On nonlinear perturbations of a periodic elliptic problem in \( \mathbb{R}^2 \) involving critical growth

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Abstract

We consider the equation 
\[-\Delta u + V(x)u = f(x,u) \text{ for } x \in \mathbb{R}^2 \]
where \( V : \mathbb{R}^2 \to \mathbb{R} \) is a positive potential bounded away from zero, and the nonlinearity \( f : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R} \) behaves like \( \exp(Z|u|^2) \) as \( |u| \to \infty \). We also assume that the potential \( V(x) \) and the nonlinearity \( f(x,u) \) are asymptotically periodic at infinity. We prove the existence of at least one weak positive solution \( u \in H^1(\mathbb{R}^2) \) by combining the mountain-pass theorem with Trudinger–Moser inequality and a version of a result due to Lions for critical growth in \( \mathbb{R}^2 \).

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1. Introduction

Coti-Zelati and Rabinowitz [5] studied a class of nonlinear Schrödinger equation in whole space \( \mathbb{R}^N, N \geq 3 \), of the form

\[-\Delta u + V(x)u = f(x,u) \text{ in } \mathbb{R}^N, \quad u \in H^1(\mathbb{R}^N), \quad u > 0, \quad (1.1)\]

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where the potential \( V : \mathbb{R}^N \to \mathbb{R} \) and the nonlinearity \( f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R} \) satisfied the following periodicity conditions

\[
V(x + p) = V(x), \quad f(x + p, s) = f(x, s), \quad \forall x \in \mathbb{R}^N, \quad p \in \mathbb{Z}^N, \quad s \in \mathbb{R}.
\]

In their work, they obtained results of existence and multiplicity by applying the mountain-pass Theorem [12] together with a sort of Concentration Compactness Principle of Lions (see book [11]), when in addition to the periodicity condition, \( f \) had subcritical growth, that is, the growth of \( f \) in \( s \) must be like \(|s|^{p-1}s; p < (N+2)/(N-2) \equiv 2^*-1 \). Later considering a small nonperiodic perturbations of \( V \) and \( f \) in the subcritical case or treating situation where \( f \) is a pure power like \(|s|^{p-1}s \) with \( p \geq 2^*-1 \) (critical and supercritical) their result were extended or complemented, for instance, by Montechiari [13], Alves-Carrion and Miyagaki [3], Zhu and Yang [15], Noussair-Swanson and Yang [14], and references therein. We recall that problem (1.1) in \( \mathbb{R} \) or \( \mathbb{R}^2 \) when \( f \) behaves such as a power \(|s|^{p-1}s \) with \( p < \infty \), it can be handled quite simply compared to the case \( \mathbb{R}^N, \ N \geq 3 \). On the other hand, in Admurth and Yadava [2] (see also [7]) motivated by Trudinger–Moser inequality, namely

\[
\exp(|u|^2) \in L^1(\Omega), \quad \forall u \in H^1_0(\Omega), \quad \alpha > 0,
\]

and

\[
\sup_{|\nabla u|_{2^*(\Omega)} \leq 1} \int_\Omega |\nabla u|^2 \leq C_2 \leq \infty, \quad \alpha \leq \alpha_2 = 4\pi,
\]

where \( \Omega \) is a bounded domain. They introduced the notion of criticality in \( \mathbb{R}^2 \); that is, \( f(s) \) has a critical growth when it behaves like \( \exp(\alpha |s|^2) \) as \( s \to +\infty \). More exactly, there exists \( \alpha_0 > 0 \) such that

\[
\lim_{s \to +\infty} \frac{f(x, s)}{\exp(\alpha_0 s^2)} = 0, \quad \forall \alpha > \alpha_0, \quad \lim_{s \to +\infty} \frac{f(x, s)}{\exp(\alpha s^2)} = +\infty, \quad \forall \alpha < \alpha_0.
\]

We would like to mention, for instance, the papers by [1,2,6,7] which contain some results of existence for problem (1.1) in a bounded domain making use of this inequality. Afterwards, Cao in [4] proved a version of Moser–Trudinger inequality in whole space \( \mathbb{R}^2 \), which was improved by do Ó in [9];

\[
\int_{\mathbb{R}^2} (\exp(|u|^2) - 1) \, dx < +\infty, \quad \forall u \in H^1(\mathbb{R}^2), \quad \alpha > 0.
\]

Moreover, if \( \alpha < 4\pi \) and \( |u|_{L^{2^*}(\mathbb{R}^2)} \leq C \), there exists a constant \( C_2 = C_2(C, \alpha) \) such that

\[
\sup_{|\nabla u|_{2^*(\mathbb{R}^2)} \leq 1} \int_{\mathbb{R}^2} (\exp(|u|^2) - 1) \, dx \leq C_2.
\]

By combining this inequality with mountain-pass theorem do Ó and Souto in [10] studied problem (1.1) when the potential \( V \) satisfies some geometric condition. Same way do Ó in [9] got some existence result for problem (1.1) involving N-Laplacian operator but imposing a coercivity condition on the potential. On the other hand, Cao
in [4] treated the situation where \( V \) and \( f \) are asymptotic to a constant function, without any coercivity condition on the potential.

We point out that in all the works mentioned above they made use of the so-called Ambrosetti–Rabinowitz condition, namely, there is a constant \( Z_{DC2} \) such that
\[
0 \leq Z_{DC2}F(x; \cdot) - sf(x, \cdot) \leq \varepsilon e^{4\pi^2},
\]
for all \( (x; \cdot) \in \mathbb{R}^2 \times [0, +\infty) \).

In this paper, by combining arguments used in [3, 8–10] together with inequality (1.2) we will discuss the existence of solution for the critical periodic and asymptotic periodic problem (1.1) in \( \mathbb{R}^2 \), that is,
\[
- \Delta u + V(x)u = f(x, u) \quad \text{in} \quad \mathbb{R}^2, \quad u \in H^1(\mathbb{R}^2), \quad u > 0,
\]
where \( V: \mathbb{R}^2 \to \mathbb{R} \) is a continuous function satisfying \( V(x) \geq \tilde{V}_0 > 0 \) for all \( x \in \mathbb{R}^2 \).

Next we shall describe the conditions on the functions \( V(x) \) and \( f(x, s) \) in a more precise way.

(H1) There exists a continuous 1-periodic function \( V_0: \mathbb{R}^2 \to \mathbb{R} \) such that
\[
\begin{align*}
(1) & \quad V_0(x) \geq V(x) \quad \text{for all} \quad x \in \mathbb{R}^2, \\
(2) & \quad V_0(x) - V(x) \to 0 \quad \text{as} \quad |x| \to \infty.
\end{align*}
\]

We assume that the nonlinearity \( f: \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R} \) is continuous and satisfies the following conditions:

(H2) there exists a continuous 1-periodic function \( f_0: \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R} \) such that
\[
\begin{align*}
(1) & \quad 0 \leq f_0(x, s) \leq f(x, s) \quad \text{for all} \quad (x, s) \in \mathbb{R}^2 \times [0, +\infty), \\
(2) & \quad \text{for all} \quad \varepsilon > 0, \quad \text{there exists} \quad \eta > 0 \quad \text{such that} \quad \text{for} \quad s \geq 0 \quad \text{and} \quad |x| \geq \eta,
\end{align*}
\]
\[
\|f(x, s) - f_0(x, s)\| \leq \varepsilon e^{4\pi^2}.
\]

(H3) \( f(x, s) = o_1(s) \) near origin uniformly with respect to \( x \in \mathbb{R}^2 \);

(H4) \( f \) has critical growth at \( +\infty \), namely,
\[
f(x, s) \leq Ce^{4\pi^2} \quad \text{for all} \quad (x, s) \in \mathbb{R}^2 \times [0, +\infty);
\]

(H5) there exist constants \( \mu \geq \theta > 2 \) such that
\[
0 \leq \theta F_0(x, s) < sf_0(x, s) \quad \text{and} \quad 0 \leq \mu F(x, s) < sf(x, s) \quad \text{for all} \quad (x, s) \in \mathbb{R}^2 \times (0, +\infty),
\]
where the functions \( F_0 \) and \( F \) are the primitive of \( f_0 \) and \( f \) respectively;

(H6) for each fixed \( x \in \mathbb{R}^2 \), the functions \( s \to f_0(x, s)/s \) and \( s \to f(x, s)/s \) are increasing;

(H7) there are constants \( p > 2 \) and \( C_p \) such that
\[
f_0(x, \cdot) \geq C_p s^{p-1} \quad \text{for all} \quad (x, s) \in \mathbb{R}^2 \times [0, +\infty),
\]
where
\[
C_p = \sqrt[p]{\frac{\theta(p - 2)}{p(\theta - 2)}} S_p,
\]
\[
S_p = \inf_{u \in H^1(\mathbb{R}^2) \setminus \{0\}} \left( \frac{\int_{\mathbb{R}^2} (|\nabla u|^2 + V_1 u^2) \, dz}{\left( \int_{\mathbb{R}^2} |u|^p \, dz \right)^{1/p}} \right)^{1/2}
\]
(1.4)
and
\[ V_1 = \max_{x \in \mathbb{R}^2} V_0(x); \]

(H8) at least one of the nonnegative continuous functions
\[ V_0(x) - V(x) \quad \text{and} \quad f(x,u) - f_0(x,u) \]

is positive on a set of positive measure.

Our first result is concerned on the existence of solutions for the following periodic critical problem:
\[ -\Delta u + V_0(x)u = f_0(x,u) \quad \text{in} \quad \mathbb{R}^2, \quad u \in H^1(\mathbb{R}^2), \quad u \geq 0. \tag{1.5} \]

**Theorem 1.1.** In addition to (H1−1) and (H2−1), suppose that (H3)−(H7) hold, then (1.5) possesses a nontrivial weak solution \( u \in H^1(\mathbb{R}^2) \).

Then, using the above result we shall prove the existence of solutions for the asymptotic periodic problem (1.3) and we have the main theorem of this paper.

**Theorem 1.2.** If (H1)−(H8) are satisfied, then (1.3) possesses a nontrivial weak solution \( u \in H^1(\mathbb{R}^2) \).

The above results complete the study made in [4] once that Cao considered the case where the potential and the nonlinearities are asymptotic to the constant functions with respect to \( x \), and the behavior at infinity between the perturbed and unperturbed nonlinearities is near linear growth. Here we work with a general class of functions which are asymptotic to a nonautonomous periodic function, the behavior at infinity between the perturbed and unperturbed nonlinearities is exponential. This work also completes the study made in [9,10], in the sense that the potential \( V \) belongs to a class different from those treated by them.

To prove our main theorems we used variational methods. An important point is a version of a Lions’ results for critical growth in \( \mathbb{R}^2 \). This results is crucial to establish some properties involving the Palais–Smale sequence.

**Remark 1.3.** We notice that in the proof of Theorem 1.1 we shall use only parts of assumptions (H1)−(H7) related with problem (1.5).

The organization of this paper is as follows: Section 2 contains preliminary results and the proof of Theorem 2. In Section 3, we prove our main result, Theorem 1.

**Notation.** In this paper we make use of the following notations:
- \( L^p, L^p_{loc}, H^1, H^1_0 \) will be used to denote standard Lebesgue, or Sobolev spaces.
- The usual norm in \( L^p(\Omega) \) will be denoted by \( | \cdot |_{L^p(\Omega)} \).
- \( C \) denotes (possibly different) positive constants;
- \( B(p,R) \) denote the open ball with the radius \( R \) centered at point \( p \) of \( \mathbb{R}^2 \);
2. The periodic problem

The main objective of this section is to study the existence of solutions for the following periodic critical problem (1.5). Since we are interested in obtaining positive solutions, it is convenient to set

\[ f_0(x,s) = f(x,s) \equiv 0 \text{ for all } (x,s) \in \mathbb{R}^2 \times (-\infty,0]. \]

Let \( E_0 \) denote the Sobolev space \( H^1(\mathbb{R}^2) \) endowed with the equivalent norm

\[ \|u\|_0 = \left\{ \int_{\mathbb{R}^2} \left[ |\nabla u|^2 + V_0(x)|u|^2 \right] \, dx \right\}^{1/2}. \]

From assumptions (H2) and (H4), given \( Z_{SI}^0 \) there exist positive constants \( C_{ZSI} \) and \( Z_{FF}^1 \) such that

\[ F_0(x,s) \leq \epsilon \frac{s^2}{2} + C_e (e^{\beta \pi^2} - 1) \text{ for all } (x,s) \in \mathbb{R}^2 \times \mathbb{R}. \]

Thus, by a Trudinger–Moser inequality (see Lemma 1 in [9]), we have \( F_0(x,u) \in L^1(\mathbb{R}^2) \) for all \( u \in H^1(\mathbb{R}^2) \). Therefore, the functional

\[ J_0(u) = \frac{1}{2} \int_{\mathbb{R}^2} \left[ |\nabla u|^2 + V_0(x)|u|^2 \right] \, dx - \int_{\mathbb{R}^2} F_0(x,u) \, dx \]

is well defined. Furthermore, using standard arguments we see that \( J_0 \) is \( C^1 \) functional on \( E_0 \) with

\[ J'_0(u)\phi = \int_{\mathbb{R}^2} [\nabla u \nabla \phi + V_0(x)u\phi] \, dx - \int_{\mathbb{R}^2} f_0(x,u)\phi \, dx, \text{ for all } \phi \in E_0. \]

Consequently, critical points of \( J_0 \) are precisely the weak solutions of problem (1.5).

The next result concerns the mountain-pass geometry of \( J_0 \). Its proof is a consequence of our assumptions (H3)–(H4), and it can be found in [9].

Lemma 2.1 (Mountain-pass geometry). The functional \( J_0 \) satisfies the following conditions:

(i) there exist \( \rho, \beta > 0 \), such that \( J_0(u) \geq \beta \), if \( \|u\|_0 = \rho \) and

(ii) for any \( u \in H^1(\mathbb{R}^2) \setminus \{0\} \) with \( u \geq 0 \) we have \( J_0(tu) \to +\infty \) as \( t \to +\infty \).

Now, in view of Lemma 2.1, we can apply a version of Ambrosetti–Rabinowitz mountain-pass theorem without a compactness condition such as the one of Palais–Smale (see [12]), to get a Palais–Smale sequence of the functional \( J_0 \), that is, \( (u_n) \subset E_0 \) such that

\[ J_0(u_n) \to c_0 \text{ and } J'_0(u_n) \to 0, \text{ as } n \to \infty, \]

where the mountain-pass level \( c_0 \) is characterized by

\[ c_0 = \inf_{\gamma \in I_0} \max_{0 \leq t \leq 1} J_0(\gamma(t)) \quad (2.1) \]
and
\[ \Gamma_0 = \{ \gamma \in C([0, 1], H^1(\mathbb{R}^2)) : J_0(\gamma(0)) \leq 0 \text{ and } J_0(\gamma(1)) \leq 0 \}. \]

**Lemma 2.2.** The mountain-pass level \( c_0 \) satisfies \( c_0 \in [\beta, (\theta - 2)/2\theta) \). Moreover, the \((PS)_{c_0}\) sequence \((u_n)\) is bounded and its weak limit denoted by \( u_0 \) satisfies \( J'_0(u_0) = 0 \).

**Proof.** In view of Lemma 2.1 we see that \( c_0 \geq \beta \). In order to prove the other inequality, we fix a positive radial function \( v_\rho \in H^1(\mathbb{R}^2) \) such that
\[
S_\rho = \frac{\int_{\mathbb{R}^2} [\|\nabla v_\rho\|^2 + V_1 v_\rho^2] \, dx}{\int_{\mathbb{R}^2} |v_\rho|^p \, dx}^{1/p}.
\]
Notice that
\[
c_0 \leq \max_{t \geq 0} J_0(t v_\rho)
\]
\[
\leq \max_{t \geq 0} \left[ \frac{t^2}{2} \int_{\mathbb{R}^2} [\|\nabla v_\rho\|^2 + V_1 v_\rho^2] \, dx - C_p \int_{\mathbb{R}^2} v_\rho^p \, dx \right]
\]
\[
= \frac{(p - 2) S_\rho^{2p/(p-2)}}{2p} \frac{S_\rho^{2p/(p-2)}}{C_p^{2/(p-2)}}.
\]
On the other hand by \((H7)\),
\[
\frac{(p - 2) S_\rho^{2p/(p-2)}}{2p} \frac{S_\rho^{2p/(p-2)}}{C_p^{2/(p-2)}} < \frac{(\theta - 2)}{2\theta}.
\]
So \( c_0 < ((\theta - 2)/2\theta) \).

Using well-known arguments it is not difficult to check that \( \{u_n\} \) is a bounded sequence. Thus for a subsequence still denoted by \((u_n)\) there is \( u_0 \in E_0 \) such that \( u_n \rightharpoonup u_0 \) in \( E_0 \), \( u_n \to u_0 \) in \( L^s_{loc}(\mathbb{R}^2) \) for all \( s \geq 1 \) and \( u_n(x) \to u_0(x) \) almost everywhere in \( \mathbb{R}^2 \). Now, from \((H5)\),
\[
c_0 = \lim J_0(u_n)
\]
\[
= \lim \left[ J_0(u_n) - \frac{1}{\theta} J'_0(u_n)u_n \right]
\]
\[
\geq \frac{\theta - 2}{2\theta} \limsup_{n \to +\infty} \|u_n\|_0^2,
\]
which implies
\[
\limsup_{n \to +\infty} \|u_n\|_0^2 = m \leq \frac{2\theta c_0}{\theta - 2} < 1. \tag{2.2}
\]

From Trudinger–Moser inequality, (see Lemma 1 in [9]) that there exists \( \gamma, q > 1 \) sufficiently close to 1 such that sequence
\[
h_n(x) = e^{4\gamma u_0^q(x)} - 1
\]
belongs to $L^q(\mathbb{R}^2)$ and there exists $C > 0$ such that $|h_n|_q \leq C$ for all $n \in \mathbb{N}$. These informations above imply the following limit:

$$\int_{\mathbb{R}^2} f(x, u_n)v \, dx \to \int_{\mathbb{R}^2} f(x, u_0)v \, dx \quad \forall v \in E_0.$$ 

From these and keeping in mind that $J'_0(u_n)v = o_n(1)$ for all $v \in E_0$, taking the limit we prove that $J'_0(u_0)v = 0$ for all $v \in E_0$.

The next proposition is a version of a Lions’ results to critical growth in $\mathbb{R}^2$.

**Proposition 2.3.** Let $(u_n) \subset H^1(\mathbb{R}^2)$ be a sequence with $u_n \to 0$ and

$$\limsup_{n \to \infty} \|u_n\|^2 \leq m < 1.$$ 

If there exists $R > 0$ such that

$$\liminf_{n \to \infty} \sup_{y \in \mathbb{R}^2} \int_{B_R(y)} |u_n|^2 \, dx = 0,$$

and (H4)–(H5) hold, we have

$$\int_{\mathbb{R}^2} F(x, u_n) \, dx, \int_{\mathbb{R}^2} u_n f(x, u_n) \, dx \to 0 \quad \text{as} \quad n \to \infty.$$ 

**Proof.** By hypothesis

$$\liminf_{n \to +\infty} \sup_{y \in \mathbb{R}^2} \int_{B_R(y)} |u_n|^2 \, dx = 0.$$ 

Thus, by Lemma 8.4 in [11], we get

$$u_n \to 0 \quad \text{in} \quad L^q(\mathbb{R}^2) \quad \text{for all} \quad q \in (2, +\infty).$$ 

This fact together with Trudinger–Moser inequality (see Lemma 1 in [9]) imply that

$$\int_{\mathbb{R}^2} (e^{4\pi \gamma u_n^2} - 1) \, dx \leq C,$$

if we choose $\gamma > 1$ sufficiently close to 1. Therefore, by assumptions (H2) and (H4), given $\varepsilon > 0$ there exist positive constants $C_\varepsilon$ and $q$, $\gamma > 1$ sufficiently close to 1 such that

$$\int_{\mathbb{R}^2} u_n f_0(x, u_n) \, dx \leq \varepsilon \int_{\mathbb{R}^2} u_n^2 \, dx + C_\varepsilon \int_{\mathbb{R}^2} u_n (e^{4\pi \gamma u_n^2} - 1) \, dx$$

$$\leq C \left\{ \int_{\mathbb{R}^2} |u_n|^{q'} \, dx \right\}^{1/q'} \left\{ \int_{\mathbb{R}^2} (e^{4\pi \gamma q u_n^2} - 1) \, dx \right\}^{1/q} + \varepsilon C$$

$$= C \left\{ \int_{\mathbb{R}^2} |u_n|^{q'} \, dx \right\}^{1/q'} + \varepsilon C,$$
which together with (2.4) leads to
\[ \lim_{n \to +\infty} \int_{\mathbb{R}^2} u_n f_0(x, u_n) = 0. \]

Thus, in view of assumption (H5) we conclude that
\[ \lim_{n \to \infty} \int_{\mathbb{R}^2} F_0(x, u_n) \, dx = 0. \]

Thus, the proof of Proposition 2.3 is complete. \( \Box \)

2.1. Proof of Theorem 1.1

Using Lemma 2.2 we have that the weak limit \( u_0 \) of the \((PS)_{c_0}\) sequence satisfies \( J'_0(u_0) = 0 \), thus if \( u_0 \) is nontrivial the theorems is proved. If \( u_0 = 0 \), we have the following claim:

Claim 1. There is a sequence \((y_n) \subset \mathbb{R}^2\), and \( R, \alpha > 0 \) such that
\[ \liminf_{n \to +\infty} \sup_{y \in \mathbb{R}^2} \int_{B_R(y_n)} |u_n|^2 \, dx > \alpha. \] (2.5)

This claim is true, because for the contrary case, using the version of Lions’ proved at Proposition 2.3, we have
\[ \int_{\mathbb{R}^2} u_n f(x, u_n) \, dx \to 0, \]
which implies that \( \{u_n\} \) converges strongly to zero. This limit is absurd, because it implies \( c_0 = 0 \). Thus the claim is proved.

It is clear that we may assume, without loss of generality, that \((y_n) \subset \mathbb{Z}^N\). Now, letting \( \tilde{u}_n(x) = u_n(x - y_n) \), since \( V, f(\cdot, s) \) and \( F(\cdot, s) \) are 1-periodic functions, by a routine calculus we obtain
\[ \|u_n\|_0 = \|\tilde{u}_n\|_0, \ J_0(u_n) = J_0(\tilde{u}_n) \quad \text{and} \quad J'_0(\tilde{u}_n) \to 0. \]

Then, there exists \( \tilde{u}_0 \in E_0 \) such that \( \tilde{u}_n \rightharpoonup \tilde{u}_0 \) weakly in \( E_0 \) and as before it follows that \( J'_0(\tilde{u}_0)v = 0 \) for all \( v \in E_0 \). Now, by (2.5), taking a subsequence and \( R \) bigger we get
\[ a^{1/2} \leq |\tilde{u}_n|_{L^2(B_R(0))} \leq |\tilde{u}_0|_{L^2(B_R(0))} + |\tilde{u}_n - \tilde{u}_0|_{L^2(B_R(0))}. \] (2.6)

Thus, from the Rellich imbedding theorem we conclude that \( \tilde{u}_0 \) is nontrivial.

Since \( f_0(x,s) = 0 \) for all \( s \leq 0 \) and \( J'_0(u_0)v \) for all \( v \in E_0 \), choosing the test function \( v = u_0^- = \max\{-u_0, 0\} \in E_0 \), we conclude that \( \|u_0^-\|_0 = 0 \). Thus, \( u_0 \) is a nonnegative function. Now using that \( u_0 \geq 0 \) is nontrivial and the maximum principle we conclude that \( u_0 > 0 \).
3. The nonperiodic problem: Proof of Theorem 1.2

Let \( E \) denote the Sobolev space \( H^1(\mathbb{R}^2) \) endowed with the equivalent norm

\[
\|u\| = \left\{ \int_{\mathbb{R}^2} \left[ |\nabla u|^2 + V(x)|u|^2 \right] \, dx \right\}^{1/2}.
\]

In this section, we are going to prove the existence of solutions of problem (1.3) as critical point of the associated \( C^1 \) functional on \( E \) given by

\[
J(u) = \frac{1}{2} \int_{\mathbb{R}^2} \left[ |\nabla u|^2 + V(x)|u|^2 \right] \, dx - \int_{\mathbb{R}^2} F(x,u) \, dx.
\]

As in the last section, we may check that the functional energy \( J \) has the geometry of the mountain-pass theorem. Therefore applying the mountain-pass theorem without Palais–Smale condition together with the arguments from the last section, we obtain a bounded sequence \( (v_n) \subset E \) such that

\[
J(v_n) \to c_1 \quad \text{and} \quad J'(v_n) \to 0 \quad \text{as} \quad n \to +\infty,
\]

where \( c_1 \) is the minimax level of the functional \( J \) given by

\[
c_1 = \inf_{v \in E \setminus \{0\}} \max_{t \geq 0} J(tv).
\]

Furthermore, from (H7) we see that \( c_1 \in (\beta', (\theta - 2)/2\theta) \) for positive constant \( \beta' \) and \( v_n \rightharpoonup v_0 \) in \( E \) (see the proof of Lemma 2.2). Therefore, \( v_0 \) is a critical point of functional \( J \) and \( v_0 > 0 \).

Using the same argument of Section 2, we get that

\[
\limsup_{n \to +\infty} \|v_n\|^2 < 1,
\]

which implies, for all \( n \),

\[
\int_{\mathbb{R}^2} \left( e^{4\pi^2 v_n^2} - 1 \right) \, dx \leq C,
\]

if we take \( \gamma > 1 \) sufficiently close to 1. From now on, we shall be working in order to prove that \( v_0 \) is nontrivial.

**Assertion 1.** \( v_0 \) is nontrivial.

**Proof.** Assume for the sake of contradiction that \( v_0 \) is trivial. Thus, from (H1) and the Sobolev compact embedding theorem,

\[
\int_{\mathbb{R}^2} [V_0(x) - V(x)] v_n^2 \, dx \to 0, \quad \text{as} \quad n \to +\infty.
\]
Next, we assume the following result, which will be proved later.

\[
\lim_{n \to +\infty} \int_{\mathbb{R}^2} |F_0(x, v_n) - F(x, v_n)| \, dx = \lim_{n \to +\infty} \int_{\mathbb{R}^2} |[f_0(x, v_n) - f(x, v_n)]v_n| \, dx = 0
\]  

(3.9)

From (3.8)–(3.9) we conclude that

\[|J_0(v_n) - J(v_n)| \to 0 \text{ and } \|J'_0(v_n) - J(v_n)\| \to 0 \text{ as } n \to +\infty.\]

Therefore,

\[J_0(v_n) \to c_1 \text{ and } J'_0(v_n) \to 0 \text{ as } n \to +\infty.\]

As in Proposition 2.3, there is a sequence \((y_n) \subset Z^2\), and \(R, a > 0\) such that

\[\liminf_{n \to +\infty} \sup_{y \in \mathbb{R}^2} \int_{B_R(y_n)} |v_n|^2 \, dx > a\]

and letting \(\tilde{v}_n(x) = v_n(x - y_n)\), since \(V, f_0(\cdot, s)\) and \(F_0(\cdot, s)\) are 1-periodic functions,

\[\|v_n\| = \|	ilde{v}_n\|, \quad J_0(v_n) = J_0(\tilde{v}_n) \text{ and } J'_0(\tilde{v}_n) \to 0.\]

Then, there exists \(\tilde{v}_0 \in E_0\) such that \(\tilde{v}_n \rightharpoonup \tilde{v}_0\) weakly in \(E_0\) and \(J'_0(\tilde{v}_0) = 0\). Moreover, \(J_0(\tilde{v}_0) \leq c_1\) since by Fatou’s lemma,

\[J_0(\tilde{v}_0) = J_0(\tilde{v}_0) - \frac{1}{2} J'_0(\tilde{v}_0) \tilde{v}_0 = \frac{1}{2} \int_{\mathbb{R}^2} \left[ f_0(x, \tilde{v}_0) \tilde{v}_0 - 2F(x, \tilde{v}_0) \right] \]

\[\leq \liminf_{n \to +\infty} \frac{1}{2} \int_{\mathbb{R}^2} \left[ f_0(x, \tilde{v}_n) \tilde{v}_n - 2F(x, \tilde{v}_n) \right] \]

\[= \lim_{n \to +\infty} \left[ J_0(\tilde{v}_n) - \frac{1}{2} J'_0(\tilde{v}_n) \tilde{v}_n \right] = c_1.\]

Arguing as in (2.6) we conclude that \(\tilde{v}_0\) is nontrivial and

\[c_1 \geq J_0(\tilde{v}_0) = \max_{t \geq 0} J_0(t\tilde{v}_0) \geq c_0.\]

(3.10)

On the other hand, from \((H_1), (H_2),\) and \((H_8),\)

\[c_1 \leq \max_{t \geq 0} J(tu_0) = J(t_1 u_0) = J_0(t_1 u_0) \leq \max_{t \geq 0} J_0(tu_0) = J_0(u_0) = c_0,\]

that is, \(c_1 < c_0\), which is a contradiction with (3.10). Therefore, \(v_0\) is nontrivial.

Next we prove (3.9). Using condition \((H_2),\) given \(\varepsilon > 0\) there exists \(\eta > 0\) such that

\[\int_{|x| \geq \eta} |f(x, v_n) - f_0(x, v_n)||v_n| \, dx \leq \varepsilon \int_{|x| \geq \eta} (e^{4\pi^2 \gamma^2 - 1} v_n) \, dx \]

\[\leq \varepsilon \left\{ \int_{\mathbb{R}^2} (e^{4\pi^2 \gamma^2 - 1})^q \, dx \right\}^{1/q} \left\{ \int_{\mathbb{R}^2} |v_n|^q \right\}^{1/q} \leq C\varepsilon,

where we have used that \(h_n(x) = e^{4\pi^2 \gamma^2 x} - 1\) belongs to \(L^q(\mathbb{R}^2)\) with \(\gamma, q > 1, 1/q + 1/q' = 1,\) and \(|h_n|^q \leq C, \forall n \in \mathbb{N}.\)
On the other hand, using the Sobolev compact embedding theorem and (3.7),

\[
\int_{|x| \leq \eta} |f(x,v_n) - f_0(x,v_n)| |v_n| \, dx \leq \left\{ \int_{\mathbb{R}^2} |(e^{4\pi^2 v_n^2} - 1)|^q \, dx \right\}^{1/q} \\
\left\{ \int_{|x| \leq \eta} |v_n|^{q'} \right\}^{1/q'} + \epsilon \int_{\mathbb{R}^2} |v_n|^2.
\]

Thus

\[
\int_{|x| \leq \eta} |f(x,v_n) - f_0(x,v_n)| |v_n| \, dx \to 0 \quad \text{as} \quad n \to \infty.
\]

Finally, these facts together with assumptions (H2)–(H5) conclude our proof.  

\[\square\]

References