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DEGREE THEORY AND BMO; PART II: COMPACT MANIFOLDS WITH BOUNDARIES

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(and an Appendix with Petru Mironescu)

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II.0. Introduction

This is a continuation of H. Brezis and L. Nirenberg [1] (= [BNI]), and we will often refer to concepts and results in that paper. There, we extended degree theory to VMO maps between compact n-dimensional oriented manifolds without boundaries. In this paper we consider a class of maps u from a bounded domain $\Omega \subset \mathbb{R}^n$ into $\mathbb{R}^n$. In classical degree theory, for $u \in C^0(\overline{\Omega}, \mathbb{R}^n)$, the degree of u at a point

$$p \notin u(\partial\Omega) \tag{0.1}$$

is defined; it is denoted by $\deg(u, \Omega, p)$.

The larger class of maps we consider, as in [BNI], is the class $\mathrm{VMO}(\Omega, \mathbb{R}^n)$ satisfying an appropriate variant of (0.1). To define VMO in a domain Ω, we have first to define BMO. There are several possible definitions; they turn out, however, to be equivalent. Here is one:

Definition. A real function f in $L^1_{\mathrm{loc}}(\Omega)$ is in $\mathrm{BMO}(\Omega)$ if

$$\|f\|_{\mathrm{BMO}(\Omega)} := \sup_B \fint_B |f - \fint_B f| < \infty, \tag{0.2}$$

where sup is taken over all Euclidean balls with closure in Ω.

In fact, one may use balls in any norm in $\mathbb{R}^n$—though this is far from obvious—see Corollary A1.1. Furthermore, one may consider the sup in (0.2) over the class of balls B lying "well inside" Ω, i.e., say $B = B_r(x)$ with $r \le \dfrac{1}{2}\mathrm{dist}(x, \partial\Omega)$. The resulting norm is smaller than that in (0.2), but is equivalent to it (see Theorem A1.1).

Now VMO is the closure of $C^0(\overline{\Omega})$ in the BMO norm of (0.2). A useful characterization of VMO(Ω) is

$$\lim_{\substack{\varepsilon\to 0\\ \varepsilon\le\frac{1}{2}\mathrm{dist}(x,\partial\Omega)}} \fint_{B_\varepsilon(x)} |f-\overline{f}_\varepsilon(x)| = 0 \quad \text{uniformly in } x.$$

Here

$$\overline{f}_\varepsilon(x) = \fint_{B_\varepsilon(x)} f.$$

This is the analogue of Sarason's characterization of VMO in $\mathbb{R}^n$; see D. Sarason [1]. A surprising fact about VMO(Ω) is that it is the closure in the BMO norm of $C_0^\infty(\Omega)$, C^∞ functions with compact support in Ω (see Theorem 1; it is proved in Appendix 1).

The facts above about BMO and VMO in Ω are due to Peter Jones; the proofs given here are slight modifications of his.

In addition to bounded domains in $\mathbb{R}^n$ we also consider domains Ω in a smooth open n-dimensional Riemannian manifold X_0, where $\overline{\Omega}$ is compact in X_0. BMO(Ω) is defined as in (0.2); the sup is now taken over geodesic balls $B_\varepsilon(x)$ with $\varepsilon < r_0$, the injectivity radius of $\overline{\Omega}$. As in $\mathbb{R}^n$, the various possible alternate definitions of BMO(Ω) are equivalent. Furthermore, the space BMO(Ω) is independent of the Riemannian metric on X_0 (see Lemma 2 in §II.1). VMO is defined as above. We then consider VMO maps of Ω into an n-dimensional smooth open manifold Y (which is smoothly embedded in some $\mathbb{R}^N$). If X_0 and Y are oriented, and $p \in Y$ is such that (0.1) holds—in a suitable sense—then we define

$$\deg(u,\Omega,p);$$

this is done again by approximation.

In dealing with manifolds one has to consider the effect of change of local coordinates. A result used here, but which more properly fits in [BNI], asserts that if the manifold X_0 is compact (without boundary), and if H is a smooth diffeomorphism of a ball B_R in $\mathbb{R}^n$ onto a subset of X_0, then there are positive constants C, ε_0, such that

$$|\overline{(f\circ H)_\varepsilon}(y) - \overline{f}_\varepsilon(H(y))| \le C\|f\|_{\mathrm{BMO}} \tag{0.3}$$

for every $f \in \mathrm{BMO}(X_0), |y| \le R/2$, and $\varepsilon < \varepsilon_0$. This is essentially Lemma A3.3.

In II.1 BMO and VMO are introduced and their invariance under choice of norms, as described above, is presented as well as associated properties.

Section II.2 takes up the definition of the degree. The analogue we use of condition (0.1) is that there exist a neighbourhood U in Ω of $\partial\Omega$, and a number $d_0 > 0$ such that

$$\fint_{B_\varepsilon(x)} \mathrm{dist}(u,p) \ge d_0 \quad \forall B_\varepsilon(x) \text{ in } U \text{ with } \varepsilon = \frac{1}{2}\mathrm{dist}(x,\partial\Omega). \tag{0.4}$$

Various properties of degree are then established, including (Corollary 1), the invariance of degree under continuous deformation in the BMO topology, provided that, under the deformation, (0.4) holds uniformly for all maps considered, with the same U and d_0. In Remark 4, an example is given in which the stability of degree fails in case this uniformity is dropped.

In general, functions in VMO(Ω) do not have a well defined trace on $\partial\Omega$. In II.3, in case $\partial\Omega$ is smooth, we introduce a subclass of VMO(Ω) which does: Suppose $\varphi \in \mathrm{VMO}(\partial\Omega)$; we may then extend φ inside Ω to a function $\widetilde{\varphi}$ belonging to VMO(Ω) with

$$\widetilde{\varphi}(x) = \varphi(P(x)) \qquad \text{near } \partial\Omega.$$

Here P is the projection to the nearest point on $\partial\Omega$. We then say that a function $f \in \mathrm{VMO}(\Omega)$ has φ as trace on $\partial\Omega$, written as

$$f \in \mathrm{VMO}_\varphi(\Omega)$$

provided the function

$$g = \begin{cases} f - \widetilde{\varphi} & \text{in } \Omega \\ 0 & \text{outside } \Omega, \end{cases}$$

belongs to VMO on a neighbourhood of $\overline{\Omega}$.

Theorem 2 asserts that for f in VMO(Ω),

$$f \in \mathrm{VMO}_\varphi(\Omega) \Longleftrightarrow \lim_{\substack{x \to \partial\Omega \\ \varepsilon = \frac{1}{2}\mathrm{dist}(x,\partial\Omega)}} \fint_{B_\varepsilon(x)} |f - \widetilde{\varphi}| = 0. \tag{0.5}$$

Various examples of $\mathrm{VMO}_\varphi(\Omega)$ are presented in §II.3. Example 2 states that $W^{1,n}(\Omega) \subset \mathrm{VMO}_\varphi(\Omega)$. Lemma 7 asserts that for x near $\partial\Omega$, if $d(x) = \mathrm{dist}(x, \partial\Omega)$, the function

$$f(x) = \overline{\varphi}_{d(x)}(P(x))$$

—then extended inside in the rest of Ω by smooth cutoff—belongs to $\mathrm{VMO}_\varphi(\Omega)$. Lemma 8 says that the harmonic extension of φ inside Ω belongs to $\mathrm{VMO}_\varphi(\Omega)$; this is proved in Appendix 3.

Recently, L. Greco, T. Iwaniec, C. Sbordone and B. Stroffolini [1] introduced a notion of degree for a class of Sobolev maps which is weaker than $W^{1,n}$ and is not contained in VMO.

Finally, in II.3, a question of H. Amann is answered. In [BNI], if X, Y are compact oriented n-manifolds without boundaries, and $\varphi, \psi \in \mathrm{VMO}(X, Y)$ are connected by some homotopy H which is continuous in a parameter t on [0,1], with values in $\mathrm{VMO}(X, Y)$, then (Corollary 6 in [BNI]) $\deg\varphi = \deg\psi$. Amann asked whether the conclusion still holds in case

$$H \in \mathrm{VMO}(X \times [0,1], Y).$$

Under suitable conditions on H for t near 0 and 1, Corollary 3 asserts that the answer is yes.

Section II.4 extends to $\mathrm{VMO}_\varphi(\Omega)$ a standard result for continuous maps $u : \overline{\Omega} \to \mathbb{R}^n$, with $u\big|_{\partial\Omega} = \varphi$. Namely, if $\varphi \neq p \quad \forall x \in \partial\Omega$, then

$$\deg(u, \Omega, p) = \deg(\frac{\varphi - p}{|\varphi - p|}, \partial\Omega, S^{n-1}). \tag{0.6}$$

Appendix 1 proves a number of results of II.1.

In Appendix 2, written with P. Mironescu, we consider Toeplitz operators on S^1. For any continuous complex-valued function φ on S^1, with $\varphi \neq 0$ everywhere, there is, classically, an associated Toeplitz operator T_φ. It is a Fredholm operator in $\mathcal{H}^2$ and

$$\mathrm{index}(T_\varphi) = -\deg\left(\frac{\varphi}{|\varphi|}, S^1, S^1\right).$$

In Theorem A2.1 a similar result is proved for φ satisfying

$$\varphi \in \mathrm{VMO}(S^1, \mathbb{C}) \cap L^\infty, |\varphi| \geq a > 0 \text{ on } S^1.$$

This result is essentially contained in Theorem 7.36 in R. G. Douglas [1]; the proof here is different and is pretty much self contained—though we use the fundamental $\mathcal{H}^1$-BMO duality of C. Fefferman [1] (see also C. Fefferman and E. Stein [1]).

Appendix 3 deals with properties of the harmonic extension of BMO and VMO maps.

The plan of the paper is:

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II.1. BMO and VMO on domains

Let Ω be a bounded domain (open connected set) in $\mathbb{R}^n$. Later we will consider domains in a manifold.

There are several natural notions of $\mathrm{BMO}(\Omega)$.

Definition 1. A locally integrable real function f on Ω belongs to $\mathrm{BMO}(\Omega)$ if

$$\|f\|_{\mathrm{BMO}} := \sup_{B\in\mathcal{C}} \frac{1}{|B|}\int_B |f-\bar{f}_B| < \infty, \tag{1.1}$$

where $\mathcal{C}$ is the class of all open balls B whose closures lie in Ω, and

$$\bar{f}_B = \fint_B f,$$

the average of f over B.

$\mathrm{BMO}(\Omega)$ so defined forms a Banach space modulo constants. Similarly a map $u:\Omega\to\mathbb{R}^N$ belongs to $\mathrm{BMO}(\Omega,\mathbb{R}^N)$ if each component of u is in $\mathrm{BMO}(\Omega)$. Its BMO norm is also given by (1.1) where $|\ \ |$ denotes the Euclidean norm in $\mathbb{R}^N$. As in [BNI] an equivalent norm is

$$\|u\|_\star = \sup_{B\in\mathcal{C}} \fint_B \fint_B |u(y)-u(z)|d\sigma(y)d\sigma(z); \tag{1.2}$$

in fact

$$\|u\|_{\mathrm{BMO}} \le \|u\|_\star \le 2\|u\|_{\mathrm{BMO}}. \tag{1.3}$$

Definition 2. For $0<k<1$ let $\mathcal{C}_k$ denote all balls $B_r(x)\subset\Omega$ satisfying

$$r \le k\ \mathrm{dist}(x,\partial\Omega).$$

Such balls are called "well inside" Ω. Using $\mathcal{C}_k$ instead of $\mathcal{C}$ in (1) we obtain a different smaller norm

$$\|f\|_{\mathrm{BMO},k}.$$

It is not difficult to see that for $0<k_1,k_2<1$, the norms

$$\|f\|_{\mathrm{BMO},k_1} \quad \text{and } \|f\|_{\mathrm{BMO},k_2} \text{ are equivalent.}$$

(see Lemma A1.1 in Appendix 1). A more striking fact is that each of these is equivalent to the norm (1.1), even if no regularity of $\partial\Omega$ is required. As we show in Theorem A1.1, this equivalence holds not just for the Euclidean norm but for any norm on $\mathbb{R}^n$. This fact is far from trivial and is due to Peter Jones. We present a slight modification of his proof; see Theorem A1.1.

It is more convenient to work with Definition 2. *From now on we take that as our definition of BMO, with k fixed as* $1/2$, *and we simply write*

$$\|f\|_{\mathrm{BMO},1/2} \textit{ as } \|f\|_{\mathrm{BMO}} \textit{ and } \mathcal{C}_{1/2}=\mathcal{C}.$$

We use formula (1.2) as well with balls B well inside (with $k = 1/2$).

Remark 1. In Definition 2, if we restrict the class $\mathcal{C}_k$ to all balls $B_r(x)$ satisfying

$$r \leq \min\{k \operatorname{dist}(x, \partial\Omega), r_0\}$$

for some given $r_0 > 0$, we get a smaller norm which is, however, equivalent to the original one. This is easily seen by a trivial covering argument.

Remark 2. Another possible definition of $\mathrm{BMO}(\Omega)$ is to take as $\mathcal{C}$ the class of all cubes with closures in Ω, or all those with edges parallel to the axes, or with cubes "well inside" Ω. The corresponding norms are all equivalent to the BMO norm above (see the discussion after Theorem A1.1 in Appendix 1).

Clearly $L^\infty(\Omega) \subset \mathrm{BMO}(\Omega)$ with continuous injection:

$$\|f\|_{\mathrm{BMO}} \leq 2\|f\|_{L^\infty}.$$

In particular $C^0(\overline{\Omega}) \subset \mathrm{BMO}(\Omega)$.

We now define $\mathrm{VMO}(\Omega)$. It was first introduced by D. Sarason [1] in all of $\mathbb{R}^n$.

Definition. $\mathrm{VMO}(\Omega)$ is the closure in $\mathrm{BMO}(\Omega)$ of $C^0(\overline{\Omega})$, i.e., $f \in \mathrm{VMO}(\Omega)$ if there is a sequence (f_j) in $C^0(\overline{\Omega})$ converging to f in $\mathrm{BMO}(\Omega)$.

In view of Lemma 1 below, if $f \in \mathrm{VMO}(\Omega)$ then there is a sequence (f_j) in $C^0(\overline{\Omega})$ converging to f in $\mathrm{BMO}(\Omega)$, in $L^1_{\mathrm{loc}}(\Omega)$, and a.e.

Lemma 1. *Given a compact set K in Ω, there is a constant C_K such that*

$$\|f - \overline{f}_K\|_{L^1(K)} \leq C_K \|f\|_{\mathrm{BMO}}$$

for every $f \in BMO(\Omega)$.

The proof of Lemma 1 is similar to that of Lemma A.1 in [BNI].

To prove the assertion before the lemma, observe first that given any $\varepsilon > 0$ and any compact set $K \subset \Omega$, there is a $g \in C^0(\overline{\Omega})$ such that

$$\|f - g\|_{\mathrm{BMO}} < \varepsilon, \quad \|f - g\|_{L^1(K)} < \varepsilon.$$

This uses Lemma 1. The assertion then follows by choosing $\varepsilon = \dfrac{1}{j}, j = 1, 2, \ldots$, and

$$K = \{x \in \Omega; \operatorname{dist}(x, \partial\Omega) \geq \frac{1}{j}\}.$$

.

It is clear that if $f \in \mathrm{VMO}(\Omega)$ then

$$\lim_{\varepsilon \to 0} \fint_{B_\varepsilon(x)} |f - \overline{f}_\varepsilon(x)| = 0 \qquad \text{“uniformly in } x\text{”.} \tag{1.4}$$

where

$$\overline{f}_\varepsilon(x) = \fint_{B_\varepsilon(x)} f.$$

More precisely, (1.4) means that for every $\delta > 0$, there exists ε_0 such that, for all $x \in \Omega$,

$$\fint_{B_\varepsilon(x)} |f - \bar{f}_\varepsilon(x)| < \delta$$

for all $\varepsilon \leq \min\{\varepsilon_0, \frac{1}{2}\,\mathrm{dist}(x, \partial\Omega)\}$.

The converse is true; this is far from obvious. In fact, a much stronger result holds. It is due to Peter Jones (private communication):

Theorem 1 (P. Jones). *The following are all equivalent for f in $BMO(\Omega)$:*

$$f \in VMO(\Omega). \tag{1.5}$$

$$\lim_{\substack{\varepsilon \to 0 \\ \varepsilon \leq \frac{1}{2}\mathrm{dist}(x,\partial\Omega)}} \fint_{B_\varepsilon(x)} |f - \bar{f}_\varepsilon(x)| = 0 \quad \textit{uniformly in } x \tag{1.6}$$

in the sense above.

(1.7) *There exists a sequence (f_j) in $C_0^\infty(\Omega)$ converging to f in $BMO(\Omega) \cap L^1_{\mathrm{loc}}(\Omega)$.*

The proof of Theorem 1 is in Appendix 1.

Example 1. $W^{1,n}(\Omega) \subset \mathrm{VMO}(\Omega)$.

To see this, observe first that $W^{1,n}(\Omega) \subset \mathrm{BMO}(\Omega)$, with continuous injection. This follows from Poincaré's inequality in any ball $\overline{B} \subset \Omega$,

$$\fint_B |f - \overline{f}_B| \leq C(n) \left(\int_B |\nabla f|^n \right)^{1/n}. \tag{1.8}$$

This implies that (1.6) holds and thus, by Theorem 1, f is in $\mathrm{VMO}(\Omega)$.

Remark 3. Theorem 1 asserts that $C_0^\infty(\Omega)$ is dense in $\mathrm{VMO}(\Omega)$. Recall that it is *not* dense in $W^{1,n}(\Omega)$.

More generally, we have as in [BNI]:

Example 2. $W^{s,p}(\Omega) \subset \mathrm{VMO}(\Omega)$ in the limiting case of the Sobolev embedding: $sp = n, 0 < s < n$, (s may or may not be an integer).

In [BNI] we discussed functions involving $\log|x|$:
(a) $\log|x|$ is in BMO(Ω) but not in VMO(Ω) if $0 \in \Omega$,
(b) $\log|\log|x||$ is in VMO(Ω).
(c) $|\log|x||^\alpha, 0 < \alpha < 1$, is in VMO($\Omega$).

Consider now a domain Ω, having compact closure in a smooth manifold X without boundary. In order to define BMO(Ω) and VMO(Ω), one first puts a smooth Riemannian metric on X, the notions above of BMO(Ω) and VMO(Ω) extend except that we use geodesic balls $B_\varepsilon(x)$ and always assume that $\varepsilon < r_0$, the injectivity radius of $\overline{\Omega}$. The definitions are independent of the choice of metric. In fact, there is a more general result:

Lemma 2. *Let Ω_1, Ω_2 be two bounded domains in $\mathbb{R}^n$ and let H be a C^1 diffeomorphism of a neighbourhood of $\overline{\Omega}_1$ onto a neighbourhood of $\overline{\Omega}_2$. If $f \in BMO(\Omega_2)$ (respectively $VMO(\Omega_2)$) then $f \circ H$ is in $BMO(\Omega_1)$ (respectively $VMO(\Omega_1)$) and*

$$\|f \circ H\|_{\mathrm{BMO}} \le C\|f\|_{\mathrm{BMO}}.$$

This is proved in Appendix 1. Furthermore, Theorem 1 holds in this situation, with no change.

Example 3. Let Ω be such a domain on a manifold X.

Lemma 3. *The function*

$$\varphi(x) = \log(1/\mathrm{dist}(x, \partial\Omega))$$

is in $BMO(\Omega)$. Here dist *could be measured using any metric on Ω which is equivalent to the Riemannian metric.*

Lemma 4. *With φ as in Lemma 3, $|\varphi|^\alpha \in VMO(\Omega)$ for* $0 < \alpha < 1$.

Lemmas 3 and 4 are proved in Appendix 1.

II.2. Degree of maps on domains

Let Ω be a general bounded domain in $\mathbb{R}^n$, let $u \in \mathrm{VMO}(\Omega, \mathbb{R}^n)$ and let p be a point in $\mathbb{R}^n$. Our goal is to define $\deg(u, \Omega, p)$ and prove that it has the standard properties of a degree.

In the usual case, when $u \in C^0(\overline{\Omega})$, one assumes that

$$p \notin u(\partial\Omega). \tag{2.1}$$

General functions in VMO(Ω) have no trace on the boundary. (Later we shall introduce a subclass of VMO functions with a trace—the notion of trace is delicate and the subclass is somewhat restricted.) Thus the condition (2.1) has to be given a different form.

Notation. We denote by $\mathcal{D}$ the class of balls $B_\varepsilon(x)$ in Ω with

$$\varepsilon = \frac{1}{2}\operatorname{dist}(x, \partial\Omega).$$

In place of (2.1) we use the condition:

$$\text{(2.2)} \qquad \begin{cases} \text{there exist } d_0 > 0, \text{ and a neighbourhood } U \text{ in } \Omega \text{ of } \partial\Omega, \text{ such that} \\ \qquad \displaystyle\fint_B |u - p| \geq d_0 \quad \forall B \subset U, B \in \mathcal{D}. \end{cases}$$

In particular, (2.2) holds if $|u - p| \geq d_0$ a.e. in some neighbourhood U of $\partial\Omega$. Clearly for $u \in C^0(\overline{\Omega})$, (2.1) and (2.2) are equivalent.

Notation. For $\varepsilon > 0$, set

$$\Omega_\varepsilon = \{x \in \Omega;\ \operatorname{dist}(x, \partial\Omega) > \varepsilon\}.$$

Definition of degree for $u \in$ VMO satisfying (2.2):

Given $u \in \mathrm{VMO}(\Omega)$, we choose $\varepsilon_0 > 0$ so that for all $x \in \Omega$,

$$\text{(2.3)} \qquad \fint_{B_\varepsilon(x)} |u - \overline{u}_\varepsilon(x)| \leq d_0/2$$

for all $\varepsilon \leq \varepsilon_0$ and $\varepsilon \leq \frac{1}{2}\operatorname{dist}(x, \partial\Omega)$. This is possible in view of (1.4). We may also take ε_0 to satisfy

$$\{x \in \Omega;\ \operatorname{dist}(x, \partial\Omega) \leq 3\varepsilon_0\} \subset U,$$

with U as in (2.2).

Combining (2.2) and (2.3) we have

$$\text{(2.4)} \qquad |\overline{u}_\varepsilon(x) - p| \geq d_0/2 \quad \text{if } x \in \partial\Omega_{2\varepsilon} \text{ and } \varepsilon \leq \varepsilon_0.$$

Hence

$$\deg(\overline{u}_\varepsilon, \Omega_{2\varepsilon}, p) \quad \text{is defined for every } \varepsilon \leq \varepsilon_0.$$

Claim: This degree is independent of ε for $0 < \varepsilon \leq \varepsilon_0$.

We then define

$$\deg(u, \Omega, p) = \deg(\overline{u}_\varepsilon, \Omega_{2\varepsilon}, p) \quad \text{for } \varepsilon \leq \varepsilon_0.$$

Proof of Claim: We may suppose $p = 0$.

We shall prove that for any given ε in $(0, \varepsilon_0]$, there exists δ depending on ε such that

$$\deg(\overline{u}_t, \Omega_{2t}, 0) = \deg(\overline{u}_\varepsilon, \Omega_{2\varepsilon}, 0) \quad \text{for } |t-\varepsilon| < \delta. \tag{2.5}$$

This yields the claim.

The map $\overline{u}_t$ is continuous in x and t where it is defined. Using (2.4) we see that there exists $\delta > 0$ such that

$$|\overline{u}_t(x)| \geq \frac{d_0}{4} \quad \text{if } |t-\varepsilon| < \delta \text{ and } \operatorname{dist}(x, \partial\Omega_{2\varepsilon}) < \delta. \tag{2.6}$$

Therefore

$$\deg(\overline{u}_t, \Omega_{2\varepsilon}, 0) \quad \text{is defined for } |t-\varepsilon| < \delta.$$

By homotopy invariance and (2.6), this degree is independent of t, and so

$$\deg(\overline{u}_t, \Omega_{2\varepsilon}, 0) = \deg(\overline{u}_\varepsilon, \Omega_{2\varepsilon}, 0) \quad \text{for } |t-\varepsilon| < \delta.$$

Finally, by excision, and (2.6) again,

$$\deg(\overline{u}_\varepsilon, \Omega_{2\varepsilon}, 0) = \deg(\overline{u}_t, \Omega_{2t}, 0),$$

and the claim is proved.

Consequently, $\deg(u, \Omega, p)$ is defined. It is clear that if $u \in C^0(\overline{\Omega})$ then the degree just defined agrees with the usual degree.

We verify now some of the standard properties of degree:

Property 1. If $u \in \mathrm{VMO}(\Omega, \mathbb{R}^n)$ satisfies (2.2) and

$$\deg(u, \Omega, p) \neq 0,$$

then

$$p \in \operatorname{ess} R(u).$$

(The essential range of a map u, $\operatorname{ess} R(u)$, is defined in §I.4 of [BNI]). In fact

$$B_{d_0}(p) \subset \operatorname{ess} R(u).$$

The proof follows that of Property 1 in §I.4 of [BNI].

Property 2. (Stability of degree in the BMO topology).

Let (u_j) and u belong to $\mathrm{VMO}(\Omega)$ and satisfy

$$u_j \longrightarrow u \quad \text{in } \mathrm{BMO}(\Omega) \cap L^1_{\mathrm{loc}}(\Omega) \tag{2.7}$$

and

$$
(2.8) \qquad \begin{cases} \text{for some } p \in \mathbb{R}^n \text{, there exist a } d_0 > 0 \text{ and a neighbourhood } U \text{ of } \partial\Omega \text{ in } \Omega, \\ \text{such that } \displaystyle⨍_B |u_j - p| \geq d_0 \quad \forall j, \quad \forall B \subset U, B \in \mathcal{D}, \end{cases}
$$

(in view of (2.7), the same holds for u).

Then

$$
\deg(u_j, \Omega, p) = \deg(u, \Omega, p)
$$

for all j sufficiently large.

Proof. We may take $p = 0$. As in Lemma 4 of I.1 of [BNI] we see that

$$
(2.9) \qquad \lim_{\substack{|B| \to 0 \\ B \in \mathcal{C}}} ⨍_B |u_j - \overline{u}_{j,B}| = 0 \qquad \text{uniformly in } j.
$$

(Recall that $B \in \mathcal{C}$ means that if $B = B_r(x)$, then $r \leq \frac{1}{2} \operatorname{dist}(x, \partial\Omega)$.) It is here that we use the assumption that $u_j \to u$ in $\mathrm{BMO}(\Omega)$. (2.8) and (2.9) imply that there exists ε_0 such that for all $\varepsilon \in (0, \varepsilon_0)$,

$$
\left| ⨍_{B_\varepsilon(x)} u_j \right| \geq \frac{d_0}{2} \qquad \forall j, \quad \forall x \in \partial\Omega_{2\varepsilon}.
$$

Fix some $\varepsilon \in (0, \varepsilon_0)$. Since $u_j \to u$ in $L^1_{\mathrm{loc}}(\Omega)$, we have

$$
\overline{u}_{j,\varepsilon} \longrightarrow \overline{u}_\varepsilon \qquad \text{uniformly in } \overline{\Omega}_{2\varepsilon}.
$$

Thus

$$
\deg(\overline{u}_{j,\varepsilon}, \Omega_{2\varepsilon}, 0) = \deg(\overline{u}_\varepsilon, \Omega_{2\varepsilon}, 0)
$$

for j sufficiently large. By our definition of degree we obtain the desired result.

Remark 4. In the argument above it is essential that (2.8) holds *uniformly* in j. Here is an illuminating example in which uniformity in (2.8) is dropped and the conclusion fails. Let $\Omega = (0, 1)$, and set

$$
u_j(x) = f_j(x) - \frac{1}{2}
$$

where f_j is the sequence defined in Example 6 of §I.2 in [BNI]. Since $u_j(0) = \frac{1}{2}$ and $u_j(1) = -\frac{1}{2}$, $\deg(u_j, \Omega, 0) = -1$.

On the other hand, $u_j \to u \equiv -\frac{1}{2}$ in BMO and in L^1, and $\deg(u, \Omega, 0) = 0$.

An immediate corollary of the above is the invariance under suitable homotopy:

Corollary 1. *Let $H_t(\cdot)$ be a one-parameter family of VMO maps from Ω to $\mathbb{R}^n$, depending continuously—in the $BMO \cap L^1_{\text{loc}}$ topology—on the parameter t. Assume in addition, that (2.8) holds uniformly in t, i.e., the same d_0 and U for all H_t. Then*

$$\deg(H_t, \Omega, p) \quad \textit{is independent of } t.$$

Corollary 2. *Suppose u, v are VMO maps from Ω into $\mathbb{R}^n$ both satisfying (2.2). Suppose that for some $d_1 < d_0$,*

$$⨍_B |u - v| \le d_1, \quad \forall B \subset U, \ B \in \mathcal{D}.$$

Then

$$\deg(v, \Omega, p) = \deg(u, \Omega, p).$$

To prove this, just use the homotopy $H_t = tv + (1-t)u, \quad 0 \le t \le 1$, and apply the preceding corollary.

Property 3 (Borsuk). Suppose $u \in \mathrm{VMO}(\Omega, \mathbb{R}^n)$ and (2.2) holds with $p = 0$. If $0 \in \Omega$, Ω is symmetric about the origin and u is odd near $\partial\Omega$, then

$$\deg(u, \Omega, 0) \quad \text{is odd.}$$

This is an immediate consequence of our definition of degree—via Borsuk's theorem for continuous maps.

Remark 5. The definition of degree extends in a straightforward way to VMO maps from a domain Ω, with compact closure, in a smooth oriented Riemannian manifold X, with values in another oriented smooth manifold Y, $\dim Y = \dim X$. Namely for $u \in \mathrm{VMO}(\Omega, Y)$, and for $p \in Y$ such that (2.2) holds, where $|u(z) - p|$ is replaced by $\mathrm{dist}(u(z), p)$, one defines

$$\deg(u, \Omega, p)$$

as in the Euclidean case.

II.3. VMO functions having a VMO trace; VMO$_\varphi$

In general, VMO functions on a domain Ω do not have a well defined trace on $\partial\Omega$—even if $\partial\Omega$ is smooth. An example for $\Omega = (0, 1)$ is the function $\cos(\log|\log x|)$. It is in VMO—even in $H^{1/2}$—but has no trace at 0.

It is useful to introduce a subclass which *does* admit a trace on $\partial\Omega$ belonging to $\mathrm{VMO}(\partial\Omega)$. As usual, Ω is a bounded domain in $\mathbb{R}^n$.

Definition of $\mathbf{VMO_0}$. A function $f \in \mathrm{VMO}(\Omega)$ belongs to $\mathrm{VMO}_0(\Omega)$ if its extension g outside Ω as identically zero, belongs to $\mathrm{VMO}(B)$, where B is an open ball containing $\overline{\Omega}$.

Remark 6. A function $f \in \mathrm{VMO}(\Omega)$ which is identically zero near $\partial\Omega$ belongs to $\mathrm{VMO}_0(\Omega)$. Indeed its extension g, by zero outside Ω lies in $\mathrm{BMO}(B)$, as is clear by Remark 1. That it lies in $\mathrm{VMO}(B)$ is a consequence of Theorem 1.

A simple characterization in case $\partial\Omega$ is *smooth* is given by

Theorem 2. *$f \in VMO(\Omega)$ belongs to $VMO_0(\Omega)$ iff*

$$\lim_{\substack{|B|\to 0\\ B\in\mathcal{D}}} ⨍_B |f| = 0. \tag{3.1}$$

Condition (3.1) means that the average of $|f|$ over balls $B_\varepsilon(x)$ tends to zero as $x \to \partial\Omega$ provided $\varepsilon = \dfrac{1}{2}\,\mathrm{dist}(x, \partial\Omega)$.

Proof.

1. Proof that $f \in \mathrm{VMO}_0(\Omega) \Rightarrow$ (3.1) if $\partial\Omega$ is smooth: To see this, consider a ball $B_\varepsilon(x) \in \mathcal{D}$, i.e., $\varepsilon = \dfrac{1}{2}\,\mathrm{dist}(x, \partial\Omega)$. Let z be a closest point on $\partial\Omega$ to x. Since $\partial\Omega$ is smooth, there is some $\alpha > 0$, and some $\varepsilon_0 > 0$ such that

$$|B_{3\varepsilon}(y) \cap \Omega^c| \geq \alpha |B_{3\varepsilon}(y)| \quad \forall y \in \partial\Omega, \quad \forall \varepsilon \leq \varepsilon_0. \tag{3.2}$$

Since $g \in \mathrm{VMO}(B)$, given $\delta > 0$, there exists $\varepsilon_1 > 0$ such that for $\varepsilon < \varepsilon_1$,

$$⨍_{B_{3\varepsilon}(z)} |g - \bar{g}_{3\varepsilon}(z)| < \delta. \tag{3.3}$$

It follows that

$$|\bar{g}_{3\varepsilon}(z)| \frac{|B_{3\varepsilon}(z) \cap \Omega^c|}{|B_{3\varepsilon}(z)|} < \delta,$$

so that

$$|\bar{g}_{3\varepsilon}(z)| \leq \frac{\delta}{\alpha} \quad \forall \varepsilon < \varepsilon_1. \tag{3.4}$$

By Lemma A.4 in [BNI]

$$|\bar{g}_\varepsilon(x) - \bar{g}_{3\varepsilon}(z)| \leq 3^n ⨍_{B_{3\varepsilon}(z)} |g - \bar{g}_{3\varepsilon}(z)| \leq 3^n \delta.$$

Using (3.4) we find that

$$|\bar{g}_\varepsilon(x)| \le (3^n + \frac{1}{\alpha})\delta. \tag{3.5}$$

Since g is in VMO(B), there is $\varepsilon_2 < \varepsilon_1$ such that

$$\fint_{B_\varepsilon(x)} |g - \bar{g}_\varepsilon(x)| \le \delta \quad \text{for } \varepsilon < \varepsilon_2.$$

Combining this with (3.5) we obtain the desired result.

It is clear from the proof that what is required of Ω is simply (3.2) rather than regularity. Thus $\partial\Omega$ might merely be a Lipschitz boundary. However, some regularity of $\partial\Omega$ is necessary. For example if $\Omega =$ unit disc in $\mathbb{R}^2$ minus the origin, and f is smooth in Ω with $f = 1$ in $0 < |x| < \frac{1}{2}$ and $f = 0$ for $|x| > 3/4$, then $f \in \mathrm{VMO}_0(\Omega)$ but does not satisfy (3.1).

One also observes from the proof above that $f \in \mathrm{VMO}_0(\Omega)$ implies

$$\lim_{\varepsilon \to 0} \fint_{B_\varepsilon(x)} |f| = 0$$

where $\varepsilon = \mathrm{dist}(x, \partial\Omega)$.

2. Proof that (3.1) $\Rightarrow f \in \mathrm{VMO}_0(\Omega)$. This is true for any bounded domain Ω.

We have to show that given any $\delta > 0$ there is some $\varepsilon_0 > 0$ such that, for $\varepsilon < \varepsilon_0$,

$$\fint_{B_\varepsilon(x)} \fint_{B_\varepsilon(x)} |g(y) - g(z)| < 2\delta$$

where $B_\varepsilon(x)$ is any ball in $\mathbb{R}^n$. If $B_\varepsilon(x)$ is in Ω^c or if $B_\varepsilon(x)$ is "well inside" Ω, this is clear. Thus we may assume that

$$B_\varepsilon(x) \cap \Omega \ne \emptyset \quad \text{and } B_{2\varepsilon}(x) \cap \Omega^c \ne \emptyset,$$

and in particular $\mathrm{dist}(x, \partial\Omega) \le 2\varepsilon$.

Set $A = B_\varepsilon(x) \cap \Omega$. It suffices to prove that, for $\varepsilon <$ some ε_0,

$$\frac{1}{|B_\varepsilon(x)|} \int_A |f| < \delta.$$

Consider a maximal family of disjoint open balls $B_{\varepsilon_{i/3}}(x_i)$ with centres $x_i \in A$ and $\varepsilon_i = \dfrac{1}{2}\,\mathrm{dist}(x_i, \partial\Omega)$. Since $x_i \in B_\varepsilon(x)$ we have

$$\varepsilon_i \le \frac{1}{2}(\mathrm{dist}(x_i, x) + \mathrm{dist}(x, \partial\Omega)) \le \frac{3}{2}\,\varepsilon \quad \forall i.$$

Claim: $G = \bigcup_i B_{\varepsilon_i}(x_i)$ covers A.

Suppose not. Suppose some $y \in A$, $y \notin G$. Set $\gamma = \dfrac{1}{2}\mathrm{dist}(y, \partial\Omega)$; by maximality there exists some i such that

$$B_{\varepsilon_i/3}(x_i) \quad \text{intersects } B_{\gamma/3}(y).$$

Then

$$\varepsilon_i \le \mathrm{dist}(y, x_i) \le \frac{1}{3}(\gamma + \varepsilon_i),$$

so that

$$2\varepsilon_i \le \gamma.$$

But

$$\begin{aligned} 2\gamma = \mathrm{dist}(y, \partial\Omega) &\le \mathrm{dist}(y, x_i) + \mathrm{dist}(x_i, \partial\Omega) \\ &\le \frac{1}{3}(\gamma + \varepsilon_i) + 2\varepsilon_i, \end{aligned}$$

i.e., $5\gamma \le 7\varepsilon_i \le \dfrac{7}{2}\gamma$. Impossible.

This proves the claim; we return to the proof of the theorem. We have

$$\int_A |f| \le \sum_i \int_{B_{\varepsilon_i}(x_i)} |f| = 3^n \sum_i |B_{\varepsilon_i/3}(x_i)| \fint_{B_{\varepsilon_i}(x_i)} |f|. \tag{3.6}$$

By (3.1) we may find $r_0 > 0$ such that, for every ball $B_r(a)$ with $r = \dfrac{1}{2}\,\mathrm{dist}(a, \partial\Omega) < r_0$,

$$\fint_{B_r(a)} |f| < \delta/6^n.$$

We take $\varepsilon_0 = \dfrac{2}{3}r_0$ and thus, for $\varepsilon < \varepsilon_0$, we have

$$\varepsilon_i \le \frac{3}{2}\varepsilon \le r_0 \qquad \forall i$$

and hence

$$\fint_{B_{\varepsilon_i}(x_i)} |f| < \delta/6^n \qquad \forall i.$$

Consequently, by (3.6),

$$\int_A |f| \le \frac{\delta}{2^n} \sum_i |B_{\varepsilon_i/3}(x_i)|.$$

The balls $B_{\varepsilon_i/3}(x_i)$ are disjoint and they are all contained in $B_{2\varepsilon}(x)$; it follows that

$$\sum_i |B_{\varepsilon_i/3}(x_i) \leq |B_{2\varepsilon}(x)| = 2^n |B_\varepsilon(x)|.$$

We conclude that

$$\frac{1}{|B_\varepsilon(x)|} \int_A |f| < \delta.$$

□

Remark 7. One may think that $\mathrm{VMO}_0(\Omega)$ is a closed subspace of $\mathrm{VMO}(\Omega)$ but this is not true. In fact, it is dense in $\mathrm{VMO}(\Omega)$; see Remark 3.

Remark 8. The space $W_0^{1,n}(\Omega)$ is contained in $\mathrm{VMO}_0(\Omega)$. This is clear from the definition of VMO_0, for the extension of $u \in W_0^{1,n}(\Omega)$ as zero outside Ω is in $W^{1,n}(B) \subset \mathrm{VMO}(B)$—see Example 1.

Next we are going to define a class $\mathrm{VMO}_\varphi(\Omega)$ where φ is a given function in $\mathrm{VMO}(\partial\Omega)$, assuming $\partial\Omega$ is smooth. $\mathrm{VMO}_\varphi(\Omega)$ will consist of functions having "trace" φ on $\partial\Omega$. First we need

Lemma 5. *Let Ω be a smooth bounded domain and let $\varphi \in VMO(\partial\Omega)$. There exists a function $\widetilde{\varphi}$ defined on a neighbourhood $\widetilde{\Omega}$ of $\overline{\Omega}$ such that $\widetilde{\varphi} \in VMO(\widetilde{\Omega})$, and for x close to $\partial\Omega$,*

$$\widetilde{\varphi}(x) = \varphi(P(x)) \tag{3.7}$$

where P is the projection to the closest point in $\partial\Omega$.

Proof. We first define $\widetilde{\varphi}$ by (3.7) in a tubular neighbourhood U of $\partial\Omega$,

$$U = \{x \in \mathbb{R}^n; \mathrm{dist}(x, \partial\Omega) < \delta\}.$$

Claim: $\widetilde{\varphi} \in \mathrm{VMO}(U)$.

In view of Lemma A.10 in [BNI] it suffices to prove the claim when the boundary is on $\{x_n = 0\}$, for $\widetilde{U} = \{x \in \mathbb{R}^n; |x_n| < \delta\}$. If Q is a cube with edges parallel to the axes, then it is clear that

$$⨍_Q |\widetilde{\varphi} - ⨍_Q \widetilde{\varphi}| \leq \|\varphi\|_{\mathrm{BMO}(\partial\Omega)}.$$

If B is a ball in $\widetilde{U}$, then it lies in such a cube Q, with side length $= \mathrm{diam} B$, and then the inequality

$$⨍_B |\widetilde{\varphi} - ⨍_B \widetilde{\varphi}| \leq C\|\varphi\|_{\mathrm{BMO}(\partial\Omega)}$$

follows with the aid of Lemma A.4 of [BNI]. We have proved that $\widetilde{\varphi} \in \mathrm{BMO}(U)$; that it is in $\mathrm{VMO}(U)$ is proved either by approximation or repeating the computation above, and letting $|B| \to 0$. The claim is proved.

To complete the proof of the lemma we simply multiply $\widetilde{\varphi}$ by a smooth cutoff function; here we rely on Lemma B.8 of [BNI].

$\square$

Now, the

Definition of $\mathbf{VMO}_\varphi$. Let Ω and φ be as above, and let $f \in \mathrm{VMO}(\Omega)$. We say that f has trace φ on $\partial\Omega$, i.e., $f \in \mathrm{VMO}_\varphi$, provided

$$(f - \widetilde{\varphi}) \quad \text{is in } \mathrm{VMO}_0(\Omega).$$

This definition also makes sense if $\Omega \subset X$, a Riemannian manifold.

Remark 9. Though $\widetilde{\varphi}$ is not quite unique—it depends on the choice of cutoff—the notion of VMO_φ is independent of our choice. This follows immediately with the aid of Remark 6. Furthermore, it is clear that $f \in \mathrm{VMO}_\varphi \Leftrightarrow$ the following function $\widetilde{f}$ belongs to $\mathrm{VMO}(\widetilde{\Omega})$:

$$\widetilde{f} = \begin{cases} f & \text{in } \Omega \\ \widetilde{\varphi} & \text{in } \widetilde{\Omega} \backslash \Omega. \end{cases}$$

Remark 10. It follows from Theorem 1 that for any fixed $\varphi \in \mathrm{VMO}(\partial\Omega)$, the space $\mathrm{VMO}_\varphi(\Omega)$ is dense in $\mathrm{VMO}(\Omega)$ in the BMO topology.

The notion of VMO_φ is invariant under diffeomorphisms. In particular, if Ω is a domain (with compact closure) in a smooth manifold X, the notion of VMO_φ is independent of the choice of Riemannian metric on X. We have namely

Lemma 6. *Let X_1, X_2 be smooth Riemannian manifolds without boundaries and let Ω_1, Ω_2 be subdomains, respectively, with compact closures and smooth boundaries. Let H be a C^1 diffeomorphism from $\overline{\Omega}_1$ onto $\overline{\Omega}_2$; H maps $\partial\Omega_1$ onto $\partial\Omega_2$ as a C^1 diffeomorphism. Let $\varphi \in VMO(\partial\Omega_2)$ and let $f \in VMO_\varphi(\Omega_2)$. Then*

$$f \circ H \quad \textit{belongs to } VMO_{\varphi\circ H}(\Omega_1).$$

Proof. For $i = 1, 2$, let $\widetilde{\Omega}_i$ be a neighbourhood of $\overline{\Omega}_i$ so that for every $x \in \widetilde{\Omega}_1 \backslash \Omega_1$ there is a unique closest point $P(x)$ on $\partial\Omega_1$. We define an extension $\widetilde{H}$ of H to $\widetilde{\Omega}_1$ as follows: For $x \in \widetilde{\Omega}_1 \backslash \overline{\Omega}_1$, we set

$$\widetilde{H}(x) = y \in \widetilde{\Omega}_2 \backslash \Omega_2$$

where y is the unique point there with $P(y) = H(P(x))$, and $\mathrm{dist}(y, H(P(x)) = \mathrm{dist}(x, \partial\Omega_1)$. To define y we may have to shrink $\widetilde{\Omega}_1$. Clearly $\widetilde{H}$ is a bi-Lipschitz map of $\widetilde{\Omega}$ onto a neighbourhood $\widetilde{\Omega}_2$ of $\overline{\Omega}_2$.

Turning to the function f, set, as in Remark 9,

$$\widetilde{f} = \begin{cases} f & \text{in } \Omega_2 \\ \widetilde{\varphi} & \text{in } \widetilde{\Omega}_2 \backslash \Omega_2, \end{cases}$$

so that $\widetilde{f} \in \mathrm{VMO}(\widetilde{\Omega}_2)$. Consider now $\widetilde{f} \circ \widetilde{H}$; it is defined on $\widetilde{\Omega}_1$.

Claim: $\widetilde{f} \circ \widetilde{H} \in \mathrm{VMO}(\omega)$ where ω is any open set with compact closure in $\widetilde{\Omega}_1$.

Once the claim is proved, we are through, for if $x \in \widetilde{\Omega}_2 \backslash \overline{\Omega}_2$, then $\widetilde{f} \circ \widetilde{H}(x) = (\varphi \circ H)(P(x))$.

Proof of Claim: Let $B_\varepsilon(x)$ be a ball in ω with $\varepsilon \leq \frac{1}{2}$ dist$(x, \partial\omega)$. Consider

$$\begin{aligned} I &= \fint_{B_\varepsilon(x)} \fint_{B_\varepsilon(x)} |\widetilde{f} \circ \widetilde{H}(y) - \widetilde{f} \circ \widetilde{H}(z)| \\ &\leq \frac{C}{|B_\varepsilon(x)|^2} \int_{\widetilde{H}(B_\varepsilon(x))} \int_{\widetilde{H}(B_\varepsilon(x))} |\widetilde{f}(\eta) - \widetilde{f}(\zeta)| \end{aligned}$$

since $(\widetilde{H})^{-1}$ is Lipschitz. Hence

$$I \leq \frac{C}{|B_\varepsilon(x)|^2} \int_{B_{\varepsilon K}(\widetilde{H}(x))} \int_{B_{\varepsilon K}(\widetilde{H}(x))} |\widetilde{f}(\eta) - \widetilde{f}(\zeta)|$$

since $\widetilde{H}$ is Lipschitz with Lipschitz constant K. We also require that

$$\varepsilon < \frac{1}{2K} \operatorname{dist}(\widetilde{H}(\omega), \partial\widetilde{\Omega}_2) =: r_0.$$

Clearly $I \leq C\|\widetilde{f}\|_{\mathrm{BMO}}$. By Remark 1 we see that

$$\|\widetilde{f} \circ \widetilde{H}\|_{\mathrm{BMO}(\omega)} \leq C\|\widetilde{f}\|_{\mathrm{BMO}(\widetilde{\Omega}_2)}.$$

By density we conclude that $\widetilde{f} \circ \widetilde{H}$ is in $\mathrm{VMO}(\omega)$.

$\square$

Next, we present some examples of functions in VMO_φ.

Example 1. If $f \in C(\overline{\Omega})$, and $\varphi = f_{|\partial\Omega}$, then $f \in \mathrm{VMO}_\varphi(\Omega)$.

Example 2. If $f \in W^{1,n}(\Omega)$ and $\varphi = f_{|\partial\Omega}$ then $f \in \mathrm{VMO}_\varphi(\Omega)$. Recall that $f \in \mathrm{VMO}(\Omega)$ and $\varphi = f_{|\partial\Omega} \in W^{1-\frac{1}{n},n}(\partial\Omega)$ also lies in $\mathrm{VMO}(\partial\Omega)$, by Example 2 in §I.1 in [BNI].

Proof. Since both conditions $f \in W^{1,n}$ and $f \in \mathrm{VMO}_\varphi$ are invariant under diffeomorphisms, we may locally flatten the boundary $\partial\Omega$. In addition we may suppose that the metric is locally Euclidean near the flat portion of boundary. Near the origin in the flat boundary, we may use coordinates (x', x_n), $x' \in \mathbb{R}^{n-1}$, with $x_n > 0$ in $\Omega, x_n = 0$ on $\partial\Omega$. In view of Theorem 2 it suffices to show that

$$\lim_{\substack{|B|\to 0\\ B\in\mathcal{D}}} \fint_B |f(x', x_n) - f(x', 0)| dx = 0.$$

For $B \in \mathcal{D}$, let $Q = Q' \times (\varepsilon, 3\varepsilon)$ be the smallest cube with edges parallel to the axes containing B. Then

$$\begin{aligned}\fint_Q |f(x', x_n) - f(x', 0)| &\leq \frac{2\varepsilon}{(2\varepsilon)^n} \int_{Q'\times(0,3\varepsilon)} |f_{x_n}| \\ &\leq C(\int_{Q'\times(0,3\varepsilon)} |f_{x_n}|^n)^{1/n} \to 0\end{aligned}$$

as $\varepsilon \to 0$.

Example 3. Consider, as usual, a domain Ω having compact closure in a smooth Riemannian manifold X without boundary; $\partial\Omega$ is smooth. Let φ belong to $\mathrm{VMO}(\partial\Omega)$. The following particular extension f of φ inside Ω belongs to $\mathrm{VMO}_\varphi(\Omega)$. Let $U = \{x \in \Omega; \mathrm{dist}(x, \partial\Omega) < \delta\}$ with δ so small that any point x in U has a unique closest point $P(x)$ on $\partial\Omega$. The geodesics starting on $\partial\Omega$ and orthogonal to $\partial\Omega$ cover U simply. Denote $\mathrm{dist}(x, \partial\Omega)$ by $d(x)$. For x in U, define

$$f(x) = \overline{\varphi}_{d(x)}(P(x))$$

i.e., $f(x)$ is the average of φ on a ball on $\partial\Omega$ centred at $P(x)$, having radius $d(x)$. We extend f to all of Ω by multiplying it by a smooth cutoff function with support in U and which is identically one near $\partial\Omega$, and we continue to denote by f the extension to all of Ω.

Lemma 7. *f belongs to $VMO_\varphi(\Omega)$.*

Proof. By Lemma 6, the property of belonging to VMO_φ is independent of the particular metric on X. It is convenient to replace the given Riemannian metric on $\overline{\Omega}$ by a different one. We describe the new metric just in U; it is easily extended to Ω. The new metric preserves all geodesics starting on $\partial\Omega$ and orthogonal to $\partial\Omega$, and preserves arc length on them. But it is a product metric. Namely, if $x' = (x_1, \dots, x_{n-1})$ are local coordinates near a point $\bar{y}$ on $\partial\Omega$, with $x' = 0$, $t = 0$ at $\bar{y}$, and $t > 0$ in U, the lines $x' =$ constant, $0 < t < \delta$, correspond to our special geodesics orthogonal to $\partial\Omega$. The new metric has the form

$$\widetilde{ds}^2 = dt^2 + ds'^2 \tag{3.8}$$

where ${ds'}^2 = ds^2_{|\partial\Omega}$.

The function f is continuous in Ω. Therefore, to prove the lemma we need only consider balls $B_\varepsilon(x)$ in U belonging to our family $\mathcal{C}$. We have to show that

$$\fint_{B_\varepsilon(x)} \fint_{B_\varepsilon(x)} |f(y) - f(z)| \le C\|\varphi\|_{\mathrm{BMO}}, \tag{3.9}$$

with C a fixed constant independent of the ball; by density this proves that f is in VMO(Ω). To verify that f is in VMO_φ we have to show that

$$\fint_{B_\varepsilon(x)} |f(y) - \varphi(P(y))| \quad \text{is small for } \varepsilon = \frac{1}{2}d(x) \text{ small.} \tag{3.10}$$

We may use the local coordinates (x', t) described above, and suppose that $B_\varepsilon(x)$ is the ball

$$B_\varepsilon(x) = B_\varepsilon(0, \tau) \quad \text{with } 2\varepsilon \le \tau \le \delta.$$

Denote the ball in $\partial\Omega$, i.e., on $t = 0$, centred at $P(x)$, which in our local coordinates is the origin, and having radius ε by $B' = B'_\varepsilon(0)$. Now $B_\varepsilon(0, \tau)$ lies in the cylinder

$$D = B'_\varepsilon \times (\tau - \varepsilon, \tau + \varepsilon),$$

and since $|D| \le C|B_\varepsilon(x)|$, to prove (3.9) it suffices to prove that

$$I = \fint_D \fint_D |f(y) - f(z)| \le C\|\varphi\|_{\mathrm{BMO}}.$$

Now if B' is the ball in $\mathbb{R}^{n-1}$ with centre 0 and radius ε (measured in our metric ds'), we have

$$I = \fint_{\substack{y' \in B' \\ \tau-\varepsilon<s<\tau+\varepsilon}} \fint_{\substack{z' \in B' \\ \tau-\varepsilon<t<\tau+\varepsilon}} |\overline{\varphi}_s(y') - \overline{\varphi}_t(z')|.$$

If $B'_s(y')$ is the ball (in our metric ds') about y' of radius s then

$$B'_t(z'), B'_s(y') \subset B'_{\tau+2\varepsilon}(0), \qquad \text{if } \tau - \varepsilon < s, t < \tau + \varepsilon.$$

Since

$$\frac{\tau + 2\varepsilon}{t}, \frac{\tau + 2\varepsilon}{s} \le \frac{\tau + 2\varepsilon}{\tau/2} \le C \quad \text{independent of } \tau \text{ and } \varepsilon \le \frac{1}{2}\tau,$$

we see with the aid of Lemma A.4 in [BNI] that

$$|\overline{\varphi}_s(y') - \overline{\varphi}_{\tau+2\varepsilon}(0)|,\ |\overline{\varphi}_t(z') - \overline{\varphi}_{\tau+2\varepsilon}(0)| \le C\|\varphi\|_{\mathrm{BMO}}$$

if τ is small; thus

$$|\overline{\varphi}_s(y') - \overline{\varphi}_t(z')| \leq C\|\varphi\|_{\mathrm{BMO}}.$$

Inserting this in I above we obtain (3.9).

Turning to the proof of (3.10), we consider again the cylinder D, with now, $\tau = 2\varepsilon$. It suffices to prove that

$$⨍_D |f(y) - \varphi(P(y))| \quad \text{is small},$$

i.e., that for ε small,

$$J := ⨍_{\substack{x' \in B' \\ \varepsilon < t < 3\varepsilon}} |\overline{\varphi}_t(x') - \varphi(x')| \quad \text{is small.}$$

Since φ is in VMO,

$$⨍_{B'} |\varphi - \overline{\varphi}_\varepsilon(0)| \quad \text{is small for } \varepsilon \text{ small.} \tag{3.11}$$

With the aid of Lemma A.4 in [BNI], we see, as above, that for ε small,

$$|\overline{\varphi}_t(x') - \overline{\varphi}_\varepsilon(0)| \quad \text{is small if } x' \in B' \text{ and } \varepsilon \leq t \leq 3\varepsilon. \tag{3.12}$$

Thus

$$J \leq ⨍_{\substack{x \in B' \\ \varepsilon < t < 3\varepsilon}} |\overline{\varphi}_t(x') - \overline{\varphi}_\varepsilon(0)| + ⨍_{B'} |\overline{\varphi}_\varepsilon(0) - \varphi(x')|.$$

The first term on the right is small by (3.12), and the second, by (3.11).

□

Example 4. Consider Ω, X and φ as in Example 3, $\varphi \in \mathrm{VMO}(\partial\Omega)$.

Lemma 8. *The harmonic function in Ω, which equals φ on $\partial\Omega$, belongs to $VMO_\varphi(\Omega)$.*

The proof is given in Appendix 3, see Theorem A3.1.

Example 5. Consider $u = (\log|x|) * f$ in $\mathbb{R}^n$, $n \geq 2$, where $f \in L^1(\mathbb{R}^n)$ with compact support (for simplicity). Let $\Omega \subset \mathbb{R}^n$ be a smooth bounded domain. Clearly, $u \in W^{1,p}(\Omega)$ $\forall p < n$, but it need not belong to $W^{1,n}(\Omega)$. Hence u has a trace on $\partial\Omega$, say φ.

Lemma 9. *φ belongs to VMO$(\partial\Omega)$ and u belongs to VMO$_\varphi(\Omega)$.*

Proof. First, note that $u \in \mathrm{VMO}(\Omega)$. Indeed, by density, this follows from the fact that

$$\|u\|_{\mathrm{BMO}(\Omega)} \leq C\|f\|_{L^1}.$$

Next, that φ belongs to VMO$(\partial\Omega)$ follows from the estimate

$$\|\varphi\|_{\mathrm{BMO}(\partial\Omega)} \leq C\|f\|_{L^1}.$$

This is derived in turn from the inequality

$$\|\log|x-a|\|_{\mathrm{BMO}(\partial\Omega)} \leq C \quad \forall a \in \mathbb{R}^n$$

where C depends only on Ω. To prove the last inequality we need only establish for ε small,

$$J := \fint_{B'_\varepsilon(x)} \fint_{B'_\varepsilon(x)} \big|\log|y-a| - \log|z-a|\big| d\sigma(y) d\sigma(z) \leq C \quad \forall a \in \mathbb{R}^n, \tag{3.13}$$

where C depends only on Ω. Here $x \in \partial\Omega$ and $B'_\varepsilon(x)$ is the geodesic ball on $\partial\Omega$ centred at x. We consider two cases:

(i) $|x-a| \geq 6\varepsilon$,

(ii) $|x-a| < 6\varepsilon$.

Case (i) is obvious, since for ε small, if $y, z \in B'_\varepsilon(x)$,

$$|x-y| < \varepsilon, \quad |x-z| < \varepsilon$$

and thus

$$\frac{1}{2} \leq \frac{|y-a|}{|z-a|} \leq 2.$$

In Case (ii) we have

$$J \leq 2\fint_{B'_\varepsilon(x)} \Big|\log\frac{|y-a|}{\varepsilon}\Big| d\sigma(y) \leq C(\Omega). \tag{3.14}$$

Finally, we prove that $u \in \mathrm{VMO}_\varphi(\Omega)$. By Theorem 2 it suffices to show that

$$\lim_{\varepsilon\to 0} \fint_{B_\varepsilon(a)} |u - \widetilde{\varphi}| = 0$$

where $\varepsilon = \dfrac{1}{2}\operatorname{dist}(a, \partial\Omega)$ and $\widetilde{\varphi}$ is as in (3.7). By density (as in the proofs of Theorem A3.1 and A3.2) it suffices to establish that

$$\fint_{B_\varepsilon(a)} |u - \widetilde{\varphi}| \leq C\|f\|_{L^1} \tag{3.15}$$

for ε small, where C depends only on Ω.

Inequality (3.15) follows from

$$\fint_{B_\varepsilon(a)} \big|\log|x-y| - \log|P(x)-y|\big| dx \leq C(\Omega) \tag{3.16}$$

for every $y \in \mathbb{R}^n$ and for every $\varepsilon <$ some ε_0. To prove (3.16) we consider, as before, two cases:

(i) $|y - a| \geq 6\varepsilon$,

(ii) $|y - a| < 6\varepsilon$.

Case (i) is obvious since, for $x \in B_\varepsilon(a)$,

$$\frac{1}{3} \leq \frac{|x-y|}{|P(x)-y|} \leq 3.$$

In Case (ii) one shows, in fact, that

$$\fint_{B_\varepsilon(a)} \Big|\log\frac{|x-y|}{\varepsilon}\Big| dx \leq C_n \tag{3.17}$$

and

$$J := \fint_{B_\varepsilon(a)} \Big|\log\frac{|P(x)-y|}{\varepsilon}\Big| dx \leq C(\Omega). \tag{3.18}$$

Inequality (3.17) is clear. To verify (3.18) one has, first, as in (3.14), that for ε small,

$$J \leq C \fint_{B'_{2\varepsilon}(P(a))} \Big|\log\frac{|\xi-y|}{\varepsilon}\Big| d\sigma(\xi)$$

where $B'_{2\varepsilon}(P(a))$ is the geodesic ball on $\partial\Omega$ centred at $P(a)$. Now, for $\xi \in B'_{2\varepsilon}(P(a))$,

$$|\xi - y| \leq 10\varepsilon.$$

Furthermore, for ε small, one sees that for such ξ,

$$|\xi - y| \geq \frac{1}{2}|\xi - P(y)|.$$

Hence

$$
\begin{aligned}
J &\le C + C \fint_{B'_{2\varepsilon}(P(a))} \Big| \log \frac{|\xi - P(y)|}{20\varepsilon} \Big| d\sigma(\xi) \\
&\le C + C \fint_{B'_{2\varepsilon}(P(a))} \Big| \log \frac{|\xi - P(y)|}{20\varepsilon} \Big| d\sigma(\xi) \le C(\Omega)
\end{aligned}
$$

since the last integral is bounded by a constant depending only on Ω. □

We conclude this section with an answer to a question raised by H. Amann. Let X, Y be smooth n-dimensional compact oriented manifolds without boundaries; Y is smoothly embedded in some $\mathbb{R}^N$. Consider two maps $\varphi, \psi \in \mathrm{VMO}(X, Y)$; by [BNI] the degrees are well defined. Suppose φ and ψ are connected by some homotopy $H(x,t), 0 \le t \le 1$. In Corollary 6 of [BNI] it was shown that if H is continuous in $[0,1]$ with values in $\mathrm{VMO}(X,Y)$ then $\deg \varphi = \deg \psi$. Amann's question was whether the same conclusion holds in case

$$
H \in \mathrm{VMO}(X \times (0,1), Y). \tag{3.19}
$$

The answer is yes, provided one makes a slightly stronger assumption on H for t near 0 and 1. In fact, under condition (3.19) it is not clear what is meant by saying that $H(\cdot\,,0) = \varphi, H(\cdot\,,1) = \psi$.

Corollary 3. *Assume in addition to (3.19) that*

$$
\begin{aligned}
&\fint_0^h \fint_{B_h(x)} |H(y,t) - \varphi(y)| d\sigma(y) dt \to 0 \quad \text{as } h \to 0, \text{ uniformly in } x \in X \\
&\fint_{1-h}^1 \fint_{B_h(x)} |H(y,t) - \psi(y)| d\sigma(y) dt \to 0 \quad \text{as } h \to 0, \text{ uniformly in } x \in X.
\end{aligned} \tag{3.20}
$$

Then

$$
\deg(\varphi, X, Y) = \deg(\psi, X, Y).
$$

Proof. Consider the manifold $\widetilde{X} = X \times \mathbb{R}$ with the product metric, and set $\Omega = X \times (-1,2)$ in $\widetilde{X}$,

$$
\widetilde{H}(x,t) = \begin{cases} \varphi(x) & \text{for } t \le 0 \\ H(x,t) & \text{for } 0 < t < 1 \\ \psi(x) & \text{for } t \ge 1. \end{cases} \tag{3.21}
$$

By Theorem 2, conditions (3.20) imply that $\widetilde{H} \in \mathrm{VMO}(\Omega, Y)$. (It is easy to see that (3.20) is, in fact, equivalent to the property that $\widetilde{H} \in \mathrm{VMO}(\Omega)$.) As in [BNI] we now define

$$
\widetilde{H}_\varepsilon(x,t) = P \fint_{B_\varepsilon(x,t)} \widetilde{H}
$$

where P is the projection to the closest point in Y. In view of Lemma A.4 of [BNI] we may also work with

$$G_\varepsilon(x,t) = P \fint_{Q_\varepsilon(x,t)} \widetilde{H}$$

where $Q_\varepsilon(x,t)$ is the cylinder $B_\varepsilon(x) \times (t-\varepsilon, t+\varepsilon)$, for by Lemma A.4 of [BNI],

$$\sup_{\substack{x\in X\\ t\in\mathbb{R}}} |\widetilde{H}_\varepsilon(x,t) - G_\varepsilon(x,t)| \to 0 \quad \text{as } \varepsilon \to 0.$$

Clearly for $t < -\varepsilon$, $G_\varepsilon(x,t) = \varphi_\varepsilon(x) = P\overline{\varphi}_\varepsilon(x)$, and for $t > 1+\varepsilon$, $G_\varepsilon(x,t) = \psi_\varepsilon(x) = P\overline{\psi}_\varepsilon(x)$. By standard homotopy

$$\deg(G_\varepsilon(\cdot\,,t), X, Y) \qquad \text{is independent of } t.$$

Thus, for ε small, $\deg(\varphi, X, Y) = \deg(\varphi_\varepsilon, X, Y) = \deg(\psi_\varepsilon, X, Y) = \deg(\psi, X, Y)$.

$\square$

Remark 11. In connection with (3.19), a word of warning: If $f \in C([-1,1], \mathrm{VMO}(X) \cap L^1(X))$ one might think that f is in $\mathrm{VMO}(X \times [-\frac{1}{2}, \frac{1}{2}])$. This need not be the case; here is an example. Take $X = [-1,1]$ in $\mathbb{R}$. For $t > 0$ consider

$$f(x,t) = \begin{cases} 1 & \text{if } |x| \le t \\ -1 + 2\dfrac{\log|x|}{\log t} & \text{if } t < |x| < \sqrt{t} \\ 0 & \text{if } |x| \ge \sqrt{t} \end{cases}$$

and for $t < 0$, $f(x,t) \equiv 0$. By Example 6 in §I.2 of [BNI], $f \in C([-1,1], \mathrm{VMO}(X))$. Continuity with values in L^1 is clear. But f does not belong to $\mathrm{VMO}(X \times [-\frac{1}{2}, \frac{1}{2}])$, for

$$\fint_{Q_h} \fint_{Q_h} |f(x,t) - f(\xi,\tau)| dx dt d\xi d\tau \ge \frac{1}{4},$$

where $Q_h = [-h, +h] \times [-h, +h]$.

II.4. For $u \in \mathbf{VMO}_\varphi, \deg(u, \Omega, p) =$ a boundary degree

Recall the standard result that for a *continuous* map $u : \overline{\Omega} \to \mathbb{R}^n$, with $u_{|\partial\Omega} = \varphi$, and with $\varphi \ne p$ everywhere on $\partial\Omega$ for some point $p \in \mathbb{R}^n$,

$$\deg(u, \Omega, p) = \deg\left(\frac{\varphi - p}{|\varphi - p|}, \partial\Omega, S^{n-1}\right). \tag{4.1}$$

Here we extend this result to maps $u \in \mathrm{VMO}_\varphi$ provided $|\varphi - p| \ge d_0 > 0$ a.e. on $\partial\Omega$.

Theorem 3. *Assume the above, with $\varphi \in VMO(\partial\Omega)$. Then there is a neighbourhood U of $\partial\Omega$ in Ω such that*

$$\fint_B |u-p| \geq \frac{d_0}{2} \quad \forall B \subset U, B \in \mathcal{D}$$

—so that $\deg(u,\Omega,p)$ *is defined. Furthermore, (4.1) holds.*

Proof. We may take $p=0$. Set $\widetilde{\varphi}(x)=\varphi(Px)$ where P is the nearest point projection on $\partial\Omega, \widetilde{\varphi}$ is defined in a neighbourhood U of $\partial\Omega$. Let ζ be a cutoff function with support in U, and $\zeta \equiv 1$ near $\partial\Omega$. Set $\overline{\varphi}(x)=\zeta(x)\widetilde{\varphi}(x)$, so that $\overline{\varphi} \in \mathrm{VMO}_\varphi(\Omega)$; since $u \in \mathrm{VMO}_\varphi$,

$$\lim_{\substack{|B|\to 0\\ B\in\mathcal{D}}} \fint_B |u-\overline{\varphi}| = 0.$$

But

$$\begin{aligned} \fint_B |u-\overline{\varphi}| &\geq \fint_B |\overline{\varphi}| - \fint_B |u| \\ &\geq d_0 - \fint_B |u| \end{aligned}$$

for $|B|$ small, since $|\overline{\varphi}| \geq d_0$ near $\partial\Omega$. Hence there exists a neighbourhood U of $\partial\Omega$ such that

$$\fint_B |u| \geq \frac{d_0}{2} \quad \forall B \subset U, B \in \mathcal{D}.$$

We have proved the first assertion of the theorem. To verify (4.1) we make use of

Lemma 10. *Assume $\psi \in VMO(\partial\Omega,\mathbb{R}^n)$ and $|\psi|=1$ a.e. on $\partial\Omega$. For $x \in \Omega$, let $\overline{\psi}(x)=\zeta(x)\psi(Px)$ as above. Then*

$$\deg(\overline{\psi},\Omega,0)=\deg(\psi,\partial\Omega,S^{n-1})$$

Proof. We know (see Corollary 5 in [BNI]) that there exists a sequence $\psi_j \in C^\infty(\partial\Omega,S^{n-1})$ such that $\psi_j \to \psi$ in BMO and a.e. By (4.1) for continuous maps,

$$\deg(\psi_j,\partial\Omega,S^{n-1})=\deg(\overline{\psi}_j,\Omega,0)$$

where

$$\overline{\psi}_j(x)=\zeta(x)\psi_j(Px).$$

As $j\to\infty$, $\deg(\psi_j,\partial\Omega,S^{n-1}) \to \deg(\psi,\partial\Omega,S^{n-1})$ (by Theorem 1 in [BNI]).

On the other hand we claim that $\overline{\psi}_j \to \overline{\psi}$ in both $L^1(\Omega)$ and $\mathrm{BMO}(\Omega)$. Indeed, convergence in L^1 follows from dominated convergence. Convergence in BMO uses the easily verified fact that $\psi_j(Px) \to \psi(Px)$ in $\mathrm{BMO}(U)$, and the estimate for products, namely Lemma B.8 in [BNI]. Moreover

$$|\overline{\psi}_j(x)| \equiv 1 \quad \text{in some } \textit{fixed} \text{ (uniform) neighbourhood of } \partial\Omega.$$

Hence, by the stability of degree in the BMO topology (Property 2 in II.2), we see that

$$\deg(\overline{\psi}_j, \Omega, 0) \to \deg(\overline{\psi}, \Omega, 0).$$

□

Proof of Theorem 3. Set $\psi(x) = \dfrac{\varphi(x)}{|\varphi(x)|}, x \in \partial\Omega$, so that $\psi \in \mathrm{VMO}$ (by Lemma A.7 in [BNI], on compositions of VMO maps with Lipschitz maps). Thus, by the previous lemma, with $\overline{\psi}$ as defined there,

$$\deg(\overline{\psi}, \Omega, 0) = \deg(\frac{\varphi}{|\varphi|}, \partial\Omega, S^{n-1}).$$

Next we have

$$\deg(\overline{\psi}, \Omega, 0) = \deg(\overline{\varphi}, \Omega, 0) \tag{4.2}$$

where $\overline{\varphi}(x) = \zeta(x)\varphi(Px)$. Indeed we may consider the homotopy

$$H(x,t) = t\overline{\psi}(x) + (1-t)\overline{\varphi}(x) = \zeta(x)\varphi(Px)\left[\frac{t}{|\varphi(Px)|} + (1-t)\right]$$

and note that for every x in some fixed neighbourhood of $\partial\Omega$,

$$|H(x,t)| \geq \left[t + (1-t)d_0\right] \geq \min(d_0, 1) \qquad \forall t \in [0,1].$$

Applying Property 2 once more, we obtain (4.2).

Finally, it remains to prove that

$$\deg(\overline{\varphi}, \Omega, 0) = \deg(u, \Omega, 0). \tag{4.3}$$

Recall that since $u \in \mathrm{VMO}_\varphi$ we have

$$\lim_{\substack{|B|\to 0 \\ B\in\mathcal{D}}} \fint_B |u - \overline{\varphi}| = 0.$$

Assertion (4.3) then follows from

Lemma 11. *Assume $u, v \in VMO(\Omega, \mathbb{R}^n)$ and*

$$\lim_{\substack{|B|\to 0\\ B\in\mathcal{D}}} \fint_B |u-v| = 0.$$

Assume that, for some neighbourhood U of $\partial\Omega$ in Ω,

$$\fint_B |u| \geq d_0 > 0 \qquad \forall B \subset U, B \in \mathcal{D}$$

so that $\deg(u, \Omega, 0)$ *is defined. Then there is a neighbourhood U' of $\partial\Omega$ in Ω such that*

$$\fint_B |v| \geq \frac{d_0}{2} \qquad \forall B \subset U', B \in \mathcal{D}.$$

Moreover

$$\deg(v, \Omega, 0) = \deg(u, \Omega, 0).$$

Proof. The existence of U' is clear. Recall that, by definition (see §II.2),

$$\deg(u, \Omega, 0) = \deg(\overline{u}_\varepsilon, \Omega_{2\varepsilon}, 0)$$

and

$$\deg(v, \Omega, 0) = \deg(\overline{v}_\varepsilon, \Omega_{2\varepsilon}, 0)$$

for $\varepsilon < \varepsilon_0$.

But we may fix ε so small that (see (2.4))

$$|\overline{u}_\varepsilon(x)| \geq \frac{d_0}{2}, \quad |\overline{v}_\varepsilon(x)| \geq \frac{d_0}{2} \quad \forall x \in \partial\Omega_{2\varepsilon}$$

and, similarly,

$$|\overline{u}_\varepsilon(x) - \overline{v}_\varepsilon(x)| \leq \frac{d_0}{4} \qquad \forall x \in \partial\Omega_{2\varepsilon}$$

(since $\lim\limits_{\substack{|B|\to 0\\ B\in\mathcal{D}}} \fint_B |u-v| = 0$). Hence, by linear homotopy for the continuous maps,

$$\deg(\overline{u}_\varepsilon, \Omega_{2\varepsilon}, 0) = \deg(\overline{v}_\varepsilon, \Omega_{2\varepsilon}, 0).$$

This proves Lemma 11 and completes the proof of Theorem 3.

An application. Consider the equation

$$\begin{aligned} \Delta u &= f \quad \text{in } \Omega, \\ u &= \varphi \quad \text{on } \partial\Omega, \end{aligned}$$

where $\Omega \subset \mathbb{R}^n$ is a smooth bounded domain with $n \geq 2$. Assume

$$f \in L^{n/2}(\Omega, \mathbb{R}^n), \tag{4.4}$$

$$\varphi \in \mathrm{VMO}(\partial\Omega, S^{n-1}), \tag{4.5}$$

with

$$\deg(\varphi, \partial\Omega, S^{n-1}) \neq 0. \tag{4.6}$$

Corollary 4. *Under the conditions above*

$$\mathrm{ess}R(u) \supset B_1(0).$$

For the definition of $\mathrm{ess}R(u)$, see §I.4 in [BNI].

Proof. We claim that

$$u \in \mathrm{VMO}_\varphi(\Omega). \tag{4.7}$$

The assertion in the corollary then follows from Theorem 3 and Property 1 in II.2. To prove (4.7) we distinguish two cases:

Case (i): $n \geq 3$,

Case (ii): $n = 2$.

In Case (i) we write $u = v + w$ where v is the solution of

$$\begin{aligned} \Delta v &= f \qquad \text{in } \Omega \\ v &= 0 \qquad \text{on } \partial\Omega \end{aligned}$$

and w the harmonic extension of φ in Ω. Since $v \in W^{2,n/2}(\Omega)$, then $v \in W^{1,n}(\Omega)$, and in fact in $W_0^{1,n}(\Omega)$. Thus $v \in \mathrm{VMO}_0(\Omega)$ by Example 2 in §II.3. On the other hand $w \in \mathrm{VMO}_\varphi(\Omega)$ by Theorem A3.1 in Appendix 3. Thus $u = v + w \in \mathrm{VMO}_\varphi(\Omega)$.

In Case (ii), we use the same decomposition $u = v + w$. But here we cannot assert that $v \in W^{1,2}$. Set

$$\widetilde{v} = c(\log|x|) * f$$

(here f is extended as 0 outside Ω) so that $\Delta\widetilde{v} = f$.

By Lemma 9, $\tilde{v} \in \mathrm{VMO}_\psi(\Omega)$ where $\psi = \tilde{v}_{|\partial\Omega} \in \mathrm{VMO}(\partial\Omega)$. We have

$$\begin{aligned} \Delta(\tilde{v} - v) &= 0 \quad \text{in } \Omega \\ \tilde{v} - v &= \psi \quad \text{on } \partial\Omega. \end{aligned}$$

Hence $\tilde{v} - v \in \mathrm{VMO}_\psi(\Omega)$ by Theorem A3.1. Thus $u = v + w = (v - \tilde{v}) + \tilde{v} + w \in \mathrm{VMO}_\varphi(\Omega)$.

□

Remark 12. If $n \geq 3$, condition (4.4) is sharp in the sense that if $f \in L^{(n/2)-\varepsilon}$ (any $\varepsilon > 0$), the conclusion of Corollary 4 need not hold. This may be easily seen on $\Omega = B_1(0)$; the function $u(x) = x/|x|$ satisfies (4.4) with $f \in L^p(\Omega)$, for all $p < n/2$, but not with $p = n/2$.

Appendix 1. Some properties of BMO and VMO in domains

We present the proofs of a number of results in §II.1. In particular, the equivalence of various notions of BMO is established—for general bounded open sets Ω. In addition we show that $C_0^\infty(\Omega)$ is dense in $\mathrm{BMO}(\Omega)$. These results are due to Peter Jones and some are implicit in P. Jones [1].

We start with an easy result; we use the definitions of §II.1 and do not repeat them here.

Lemma A1.1. *Consider* $0 < k_1 < k_2 < 1$. *Then*

$$\| \ \|_{\mathrm{BMO},k_1} \leq \| \ \|_{\mathrm{BMO},k_2} \leq C \| \ \|_{\mathrm{BMO},k_1} \tag{A1.1}$$

where C *may depend on* n, k_1, *and* k_2.

A deeper result, Theorem A1.1, implies that the constant C depends only on k_1.

Proof of Lemma A1.1. Throughout the proof, C denotes various constants depending on n, k_1, k_2. Fix a ball $B_r(x)$ in Ω with

$$r \leq k_2 \ \mathrm{dist}(x, \partial\Omega).$$

Our aim is to show that

$$I = \fint_{B_r(x)} \fint_{B_r(x)} |f(y) - f(z)| \leq C \|f\|_{\mathrm{BMO},k_1}. \tag{A1.2}$$

We use a covering argument similar to one in the proof of Lemma A.14 in [BNI].

Consider a maximal family of disjoint open balls $B_\rho(x_i)$, with centres $x_i \in B_r(x)$, and radius

$$\rho = Ar \qquad \text{with } A = \frac{k_1(1-k_2)}{2k_2} < 1. \tag{A1.3}$$

Each ball of double radius, $B_{2\rho}(x_i)$, belongs to the class $\mathcal{C}_{k_1}$. Indeed

$$\begin{aligned} r \le k_2 \ \mathrm{dist}(x, \partial\Omega) &\le k_2(|x - x_i| + \ \mathrm{dist}(x_i, \partial\Omega)) \\ &\le k_2 r + k_2 \ \mathrm{dist}(x_i, \partial\Omega), \end{aligned}$$

so that

$$r \le \frac{k_2}{1-k_2} \ \mathrm{dist}(x_i, \partial\Omega)$$

and

$$2\rho = 2Ar \le \frac{2Ak_2}{1-k_2} \mathrm{dist}(x_i, \partial\Omega) = k_1 \ \mathrm{dist}(x_i, \partial\Omega).$$

Furthermore, clearly,

$$B_r(x) \subset \bigcup_i B_{2\rho}(x_i).$$

Thus

$$I \le \frac{C}{r^{2n}} \Big[\sum_i \int_{B_{2\rho}(x_i)} \int_{B_{2\rho}(x_i)} |f(y) - f(z)| + \sum_{i \ne j} \int_{B_{2\rho}(x_i)} \int_{B_{2\rho}(x_j)} |f(y) - f(z)| \Big]. \tag{A1.4}$$

The first sum is bounded by

$$\begin{aligned} 2\|f\|_{\mathrm{BMO},k_1} \sum_i |B_{2\rho}(x_i)|^2 &\le C\|f\|_{\mathrm{BMO},k_1} \sum |B_\rho(x_i)|^2 \\ &\le C\|f\|_{\mathrm{BMO},k_1} |B_{r+\rho}(x)|^2 \\ &\le C\|f\|_{\mathrm{BMO},k_1} r^{2n}. \end{aligned} \tag{A1.5}$$

To estimate the second sum in (A1.4) we have

$$\begin{aligned} J &= \sum_{i\ne j} \int_{B_{2\rho}(x_i)} \int_{B_{2\rho}(x_j)} |f(y) - f(z)| \\ &\le \sum_{i\ne j} \int_{B_{2\rho}(x_i)} \int_{B_{2\rho}(x_j)} [|f(y) - \overline{f}_{2\rho}(x_i)| + |\overline{f}_{2\rho}(x_i) - \overline{f}_{2\rho}(x_j)] + [\overline{f}_{2\rho}(x_j) - f(z)] \\ &\le C\|f\|_{\mathrm{BMO},k_1} \sum_{i\ne j} |B_\rho(x_i)|\,|B_\rho(x_j)| + C \sum_{i \ne j} |B_\rho(x_i)|\,|B_\rho(x_j)|\,|\overline{f}_{2\rho}(x_i) - \overline{f}_{2\rho}(x_j)|. \end{aligned}$$

We now claim that for $i \neq j$

$$|\overline{f}_{2\rho}(x_i) - \overline{f}_{2\rho}(x_j)| \le C\|f\|_{\mathrm{BMO},k_1}. \tag{A1.6}$$

Assuming (A1.6) we see that

$$\begin{aligned} J &\le C\|f\|_{\mathrm{BMO},k_1}(\sum |B_\rho(x_i)|)^2 \\ &\le C\|f\|_{\mathrm{BMO},k_1}|B_{r+\rho}(x)|^2 \\ &\le Cr^{2n}\|f\|_{\mathrm{BMO},k_1}. \end{aligned} \tag{A1.7}$$

If we combine this with (A1.5) and (A1.4) we obtain (A1.2).

Proof of (A1.6). This is done as in [BNI] (proof of inequality (A.12)). Namely, for any two points y, z in $B_r(x)$,

$$|\overline{f}_{2\rho}(y) - \overline{f}_{2\rho}(z)| \le C\frac{r}{\rho}\ \|f\|_{\mathrm{BMO},k_1} \tag{A1.8}$$

In view of (A1.3), we then obtain (A1.6). To verify (A1.8) consider a chain of points $y, y_1, \dots, y_{\ell-1}, z$ in $B_r(x)$ such that the distance between any two successive ones is bounded by ρ, and with $\ell \le Cr/p$. For any two successive points of the chain, say y_i, y_{i+1}, we see, using Lemma A.4 of [BNI], that

$$\begin{aligned} \left|\overline{f}_{2\rho}(y_i) - \overline{f}_{2\rho}(y_{i+1})\right| &\le C \fint_{B_{3\rho}(y_i)} |f - \overline{f}_{3\rho}(y_i)| \\ &\le C\|f\|_{\mathrm{BMO},k_1}. \end{aligned}$$

Consequently, adding these inequalities for all successive points we obtain (A1.8). ☐

Remark A1.1. The definition of $\|\ \ \|_{\mathrm{BMO},k}$ involves balls in $\mathbb{R}^n$, and we have only spoken of Euclidean balls. The reader may verify that Lemma A1.1 holds if we replace the Euclidean metric by any norm on $\mathbb{R}^n$.

Using Lemma A1.1 it is easy to give the

Proof of Lemma 2. Consider a ball $B_r(x)$ in Ω_1 with $r < k \operatorname{dist}(x, \partial\Omega_1)$, k to be chosen. We wish to estimate

$$I = \fint_{B_r(x)} \fint_{B_r(x)} |f(H(y)) - f(H(z))|.$$

In view of Lemma A1.1 it suffices to consider any k in $(0, 1)$. We have

$$I \le \frac{C}{r^{2n}} \int_{H(B_r(x))} \int_{H(B_r(x))} |f(\eta) - f(\zeta)|;$$

C depends on a bound for the Jacobian of H^{-1}. Note that

$$H(B_r(x)) \subset B_{\alpha r}(H(x))$$

for suitable α depending on the Lipschitz constant of H. Furthermore,

$$\operatorname{dist}(H(x), \partial\Omega_2) \geq \delta \operatorname{dist}(x, \partial\Omega_1)$$

where δ depends on the Lipschitz constant of H^{-1}. Thus

$$\alpha r < \alpha k \operatorname{dist}(x, \partial\Omega_1) < \frac{\alpha k}{\delta} \operatorname{dist}(H(x), \partial\Omega_2).$$

We now fix k so that, for example, $\alpha k/\delta = 1/2$. Then we find

$$I \leq C\|f\|_{\mathrm{BMO}(\Omega_2)}.$$

$\square$

We now come to one of the main results in this Appendix, the equivalence, due to Peter Jones, of the various notions of BMO, i.e., using all balls or just balls well inside the domain. In fact the balls need not be Euclidean ones. They may be balls in any norm on $\mathbb{R}^n$. In the statement of Theorem A1.1, and in the proof, the balls and distance may be measured in any given norm.

Theorem A1.1. *Let Ω be an open bounded set in $\mathbb{R}^n$. For any real function $f \in L^1_{\mathrm{loc}}(\Omega)$, consider two (semi) norms*

$$\|f\| = \|f\|_{\mathrm{BMO}} = \sup_{\substack{x\in\Omega \\ \varepsilon \leq \frac{1}{2}\operatorname{dist}(x,\partial\Omega)}} ⨍_{B_\varepsilon(x)} |f - ⨍_{B_\varepsilon(x)} f|,$$

$$\|f\|' = \|f\|'_{\mathrm{BMO}} = \sup_{\substack{x\in\Omega \\ \varepsilon < \operatorname{dist}(x,\partial\Omega)}} ⨍_{B_\varepsilon(x)} |f - ⨍_{B_\varepsilon(x)} f|.$$

There is a constant $\overline{C}$ depending only on n and the choice of norm on $\mathbb{R}^n$, such that

(A1.9) $$\|f\|_{\mathrm{BMO}} \leq \|f\|'_{\mathrm{BMO}} \leq \overline{C}\|f\|_{\mathrm{BMO}}.$$

The proof of Theorem A1.1 relies on the following two lemmas.

Lemma A1.2. *There is a covering of $B = B_1(0)$ by balls $B_i = B_{r_i}(x_i)$ with $r_i = \frac{1}{2}(1 - |x_i|) > 0$ such that for every $\gamma > (n-1)/n$,*

$$\sum_i |B_i|^\gamma = c_\gamma < \infty.$$

In particular,

$$\sum_i |B_i| \, |\log |B_i|| < \infty. \tag{A1.10}$$

Proof. Let $0 < b < 1$ and set, for $j = 1, 2, \dots$,

$$A_j = \{x \in B;\ \ 1 - b^{j-1} \le |x| \le 1 - b^j\}.$$

Note that

$$B = \bigcup_{j=1}^\infty A_j.$$

For each fixed j, consider a maximal family F_j of disjoint balls $B_\rho(x_i)$ with $x_i \in A_j \quad \forall i$, and $\rho = \frac{1}{4} b^j$. Clearly, the family $B_{2\rho}(x_i)$ covers A_j. The corresponding family $B_i = B_{r_i}(x_i)$ with $r_i = \frac{1}{2}(1 - |x_i|) \ge 2\rho$ also covers A_j. Moreover

$$\bigcup_{i \in F_j} B_\rho(x_i) \subset A = \left\{x;\ \ 1 - b^{j-1} - \rho < |x| < 1 - b^j + \rho\right\}$$

and so

$$\sum_{i \in F_j} |B_\rho(x_i)| \le |A| \le C b^j,$$

where C is independent of j. It follows that

$$\mathrm{card} F_j \le C b^{j(1-n)}.$$

Thus we obtain, since $r_i \le C b^j \quad \forall i$,

$$\sum_{i \in F_j} |B_i|^\gamma \le C b^{j(1-n)} b^{nj\gamma} \le C d^j,$$

where $d = b^{n\gamma - n + 1} < 1$. Consequently

$$\sum_{j=1}^\infty \sum_{i \in F_j} |B_i|^\gamma \le C \sum_{j=1}^\infty d^j < \infty.$$

$\square$

Lemma A1.3. *There is a constant C depending only on n and the choice of norm on $\mathbb{R}^n$ such that*

$$|\overline{f}_r(x) - \overline{f}_{1/2}(0)| \leq C\|f\|_{\mathrm{BMO}} \log(1/r) \quad \forall x \in B_1(0) \tag{A1.11}$$

with

$$r = \frac{1}{2}(1 - |x|).$$

Assuming Lemma A1.3 it is easy to derive Theorem A1.1.

Proof of Theorem A1.1. It suffices to show that, for any ball $B = B_\delta(x_0) \subset \overline{B_\delta(x_0)} \subset \Omega$,

$$I := \fint_B |f - f_0| \leq \overline{C} \sup_{\substack{x \in B \\ \varepsilon \leq \frac{1}{2}\mathrm{dist}(x, \partial B)}} \fint_{B_\varepsilon(x)} |f - \overline{f}_\varepsilon(x)|$$

for some constant f_0 and some constant $\overline{C}$ depending only on n and the given norm on $\mathbb{R}^n$. Without loss of generality we may suppose $B = B_1(0)$ and $\|f\|_{\mathrm{BMO}(B)} = 1$.

Consider a covering $B_i = B_{r_i}(x_i)$ of B as in Lemma A1.2. Set

$$f_0 = \overline{f}_{1/2}(0)$$

and

$$f_i = \overline{f}_{r_i}(x_i).$$

We deduce from (A1.11) that, for all i,

$$|f_i - f_0| \leq C \log \frac{1}{r_i} \leq C|\log |B_i|| + C. \tag{A1.12}$$

Therefore

$$\begin{aligned} I \leq \frac{1}{|B|} \sum_i \int_{B_i} |f - f_0| &\leq \frac{1}{|B|} \sum_i |B_i| \fint_{B_i} |f - f_i| + \frac{1}{|B|} \sum_i |B_i||f_i - f_0| \\ &\leq C \end{aligned}$$

by (A1.12) and Lemma A1.2.

□

We now return to the

Proof of Lemma A1.3. The line from 0 to x is identified with $\mathbb{R}$ and we assume $0 \leq x < 1$. Consider the sequence $B_{r_k}(x_k)$ of balls centred on that line with

$$x_k = 1 - (1 - x)2^{k-1}$$

and

$$r_k = \frac{1}{2}(1 - x_k) = (1 - x)2^{k-2}.$$

Let k_0 be the largest integer such that $x_k \geq 0$. We always assume that $k \leq k_0$, so that $\overline{B_{r_k}(x_k)} \subset B_1(0)$.

It is easy to check that

$$B_{r_k/2}\left(x_k - \frac{r_k}{2}\right) \subset B_{r_k}(x_k) \cap B_{r_{k+1}}(x_{k+1}),$$

and thus, by Lemma A.4 of [BNI], we have

$$|\overline{f}_{r_k}(x_k) - \overline{f}_{r_k/2}\left(x_k - \frac{r_k}{2}\right)| \leq C\|f\|_{\mathrm{BMO}}$$
$$|\overline{f}_{r_{k+1}}(x_{k+1}) - \overline{f}_{r_k/2}\left(x_k - \frac{r_k}{2}\right)| \leq C\|f\|_{\mathrm{BMO}}.$$

Set

$$f_k = f_{r_k}(x_k);$$

we infer that

$$|f_k - f_{k+1}| \leq C\|f\|_{\mathrm{BMO}} \quad \forall k \leq k_0 - 1.$$

Adding these inequalities we find

$$|f_1 - f_{k_0}| \leq C\|f\|_{\mathrm{BMO}}(k_0 - 1) \leq C\|f\|_{\mathrm{BMO}} \log\frac{1}{r}. \tag{A1.13}$$

Note that

$$f_1 = \overline{f}_r(x) \quad \text{with} \quad r = \frac{1}{2}(1 - x).$$

Finally we claim that

$$|f_{k_0} - \overline{f}_{1/2}(0)| \leq C\|f\|_{\mathrm{BMO}}. \tag{A1.14}$$

The desired conclusion (A1.11) then follows from (A1.13) and (A1.14).

Proof of (A1.14). Since $x_{k_0+1} \leq 0$ we have $x_{k_0} \leq \frac{1}{2}$ and $r_{k_0} = \frac{1}{2}(1 - x_{k_0}) \geq \frac{1}{4}$. It follows that $B_{1/2}(0) \cap B_{r_{k_0}}(x_{k_0})$ contains the ball $\hat{B} = B_{1/8}(x_{k_0} - \frac{1}{8})$. Applying Lemma A.4 of [BNI] once more we obtain

$$|\overline{f}_{1/2}(0) - \overline{f}_{\hat{B}}| \leq C\,\|f\|_{\mathrm{BMO}}$$
$$|f_{k_0} - \overline{f}_{\hat{B}}| \leq C\,\|f\|_{\mathrm{BMO}}$$

and thus (A1.14) is established.

Remark A1.2. Theorem A1.1 holds for any open set Ω, with $\overline{\Omega}$ compact in a smooth open Riemannian manifold X_0. In the definitions of the norms $\|\ \ \|$ and $\|\ \ \|'$, one also restricts the radii of the balls to be less than the injectivity radius r_0 of X_0—assumed to be positive. The constant $\overline{C}$ in (A1.9) then depends on the Riemannian metric on X_0. The proof of this more general result proceeds as in the proof above with minor modifications.

Here are some consequences of the above results.

Corollary A1.1. *Let Ω be an open bounded set in $\mathbb{R}^n$. Suppose $\| \ \|_1$ and $\| \ \|_2$ are two norms on $\mathbb{R}^n$. Associated with these are two notions of BMO(Ω):*

$$\|f\|_{\mathrm{BMO}_i} = \sup_{\overline{B^i_\varepsilon(x)}\subset\Omega} \fint_{B^i_\varepsilon(x)} |f - \fint_{B^i_\varepsilon(x)} f| \qquad i = 1, 2.$$

Here the ball $B^i_\varepsilon(x)$ is measured in the norm $\| \ \|_i$. Then the two BMO norms are equivalent (and the equivalence constants depend only on n and the equivalence constants for $\| \ \|_1$ and $\| \ \|_2$).

Next we take up the

Proof of Lemma 3 in §II.1. Consider the function

$$\varphi(x) = \log \frac{1}{\widetilde{d}(x, \partial\Omega)}$$

where $\widetilde{d}$ is the distance measured in some metric equivalent to the Riemannian one. For any ball $B_\varepsilon(x)$ in Ω, with $\varepsilon \leq \frac{1}{2}$ dist$(x, \partial\Omega)$—here dist refers to our Riemannian metric—we have to estimate

$$J = \fint_{B_\varepsilon(x)} \fint_{B_\varepsilon(x)} |\varphi(y) - \varphi(z)|.$$

Clearly if $y \in B_\varepsilon(x)$, $\mathrm{dist}(y, \partial\Omega) > \varepsilon$, and thus $\widetilde{d}(y, \partial\Omega) > \alpha\varepsilon$ for some constant α. Hence for $y, z \in B_\varepsilon(x)$,

$$\text{(A1.15)} \qquad |\varphi(y) - \varphi(z)| \leq \frac{C}{\varepsilon}\widetilde{d}(y, z) \leq \frac{C}{\varepsilon} \,\mathrm{dist}(y, z) \leq C.$$

Consequently $J \leq C$.

□

With the aid of Theorem 1 we also give the

Proof of Lemma 4. Consider a ball $B_\varepsilon(x)$ in Ω with $\varepsilon \leq \frac{1}{2}\mathrm{dist}(x, \partial\Omega)$. In view of Theorem 1 we have to show that given $\delta > 0$, there exists $\varepsilon_0 > 0$ such that

$$J = \fint_{B_\varepsilon(x)} \fint_{B_\varepsilon(x)} |\varphi^\alpha(y) - \varphi^\alpha(z)| < \delta \quad \forall x \in \Omega,$$

and for all $\varepsilon \leq \min\{\varepsilon_0, \frac{1}{2}\mathrm{dist}(x, \partial\Omega)\}$. Since φ is continuous in Ω we need only consider such balls with $\mathrm{dist}(x, \partial\Omega)$ small. We have

$$|\varphi^\alpha(y) - \varphi^\alpha(z)| \leq \alpha \frac{|\varphi(y) - \varphi(z)|}{\min\{\varphi(y), \varphi(z)\}^{1-\alpha}};$$

by (A1.15),

$$|\varphi^\alpha(y) - \varphi^\alpha(z)| \le \frac{C}{\varepsilon} \, \frac{\operatorname{dist}(y,z)}{|\min\limits_{B_\varepsilon(x)} \varphi|^{1-\alpha}}.$$

Consequently

$$J \le C|\min_{B_\varepsilon(x)} \varphi|^{\alpha-1}$$

with C independent of x and ε. Thus for $\operatorname{dist}(x,\partial\Omega)$ small, $\min\limits_{B_\varepsilon(x)} \varphi$ is as large as wanted, so that J is small.

□

We turn finally to the proof of Theorem 1. The proof we present is a slight modification of one shown to us by Peter Jones.

Proof of Theorem 1. We need only prove that (1.6) implies (1.7), namely, if $f \in \mathrm{BMO}(\Omega)$ and satisfies

$$\lim_{\substack{\varepsilon\to 0\\ \varepsilon\le\frac{1}{2}\operatorname{dist}(x,\partial\Omega)}} \fint_{B_\varepsilon(x)} |f - \overline{f}_\varepsilon(x)| = 0 \quad \text{uniformly in } x, \tag{A1.16}$$

then

(A1.17) there exists a sequence (f_j) in $C_0^\infty(\Omega)$ converging to f in $\mathrm{BMO}(\Omega) \cap L^1_{\mathrm{loc}}(\Omega)$.

The proof makes use of the following simple

Lemma A1.4. *Assume that f is in BMO(Ω) and satisfies (A1.16). Then each truncation*

$$f^k(x) = \begin{cases} k & \text{if } f(x) \ge k \\ f(x) & \text{if } -k < f(x) < k \\ -k & \text{if } f(x) \le -k \end{cases}$$

also satisfies (A1.16) and moreover,

$$f^k \to f \quad \text{in BMO}\cap L^1_{\mathrm{loc}} \text{ as } k \to \infty. \tag{A1.18}$$

The lemma is a variant of Lemma A.17 in [BNI] and is proved in the same way.

In view of Lemma A1.4 we may assume that our f satisfying (A1.16) is in L^∞. The main step is to show that f may then be approximated in BMO$\cap L^1$ by L^∞ functions F satisfying (A1.16) and which, furthermore, have compact support in Ω. Once this is done it is easy to complete the proof of the theorem: We may think of Ω as lying in a compact manifold X_0, without boundary and consider F defined on X_0 to be zero outside Ω. By Sarason's result (see Lemma 3 in [BNI]) F belongs to VMO(X_0). By Corollary 1 in [BNI], $\overline{F}_\varepsilon$ is close to F in BMO$\cap L^1$, if ε is small. But for ε small, $\overline{F}_\varepsilon$ also has compact support in Ω. Since $\overline{F}_\varepsilon$ is continuous, it may be approximated in the L^∞ norm—and hence in BMO$\cap L^1$—by smooth functions with compact support in Ω. The proof of Theorem 1 would then be complete.

As usual, it is convenient to replace the BMO norm by an equivalent one:

$$\|f\|_\star = \sup_{\substack{x \\ \varepsilon<\frac{1}{2}\operatorname{dist}(x,\partial\Omega)}} \fint_{B_\varepsilon(x)} \fint_{B_\varepsilon(x)} |f(y)-f(z)|,$$

and to rewrite (A1.16) as

$$(A1.16)' \qquad \lim_{\substack{\varepsilon\to 0 \\ \varepsilon\le\frac{1}{2}\operatorname{dist}(x,\partial\Omega)}} \fint_{B_\varepsilon(x)} \fint_{B_\varepsilon(x)} |f(y)-f(z)| = 0 \quad \text{uniformly in } x.$$

To carry out the main step, consider f satisfying $(A1.16)'$ with $|f|\le k$. Using suitable cutoff functions we will construct the approximating functions F_j.

Recalling the function of Lemma 3,

$$\varphi(x) = \log\frac{1}{\operatorname{dist}(x,\partial\Omega)},$$

without loss of generality, we may always assume that for all x, $\operatorname{dist}(x,\partial\Omega)\le 1$, so that $\varphi(x)\ge 0$. For $j=1,2,\dots$, set

$$h_j(x) = \Big(1-\frac{1}{j}\varphi(x)\Big)^+$$

and

$$F_j = h_j f.$$

We claim that the F_j have all the desired properties:

(i) $F_j\in L^\infty$,

(ii) each F_j satisfies (A1.16),

(iii) the F_j have compact support,

(iv) $F_j\to f$ in L^1,

(v) $F_j\to f$ in BMO.

Clearly (i), (iii) and (iv) are trivial.

Proof of (ii). Each function h_j is Lipschitz on Ω, with Lipschitz constant k_j. Then, for our usual balls $B_\varepsilon(x)$,

$$\begin{aligned}
&\fint_{B_\varepsilon(x)} \fint_{B_\varepsilon(x)} |h_j(y)f(y) - h_j(z)f(z)| \\
&\qquad\qquad \leq \fint_{B_\varepsilon(x)} \fint_{B_\varepsilon(x)} |f(y) - f(z)| + k_j \|f\|_{L^\infty} \fint_{B_\varepsilon(x)} \fint_{B_\varepsilon(x)} \mathrm{dist}(y,z) \\
&\qquad\qquad \longrightarrow 0 \quad \text{as} \quad \varepsilon \longrightarrow 0 \qquad \text{by } (A1.16)'.
\end{aligned}$$

Proof of (v). Given $\delta > 0$, there exists $\varepsilon_0 > 0$ such that

$$\fint_{B_\varepsilon(x)} \fint_{B_\varepsilon(x)} |f(y) - f(z)| < \frac{\delta}{3} \quad \text{for } \varepsilon < \varepsilon_0, \varepsilon \leq \frac{1}{2}\mathrm{dist}(x, \partial\Omega).$$

Consider

$$I = \fint_{B_\varepsilon(x)} \fint_{B_\varepsilon(x)} |h_j(y)f(y) - h_j(z)f(z) - (f(y) - f(z))|.$$

We will prove that $I < \delta$ for j sufficiently large (independent of x and ε). As usual, we distinguish two cases.

(a) If $\varepsilon < \varepsilon_0$ then

$$\begin{aligned}
I &\leq 2 \fint_{B_\varepsilon(x)} \fint_{B_\varepsilon(x)} |f(y) - f(z)| + \|f\|_{L^\infty} \fint_{B_\varepsilon(x)} \fint_{B_\varepsilon(x)} |h_j(y) - h_j(z)| \\
&< \frac{2\delta}{3} + \frac{1}{j}\|f\|_{L^\infty} \fint_{B_\varepsilon(x)} \fint_{B_\varepsilon(x)} |\varphi(y) - \varphi(z)| \\
&\leq \frac{2\delta}{3} + \frac{C}{j}\|f\|_{L^\infty} \qquad \text{by Lemma 3,} \\
&< \delta \quad \text{for } j \text{ sufficiently large.}
\end{aligned}$$

(b) If $\varepsilon \geq \varepsilon_0$ then

$$I \leq 2 \fint_{B_\varepsilon(x)} |h_j(y) - 1|\,|f(y)| \leq C\|f\|_{L^\infty} \int_\Omega (1 - h_j).$$

This can be made less than δ for h_j large, by dominated convergence.

$\square$

Appendix 2 (with P. Mironescu). Toeplitz operators and VMO

In this appendix we discuss Toeplitz operators on the circle S^1. Let us first recall the classical Toeplitz operators. Consider complex valued L^2-functions on S^1 and the closed subspace

$$\mathcal{H}^2 = \left\{ f \in L^2(S^1); \int_{S^1} e^{in\theta} f(\theta) d\theta = 0, \quad n = 1, 2, \dots \right\},$$

and more generally, for p in $[1, \infty]$,

$$\mathcal{H}^p = \left\{ f \in L^p(S^1); \int_{S^1} e^{in\theta} f(\theta) d\theta = 0, \quad n = 1, 2, \dots \right\}.$$

Let P be the orthogonal projection from L^2 onto $\mathcal{H}^2$.

Given a function $\varphi \in L^\infty(S^1, \mathbb{C})$ we denote by M_φ the operation on L^2 of multiplication by φ. The associated Toeplitz operator (with symbol φ), is

$$T_\varphi = P M_\varphi P; \tag{A2.1}$$

the associated Hankel operator is

$$H_\varphi = (I - P) M_\varphi P. \tag{A2.2}$$

T_φ is often considered as an operator from $\mathcal{H}^2$ to $\mathcal{H}^2$.

A classical result is that if φ is continuous and nowhere zero, then T_φ is a Fredholm operator and

$$\text{index } (T_\varphi) = -\deg \left(\frac{\varphi}{|\varphi|}, S^1, S^1 \right). \tag{A2.3}$$

See, for example, R. G. Douglas [1], Theorem 7.26 and R. G. Douglas [2]; further references and history may be found there. A number of authors have extended this result to other classes of functions φ, not necessarily continuous. See for example Theorem 7.36 in R. G. Douglas [1] and D. Sarason [1],[2],[3],[4], and the recent book by I. Gohberg and N. Krupnik [1].

Since the right hand side of (A2.3) makes sense for $\varphi \in \text{VMO}(S^1)$ with $|\varphi| \geq a > 0$—by [BNI], it is natural to extend the classical result above to functions φ satisfying

$$\varphi \in \text{VMO}(S^1) \cap L^\infty(S^1), \quad |\varphi| \geq a > 0. \tag{A2.4}$$

We present such a result

Theorem A2.1. *Let φ satisfy (A2.4). Then T_φ is Fredholm and (A2.3) holds.*

This follows, in fact, from Theorem 7.36 in R. G. Douglas [1]. His result is more general: it asserts that if φ is in $\mathcal{H}^\infty + C^0$ and if $\hat{\varphi}$, the harmonic extension of φ to the unit disc D, satisfies

$$|\hat{\varphi}(re^{i\theta})| \geq \alpha > 0 \quad \text{for } 1-\delta < r < 1, \tag{A2.5}$$

then T_φ is Fredholm. Moreover,

$$\text{index } (T_\varphi) = -\deg\left(\frac{\hat{\varphi}(re^{i\theta})}{|\hat{\varphi}(re^{i\theta}|}, S^1, S^1\right) \quad \text{for every } r \text{ in } (1-\delta, 1). \tag{A2.6}$$

To derive Theorem A2.1 from Douglas' result one uses two facts:

(i) If $\varphi \in \text{VMO} \cap L^\infty$, then $\varphi \in \mathcal{H}^\infty + C^0$. More precisely,

$$\varphi \in \text{VMO } \cap L^\infty \Longleftrightarrow \varphi \text{ and } \overline{\varphi} \text{ belong to } \mathcal{H}^\infty + C^0.$$

This result is due to D. Sarason [1]. The space VMO$\cap L^\infty$ is sometimes called QC (quasi continuous);

(ii) If $\varphi \in$ VMO and $|\varphi| \geq a > 0$ then its harmonic extension $\hat{\varphi}$ satisfies (A2.5); see Lemma 5 in D. Sarason [3], and also Theorem A3.2 in Appendix 3 here.

It seems worthwhile to present here a different proof which is more or less self contained. It is elementary except for the Fefferman inequality (see (A2.10) below).

We derive Theorem A2.1 from the classical case—for φ continuous—by approximation. The convergence of the right hand side of (A2.3), in the approximation, holds by stability of degree in VMO, see Theorem 1 in [BNI]. The convergence of the left hand side is more subtle since T_φ does *not* depend continuously in the operator norm on the BMO norm of φ; see Remark A2.1. It turns out that H_φ has that property:

Lemma A2.1. *There is a constant C such that*

$$\|H_\varphi\| \leq C\|\varphi\|_{\text{BMO}} \qquad \forall \varphi \in L^\infty(S^1). \tag{A2.7}$$

Proof. Clearly $H_\psi = 0$ if $\psi \in \mathcal{H}^\infty$. Thus for any $\psi \in \mathcal{H}^\infty$,

$$\|H_\varphi\| = \|(I-P)M_{\varphi-\psi}P\| \leq \|M_{\varphi-\psi}\| \leq \|\varphi - \psi\|_{L^\infty}.$$

Hence

$$\|H_\varphi\| \leq \inf_{\psi \in \mathcal{H}^\infty} \|\varphi - \psi\|_{L^\infty} = \text{dist}(\varphi, \mathcal{H}^\infty) \text{ in } L^\infty.$$

(In fact equality holds by Nehari's theorem; see D. Sarason [4], page 100.)

The assertion of the lemma follows from the

Claim:

(A2.8)
$$\operatorname{dist}(\varphi, \mathcal{H}^\infty) \le C\|\varphi\|_{\mathrm{BMO}} \quad \text{for } \varphi \in L^\infty.$$

Proof of Claim. Recall that if X is a real Banach space, and M is a linear subspace of X then for any $f \in X^\star$,

(A2.9)
$$\sup_{\substack{u \in M \\ \|u\| \le 1}} \langle f, u\rangle = \operatorname{dist}(f, M^\perp),$$

where $M^\perp$ is the set of points in $X^\star$ which annihilate M. We take $X = L^1(S^1, \mathbb{C}) \simeq L^1(S^1, \mathbb{R}^2)$, M =the set of finite linear combinations (over $\mathbb{C}$) of $e^{-in\theta}, n = 1, 2, \dots$. Feffermans' inequality (see C. Feffermann [1]; see also C. Fefferman and E. Stein [1], and E. Stein [1]) implies that for $u \in M$,

(A2.10)
$$\left|\int_{S^1} fu\right| \le C\|f\|_{\mathrm{BMO}}\|u\|_{L^1}$$

By definition, $M^\perp = \mathcal{H}^\infty$, and (A2.7) then follows from (A2.9) and (A2.10).

□

Remark A2.1. There is no estimate of the form

(A2.11)
$$\|T_\varphi\| \le C\left(\|\varphi\|_{\mathrm{BMO}} + \|\varphi\|_{L^1}\right) \qquad \forall \varphi \in L^\infty.$$

Proof. Write $f \in L^2$ as

(A2.12)
$$f = Pf + (I - P)f = Pf + \overline{P\overline{f}} - \fint f.$$

Since $H_\varphi + T_\varphi = M_\varphi P$ we may write, for any $f \in L^2$,

$$\begin{aligned}
M_\varphi f &= M_\varphi(Pf) + M_\varphi((I-P)f) \\
&= M_\varphi(Pf) + M_\varphi(\overline{P\overline{f}}) - (\fint f)\varphi \\
&= M_\varphi(Pf) + \overline{M_{\overline{\varphi}}P\overline{f}} - (\fint f)\varphi \\
&= H_\varphi(f) + T_\varphi(f) + \overline{H_{\overline{\varphi}}(\overline{f})} + \overline{T_{\overline{\varphi}}(\overline{f})} - (\fint f)\varphi.
\end{aligned}$$

Thus, if (A2.11) were to hold, by (A2.11) and Lemma A2.1,

$$\|M_\varphi f\|_{L^2} \le C(\|\varphi\|_{\mathrm{BMO}} + \|\varphi\|_{L^1})\|f\|_{L^2} + |\fint f|\|\varphi\|_{L^2}.$$

In particular,

$$\|M_\varphi\| \le C(\|\varphi\|_{\mathrm{BMO}} + \|\varphi\|_{L^2}).$$

But $\|M_\varphi\| = \|\varphi\|_{L^\infty}$. This yields a contradiction if we choose for φ the truncations of a function in BMO which is not in L^∞.

□

Lemma A2.2. *For $\varphi \in VMO \cap L^\infty$,*

$$H_\varphi \text{ is compact from } L^2 \text{ into itself.}$$

Proof. There is a sequence (φ_j) of functions in C^0 such that $\varphi_j \to \varphi$ in BMO; see D. Sarason [1]. By Lemma A2.1

$$\|H_{\varphi_j} - H_\varphi\| \le C\|\varphi_j - \varphi\|_{\mathrm{BMO}} \to 0. \tag{A2.13}$$

On the other hand, for every continuous ψ, H_ψ is compact. This fact is classical and is easily verified by noting that for every ψ of the form

$$\psi(\theta) = \sum_{n=-N}^{+N} a_n e^{in\theta}$$

H_ψ is a finite rank operator.

□

Corollary A2.1. *For $\varphi \in L^\infty$ and $\psi \in VMO \cap L^\infty$*

$$T_\varphi T_\psi - T_{\varphi\psi} \quad \text{is compact}$$

Proof. Just write

$$T_\varphi T_\psi - T_{\varphi\psi} = -PM_\varphi H_\psi \tag{A2.14}$$

and apply Lemma A2.2.

Lemma A2.3. *Assume (A2.4), then T_φ is Fredholm in $\mathcal{H}^2$.*

Proof. By Lemma 2′ in [BNI] we know that $\varphi^{-1} \in \mathrm{VMO} \cap L^\infty$ and so, by Corollary A2.1, we have, on $\mathcal{H}^2$

$$T_\varphi T_{\varphi^{-1}} = I + K, \quad K \text{ compact.}$$

Similarly, we have, on $\mathcal{H}^2$,

$$T_{\varphi^{-1}} T_\varphi = I + K', \quad K' \text{ compact.}$$

It follows that (see e.g. S. Lang[1]) T_φ is Fredholm.

Before continuing with the proof of Theorem A2.1, it is convenient to introduce the class

$$A = \{\varphi \in \mathrm{VMO};\ \ \varphi \in L^\infty \text{ and } \varphi^{-1} \in L^\infty\}.$$

Note that if $\varphi \in A$, then $\varphi^{-1} \in \mathrm{VMO}$; see Lemma 2′ in [BNI].

Lemma A2.4. *Let* (ψ_j) *be a sequence in* A *such that* $\|\psi_j\|_{L^\infty} \le C$, $\|\psi_j^{-1}\|_{L^\infty} \le C$ *and* $\|\psi_j\|_{\mathrm{BMO}} \to 0$. *Then* T_{ψ_j} *is invertible in* $\mathcal{H}^2$ *for* j *sufficiently large.*

Proof. By (A2.14) we have, in $\mathcal{H}^2$,

$$T_{\psi_j} T_{\psi_j^{-1}} = I - PM_{\psi_j} H_{\psi_j^{-1}} \tag{A2.15}$$

and

$$T_{\psi_j^{-1}} T_{\psi_j} = I - PM_{\psi_j^{-1}} H_{\psi_j}. \tag{A2.16}$$

Passing to a subsequence, we may always assume (by Lemma A.1 in [BNI]) that $\psi_j \to c$, for some constant c, in L^1. It follows (by Lemma A.7 in [BNI]) that $\psi_j^{-1} \to 0$ in BMO. Applying Lemma A2.1 we conclude that

$$\|PM_{\psi_j} H_{\psi_j^{-1}}\| \to 0 \text{ and } \|PM_{\psi_j^{-1}} H_{\psi_j}\| \to 0.$$

Hence $I - PM_{\psi_j} H_{\psi_j^{-1}}$ and $I - PM_{\psi_j^{-1}} H_{\psi_j}$ are invertible for j sufficiently large; the conclusion of the lemma follows easily from (A2.15) and (A2.16).

Next, a useful lemma about the product of functions in BMO.

Lemma A2.5. *Let* $g \in VMO \cap L^\infty$. *Then for every* $\delta > 0$ *there exists a constant* C_δ *(depending on* δ **and** g*) such that*

$$\|fg\|_{\mathrm{BMO}} \le \delta \|f\|_{L^\infty} + C_\delta(\|f\|_{\mathrm{BMO}} + \|f\|_{L^1}) \quad \forall f \in L^\infty.$$

Proof. Recall (see (1″) in [BNI]) that

$$\|fg\|_{\mathrm{BMO}} \le \sup_{\varepsilon, x} \fint_{B_\varepsilon(x)} \fint_{B_\varepsilon(x)} |f(y)g(y) - f(z)g(z)|.$$

But

$$\begin{aligned} \fint_{B_\varepsilon(x)} \fint_{B_\varepsilon(x)} |f(y)g(y) - f(z)g(z)| &\le L + \fint_{B_\varepsilon(x)} \fint_{B_\varepsilon(x)} |f(y) - f(z)|\,|g(z)| \\ &\le L + 2\|g\|_{L^\infty} \|f\|_{\mathrm{BMO}}, \end{aligned}$$

where

$$L = \fint_{B_\varepsilon(x)} \fint_{B_\varepsilon(x)} |f(y)(g(y) - g(z))|.$$

Clearly, two estimates hold for L:

$$L \leq 2\|g\|_{L^\infty} \fint_{B_\varepsilon(x)} |f|, \quad \text{and } L \leq \|f\|_{L^\infty} \fint_{B_\varepsilon(x)} \fint_{B_\varepsilon(x)} |g(y) - g(z)|. \tag{A2.17}$$

Since $g \in \mathrm{VMO}$, there exists ε_0 depending only on g such that

$$\fint_{B_\varepsilon(x)} \fint_{B_\varepsilon(x)} |g(y) - g(z)| \leq \delta \qquad \text{if } \varepsilon \leq \varepsilon_0,$$

and thus $L \leq \delta\|f\|_{L^\infty}$ by the second estimate in (A2.17). For $\varepsilon > \varepsilon_0$ we use the first estimate in (A2.17), namely

$$L \leq 2\frac{\|g\|_{L^\infty}}{\varepsilon_0} \int_{B_\varepsilon(x)} |f| \leq C\|f\|_{L^1}$$

and the conclusion of the lemma follows.

Lemma A2.6. *Let $\varphi \in A$ and (φ_j) be a sequence in A such that $\|\varphi_j\|_{L^\infty} \leq C$, $\|\varphi_j^{-1}\|_{L^\infty} \leq C$ and $\varphi_j \to \varphi$ in BMO$\cap L^1$. Then*

$$\textit{index } (T_{\varphi_j}) = \textit{index } (T_\varphi) \textit{ for } j \textit{ sufficiently large.}$$

Proof. Lemma A2.5 (applied to $f = \varphi_j - \varphi$ and $g = \varphi^{-1}$) implies that

$$\left\| \frac{\varphi_j}{\varphi} \right\|_{\mathrm{BMO}} \longrightarrow 0 \quad \text{as } j \to \infty.$$

We deduce from Lemma A2.4 that $T_{\varphi_j/\varphi}$ is invertible in $\mathcal{H}^2$ for j sufficiently large.

By Corollary A2.1 we have, in $\mathcal{H}^2$,

$$\begin{aligned} T_{\varphi_j/\varphi} &= T_{\varphi_j} T_{1/\varphi} + K \\ T_{1/\varphi} T_\varphi &= I + K' \end{aligned}$$

where K and K' are compact. Applying the standard properties of the index (see e.g. S. Lang [1]) we conclude that, for j sufficiently large

$$\begin{aligned} 0 = \mathrm{index}(T_{\varphi_j/\varphi}) &= \mathrm{index}(T_{\varphi_j} T_{1/\varphi}) = \\ &= \mathrm{index}(T_{\varphi_j}) + \mathrm{index}(T_{1/\varphi}) = \mathrm{index}(T_{\varphi_j}) - \mathrm{index}(T_\varphi). \end{aligned}$$

We may now prove Theorem A2.1 by approximation using (A2.3) for continuous φ.

Proof of Theorem A2.1. Given $\varphi \in A$ there is a sequence (φ_j) of continuous functions such that $\|\varphi_j\|_{L^\infty} \le C$, $\|\varphi_j^{-1}\|_{L^\infty} \le C$ and $\varphi_j \to \varphi$ in $\mathrm{BMO}\cap L^1$; see e.g. Corollary 4 in [BNI]. We have

$$\mathrm{index} T_{\varphi_j} = -\deg(\varphi_j/|\varphi_j|).$$

For j sufficiently large, the left hand side equals index T_φ (by Lemma A2.6) and the right hand side equals $\deg(\varphi/|\varphi|)$ by Theorem 1 in [BNI]. □

Here is an alternative proof of Theorem A2.1 which does not make use of Lemmas A2.4, A2.5 and A2.6. It is slightly shorter, but it relies on an additional ingredient: the lifting property for maps in $\mathrm{VMO}(S^1, S^1)$ with degree zero (see Theorem 3 in Section I.6 of [BNI]). On the other hand this proof is totally self contained—it does not rely on the classical case (φ continuous). The key observation is the following:

Lemma A2.7. *Consider a map $m\colon A \to \mathbb{Z}$ satisfying*

$$m(\varphi\psi) = m(\varphi) + m(\psi) \quad \forall \varphi, \psi \in A.$$

Then there is an integer k such that

$$m(\varphi) = k \deg\left(\frac{\varphi}{|\varphi|}, S^1, S^1\right) \quad \forall \varphi \in A. \tag{A2.18}$$

Remark A2.2. Surprisingly, in Lemma A2.7, no continuity is required of m. The condition on m is purely algebraic.

Proof. We first claim that

$$m(\psi) = 0 \quad \forall \psi \in A \quad \text{with } \deg\left(\frac{\psi}{|\psi|}\right) = 0. \tag{A2.19}$$

Indeed we may write, by Theorem 3 in §I.6 of [BNI],

$$\psi = |\psi| e^{i\sigma}$$

for some function $\sigma \in \mathrm{VMO}(S^1, \mathbb{R})$. For every integer n, let

$$\psi_n = |\psi|^{\frac{1}{n}} e^{i\sigma/n} \in A,$$

so that

$$m(\psi) = m(\psi_n^n) = n m(\psi_n).$$

Thus, if $m(\psi) \neq 0$, $|m(\psi)| \ge n \quad \forall n$—impossible; (A2.19) is proved.

For $\varphi \in A$ let

$$d = \deg\left(\frac{\varphi}{|\varphi|}, S^1, S^1\right)$$

and write

$$\varphi = e^{di\theta}|\varphi|e^{i\eta}, \eta \in \mathrm{VMO}(S^1, \mathbb{R})$$

(see Remark 10 in §I.6 of [BNI]). Then

$$m(\varphi) = m(e^{di\theta}) + m(|\varphi|e^{i\eta}) = dm(e^{i\theta})$$

by (A2.19). This proves (A2.18) with $k = m(e^{i\theta})$.

Proof of Theorem A2.1. For every $\varphi \in A$ we know that T_φ is Fredholm by Lemma A2.3. Set

$$m(\varphi) = \mathrm{index}(T_\varphi)$$

We have, by Corollary A2.1, for some compact operator K,

$$m(\varphi\psi) = \mathrm{index}(T_{\varphi\psi}) = \mathrm{index}(T_\varphi T_\psi + K) = \mathrm{index}(T_\varphi T_\psi) = m(\varphi) + m(\psi).$$

by standard properties of Fredholm operators. Applying Lemma A2.4 we conclude that

$$m(\varphi) = k \deg\left(\frac{\varphi}{|\varphi|}, S^1, S^1\right)$$

for some integer k. Choosing $\varphi(\theta) = e^{i\theta}$ we see that $k = -1$.

□

Appendix 3. The harmonic extension of VMO maps

In this appendix we discuss properties of the harmonic extension u of a BMO (or VMO) map φ defined on the boundary $\partial\Omega$ of a domain $\Omega \subset \mathbb{R}^n$; throughout we assume that Ω is smooth and bounded.

The two main properties which are related to the core of our paper are the following:

Theorem A3.1. *Assume φ is a function in VMO($\partial\Omega$). Then its harmonic extension u belongs to VMO$_\varphi(\Omega)$.*

Theorem A3.2. *Assume $\varphi \in$ VMO$(\partial\Omega, \mathbb{R}^N)$ and $\varphi(x) \in \Sigma$ a.e. on $\partial\Omega$, where Σ is a closed set in $\mathbb{R}^N$. Then, for any $\delta > 0$ there is a neighbourhood U of $\partial\Omega$ in Ω such that*

$$\mathrm{dist}(u(x), \Sigma) \le \delta \qquad \forall x \in U. \tag{A3.1}$$

Remark A3.1. The two theorems above hold in the general setting where Ω is a domain on a manifold; the proofs carry over.

First some notation. Fix a neighbourhood V of $\partial\Omega$ in Ω such that every point $x \in V$ has a unique projection $P(x)$ on $\partial\Omega$. Set

$$d(x) = \operatorname{dist}(x, \partial\Omega).$$

Clearly, there is a constant C such that

$$\text{(A3.2)} \quad C^{-1}(d^2(x) + |P(x) - \xi|^2) \le |x - \xi|^2 \le C(d^2(x) + |P(x) - \xi|^2) \quad \forall x \in V, \quad \forall \xi \in \partial\Omega.$$

Given a function φ defined on $\partial\Omega$, consider (as in §II.3, Example 3), for $x \in V$,

$$\overline{u}(x) = \overline{\varphi}_{d(x)}(P(x)) = \fint_{B_{d(x)}(P(x))} \varphi.$$

The next result provides a useful connection between the harmonic extension u of φ and the function $\overline{u}$; it will allow us to derive, easily, Theorems A3.1 and A3.2 from the corresponding properties of $\overline{u}$.

Lemma A3.1. *There is a constant C such that*

$$\text{(A3.3)} \qquad \|u - \overline{u}\|_{L^\infty(V)} \le C\|\varphi\|_{\mathrm{BMO}(\partial\Omega)}.$$

The proof of Lemma A3.1 relies on the following two lemmas; the first one is a variant of an observation due to C. Fefferman and E. Stein [1]:

Lemma A3.2. *There is a constant C, depending only on n, such that*

$$\text{(A3.4)} \qquad \int_{y \in B_R} \frac{t|\psi(y) - \overline{\psi}_t(a)|}{(t^2 + |a - y|^2)^{n/2}} dy \le C\|\psi\|_{\mathrm{BMO}(B_{2R})}$$

where $B_R = \{y \in \mathbb{R}^{n-1}; |y| \le R\}, a \in B_{R/2}, 0 < t < R/2$ and

$$\overline{\psi}_t(a) = \fint_{B_t(a)} \psi.$$

Lemma A3.3. *Let H be a smooth diffeomorphism from B_R onto a subset of $\partial\Omega$. Then there are constants C and t_0 such that*

$$\text{(A3.5)} \qquad |(\overline{\varphi \circ H})_t(y) - \overline{\varphi}_t(H(y))| \le C\|\varphi\|_{\mathrm{BMO}(\partial\Omega)}$$

for all $\varphi \in BMO(\partial\Omega)$, $|y| \le R/2$ and $0 < t < t_0$.

Assuming Lemmas A3.2 and A3.3 we present the

Proof of Lemma A3.1. We may suppose that $\|\varphi\|_{\mathrm{BMO}(\partial\Omega)} = 1$ and $\int_{\partial\Omega} \varphi = 0$. Let $P(x,\xi)$ be the Poisson kernel so that

$$u(x) = \int_{\partial\Omega} P(x,\xi)\varphi(\xi)d\xi.$$

Recall (see e.g. M. Avellaneda and F. H. Lin [1], Lemma 21) the estimate

$$0 \le P(x,\xi) \le C\frac{\mathrm{dist}(x,\partial\Omega)}{|x-\xi|^n} \quad \forall x \in \Omega, \quad \forall \xi \in \partial\Omega. \tag{A3.6}$$

For every constant c we have

$$|u(x) - c| \le \int_{\partial\Omega} P(x,\xi)|\varphi(\xi) - c|d\xi. \tag{A3.7}$$

We apply (A3.7) with $c = \overline{u}(x) = \overline{\varphi}_{d(x)}(p(x))$ and set

$$t = \mathrm{dist}(x,\partial\Omega) = d(x).$$

From the estimate (A3.6) we obtain

$$|u(x) - \overline{u}(x)| \le Ct\int_{\partial\Omega} \frac{|\varphi(\xi) - \overline{\varphi}_t(P(x))|d\xi}{|x-\xi|^n}. \tag{A3.8}$$

Consider a finite family of smooth maps $H_i : B_{2R} \to \partial\Omega$ such that each H_i is a diffeomorphism (onto its image) and

$$\bigcup_i H_i(B_{R/2}) \quad \text{covers } \partial\Omega.$$

For each $x \in V$ there is some i such that

$$P(x) \in H_i(B_{R/2}). \tag{A3.9}$$

Thus we have

$$|u(x) - \overline{u}(x)| \le Ct\int_{H_i(B_R)} [\ \ ] + Ct\int_{H_i(B_R)^c} [\ \ ] = I_1 + I_2, \tag{A3.10}$$

where

$$[\ \ ] = \frac{|\varphi(\xi) - \overline{\varphi}_t(P(x))|}{|x-\xi|^n}.$$

To estimate I_2 note that, by (A3.2),

$$|x-\xi| \ge C^{-1/2}|P(x) - \xi| \ge \alpha > 0,$$

since $\xi \in H_i(B_R)^c$ and $p(x) \in H_i(B_{R/2})$. Therefore

$$I_2 \le Ct\left(\|\varphi\|_{L^1(\partial\Omega)} + |\overline{\varphi}_t(P(x))|\right) \le C \tag{A3.11}$$

by Lemmas A.1 and B.7 in [BNI]. We recall that Lemma B.7 implies that $\|\overline{\varphi}_t\|_{L^\infty} \le C(1 + |\log t|)$; the proof of this fact uses the John-Nirenberg inequality.

To estimate I_1, use the change of variables $\xi = H_i(y)$, so that by (A3.2),

$$I_1 \le Ct\int_{B_R} \frac{|\varphi(H_i(y)) - \overline{\varphi}_t(P(x))|}{(t^2 + |P(x) - H_i(y)|^2)^{n/2}}dy,$$

and thus

$$I_1 \le Ct\int_{B_R} \frac{|\psi(y) - \overline{\varphi}_t(P(x))|}{(t^2 + |a-y|^2)^{n/2}}dy, \tag{A3.12}$$

where $\psi = \varphi \circ H_i$ and $a = H_i^{-1}(P(x))$.

From (A3.12) we deduce that

$$\begin{aligned} I_1 &\le Ct\int_{B_R} \frac{|\psi(y) - \overline{\psi}_t(a)| + |\overline{\psi}_t(a) - \overline{\varphi}_t(P(x))|}{(t^2 + |a-y|^2)^{n/2}}dy \\ &\le C\|\psi\|_{\mathrm{BMO}(B_R)} + C\|\varphi\|_{\mathrm{BMO}(\partial\Omega)}, \end{aligned} \tag{A3.13}$$

by Lemmas A3.2 and A3.3. Note that $|a| \le R/2$ by (A3.9), and that we may choose a neighbourhood V' of $\partial\Omega$, $V' \subset V$, so that, for every $x \in V'$, $t = d(x) \le \min\{t_0, R/2\}$; here t_0 is defined in Lemma A3.3.

In view of Lemma 2 in §II.1 we obtain

$$I_1 \le C. \tag{A3.14}$$

Combining (A3.11) and (A3.14) we conclude that

$$|u(x) - \overline{u}(x)| \le C \qquad \forall x \in V'.$$

If $x \in V\backslash V'$ we have

$$|u(x)| \le C\|\varphi\|_{L^1(\partial\Omega)} \le C$$

(since u is harmonic), and clearly

$$|\overline{u}(x)| \le C\|\varphi\|_{L^1(\partial\Omega)} \le C.$$

Hence, in all cases,

$$|u(x) - \overline{u}(x)| \leq C \qquad \forall x \in V.$$

$\square$

We now return to the

Proof of Lemma A3.2. By scaling we may assume that $R = 1$. We may also suppose that

$$\|\psi\|_{\mathrm{BMO}(B_{2R})} = 1 \quad \text{and that} \int_{B_R} \psi = 0.$$

Consider the sequence of balls in $\mathbb{R}^{n-1}$,

$$B_R = B_{2^k t}(a) \qquad k = 0, 1, 2, \dots$$

and set

$$A_k = B_k \backslash B_{k-1} \qquad k = 1, 2, 3, \dots$$

Let k_0 be the largest integer k such that

$$2^k t + |a| \leq 1,$$

and set

$$b_k = \fint_{B_k} \psi \qquad \text{for } 0 \leq k \leq k_0.$$

Note that

$$b_0 = \overline{\psi}_t(a).$$

By Lemma A.4 in [BNI]—recall our definition of BMO(B_{2R})— we have

$$|b_{k+1} - b_k| \leq C \quad \text{for } 0 \leq k \leq k_0 - 1.$$

Adding these inequalities yields

$$|b_k - b_0| \leq Ck \quad \text{for } 0 \leq k \leq k_0. \tag{A3.15}$$

On the other hand, note that

$$\frac{1}{4} \leq 2^{k_0} t \leq 1. \tag{A3.16}$$

By Lemma A.4 in [BNI] we have

$$|b_{k_0} - \fint_{|y| \leq 1} \psi| \leq C$$

and thus

$$|b_{k_0}| \le C$$

since $\int_{|y|\le 1} \psi = 0$. It follows from (A3.15) and (A3.16) that

$$|b_0| \le Ck_0 \le C\log(1/t). \tag{A3.17}$$

We have to estimate

$$I = t\int_{|y|\le 1} \frac{|\psi(y)-b_0|}{(t^2+|a-y|^2)^{n/2}}dy.$$

We write

$$I = I_1 + I_2 + I_3$$

where

$$I_1 = t\int_{B_0} \frac{|\psi(y)-b_0|}{(t^2+|a-y|^2)^{n/2}},$$

$$I_2 = \sum_{k=1}^{k_0} t\int_{A_k} \frac{|\psi(y)-b_0|}{(t^2+|a-y|^2)^{n/2}} = \sum_{k=1}^{k_0} J_k$$

and

$$I_3 = t\int_{\substack{|y|\le 1\\ y\notin B_{k_0}}} \frac{|\psi(y)-b_0|}{(t^2+|a-y|^2)^{n/2}}.$$

Clearly

$$I_1 \le \frac{1}{t^{n-1}}\int_{B_0} |\psi(y)-b_0| \le C\fint_{B_0} |\psi(y)-b_0| \le C. \tag{A3.18}$$

Next, we estimate I_3; observe that if $y\notin B_{k_0}$, $|a-y|\ge 2^{k_0}t \ge 1/4$, and thus

$$I_3 \le Ct\int_{|y|\le 1} |\psi(y)-b_0| \le Ct\left(\|\psi\|_{\mathrm{BMO}(B_{2R})} + |b_0|\right).$$

Therefore, by (A3.17),

$$I_3 \le Ct\,(1+\log(1/t)) \le C. \tag{A3.19}$$

Finally, we estimate J_k. On A_k we have $|a-y| \geq 2^{(k-1)}t$ and thus

$$J_k \leq \frac{t}{(t^2+2^{2(k-1)}t^2)^{n/2}} \int_{B_k} |\psi(y)-b_0|.$$

Consequently

$$\begin{aligned} J_k &\leq \frac{1}{t^{n-1}2^{n(k-1)}} \int_{B_k} (|\psi - b_k| + |b_k - b_0|) \\ &\leq \frac{C}{t^{n-1}2^{nk}} |B_k| \left(\fint_{B_k} |\psi - b_k| + |b_k - b_0| \right) \\ &\leq \frac{C}{2^k}(1+k) \quad \text{by (A3.15).} \end{aligned}$$

It follows that

$$I_2 = \sum_{k=1}^{k_0} J_k \leq C. \tag{A3.20}$$

Combining (A3.18) - (A3.20) we obtain the desired estimate (A3.4).

□

Next, we give the

Proof of Lemma A3.3. For any constant c we have

$$\begin{aligned} \Big|\fint_{B_t(y)} \varphi(H(\xi))d\xi - c\Big| &\leq \fint_{B_t(y)} |\varphi(H(\xi)) - c| d\xi \\ &\leq \frac{C}{|B_t(y)|} \int_{H(B_t(y))} |\varphi(\eta) - c| d\eta. \end{aligned}$$

Choosing

$$c = \fint_{H(B_t(y))} \varphi(\zeta) d\zeta,$$

we find

$$\begin{aligned} &\Big|\fint_{B_t(y)} \varphi(H(\xi))d\xi - \fint_{H(B_t(y))} \varphi(\zeta)d\zeta\Big| \leq \\ &\leq \frac{C}{|B_t(y)|\,|H(B_t(y))|} \int\int_{H(B_t(y))} |\varphi(\eta) - \varphi(\zeta)| d\eta d\zeta. \end{aligned} \tag{A3.21}$$

There are constants $t_0 > 0$ and $K > 1$ such that

$$B_{t/K}(H(y)) \subset H(B_t(y)) \subset B_{tK}(H(y)) \quad \forall t < t_0,\ |y| \leq R/2. \tag{A3.22}$$

We deduce from (A3.21) and (A3.22) that

$$\tag{A3.23} \begin{aligned} \Big|\fint_{B_t(y)} \varphi(H(\xi))d\xi - \fint_{H(B_t(y))} \varphi(\zeta)d\zeta\Big| &\le C \fint\fint_{B_{tK}(H(y))} |\varphi(\eta)-\varphi(\zeta)|d\eta d\zeta \\ &\le C\|\varphi\|_{\mathrm{BMO}}. \end{aligned}$$

On the other hand, by Lemma A.4 in [BNI], we have

$$\tag{A3.24} \Big|\fint_{H(B_t(y))} \varphi - \fint_{B_{tK}(H(y))} \varphi\Big| \le C\|\varphi\|_{\mathrm{BMO}}$$

and

$$\tag{A3.25} \Big|\fint_{B_t(H(y))} \varphi - \fint_{B_{tK}(H(y))} \varphi\Big| \le C\|\varphi\|_{\mathrm{BMO}}.$$

Combining (A3.23), (A3.24) and (A3.25) we are led to the desired conclusion

$$\Big|\fint_{B_t(y)} (\varphi\circ H) - \fint_{B_t(H(y))} \varphi\Big| \le C\|\varphi\|_{\mathrm{BMO}}.$$

□

Finally, we turn to the

Proof of Theorem A3.1. Observe first that if $\varphi \in \mathrm{BMO}(\partial\Omega)$, then its harmonic extension u belongs to $\mathrm{BMO}(\Omega)$ and

$$\tag{A3.26} \|u\|_{\mathrm{BMO}(\Omega)} \le C\|\varphi\|_{\mathrm{BMO}(\partial\Omega)}.$$

In proving (A3.26) we may assume, as usual, that $\|\varphi\|_{\mathrm{BMO}(\partial\Omega)} = 1$ and that $\int_{\partial\Omega} \varphi = 0$. Let ζ be a smooth cutoff function with support in a small neighbourhood of $\partial\Omega$ and such that $\zeta \equiv 1$ near $\partial\Omega$. By Lemma A3.1 we have

$$\|\zeta u - \zeta\overline{u}\|_{L^\infty(\Omega)} \le C$$

and, in particular,

$$\|\zeta u - \zeta\overline{u}\|_{\mathrm{BMO}(\Omega)} \le C.$$

On the other hand, by (3.8) in Lemma 7 of §II.3 we have

$$\|\zeta\overline{u}\|_{\mathrm{BMO}(\Omega)} \le C$$

and therefore

$$\|\zeta u\|_{\mathrm{BMO}(\Omega)} \leq C.$$

Since we clearly have

$$\|(1-\zeta)u\|_{L^\infty(\Omega)} \leq C,$$

it follows that (A3.26) holds. The fact that $u \in \mathrm{VMO}(\Omega)$ whenever $\varphi \in \mathrm{VMO}(\partial\Omega)$ is derived from (A3.26) by a standard density argument.

Next we prove that if $\varphi \in \mathrm{VMO}(\partial\Omega)$, then $u \in \mathrm{VMO}_\varphi(\Omega)$. Since we already know that $\zeta\overline{u} \in \mathrm{VMO}_\varphi(\Omega)$ (see Lemma 7 in §II.3) it suffices to verify that

$$(u - \zeta\overline{u}) \in \mathrm{VMO}_0(\Omega).$$

Given $\delta > 0$ we have to check (see Theorem 2 in §II.3) that

$$\fint_{B_\varepsilon(x)} |u - \overline{u}| < \delta \quad \text{for } \varepsilon = \frac{1}{2}d(x) \text{ small.} \tag{A3.27}$$

Let ψ be a continuous function on $\partial\Omega$. Let v be its harmonic extension in Ω and let $\overline{v}(x) = \overline{\psi}_{d(x)}(P(x))$ for $x \in V$.

Write

$$u - \overline{u} = \big[(u - v) - (\overline{u} - \overline{v})\big] + (v - \overline{v}).$$

Application of Lemma A3.1 to $(\varphi - \psi)$ yields

$$\|u - \overline{u}\|_{L^\infty(B_\varepsilon(x))} \leq C\|\varphi - \psi\|_{\mathrm{BMO}(\partial\Omega)} + \|v - \overline{v}\|_{L^\infty(B_\varepsilon(x))} \tag{A3.28}$$

provided $\varepsilon < \varepsilon_0$ with ε_0 sufficiently small such that $B_{\varepsilon_0}(x) \subset V$.

Choose $\psi \in C^0(\partial\Omega)$ with

$$C\|\varphi - \psi\|_{\mathrm{BMO}(\partial\Omega)} < \delta/2 \tag{A3.29}$$

and then choose $\varepsilon_1 < \varepsilon_0$ sufficiently small so that

$$\|v - \overline{v}\|_{L^\infty(B_\varepsilon(x))} < \delta/2 \quad \text{for } \varepsilon < \varepsilon_1. \tag{A3.30}$$

This is clearly possible since v and $\overline{v}$ are continuous on $\overline{\Omega}$ and $v = \overline{v} = \psi$ on $\partial\Omega$. Together, (A3.28) - (A3.30) yield

$$\|u - \overline{u}\|_{L^\infty(B_\varepsilon(x))} < \delta \quad \text{for } \varepsilon = \frac{1}{2}d(x) < \varepsilon_1.$$

The desired conclusion (A3.27) follows. □

A similar procedure furnishes the

Proof of Theorem A3.2. As in the proof of Theorem A3.1 we write

$$u = \overline{u} + [(u - v) - (\overline{u} - \overline{v})] + (v - \overline{v}). \tag{A3.31}$$

Recall that, by Lemma A3.1,

$$\|(u - v) - (\overline{u} - \overline{v})\|_{L^\infty(V)} \leq C\|\varphi - \psi\|_{\mathrm{BMO}(\partial\Omega)}. \tag{A3.32}$$

Fix $\varepsilon_0 > 0$ such that $d(x) < \varepsilon_0$ implies $x \in V$. Choose $\psi \in C^0(\partial\Omega)$ such that

$$C\|\varphi - \psi\|_{\mathrm{BMO}(\partial\Omega)} < \delta/3. \tag{A3.33}$$

Next, let $\varepsilon_1 < \varepsilon_0$ be so small that

$$|v(x) - \overline{v}(x)| < \delta/3 \quad \text{if } d(x) < \varepsilon_1.$$

Finally, we may find $\varepsilon_2 < \varepsilon_1$ such that

$$\operatorname{dist}(\overline{u}(x), \Sigma) < \delta \quad \text{if } d(x) < \varepsilon_2; \tag{A3.34}$$

this can be achieved since $\varphi \in \mathrm{VMO}(\partial\Omega)$ (see (7) and Remark 3 in [BNI]).

Combining (A3.31) - (A3.34) we obtain the desired estimate

$$\operatorname{dist}(u(x), \Sigma) < \delta \qquad \text{if } d(x) < \varepsilon_2.$$

□

Remark A3.2. Theorem A3.2 asserts that if φ takes its values into some closed set Σ, the harmonic extension u has the properties that, close to $\partial\Omega$, the values of u lie near Σ. This need not be true for arbitrary extensions of φ in Sobolev spaces. For example, with $n = 2$ and $\varphi \equiv 0$: If $u \in H_0^1(\Omega)$, near the boundary, u need not be small.

Here is such a function u defined on $\Omega = \mathbb{R}^2_+ = \{(x_1, x_2), x_2 > 0\}$. Consider any decreasing sequence (ε_j) of positive numbers such that

$$\sum_{j=1}^{\infty} \varepsilon_j < \infty$$

and

$$\sum_{j=1}^{\infty} \frac{1}{|\log \varepsilon_j|} < \infty;$$

for example $\varepsilon_j = e^{-j^2}$ does it. Let (a_k) be the sequence of points on the x_2-axis defined by

$$a_k = (0, 2\sum_{j=k}^{\infty} \varepsilon_j).$$

Set

$$u(x) = \sum_{j=1}^{\infty} \psi_j \left(|x - a_j|\right)$$

where $\psi_j(r) = \log|\log r| - \log|\log \varepsilon_j|$ if $r < \varepsilon_j$ and $\psi_j(r) = 0$ if $r \geq \varepsilon_j$. Note that $\operatorname{supp} u$ is contained in the set

$$\bigcup_{j=1}^{\infty} B(a_j, \varepsilon_j),$$

and that $u \in H^1(\mathbb{R}^2)$ since

$$\int_{B(a_j,\varepsilon_j)} |\nabla u|^2 = 2\pi \int_0^{\varepsilon_j} \frac{rdr}{r^2 |\log r|^2} = \frac{2\pi}{|\log \varepsilon_j|}.$$

Clearly, $u(a_k) = +\infty \quad \forall k$ and $a_k \to 0$ as $k \to \infty$. □

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