Critical and Subcritical Elliptic Systems in Dimension Two

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ABSTRACT. In this paper we study the existence of nontrivial solutions for the following system of two coupled semilinear Poisson equations:

$$\text{(S)} \qquad \begin{cases} -\Delta u = g(v), \ v > 0 & \text{in } \Omega, \\ -\Delta v = f(u), \ u > 0 & \text{in } \Omega, \\ u = 0, \ v = 0, & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^2$ with smooth boundary $\partial\Omega$, and the functions f and g have the maximal growth which allow us to treat problem (S) variationally in the Sobolev space $H_0^1(\Omega)$. We consider the case with nonlinearities in the critical growth range suggested by the so-called Trudinger-Moser inequality.

1. INTRODUCTION

There has been recently a good amount of work on Hamiltonian systems of second order involving elliptic equations defined in subsets of $\mathbb{R}^N$, $N \geq 3$, see for example [6–8]. In this paper we study some classes of such systems, when the equations are defined in bounded subsets of $\mathbb{R}^2$. Limitations on the growth of the nonlinearities vary substantially when we come to dimension two. As it is well known, in dimensions $N \geq 3$ the nonlinearities are required to have polynomial growth at infinity, so that one can define associated functionals in Sobolev spaces. Coming to dimension two, much faster growth is allowed for the nonlinearity. In fact exponential growth can be handled, and the Trudinger-Moser estimates in $N = 2$ replaces the Sobolev imbedding theorem used in $N \geq 3$.

The case of a single semilinear elliptic equation in bounded subsets of $\mathbb{R}^2$ has been investigated by several authors, see for example [2, 3, 5, 6]. It has been observed that criticality in dimension two is connected with the imbedding of $H_0^1(\Omega)$ in an Orlicz space L_φ when $\varphi(t) = e^{\alpha t^2} - 1$, see [1, 9]. This is analogous to the phenomenon of criticality in dimension $N \geq 3$ when it occurs at the value of p (namely $p = 2^*$) such that the continuous imbedding of $H_0^1(\Omega)$ into L^p, $p > 1$ fails to be compact.

Our aim in this paper is then to establish the existence of solutions for the following class of elliptic systems

$$\begin{cases} -\Delta u = g(v), \ v > 0 & \text{in } \Omega, \\ -\Delta v = f(u), \ u > 0 & \text{in } \Omega, \\ u = 0, \ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

where Ω is a bounded open subset of $\mathbb{R}^N$, with smooth boundary $\partial\Omega$ and Δ is the Laplace operator. This class of problems allows a variational formulation. More precisely, their weak solutions are the critical points of the associated energy functional

$$\begin{gathered} I(u,v) = \int_\Omega \nabla u \nabla v \, dx - \int_\Omega [F(u) + G(v)] \, dx, \\ (u,v) \in E := H_0^1(\Omega) \times H_0^1(\Omega), \end{gathered} \tag{1.2}$$

where the functions F and G are the primitives of f and g, respectively. The norm of $u \in H_0^1(\Omega)$ is given by $\|u\| := (\int_\Omega |\nabla u|^2 \, dx)^{1/2}$. The norm of an element $z = (u,v)$ in E is defined by $\|z\| := (\|u\|^2 + \|v\|^2)^{1/2}$.

Although system (1.1) above is a special case of a general Hamiltonian system, it already contains the basic difficulties of the general case. Namely, the associated functional I, given in (1.2), is strongly indefinite, and the nonlinearities f, g treated in the present paper can have critical growth, see the definition below. We believe that once we know how to overcome these difficulties in this special case, more general cases can be treated by the same techniques.

Here we assume the following conditions:

(H_1) $f, g : [0,\infty) \to [0,\infty)$ are continuous functions;
(H_2) $f(s) = o(s)$ and $g(t) = o(t)$ near the origin;
(H_3) there exist constants $\theta > 2$ and $t_0 > 0$ such that, for all $t \geq t_0$, one has

$$0 < \theta F(t) \leq t f(t) \quad \text{and} \quad 0 < \theta G(t) \leq t g(t);$$

(H_4) there exists $M > 0$ and $t_o > 0$ such that, for all $t \geq t_o$,

$$0 < F(t) \leq M f(t) \quad \text{and} \quad 0 < G(t) \leq M g(t).$$

Motivated by the so-called Trudinger-Moser inequality, which says that if u is a $H_0^1(\Omega)$ function, then the integral $\int_\Omega e^{u^2}$ is finite, we say that g has subcritical growth at $+\infty$ if for all $\alpha > 0$

$$\lim_{t\to+\infty} \frac{g(t)}{e^{\alpha t^2}} = 0, \tag{1.3}$$

and g has critical growth at $+\infty$ if there exists $\alpha_0 > 0$, such that

$$\lim_{t\to+\infty} \frac{g(t)}{e^{\alpha t^2}} = 0 \text{ for all } \alpha > \alpha_0 \quad \text{and} \quad \lim_{t\to+\infty} \frac{g(t)}{e^{\alpha t^2}} = +\infty \text{ for all } \alpha < \alpha_0. \tag{1.4}$$

In the case of critical growth, we say that α_0 is the critical exponent of g.

Theorem 1.1 (The subcritical case)**.** *If g has subcritical growth, f has subcritical or critical growth, and* (H_1)-(H_3) *are satisfied,* (1.1) *possesses a nontrivial weak solution* $(u, v) \in E$.

We denote by d the inner radius of the set Ω, that is, d is equal to the radius of the largest open ball contained in Ω.

Theorem 1.2 (The critical case)**.** *If f and g have critical growth and* (H_1)-(H_2) *and* (H_4) *are assumed, and furthermore suppose that*

(H_5) $\lim_{t\to+\infty} tf(t)e^{-\alpha_o t^2} > 4/\alpha_o d^2$ *and* $\lim_{t\to+\infty} tg(t)e^{-\alpha_o t^2} > 4/\alpha_o d^2$;

then (1.1) *possesses a nontrivial weak solution* $(u, v) \in E$.

Remark 1.3.

(1) Condition (H_4) is stronger than (H_3), in the sense that (H_4) implies (H_3).

(2) It follows from (1.3) and (1.4) that, in any case, subcritical or critical, there exist positive constants C and β such that

$$f(t),\ g(t) \le Ce^{\beta t^2}, \quad \forall t \ge 0. \tag{1.5}$$

(3) In the critical case, we shall need a more precise estimate, namely, given $\varepsilon > 0$, there is a positive constant C_ε such that

$$f(t),\ g(t) \le C_\varepsilon e^{(\alpha_0+\varepsilon)t^2}, \quad \forall t \ge 0. \tag{1.6}$$

(4) Hypothesis (H_5) implies that f and g are critical with critical exponent α_0.

Remark 1.4.

(1) Theorem 1.2 is the extension to systems of Theorem 1.3 in [5]: indeed, condition (H_5) corresponds to condition (H_7) in [5].

(2) In higher dimensions critical growth is given by powers. In [6–8] it was shown that for $N \ge 5$ the limiting powers of f and g in (1) form a "critical hyperbola." It would be of interest to find a related "critical curve" in dimensions 2, 3 and 4. The critical case considered in Theorem 1.2 lies on the intersection of this hypothetical curve with the diagonal.

2. Abstract Framework

As mentioned in the introduction, the nonlinearities f and g are allowed to have the maximal growth which allows to treat the problem by variational methods in $H_0^1(\Omega)$. This growth is given by the so-called Trudinger-Moser inequality, which says:

(TM-1) If $u \in H_0^1(\Omega)$, then $\int_\Omega e^{u^2} < +\infty$, see N. Trudinger [14] (cf. also S. Pohozaev [13].

(TM-2) $\sup_{\|u\|\le 1} \int_\Omega e^{\beta u^2} < +\infty$, for $0 \le \beta \le 4\pi$, see J. Moser [11] (for a uniform bound for some $\beta > 0$, see also N. Trudinger [14]).

(TM-3) Let $\{u_n\} \subset H_0^1(\Omega)$ with $\|u_n\| \le 1$ and $u_n \rightharpoonup u$, and let $\alpha < 4\pi$. Then, for a subsequence, $\int_\Omega e^{\alpha u_n^2} \to \int_\Omega e^{\alpha u^2}$, see P.L. Lions [10] (adapting the proof of Th. 1.6, p.197).

We now consider the functional I given in (1.2). Since we are interested in positive solutions we define f and g to be zero on $(-\infty, 0]$. Under our assumptions we have

(i) I is well-defined, since by (5)

$$F(t) = \int_0^t f(s)\,ds \le c\int_0^t e^{\beta t^2} \le cte^{\beta t^2} \le ce^{(\beta+\delta)t^2}$$

(ii) I is C^1 with

(2.1)
$$I'(u,v)(\varphi,\psi) = \int_\Omega [\nabla u \nabla \psi + \nabla v \nabla \varphi]\,dx - \int_\Omega [f(u)\varphi + g(v)\psi]\,dx, \quad \text{for all } (\varphi,\psi) \in E;$$

indeed, given $u_n \to u$ in $H_0^1(\Omega)$ there exists a subsequence $u_{n_k}(x)$ and $h \in H_0^1$ such that $|u_{n_k}| \le h(x)$ (this is seen using the same arguments as in the proof of the Riesz-Fischer theorem, completeness of L^2, see e.g. [4], Theorem IV.8). Then, by (TM-1) we have $e^{h^2} \in L^1$, and then by the Lebesgue dominated convergence theorem $\int_\Omega f(u_n)\varphi \to \int_\Omega f(u)\varphi$, for all $\varphi \in C_0^\infty$.

Consequently, critical points of the functional I are precisely the weak solutions of (1.1).

2.1. The Geometry of the Linking Theorem. We use the following notation:

$$E^+ = \{(u,u) \mid u \in H_0^1(\Omega)\} \quad \text{and} \quad E^- = \{(u,-u) \mid u \in H_0^1(\Omega)\}.$$

Lemma 2.1. *There exist* ρ, $\sigma > 0$ *such that* $I(z) \geq \sigma$, *for all* $z \in S := \partial B_\rho \cap E^+$.

Proof. From (H_2), for given $\varepsilon_0 > 0$, there exists $r_0 > 0$ such that

$$\begin{aligned} f(t) &\leq 2\varepsilon_0 t, \\ g(t) &\leq 2\varepsilon_0 t, \end{aligned} \quad \text{for all } t \leq r_0.$$

On the other hand, it follows from (1.5) that, for a given $q > 2$, there exists a constant $C_1 > 0$ such that

$$\begin{aligned} F(t) &\leq C_1 t^q \exp(\beta t^2), \\ G(t) &\leq C_1 t^q \exp(\beta t^2), \end{aligned} \quad \text{for all } t \geq r_0.$$

From these two estimates we get

$$\begin{aligned} F(t) &\leq \varepsilon_0 t^2 + C_1 t^q \exp(\beta t^2), \\ G(t) &\leq \varepsilon_0 t^2 + C_1 t^q \exp(\beta t^2), \end{aligned} \quad \text{for all } t \geq 0,$$

which implies,

$$\begin{aligned} I(u,u) &\geq \int_\Omega |\nabla u|^2\,dx - 2\varepsilon_0 \int_\Omega u^2 - 2C_1 \int_\Omega |u|^q \exp(\beta u^2)\,dx \\ &\geq \int_\Omega |\nabla u|^2\,dx - 2\varepsilon_0 \int_\Omega u^2 \\ &\qquad - 2C_1 \left(\int_\Omega |u|^{qs'}\,dx \right)^{1/s'} \left(\int_\Omega \exp(\beta s u^2)\,dx \right)^{1/s}, \end{aligned}$$

where $1/s + 1/s' = 1$. Using the Trudinger-Moser inequality (TM-2),

$$\int_\Omega \exp(\beta s u^2)\,dx = \int_\Omega \exp\left(\|u\|^2 \beta s \left(\frac{u}{\|u\|} \right)^2 \right) dx \leq C,$$

if $\|u\| \leq \delta$, with $\delta > 0$, such that $\beta s \delta^2 \leq 4\pi$. So, by the Sobolev imbedding theorem we obtain,

$$I(u,u) \geq \|u\|^2 - c_3 \varepsilon \|u\|^2 - c_4 \|u\|^q.$$

Therefore, we can find ρ, $\sigma > 0$, ρ sufficiently small, such that $I(u,u) \geq \sigma > 0$, for $\|u\| = \rho$. □

Let $y \in H_0^1(\Omega)$ be a fixed nonnegative function with $\|y\| = 1$ and

$$Q_y = \{ r(y,y) + w \mid w \in E^-,\ \|w\| \leq R_0 \text{ and } 0 \leq r \leq R_1 \}.$$

Lemma 2.2. *There exist positive constants* R_0, R_1, *which depend on* y, *such that* $I(z) \le 0$ *for all* $z \in \partial Q_y$.

Proof. Notice that the boundary ∂Q_y of the set Q_y is taken in the space $\mathbb{R}(y,y) \oplus E^-$, and consists of three parts. On these parts the functional I is estimated as follows:

(i) If $z \in \partial Q_y \cap E^-$, we have $I(z) \le 0$ because, for all $z = (u,-u) \in E^-$,

$$I(z) = -\int_\Omega |\nabla u|^2 \, dx - \int_\Omega [F(u) + G(-u)] \, dx \le 0.$$

(ii) If $z = R_1(y,y) + (u,-u) \in \partial Q_y$ with $\|(u,-u)\| \le R_0$,

$$\begin{aligned} I(z) = R_1^2 \int_\Omega |\nabla y|^2 \, dx &- \int_\Omega |\nabla u|^2 \, dx \\ &- \int_\Omega [F(R_1 y + u) + G(R_1 y - u)] \, dx. \end{aligned} \tag{2.2}$$

It follows from assumption (H$_3$) that there exist constants c_1, $c_2 > 0$ such that

$$F(t),\ G(t) \ge c_1 t^\theta - c_2, \quad \text{for all } t \ge 0.$$

Let

$$\xi(t) = \begin{cases} t^\theta & \text{if } t \ge 0, \\ 0 & \text{if } t < 0. \end{cases}$$

We then obtain from (2.2) that

$$I(z) \le R_1^2 - \int_\Omega [\xi(R_1 y + u) + \xi(R_1 y - u)] \, dx + c_3.$$

Now, using the convexity of ξ, it follows that

$$I(z) \le R_1^2 - 2\int_\Omega \xi(R_1 y) \, dx + c_3 = R_1^2 - 2R_1^\theta \int_\Omega y^\theta \, dx + c_3.$$

Finally taking $R_1 = R_1(y)$ sufficiently large, we get $I(z) \le 0$.

(iii) If $z = r(y,y) + (u,-u) \in \partial Q_y$ with $\|(u,-u)\| = R_0$ and $0 \le r \le R_1$,

$$\begin{aligned} I(z) &= r^2 \int_\Omega |\nabla y|^2 \, dx - \int_\Omega |\nabla u|^2 \, dx - \int_\Omega [F(ry + u) + G(ry - u)] \, dx \\ &\le R_1^2 - \frac{1}{2} R_0^2. \end{aligned}$$

Thus, $I(z) \le 0$ if $\sqrt{2} R_1 \le R_0$.

So, the geometry of the linking theorem holds. □

2.2. On Palais-Smale Sequences.

Proposition 2.3. *Let* $(u_m, v_m) \in E$ *such that*

(I_1) $I(u_m, v_m) = c + \delta_m$, *where* $\delta_m \to 0$ *as* $m \to +\infty$;

(I_2) $| I'(u_m, v_m)(\varphi, \psi) | \leq \varepsilon_m \|(\varphi, \psi)\|$, *for* φ, $\psi \in \{u_m, v_m\}$, *where* $\varepsilon_m \to 0$ *as* $m \to +\infty$.

Then

$$\|u_m\| \leq C, \qquad \|v_m\| \leq C,$$

$$\int_\Omega f(u_m)u_m\,dx \leq C, \qquad \int_\Omega g(v_m)v_m\,dx \leq C,$$

$$\int_\Omega F(u_m)\,dx \leq C, \qquad \int_\Omega G(v_m)\,dx \leq C.$$

Proof. Taking $(\varphi, \psi) = (u_m, v_m)$ in (I_2), we have

$$\left|2\int_\Omega \nabla u_m \nabla v_m\,dx - \int_\Omega f(u_m)u_m\,dx - \int_\Omega g(v_m)v_m\,dx\right| \leq \varepsilon_m\|(u_m, v_m)\|, \tag{2.3}$$

which together with (I_1) and hypothesis (H_3), implies

$$\begin{aligned}
&\int_\Omega [f(u_m)u_m + g(v_m)v_m]\,dx \\
&\quad\leq 2\int_\Omega [F(u_m) + G(v_m)]\,dx + 2c + 2\delta_m + \varepsilon_m\|(u_m, v_m)\| \\
&\quad\leq \frac{2}{\theta}\int_\Omega [f(u_m)u_m + g(v_m)v_m]\,dx + c_1 + 2\delta_m + \varepsilon_m\|(u_m, v_m)\|,
\end{aligned}$$

where c_1 depends only on c and t_0 in hypothesis (H_3). Thus, for some constant c_2, we obtain

$$\int_\Omega [f(u_m)u_m + g(v_m)v_m]\,dx \leq c_2(1 + 2\delta_m + \varepsilon_m\|(u_m, v_m)\|). \tag{2.4}$$

Next taking $(\varphi, \psi) = (v_m, 0)$ and $(\varphi, \psi) = (0, u_m)$ in (I_2) we have

$$\|v_m\|^2 - \varepsilon_m\|v_m\| \leq \int_\Omega f(u_m)v_m\,dx,$$

$$\|u_m\|^2 - \varepsilon_m\|u_m\| \leq \int_\Omega g(v_m)u_m\,dx.$$

Setting $U_m = u_m/\|u_m\|$ and $V_m = v_m/\|v_m\|$ we have

$$\|v_m\| \leq \int_\Omega f(u_m)V_m\,dx + \varepsilon_m, \tag{2.5}$$

$$\|u_m\| \leq \int_\Omega g(v_m)U_m\,dx + \varepsilon_m. \tag{2.6}$$

We now rely on the following inequality whose proof is given in Lemma 2.4 below,

$$st \le \begin{cases} (e^{t^2}-1)+s(\log s)^{1/2}, & \text{for all } t \ge 0 \text{ and } s \ge e^{1/4}, \\ (e^{t^2}-1)+\dfrac{1}{2}s^2, & \text{for all } t \ge 0 \text{ and } 0 \le s \le e^{1/4}. \end{cases} \tag{2.7}$$

By using inequality (2.7) (with $t = V_m$ and $s = f(u_m)/C$, where C is the constant appearing in (1.5)), estimate (1.5), and the Trudinger-Moser inequality, we obtain

$$\begin{aligned} &C\int_\Omega \frac{1}{C}f(u_m)V_m\,dx \\ &\quad\le C\int_\Omega e^{V_m^2}\,dx + C\int_{\{x\in\Omega \mid f(u_m)(x)/C\ge e^{1/4}\}} \frac{1}{C}f(u_m)\left[\log\frac{1}{C}f(u_m)\right]^{1/2}dx \\ &\qquad\qquad + \frac{1}{2}\int_{\{x\in\Omega \mid f(u_m)(x)/C\le e^{1/4}\}} \frac{1}{C^2}[f(u_m)]^2\,dx \\ &\quad\le c_3 + \beta^{1/2}\int_\Omega f(u_m)u_m\,dx, \end{aligned}$$

for some positive constant c_3. This estimate together with (2.5) implies that, for some constant $c > 0$, we have

$$\|v_m\| \le c\left(1+\int_\Omega f(u_m)u_m\,dx + \varepsilon_m\right), \tag{2.8}$$

Using a similar argument we get from (2.6)

$$\|u_m\| \le c\left(1+\int_\Omega g(v_m)v_m\,dx + \varepsilon_m\right), \tag{2.9}$$

Now joining the estimates (2.8) and (2.9) and using (2.4) we finally obtain

$$\|(u_m,v_m)\| \le c(1+\delta_m+\varepsilon_m\|(u_m,v_m)\|+\varepsilon_m),$$

which implies that $\|(u_m,v_m)\| \le c$. From this estimate, inequality (2.4) and assumption (H_3), we obtain the other estimates in the statement of the proposition. □

Lemma 2.4. *The following inequality holds*

$$st \le \begin{cases} (e^{t^2}-1)+s(\log^+ s)^{1/2}, & \text{for all } t \ge 0 \text{ and } s \ge e^{1/4}, \\ (e^{t^2}-1)+\dfrac{1}{2}s^2, & \text{for all } t \ge 0 \text{ and } 0 \le s \le e^{1/4}. \end{cases}$$

Proof. For $s > 0$ given, consider $\sup_{t\geq 0}\{ts - (e^{t^2} - 1)\}$. Let t_s denote the (unique) point where the supremum is attained. Then $s = 2t_s e^{t_s^2}$. Consider now three cases:

Case 1: $t_s \geq \frac{1}{2}$; then $s = 2t_s e^{t_s^2} \geq e^{t_s^2}$, which implies $(\log s)^{1/2} \geq t_s$. Thus

$$\sup_{t\geq 0}\{ts - (e^{t^2} - 1)\} \leq t_s s - e^{t_s^2} \leq t_s s \leq (\log s)^{1/2} s.$$

Case 2: $0 \leq t_s \leq \frac{1}{2}$ and $s \geq e^{1/4}$; then $st_s \leq s/2$ and $s/2 \leq s(\log^+ s)^{1/2}$ iff $s \geq e^{1/4}$.

Case 3: $0 \leq t_s \leq \frac{1}{2}$ and $s \leq e^{1/4}$; in fact, the second inequality in (2.7) holds always, since

$$ts \leq \frac{1}{2}t^2 + \frac{1}{2}s^2 \leq \frac{1}{2}(e^{t^2} - 1) + \frac{1}{2}s^2.$$

Hence, the lemma is proved. ◻

2.3. Finite Dimensional Problem. Since the functional I is strongly indefinite and defined in an infinite dimensional space, no suitable linking theorem is available. We therefore approximate problem (1.1) with a sequence of finite dimensional problems (a Galerkin approximation procedure).

Associated with the eigenvalues $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_j \to \infty$ of $(-\Delta, H_0^1(\Omega))$, there exists an orthonormal basis $\{\varphi_1, \varphi_2, \dots\}$ of corresponding eigenfunctions in $H_0^1(\Omega)$. We set,

$$\begin{aligned} E_n^+ &= \operatorname{span}\{(\varphi_i, \varphi_i) \mid i = 1, \dots, n\}, \\ E_n^- &= \operatorname{span}\{(\varphi_i, -\varphi_i) \mid i = 1, \dots, n\}, \\ E_n &= E_n^+ \oplus E_n^-. \end{aligned}$$

Let $y \in H_0^1(\Omega)$ be a fixed nonnegative function with $\|y\| = 1$ and

$$Q_{n,y} = \{r(y, y) + w \mid w \in E_n^-, \ \|w\| \leq R_0, \text{ and } 0 \leq r \leq R_1\},$$

where R_0 and R_1 are given in Lemma 2.2. We recall that these constants depend of y only. We use the following notation:

$$H_{n,y} = \mathbb{R}(y, y) \oplus E_n, \quad H_{n,y}^+ = \mathbb{R}(y, y) \oplus E_n^+, \quad H_{n,y}^- = \mathbb{R}(y, y) \oplus E_n^-.$$

Furthermore, define the class of mappings

$$\Gamma_{n,y} = \{h \in C(Q_{n,y}, H_{n,y}) \mid h(z) = z \text{ on } \partial Q_{n,y}\}$$

and set

$$c_{n,y} = \inf_{h\in\Gamma_{n,y}} \max_{z\in Q_{n,y}} I(h(z)).$$

Using an intersection theorem (see Proposition 5.9 in [12]), we have

$$h(Q_{n,y}) \cap (\partial B_\rho \cap E^+) \neq \emptyset, \quad \forall h \in \Gamma_{n,y},$$

which in combination with Lemma 2.1 implies that $c_{n,y} \geq \sigma > 0$. On the other hand, an upper bound for the mini-max level $c_{n,y}$ can be obtained as follows. Since the identity mapping $\mathrm{Id} : Q_{n,y} \to H_{n,y}$ belongs to $\Gamma_{n,y}$, we have for $z = r(y,y) + (u,-u) \in Q_{n,y}$ that

$$I(z) = r^2 \int_\Omega |\nabla y|^2\,dx - \int_\Omega |\nabla u|^2\,dx - \int_\Omega [F(ry+u) + G(ry-u)]\,dx \leq R_1^2.$$

Therefore we have $0 < \sigma \leq c_{n,y} \leq R_1^2$. We remark that the upper bound does not depend of n, but it depends on y.

Let us denote by $I_{n,y}$ the functional I restricted to the finite dimensional subspace $H_{n,y}$. So, in view of Lemmas 2.1 and 2.2, we see that the geometry of a linking theorem holds for the functional $I_{n,y}$. Therefore, applying the linking theorem for $I_{n,y}$ (see Theorem 5.3 in [12]), we obtain a (PS)-sequence, which is bounded in view of Proposition 2.3. Finally, using the fact that $H_{n,y}$ is a finite dimensional space, we get the main result of this section.

Proposition 2.5. *For each $n \in \mathbb{N}$ and for each $y \in E$, a fixed nonnegative function with $\|y\| = 1$, the functional $I_{n,y}$, has a critical point at level $c_{n,y}$. More precisely, there is a $z_{n,y} \in H_{n,y}$ such that*

$$I_{n,y}(z_{n,y}) = c_{n,y} \in [\sigma, R_1^2], \quad (I_{n,y})'(z_{n,y}) = 0.$$

Furthermore, $\|z_{n,y}\| \leq C$ where C does not depend of n.

3. Subcritical Case - Proof of Theorem 1.1

In this section we assume that g has subcritical growth (see definition in (1.3)).

Let $y \in E$ be a fixed nonnegative function with $\|y\| = 1$. Applying Proposition 2.5, we have a sequence $z_{n,y} \in H_{n,y}$ bounded in E and such that

$$I_{n,y}(z_{n,y}) = c_{n,y} \in [\sigma, R_1^2], \tag{3.1}$$

$$(I_{n,y})'(z_{n,y}) = 0, \tag{3.2}$$

$$z_{n,y} := (u_{n,y}, v_{n,y}) \rightharpoonup (u_o, v_o) \quad \text{in } E, \tag{3.3}$$

$$u_{n,y} \to u_o \text{ and } v_{n,y} \to v_o \quad \text{in } L^q(\Omega),\ \forall q \geq 1, \tag{3.4}$$

$$u_{n,y}(x) \to u_o(x) \text{ and } v_{n,y}(x) \to v_o(x) \quad \text{a.e. in } \Omega. \tag{3.5}$$

Next, using Proposition 2.3 we conclude

$$\int_\Omega f(u_{n,y})u_{n,y}\,dx \leq C, \qquad \int_\Omega g(v_{n,y})v_{n,y}\,dx \leq C, \tag{3.6}$$

$$\int_\Omega F(u_{n,y})\,dx \leq C, \qquad \int_\Omega G(v_{n,y})\,dx \leq C. \tag{3.7}$$

Taking as test functions $(0, \psi)$ and $(\varphi, 0)$ in (3.2), where φ and ψ are arbitrary functions in $F_n := \operatorname{span}\{\varphi_i \mid i = 1, \dots, n\}$, we get

$$\int_\Omega \nabla u_{n,y} \nabla \psi \, dx = \int_\Omega g(v_{n,y}) \psi \, dx, \quad \forall \varphi \in F_n, \tag{3.8}$$

$$\int_\Omega \nabla v_{n,y} \nabla \varphi \, dx = \int_\Omega f(u_{n,y}) \varphi \, dx, \quad \forall \varphi \in F_n. \tag{3.9}$$

Next, using (3.6) and an argument similar to the one used in Lemma 2.1 in [6], we can prove that $f(u_{n,y}) \to f(u_o)$ and $g(v_{n,y}) \to g(u_o)$ in $L^1(\Omega)$. Thus, taking the limit in (3.8) and (3.9) and using the fact that $\bigcup_{n \in \mathbb{N}} F_n$ is dense in $H_0^1(\Omega)$, it follows that

$$\left.\begin{aligned} \int_\Omega \nabla u_o \nabla \psi \, dx &= \int_\Omega g(v_o) \psi \, dx, \\ \int_\Omega \nabla v_o \nabla \varphi \, dx &= \int_\Omega f(u_o) \varphi \, dx, \end{aligned}\right. \qquad \forall \varphi, \psi \in H_0^1(\Omega).$$

Since $f(u_o), g(v_o) \in L^2(\Omega)$ we conclude that $u_o, v_o \in H^2(\Omega)$ and

$$-\Delta u_o = g(v_o) \quad \text{and} \quad -\Delta v_o = f(u_o)$$

in the strong sense.

Finally, it only remains to prove that u_o and v_o are nontrivial. Assume by contradiction that $u_o \equiv 0$. This implies that $v_o \equiv 0$. Since g has subcritical growth, we see that for all $\delta > 0$ there exists $C_\delta > 0$ such that

$$g(t) \le C_\delta e^{\delta t^2}, \quad \forall t \in \mathbb{R}.$$

Now, using Hölder inequality we get

$$\left| \int_\Omega g(v_{n,y}) u_{n,y} \, dx \right| \le C_\delta |u_{n,y}|_{L^{q'}} |e^{\delta v_{n,y}^2}|_{L^q} \le \tilde{C}_\delta |u_{n,y}|_{L^{q'}},$$

since by the Trudinger-Moser inequality (TM-2) we have

$$|e^{\delta v_{n,y}^2}|_{L^q}^q = \int_\Omega e^{\delta q v_{n,y}^2} \, dx \le C.$$

Indeed, we can take δ and q such that $q\delta \|v_{n,y}\|^2 \le 4\pi$. Now it follows from (3.2) that

$$\int_\Omega |\nabla u_{n,y}|^2 \, dx = \int_\Omega g(v_{n,y}) u_{n,y} \, dx \le \tilde{C}_\delta |u_{n,y}|_{L^{q'}},$$

and so we conclude that $u_{n,y} \to 0$ strongly in $H_0^1(\Omega)$, because $u_{n,y} \to 0$ in $L^{q'}(\Omega)$. This implies that

$$\lim_{n\to+\infty} \int_\Omega \nabla u_{n,y} \nabla v_{n,y}\, dx = 0. \tag{3.10}$$

Then we obtain by (3.2)

$$\lim_{n\to+\infty} \int_\Omega u_{n,y} f(u_{n,y})\, dx = 0 \quad \text{and} \quad \lim_{n\to+\infty} \int_\Omega v_{n,y} g(v_{n,y})\, dx = 0.$$

Using these limits and (H_3) it follows that

$$\lim_{n\to+\infty} \int_\Omega F(u_{n,y})\, dx = 0 \quad \text{and} \quad \lim_{n\to+\infty} \int_\Omega G(u_{n,y})\, dx = 0. \tag{3.11}$$

Observe that in this conclusion we have used the fact that

$$\int_{\{x\in\Omega | u_{n,y}(x) \le t_0\}} F(u_{n,y})\, dx \to 0.$$

Finally, using (3.10) and (3.11) we see that $c_{n,y} \to 0$, which is a contradiction to (3.1). Consequently, we have a nontrivial critical point of I, and thereby conclude the proof of the Theorem 1.1.

4. Critical Case - Proof of Theorem 1.2

In this section we assume that f and g have critical growth (see definition in (1.4)).

Let d be the inner radius of Ω, that is, it is the radius of the largest open ball contained in Ω. So $B_d(x_0) \subset \Omega$ for some $x_0 \in \Omega$. We may assume that $x_0 = 0$.

We start by introducing the following concentrating functions

$$y_k(x) = \bar{\omega}_k\left(\frac{x}{d}\right), \quad k \in \mathbb{N},$$

where

$$\bar{\omega}_k(x) = \frac{1}{\sqrt{2\pi}} \begin{cases} (\log k)^{1/2}, & |x| \le 1/k, \\ \dfrac{\log(1/|x|)}{(\log k)^{1/2}}, & 1/k \le |x| \le 1, \\ 0, & |x| \ge 1. \end{cases} \tag{4.1}$$

We also consider the sets

$$Q_{n,k} := Q_{n,y_k} = \{r(y_k, y_k) + w \mid w \in E_n^-,\ \|w\| \le R, \text{ and } 0 \le r \le R_1\}.$$

Next we assume the following result, which will be proved later.

Proposition 4.1. *There exists $k \in \mathbb{N}$ such that*

$$\sup_{\mathbb{R}(y_k, y_k) \oplus E^-} I < \frac{4\pi}{\alpha_0}.$$

This proposition implies that there is $\delta > 0$ such that for all n we have $c_n := c_{n,y_k} \leq 4\pi/\alpha_o - \delta$, where c_{n,y_k} is defined in Proposition 2.5. In fact, in view of Propositions 2.3, 2.5 and 4.1, there exists $\delta > 0$ such that for each n we have $z_n := z_{n,k} \in H_{n,k}$ such that

$$\|z_n\| \leq C \quad \text{in } E, \tag{4.2}$$

$$I_{n,k}(z_n) = c_n \in \left[\sigma, \frac{4\pi}{\alpha_o} - \delta\right), \tag{4.3}$$

$$(I_{n,k})'(z_n) = 0, \tag{4.4}$$

$$z_n := (u_n, v_n) \rightharpoonup (u_o, v_o) \quad \text{in } E, \tag{4.5}$$

$$u_n \to u_o \text{ and } v_n \to v_o \quad \text{in } L^q(\Omega),\ \forall q \geq 1, \tag{4.6}$$

$$u_n(x) \to u_o(x) \text{ and } v_n(x) \to v_o(x) \quad \text{a.e. in } \Omega. \tag{4.7}$$

Using Proposition 2.3 we conclude

$$\int_\Omega f(u_n)u_n\,dx \leq C, \quad \int_\Omega g(v_n)v_n\,dx \leq C, \tag{4.8}$$

$$\int_\Omega F(u_n)\,dx \leq C, \quad \int_\Omega G(v_n)\,dx \leq C. \tag{4.9}$$

Taking the test functions $(0, \psi)$ and $(\varphi, 0)$ in (4.4),

$$\int_\Omega \nabla u_n \nabla \psi\,dx = \int_\Omega g(v_n)\psi\,dx, \quad \forall \varphi \in F_n, \tag{4.10}$$

$$\int_\Omega \nabla v_n \nabla \varphi\,dx = \int_\Omega f(u_n)\varphi\,dx, \quad \forall \varphi \in F_n. \tag{4.11}$$

Arguing as in the subcritical case, taking the limits in (4.10) and (4.11) and using that $\bigcup_{n\in\mathbb{N}} F_n$ is dense in $H_0^1(\Omega)$, it follows that

$$\int_\Omega \nabla u_o \nabla \psi\,dx = \int_\Omega g(v_o)\psi\,dx \quad \text{and} \quad \int_\Omega \nabla v_o \nabla \varphi\,dx = \int_\Omega f(u_o)\varphi\,dx, \qquad \forall \varphi, \psi \in H_0^1(\Omega).$$

So, it remains to prove that u_o and v_o are nontrivial. Assume by contradiction that $u_o \equiv 0$. This implies that $v_o \equiv 0$. Now, if $\|u_n\| \to 0$, then we get directly

(4.15) below, and then a contradiction. Thus, assume that $\|u_n\| \geq b > 0$, $\forall n$, and consider

$$\|u_n\|^2 = \int_\Omega g(v_n)u_n\,dx. \tag{4.12}$$

Setting $\bar{u}_n = (4\pi/\alpha_0 - \delta)^{1/2}u_n/\|u_n\|$, and using inequality (2.7) with $s = g(v_n)/\sqrt{\alpha_0}$ and $t = \sqrt{\alpha_0}\bar{u}_n$ we have

$$\begin{aligned}\left(\frac{4\pi}{\alpha_0} - \delta\right)^{1/2}\|u_n\| &= \int_\Omega g(v_n)\bar{u}_n\,dx\\ &\leq \int_\Omega (e^{\alpha_0\bar{u}_n^2} - 1)\,dx\\ &\quad + \int_{\{x\in\Omega | g(v_n)(x)/\sqrt{\alpha_0}\geq e^{1/4}\}} \frac{g(v_n)}{\sqrt{\alpha_0}}\left[\log\left(\frac{g(v_n)}{\sqrt{\alpha_0}}\right)\right]^{1/2} dx\\ &\quad + \frac{1}{2}\int_{\{x\in\Omega | g(v_n)(x)/\sqrt{\alpha_0}\leq e^{1/4}\}} \frac{[g(v_n)]^2}{\alpha_0}\,dx.\end{aligned}$$

Since $\|\bar{u}_n\|^2 = 4\pi/\alpha_0 - \delta$, we know by (TM-3) that the first term tends to zero, while the third term tends to zero by Lebesgue's dominated convergence theorem. Using Remark 1.3(3) we can estimate the second term by

$$\begin{aligned}&\int_\Omega \frac{1}{\sqrt{\alpha_0}}g(v_n)\left(\log\left(\frac{C_\varepsilon}{\sqrt{\alpha_0}}e^{(\alpha_0+\varepsilon)v_n^2}\right)\right)^{1/2}\\ &\quad\leq \frac{1}{\sqrt{\alpha_0}}\int_\Omega g(v_n)\left(\log\left(\frac{C_\varepsilon}{\sqrt{\alpha_0}}\right)^{1/2} + (\alpha_0+\varepsilon)^{1/2}v_n\right)\\ &\quad\leq o(1) + \left(1 + \frac{\varepsilon}{\alpha_0}\right)^{1/2}\int_\Omega g(v_n)v_n,\end{aligned}$$

and hence we obtain

$$\left(\frac{4\pi}{\alpha_0} - \delta\right)^{1/2}\|u_n\| \leq o(1) + \left(1 + \frac{\varepsilon}{\alpha_0}\right)^{1/2}\int_\Omega g(v_n)v_n\,dx. \tag{4.13}$$

Repeating the same argument with

$$\|v_n\|^2 = \int_\Omega f(u_n)v_n\,dx,$$

we see that also

$$\left(\frac{4\pi}{\alpha_0} - \delta\right)^{1/2}\|v_n\| \leq o(1) + \left(1 + \frac{\varepsilon}{\alpha_0}\right)^{1/2}\int_\Omega f(u_n)u_n\,dx. \tag{4.14}$$

Since

$$\lim_{n\to+\infty}\int_\Omega F(u_n)\,dx = 0 \quad \text{and} \quad \lim_{n\to+\infty}\int_\Omega G(u_n)\,dx = 0,$$

we conclude from (4.3) that

$$\left|\int_\Omega \nabla u_n \nabla v_n\,dx\right| \le o(1) + \frac{4\pi}{\alpha_0} - \delta,$$

which together with (4.4) implies that

$$\int_\Omega f(u_n)u_n\,dx + \int_\Omega g(v_n)v_n\,dx \le o(1) + 2\left(\frac{4\pi}{\alpha_0} - \delta\right).$$

So, from (4.13)-(4.14) we obtain

$$\|u_n\| + \|v_n\| \le o(1) + 2\left(1+\frac{\varepsilon}{\alpha_0}\right)^{1/2}\left(\frac{4\pi}{\alpha_0} - \delta\right)^{1/2} \le 2\left(\frac{4\pi}{\alpha_0} - \frac{\delta}{2}\right)^{1/2},$$

for $\varepsilon > 0$ sufficiently small and n sufficiently large. It follows that there is a subsequence of (u_n) or (v_n) (without loss of generality assume it is (v_n)) such that

$$\|v_n\| \le \left(\frac{4\pi}{\alpha_0} - \frac{\delta}{2}\right)^{1/2}.$$

Thus, using (1.6) with $\varepsilon > 0$ and the Hölder inequality with $q > 1$ such that $(\alpha_0+\varepsilon)(4\pi/\alpha_0 - \delta/2)q \le 4\pi$, we get

$$\left|\int_\Omega g(v_n)v_n\,dx\right| \le C_\varepsilon |v_n|_{L^{q'}} |e^{(\alpha_0+\varepsilon)v_n^2}|_{L^q} \le C|v_n|_{L^{q'}}.$$

Since $|v_n|_{L^{q'}} \to 0$ we conclude by (4.12) that $u_n \to 0$ strongly in $H_0^1(\Omega)$, and hence

$$\lim_{n\to+\infty}\int_\Omega \nabla u_n \nabla v_n\,dx = 0. \tag{4.15}$$

By (3.10) and (3.11) we conclude that $c_n \to 0$, which is a contradiction to (4.3).

4.1. On the mini-max level – Proof of Proposition 4.1.

Proof. Suppose by contradiction that for all k,

$$\sup_{\mathbb{R}(y_k,y_k)\oplus E^-} I \ge \frac{4\pi}{\alpha_0}.$$

So, for all fixed k, there exists $\delta_n \to 0$ as $n \to \infty$ and

$$\eta_{n,k} = \tau_{n,k}(y_k, y_k) + (u_{n,k}, -u_{n,k}) \in Q_{n,k}$$

such that

$$I(\eta_{n,k}) \geq \frac{4\pi}{\alpha_0} - \delta_n.$$

Let $h : [0, \infty) \to \mathbb{R}$, $h(t) := I(t\eta_{n,k})$; as $h(0) = 0$ and $\lim_{t\to+\infty} h(t) = -\infty$, there exists a maximum point $\bar{t}\eta_{n,k}$ with $I(\bar{t}\eta_{n,k}) \geq 4\pi/\alpha_0 - \delta_n$. We may assume that $\eta_{n,k}$ is this point, and thus $I'(\eta_{n,k})\eta_{n,k} = 0$. Let us write in detail $I(\eta_{n,k}) \geq 4\pi/\alpha_0 - \delta_n$ and $I'(\eta_{n,k})\eta_{n,k} = 0$:

$$\tau_{n,k}^2 - \int_\Omega |\nabla u_{n,k}|^2 - \int_\Omega [F(\tau_{n,k}y_k + u_{n,k}) + G(\tau_{n,k}y_k - u_{n,k})] \geq \frac{4\pi}{\alpha_0} - \delta_n; \tag{4.16}$$

$$\tau_{n,k}^2 - \int_\Omega |\nabla u_{n,k}|^2 = \int_\Omega [f(\tau_{n,k}y_k + u_{n,k})(\tau_{n,k}y_k + u_{n,k}) + g(\tau_{n,k}y_k - u_{n,k})(\tau_{n,k}y_k - u_{n,k})]. \tag{4.17}$$

From (4.16), we get $4\pi/\alpha_0 + s_{n,k} := \tau_{n,k}^2 \geq 4\pi/\alpha_0 - \delta_n$. From ($H_5$), there exists $\beta_0 > 4/\alpha_0 d^2$ such that

$$\lim_{t\to+\infty} tf(t)e^{-\alpha_0 t^2} \geq \beta_0 \quad \text{and} \quad \lim_{t\to+\infty} tg(t)e^{-\alpha_0 t^2} \geq \beta_0. \tag{4.18}$$

So, given $\varepsilon > 0$, there exists R_ε such that

$$tf(t),\ tg(t) \geq (\beta_0 - \varepsilon)e^{\alpha_0 t^2} \quad \text{for all } t \geq R_\varepsilon.$$

Next, choosing k sufficiently large such that $\tau_{n,k}(\log k/2\pi)^{1/2} \geq R_\varepsilon$, we get that $\max\{\tau_{n,k}y_k + u_{n,k}, \tau_{n,k}y_k - u_{n,k}\} \geq R_\varepsilon$ for all $x \in B_{d/k}(0)$. So, on $B_{d/k}(0)$,

$$\begin{aligned} \frac{4\pi}{\alpha_0} + s_{n,k} &\geq (\beta_0 - \varepsilon)\int_{B_{d/k}(0)} e^{\alpha_0 \tau_{n,k}^2 \log k/2\pi} \\ &\geq (\beta_0 - \varepsilon)\frac{\pi d^2}{k^2} e^{\alpha_0(4\pi/\alpha_0 + s_{n,k})\log k/2\pi} \\ &= (\beta_0 - \varepsilon)\pi d^2 e^{\alpha_0 s_{n,k}\log k/2\pi}. \end{aligned} \tag{4.19}$$

Note that by (4.16) we have for each fixed k, $\lim_{n\to\infty} s_{n,k} \to s_{0,k} \geq 0$, and then (4.19) implies that $s_{0,k} = 0$. Thus we see that

$$\frac{4\pi}{\alpha_0} \geq (\beta_0 - \varepsilon)\pi d^2.$$

This contradicts (4.18), since $\varepsilon > 0$ is arbitrary. □

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