Soliton solutions for quasilinear Schrödinger equations: the critical exponential case

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Abstract

Quasilinear elliptic equations in $\mathbb{R}^2$ of second order with critical exponential growth are considered. By using a change of variable, the quasilinear equations are reduced to semilinear equations, whose respective associated functionals are well defined in $H^1(\mathbb{R}^2)$ and satisfy the geometric hypotheses of the mountain pass theorem. Using this fact, we obtain a Cerami sequence converging weakly to a solution v. In the proof that v is nontrivial, the main tool is the concentration-compactness principle [14] combined with test functions connected with optimal Trudinger-Moser inequality.

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1 Introduction

Recently, there has been growing interest in the study of quasilinear elliptic equations of the form

$$-\Delta u + V(x)u - (\Delta(|u|^2))u = h(u) \quad \text{in } \mathbb{R}^N. \tag{1.1}$$

These equations are related to existence of standing wave solutions for quasilinear Schrödinger equations of the form

$$iz_t = -\Delta z + V(x)z - h(|z|^2)z - \kappa\Delta g(|z|^2)g'(|z|^2)z \quad \text{in } \mathbb{R}^N, \tag{1.2}$$

where V is a given potential, κ is a real constant, and g and h are real functions. The related semilinear equations for $\kappa = 0$ have been intensively studied (see e.g. [2], [6], [7], [10], [11], [12], [19], [22], as well as their references). Quasilinear equations such as (1.1) have been accepted as a models of several physical phenomena corresponding to various types of g. We refer the reader to the Introduction in [15] and the references therein for a discussion on the subject. Recent mathematical studies have focused on the existence of solutions for (1.1) with $h(s) = |s|^{p-1}s$, with $4 \leq p+1 < 4N/(N-2)$, $N \geq 3$, for example, in [15], [16], and [18]. The existence of a positive ground state solution has been proved by Poppenberg, Schmitt and Wang [18] and Liu and Wang [16] by using a constrained minimization argument, which gives a solution of (1.1) with an unknown Lagrange multiplier λ in front of the nonlinear term. In [15], by a change of

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variable the quasilinear problem was reduced to a semilinear one and an Orlicz space framework was used to prove the existence of a positive solution of (1.1) for every positive λ via mountain pass theorem. In [8], Colin and Jeanjean also made use a change of variable in order to reduce the equation (1.1) to semilinear one. By using the Sobolev space $H^1(\mathbb{R}^N)$, they proved the existence of solutions from classical results given by Berestycki and Lions [6] when $N = 1$ or $N \geq 3$, and Berestycki, Gallouët and Kavian [5] when $N = 2$.

Although considerable research has been devoted to the case $N \geq 3$ rather less attention has been paid to the case $N = 2$. In [15], the authors established the existence of solutions for (1.1) in $\mathbb{R}^2$ when the potential function V is radially symmetric and $h(s) = |s|^{p-1}s$, with $4 \leq p+1 < \infty$. In [8], Colin and Jeanjean treated, among other situations, the case where h satisfies the assumption: for any $\alpha > 0$ there exists positive constant C_α such that

$$|h(s)| \leq C_\alpha e^{\alpha s^2} \quad \forall s \geq 0. \tag{1.3}$$

In the literature [1, 9, 11, 23], the assumption (1.3) says that h has subcritical growth. We recall that h satisfies the critical growth condition if there exists $\alpha_0 > 0$ such that

$$\lim_{s\to\infty} \frac{|h(s)|}{\exp(\alpha s^2)} = \begin{cases} 0 & \forall\, \alpha > \alpha_0, \\ +\infty & \forall\, \alpha < \alpha_0. \end{cases}$$

We note that such notion is motivated by Trudinger-Moser estimates [17, 24] which provide

$$\exp(\alpha|u|^2) \in L^1(\Omega),\ \forall\, u \in H_0^1(\Omega),\ \forall\, \alpha > 0, \tag{1.4}$$

and

$$\sup_{\|u\|_{H_0^1} \leq 1} \int_\Omega \exp(\alpha|u|^2)\, dx \leq C,\ \forall\, \alpha \leq 4\pi, \tag{1.5}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded smooth domain. Subsequently, Cao [7] proved a version of Trudinger-Moser inequality in whole space, which was improved by do Ó [11], namely,

$$\exp(\alpha|u|^2) - 1 \in L^1(\mathbb{R}^2),\ \forall\, u \in H^1(\mathbb{R}^2),\ \forall\, \alpha > 0. \tag{1.6}$$

Moreover, if $\alpha < 4\pi$ and $|u|_{L^2(\mathbb{R}^2)} \leq C$, there exists a constant $C_2 = C_2(C, \alpha)$ such that

$$\sup_{\|\nabla u\|_{L^2(\mathbb{R}^2)} \leq 1} \int_{\mathbb{R}^2} (\exp(\alpha|u|^2) - 1)\, dx \leq C_2. \tag{1.7}$$

The main purpose of the present paper is to obtain standing wave solutions for quasilinear Schrödinger type problems (1.1) when $N = 2$ and h satisfies the new critical growth condition:

$(c)_{\alpha_0}$ There exists $\alpha_0 > 0$ such that $\displaystyle\lim_{s\to\infty} \frac{|h(s)|}{\exp(\alpha s^4)} = \begin{cases} 0 & \forall\, \alpha > \alpha_0, \\ +\infty & \forall\, \alpha < \alpha_0. \end{cases}$

We believe that the exponential growth above is the critical growth for this kind of problem when $N = 2$, according to the case $N \geq 3$ whose the critical exponent is $22^* = 4N/(N-2)$ (see [15]).

In this article, we study the existence of solutions for (1.1) assuming that $V : \mathbb{R}^2 \to \mathbb{R}$ is a continuous function bounded from below away from zero, that is there exists $V_0 > 0$ such that

(V_1) $$V(x) \geq V_0 > 0,\quad \forall\, x \in \mathbb{R}^2,$$

and satisfying the asymptotic condition

(V_2) $$V(x) \leq \lim_{|x|\to\infty} V(x) \doteq V_\infty < \infty,$$

with $V(x) \not\equiv V_\infty$, and $h \in C(\mathbb{R}, \mathbb{R})$ satisfies

(h_1) $\lim_{s \to 0} \frac{h(s)}{s} = 0.$

(h_2) There exists $\mu > 4$ such that $0 < \mu H(s) \leq h(s)s$, for all $s > 0$, with $H(s) = \int_0^s h(t)\, dt$.

(h_3) There exists $\beta_0 > 0$ such that

$$\liminf_{s \to +\infty} \frac{sh(s)}{\exp(\alpha_0 s^4)} \geq \beta_0 > 0,$$

where α_0 is given by condition $(c)_{\alpha_0}$.

Our main result is:

Theorem 1.1 *Suppose $V(x)$ verifies $(V_1) - (V_2)$ and $h(s)$ satisfies $(h_1) - (h_3)$ and $(c)_{\alpha_0}$. Then problem (1.1), with $N = 2$, possesses a positive solution.*

Remark 1.2 *We observe that typical and motivating examples for the study of problem (1.1) are given in the following problems, where the nonlinearities satisfy the assumptions $(h_1) - (h_3)$ and $(c)_{\alpha_0}$ with $\alpha_0 = 4\pi$:*

$$-\Delta u + V(x)u - (\Delta(|u|^2))u = \exp(4\pi u^4) - 1 \quad \text{in } \mathbb{R}^2$$

and

$$-\Delta u + V(x)u - (\Delta(|u|^2))u = h(u) \quad \text{in } \mathbb{R}^2,$$

where the nonlinear term is given by $h(u) = H'(u)$ and $H(u) := u^7 \exp(4\pi u^4)$.

In order to prove Theorem 1.1, motivated by the argument used in [8] and [15], we also use a change of variable to reformulate the problem obtaining a semilinear problem which has an associated functional well defined in the Sobolev space $H^1(\mathbb{R}^2)$ and satisfies the geometric hypotheses of the mountain pass theorem (see [3]). Using this fact, we obtain a Cerami sequence converging weakly to a solution v. In order to prove that v is nontrivial, we combine Lions's compactness lemma with test functions connected with optimal Trudinger-Moser inequality to establish that the Cerami sequence has a non-vanishing behavior. Finally, arguing by contradiction that $v = 0$, a translated Cerami sequence converges to a nonzero critical point of an associated *functional at infinity*. Then, this critical point is used to construct a path related to mountain pass theorem to find a contradiction with definition the mountain pass critical value. Since we deal with exponential case, some difficulties appear mainly due to the lack of homogeneity of the nonlinearity. In addition, in the critical exponential case, the Trudinger-Moser inequality has a restricted use.

Notation: In the rest of the paper we will make use of the following notations: $\int_{\mathbb{R}^2} f(x)dx$ and $\int_D g(x)dx$ will be denoted by $\int f$ and $\int_D g$ respectively; $|.|_p$ denotes the norm in $L^p(D)$ spaces; C denotes (possibly different) positive constants.

The organization of this paper is as follows: In Section 2, we introduce the variational framework associated with (1.1). In Section 3, we verify the geometric conditions of the mountain pass theorem. In Section 4, the existence of the solution for (1.1) is established.

2 Adjust of the variational setting

Observing that $u \equiv 0$ is a (trivial) solution of (1.1), our objective in this article is to apply minimax methods to study the existence on nontrivial solution for (1.1). However, it should be pointed out that we may not apply directly such methods since the natural associated functional, namely

$$J(u) = \frac{1}{2}\int (1+u^2)|\nabla u|^2 + \frac{1}{2}\int V(x)u^2 - \int H(u)$$

where $H(s) = \int_0^s h(t)\,dt$ is not well defined in general, for instance, in $H^1(\mathbb{R}^2)$. To overcome this difficulty, we employ an argument developed by Liu, Wang and Wang in [15] (see also [8, Lemma 2.1]). We make the change of variables $v = f^{-1}(u)$, where f is defined by

$$f'(t) = \frac{1}{\sqrt{1+2f^2(t)}}$$

on $[0,+\infty)$, $f(0) = 0$, and $f(-t) = -f(t)$ on $(-\infty, 0]$. Also f satisfies

(f_0) $|f'(t)| \le 1 \quad \forall\, t \in \mathbb{R}$,

(f_1) $\frac{f(t)}{t} \to 1$ as $t \to 0$,

(f_2) $\frac{f(t)}{\sqrt{t}} \to 2^{1/4}$ as $t \to \infty$,

(f_3) $\frac{1}{2}f(t) \le \frac{t}{\sqrt{1+2f^2(t)}} \le f(t), \quad \forall\, t \in \mathbb{R}$.

Thus, we can write $J(u)$ as

$$I(v) = \frac{1}{2}\int |\nabla v|^2 + \frac{1}{2}\int V(x)f^2(v) - \int H(f(v)).$$

From these properties of f, the funtional I is well defined and $I \in C^1(H^1(\mathbb{R}^2), \mathbb{R})$. In fact, by definition of f and from $(f_2), (h_1), (h_2)$ together with a version of the Trudinger and Moser inequality ([7], [11]) it follows that

$$\int H(f(v)) < \infty \quad \text{and} \quad \int f'(v)h(f(v))w < \infty, \ \forall\, v, w \in H^1(\mathbb{R}^2).$$

As in [8], we observe that if $v \in H^1(\mathbb{R}^2) \cap C^2(\mathbb{R}^2)$ is a critical point of I, that is, $I'(v)w = 0$ for every $w \in H^1(\mathbb{R}^2)$, where

$$I'(v)w = \int \nabla v \cdot \nabla w + \int f'(v)(V(x)f(v)w - h(f(v))w),$$

then v is a solution of problem

$$-\Delta v = g(x, v) \quad \text{in } \mathbb{R}^2,$$

where

$$g(x,s) \doteq \frac{1}{\sqrt{1+2f^2(s)}}\left(-V(x)f(s) + h(f(s))\right), \quad \forall\, x \in \mathbb{R}^2,\ s \in \mathbb{R}. \tag{2.1}$$

Then, setting $u = f(v)$ ($v = f^{-1}(u)$) and since $(f^{-1})'(t) = \frac{1}{f'(f^{-1}(t))} = \sqrt{1+2t^2}$ we conclude that u is a nonnegative solution of problem

$$-\Delta u + V(x)u - \Delta(u^2)u = h(u) \quad \text{in } \mathbb{R}^2.$$

From (V_1) and (V_2), we have

$$g_1(s) \le g(x,s) \le g_2(s) \quad \forall\, x \in \mathbb{R}^2,\ \forall\, s \in \mathbb{R}$$

where

$$g_1(s) = f'(s)\,(-V_\infty f(s) + h(f(s))) \qquad \text{and} \qquad g_2(s) = f'(s)\,(-V_0 f(s) + h(f(s)))\,.$$

For $i \in \{1,2\}$, the functions g_i satisfy:

- (g_0) $g \in C(\mathbb{R},\mathbb{R})$ and $g(0) = 0$,
- (g_1) There exists $\nu < 0$ such that $\lim_{s\to 0} \dfrac{g(s)}{s} = \nu$.
- (g_2) Given $\alpha > \alpha_0$ and $\beta \ge \sqrt[4]{2}\alpha$, there exist positive constants C and R such that

$$|g(s)| \le C(\exp(\beta s^2) - 1), \quad \forall\, |s| > R.$$

Moreover,

- (g_3) There exists $s_0 > 0$ such that $G_1(s_0) > 0$, where $G_1(s) = \int_0^s g_1(t)dt$.

The property (g_0) is obvious, while condition (g_1) follows from the limit (f_1) and (g_2) is a consequence of $(c)_{\alpha_0}$ and (f_2). Finally, in order to verify (g_3), fix $\alpha \in (0,\alpha_0)$. From $(c)_{\alpha_0}$, there exists $s^* > 0$ such that

$$\frac{h(s)}{\exp(\alpha s^4) - 1} \ge 1, \quad \forall\, s \ge s^*. \tag{2.2}$$

From (2.2), (V_2), and the continuity of f and h, there exists a constant m such that

$$\begin{aligned}
G_1(s) &\ge m + \int_{s^*}^{s} [-V_\infty \frac{f(t)}{\sqrt{1+2f^2(t)}} + \frac{\exp(\alpha f^4(t)) - 1}{\sqrt{1+2f^2(t)}}]dt \\
&\ge m - V_\infty \int_{s^*}^{s} \frac{f(t)}{\sqrt{1+2f^2(t)}}dt + \int_{s^*}^{s} \frac{\alpha f^2(t)}{\sqrt{1+2f^2(t)}}dt \\
&= m - V_\infty \int_{f(s^*)}^{f(s)} u du + \int_{f(s^*)}^{f(s)} \alpha u^2 du \\
&= m - \frac{V_\infty (f(s)^2 - f(s^*)^2)}{2} + \frac{f(s)^3 - f(s^*)^3}{3\alpha}.
\end{aligned}$$

Then, from (f_2), there exists $s_0 > 0$ sufficiently large such that $G_1(s_0) > 0$. ■

3 Mountain pass geometry

In section we establish the geometric hypotheses of the mountain pass theorem.

Proposition 3.1 *The functional $I : H^1(\mathbb{R}^2) \longrightarrow \mathbb{R}$ satisfies*

(1.) There exist positive constant b and ρ such that $I(v) \ge b$, $\|v\| = \rho$, where $\|\cdot\|$ denotes the usual norm in $H^1(\mathbb{R}^2)$.

(2.) There exists a path $\gamma \in C([0,1], H^1(\mathbb{R}^2))$ verifying $\gamma(0) = 0$, $\gamma(1) \ne 0$ with $I(\gamma(1)) < 0$.

Proof. First, we show that I satisfies *(1.)*. We claim that given $\alpha > \alpha_0$, there exists $C > 0$ such that

$$G(x,s) \leq -\frac{V_0}{4}s^2 + C[\exp(\alpha s^2) - 1]s^3, \ \forall\, s \geq 0 \ \text{ and } \ \forall\, x \in \mathbb{R}^2, \tag{3.1}$$

where $G(x,s) = \int_0^s g(x,t)\,dt$. In fact, since $g(x,s) \leq g_2(s)$ and g_2 satisfies $(g_0) - (g_2)$, given $\alpha > \alpha_0$, there exist $\delta > 0$ and $C > 0$ independent of x such that

$$g(x,s) \leq -\frac{V_0}{2}s + C\exp(\alpha s^2)s^4.$$

Then, to obtain (3.1), it suffices to note that

$$\begin{aligned}\int_0^s t^4\exp(\alpha t^2)dt &= \frac{1}{2\alpha}s^3\exp(\alpha s^2) - \int_0^s \frac{3}{2\alpha}t^2\exp(\alpha t^2)dt \\ &= \frac{1}{2\alpha}s^3(\exp(\alpha s^2) - 1) + \frac{s^3}{2\alpha} - \int_0^s \frac{3}{2\alpha}t^2\exp(\alpha t^2)dt \\ &\leq \frac{1}{2\alpha}s^3(\exp(\alpha s^2) - 1),\end{aligned}$$

because

$$\mu(s) = \frac{s^3}{2\alpha} - \int_0^s \frac{3}{2\alpha}t^2\exp(\alpha t^2)dt$$

satisfies $\mu(0) = 0$ and $\mu'(s) \leq 0$.

Consequently, from (3.1), we obtain

$$\begin{aligned}I(v) &= \frac{1}{2}\int|\nabla v|^2 - \int G(x,v) \\ &\geq \frac{1}{2}\int|\nabla v|^2 + \frac{V_0}{4}\int v^2 - C\int v^3(\exp(\alpha v^2) - 1) \\ &\geq C_1||v||^2 - C_2||v||^3, \quad \forall\, v \in H^1(\mathbb{R}^2),\end{aligned}$$

where the last inequality we made use of the following estimate (see [11] for a proof):

$$\int v^3(\exp(\alpha v^2) - 1) \leq C||v||^3, \ \text{where } C = C(\alpha) > 0,$$

provided $v \in H^1(\mathbb{R}^2)$, $\|v\| = \rho$, for $\rho > 0$ sufficiently small.

Hence, we can choose positive constants ρ and b such that

$$I(v) \geq b > 0, \ \ \forall\, v \in H^1(\mathbb{R}^2), \ ||v|| = \rho.$$

To prove the second part of Proposition 3.1, we start arguing as in [6]. Let $R > 1$ and define

$$w_R(x) = \begin{cases} s_0 & \text{if } \ |x| \leq R \\ s_0(R + 1 - |x|) & \text{if } \ |x| \in [R, R+1) \\ 0 & \text{if } \ |x| \geq R + 1, \end{cases}$$

where s_0 is given by (g_3). Let

$$w_t(x) = \begin{cases} w_R(\frac{x}{t}) & \text{if } \ t > 0 \\ 0 & \text{if } \ t = 0, \end{cases}$$

then

$$\int |\nabla w_t|^2 = \int |\nabla w_R|^2 \qquad \text{and} \qquad \int G_1(w_t) = t^2 \int G_1(w_R).$$

By taking $\gamma(t) \equiv w_t(\cdot)$, that is, $\gamma(t)(x) = w_t(x)$, we have

$$\begin{aligned} I(\gamma(t)) &= \int |\nabla w_t|^2 - \int G(x, w_t) \\ &\leq \int |\nabla w_t|^2 - \int G_1(w_t) \\ &= \int |\nabla w_R|^2 - t^2 \int G_1(w_R) \ \to -\infty, \text{ as } t \to +\infty, \end{aligned}$$

because

$$\begin{aligned} \int G_1(w_R) &= G_1(s_0)|B_R| + \int_{B_{R+1}\setminus B_R} G_1(w_R) \\ &\geq G_1(s_o)\pi R^2 - |B_{R+1} \setminus B_R| \max_{s\in[0,s_o]} G_1(w_R) \\ &\geq G_1(s_o)\pi R^2 - 3\pi R \max_{s\in[0,s_o]} G_1(w_R) > 0, \end{aligned}$$

for R suficiently large. Hence, there exists $L > 0$ such that $I(\gamma(L)) < 0$ and $\gamma(L) \neq 0$. Therefore, after a suitable scale change in t, we obtain desired path γ. This proves Proposition 3.1. ■

4 Existence

In consequence of Proposition 3.1 and of a version of Ambrosetti-Rabinowitz Mountain Pass Theorem [3], see also [4, 20, 21], for the constant

$$c_0 = \inf_{\gamma\in\Gamma} \sup_{t\in[0,1]} I(\gamma(t)) > 0,$$

where

$$\Gamma = \{\gamma \in C([0,1], H^1(\mathbb{R}^2));\ \gamma(0) = 0, \gamma(1) \neq 0,\ I(\gamma(1)) < 0\},$$

there exists a Cerami sequence (v_n) in $H^1(\mathbb{R}^2)$ at the level c_0, that is

$$I(v_n) \to c_0 \qquad \text{and} \qquad (1 + \|v_n\|)\|I'(v_n)\| \to 0, \quad \text{as } n \to \infty.$$

Lemma 4.1 *The sequence (v_n) is bounded in $H^1(\mathbb{R}^2)$.*

Proof. Since (v_n) satisfies

$$I(v_n) = \frac{1}{2}\int |\nabla v_n|^2 + \frac{1}{2}\int V(x)f^2(v_n) - \int H(f(v_n)) \to c_0 \quad \text{as } n \to \infty, \tag{4.1}$$

and, for every $w \in H^1(\mathbb{R}^2)$,

$$\begin{aligned} (1 + \|v_n\|)I'(v_n)w &= \int \nabla v_n \cdot \nabla w + \int f'(v_n)(V(x)f(v_n)w - h(f(v_n))w) \\ &= \varepsilon_n\|w\|, \end{aligned} \tag{4.2}$$

where $\varepsilon_n \to 0$ as $n \to \infty$, by choosing $w = w_n \equiv f(v_n)/f'(v_n)$ and inserting in (4.2) we obtain

$$\begin{aligned}
&(1+\|v_n\|)I'(v_n)w_n \\
&\quad = \int (1+\frac{2f^2(v_n)}{1+2f^2(v_n)})|\nabla v_n|^2 + \int (V(x)f^2(v_n) - h(f(v_n))f(v_n)) \\
&\quad = \varepsilon_n \|w_n\|. \qquad (4.3)
\end{aligned}$$

Notice that w_n verifies

$$|w_n|_2 \le C|v_n|_2, \quad |\nabla w_n| \le 2|\nabla v_n|, \quad \text{and } ||w_n|| \le C||v_n||.$$

In consequence,

$$\begin{aligned}
&I'(v_n)w_n \\
&\quad = \int (1+\frac{2f^2(v_n)}{1+2f^2(v_n)})|\nabla v_n|^2 + \int V(x)f^2(v_n) - h(f(v_n))f(v_n)) \\
&\quad = \varepsilon_n. \qquad (4.4)
\end{aligned}$$

Combining (4.1), (4.4) and (h_2), we infer that

$$\int (\frac{1}{2} - \frac{1}{\mu}(1+\frac{2f^2(v_n)}{1+2f^2(v_n)}))|\nabla v_n|^2 + \frac{1}{4}\int V(x)f^2(v_n) \le c_0 + \delta_n + \varepsilon_n,$$

where δ_n is given by (4.1).

Since $\mu > 4$, we can conclude that the term

$$\int (|\nabla v_n|^2 + V(x)f^2(v_n))$$

is bounded. Then, to conclude that (v_n) is bounded in $H^1(\mathbb{R}^2)$, it remains to show that (v_n) is bounded in $L^2(\mathbb{R})$. To verify this we start splitting

$$\int v_n^2 = \int_{\{x:|v_n(x)|\le 1\}} v_n^2 + \int_{\{x:|v_n(x)|>1\}} v_n^2.$$

Notice that there exists $C > 0$ such that $H(s) \ge Cs^4$, for every $s \ge 1$. Then, from (f_2) we have $H(f(s)) \ge Cs^2$, for every $s \ge 1$. Therefore

$$\int_{\{x:|v_n(x)|>1\}} v_n^2 \le \frac{1}{C}\int_{\{x:|v_n(x)|>1\}} H(f(v_n)) \le \frac{1}{C}\int H(f(v_n)).$$

By using that $f(s) \ge Cs$, for some $C > 0$, we have

$$\int_{\{x:|v_n(x)|\le 1\}} v_n^2 \le \frac{1}{C}\int_{\{x:|v_n(x)|\le 1\}} f^2(v_n) \le \frac{1}{C}\int f^2(v_n).$$

Hence v_n is bounded in $L^2(\mathbb{R}^2)$. This proves Lemma 4.1. ■

From Lemma 4.1, there exists $v \in H^1(\mathbb{R}^2)$ such that $v_n \rightharpoonup v$ weakly in $H^1(\mathbb{R}^2)$ and $I'(v)\phi = 0$ for every $\phi \in C_o^\infty(\mathbb{R}^2)$, that is, v is a weak solution. In fact, recalling the definition of the function g given by (2.1), it suffices to prove that

$$\int_{\mathbb{R}^2} g(x, v_n)\phi \longrightarrow \int_{\mathbb{R}^2} g(x, v)\phi, \ \forall\, \phi \in C_o^\infty(\mathbb{R}^2).$$

In order to verify this convergence, given $\phi \in C_o^\infty(\mathbb{R}^2)$, we denote by Ω the support set of ϕ. Since (v_n) is bounded in $H^1(\mathbb{R}^2)$, we may take a subsequence denoted again by (v_n) such that

$$v_n \rightharpoonup v \text{ in } H^1(\mathbb{R}^2); \quad v_n \to v \text{ in } L^q(\Omega),\ \forall\, q \geq 1; \quad v_n(x) \to v(x) \text{ a.e. in } \Omega.$$

Moreover, from (4.2), the sequence $(\int g(x, v_n)\phi v_n)$ is bounded. Then, invoking Lemma 2.1 [9], we have

$$\int_{\mathbb{R}^2} g(x, v_n)\phi = \int_\Omega g(x, v_n)\phi \longrightarrow \int_\Omega g(x, v)\phi = \int_{\mathbb{R}^2} g(x, v)\phi.$$

Hence, v is a weak solution of (1.1).

In order to complete the proof of Theorem 1.1, we must show v is nontrivial. The proof of this fact will be carried out in a series of steps. First, we suppose, by contradiction, that $v \equiv 0$. In consequence, we prove that the Cerami sequence (v_n) is a Palais-Smale sequence of an associated functional at infinity, I_∞, and it has a non-vanishing behavior. After a translation, this sequence converge weakly to a nonzero critical point of I_∞. Finally, we use this critical point to construct a path which allows us to obtain a contradiction with the definition of mountain pass level c_0.

We start introducing some notations and facts. Let V_∞ given by condition (V_2). Consider the Sobolev space $H^1(\mathbb{R}^2)$ endowed with the equivalent norm

$$\|v\| = \left(\int |\nabla v|^2 + V_\infty |v|^2 \right)^{1/2}, \quad \forall\, v \in H^1(\mathbb{R}^2).$$

Define the functional $J_\infty : H^1(\mathbb{R}^2) \to \mathbb{R}$ given by

$$J_\infty(v) = \frac{1}{2} \int (|\nabla v|^2 + V_\infty v^2) - \int H(f(v)).$$

Working with the analogue of I, the functional J_∞ is well defined and belongs to $C^1(H^1(\mathbb{R}^2), \mathbb{R})$.

Now, take β_0 given by (h_3) and let $r > 0$ be such that

$$\beta_0 > \frac{8}{\alpha_0 r^2}. \tag{4.5}$$

We consider the Moser sequence [17] defined by

$$\widetilde{M}_n(x, r) \equiv \widetilde{M}_n = \frac{1}{\sqrt{2\pi}} \begin{cases} (\log n)^{1/2} & \text{if } \ |x| \leq \frac{r}{n}, \\ (\log(r/|x|)/(\log n)^{1/2} & \text{if } \ \frac{r}{n} \leq |x| \leq r, \\ 0 & \text{if } \ |x| > r, \end{cases}$$

which satisfies

$$\widetilde{M}_n \in H^1(\mathbb{R}^2) \quad \text{and} \quad ||\widetilde{M}_n||^2 = 1 + O((\log n)^{-1}) \text{ as } \ n \to \infty.$$

Also we have $M_n^2(x, r) \equiv M_n^2 = (2\pi)^{-1/2} \log n + d_n$, where $M_n = \widetilde{M}_n / ||\widetilde{M}_n||$ and d_n is a bounded real sequence.

Thus we have the following estimate, whose the proof is based on the argument used in [11, Lemma 5].

Proposition 4.1 *Suppose $h(s)$ satisfies $(c)_{\alpha_0}$ and (h_3). Then, for every $\sigma > \sqrt[4]{2}$, there exists $n \in \mathbb{N}$ such that*

$$\max\{J_\infty(tM_n) : \ t \geq 0\} < C^* \equiv \frac{4\pi\mu_0}{\alpha_0 r \sigma^4},$$

where $\mu_0 = (\mu - 4)/2\mu$ and μ is given by (h_2).

Proof. Suppose, by contradiction, that for all n we have

$$\max\{J_\infty(tM_n):\ t\geq 0\}\geq C^*.$$

Thus, there exists $t_n>0$ such that

$$J_\infty(t_nM_n)=\max\{J_\infty(tM_n):\ t\geq 0\}.$$

Then,

$$J_\infty(t_nM_n)=\frac{t_n^2}{2}-\int H(f(t_nM_n))\geq C^*,$$

that is

$$t_n^2\geq 2C^*. \tag{4.6}$$

Since $\frac{d}{dt}J_\infty(tM_n)=0$ at $t=t_n$, it follows that

$$t_n^2=\int_{|x|\leq r}t_nM_nh(f(t_nM_n))f'(t_nM_n). \tag{4.7}$$

From (h_3), given $\epsilon>0$ there exists $R_\epsilon>0$ such that for all $s\geq R_\epsilon$ and for all $|x|\leq r$,

$$sh(s)\geq(\beta_0-\epsilon)\exp\left(\alpha_0s^4\right)\ \text{ and }\ M_n(x)\geq R_\epsilon. \tag{4.8}$$

Combining (f_3) with $(4.7)-(4.8)$, for large n, we obtain

$$\begin{aligned} t_n^2 &\geq \frac{(\beta_0-\epsilon)}{2}\int_{|x|\leq\frac{r}{n}}\exp\left(\alpha_0(f(t_nM_n)^4\right)\\ &\geq \frac{(\beta_0-\epsilon)}{2}\int_{|x|\leq\frac{r}{n}}\exp\left(\alpha_0(t_nM_n)^2\right)\\ &\geq \frac{(\beta_0-\epsilon)}{2}\pi(\frac{r}{n})^2\exp\left(\alpha_0t_n^2(2\pi)^{-1}\text{log}n+\alpha_0t_n^2d_n\right), \end{aligned}$$

where we used (f_2) and (f_3).

Thus

$$1\geq\frac{(\beta_0-\epsilon)}{2}\pi r^2\exp\left(\alpha_0t_n^2(2\pi)^{-1}\text{log}n+\alpha_0t_n^2d_n-2\text{log}n-2\text{log}t_n\right)$$

which implies that t_n is bounded.

By (4.6) and

$$t_n^2\geq\frac{(\beta_0-\epsilon)}{2}\pi r^2\exp\left((\alpha_0t_n^2(2\pi)^{-1}-2)\text{log}n+\alpha_0t_n^2d_n\right)$$

it follows that

$$t_n^2\longrightarrow\frac{4\pi}{\alpha_0}. \tag{4.9}$$

Now consider the sets

$$A_n=\{x:t_nM_n\geq R_\epsilon,|x|\leq r\}\text{ and }B_n=\{x:t_nM_n<R_\epsilon,|x|\leq r\}.$$

From (4.7) and (4.8) we achieve

$$\begin{aligned} t_n^2 &\geq \frac{(\beta_0-\epsilon)}{2}\int_{|x|\leq r}\exp\left(\alpha_0(t_nM_n)^2\right)\\ &- \frac{(\beta_0-\epsilon)}{2}\int_{B_n}\exp\left(\alpha_0(t_nM_n)^2\right)+\int_{B_n}t_nM_nh(f(t_nM_n)f'(t_nM_n) \end{aligned}$$

Arguing once more as in [11], by using (4.9) we conclude that

$$\frac{4\pi}{\alpha_0} \geq \frac{(\beta_0 - \epsilon)\pi r^2}{2},$$

which implies that

$$\beta_0 \leq \frac{8}{\alpha_0 r^2},$$

contrary to (4.5). Thus Proposition 4.1 is proved. ■

Remark 4.2 *We observe that Proposition 4.1 implies that* $c_0 < 4\pi\mu_0/\alpha_0 r\sigma^4$*. Indeed, from* (f_0)*,* (V_2)*, and mean value theorem, we have* $I(v) \leq J_\infty(v)$ *for every* $v \in H^1(\mathbb{R}^2)$*. Then, applying Proposition 4.1 we conclude this estimate.*

The following lemma shows that the Cerami sequence (v_n) has a "non-vanishing" behavior.

Lemma 4.3 *There exist positive constants* a *and* R*, and a sequence* $(y_n) \subset \mathbb{R}^2$ *such that*

$$\lim_{n\to\infty} \int_{B_R(y_n)} f^2(v_n) \geq a > 0, \tag{4.10}$$

where $B_R(z)$ *denotes a ball of radius* R *centered at the point* z*.*

Proof. Suppose by contradiction that (4.10) does not occur. Then

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^2} \int_{B_R(y)} f^2(v_n) = 0, \tag{4.11}$$

From (4.11) and applying a Lions compactness Lemma ([14]) we obtain as $n \to \infty$,

$$f(v_n) \to 0, \quad \text{in} \quad L^q(\mathbb{R}^2), \ \forall\, q \in (2, \infty). \tag{4.12}$$

Then, we can show the crucial part of this proof, which is the following:

$$\int h(f(v_n))f(v_n) \to 0, \quad \text{as } n \to \infty. \tag{4.13}$$

To prove such convergence, we start arguing as in the proof of Lemma 4.1, making $w = f(v_n)\sqrt{1 + 2f^2(v_n)})$ in (4.2). Thus, given $\eta > 0$ we take n_0 such that

$$|\nabla v_n|_2^2 \leq (1+\eta)\frac{c_0}{\mu_0} \quad \forall\, n \geq n_0, \tag{4.14}$$

where $\mu_0 = (\mu - 4)/2\mu$.

From (f_2), there exist $\sigma > \sqrt[4]{2}$ and $R > 0$ such that

$$f(s) < \sigma\sqrt{s}, \quad \forall\, s > R. \tag{4.15}$$

Now, we take $\alpha > \alpha_0$. From (h_1) and $(c)_{\alpha_0}$, given $\epsilon > 0$, there exists a positive constant $C = C(\epsilon, \alpha, q)$ such that

$$h(s) \leq \epsilon s + C(\exp(\alpha s^4) - 1)s^3, \quad \forall\, s \geq 0. \tag{4.16}$$

Thus, using (4.14)-(4.16), we get for every $n \geq n_0$

$$\begin{aligned}
0 &\leq \int h(f(v_n))f(v_n) \\
&\leq \epsilon \int f^2(v_n) + C\int (\exp(\alpha f^4(v_n)) - 1)f(v_n)^4 \\
&= \epsilon \int f^2(v_n) + C\{\int_{\{x;|v_n(x)|\leq R\}} + \int_{\{x;|v_n(x)|\geq R\}}\}(\exp(\alpha f^4(v_n)) - 1)f(v_n)^4 \\
&\leq \epsilon \int f^2(v_n) + \widetilde{C}\int f^4(v_n) + C(\int (\exp(\alpha\, r f^4(v_n)) - 1))^{1/r}(\int f(v_n)^{4r'})^{1/r'} \\
&\leq \epsilon \int f^2(v_n) + \widetilde{C}\int f^4(v_n) + C(\int [\exp(\alpha r\sigma^4(1+\eta)\frac{c_0}{\mu_0}(\frac{v_n}{|\nabla v_n|_2})^2) - 1])^{1/r}(\int f(v_n)^{4r'})^{1/r'},
\end{aligned}$$

where r satisfies (4.5) and $1/r + 1/r' = 1$. By Proposition 4.1 (see also Remark 4.2), we may take $\alpha > \alpha_0$ and $\eta > 0$ such that $\alpha\, r\, \sigma^4\, (1+\eta)c_0 < 4\pi\mu_0$. Then, from (1.7), the last integral is bounded uniformly. Hence, from (4.12), we conclude that (4.13) holds.

Now, we are ready to conclude the proof of Lemma 4.3. Taking again $w_n = f(v_n)/f'(v_n)$ in the equation (4.2) we have

$$\begin{aligned}
o(1) &= I'(v_n)w_n \\
&= \int (1 + \frac{2f^2(v_n)}{1+2f^2(v_n)})|\nabla v_n|^2 + \int (V(x)f^2(v_n) - h(f(v_n))f(v_n)) \\
&\geq \int |\nabla v_n|^2 + \int (V(x)f^2(v_n) - h(f(v_n))f(v_n)).
\end{aligned}$$

Then from (4.13), we conclude that

$$\int |\nabla v_n|^2 + \int V(x)f^2(v_n) \to 0, \quad \text{as} \quad n \to \infty. \tag{4.17}$$

By combining (h_2) and (4.13) we obtain

$$\int H(f(v_n)) \to 0, \quad \text{as} \quad n \to \infty. \tag{4.18}$$

By using (4.17) and (4.18) in (4.1), we reach a contradiction because

$$0 < c_0 = \lim_{n\to\infty} I(v_n) = 0.$$

The proof of Lemma 4.3 is completed. ■

In the following we consider the functional at infinity I_∞ associated with I. We define $I_\infty : H^1(\mathbb{R}^2) \to \mathbb{R}$ by

$$I_\infty(v) = \frac{1}{2}\int |\nabla v|^2 + \frac{1}{2}\int V_\infty f^2(v) - \int H(f(v)).$$

Lemma 4.4 *The Cerami sequence $\{v_n\}$ is a Palais-Smale sequence for I_∞ at level c_0.*

Proof. From (V_2), given $\epsilon > 0$ there exists $R > 0$ such that

$$|V(x) - V_\infty| < \epsilon, \quad \forall\, |x| \geq R.$$

Thus,

$$\begin{aligned} & |I_\infty(v_n) - I(v_n)| \\ = \ & \frac{1}{2}\int_{B_R(0)} |V_\infty - V(x)| f^2(v_n) + \frac{1}{2}\int_{\mathbb{R}^2\setminus B_R(0)} |V_\infty - V(x)| f^2(v_n) \\ \le \ & \frac{1}{2}|V_\infty - V(x)|_\infty \int_{B_R(0)} f^2(v_n) + \frac{1}{2}\epsilon \int_{\mathbb{R}^2\setminus B_R(0)} f^2(v_n) \\ \le \ & o(1), \ \text{as } n \to \infty, \end{aligned}$$

where in the last inequality we made use that

$$\int_{B_R(0)} f^2(v_n) \to 0, \ \text{as } n \to \infty,$$

since $f(v_n) \in H^1(\mathbb{R}^2)$ and the embedding $H^1(\mathbb{R}^2)$ into $L^q(\mathbb{R}^2)$, $q > 1$, is locally compact and $v_n \rightharpoonup v \equiv 0$ weakly in $H^1(\mathbb{R}^2)$.
Therefore,

$$I_\infty(v_n) \to c_0, \quad \text{as } n \to \infty.$$

Similarly

$$\sup_{||\phi||<1} |(I'_\infty(v_n) - I'(v_n), \phi)| \ = \ \sup_{||\phi||<1} |\int_{\mathbb{R}^2} (V_\infty - V(x)) f(v_n) f'(v_n)\phi| = o(1), \quad \text{as } n \to \infty.$$

Hence $I'_\infty(v_n) \to 0$, as $n \to \infty$. This proves Lemma 4.4. ■

Define

$$\tilde{v}_n(x) = v_n(x + y_n),$$

where $\{y_n\}$ is the sequence defined in Lemma 4.3. Then, $\tilde{v}_n$ is a bounded sequence in $H^1(\mathbb{R}^2)$ and it verifies, as $n \to \infty$,

$$I_\infty(\tilde{v}_n) = I_\infty(v_n) \to c_0 \text{ and } I'_\infty(\tilde{v}_n) \to 0,$$

also $\tilde{v}_n \rightharpoonup \tilde{v}$ and $\tilde{v}$ is a critical point of I_∞. Taking an odd extension of h from $\mathbb{R}^-$ to $\mathbb{R}$ if necessary, we may replace $\tilde{v}_n$ by $|\tilde{v}_n|$. Thus, we may assume $\tilde{v} \ge 0$ in $\mathbb{R}^2$. By elliptic regularity theory, $\tilde{v}$ is of C^2 class. To see that $\tilde{v} > 0$ in $\mathbb{R}^2$, the strong maximum principle will be employed. We observe that $\tilde{v}$ is also solution of the problem

$$-\Delta v + cv = (h(f(v)) - V_\infty f(v)) f'(v) + cv \quad \text{in } \mathbb{R}^2,$$

where $c \ge 0$ is such that the term on the right is nonnegative for $x \in \mathbb{R}^2$; the existence of this c follows from $(g_0) - (g_2)$. Thus, by strong maximum principle, $\tilde{v} > 0$.

We also remark that

$$\tilde{v}(x) \to 0 \text{ as } |x| \to \infty. \tag{4.19}$$

Effectively, $\tilde{v}$ is a weak solution of

$$-\Delta v = g(v), \quad \text{in } \mathbb{R}^2,$$

where $g(s) \doteq (h(f(v)) - V_\infty f(v)) f'(v)$. By the Sobolev embedding theorem and Trudinger inequality (1.6), $g(\tilde{v}) \in L^p(\mathbb{R}^2)$ for every $p \ge 2$. Thus, we infer by interior elliptic estimates that $\tilde{v} \in W^{2,p}_{loc}$ and moreover

$$\|\tilde{v}\|_{W^{2,p}(\Omega')} \le C(|g(\tilde{v})|_{L^p(\Omega)} + |\tilde{v}|_{L^p(\Omega)}),$$

where $\Omega' \subset\subset \Omega$, Ω is an open bounded set of $\mathbb{R}^2$ and C depends only on the diameter of Ω and the measure of $\Omega \setminus \Omega'$.

Let $x_0 \in \mathbb{R}^2$ and denote by $B_r \subset \mathbb{R}^2$ the open ball of radius $r > 0$ centered at x_0. Then,

$$\|\tilde{v}\|_{W^{2,p}(B_1)} \leq C(|g(\tilde{v})|_{L^p(B_2)} + |\tilde{v}|_{L^p(B_2)}),$$

where C depends only on the diameter of B_2 and the measure of $B_2 \setminus B_1$.

Since $W^{2,p}(B_2) \subset C(\overline{B_1})$, because $p \geq 2$, we obtain

$$\|\tilde{v}\|_{L^\infty(B_1)} \leq C(|g(\tilde{v})|_{L^p(B_2)} + |\tilde{v}|_{L^p(B_2)}).$$

In particular,

$$|\tilde{v}(x_0)| \leq C(|g(\tilde{v})|_{L^p(B_2)} + |\tilde{v}|_{L^p(B_2)})$$

and since $g(\tilde{v})$ and $\tilde{v}$ belong to $L^p(\mathbb{R}^2)$, we have

$$|g(\tilde{v})|_{L^p(B_2)} + |\tilde{v}|_{L^p(B_2)} \to 0 \quad \text{as } |x_0| \to \infty$$

so that $|\tilde{v}(x)| \to 0$ as $|x| \to \infty$ and the verification of (4.19) is complete.

We assert now that

$$c_\infty \leq I_\infty(\tilde{v}) \leq c_0, \tag{4.20}$$

where c_∞ is the Mountain Pass level given by

$$c_\infty = \inf_{\gamma \in \Gamma} \sup_{t \in [0,1]} I_\infty(\gamma(t)),$$

and

$$\Gamma_\infty = \{\gamma \in C([0,1], H^1(\mathbb{R}^2));\ \gamma(0) = 0, \gamma(1) \neq 0,\ I_\infty(\gamma(1)) < 0\}.$$

We start the verification of (4.20) showing that $I_\infty(\tilde{v}) \leq c_0$. Indeed by (f_3):

$$f^2(\tilde{v}_n) - f(\tilde{v}_n)f'(\tilde{v}_n)\tilde{v}_n \geq 0,\ \forall\, n \in \mathbb{N}. \tag{4.21}$$

Now, from (f_3) again and (h_2), we obtain

$$\frac{1}{2}h(f(\tilde{v}_n))f'(\tilde{v}_n)\tilde{v}_n - H(f(\tilde{v}_n)) \geq \frac{1}{\mu}h(f(\tilde{v}_n))f(\tilde{v}_n) - H(f(\tilde{v}_n)) \geq 0, \quad \forall n \in \mathbb{N}. \tag{4.22}$$

Hence by Fatou Lemma combined with (4.21) and (4.22), we have

$$\begin{aligned}
c_0 &= \limsup_{n\to\infty} \{I_\infty(\tilde{v}_n) - \frac{1}{2}I'_\infty(\tilde{v}_n)\tilde{v}_n\} \\
&= \limsup_{n\to\infty} \int \{\frac{1}{2}[(f^2(\tilde{v}_n) - f(\tilde{v}_n)f'(\tilde{v}_n)\tilde{v}_n)V_\infty] + \frac{1}{2}h(f(\tilde{v}_n))f'(\tilde{v}_n)\tilde{v}_n - H(f(\tilde{v}_n))\} \\
&\geq \int \frac{1}{2}(f^2(\tilde{v}) - f(\tilde{v})f'(\tilde{v})\tilde{v})V_\infty + \int \frac{1}{2}h(f(\tilde{v}))f'(\tilde{v})\tilde{v} - H(f(\tilde{v})) \\
&= I_\infty(\tilde{v}) - \frac{1}{2}I'_\infty(\tilde{v})\tilde{v} = I_\infty(\tilde{v}).
\end{aligned}$$

Thus $I_\infty(\tilde{v}) \leq c_0$. Now, in order to show $c_\infty \leq I_\infty(\tilde{v})$, we slightly modify an argument used in [13] to get a path $\gamma : [0,1] \to H^1(\mathbb{R}^2)$ such that

$$\begin{cases} \gamma(0) = 0, \quad I_\infty(\gamma(1)) < 0, \quad \tilde{v} \in \gamma([0,1]), \\ \gamma(t)(x) > 0 \quad \forall\, x \in \mathbb{R}^2,\ t \in (0,1], \\ \max_{t\in[0,1]} I_\infty(\gamma(t)) = I_\infty(\tilde{v}). \end{cases} \tag{4.23}$$

Indeed, define

$$\tilde{v}_t(x) = \begin{cases} \tilde{v}(x/t) & \text{if} \quad t > 0, \\ 0 & \text{if} \quad t = 0. \end{cases}$$

Choose three points $t_o \in (0,1)$, $t_1 \in (1,\infty)$ and $\theta_1 > t_1$ such that the path γ defined by three pieces, namely, $\gamma_1 : [0,1] \to H^1(\mathbb{R}^2)$, $\gamma_1(\theta) = \theta \tilde{v}_{t_o}$, $\gamma_2 : [t_o, t_1] \to H^1(\mathbb{R}^2)$, $\gamma_2(t) = \tilde{v}_t$, and $\gamma_3 : [1, \theta_1] \to H^1(\mathbb{R}^2)$, $\gamma_3(\theta) = \theta \tilde{v}_{t_1}$, it is desired path. Effectively, because of $\tilde{v}$ is a critical point of I_∞, the function $\tilde{v}$ is a weak positive solution of

$$-\Delta \tilde{v} = g(\tilde{v}), \quad \text{in} \ \ \mathbb{R}^2.$$

Then

$$\int g(\tilde{v})\tilde{v} = ||\nabla \tilde{v}||^2 > 0,$$

where $g(s) = (h(f(s)) - V_\infty f(s))f'(s)$. Thus, there exists $\theta_1 > 0$ such that

$$\int g(\theta \tilde{v})\tilde{v} > 0, \quad \forall\, \theta \in [1, \theta_1]. \tag{4.24}$$

Let $\Phi(s) = \frac{g(s)}{s}$ for $s > 0$. By (4.24) we infer that

$$\int \Phi(\theta \tilde{v})\tilde{v}^2 > 0, \quad \forall\, \theta \in [1, \theta_1]. \tag{4.25}$$

On the other hand, from

$$\frac{d}{d\theta} I_\infty(\theta \tilde{v}_t) = \theta \left(||\nabla \tilde{v}||_2^2 - t^2 \int \Phi(\theta v_t) v^2 \right)$$

there exists $t_o \in (0,1)$ such that

$$||\nabla \tilde{v}||_2^2 - t_o^2 \int \Phi(\theta \tilde{v}_t)\tilde{v}^2 > 0, \quad \forall\, \theta \in [0,1]. \tag{4.26}$$

From (4.25) there exists $t_1 > 1$ such that

$$||\nabla \tilde{v}||_2^2 - t_1^2 \int \Phi(\theta \tilde{v}_t)\tilde{v}^2 < \frac{-2}{\theta_1^2 - 1} \|\nabla \tilde{v}\|_2^2, \quad \forall\, \theta \in [1, \theta_1]. \tag{4.27}$$

From (4.26), by along of the path γ_1, $I_\infty(\theta \tilde{v}_{t_o})$ decreases and it takes its maximum value at $\theta = 1$. Since $\int G(\tilde{v}) = 0$, by Pohozaev identity we obtain

$$I_\infty(\tilde{v}_t) = I_\infty(\tilde{v}) = \frac{1}{2}\|\nabla \tilde{v}\|_2^2$$

along the path γ_2. From (4.27), $I_\infty(\theta \tilde{v}_{t_1})$ decreases along the path γ_3. Thus,

$$I_\infty(\gamma_1(t)) \le I_\infty(\tilde{v}_t) = I_\infty(\tilde{v}),$$

on the other hand

$$I_\infty(\tilde{v}) = I_\infty(\tilde{v}_t) \ge I_\infty(\theta \tilde{v}_{t_1}), \quad \forall\, \theta \in [0, \theta_1].$$

Therefore

$$\max_{t \in [0, \theta_1]} I_\infty(\gamma(t)) = I_\infty(\tilde{v}).$$

Moreover, from (4.27) and the fact $I_\infty(\theta\tilde{v}_{t_1})$ decreases along γ_3 we have

$$\begin{aligned} I_\infty(\theta_1\tilde{v}_{t_1}) &= I_\infty(\tilde{v}_{t_1}) + \int_1^{\theta_1} \frac{d}{d\theta} I_\infty(\theta\tilde{v}_{t_1})\, d\theta \\ &\leq \frac{1}{2}\|\nabla\tilde{v}\|_2^2 - \int_1^{\theta_1} \frac{2\theta}{\theta_1^2 - 1}\|\nabla\tilde{v}\|_2^2\, d\theta \\ &= -\frac{1}{2}\|\nabla\tilde{v}\|_2^2 < 0. \end{aligned}$$

Hence we obtain the desired path (4.23).

The path (4.23) together with the definition of c_∞ imply that

$$c_\infty \leq \max_{t\in[0,1]} I_\infty(\gamma(t)) = I_\infty(\tilde{v}).$$

Thus, $c_\infty \leq I_\infty(\tilde{v})$ and the verification of (4.20) is complete.

Finally, we may conclude the proof of Theorem 1.1. Take again the path γ given by (4.23). Since $\gamma \in \Gamma_\infty \subset \Gamma$, $\gamma(t)(x) > 0$, and $V(x) \leq V_\infty$, with $V \neq V_\infty$, from (4.20) we obtain

$$\begin{aligned} c_0 &\leq \sup_{t\in[0,1]} I(\gamma(t)) = I(\gamma(\bar{t})) \\ &< I_\infty(\gamma(\bar{t}) \leq \max_{t\in[0,1]} I_\infty(\gamma(t)) \\ &= I_\infty(\tilde{v}) \leq c_0, \end{aligned}$$

which is contradiction. Therefore, v is nontrivial. Theorem 1.1 is proved. ∎

Remark 4.5 *1. By a similar argument we can prove a version of Theorem* 1.1 *in the asymptotic case to a periodic function V_p, that is, when V satisfies*

$$V_p(x) \doteq \lim_{|x|\to\infty} V(x),\ V_p(x+1) = V_p(x),\ \forall\, x \in \mathbb{R}^2,\ \text{and}$$

$$V(x) \leq V_p(x),\ \forall\, x \in \mathbb{R}^2,$$

where the last inequality is strict on a positive Lebesgue measure set of $\mathbb{R}^2$.

2. *We can establish Theorem* 1.1, *in the compact-coercive case, that is, when* $\lim_{|x|\to\infty} V(x) = +\infty$, *and its proof follows easily because the map $v \to f(v)$ from $H^1(\mathbb{R}^2)$ into $L^q(\mathbb{R}^2)$ is compact for* $2 \leq q < \infty$. *(See [19] also [15]).*

3. *Theorem* 1.1 *still holds in the radially symmetric case, namely $V(x) = V(|x|)$, $\forall\, x \in \mathbb{R}^2$. The proof can be handled as above by using that the map $v \to f(v)$ from $H^1(\mathbb{R}^2)$ into $L^q(\mathbb{R}^2)$ is compact for* $2 < q < \infty$. *(See [22] also [15]).*

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