consider the properties (A-1), (A-2) and (A-3) in [NB13, Hypothesis 2.3]. In both works, R is a commutative Noetherian regular ring that contains a field of characteristic zero. It is worth noting that the conditions (A-2) and (A-3)' imply that the ring R is excellent, see [Mat80, Theorem 102].

In [NB13, Proposition 2.6], it was proved that any strong differentiable admissible k-algebra is differentiable admissible (see also [NM14]). Although the latter class of k-algebras is greater in general than the former as seen in [NB13, Remark 2.8], these classes coincide in the local case as a consequence of a theorem due to M. Nomura ([Mat86, Theorem 30.6]) which we now state.

Theorem 6.2. *Let $(R, \mathfrak{m}, K)$ be a Noetherian regular k-algebra of dimension n. Suppose that K is an algebraic extension of k. Let $\hat{R}$ denote the completion of R with respect to $\mathfrak{m}$. Let $x_1, \ldots, x_n$ be a regular system of parameters of R. Then $\hat{R} = K[[x_1, \ldots, x_n]]$*

is the power series ring with coefficients in K and if we write $\partial/\partial x_i$ for the partial derivatives in this representation, then $Der_k(\hat{R}) = Der_K(\hat{R})$ is the free $\hat{R}$-module with basis $\partial/\partial x_1, \ldots, \partial/\partial x_n$. Moreover, the following conditions are equivalent:

(i) $\partial/\partial x_i$ *maps R into R for $1 \leq i \leq n$, so that every $\partial/\partial x_i$ can be considered as an element of $Der_k(R)$;*

(ii) there exist derivations $D_1, \ldots, D_n \in Der_k(R)$ and elements $a_1, \ldots, a_n \in R$ such that $D_i a_j = \delta_{ij}$;

(iii) there exist derivations $D_1, \ldots, D_n \in Der_k(R)$ and elements $a_1, \ldots, a_n \in R$ such that $\det(D_i a_j) \notin \mathfrak{m}$;

(iv) $Der_k(R)$ is a free R-module of rank n;

(v) $\operatorname{rank}(Der_k(R)) = n$.

Remark 6.3. Observe that localisations on maximal ideals of differentiable admissible k-algebras are also differentiable admissible. Completions of local differentiable admissible k-algebras are also differentiable admissible. When a differentiable admissible k-algebra is a domain, then any quotient over principal ideals is also differentiable admissible. Interesting examples of differentiable admissible k-algebras can be found in [NM14]. Remarkable examples of strong differentiable admissible k-algebras are the polynomial rings, the power series rings and the rings of convergent power series.

When R is a strong differentiable admissible k-algebra, it is easy to describe the ring of k-linear differential operators over R. In this case, Theorem 6.2 guarantees that the R-module of k-derivations $Der_k(R)$ of R is free of rank n and $D_1, \ldots, D_n$ is a basis. Moreover, the left R-module $\mathscr{D}^i_{R/k}$ is free with basis

$$\{D^\alpha := D_1^{\alpha_1} \cdots D_n^{\alpha_n}, |\alpha| = \alpha_1 + \cdots + \alpha_n \leq i\}.$$

Therefore, every element $P \in \mathscr{D}_{R/k}$ can be written in a unique form as a finite sum $P = \sum_\alpha r_\alpha D^\alpha$, where $r_\alpha \in R$. Thus $\mathscr{D}_{R/k}$ coincides with the k-subalgebra of $End_k(R)$ generated by R and $Der_k(R)$, that is, $\mathscr{D}_{R/k} = R\langle D_1, \ldots, D_n\rangle$. In particular, $\mathscr{D}_{R/k}$ has no zero-divisors when R is a domain. On the other hand, if $R[y_1, \ldots, y_n]$ is the polynomial ring with coefficients in R and variables $y_1, \ldots, y_n$, the R-algebra map $\psi : R[y_1, \ldots, y_n] \to \operatorname{gr}^\Gamma(\mathscr{D}_{R/k})$ defined by $\psi(y_i) = \sigma_1(D_i)$, where σ_1 is the quotient map $\mathscr{D}^1_{R/k} \to \mathscr{D}^1_{R/k}/\mathscr{D}^0_{R/k}$, is an isomorphism of graded rings.

Suppose now that R is a differentiable admissible k-algebra of dimension n. Then $\mathscr{D}_{R/k}$ is left and right Noetherian (see [NB13, Corollary 2.14]). Moreover, the global dimension or homological dimension of $\mathscr{D}_{R/k}$ coincides with the Krull dimension of R, that is,

$$\operatorname{gl.dim}(\mathscr{D}_{R/k}) = \dim(R) = n. \tag{6.1}$$

See [NB13, Proposition 2.15].

Now let A be a filtered ring with filtration Γ and let M be a left A-module. A *filtration* Σ of M consists of an ascending sequence $\Sigma = \{\Sigma_i,\ i \in \mathbb{Z}\}$ of additive subgroups of M such that $\Sigma_i = 0$ for $i \ll 0$, $\bigcup\limits_{i\in\mathbb{Z}} \Sigma_i = M$ and $\Gamma_i \Sigma_j \subset \Sigma_{i+j}$ for all $i, j \in \mathbb{Z}$. In particular, each Σ_i is an Γ_0-module.

Suppose that A is a filtered ring such that $\operatorname{gr}^\Gamma(A)$ is a commutative Noetherian ring. A filtration Σ of a left A-module M is a *good filtration* if the *associated graded module* $\operatorname{gr}^\Sigma(M) := \bigoplus_{i\in\mathbb{Z}} \Sigma_i/\Sigma_{i-1}$ is finitely generated over $\operatorname{gr}^\Gamma(A)$.

Under the former conditions, J.-E. Björk proved in [Bjö79, Chapter 2, Proposition 6.1] that A is both left and right Noetherian ring. Moreover, a left A-module M has a good filtration if and only if M is finitely generated. If Σ and Σ' are two good filtrations of M, then there are non-negative integers j and k such that $\Sigma_i \subset \Sigma'_{i+k}$ and $\Sigma'_i \subset \Sigma_{i+j}$ for every i. Thus the Krull dimension of the $\operatorname{gr}^\Gamma(A)$-module $\operatorname{gr}^\Sigma(M)$ does not depend on the choice of the good filtration Σ of M. We call this number the *dimension* of M and we denote it by $d(M)$.

The next invariant plays an important role in this work.

Definition 6.4. Let A be an associative ring with identity. The *grade* $j_A(M)$ of a left A-module M is defined by $j_A(M) := \min\left\{j \geq 0 : \operatorname{Ext}^j_A(M, A) \neq 0\right\}$.

Remark 6.5. Every short exact sequence of A-modules $0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$ leads to an exact sequence $0 \longrightarrow \operatorname{Hom}_A(N, A) \longrightarrow \operatorname{Hom}_A(M, A) \longrightarrow \operatorname{Hom}_A(L, A) \longrightarrow \operatorname{Ext}^1_A(N, A) \longrightarrow \cdots \longrightarrow \operatorname{Ext}^n_A(L, A) \longrightarrow \operatorname{Ext}^{n+1}_A(N, A) \longrightarrow \operatorname{Ext}^{n+1}_A(M, A) \longrightarrow \cdots$. Hence $j_A(M) \geq \min\{j_A(L), j_A(N)\}$ and $j_A(N) \geq \min\{j_A(L), j_A(M)\}$.

According to [MNM91], a filtered ring A is a ring of *differentiable type* if its associated graded ring is a commutative Noetherian regular ring and all its maximal graded ideals have the same height. For example, if R is a differentiable admissible k-algebra, then $\mathscr{D}_{R/k}$ is a ring of differentiable type (see [NB13, Theorem 2.12]).

Let A be a ring of differentiable type and M be a non-zero finitely generated left or right A-module. Z. Mebkhout and L. Narváez-Macarro proved in [MNM91, Théorème 1.2.2] that

$$d(M) + j_A(M) = \dim(\mathrm{gr}^\Gamma(A)). \tag{6.2}$$

In particular,

$$d(M) \geq \dim(\mathrm{gr}^\Gamma(A)) - \mathrm{gl.}\dim(A). \tag{6.3}$$

Recall that a finitely generated left or right A-module M is said to be *holonomic* or to be in the left or right *Bernstein class* when the equality holds in equation (6.3), this is, $d(M) = \dim(\mathrm{gr}^\Gamma(A)) - \mathrm{gl.}\dim(A)$.

In the special case when R is a differentiable admissible k-algebra, we already observed that $\mathrm{gr}^\Gamma(\mathscr{D}_{R/k}) \cong R[y_1, \ldots, y_n]$. Also $\mathrm{gl.}\dim(\mathscr{D}_{R/k}) = \dim(R) = n$ by equation (6.1). Thus a finitely generated left or right $\mathscr{D}_{R/k}$-module is holonomic if and only if $d(M) = \dim(R[y_1, \ldots, y_n]) - \dim(R) = 2n - n = n$.

6.2 Quasi-holonomic $\mathscr{D}$-modules

For the remainder of this work, R is considered a differentiable admissible k-algebra over a field k of characteristic zero and Krull dimension n and we denote by $\mathscr{D}$ the ring $\mathscr{D}_{R/k}$ of k-linear differential operators on R.

For any $\mathscr{D}$-module M, we set $\tau(M) = \inf\{j_{\mathscr{D}}(N) : N \text{ is a } \mathscr{D}\text{-submodule of } M\}$. If $\mathrm{proj.}\dim(M)$ is the projective dimension of M, then $\mathrm{proj.}\dim(M) \leq \mathrm{gl.}\dim(\mathscr{D}) = n$.

We recall the following property appearing in [CE56, Chapter VI, exercise 9]. It allows us to conclude that $\tau(M) \leq n$.

Lemma 6.6. *Let A be an associative ring and M be a non-zero finitely generated left A-module. Suppose that A is Noetherian and M has finite projective dimension* $\mathrm{proj.}\dim_A(M) = r$. *Then* $\mathrm{Ext}^r_A(M, A) \neq 0$. *In particular,* $j_A(M) \leq r$.

Proof. Since $\mathrm{proj.}\dim_A(M) = r$, $\mathrm{Ext}^{r+1}_A(M, G) = 0$ for every left A-module G. Moreover, there exists a left A-module N such that $\mathrm{Ext}^r_A(M, N) \neq 0$. For this module there are left A-modules F and L such that F is free and the sequence $0 \to L \to F \to N \to 0$ is exact. This sequence induces a long exact sequence

$$\cdots \to \mathrm{Ext}^r_A(M, F) \to \mathrm{Ext}^r_A(M, N) \to \mathrm{Ext}^{r+1}_A(M, L) \to \mathrm{Ext}^{r+1}_A(M, F) \to \cdots$$

As $\mathrm{Ext}^{r+1}_A(M, L) = 0$ and $\mathrm{Ext}^r_A(M, N) \neq 0$, we have that $\mathrm{Ext}^r_A(M, F) \neq 0$.

Now we write $F = \bigoplus_{\lambda \in \Lambda} A$ and since A is Noetherian and M is finitely generated, we have that $\mathrm{Ext}^r_A(M, F) = \mathrm{Ext}^r_A(M, \bigoplus_{\lambda\in\Lambda} A) \cong \bigoplus_{\lambda\in\Lambda} \mathrm{Ext}^r_A(M, A)$. Therefore, $\mathrm{Ext}^r_A(M, A) \neq 0$ and $j_A(M) \leq r$. □

The $\mathscr{D}$-modules M such that $\tau(M)$ is maximal are of special importance in our work.

Definition 6.7. A left $\mathscr{D}$-module M is *quasi-holonomic* if $\tau(M) = n$.

Remark 6.8. Note that a $\mathscr{D}$-module M is holonomic if and only it is finitely generated and quasi-holonomic. In fact, if M is holonomic, then every non-zero submodule N of M is holonomic. Thus $d(N) = n$ and hence $j_{\mathscr{D}}(N) = n$ by equation (6.2). Therefore $\tau(M) = n$. Conversely, if $\tau(M) = n$, then $j_{\mathscr{D}}(M) \geq n$. On the other hand, Lemma 6.6 implies that $j_{\mathscr{D}}(M) \leq n$. Therefore, $j_{\mathscr{D}}(M) = n$. By equation (6.2) we have that $d(M) = n$.

Remark 6.9. Every holonomic $\mathscr{D}$-module is Artinian. Furthermore, every holonomic $\mathscr{D}$-module has finite length by [MNM91, Proposition 1.2.5]. On the other hand, if $\mathscr{D}$ is a simple ring (e.g., if $\mathscr{D} = R\langle \partial_1, \dots, \partial_n\rangle$, where $R = k[x_1, \dots, x_n]$ or $R = k[[x_1, \dots, x_n]]$ or $R = \mathbb{C}\{x_1, \dots, x_n\}$ and $\partial_1, \dots, \partial_n$ are the usual derivations over R), every holonomic $\mathscr{D}$-module is cyclic. This is a consequence of the following result due to J. T. Stafford: if A is a simple ring of infinite length as a left A-module, then every left A-module with a finite length is cyclic (see [Bjö79, Chapter 1, Theorem 8.18]). Consequently, if M is a quasi-holonomic $\mathscr{D}$-module, then M is *locally Artinian*, i.e., every finitely generated submodule of M is Artinian. Moreover, if $\mathscr{D}$ is a simple ring, M is *locally cyclic*, that is, every finitely generated submodule of M is cyclic.

We readily get the following consequence from Remark 6.5.

Lemma 6.10. *The class T of quasi-holonomic $\mathscr{D}$-modules is a full subcategory of the category of $\mathscr{D}$-modules closed under the following operations: taking submodules, quotients and extensions.*

The class T is also closed for inductive limits. More precisely,

Theorem 6.11. *Suppose that $\{M_\lambda, \lambda \in \Lambda\}$ is an inductive system of quasi-holonomic $\mathscr{D}$-modules and consider $M := \varinjlim_{\lambda\in\Lambda} M_\lambda$. Then M is quasi-holonomic.*

Proof. Set $M' := \bigoplus_{\lambda\in\Lambda} M_\lambda$. Since M is a quotient of M', it is enough to prove that M' is quasi-holonomic by Lemma 6.10. For this, consider $\lambda_0 \in \Lambda$ and the canonical injection $\imath_0 : M_{\lambda_0} \to M'$. We set $M_0 := \operatorname{im} \imath_0$, then M_0 is a quasi-holonomic $\mathscr{D}$-module.

In particular, $j_{\mathscr{D}}(M_0) \geq n$. Let us prove at first instance that $j_{\mathscr{D}}(N) \geq n$ for every submodule N of M' such that $M_0 \subset N$. Then $N/M_0 \cong \bigoplus_{\lambda \in \Lambda'} M_\lambda$, where $\Lambda' = \Lambda - \{\lambda_0\}$. On the other hand, $j_{\mathscr{D}}(M_\lambda) \geq n$ for each $\lambda \in \Lambda'$. This implies that $\mathrm{Ext}^i_{\mathscr{D}}(M_\lambda, \mathscr{D}) = 0$ for all $0 \leq i < n$ and for each $\lambda \in \Lambda'$. Since $\mathrm{Ext}^i_{\mathscr{D}}(\bigoplus_{\lambda \in \Lambda'} M_\lambda, \mathscr{D}) \cong \prod_{\lambda \in \Lambda'} \mathrm{Ext}^i_{\mathscr{D}}(M_\lambda, \mathscr{D})$ by [Rot09, Theorem 7.13], we have that $\mathrm{Ext}^i_{\mathscr{D}}(N/M_0, \mathscr{D}) = 0$ for all $0 \leq i < n$ and thus $j_{\mathscr{D}}(N/M_0) \geq n$. By Remark 6.5, $j_{\mathscr{D}}(N) \geq n$.

Now consider any submodule N of M'. As proved before, $j_{\mathscr{D}}((N + M_0)/M_0) \geq n$. Hence $j_{\mathscr{D}}(N/(N \cap M_0)) \geq n$. Since $N \cap M_0$ is quasi-holonomic, $j_{\mathscr{D}}(N \cap M_0) \geq n$. Therefore, we have that $j_{\mathscr{D}}(N) \geq n$ for every submodule N of M' by Remark 6.5. Thus $\tau(M') \geq n$ and consequently M' is quasi-holonomic. □

Corollary 6.12. *There is a quasi-holonomic $\mathscr{D}$-module M such that for every quasi-holonomic $\mathscr{D}$-module N there is a submodule L of $M^{(|N|)}$ such that $N \hookrightarrow M^{(|N|)}/L$.*

Proof. We select any set $\{M_\lambda, \lambda \in \Lambda\}$ of representatives of isomorphy classes of cyclic submodules of quasi-holonomic $\mathscr{D}$-modules. Hence, for a quasi-holonomic module N and $m \in N$, there is $\lambda \in \Lambda$ such that $\mathscr{D}m \cong M_\lambda$. We set $M := \bigoplus_{\lambda \in \Lambda} M_\lambda$. Since each M_λ is quasi-holonomic (even more, holonomic), M is quasi-holonomic by Theorem 6.11. If ϕ is the sum map $\bigoplus_{x \in N} \mathscr{D}x \to N = \sum_{x \in N} \mathscr{D}x$ and $V = \ker \phi$, then $N \cong \bigoplus_{x \in N} \mathscr{D}x/V$. Since $\mathscr{D}x \hookrightarrow M$ for each $x \in N$, we have that $N \hookrightarrow M^{(|N|)}/L$ for some submodule L of M. □

Let M be a left $\mathscr{D}$-module and consider an element $m \in M$. Recall that m is a *torsion element* if $\mathrm{Ann}_{\mathscr{D}}(m) := \{r \in \mathscr{D} : rm = 0\}$ is a non-zero left ideal of $\mathscr{D}$. If every element of M is torsion, then M is called a *torsion module*.

Proposition 6.13. *Every quasi-holonomic $\mathscr{D}$-module is a torsion module. Conversely, if $\mathscr{D}$ has no zero-divisors and $n = 1$, then every torsion $\mathscr{D}$-module is quasi-holonomic.*

Proof. Let M be a quasi-holonomic $\mathscr{D}$-module and m be a non-zero element of M. Consider the map $\varphi : \mathscr{D} \to M$ defined by $\varphi(r) = rm$. If $\mathrm{Ann}_{\mathscr{D}}(m) = \ker \varphi = 0$, then $0 = j_{\mathscr{D}}(\mathscr{D}) = j_{\mathscr{D}}(\mathrm{im}\, \varphi) \geq n$, which is a contradiction. Therefore, $\mathrm{Ann}_{\mathscr{D}}(m) \neq 0$.

Conversely, suppose that $\mathscr{D}$ has no zero-divisors and $n = 1$. Let a be a non-zero element of $\mathscr{D}$. Then $\mathrm{Hom}_{\mathscr{D}}(\mathscr{D}/\mathscr{D}a, \mathscr{D}) = 0$. Therefore $j_{\mathscr{D}}(\mathscr{D}/\mathscr{D}a) \geq 1$ and consequently $d(\mathscr{D}/\mathscr{D}a) = 1$, i.e., $\mathscr{D}/\mathscr{D}a$ is holonomic. Now let M be a torsion $\mathscr{D}$-module and let N be a finitely generated submodule of M. Suppose that N is generated by $m_1, \ldots, m_r$. Since M is torsion, for each $j = 1, \ldots, r$ there exists $0 \neq a_j \in \mathscr{D}$ such that $a_j m_j = 0$. Hence $\mathscr{D}m_j$ is a quotient of $\mathscr{D}/\mathscr{D}a_j$. Thus each $\mathscr{D}m_j$ is holonomic. Since N is the sum of all the $\mathscr{D}m_j$, it is holonomic. Finally, since every module is the inductive limit of its finitely generated submodules, we have that M is quasi-holonomic by Theorem 6.11. □

The ring R has a natural structure of $\mathscr{D}$-module. The action of each element of R is by straightforward multiplication, whilst the action of each derivation $\delta \in Der_k(R)$ over an element f of R is $\delta \cdot f = \delta(f)$. It is not hard to show that R is a holonomic $\mathscr{D}$-module, whence torsion.

Proposition 6.14. *If R is strong differentiable admissible and $\mathscr{D}$ is a simple ring, then R is an irreducible $\mathscr{D}$-module. In this case,*

$$R \cong \mathscr{D} / \sum_{i=1}^{n} \mathscr{D} D_i$$

Proof. If I is a non-zero submodule of R, then the two-sided ideal $\mathfrak{a}$ of $\mathscr{D}$ generated by I coincides with $\mathscr{D}$. But $I = \mathfrak{a} \cap R$. Then $I = \mathscr{D} \cap R = R$.

The element $1 \in R$ is annihilated by $D_1, \ldots, D_n$. Hence the left ideal J of $\mathscr{D}$ generated by $D_1, \ldots, D_n$ is contained in $\mathrm{Ann}_{\mathscr{D}}(1)$. Conversely, consider $P \in \mathrm{Ann}_{\mathscr{D}}(1)$. Then P may be written in the form $f + Q$, where $Q \in J$ and $f \in R$. Thus

$$0 = P \cdot 1 = f \cdot 1 = f.$$

Therefore $P = Q \in J$. Consequently, $J = \mathrm{Ann}_{\mathscr{D}}(1)$. Consider now the map of $\mathscr{D}$-modules $\phi : \mathscr{D} \to R$ defined by $\phi(1) = 1$. Since $0 \neq 1 \in R$ and R is irreducible, ϕ is surjective. On the other hand, $\ker \phi = \mathrm{Ann}_{\mathscr{D}}(1) = J$. Therefore, $\mathscr{D}/J \cong R$. $\square$

Proposition 6.15. *Let M be a quasi-holonomic $\mathscr{D}$-module. Then the flat dimension* $\mathrm{fd}(M)$ *of M is at most n. Furthermore, if* $\mathrm{fd}(M) = n$, *then* $\mathrm{proj.\,dim}(M) = n$.

Proof. Let us assume first that M is finitely generated (hence holonomic by Remark 6.8). If $\mathrm{proj.\,dim}(M) \leq n-1$, then $\mathrm{Ext}^n_{\mathscr{D}}(M, \mathscr{D}) = 0$. This contradicts the fact that $j_{\mathscr{D}}(M) = n$. Therefore, $\mathrm{proj.\,dim}(M) \geq n$. Since $\mathrm{proj.\,dim}(M) \leq \mathrm{gl.\,dim}(\mathscr{D}) = n$, we have that $\mathrm{proj.\,dim}(M) = n$. Since $\mathscr{D}$ is Noetherian and M is finitely generated, $\mathrm{fd}(M) = \mathrm{proj.\,dim}(M) = n$.

In general, M is the inductive limit of its finitely generated submodules, i.e., $M = \varinjlim_{\lambda \in \Lambda} M_\lambda$. On the other hand, if $0 \to (F_n)_\lambda \to \cdots \to (F_1)_\lambda \to (F_0)_\lambda \to M_\lambda$ is a flat resolution of each M_λ, then $0 \to \varinjlim_{\lambda \in \Lambda} (F_n)_\lambda \to \cdots \to \varinjlim_{\lambda \in \Lambda} (F_1)_\lambda \to \varinjlim_{\lambda \in \Lambda} (F_0)_\lambda \to \varinjlim_{\lambda \in \Lambda} M_\lambda$ is a flat resolution of M. Therefore $\mathrm{fd}(M) \leq n$.

The last statement follows from $\mathrm{fd}(M) \leq \mathrm{proj.\,dim}(M) \leq \mathrm{gl.\,dim}(\mathscr{D}) = n$. $\square$

Quasi-holonomic $\mathscr{D}$-modules which are direct sums of holonomic $\mathscr{D}$-modules have projective and flat dimension equal to n, because $\mathrm{proj.\,dim}(\oplus M_\lambda) = \sup\{\mathrm{proj.\,dim}(M_\lambda)\}$ and $\mathrm{fd}(\oplus M_\lambda) = \sup\{\mathrm{fd}(M_\lambda)\}$.

Now let S be a multiplicative subset of R. For any $\mathscr{D}$-module M, the R-module $S^{-1}M$ has structure of $\mathscr{D}$-module in such a way that the natural map $M \to S^{-1}M$ is a homomorphism of $\mathscr{D}$-modules. In order to extend the action of $\delta \in Der_k(R)$ to $S^{-1}M$, we define it by the usual quotient rule for differentiation, i.e.,

$$\delta \cdot \left(\frac{m}{f}\right) = \frac{f \cdot \delta m - \delta(f) \cdot m}{f^2}.$$

Remark 6.16. J.-E. Björk proved in [Bjö79, Chapter 3, Theorem 4.1] the non-trivial result that the localisation by an element of a power series ring $A = k[[x_1, \dots, x_n]]$ of a holonomic $\mathscr{D}_{A/k}$-module is also a holonomic $\mathscr{D}_{A/k}$-module. Later, Z. Mebkhout and L. Narváez-Macarro extended this result for strong differentiable admissible k-algebras in [MNM91, Théorème 3.2.1] and L. Núñez-Betancourt did the proper for differentiable admissible k-algebras in [NB13, Corollary 3.12].

Now, let M be an R-module and let S be a multiplicative subset of R. In S we define the following relation: given $s, t \in S$, $s \leq t$ if there is $r \in R$ such that $t = rs$. It is not hard see that $\leq$ is a partial order over S. With this order, S becomes a directed set. For $s \leq t$, we consider the map $\varphi_{s,t} : M_s \to M_t$ defined by $\varphi_{s,t}(\frac{m}{s}) = \frac{rm}{t}$, where $t = rs$. Then, $\{M_s, \varphi_{s,t} : M_s \to M_t\}$ is an inductive system of R-modules such that $\varinjlim_{s\in S} M_s = S^{-1}M$ in the category of R-modules with the natural R-homomorphisms $i_s : M_s \to S^{-1}M$ as insertion morphisms.

The next result follows immediately from the constructions above.

Proposition 6.17. *Suppose that M is a $\mathscr{D}$-module and that S is a multiplicative subset of R. Then*

(i) M_s is a $\mathscr{D}$-module for each $s \in S$ and for $s \leq t$, the map $\varphi_{s,t} : M_s \to M_t$ is a homomorphism of $\mathscr{D}$-modules. Moreover, $\{M_s, \varphi_{s,t} : M_s \to M_t\}$ is an inductive system of $\mathscr{D}$-modules.

(ii) For each $s \in S$, the natural map $i_s : M_s \to S^{-1}M$ is a homomorphism of $\mathscr{D}$-modules such that $\varinjlim_{s\in S} M_s = S^{-1}M$ is the inductive limit of the system above in the category of $\mathscr{D}$-modules.

Corollary 6.18. *Let S be a multiplicative subset of R. If M is a quasi-holonomic $\mathscr{D}$-module, then $S^{-1}M$ is a quasi-holonomic $\mathscr{D}$-module.*

Proof. We shall see that M_f is quasi-holonomic for all $f \in R$. In fact, $M \cong \varinjlim M^\lambda$ where M^λ is a finitely generated $\mathscr{D}$-module for every λ. Since localisation at f commutes with inductive limits, we have that the R-modules M_f and $\varinjlim M_f^\lambda$ are isomorphic. Therefore, using Proposition 6.17 they are also isomorphic as $\mathscr{D}$-modules. Thus

M_f is a quasi-holonomic $\mathscr{D}$-module for every $f \in R$ by Theorem 6.11. The statement follows by Proposition 6.17, (2) and Theorem 6.11. $\square$

6.3 Local cohomology and Bass numbers

Consider two families $\beta \subseteq \alpha$ of R and an R-module M. The cohomology modules $H^i_{\alpha/\beta}(M)$ are defined by the right derived functors arising from the exact sequence of R-modules

$$0 \longrightarrow \Gamma_\beta(M) \xrightarrow{\iota} \Gamma_\alpha(M) \xrightarrow{\pi} \Gamma_\alpha(M)/\Gamma_\beta(M) \longrightarrow 0.$$

See [Har66, pp. 219–221]. A functor $\mathscr{T}(-)$ is called a *Lyubeznik functor* (cf. [Lyu93, 1] and [NB13, Definition 4.1]) if it is a composition of cohomology functors or kernels of the induced long exact sequence

$$\cdots \xrightarrow{\iota^i} H^i_\alpha(-) \xrightarrow{\pi^i} H^i_{\alpha/\beta}(-) \xrightarrow{\partial^i} H^{i+1}_\beta(-) \xrightarrow{\iota^{i+1}} \cdots . \tag{6.4}$$

Theorem 6.19. *Let M be a $\mathscr{D}$-module and let α be a family of supports on* $\operatorname{Spec} R$. *Then the local cohomology modules $H^i_\alpha(M)$ all have the structure of $\mathscr{D}$-modules. Moreover, if M is quasi-holonomic, then $H^i_\alpha(M)$ is quasi-holonomic.*

Proof. The i-th cohomology functor $H^i_\alpha(-)$ is an additive functor for every i. Hence, by [Lyu93, Example 2.1, (iii)], $H^i_\alpha(M)$ is a $\mathscr{D}$-module for every i. By Theorem 1.11 we have that $H^i_\alpha(M) \cong \varinjlim_{I \in \langle\alpha\rangle} H^i_I(M)$ as R-modules. We claim that this is also an isomorphism of $\mathscr{D}$-modules. In fact, we have that $H^i_I(M) \cong H^i(C^\bullet_{\mathbf{f}}(M))$ as $\mathscr{D}$-modules, where $\mathbf{f}$ is a set of generators of I and $C^\bullet_{\mathbf{f}}(M)$ is the Čech complex of M. The direct system $\{H^i_I(M), \iota_{JI}\}_{I\in\langle\alpha\rangle}$ of R-modules is also a direct system of $\mathscr{D}$-modules because if $I \supseteq J$, then we can choose generators $\mathbf{g}$ of J and complete it to a set of generators $\mathbf{g}, \mathbf{f}$ of I, in which case the morphisms from $C^\bullet_{\mathbf{g},\mathbf{f}}(M)$ to $C^\bullet_{\mathbf{g}}(M)$ are given by projections. Hence they are $\mathscr{D}$-homomorphisms. Thus ι_{JI} is also a $\mathscr{D}$-homomorphism and this proves the claim. From Theorem 6.11 we have the statement. $\square$

Corollary 6.20. *Let M be a $\mathscr{D}$-module. Then $\mathscr{T}(M)$ has a structure of $\mathscr{D}$-module. Moreover, if M is quasi-holonomic, then $\mathscr{T}(M)$ is quasi-holonomic.*

Proof. It suffices to show the second statement for $H^i_{\alpha/\beta}(M)$, but this follows from the exact sequence $0 \longrightarrow H^i_\alpha(M)/\ker \pi^i \longrightarrow H^i_{\alpha/\beta}(M) \longrightarrow \operatorname{im} \partial^i \longrightarrow 0$ which is induced by equation (6.4). $\square$

Corollary 6.21. *For every i, the local cohomology module $H^i_\alpha(M)$ is a locally Artinian, torsion $\mathscr{D}$-module. Moreover, if $\mathscr{D}$ is a simple ring, then $H^i_\alpha(M)$ is a locally cyclic $\mathscr{D}$-module.*

Proof. This follows from Theorem 6.19, Remark 6.9 and Corollary 6.13. □

Before presenting our next results, we need the following statement.

Proposition 6.22. *Let $(R, \mathfrak{m}, K)$ be a local differentiable admissible k-algebra. If M is a $\mathscr{D}$-module, then $\hat{R} \otimes_R M$ is a $\hat{\mathscr{D}}$-module, where $\hat{\mathscr{D}} = \mathscr{D}_{\hat{R}/K}$. Moreover, if M is a holonomic $\mathscr{D}$-module, then $\hat{R} \otimes_R M$ is a holonomic $\hat{\mathscr{D}}$-module.*

Proof. By Theorem 6.2, we have that $\hat{R} = K[[x_1, \ldots, x_n]]$. Therefore, if $\partial_1, \ldots, \partial_n$ are the usual derivations over $K[[x_1, \ldots, x_n]]$, then $\hat{\mathscr{D}} = K[[x_1, \ldots, x_n]]\langle \partial_1, \ldots, \partial_n \rangle$ and $\partial_i \in Der_k(R)$. Now we define the actions of the elements $f \in \hat{R}$ and the derivations ∂_i, $i = 1, \ldots, n$ over the elements $g \otimes m$ of the R-module $\hat{R} \otimes_R M$ by $f \cdot (g \otimes m) := fg \otimes m$ and $\partial_i \cdot (g \otimes m) := \partial_i(g) \otimes m + g \otimes \partial_i \cdot m$.

Since $\partial_i \in Der_k(R)$ and $[\partial_i, f] \cdot (g \otimes m) = \partial_i(f) \cdot (g \otimes m)$ for all $i = 1, \ldots, n$ and for all $f, g \in \hat{R}$ and $m \in M$, the action can be extended to all elements of $\hat{\mathscr{D}}$. Consequently, $\hat{R} \otimes_R M$ is a $\hat{\mathscr{D}}$-module.

On the other hand, by Theorem 6.2, $Der_k(\hat{R}) = Der_K(\hat{R})$. Thus $\mathscr{D}_{\hat{R}/k} = \hat{\mathscr{D}}$. Consider now the map $\hat{R} \otimes_R M \to \mathscr{D}_{\hat{R}/k} \otimes_{\mathscr{D}} M$ defined by $f \otimes m \mapsto f \otimes m$. It is not hard to prove that this is an injective map of $\mathscr{D}_{\hat{R}/k}$-modules. Since $\mathscr{D}_{\hat{R}/k} \otimes_{\mathscr{D}} M$ is a holonomic $\mathscr{D}_{\hat{R}/k}$-module (see [MNM91, Remarque 2.2.5]), it follows that $\hat{R} \otimes_R M$ is a holonomic $\mathscr{D}_{\hat{R}/k}$-module and the statement follows. □

Corollary 6.23. *Let $(R, \mathfrak{m}, K)$ be a local differentiable admissible k-algebra and M be a $\mathscr{D}$-module. Then* $\operatorname{inj.dim}_{\hat{R}}(\hat{R} \otimes_R M) \leq \dim_{\hat{R}}(\hat{R} \otimes_R M)$. *If M is holonomic, then the set of associated primes of the R-module M is finite.*

Proof. We have that $\hat{R} \otimes_R M$ is a $\hat{\mathscr{D}}$-module by Proposition 6.22. Since the completion $\hat{R}$ is the power series ring $K[[x_1, \ldots, x_n]]$, we obtain from [Lyu93, Theorem 2.4, (b)] that $\operatorname{inj.dim}_{\hat{R}}(\hat{R} \otimes_R M) \leq \dim_{\hat{R}}(\hat{R} \otimes_R M)$. If M is holonomic, then $\hat{R} \otimes_R M$ is holonomic by Proposition 6.22. Hence the set of associated primes of $\hat{R} \otimes_R M$ as $\hat{R}$-module is finite by [Lyu93, Theorem 2.4, (c)]. Since every associated prime of the R-module M is restriction of an associated prime of the $\hat{R}$-module $\hat{R} \otimes_R M$, we have that the set of associated primes of M as R-module is finite. □

Theorem 6.24. *Let R be a differentiable admissible k-algebra and let M be a left $\mathscr{D}$-module.*

(a) For any maximal ideal $\mathfrak{m}$ of R, $H^j_{\mathfrak{m}}(M)$ is an injective R-module.

(b) If $\dim_R(M) = 0$, then M is an injective R-module.

(c) *Suppose that M is a quasi-holonomic $\mathscr{D}$-module. If N is a finitely generated $\mathscr{D}$-submodule of M, then the set of associated primes of N is finite.*

(d) *Suppose that M is a holonomic $\mathscr{D}$-module. Then $\mu^i(\mathfrak{m}, M)$ is finite for every maximal ideal $\mathfrak{m}$ and every i.*

Proof. (a) Note that $H^j_{\mathfrak{m}}(M)$ is an injective R-module if and only if $(H^j_{\mathfrak{m}}(M))_{\mathfrak{n}}$ is an injective $R_{\mathfrak{n}}$-module for every maximal ideal $\mathfrak{n}$ of R. Since $(H^j_{\mathfrak{m}}(M))_{\mathfrak{n}} = H^j_{\mathfrak{m}R_{\mathfrak{n}}}(M_{\mathfrak{n}})$ by Lemma 1.18, we can assume that R is local with maximal ideal $\mathfrak{m}$. Let $\hat{R}$ be the completion of R with respect to $\mathfrak{m}$. Again, by Lemma 1.18, we have that $\hat{R} \otimes_R H^j_{\mathfrak{m}}(M) \cong H^j_{\mathfrak{m}\hat{R}}(\hat{R} \otimes_R M)$. Since $H^j_{\mathfrak{m}}(M)$ is supported only at $\mathfrak{m}$, $\hat{R} \otimes_R H^j_{\mathfrak{m}}(M) = H^j_{\mathfrak{m}}(M)$. Therefore, $H^j_{\mathfrak{m}\hat{R}}(\hat{R} \otimes_R M) \cong H^j_{\mathfrak{m}}(M)$. In view of Proposition 6.22, $\hat{R} \otimes_R M$ is a $\mathscr{D}_{\hat{R}/K}$-module, where $K = R/\mathfrak{m}$ and $\mathscr{D}_{\hat{R}/K}$ is the ring of differential operators $K[[x_1, \ldots, x_n]]\langle \partial_1, \ldots, \partial_n \rangle$. Since the dimension of $H^j_{\mathfrak{m}\hat{R}}(\hat{R} \otimes_R M)$ is zero, we have that $H^j_{\mathfrak{m}\hat{R}}(\hat{R} \otimes_R M)$ is a direct sum of copies of $E_{\hat{R}}(\hat{R}/\mathfrak{m}\hat{R}) = E_R(R/\mathfrak{m})$ by [Lyu93, Proposition 2.3 and Theorem 2.4]. Hence $H^j_{\mathfrak{m}\hat{R}}(\hat{R} \otimes_R M)$ is an injective R-module and $H^j_{\mathfrak{m}}(M)$ is also an injective R-module.

(b) It suffices to show that $M_{\mathfrak{m}}$ is $R_{\mathfrak{m}}$-injective for every maximal ideal $\mathfrak{m}$ of R. For each maximal ideal $\mathfrak{m}$ of R, we have that $\operatorname{Supp}_{R_{\mathfrak{m}}}(M_{\mathfrak{m}}) \subseteq \{\mathfrak{m}R_{\mathfrak{m}}\}$. Therefore, $H^0_{\mathfrak{m}R_{\mathfrak{m}}}(M_{\mathfrak{m}}) = \Gamma_{\mathfrak{m}R_{\mathfrak{m}}}(M_{\mathfrak{m}}) = M_{\mathfrak{m}}$ and this implies that $M_{\mathfrak{m}}$ is an injective $R_{\mathfrak{m}}$-module by (a). Thus the statement follows.

(c) Let N be a finitely generated submodule of M. Since M is quasi-holonomic, we have that N is holonomic by Remark 6.8. Now, the result follows from [NB13, Lemma 4.3].

(d) Let $\mathfrak{m}$ be a maximal ideal of R. Then $\mu^i(\mathfrak{m}, M) = \mu^i(\mathfrak{m}R_{\mathfrak{m}}, M_{\mathfrak{m}})$. Therefore, we can assume that R is local and $\mathfrak{m}$ is the maximal ideal of R. By part (a) and [Lyu93, Lemma 1.4] we have that $\mu^i(\mathfrak{m}, M) = \mu^0(\mathfrak{m}, H^i_{\mathfrak{m}}(M))$. Therefore it is sufficient to prove that $\mu^0(\mathfrak{m}, H^i_{\mathfrak{m}}(M))$ is finite. Let $\hat{R}$ be the completion of R with respect to the maximal ideal $\mathfrak{m}$. Then $\hat{R} = K[[x_1, ..., x_n]]$, where $K = R/\mathfrak{m}$. By Lemma 1.18 we obtain that $H^i_{\mathfrak{m}\hat{R}}(\hat{R} \otimes_R M) = \hat{R} \otimes_R H^i_{\mathfrak{m}}(M)$. But $\hat{R} \otimes_R H^i_{\mathfrak{m}}(M) = H^i_{\mathfrak{m}}(M)$ because $\dim_R(H^i_{\mathfrak{m}}(M)) = 0$. We conclude in this way that $H^i_{\mathfrak{m}\hat{R}}(\hat{R} \otimes_R M) = H^i_{\mathfrak{m}}(M)$ and $\mu^0(\mathfrak{m}, H^i_{\mathfrak{m}}(M)) = \mu^0(\mathfrak{m}\hat{R}, H^i_{\mathfrak{m}\hat{R}}(\hat{R} \otimes_R M))$.

By Proposition 6.22, $\hat{R} \otimes_R M$ is a holonomic $\hat{\mathscr{D}}$-module. Hence $H^i_{\mathfrak{m}\hat{R}}(\hat{R} \otimes_R M)$ is holonomic. Therefore, $\mu^0(\mathfrak{m}\hat{R}, H^i_{\mathfrak{m}\hat{R}}(\hat{R} \otimes_R M))$ is finite by [Lyu93, Theorem 2.4, (d)] and the statement follows.

□

Corollary 6.25. *Let R be a differentiable admissible k-algebra and M be a $\mathscr{D}$-module.*

(a) If $\dim_R(\mathscr{T}(M)) = 0$*, then* $\mathscr{T}(M)$ *is an injective* R*-module. In particular,* $H^j_\alpha(\mathscr{T}(M))$ *is an injective* R*-module for every* j *and every good family* $\langle\alpha\rangle$ *of* R *generated by maximal ideals.*

(b) Suppose that M *is a quasi-holonomic* $\mathscr{D}$*-module. If* N *is a finitely generated* $\mathscr{D}$*-submodule of* $\mathscr{T}(M)$*, then the set of associated primes of* N *is finite.*

(c) If M *is quasi-holonomic, then every finitely generated* $\mathscr{D}$*-submodule of* $\mathscr{T}(M)$ *has finite Bass numbers with respect to the maximal ideals.*

Proof. Since $\mathscr{T}(M)$ is a $\mathscr{D}$-module, the first part of (a) follows from Theorem 6.24, (b). The second statement follows from the first one by taking the Lyubeznik functor $\tilde{\mathscr{T}}(-) = H^j_\alpha \circ \mathscr{T}(-)$ because $\dim(R/I) = 0$ for every $I \in \langle\alpha\rangle$.

Since $\mathscr{T}(M)$ is quasi-holonomic by Corollary 6.20, item (b) follows directly from Theorem 6.24, (c).

For (c), note that $\mathscr{T}(M)$ is quasi-holonomic because M is, hence every finitely generated submodule of $\mathscr{T}(M)$ is holonomic. $\square$

Appendix

Appendix A

Complementary results

We show here a particular version of Grothendieck's spectral sequences. This proof is a suitable modification of the proof of [Rot09, Theorem 10.47].

Theorem A.1. *Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be three abelian categories, $F : \mathcal{B} \to \mathcal{C}$ and $G : \mathcal{A} \to \mathcal{B}$ two additive functors. Suppose that $\mathcal{A}$ has enough injectives, $\mathcal{B}$ has enough projectives, F is right exact, G is contravariant and $G(E)$ is left F-acyclic for every injective object E of $\mathcal{A}$. Then, for every object A of $\mathcal{A}$, there exists a first quadrant spectral sequence*

$$E^2_{p,q} = (L_pF)(L_qG)(A) \Rightarrow (L_n(FG))(A).$$

Proof. We shall construct a double complex such that its iterated homology gives the desired spectral sequence. Consider an injective resolution $(E^*(A), d^*) = (E^i, d^i)_{i\geq 0}$ of A and apply the contravariant functor G in order to obtain the complex $(G(E^i), \delta_i)_{i\geq 0}$ where $\delta_i = G(d^{i-1}) : G(E^i) \to G(E^{i-1})$. Next we construct a *Cartan-Eilenberg projective resolution* for this complex: for every non-negative integer p, there are two exact sequences $0 \to B_p \to Z_p \to H_p(G(E^*)) \to 0$ and $0 \to Z_p \to G(E^p) \to B_{p-1} \to 0$ where $Z_p = \ker \delta_p$ and $B^p = \operatorname{im} \delta_{p+1}$. Take a projective resolution $B_{p,*}$ of B_p and another projective resolution $H_{p,*}$ of $H_p(G(E^*))$. Then we obtain projective resolutions, $Z_{p,*}$ of Z_p and $M_{p,*}$ of $G(E^p)$, and exact sequences of complexes $0 \to B_{p,*} \to Z_{p,*} \to H_{p,*} \to 0$ e $0 \to Z_{p,*} \to M_{p,*} \to B_{p-1,*} \to 0$. Define chain maps $d_{p,q} : M_{p,q} \to M_{p-1,q}$ as compositions $M_{p,q} \to B_{p-1,q} \to Z_{p-1,q} \to M_{p-1,q}$. In this way, the commutative diagram

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
\cdots \arrow[r] & M_{22} \arrow[r, "d_{22}"] \arrow[d] & M_{12} \arrow[r, "d_{12}"] \arrow[d] & M_{02} \arrow[r] \arrow[d] & 0 \\
\cdots \arrow[r] & M_{21} \arrow[r, "d_{21}"] \arrow[d] & M_{11} \arrow[r, "d_{11}"] \arrow[d] & M_{01} \arrow[r] \arrow[d] & 0 \\
\cdots \arrow[r] & M_{20} \arrow[r, "d_{20}"] \arrow[d] & M_{10} \arrow[r, "d_{10}"] \arrow[d] & M_{00} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

is a projective resolution of the complex $(G(E^i), \delta_i)_{i\geq 0}$. Denote the associated double complex by M. By calculating $F(M)$ we obtain the diagram

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
\cdots \arrow[r] & F(M_{22}) \arrow[r, "F(d_{22})"] \arrow[d] & F(M_{12}) \arrow[r, "F(d_{12})"] \arrow[d] & F(M_{02}) \arrow[r] \arrow[d] & 0 \\
\cdots \arrow[r] & F(M_{21}) \arrow[r, "F(d_{21})"] \arrow[d] & F(M_{11}) \arrow[r, "F(d_{11})"] \arrow[d] & F(M_{01}) \arrow[r] \arrow[d] & 0 \\
\cdots \arrow[r] & F(M_{20}) \arrow[r, "F(d_{20})"] \arrow[d] & F(M_{10}) \arrow[r, "F(d_{10})"] \arrow[d] & F(M_{00}) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Since $M_{p,*}$ is a projective resolution of $G(E^p)$, we have that $F(M_{p,*})$ is a complex such that its q-th homology equals $(L_qF)(G(E^p))$. As $G(E^p)$ is left F-acyclic, we have that $(L_qF)(G(E^p)) = 0$ for $q \geq 1$. Now F is right exact. Then $(L_0F)(G(E^p)) = FG(E^p)$ for each p. Thus the horizontal filtration of M generated a spectral sequence whose second page has terms ${}^{I}E^2_{p,0} = (L_p(FG))(A)$ and ${}^{I}E^2_{p,q} = 0$ when $q \neq 0$. Hence this spectral sequence collapses at $q = 0$ and $H_n(\mathrm{Tot}(F(M))) \cong (L_n(FG))(A)$.

It is time to calculate the second iterated homology: notice that

$$H_q(F(M_{*,p})) = \frac{\ker F(d_{q,p})}{\operatorname{im} F(d_{q+1,p})}.$$

We have thus a commutative diagram, being the row a complex,

$$\begin{tikzcd}
0 \arrow[r] & F(Z_{q,p}) \arrow[r, "F(\iota)"] & F(M_{q,p}) \arrow[r, "F(\pi)"] \arrow[dr, "F(d)"'] & F(B_{q-1,p}) \arrow[r] \arrow[d, "F(\iota)F(j)"] & 0 \\
 & & & F(M_{q-1,p}) &
\end{tikzcd}$$

Since $B_{q-1,p}$ is projective, the row is a split exact sequence. Now $j^{q,p} : B^{q,p} \to Z^{q,p}$ is an inclusion and $B^{q,p}$ is injective. Thus $F(j)$ is injective. We also have that $d = \iota j \pi$, whence $F(d) = F(\iota)F(j)F(\pi)$. Since $F(\iota)$ and $F(j)$ are monomorphisms, we have that $\ker F(d) = \ker F(\pi) = \operatorname{im} F(\iota) = F(\iota)(F(Z))$. Now

$$\operatorname{im} F(d) = F(d)(F(M)) = F(\iota)F(j)F(\pi)(F(M)) = F(\iota)F(j)(F(B)).$$

Again by the injectivity of $F(\iota) : F(Z) \to F(M)$ and $F(j) : F(B) \to F(Z)$ we obtain an isomorphism $\frac{F(Z)}{F(j)(F(B))} \cong \frac{F(\iota)(F(Z))}{F(\iota)F(j)(F(B))}$. Thus

$$\frac{\ker F(d)}{\operatorname{im} F(d)} = \frac{F(\iota)(F(Z))}{F(\iota)F(j)(F(B))} \cong \frac{F(Z)}{F(j)(F(B))}.$$

But $\frac{F(Z)}{F(j)(F(B))} \cong \operatorname{coker} F(j) \cong F(H)$ because the sequence

$$0 \longrightarrow F(B) \xrightarrow{F(j)} F(Z) \longrightarrow F(H) \longrightarrow 0$$

is exact as B is injective. Hence $H^q(F(M_{*,p})) = F(H_{q,p})$, this is, F commutes with H_q. Now $\cdots \to H_{q,1} \to H_{q,0} \to 0$ is a projective resolution of $H_q(G(E^*)) = (L_qG)(A)$ by the construction of the Cartan-Eilenberg projective resolution and by the same reason we have that $H_q(M_{*,p}) = H_{q,p}$. Hence $(H_q(M_{*,p}))$ is a projective resolution of $(L_qG)(A)$ and the generic term of the second page of the spectral sequence generated by the vertical filtration of $F(M)$ is

$${}^{II}E^2_{p,q} = H_pH_q(F(M)) = H_p(F(H_q(M))) = (L_pF)(L_qG)(A).$$

Since both filtrations converge to the homology of the total complex of $F(M)$, it follows that $(L_pF)(L_qG)(A) \Rightarrow L_n(FG)(A)$. $\square$

Among other results envolving duality, C. Huneke proved in [Hun07] the next statement. We write it just as it appears in [MZ14].

Lemma A.2. *Let R be a Noetherian local ring and M, N be R-modules. The following statements hold for every integer i:*

(i) $\operatorname{Ext}^i_R(N, M^\vee) \cong \operatorname{Tor}_i^R(N, M)^\vee$.

(ii) If N is finitely generated, then $\operatorname{Ext}^i_R(N, M)^\vee \cong \operatorname{Tor}_i^R(N, M^\vee)$.

Proof. The first isomorphism follows from the adjoint isomorphism

$$\operatorname{Hom}_R(P \otimes_R M, E) \cong \operatorname{Hom}_R(P, \operatorname{Hom}_R(M, E))$$

while the second one follows from the natural transformation (4.1). □

The next result appears in [Sch07] and it displays a constructive proof of [Bou89, Proposition 4, p. 263].

Lemma A.3. *Let R be a commutative Noetherian ring and M be a finitely generated R-module. Consider a subset $S = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_s\}$ of $\operatorname{Ass}(M) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_t\}$ and a minimal primary decomposition $0 = \bigcap_{i=1}^{t} Q_{\mathfrak{p}_i}$ for the zero submodule of M. If $N = \bigcap_{\mathfrak{p} \in S} Q_{\mathfrak{p}}$, then $\operatorname{Ass}(M/N) = S$ and $\operatorname{Ass}(N) = \operatorname{Ass}(M) - S$.*

Proof. Since $N = \bigcap_{\mathfrak{p} \in S} Q_{\mathfrak{p}}$ is a minimal primary decomposition, we have that

$$\operatorname{Ass}(M/N) = S.$$

Consider $N' = \bigcap_{\mathfrak{p} \in \operatorname{Ass}(M) - S} Q_{\mathfrak{p}}$. Since $N \cap N' = 0$, we have that $N \cong (N + N')/N'$. Thus $\operatorname{Ass}(N) \subseteq \operatorname{Ass}(M/N') = \operatorname{Ass}(M) - S$. If $\mathfrak{p} \in \operatorname{Ass}(M) - S$, then

$$0 \neq N/(N \cap Q_{\mathfrak{p}}) \cong (N + Q_{\mathfrak{p}})/Q_{\mathfrak{p}}$$

and $\operatorname{Ass}(N/(N \cap Q_{\mathfrak{p}})) = \{\mathfrak{p}\}$. On the other hand, $N \cap Q_{\mathfrak{p}}$ is part of a minimal primary decomposition of the zero submodule of N. Hence $\mathfrak{p} \in \operatorname{Ass}(N)$ as desired. □

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