1. Introduction

A classical result about composition in Sobolev spaces asserts that if \( u \in W^{k,p}(\Omega) \cap L^\infty(\Omega) \) and \( \Phi \in C^k(\mathbb{R}) \), then \( \Phi \circ u \in W^{k,p}(\Omega) \). Here \( \Omega \) denotes a smooth bounded domain in \( \mathbb{R}^N \), \( k \geq 1 \) is an integer and \( 1 \leq p < \infty \). This result was first proved in [13] with the help of the Gagliardo-Nirenberg inequality [14]. In particular if \( u \in W^{k,p}(\Omega) \) with \( kp > N \) and \( \Phi \in C^k(\mathbb{R}) \) then \( \Phi \circ u \in W^{k,p} \) since \( W^{k,p} \subset L^\infty \) by the Sobolev embedding theorem. When \( kp = N \) the situation is more delicate since \( W^{k,p} \) is not contained in \( L^\infty \). However the following result still holds (see [2],[3])

**Theorem 1.** Assume \( u \in W^{k,p}(\Omega) \) where \( k \geq 1 \) is an integer, \( 1 \leq p < \infty \), and

\[
(1) \quad kp = N.
\]

Let \( \Phi \in C^k(\mathbb{R}) \) with

\[
(2) \quad D^j\Phi \in L^\infty(\mathbb{R}) \quad \forall j \leq k.
\]

Then

\[
\Phi \circ u \in W^{k,p}(\Omega)
\]

The proof is based on the following

**Lemma 1.** Assume \( u \in W^{k,p}(\Omega) \cap W^{1,kp}(\Omega) \) where \( k \geq 1 \) is an integer and \( 1 \leq p < \infty \). Assume \( \Phi \in C^k(\mathbb{R}) \) satisfies (2). Then

\[
\Phi \circ u \in W^{k,p}(\Omega).
\]
Proof of Theorem 1. Since \( u \in W^{k,p} \) we have
\[ Du \in W^{k-1,p} \subset L^q \]
by the Sobolev embedding with
\[ \frac{1}{q} = \frac{1}{p} - \frac{k-1}{N}. \]
Applying assumption (1) we find \( q = N = kp \) and thus \( u \in W^{1,kp} \). We deduce from Lemma 1 that \( \Phi \circ u \in W^{k,p} \).

Proof of Lemma 1. Note that if \( u \in W^{k,p} \cap L^\infty \) with \( k \geq 1 \) integer and \( 1 \leq p < \infty \) then
\( u \in W^{1,kp} \) by the Gagliardo - Nirenberg inequality [14]. Thus, Lemma 1 is a generalization of the standard result about composition. In fact, it is proved exactly in the same way as in the standard case (when \( u \in W^{k,p} \cap L^\infty \)). When \( k = 2 \) the conclusion is trivial.

Assume, for example that, \( k = 3 \), then
\[ W^{3,p} \cap W^{1,3p} \subset W^{2,3p/2} \]
by the Gagliardo - Nirenberg inequality. Then
\[ D^3(\Phi \circ u) = \Phi'(u)D^3u + 3\Phi''(u)D^2uD + \Phi'''(u)(Du)^3, \]
and thus \( \Phi \circ u \in W^{3,p} \) since
\[ \int |D^2u|^p |Du|^p \leq \left( \int |D^2u|^{3p/2} \right)^{2/3} \left( \int |Du|^{3p} \right)^{1/3} \leq C\|u\|_{W^{3,p}}^{p/2}\|u\|_{W^{1,3p}}^{3p/2}. \]
A simular argument holds for any \( k \geq 4 \).

Starting in the mid-60’s a number of authors considered composition in various classes of “Sobolev spaces” \( W^{s,p} \), where \( s > 0 \) is a real number and \( 1 \leq p < \infty \). The most commonly used are the Bessel potential spaces \( L^{s,p}(\mathbb{R}^N) = \{ f = G_s * g; g \in L^p(\mathbb{R}^N) \} \) where \( G_s = (1 + |\xi|^2)^{-s/2} \) and the Besov spaces \( B^{s,p}_{r}(\mathbb{R}^N) \) (who’s definition is recalled below when \( s \) is not an integer). It is well-known (see e.g. [1], [19] and [20]) that if \( k \) is an integer, \( L^{k,p} \) coincides with the standard Sobolev space \( W^{k,p} \); also if \( p = 2 \), the Bessel potential spaces \( L^{s,2} \) and the Besov spaces \( B^{s,2}_{2} \) coincide for every \( s \) non-integer and they are usually denoted by \( H^s \). When \( p \neq 2 \) the spaces \( L^{s,p} \) and \( B^{s,p}_{r} \) are distinct.

The first result about composition in fractional Sobolev spaces seems to be due to Mizohata [12] for \( H^s, s > N/2 \). In 1970 Peetre [15] considered \( B^{s,p}_{r} \cap L^\infty \) using interpolation...
techniques; a very simple direct argument for the same class, $B_p^{s,p} \cap L^\infty$, was given by M. Escobedo [10] (see the proof of Lemma 2 below).

Starting in 1980 techniques of dyadic analysis and Littlewood-Paley decomposition à la Bony [5] were introduced. For example, Y. Meyer [11] considered composition in $L^{s,p}$ for $sp > N$; see also [16],[4],[9] for $H^s$ with $s > N/2$ or for $H^s \cap L^\infty$, any $s > 0$. We refer to [17],[6],[7],[18] and their bibliographies for other directions of research concerning composition in Sobolev spaces.

In what follow we denote by $W^{s,p}(\Omega)$ the restriction of $B_p^{s,p}(\mathbb{R}^N)$ to $\Omega$ when $s$ is not an integer. Our main result is the following

**Theorem 2.** Assume $u \in W^{s,p}(\Omega)$ where $s > 1$ is a real number, $1 < p < \infty$, and

\[
sp = N.
\]

Let $\Phi \in C^k(\mathbb{R})$, where $k = \lfloor s \rfloor + 1$, be such that

\[
D^j\Phi \in L^\infty(\mathbb{R}) \quad \forall j \leq k.
\]

Then

$\Phi \circ u \in W^{s,p}(\Omega)$.

The proof of Theorem 2 relies on a variant of Lemma 1 for fractional Sobolev spaces.

**Lemma 2.** Let $u \in W^{s,p}(\Omega)$, where $s > 1$ is a real number and $1 < p < \infty$. Assume, in addition, that $u \in W^{\sigma,q}$ for some $\sigma \in (0,1)$ with

\[
q = sp/\sigma.
\]

Let $\Phi \in C^k(\mathbb{R})$, where $k = \lfloor s \rfloor + 1$, be such that (4) holds. Then

$\Phi \circ u \in W^{s,p}$

**Proof of Theorem 2.** By the Sobolev embedding theorem we have

$W^{s,p} \subset W^{r,q}$

with $r < s$ and

\[
\frac{1}{q} = \frac{1}{p} - \frac{(s - r)}{N}.
\]

In view of assumption (3) we find

$q = N/r$. 
In particular,

\[ u \in W^{\sigma,q} \]

for all \( \sigma \in (0,1) \) with

\[ q = \frac{N}{\sigma} = \frac{sp}{\sigma}. \]

Thus we may apply Lemma 2 and conclude that \( \Phi \circ u \in W^{s,p} \).

**Remark 1.** Theorem 2 is known to be true when the Sobolev spaces \( W^{s,p} \) are replaced by the Bessel potential spaces \( L^{s,p} \) with \( sp = N \); see D. Adams and M. Frazier [3]. Even though the two results are closely related it does not seem possible to deduce one from the other. Their argument relies on a variant of Lemma 2 for Bessel potential spaces:

Let \( u \in L^{s,p} \cap L^{1,sp} \) where \( s > 1 \) is a real number and \( 1 < p < \infty \). Let \( \Phi \) be as in Lemma 2. Then \( \Phi \circ u \in L^{s,p} \).

**Remark 2.** The assumption in Lemma 2, \( u \in W^{s,p} \cap W^{\sigma,q} \), with \( q = sp/\sigma \) for some \( \sigma \in (0,1) \), is **weaker** than the assumption \( u \in W^{s,p} \cap L^{\infty} \) but it is **stronger** than the assumption \( u \in W^{1,sp} \); this is a consequence of Gagliardo - Nirenberg type inequalities (see e.g. the proof of Lemma D.1 in the Appendix D of [8]). It is therefore natural to raise the following:

**Open Problem.** Is the conclusion of Lemma 2 valid if one assumes only \( u \in W^{s,p} \cap W^{1,sp} \) where \( s > 1 \) is a (non-integer) real number?

Before giving the proof of Lemma 2 we recall some properties of \( W^{s,p} \) when \( s \) is not an integer.

When \( 0 < \sigma < 1 \) and \( 1 < p < \infty \) the standard definition of \( W^{\sigma,p} \) is

\[ W^{\sigma,p}(\Omega) = \{ f \in L^p(\Omega); \quad \int \int \frac{|f(x) - f(y)|^p}{|x-y|^{N+\sigma p}} dxdy < \infty \}. \]

If \( s > 1 \) is not an integer write \( s = [s] + \sigma \) where \([s]\) denotes the integer part of \( s \) and \( 0 < \sigma < 1 \). Then

\[ W^{s,p}(\Omega) = \{ f \in W^{[s],p}(\Omega), D^\alpha f \in W^{\sigma,p} \quad \text{for} \quad |\alpha| = [s] \}. \]

There is a very useful characterization of \( W^{s,p} \) in terms of finite differences (see Triebel [20], p.110). Here it is more convenient to work with functions defined on all of \( \mathbb{R}^N \) and to consider their restrictions to \( \Omega \). Set

\[ (\delta_h u)(x) = u(x + h) - u(x), \quad h \in \mathbb{R}^N, \]
so that
\[(\delta_h^2 u)(x) = u(x + 2h) - 2u(x + h) + u(x), \text{ etc}..\]

Given \(s > 0\) not integer, fix any integer \(M > s\). Then
\[
W^{s,p} = \{ f \in L^p; \int \int |\delta_h^M f(x)|^p |h|^{N+sp}dxdh < \infty \}.
\]

**Proof of Lemma 2.** It suffices to consider the case where \(s\) is not an integer. For simplicity we treat just the case where \(1 < s < 2\). The same argument extends to general \(s > 2\), \(s\) noninteger, using the same type of computations as in Escobedo [10].

The key observation is that \(\delta_h^2 (\Phi \circ u)\) can be expressed in terms of \(\delta_h^2 u\) and \(\delta_h u\). This is the purpose of our next computation.

Set
\[
X = u(x + 2h) \\
Y = u(x + h) \\
Z = u(x).
\]

Since \(\Phi'' \in L^\infty(\mathbb{R})\) we have
\[
\Phi(X) - \Phi(Y) = \Phi'(Y)(X - Y) + 0(|X - Y|^2)
\]
and since \(\Phi' \in L^\infty(\mathbb{R})\) we also have
\[
\Phi(X) - \Phi(Y) = \Phi'(Y)(X - Y) + 0(|X - Y|).
\]
Combining (6) and (7) we find
\[
\Phi(X) - \Phi(Y) = \Phi'(Y)(X - Y) + 0(|X - Y|^a)
\]
for any \(1 \leq a \leq 2\) (we will choose a specific value of \(a\) later) Similarly
\[
\Phi(Z) - \Phi(Y) = \Phi'(Y)(Z - Y) + 0(|Z - Y|^a)
\]
Since
\[
\delta_h^2(\Phi \circ u)(x) = (\Phi(X) - \Phi(Y)) + (\Phi(Z) - \Phi(Y)),
\]
one finds
\[
|\delta_h^2(\Phi \circ u)(x)| \leq C(|\delta_h^2 u(x)| + |\delta_h u(x + h)|^a + |\delta_h u(x)|^a).
\]
This yields

\[ \int \int \frac{|\delta^2_h(\Phi \circ u)(x)|^p}{|h|^{N+sp}} \, dx \, dh \leq C \int \int \frac{|\delta^2_h u(x)|^p}{|h|^{N+sp}} \, dx \, dh + C \int \int \frac{|\delta_h u(x)|^{ap}}{|h|^{N+sp}} \, dx \, dh. \]

The first integral on the right-hand side of (9) is finite since \( u \in W^{s,p} \). To handle the second integral we argue as follows. From the assumption \( u \in W^{s,p} \cap W^{\sigma,q} \) with \( \sigma \in (0,1) \) and \( q \) given by (5) we know that

\[ \int \int \frac{|\delta^2_h u(x)|^p}{|h|^{N+sp}} \, dx \, dh < \infty \quad \text{and} \quad \int \int \frac{|\delta^2_h u(x)|^q}{|h|^{N+sp}} \, dx \, dh < \infty. \]

From (10) and Hölder’s inequality we derive that

\[ \int \int \frac{|\delta^2_h u(x)|^r}{|h|^{N+sp}} \, dx \, dh < \infty \]

for all \( r \in [p,q] \), i.e., \( u \in W^{r,r} \) with \( r = \frac{sp}{t} \). We now choose

\[ a = \min\{2, \frac{s}{\sigma}\}, \quad \text{so that} \quad a \in [1,2] \]

and \( r = ap \in [p,q] \). It follows that

\[ \int \int \frac{|\delta_h u(x)|^{ap}}{|h|^{N+sp}} \, dx \, dh < \infty, \]

which is the desired inequality.

**Remark 3.** There could be another natural proof of Theorem 2 by induction on \([s]\). One might attempt to prove that

\[ D(\Phi \circ u) = \Phi'(u)Du \in W^{s-1,p}. \]

Note that \( u \in W^{(s-1),N/(s-1)} \) and thus (by induction) we would have \( \Phi'(u) \in W^{(s-1),N/(s-1)} \). On the other hand \( Du \in W^{s-1,p} \). In order to conclude we need a lemma about products, but we are not aware of any such tool.

**Remark 4.** When \( s \) (or equivalently \( p \)) is a **rational** number, and \( \Phi \in C^\infty \) with \( D^j \Phi \in L^\infty \) \( \forall j \), there is a simple proof of Theorem 2 based on trace theory and Theorem 1. Assume for simplicity that \( \Omega = \mathbb{R}^N \). Suppose that \( s \) is not an integer, but that \( s_1 = s + 1/p \) is an integer. Then \( u \) is the trace of some function \( u_1 \in W^{s_1,p}(\mathbb{R}^{N+1}) \). Then \( s_1 p = N + 1 \) and by Theorem 1 we deduce that \( \Phi \circ u_1 \in W^{s_1,p}(\mathbb{R}^{N+1}) \). Taking traces we find \( \Phi \circ u \in W^{s,p}(\mathbb{R}^N) \). If \( s_1 \) is not an integer we keep extending \( u_1 \) to higher dimensions and stop at the first integer \( k \) such that \( s_k = s + k/p \) is an integer \( ( \text{this is possible since} \ p \ \text{is rational and} \ s + k/p = (N + k)/p \ \text{becomes an integer for some integer} \ k ) \). We have an extension \( u_k \in W^{s_k,p}(\mathbb{R}^{N+k}) \) of \( u \). Then \( \Phi \circ u_k \in W^{s_k,p}(\mathbb{R}^{N+k}) \) by Theorem 1. Taking back traces yields \( u \in W^{s,p} \).
References


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