Relatório parcial de estágio pós doutoral

Este relatório descreve os resultado obtidos e os estudos em progresso relativos aos trabalhos por mim, Abiel Costa Macedo, realizados em parceria com o Professor João Marcos Bezerra do Ó durante o meu estágio pós doutoral. Neste o estágio obtivemos relevantes avanços nos projetos que foram propostos dentre os quais destacamos os seguintes resultados:

- Relativo ao projeto 1 produzimos o trabalho intitulado *"Adams inequalities and extremal functions on unbounded domains"*, em anexo, e outro em fase de produção. Faremos agora uma breve explanação sobre os resultados obtidos.

  Para $n > m \geq 2$ inteiros e

$$\Phi(t) := e^t - \sum_{j=0}^{j_{m,n}-2} \frac{t^j}{j!}, \tag{1}$$

onde $j_{m,n} := \min\{j \in \mathbb{N} : j \geq \frac{n}{m}\}$, provamos a seguinte desigualdade do tipo Adams da forma escalar invariante: Dado $\beta \in (0, \beta_0)$ existe $C_{\beta,m,n} = C(\beta, m, n)$ tal que

$$\int_{\mathbb{R}^n} \Phi\left(\beta\left(\frac{|u|}{\|\nabla^m u\|_{n/m}}\right)^{n/(n-m)}\right) \, dx \leq C_{\beta,m,n} \frac{\|u\|_{n/m}^{n/m}}{\|\nabla^m u\|_{n/m}^{n/m}}, \quad \forall\, u \in W^{m,n/m}_{rad}(\mathbb{R}^n), \tag{2}$$

onde β_0 é a melhor constante de Adams. Provamos ainda a existência de extremais para o seguinte problema variacional associado à desigualdade (2)

$$\mu_{\beta,n,m} := \sup_{\substack{u \in W^{m,n/m}_{rad}(\mathbb{R}^n) \\ \|\nabla^m u\|_{\frac{n}{m}} = 1}} \frac{1}{\|u\|_{n/m}^{n/m}} \int_{\mathbb{R}^n} \Phi\left(\beta\left(|u|\right)^{n/(n-m)}\right) \, dx.$$

Usando a desigualdade (2) provamos que a seguinte desigualdade do tipo Adams provado por B. Ruf, F. Sani [8] e N. Lam, G. Lu [4]

Teorema A (B. Ruf and F. Sani, 2013) *Seja m um inteiro positivo e par, i.e., $m = 2k$ para algum $k \in \mathbb{N}$. Então existe uma constante $C_{m,n} > 0$ tal que*

$$\sup_{\substack{u \in W^{m,n/m}_0(\Omega) \\ \|(-\Delta+I)^k u\|_{n/m} \leq 1}} \int_\Omega \Phi(\beta_0 |u|^{n/(n-m)}) \, dx \leq C_{m,n}, \tag{3}$$

para todo domínio $\Omega \subset \mathbb{R}^n$. Mais ainda, o supremo acima se torna infinito se β_0 for trocado por $\beta > \beta_0$.

Teorema B (N. Lam and G. Lu, 2012) *Seja m um inteiro positivo e impar, i.e., $m = 2k + 1$ para algum $k \in \mathbb{N}$. Então existe uma constante $C_{m,n} > 0$ tal que*

$$\sup_{\substack{u \in W^{m,n/m}_0(\Omega) \\ \|\nabla(-\Delta+I)^k u\|_{n/m}^{n/m} + \|(-\Delta+I)^k u\|_{n/m}^{n/m} \leq 1}} \int_\Omega \Phi(\beta_0 |u|^{n/(n-m)}) \, dx \leq C_{m,n}, \tag{4}$$

para todo domínio $\Omega \subset \mathbb{R}^n$. Mais ainda, o supremo acima se torna infinito se β_0 for trocado por $\beta > \beta_0$.

não possui extremais para $\Omega = \mathbb{R}^n$ no caso subcrítico como $n/m = 2$.

Para mais detalhes ver o artigo em anexo.

No segundo trabalho provamos a existência de extremais para as desigualdades (3) e (4) no caso crítico para $m = 2$ com $\Omega = \mathbb{R}^2$.

- Relativo ao projeto 2 temos dois trabalhos em produção *"Weighted Sobolev inequalities and critical equations for the high order elliptic operator"* e *"Adams inequalities for weighted Sobolev spaces"*. No que segue daremos uma pequena explanação, em inglês, sobre o assunto dos artigos.

  Let $AC^1_{loc}(0,R]$ be set of the all functions locally absolutely continuous $u:(0,R]\to\mathbb{R}$ such that u' is still locally absolutely continuous. For $0<R<\infty$, $p\geq 1$, $\alpha,\beta\geq -1$, we set $X^{2,p}_R(\alpha,\beta)$ the space of $u\in AC^1_{loc}(0,R]$ such that $u(R)=u'(R)=0$ and

  $$\int_0^R |\Delta_\alpha u|^p r^\beta dr < +\infty, \text{ with } \Delta_\alpha = r^{-\alpha}(r^\alpha u')'.$$

  Then $X^{2,p}_R(\alpha,\beta)$ is a Banach space under the norm

  $$\|u\|_{X^{2,p}_R} = \left(\int_0^R |\Delta_\alpha u|^p r^\beta dr\right)^{\frac{1}{p}}.$$

  The $X^{2,p}_R(\alpha,\beta)$ spaces have interesting properties which turn out be appropriated for study of the following class of equations

  $$\Delta^2_\alpha u := \Delta_\alpha(\Delta_\alpha u) = f(u) \quad \text{on} \quad (0,R] \quad \text{and} \quad u(R)=u'(R)=0. \tag{5}$$

  Let $q\geq 1$ and $\gamma>-1$. Let R be such that $0<R<\infty$. Denote by $L^q_\gamma = L^q(0,R)$ the Banach space of Lebesgue measurable functions $u:(0,R)\to\mathbb{R}$ such that

  $$\|u\|_{L^q_\gamma} := \left(\int_0^R |u|^q r^\gamma dr\right)^{1/q} < \infty.$$

  Accordingly, we can derive the following embeddings for $X^{2,p}_R(\alpha,\beta)$ spaces.

  Theorem 1 *Let $p\geq 1$, $\alpha,\beta,\nu>-1$ be real numbers. Suppose $\beta-2p+1>0$, then hold the following continuous embeddings*

  $$X^{2,p}_R \hookrightarrow L^q_\nu, \;\; 1\leq q\leq p^*$$

  where

  $$p^* = p^*(\nu,p,\beta) = \frac{(\nu+1)p}{\beta-2p+1}.$$

  Moreover, if $q<p^*$holds then these embeddings are compact.

  We study the problem (5) for the critical case. We also generalized the concept of the space $X^{2,p}_R(\alpha,\beta)$ for any arbitrary order $X^{m,p}_R(\alpha,\beta)$, extend the Theorem 1 and prove an Adams tipo inequality for this space.

Referências

[1] Adachi, S., Tanaka, K.: *Trudinger type inequalities in $\mathbb{R}^N$ and their best exponents*. Proc. Amer. Math. Soc. **128**, 2051–2057 (2000).

[2] Adams, D. R.: *A sharp inequality of J. Moser for higher order derivatives*. Ann. Math. **128**, 385–398 (1988).

[3] Cao, D. M.: *nontrivial solution of semilinear elliptic equation with critical exponent in $\mathbb{R}^2$*. Comm. Partial Differential Equations **17**, 407–435 (1992).

[4] Lam, N., Lu, G.: *Sharp Adams type inequalities in Sobolev spaces* $W^{m,n}(\mathbb{R}^n)$ *for arbitrary integer* m . J. Differential Equations **253**, 1143–1171 (2012).

[5] Lions, P.-L.: *The concentration compactness principle in the calculus of variation, the limit case, part I.* Rev. Mat. Iberoamericana **1**, 145–201 (1985).

[6] Lu, G., Yang, Y.: *Adams' inequalities for bi-Laplacian and extremal functions in dimension four.* Adv. Math. 220, 1135–1170 (2009).

[7] Moser, J.: *A sharp form of an inequality by N. Trudinger.* Ind. Univ. Math. J. **20**, 1077–1092 (1971).

[8] Ruf, B, Sani, F.: *sharp Adams-Type Inequalities in* $\mathbb{R}^n$. Trans. Amer. Math. Soc. **365**, 645–670 (2013).

[9] Trudinger, N. S.: *On embeddings into Orlicz spaces and some applications.* J. Math. Mech. **17**, 473–483 (1967).

[10] Yang, Y.: *Extremal functions for Moser-Trudinger inequalities on 2-dimensional compact Riemannian manifolds with boundary.*. Internat. J. Math. 17, 313–330 (2006).

[11] Yang, Y.: *Trudinger-Moser inequalities on complete noncompact Riemannian manifolds.*. J. Funct. Anal. 263, 1894–1938 (2012).

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Adams inequalities and extremal functions on unbounded domains ☆

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Abstract

In this paper we establish a sharp Adams type inequality of the scaling invariant form and prove the existence of maximizer for the associated variational problem. Using this scaling invariant inequality we prove that the Adams type inequality proved by B. Ruf, F. Sani [32] and N. Lam, G. Lu [21] has no extremal functions in the subcritical case on $W^{m,2}(\mathbb{R}^{2m})$, for $m \geq 2$. Moreover, in line with the Concentration Compactness Principle due to P.-L. Lions [25], we will obtain an improvement for Adams' exponent in certain class of sequences in $W_0^{m,n/m}(\Omega)$, for any domain $\Omega \subset \mathbb{R}^n$, $n > 2$.

Keywords: Trudinger-Moser inequality, Adams' inequality, Exponential growth, Concentration compactness principle, Extremal functions.
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1. Introduction

This paper is concerned on the problem of finding optimal Sobolev inequalities and the attainability to the associated variational problem for the borderline case known nowadays as Trudinger-Moser case. These inequalities play an important role in the geometric analysis, partial differential equations and have been a source of inspiration of many research works in recent years. In order to motivate our work, let us introduce now a brief history of some results on these class of problems.

Let $\Omega \subset \mathbb{R}^n (n \geq 2)$ be a smooth bounded domain and $W_0^{1,p}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,p}(\Omega)$ under the usual Sobolev norm. It is well known that the Sobolev embedding $W^{1,p}(\Omega) \hookrightarrow L^s(\Omega)$ holds for any $1 \leq s \leq p^* = np/(n-p)$ if $1 \leq p < n$. For the limiting case $p = n$, formally $p^* = +\infty$ and it was proved by V. Yudovich [41], S. Pohozaev [29], J. Peetre [28], N. Trudinger [39] and J. Moser [30] that the optimal embedding of the Sobolev space $W_0^{1,n}(\Omega)$ is into an Orlicz space, that we state as follows:

$$\sup_{\substack{u \in W_0^{1,n}(\Omega) \\ \|\nabla u\|_n \leq 1}} \int_\Omega e^{\alpha |u|^{n/(n-1)}} \, \mathrm{d}x \begin{cases} < \infty & \text{for} \quad \alpha \leq \alpha_n, \\ = \infty & \text{for} \quad \alpha > \alpha_n, \end{cases} \tag{1.1}$$

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where $\alpha_n := n\omega_{n-1}^{1/(n-1)}$ and ω_{n-1} is the area of the surface of the unit n-ball in $\mathbb{R}^n$, for $\|\cdot\|_n$ denoting the standard norm in the Lebesgue space $L^n(\Omega)$. Later P.-L. Lions [25] proved that the exponent α_n can be improved along certain sequences. More precisely, if $(u_i) \subset W_0^{1,n}(\Omega)$ with $\|\nabla u_i\|_n = 1$ and $u_i \rightharpoonup u_0$ in $W_0^{1,n}(\Omega)$, then

$$\sup_i \int_\Omega e^{\gamma\alpha|u_i|^{n/(n-1)}}\,\mathrm{d}x < \infty, \quad \text{for any} \quad \gamma < 1/(1-\|\nabla u_0^*\|_n^n)^{1/(n-1)},$$

where u_0^* is the spherically symmetric decreasing rearrangement of u_0, see section 5.

The Trudinger-Moser inequality for unbounded domains were proposed by D. M. Cao [8], for the case $n=2$, J. M. do Ó [15], R. Panda [27] and S. Adachi and K. Tanaka [1], for the general case $n \geq 2$, which we state now its scaling invariant form: Given $\alpha \in (0,\alpha_n)$ there exists a constant $C_{\alpha,n}$ depending only on α and n such that

$$\int_{\mathbb{R}^n} \Psi\left(\alpha\left(\frac{|u|}{\|\nabla u\|_n}\right)^{n/(n-1)}\right)\mathrm{d}x \leq C_{\alpha,n}\frac{\|u\|_n^n}{\|\nabla u\|_n^n}, \quad \forall\, u \in W^{1,n}(\mathbb{R}^n)\setminus\{0\}, \tag{1.2}$$

where

$$\Psi(t) := e^t - \sum_{j=0}^{n-2}\frac{t^j}{j!}.$$

Moreover, inequality (1.2) fail for $\alpha \geq \alpha_n$. All these results treated the subcritical case $0<\alpha<\alpha_n$. Later B. Ruf [31], for the case $n=2$, and Y. Li, B. Ruf [23], for the general case $n \geq 2$, studied the critical case $\alpha = \alpha_n$ and proved that the result of J. Moser can be fully extended to unbounded domains if the Dirichlet norm $\|\nabla u\|_n$ is replaced by the full norm $(\|\nabla u\|_n^n + \|u\|_n^n)^{1/n}$. More precisely, they proved that

$$\sup_{\substack{u\in W^{1,n}(\mathbb{R}^n)\\ \|\nabla u\|_n^n+\|u\|_n^n\leq 1}} \int_{\mathbb{R}^n} \Psi(\alpha|u|^{n/(n-1)})\,\mathrm{d}x < +\infty, \quad \forall\, \alpha \leq \alpha_n. \tag{1.3}$$

Moreover, inequality (1.3) became infinite if $\alpha > \alpha_n$.

Another interesting question about Trudinger-Moser inequalities is whether extremal function exists, or not. The first result in this direction belongs to L. Carleson and A. Chang [9] who proved that if $\Omega \subset \mathbb{R}^n$ is the ball $B_1(0)$, then the supremum in (1.1) is achieved when $\alpha \leq \alpha_n$. Later, M. Struwe [34] studied the existence of extremal functions for a class of nonsymmetric domains. He obtained a sufficient condition for these class of domains in $\mathbb{R}^2$ using blow-up analysis. M. Flucher [17] introduced another method, the conformal rearrangement, and derived an isoperimetric inequality, which implies the existence of extremal functions to any smooth bounded domain in $\mathbb{R}^2$. At last, K. Lin [24] generalized the existence of extremal function to any smooth bounded domain in $\mathbb{R}^n (n \geq 2)$. It should be mentioned that the existence of extremal for (1.1) correspond to the existence of solutions to an associated Euler-Lagrange equation involving critical growth. Thus, these class of problems is harder or more difficult than subcritical ones and the lack of compactness makes the proofs more involved, as one can see in very intricate analysis given in the papers [9], [17], [34]. For works related to (1.1) and applications, we refer to [5, 10, 11, 35] and references therein. At this point we mention that existence of extremal for (1.3) were first analyzed in [31, 23] and complemented by M. Ishiwata [18]. Recently M. Ishiwata [19] obtained an weighted Trudinger-Moser type inequality of the scaling invariant form and studied the existence of extremal for the associated variational problem.

In the case of Sobolev spaces with higher order derivatives $W_0^{m,p}(\Omega)$, D. Adams (cf. [2]) obtained the following version of the Trudinger-Moser inequality (1.1).

Theorem A (D. Adams, 1988)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded smooth domain and m a positive integer with $m < n$. Then there exists a constant $C_{m,n}$ such that*

$$\sup_{\substack{u\in W_0^{m,n/m}(\Omega),\\ \|\nabla^m u\|_{n/m}\le 1}} \int_\Omega e^{\beta|u|^{n/(n-m)}}\,\mathrm{d}x \le C_{m,n}|\Omega|, \qquad \beta \le \beta_0, \tag{1.4}$$

where

$$\beta_0 = \beta_0(m,n) = \begin{cases} \frac{n}{\omega_{n-1}}\left[\frac{\pi^{\frac{n}{2}}2^m\Gamma\left(\frac{m+1}{2}\right)}{\Gamma\left(\frac{n-m+1}{2}\right)}\right]^{n/(n-m)}, & m \text{ odd},\\ \frac{n}{\omega_{n-1}}\left[\frac{\pi^{\frac{n}{2}}2^m\Gamma\left(\frac{m}{2}\right)}{\Gamma\left(\frac{n-m}{2}\right)}\right]^{n/(n-m)}, & m \text{ even}, \end{cases} \tag{1.5}$$

and β_0 is sharp, i.e., the supremum in (1.4) *is $+\infty$ if $\beta > \beta_0$.*

Here we are denoting $\nabla^m u$ by the mth gradient of u, i.e.,

$$\nabla^m u = \begin{cases} \Delta^{m/2}u, & m = 2,4,6,\ldots\\ \nabla\Delta^{(m-1)/2}u, & m = 1,3,5,\ldots \end{cases}$$

This inequality was extended by C. Tarsi [37, Theorem 4] to a more large space given by

$$W_{\mathcal{N}}^{m,p}(\Omega) := \{u \in W^{m,p}(\Omega) : u_{|\partial\Omega} = \Delta^j u_{|\partial\Omega} = 0 \text{ in the sense of trace}, 1 \le j < m/2\},$$

Theorem B. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and $m < n$ an integer. Then there exists a constant $C_{m,n} > 0$ such that*

$$\sup_{\substack{u\in W_{\mathcal{N}}^{m,n/m}(\Omega)\\ \|\nabla^m u\|_{n/m}\le 1}} \int_\Omega e^{\beta|u|^{n/(n-m)}}\,\mathrm{d}x \le C_{m,n}|\Omega|, \quad \forall\, 0 \le \beta \le \beta_0,$$

where β_0 was defined in (1.5)*. Moreover, β_0 is sharp, i.e., the supremum above is $+\infty$ if $\beta > \beta_0$*

Recently B. Ruf, F. Sani [32] and N. Lam, G. Lu [21] have obtained a version of the Adams inequality (1.4) for domain not necessarily bounded. Let

$$\Phi(t) := e^t - \sum_{j=0}^{j_{m,n}-2}\frac{t^j}{j!}, \qquad j_{m,n} := \min\{j \in \mathbb{N} : j \ge n/m\}, \tag{1.6}$$

be a Young function.

Theorem C (B. Ruf and F. Sani, 2013)**.** *Let m be an even positive integer, i.e., $m = 2k$ for some $k \in \mathbb{N}$. Then there exists a constant $C_{m,n} > 0$ such that*

$$\sup_{\substack{u\in W_0^{m,n/m}(\Omega)\\ \|(-\Delta+I)^k u\|_{n/m}\le 1}} \int_\Omega \Phi(\beta_0|u|^{n/(n-m)})\,\mathrm{d}x \le C_{m,n}, \tag{1.7}$$

for any domain $\Omega \subset \mathbb{R}^n$. Moreover, the supremum became infinite if β_0 is replaced for $\beta > \beta_0$.

Theorem D (N. Lam and G. Lu, 2012)**.** *Let m be an odd positive integer, i.e., $m=2k+1$ for some $k\in\mathbb{N}$. Then there exists a constant $C_{m,n}>0$ such that*

$$\sup_{\substack{u\in W_0^{m,n/m}(\Omega)\\ \|\nabla(-\Delta+I)^k u\|_{n/m}^{n/m}+\|(-\Delta+I)^k u\|_{n/m}^{n/m}\le 1}} \int_\Omega \Phi(\beta_0|u|^{n/(n-m)})\,\mathrm{d}x \le C_{m,n}, \tag{1.8}$$

for any domain $\Omega\subset\mathbb{R}^n$. Moreover, the supremum became infinite if β_0 is replaced for $\beta>\beta_0$.

The main purpose of this paper is three-fold: First we obtain a scaling invariant inequality for the higher order Sobolev space of radially symmetric functions and prove the existence of extremal to the associated variacional problem. Secondly we prove a result about nonexistence of extremals for Adams type inequality (1.7) and (1.8) in the Hilbert case. Thirdly, in line with the Concentration Compactness Principle due to P.-L. Lions [25], we will obtain an improvement for Adams exponent in certain classes of sequence on $W_0^{m,n/m}(\Omega)$, for any arbitrary domain.

First we establish the following Adams type inequality of the scaling invariant form.

Theorem 1.1. *Let $n>m\ge 2$ be integers. Then given $\beta\in(0,\beta_0)$ there exists $C_{\beta,m,n}=C(\beta,m,n)$ depending only on β, m and n such that*

$$\int_{\mathbb{R}^n}\Phi\left(\beta\left(\frac{|u|}{\|\nabla^m u\|_{n/m}}\right)^{n/(n-m)}\right)\mathrm{d}x \le C_{\beta,m,n}\frac{\|u\|_{n/m}^{n/m}}{\|\nabla^m u\|_{n/m}^{n/m}},\quad \forall\, u\in W_{rad}^{m,n/m}(\mathbb{R}^n)\setminus\{0\}, \tag{1.9}$$

where Φ was defined in (1.6)*, β_0 was given in* (1.5) *and $W_{rad}^{m,n/m}(\mathbb{R}^n)$ denote the space of the radial $W^{m,n/m}(\mathbb{R}^n)$-functions. Moreover, for $\beta\in[\beta_0,\infty)$ inequality* (1.9) *fail, i.e., there exists $(u_i)\subset W_{rad}^{m,n/m}(\mathbb{R}^n)$ such that*

$$\frac{\|\nabla^m u_i\|_{n/m}^{n/m}}{\|u_i\|_{n/m}^{n/m}}\int_{\mathbb{R}^n}\Phi\left(\beta\left(\frac{|u_i|}{\|\nabla^m u_i\|_{n/m}}\right)^{n/(n-m)}\right)\mathrm{d}x\to\infty. \tag{1.10}$$

We also study the existence of extremal associated to the scaling invariant Adams type inequality (1.9) in the following sense. Let

$$\mu_{\beta,n,m}:=\sup_{\substack{u\in W_{rad}^{m,n/m}(\mathbb{R}^n)\setminus\{0\}\\ \|\nabla^m u\|_{\frac{n}{m}}=1}}\frac{1}{\|u\|_{n/m}^{n/m}}\int_{\mathbb{R}^n}\Phi\left(\beta\,(|u|)^{n/(n-m)}\right)\mathrm{d}x.$$

By Theorem 1.1,we have that $\mu_{\beta,n,m}<\infty$ for $\beta\in(0,\beta_0)$ and $\mu_{\beta,n,m}=\infty$ for $\beta\in[\beta_0,\infty)$. Under this notation we obtain

Theorem 1.2. *$\mu_{\beta,n,m}$ is attained for all $\beta\in(0,\beta_0)$, i.e., there exists $u\in W_{rad}^{m,n/m}(\mathbb{R}^n)$ such that $\|\nabla^m u\|_{\frac{n}{m}}=1$ and*

$$\mu_{\beta,n,m}=\frac{1}{\|u\|_{n/m}^{n/m}}\int_{\mathbb{R}^n}\Phi\left(\beta\,(|u|)^{n/(n-m)}\right)\mathrm{d}x.$$

Secondly, we will consider the Adams type inequality proved in Theorems C and D for $\Omega = \mathbb{R}^n$. For $n, m \geq 2$ integers and $1 < q < \infty$ a real number, we consider the Sobolev space $W^{m,q}(\mathbb{R}^n)$ endowed with the norm

$$\|u\|_{m,n,q}^q = \begin{cases} \|\nabla(-\Delta+I)^k u\|_q^q + \|(-\Delta+I)^k u\|_q^q, & \text{for} \quad m = 2k+1; \\ \\ \|(-\Delta+I)^k u\|_q^q, & \text{for} \quad m = 2k, \end{cases} \tag{1.11}$$

which is equivalent to the usual Sobolev norm in $W^{m,q}(\mathbb{R}^n)$. Now we denote the extremal constant for the Adams type inequalities (1.7) and (1.8) by

$$\eta_{\beta,n,m} := \sup_{\substack{u \in W^{m,\frac{n}{m}}(\mathbb{R}^n) \\ \|u\|_{m,n,\frac{n}{m}} \leq 1}} \int_{\mathbb{R}^n} \Phi(\beta |u|^{n/(n-m)}) \, \mathrm{d}x. \tag{1.12}$$

By Theorems C and D, we know that $\eta_{\beta,n,m}$ is bounded for $\beta \leq \beta_0$ and infinite for $\beta > \beta_0$. We will prove that in the Hilbert case, $n/m = 2$, there is not extremal function for $\eta_{\beta,n,m}$ provided that β is sufficiently small. Note that in the Hilbert case $\beta_0 = \beta_0(m, 2m) = (4\pi)^m m!$. So we prove the following theorem

Theorem 1.3. *$\eta_{\beta,n,m}$ is not attained when $n/m = 2$ and $0 < \beta \ll (4\pi)^m m!$.*

Our strategy to prove Theorem 1.3 is reduce the study of attainability of the supremum (1.12) to the radial case and so apply inequality (1.9). This reduction follows as an corollary of the following result

Theorem 1.4. *Given any $u \in W^{m,n/m}(\mathbb{R}^n)$ we can find $v \in W^{m,n/m}_{rad}(\mathbb{R}^n)$ such that $\|v\|_{m,n,\frac{n}{m}} \leq \|u\|_{m,n,\frac{n}{m}}$ and $u^* \leq v$.*

Then, using this result we can prove the following corollary.

Corollary 1.5. *The supremum in (1.12) can be taken on radially symmetric functions, i.e.,*

$$\eta_{\beta,n,m} = \sup_{\substack{u \in W^{m,n/m}_{rad}(\mathbb{R}^n) \\ \|u\|_{m,n,\frac{n}{m}} \leq 1}} \int_{\mathbb{R}^n} \Phi(\beta |u|^{n/(n-m)}) \, \mathrm{d}x, \quad \forall \beta > 0,$$

Moreover, if $\eta_{\beta,n,m}$ is attained then we can take a radially symmetric function as an extremal.

This result is important not only in the study here present but in the general analysis of the attainability of the optimal constant $\eta_{\beta,n,m}$.

In the last direction, we will prove that the Adams exponent β_0 in the inequalities (1.7) and (1.8) can be improved in the case of sequence which the weak limit is not an identically zero function in $W_0^{m,n/m}(\Omega)$, for any domain $\Omega \subset \mathbb{R}^n$. More precisely,

Theorem 1.6. *Let $\Omega \subset \mathbb{R}^n$ be any arbitrary domain. Assume that $u_i, u \in W_0^{m,n/m}(\Omega)$, $\|u_i\|_{m,n} \leq 1$, $u \neq 0$ and $u_i \rightharpoonup u$ in $W_0^{m,n/m}(\Omega)$. Then, given $\gamma \in [1, \eta)$, there exists a constant $C = C(\gamma, \Omega) > 0$ such that*

$$\sup_i \int_\Omega \Phi\left(\beta_0 \gamma |u_i|^{n/(n-m)}\right) \, \mathrm{d}x \leq C,$$

where $\eta = \eta_{m,n}(u) := \left(1 - \left\|(I-\Delta)^k u\right\|_{n/m}^{n/m}\right)^{-m/(n-m)}$ if $m = 2k+1$ or $m = 2k$ for some $k \in \mathbb{N}$.

This result has important consequence in the study of nonlinear elliptic problems involving exponential critical growth and even in the study of existence of extremal for Adams type inequalities (1.7) and (1.8) in the critical case. The proof of Theorem 1.6 is based on the application of comparison result proved in section 5. We observe that we can use the same idea in the proof of Proposition 5.2 to derive a similar comparison result on bounded domains. Thus, using this comparison result we can get rid of the restriction on p in the concentration-compactness result proved in [16, Theorem 1.1].

This paper is organized as follows: In section 2, we present some preliminaries results that we will need throughout the paper. In the section 3, we will obtain a scaling invariant Adams type inequality, Theorem 1.1. In section 4, we will prove the existence of extremal for the variacional problem associated to the scaling invariant Adams type inequality, Theorem 1.2. In section 5, we will prove a comparison result and Theorem 1.4. In section 6, we will prove the nonexistence result, Theorem 1.3. In section 7, we will prove Theorem 1.6.

2. Preliminaries

In this section we will discuss on some questions involving the Sobolev space $W^{m,q}(\mathbb{R}^n)$ and some equivalent norms, for $q \in (1,\infty)$. First we consider the Bessel potential G_m defined by

$$G_m(x) = \frac{1}{(4\pi)^{m/2}} \frac{1}{(k-1)!} \int_0^\infty e^{-\pi|x|^2/\delta} e^{-\delta/4\pi} \delta^{(-n+m)/2} \frac{\mathrm{d}\delta}{\delta}.$$

It is well known that G_m satisfies the following conditions (see [33])

- $G_m \in L^1(\mathbb{R}^n)$
- Given $q \in (1,\infty)$, then $u \in W^{m,q}(\mathbb{R}^n)$ if and only if $u = G_m * f$ for some $f \in L^q(\mathbb{R}^n)$.

Moreover, it is easy to see that,

Lemma 2.1. *Given $k \in \mathbb{N}$, then the operator $L_k : L^q(\mathbb{R}^n) \to W^{2k,q}(\mathbb{R}^n)$ given by*

$$L_k(f) := G_{2k} * f,$$

is an isometric isomorphism onto $W^{2k,q}(\mathbb{R}^n)$ endowed with the norm $\|\cdot\|_{2k,n,q}$. Furthermore, if $f \in W^{1,q}(\mathbb{R}^n)$ then $L_k(f) \in W^{2k+1,q}(\mathbb{R}^n)$ and L_k is an isometric isomorphism from $W^{1,q}(\mathbb{R}^n)$ onto $W^{2k,q}(\mathbb{R}^n)$, endowed with the norm $\|\cdot\|_{2k+1,n,q}$.

Now we discuss about some norm on $W^{m,q}(\mathbb{R}^n)$, for $q \in (1,\infty)$. First, note that in the Hilbert case, $q = 2$, we have

$$\|u\|_{m,n,2}^2 = \sum_{r=0}^{m} \binom{m}{r} \|\nabla^r u\|_2^2. \tag{2.1}$$

In fact, given $u \in W^{m,2}(\mathbb{R}^n)$, for $m = 2k$, we have

$$(-\Delta + I)^k u = \sum_{i=1}^{k} (-1)^{k-i} \binom{k}{i} \Delta^{k-i} u,$$

from this follows that

$$\begin{aligned}
\|u\|_{m,n,2}^2 &= \int_{\mathbb{R}^n} |(-\Delta+I)^k u|^2 \, \mathrm{d}x \\
&= \int_{\mathbb{R}^n} \sum_{0\le i,j\le k} (-1)^{k-i}(-1)^{k-j} \binom{k}{i}\binom{k}{j} \Delta^{k-i}u\Delta^{k-j}u \, \mathrm{d}x \\
&= \int_{\mathbb{R}^n} \sum_{s=0}^{2k} \sum_{i+j=s} (-1)^{2k-s} \binom{k}{i}\binom{k}{j} \Delta^{k-i}u\Delta^{k-j}u \, \mathrm{d}x \\
&= \int_{\mathbb{R}^n} \sum_{s=0}^{2k} \sum_{i+j=s} \binom{k}{i}\binom{k}{j} |\nabla^{2k-s}u|^2 \, \mathrm{d}x \\
&= \int_{\mathbb{R}^n} \sum_{s=0}^{m} \binom{m}{s} |\nabla^{2k-s}u|^2 \, \mathrm{d}x
\end{aligned}$$

where we have used the elementary fact

$$\sum_{i+j=s} \binom{k}{i}\binom{k}{j} = \binom{2k}{s}.$$

For $m = 2k+1$, we have

$$\nabla(-\Delta+I)^k u = \sum_{i=1}^{k} (-1)^{k-i} \binom{k}{i} \nabla\Delta^{k-i}u,$$

and thus

$$\begin{aligned}
\|u\|_{m,n,2}^2 &= \int_{\mathbb{R}^n} |\nabla(-\Delta+I)^k u|^2 \, \mathrm{d}x + \int_{\mathbb{R}^n} |(-\Delta+I)^k u|^2 \, \mathrm{d}x \\
&= \int_{\mathbb{R}^n} \sum_{0\le i,j\le k} (-1)^{2k-i-j} \binom{k}{i}\binom{k}{j} \left(\nabla\Delta^{k-i}u\nabla\Delta^{k-j}u + \Delta^{k-i}u\Delta^{k-j}u\right) \, \mathrm{d}x \\
&= \int_{\mathbb{R}^n} \sum_{s=0}^{2k} \binom{2k}{s} \left(|\nabla^{2k-s+1}u|^2 + |\nabla^{2k-s}u|^2\right) \, \mathrm{d}x \\
&= \int_{\mathbb{R}^n} \sum_{j=0}^{m} \binom{m}{j} |\nabla^j u|^2 \, \mathrm{d}x.
\end{aligned}$$

We also will consider the following norm on the Sobolev space $W^{m,q}(\mathbb{R}^n)$:

$$\left(\|u\|_q^q + \|\nabla^m u\|_q^q\right)^{1/q}, \quad \forall\, u \in W^{m,q}(\mathbb{R}^n). \tag{2.2}$$

This norm is equivalent to the usual Sobolev norm. This equivalence is a direct consequence of [3, Theorem 5.2] and the Riesz Potential, which we state in the following lemmas.

Lemma 2.2. *Let $n,m \geq 2$ be positive integers and $q > 1$. Then there exists $K = K(m,n,q)$ such that*

$$|u|_{j,q}^q \leq K(|u|_{m,q}^q + |u|_{0,q}^q),$$

for all $0 \le j < m$ *and* $u \in W^{m,q}(\mathbb{R}^n)$*, where*

$$|u|_{j,q}^q = \sum_{|\alpha|=j} \|D^\alpha u\|_q^q.$$

Lemma 2.3. *There exists a constant* $\tilde{K} = \tilde{K}(m,n,q)$ *such that*

$$|w|_{m,q}^q \le \tilde{K} \|\nabla^m w\|_q^q, \quad \forall\, w \in W^{m,q}(\mathbb{R}^n).$$

From this equivalence we have the following characterization of the dual space of $W^{m,q}(\mathbb{R}^n)$.

Lemma 2.4. *For every* $L \in (W^{m,q}(\mathbb{R}^n))'$ *there exists* $(w,v) \in L^{q'}(\mathbb{R}^n) \times L^{q'}\left((\mathbb{R}^n)^{r(m)}\right)$ *such that*

$$L(u) = \int_{\mathbb{R}^n} u\, w\, \mathrm{d}x + \int_{\mathbb{R}^n} \nabla^m u\, v\, \mathrm{d}x, \quad \forall\, u \in W^{m,q}(\mathbb{R}^n),$$

where $q' = q/(q-1)$ *and* $L^{q'}\left((\mathbb{R}^n)^{r(m)}\right) = L^{q'}(\mathbb{R}^n)$ *for* m *even and* $L^{q'}\left((\mathbb{R}^n)^{r(m)}\right) = L^{q'}((\mathbb{R}^n)^n) = \prod_{j=1}^n L^{q'}(\mathbb{R}^n)$ *for* m *odd.*

Proof: To prove this lemma we consider $P : W^{m,q}(\mathbb{R}^n) \to L^q(\mathbb{R}^n) \times L^q\left((\mathbb{R}^n)^{r(m)}\right)$ given by

$$P(u) = (u, \nabla^m u),$$

with $L^q(\mathbb{R}^n) \times L^q\left((\mathbb{R}^n)^{r(m)}\right)$ endowed we the product norm, which is an isometric isomorphism onto $\mathscr{W} = P(W^{m,q}(\mathbb{R}^n)) \subset L^q(\mathbb{R}^n) \times L^q\left((\mathbb{R}^n)^{r(m)}\right)$, for $W^{m,q}(\mathbb{R}^n)$ endowed with the norm (2.2) for m even and with the norm $\left(\|u\|_q^q + \sum_{j=1}^n \|\partial_{x_j} \Delta^k u\|_q^q\right)^{1/q}$ for m odd, $m = 2k+1$. Thus given $L \in (W^{m,q}(\mathbb{R}^n))'$, we have that the linear functional $\tilde{L}$ given by

$$\tilde{L}(Pu) = L(u),$$

is continuous, i.e, $\tilde{L} \in \mathscr{W}'$ and $\|\tilde{L}; \mathscr{W}'\| = \left\|L; (W^{m,q}(\mathbb{R}^n))'\right\|$. Note that, by Hahn Banach Theorem, there exists a preserving norm extension $\widehat{L} \in \left(L^q(\mathbb{R}^n) \times L^q\left((\mathbb{R}^n)^{r(m)}\right)\right)'$ of the functional $\tilde{L}$. Therefore, there exists $(w,v) \in L^{q'}(\mathbb{R}^n) \times L^{q'}\left((\mathbb{R}^n)^{r(m)}\right)$ such that

$$L(u) = \tilde{L}(Pu) = \widehat{L}(Pu) = \int_{\mathbb{R}^n} u\, w\, \mathrm{d}x + \int_{\mathbb{R}^n} \nabla^m u\, v\, \mathrm{d}x, \quad \forall\, u \in W^{m,q}(\mathbb{R}^n).$$

■

In view of the Lemma 2.1, we can consider another important characterization.

Lemma 2.5. *For every* $L \in (W^{m,q}(\mathbb{R}^n))'$ *there exist* $w \in L^{q'}(\mathbb{R}^n)$ *and* $(z,v) \in L^{q'}(\mathbb{R}^n) \times L^{q'}((\mathbb{R}^n)^n)$ *such that*

$$L(u) = \int_{\mathbb{R}^n} (I-\Delta)^k u\, w\, \mathrm{d}x, \quad \forall\, u \in W^{m,q}(\mathbb{R}^n),$$

if $m = 2k$*, for some* $k \in \mathbb{N}$*, and*

$$L(u) = \int_{\mathbb{R}^n} (I-\Delta)^k u\, z\, \mathrm{d}x + \int_{\mathbb{R}^n} \nabla(I-\Delta)^k u\, v\, \mathrm{d}x, \quad \forall\, u \in W^{m,q}(\mathbb{R}^n),$$

if $m = 2k+1$*, for some* $k \in \mathbb{N}$*.*

Proof: The proof follows by the same argument used in the previous lemma, we only need to consider the operators $P_1 : W^{2k,q}(\mathbb{R}^n) \to L^q(\mathbb{R}^n)$ given by $P_1(u) = (I-\Delta)^k u$ and $P_2 : W^{2k,q}(\mathbb{R}^n) \to L^q\left((\mathbb{R}^n)^{n+1}\right)$ given by $P_2(u) = \left((I-\Delta)^k u, \nabla(I-\Delta)^k u\right)$. ■

Another important result that we want to mention is a Radial Lemma that can be easily extended from [20], Lemma 1.1, Chapter 6, which will be useful in our analysis.

Lemma 2.6. *If $u \in W^{m,q}_{rad}(\mathbb{R}^n)$, for $1 < q < \infty$, then*

$$|u(x)| \leq \omega_{n-1}^{-1/q} q \frac{1}{|x|^{(n-1)q}} \|u\|_{W^{m,q}} \quad a.e \text{ in } \mathbb{R}^n.$$

3. Proof of the scaling invariant Adams type inequality

In this section we will prove Theorem 1.1. Before start the proof, we define two operators in $W^{m,n/m}(\mathbb{R}^n)$. Let $t,s > 0$ be real numbers. We define $I_t, J_s : W^{m,n/m}(\mathbb{R}^n) \to W^{m,n/m}(\mathbb{R}^n)$ given by

$$I_t(u)(x) := t^{m/n} u(t^{1/n} x) \tag{3.1}$$

and

$$J_s(u)(x) := u(s^{1/n} x).$$

The norms of $I_t(u)$ and $J_s(u)$ satisfy the following properties

$$\|I_t(u)\|_{n/m} = \|u\|_{n/m}, \quad \text{and} \quad \|\nabla^m I_t(u)\|_{n/m} = t^{m/n} \|\nabla^m u\|_{n/m}, \quad \forall\, t > 0,$$

and

$$\|J_s(u)\|_{n/m} = s^{-m/n} \|u\|_{n/m}, \quad \text{and} \quad \|\nabla^m J_s(u)\|_{n/m} = \|\nabla^m u\|_{n/m}, \quad \forall\, s > 0.$$

Moreover,

$$\|J_s \circ I_t(u)\|_{n/m} = s^{-m/n} \|u\|_{n/m}, \quad \text{and} \quad \|\nabla^m J_s \circ I_t(u)\|_{n/m} = t^{m/n} \|\nabla^m u\|_{n/m}, \quad \forall\, t,s > 0.$$

Now, using these two operators, we can write inequality (1.9) in the following equivalent form:

$$\sup_{\substack{u \in W^{m,n/m}_{rad}(\mathbb{R}^n) \\ \|u\|_{n/m}=1,\ \|\nabla^m u\|_{n/m}=1}} \int_{\mathbb{R}^n} \Phi\left(\beta |u|^{n/(n-m)}\right) \mathrm{d}x \leq C_{\beta,m,n}, \quad \beta \in (0, \beta_0), \tag{3.2}$$

for some constant $C_{\beta,m,n} = C(\beta, m, n)$ depending only on $\beta,$ m and $n.$ In fact, given $u \in W^{m,n/m}_{rad}(\mathbb{R}^n) \setminus \{0\}$ we take $t = \|\nabla^m u\|_{n/m}^{-n/m}$ and $s = \|u\|_{n/m}^{n/m}$ which imply

$$\|J_s \circ I_t(u)\|_{n/m} = 1, \quad \text{and} \quad \|\nabla^m J_s \circ I_t(u)\|_{n/m} = 1.$$

Then applying the inequality (3.2) to $J_s \circ I_t(u)$ and the definition of I_t and J_s we have

$$\frac{\|\nabla^m u\|_{n/m}^{n/m}}{\|u\|_{n/m}^{n/m}} \int_{\mathbb{R}^n} \Phi\left(\beta \left(\frac{|u|}{\|\nabla^m u\|_{n/m}}\right)^{n/(n-m)}\right) \mathrm{d}x = \int_{\mathbb{R}^n} \Phi\left(\beta |J_s \circ I_t(u)|^{n/(n-m)}\right) \mathrm{d}x \leq C_{\beta,m,n},$$

which gives inequality (1.9). The reciprocal is immediate.

Henceforth we will prove inequality (3.2). Let us consider

$$M := \{u \in W^{m,n/m}_{rad}(\mathbb{R}^n) : \ \|u\|_{n/m} = 1 \text{ and } \|\nabla^m u\|_{n/m} = 1\}.$$

From Lemmas 2.2 and 2.3, we have

$$\|u\|^{n/m}_{W^{m,n/m}} \le K(\|u\|^{n/m}_{n/m} + \|\nabla^m u\|^{n/m}_{n/m}) \le 2K, \quad \forall\, u \in M. \tag{3.3}$$

for some constant $K = K(m,n) > 0$ depending only on m and n, i.e, M is a bounded subset of $W^{m,n/m}(\mathbb{R}^n)$.

Now take $\beta \in (0, \beta_0)$ fixed and any arbitrary $u \in M$. For any $R_0 > 0$ we have

$$\begin{aligned}\int_{\mathbb{R}^n} \Phi\left(\beta|u|^{n/(n-m)}\right)\,\mathrm{d}x &= \int_{B_{R_0}} \Phi\left(\beta|u|^{n/(n-m)}\right)\,\mathrm{d}x + \int_{\mathbb{R}^n\setminus B_{R_0}} \Phi\left(\beta|u|^{n/(n-m)}\right)\,\mathrm{d}x\\ &\le \int_{B_{R_0}} e^{\left(\beta|u|^{n/(n-m)}\right)}\,\mathrm{d}x + \int_{\mathbb{R}^n\setminus B_{R_0}} \Phi\left(\beta|u|^{n/(n-m)}\right)\,\mathrm{d}x\\ &= A_0 + A_1,\end{aligned}$$

where

$$A_0 := \int_{B_{R_0}} e^{\left(\beta|u|^{n/(n-m)}\right)}\,\mathrm{d}x \quad \text{and} \quad A_1 := \int_{\mathbb{R}^n\setminus B_{R_0}} \Phi\left(\beta|u|^{n/(n-m)}\right)\,\mathrm{d}x.$$

In what follows we will show that it is possible to choose $R_0 = R_0(\beta, m, n)$ fixed such that the integrals A_0 e A_1 are bounded by a constant $C_{\beta,m,n}$ depending only on β, m and n.

Now we estimate A_1. By the Radial Lemma 2.6, there exist C_n depending only on n such that

$$|u(x)| = |u(|x|)| \le C_n \frac{1}{|x|^{m(n-1)/n}}\|u\|_{W^{m,n/m}} \le 2KC_n \frac{1}{|x|^{m(n-1)/n}}, \quad \text{a.e. in } \mathbb{R}^n,$$

where in last inequality we have used the inequality (3.3). Then taking $R_0 > (2KC_n)^{n/m(n-1)}$ we have

$$|u(x)| < 1, \quad \text{a.e. in } \mathbb{R}^n \setminus B_{R_0}.$$

Thus

$$A_1 = \int_{\mathbb{R}^n\setminus B_{R_0}} \Phi\left(\beta|u|^{n/(n-m)}\right)\,\mathrm{d}x \le e^{\beta}\int_{\mathbb{R}^n\setminus B_{R_0}} |u|^{n/m}\,\mathrm{d}x \le e^{\beta}\|u\|^{n/m}_{n/m} \le e^{\beta}.$$

To estimate integral A_0, we will construct an auxiliary function $w \in W^{m,n/m}_{\mathscr{N}}(B_{R_0})$ with $\|\nabla^m w\|_{n/m} = 1$ and apply Theorem B. To construct $w \in W^{m,n/m}_{\mathscr{N}}(B_{R_0})$ we consider the functions

$$g_l(|x|) := |x|^{2k-2l}, \qquad \forall x \in B_{R_0},$$

for $l = 1, 2, \ldots, k-1$ where k is the natural number such that $m = 2k+1$ or $m = 2k$. Note that $g_l \in W^{m,n/m}_{rad}(B_{R_0})$ and more

$$\Delta^j g_l(|x|) = \begin{cases} c_l^j |x|^{2k-2(l+j)} & \text{for} \quad j = 1, 2, \ldots, k-l,\\ 0 & \text{for} \quad j = k-l+1, \ldots, k.\end{cases} \quad \forall\, x \in B_{R_0},$$

where

$$c_l^j = \prod_{s=1}^{j} [n+2k-2(s+l)][2k-2(l+s-1)],$$

for $j = 1,2,\ldots,k-l$. Now we define

$$w(|x|) := u(|x|) - \sum_{l=1}^{k-1} a_l g_l(|x|) - a_k,$$

where

$$\begin{aligned} a_0 &:= \frac{\Delta^k u(R_0)}{\Delta^k g_0(R_0)} \\ a_l &:= \frac{\Delta^{k-l} u(R_0) - \sum_{s=0}^{l-1} a_s \Delta^{k-l} g_s(R_0)}{\Delta^{k-l} g_l(R_0)}, \quad \text{for } l = 1,2,\ldots,k-1, \\ a_k &:= u(R_0) - \sum_{s=0}^{k-1} a_s g_s(R_0). \end{aligned} \tag{3.4}$$

Note that $\nabla^m u = \nabla^m w$ and more $w \in W_{\mathcal{N}}^{m,n/m}(B_{R_0})$. Hence, using the following elementary inequality

$$(a+b)^q \leq (1+\delta)^q a^q + (1+1/\delta)^q b^q, \quad \forall\, \delta > 0,$$

for $a,b > 0$ and $q \geq 1$, we have

$$\begin{aligned} A_0 &= \int_{B_{R_0}} e^{\left(\beta |u|^{n/(n-m)}\right)}\, \mathrm{d}x = \int_{B_{R_0}} e^{\left(\beta \left|w + \sum_{l=1}^{k-1} a_l g_l(|x|) + a_k\right|^{n/(n-m)}\right)}\, \mathrm{d}x \\ &\leq e^{\beta (1+1/\delta)^{n/(n-m)} \left|\sum_{l=1}^{k-1} a_l g_l(|R_0|) + a_k\right|^{n/(n-m)}} \int_{B_{R_0}} e^{\left(\beta (1+\delta)^{n/(n-m)} |w|^{n/(n-m)}\right)}\, \mathrm{d}x. \end{aligned}$$

Now, since $\|\nabla^m w\|_{n/m} = \|\nabla^m u\|_{n/m} = 1$, taking $\delta > 0$ sufficiently small such that $\beta(1+\delta)^{n/(n-m)} < \beta_0$ and applying Theorem B, we can estimate

$$A_0 \leq e^{\beta (1+1/\delta)^{n/(n-m)} \left|\sum_{l=1}^{k-1} a_l g_l(|R_0|) + a_k\right|^{n/(n-m)}} C_{R_0,m,n}.$$

Furthermore, by Lemma 2.6,

$$|\Delta^j u(R_0)| \leq C_n \frac{1}{R_0^{m(n-1)/n}} \|u\|_{W^{m,n/m}} \leq 2K C_n \frac{1}{R_0^{m(n-1)/n}}, \quad \text{for } 0 \leq j \leq k,$$

which, together with (3.4), implies that $|\sum_{l=1}^{k-1} a_l g_l(|R_0|) + a_k| \leq \tilde{C}_{R_0,m,n}$. Therefore, since R_0 is fixed we conclude that

$$\int_{\mathbb{R}^n} \Phi\left(\beta |u|^{n/(n-m)}\right)\, \mathrm{d}x \leq A_0 + A_1 \leq C_{\beta,m,n},$$

for some constant $C_{\beta,m,n} > 0$ depending only on β, m and n.

Now we will construct a sequence to guarantee the optimality of inequality (1.9). We only need to find a sequence that satisfies (1.10) for $\beta = \beta_0$. The sequence is basically the same sequence used by D. Adams in [2]. Let $\Phi(t) \in C^\infty[0,1]$ such that

$$\begin{aligned} &\Phi(0) = \Phi'(0) = \cdots = \Phi^{(m-1)}(0) = 0, \\ &\Phi(1) = \Phi'(1) = 1 \qquad \Phi''(1) = \Phi'''(1) = \cdots = \Phi^{(m-1)}(1) = 0. \end{aligned}$$

For $0 < \varepsilon < \frac{1}{2}$, we define

$$H(t) = \begin{cases} \varepsilon\Phi(\frac{1}{\varepsilon}t), & \text{if} \quad t \le \varepsilon \\ t, & \text{if} \quad \varepsilon \le t \le 1-\varepsilon \\ 1 - \varepsilon\Phi(\frac{1}{\varepsilon}(1-t)), & \text{se} \quad 1-\varepsilon \le t \le 1 \\ 1, & \text{if} \quad 1 \le t, \end{cases}$$

and

$$\psi_i(r) = (\log i)^{(m-n)/n} H\left((\log i)^{-1} \log \frac{1}{r}\right).$$

Note that $\psi_i(|x|) \in W^{m,\frac{n}{m}}_{rad}(\mathbb{R}^n)$. More than that, $\psi_i(|x|) = (\log i)^{(m-n)/n}$ for $|x| \le 1/i$ and, as proved for D. Adams in [2], we have

$$\|\nabla^m \psi_i\|_{n/m} = n^{(m-n)/n} \beta_0^{(m-n)/n} A_{i,\varepsilon}^{m/n},$$

where

$$A_{i,\varepsilon} \le \left[1 + 2\varepsilon\left(\|\Phi'\|_\infty + O\left((\log i)^{-1}\right)\right)^{n/m}\right].$$

By easy calculation we also have

$$\|\psi_i\|_{n/m}^{n/m} = o(1). \tag{3.5}$$

Now, for each i, we take $\varepsilon = (\log i)^{-1}$. Then, since $m/(n-m) < 1$,

$$\begin{aligned} \int_{\mathbb{R}^n} \Phi\left(\beta_0 \frac{\psi_i^{n/(n-m)}}{\|\nabla^m \psi_i\|_{n/m}^{n/(n-m)}}\right) \,\mathrm{d}x &\ge \int_{B(1/i)} \Phi\left(\frac{n \log i}{A_i^{m/(n-m)}}\right) \,\mathrm{d}x = \frac{\omega_{n-1}}{n} \Phi\left(\frac{n\log i}{A_i^{m/(n-m)}}\right) \frac{1}{i^n} \\ &\ge \frac{\omega_{n-1}}{n} e^{-n\log i\left(1 - \frac{1}{1+2(\log i)^{-1}\left(\|\Phi'\|_\infty + O\left((\log i)^{-1}\right)\right)^{n/m}}\right)} - \sum_{j=0}^{j_p - 2} \left(\frac{n\log i}{A_i^{m/(n-m)}}\right)^j e^{-n\log i}. \end{aligned}$$

Therefore, passing the limit we have

$$\lim_{i\to\infty} \int_{\mathbb{R}^n} \Phi\left(\beta_0 \frac{\psi_i^{n/(n-m)}}{\|\nabla^m \psi_i\|_{n/m}^{n/(n-m)}}\right) \,\mathrm{d}x \ge \frac{\omega_{n-1}}{n} e^{-2n\|\Phi'\|_\infty},$$

which, together with (3.5), proves that (ψ_i) satisfies (1.10) for $\beta = \beta_0$.

4. Existence of extremal for the scaling invariant Adams type inequality

In this section we will prove Theorem 1.2. Given $\beta \in (0,\beta_0)$, from (3.2), we have that

$$\mu_{\beta,n,m} = \sup_{\substack{u\in W^{m,n/m}_{rad}(\mathbb{R}^n)\\ \|u\|_{n/m}=1,\ \|\nabla^m u\|_{n/m}=1}} \int_{\mathbb{R}^n} \Phi\left(\beta|u|^{n/(n-m)}\right)\, \mathrm{d}x \tag{4.1}$$

Then, we can take a sequence $(u_i) \subset W^{m,n/m}_{rad}(\mathbb{R}^n)$ such that $\|u_i\|_{n/m} = 1$, $\|\nabla^m u_i\|_{n/m} = 1$, for all $i \in \mathbb{N}$, and

$$\lim_{i\to\infty} \int_{\mathbb{R}^n} \Phi\left(\beta|u_i|^{n/(n-m)}\right)\, \mathrm{d}x = \mu_{\beta,n,m}. \tag{4.2}$$

From Lemmas 2.2 and 2.3, we have that (u_i) is bounded in $W^{m,n/m}_{rad}(\mathbb{R}^n)$. Thus, there exist $u \in W^{m,n/m}_{rad}(\mathbb{R}^n)$ such that

$$\begin{array}{llll} u_i \rightharpoonup u & \text{in} & W^{m,n/m}_{rad}(\mathbb{R}^n) & \\ u_i \to u & \text{in} & L^q(\mathbb{R}^n), & \text{for} \quad \frac{n}{m} < q < \infty, \\ u_i(x) \to u(x) & \text{a.e. in} & \mathbb{R}^n. & \end{array}$$

Moreover, by Lemma 2.4 and the compact embedding $u_i \rightharpoonup u$ and $\nabla^m u_i \rightharpoonup \nabla^m u$ in $L^{n/m}(\mathbb{R}^n)$ which implies that $\|u\|_{n/m} \le 1$ and $\|\nabla^m u\|_{n/m} \le 1$.

We will prove that $u \not\equiv 0$ and that u is an extremal function to $\mu_{\beta,n,m}$. In order to prove that, we will divide in two parts.

Part 1: $n/m \notin \mathbb{N}$.

Since $nj/(n-m) > n/m$ for $j \in \{j_{m,n}-1, j_{m,n}, j_{m,n}+1, j_{m,n}+2, \ldots\}$, Vitali Convergence Theorem yields

$$\int_{\mathbb{R}^n} \Phi(\beta|u_i|^{n/(n-m)})\, \mathrm{d}x = \sum_{j:=j_{m,n}-1}^{\infty} \frac{\beta^j}{j!} \|u_i\|^{nj/(n-m)}_{nj/(n-m)} \to \int_{\mathbb{R}^n} \Phi(\beta|u|^{n/(n-m)})\, \mathrm{d}x, \quad \forall\, \beta \in (0,\beta_0).$$

Then, by (4.2), we have $u \not\equiv 0$ and

$$\begin{aligned} \mu_{\beta,n,m} &= \int_{\mathbb{R}^n} \Phi\left(\beta|u|^{n/(n-m)}\right)\, \mathrm{d}x = \sum_{j:=j_{m,n}-1}^{\infty} \frac{\beta^j}{j!} \|u\|^{nj/(n-m)}_{nj/(n-m)} \\ &\le \frac{1}{\|u\|^{n/m}_{n/m}} \sum_{j:=j_{m,n}-1}^{\infty} \frac{\beta^j}{j!} \frac{\|u\|^{nj/(n-m)}_{nj/(n-m)}}{\|\nabla^m u\|^{nj/(n-m)-n/m}_{n/m}} \\ &= \frac{\|\nabla^m u\|^{n/m}_{n/m}}{\|u\|^{n/m}_{n/m}} \int_{\mathbb{R}^n} \Phi\left(\beta \left| \frac{u}{\|\nabla^m u\|_{n/m}} \right|^{n/(n-m)}\right)\, \mathrm{d}x, \end{aligned}$$

which implies that $\|u\|_{n/m} = 1$ and $\|\nabla^m u\|_{n/m} = 1$. Therefore, u is an extremal function to $\mu_{\beta,n,m}$.

Part 2: $n/m \in \mathbb{N}$.

By Vitali Convergence Theorem we have that

$$\lim_{i\to\infty}\int_{\mathbb{R}^n}\Phi(\beta|u_i|^{n/(n-m)})\,\mathrm{d}x=\int_{\mathbb{R}^n}\Phi(\beta|u|^{n/(n-m)})\,\mathrm{d}x-\int_{\mathbb{R}^n}\frac{\beta^{(n-m)/m}}{((n-m)/m)!}|u|^{n/m}\,\mathrm{d}x+\frac{\beta^{(n-m)/m}}{((n-m)/m)!},\quad(4.3)$$

for all $\beta\in(0,\beta_0)$.

First we suppose that $u\not\equiv 0$. Since $\|\nabla^m u\|_{n/m}\le 1$ we have

$$\mu_{\beta,n,m}\ge\frac{\|\nabla^m u\|_{n/m}^{n/m}}{\|u\|_{n/m}^{n/m}}\int_{\mathbb{R}^n}\Phi\left(\beta\left|\frac{u}{\|\nabla^m u\|_{n/m}}\right|^{n/(n-m)}\right)\,\mathrm{d}x\ge\frac{1}{\|u\|_{n/m}^{n/m}}\int_{\mathbb{R}^n}\Phi\left(\beta\,|u|^{n/(n-m)}\right)\,\mathrm{d}x$$

and, using (4.3),

$$\begin{aligned}\mu_{\beta,n,m}&\ge\int_{\mathbb{R}^n}\Phi\left(\beta\,|u|^{n/(n-m)}\right)\,\mathrm{d}x-\int_{\mathbb{R}^n}\frac{\beta^{(n-m)/m}}{((n-m)/m)!}|u|^{n/m}\,\mathrm{d}x+\frac{\beta^{(n-m)/m}}{((n-m)/m)!}\\&\qquad+\left(\frac{1}{\|u\|_{n/m}^{n/m}}-1\right)\left(\int_{\mathbb{R}^n}\Phi\left(\beta\,|u|^{n/(n-m)}\right)\,\mathrm{d}x-\int_{\mathbb{R}^n}\frac{\beta^{(n-m)/m}}{((n-m)/m)!}|u|^{n/m}\,\mathrm{d}x\right)\\&=\lim_{i\to\infty}\int_{\mathbb{R}^n}\Phi\left(\beta\,|u_i|^{n/(n-m)}\right)\,\mathrm{d}x\\&\qquad+\left(\frac{1}{\|u\|_{n/m}^{n/m}}-1\right)\left(\int_{\mathbb{R}^n}\Phi\left(\beta\,|u|^{n/(n-m)}\right)\,\mathrm{d}x-\int_{\mathbb{R}^n}\frac{\beta^{(n-m)/m}}{((n-m)/m)!}|u|^{n/m}\,\mathrm{d}x\right)\\&=\mu_{\beta,n,m}+\left(\frac{1}{\|u\|_{n/m}^{n/m}}-1\right)\left(\int_{\mathbb{R}^n}\Phi\left(\beta\,|u|^{n/(n-m)}\right)\,\mathrm{d}x-\int_{\mathbb{R}^n}\frac{\beta^{(n-m)/m}}{((n-m)/m)!}|u|^{n/m}\,\mathrm{d}x\right)\end{aligned}$$

Thus, since $\|u\|_{n/m}^{n/m}\le 1$, we have that $\|u\|_{n/m}^{n/m}=1$. Then,

$$\lim_{i\to\infty}\int_{\mathbb{R}^n}\Phi(\beta|u_i|^{n/(n-m)})\,\mathrm{d}x=\int_{\mathbb{R}^n}\Phi(\beta|u|^{n/(n-m)})\,\mathrm{d}x,$$

and,

$$\mu_{\beta,n,m}=\int_{\mathbb{R}^n}\Phi\left(\beta|u|^{n/(n-m)}\right)\,\mathrm{d}x\le\|\nabla^m u\|_{n/m}^{n/m}\int_{\mathbb{R}^n}\Phi\left(\beta\left|\frac{u}{\|\nabla^m u\|_{n/m}}\right|^{n/(n-m)}\right)\,\mathrm{d}x,$$

which implies that $\|\nabla^m u\|_{n/m}=1$ and u is an extremal function to $\mu_{\beta,n,m}$.

Now we will prove that $u\not\equiv 0$. If $u\equiv 0$, by (4.3), we have that

$$\lim_{i\to\infty}\int_{\mathbb{R}^n}\Phi(\beta|u_i|^{n/(n-m)})\,\mathrm{d}x=\frac{\beta^{(n-m)/m}}{((n-m)/m)!}.$$

But, given $v\in W^{m,n/m}_{rad}(\mathbb{R}^n)$ with $\|v\|_{n/m}=1$ and $\|\nabla^m v\|_{n/m}=1$

$$\mu_{\beta,n,m}\ge\int_{\mathbb{R}^n}\Phi(\beta|v|^{n/(n-m)})\,\mathrm{d}x=\sum_{j:=n/m}^{\infty}\frac{\beta^j}{j!}\|v\|_{nj/(n-m)}^{nj/(n-m)}+\frac{\beta^{(n-m)/m}}{((n-m)/m)!}>\frac{\beta^{(n-m)/m}}{((n-m)/m)!},$$

which is a contradiction with (4.1). Therefore $u\not\equiv 0$.

5. Comparison result

On this section we will prove Theorem 1.4 and Corollary 1.5. For this purpose we consider the Schwarz rearrangement. Given $A \subset \mathbb{R}^n$ any open set. In case that $|A| < \infty$, we denote by A^* the ball of radio $R > 0$ centered at the origin in $\mathbb{R}^n$ such that $|A^*| = |A|$. Otherwise, we consider $A^* = \mathbb{R}^n$. Let $u : A \to \mathbb{R}$ be a measurable function. We denote by

$$\mu(t) = |\{x \in A : |u(x)| > t\}|$$

and

$$u^{\#}(s) := \inf\{t \geq 0 : \mu(t) < s\} \quad \forall s \in [0, |A|],$$

the *distribution function* and the *decreasing rearrangement* of u, respectively, and by

$$u^*(x) := u^{\#}(\omega_{n-1}|x|^n) \quad \forall x \in A^*,$$

the *spherically symmetric decreasing rearrangement* of u. A comparison result was proved by G. Trombetti, J. L. Vazquez [38] which gave a powerful tool on the study of elliptic partial differential equations, which we enunciated here for easy reference.

Theorem E. *Let $B_R \subset \mathbb{R}^n$ be a ball of radio $R > 0$ centered in the origin. Let $f \in L^p(B_R)$, with $p \geq 2n/(n+2)$ and $u \in W^{2,p}_{\mathcal{N}}(B_R)$, the unique strong solution of*

$$\begin{cases} u - \Delta u = f & in \quad B_R \\ u = 0 & in \quad \partial B_R. \end{cases}$$

Let, $f^* \in L^p(B_R)$ and u^* be the spherically symmetric decreasing rearrangement of f and u, respectively, and $v \in W^{2,p}_{\mathcal{N}}(B_R)$ the unique strong solution of

$$\begin{cases} v - \Delta v = f^* & in \quad B_R \\ v = 0 & in \quad \partial B_R. \end{cases}$$

Then $u^* \leq v$ a.e in B_R.

This result was extended for an operator of high order derivative by B. Ruf and F. Sani in [32]. We will enunciate this result and give the proof for completeness.

Proposition 5.1. *Let $p \geq 2n/(n+2)$ and B_R the ball of radio R centered in the origin. If $f \in L^p(B_R)$ and $u \in W^{2k,p}_{\mathcal{N}}(B_R)$ is the unique strong solution of*

$$\begin{cases} (I - \Delta)^k u = f & in \quad B_R, \\ \Delta^j u = u = 0 & in \quad \partial B_R, \ j = 1, 2, \dots, k-1. \end{cases} \tag{5.1}$$

and $v \in W^{2k,p}_{\mathcal{N}}(B_R)$ is the unique strong solution of

$$\begin{cases} (I - \Delta)^k v = f^* & in \quad B_R, \\ \Delta^j v = v = 0 & in \quad \partial B_R, \ j = 1, 2, \dots, k-1. \end{cases} \tag{5.2}$$

Then $u^* \leq v$ a.e. in B_R.

Proof: When $k = 1$ the proposition is exactly the the Theorem E. Thus, for $k \geq 2$, we can rewrite the problems (5.1) and (5.2) in the following system form

$$\begin{cases} -\Delta u_1 = f & \text{in} \quad \Omega \\ u_1 = 0 & \text{in} \quad \partial\Omega. \end{cases} \qquad \begin{cases} -\Delta v_1 = f^* & \text{in} \quad \Omega^* \\ v_1 = 0 & \text{in} \quad \partial\Omega^*. \end{cases}$$

$$\begin{cases} -\Delta u_i = u_{i-1} & \text{in} \quad \Omega \\ u_i = 0 & \text{in} \quad \partial\Omega. \end{cases} \qquad \begin{cases} -\Delta v_i = v_{i-1} & \text{in} \quad \Omega^* \\ v_i = 0 & \text{in} \quad \partial\Omega^*. \end{cases}$$

For $i = 2, \ldots, k$. Note that $u_k = u$, $v_k = v$. Thus applying iteratively the Theorem E, together with Maximum Principle, we have $u_i^* \leq v_i$ a.e. in B_R for $i = 2, \ldots, k$. Therefore $u^* \leq v$ a.e. in B_R. ■

Now we will extend this comparison result for functions defined in the whole euclidean space.

Proposition 5.2. *Let $u \in W^{2k,p}(\mathbb{R}^n)$ for some $k \in \mathbb{N}$ and $p \geq 1$. Now, let $f := (I - \Delta)^k u$ and $v \in W^{2k,p}(\mathbb{R}^n)$ given by*

$$v := G_{2k} * f^*,$$

where G_{2k} is the Bessel potential. Then $u^* \leq v$ a.e. in $\mathbb{R}^n$.

Proof: In the first way, by density we can take $u_i \in C_0^\infty(\mathbb{R}^n)$ such that $u_i \to u$ in $W^{2k,p}(\mathbb{R}^n)$ and $\operatorname{supp} u_i \subset B_{R_i}$, the ball of radio R_i centered in the origin, with $R_i \to \infty$ as $i \to \infty$. We set $f_i = (I - \Delta)^k u_i$, with $\operatorname{supp} f_i \subset B_{R_i}$. Now, we take $\tilde{u}_i \in W^{2k,p}_{\mathcal{N}}(B_{R_i})$ and $v_i \in W^{2k,p}(\mathbb{R}^n)$ given by

$$\begin{cases} (I - \Delta)^k \tilde{u}_i = f_i^* & \text{in} \quad B_{R_i} \\ \tilde{u}_i = \Delta^j \tilde{u}_i = 0 & \text{in} \quad \partial B_{R_i}, \quad 1 \leq j \leq k-1. \end{cases}$$

and

$$v_i = G_{2k} * f_i^*.$$

Note also that, since $\operatorname{supp} f_i \subset B_{R_i}$, we have $f_i^* = \left(f_i|_{B_{R_i}} \right)^*$. Thus, applying the Proposition 5.1,

$$u_i^* \leq \tilde{u}_i \quad \text{a.e. in } B_{R_i}.$$

Moreover, since $(I - \Delta)^k v_i = (I - \Delta)^k \tilde{u}_i = f_i^*$ in B_{R_i} and $v_i(x), \Delta^j v_i(x) \geq 0$ for all $x \in \mathbb{R}^n$, for $1 \leq j \leq k-1$, we can apply the Maximum Principal iteratively to get that

$$u_i^* \leq \tilde{u}_i \leq v_i \quad \text{a.e. in } B_{R_i}. \tag{5.3}$$

Now, as $u_i \to u$ in $W^{2k,p}(\mathbb{R}^n)$ we have that $u_i^* \to u^*$ and $f_i^* \to f^*$, from which follows that $v_i \to v$ in $W^{2k,p}(\mathbb{R}^n)$. Therefore, from (5.3) we have

$$u^* \leq v \quad \text{a.e. in } \mathbb{R}^n,$$

which conclude the proof. ■

Now we are in position to prove the Theorem 1.4.

Proof of Theorem 1.4. Given $u \in W^{m,n/m}(\mathbb{R}^n)$, when $m = 2k+1$ or $m = 2k$ for some $k \in \mathbb{N}$, we set $f := (I-\Delta)^k u$ and take $v \in W^{2k,n/m}_{rad}(\mathbb{R}^n)$ such that

$$v := G_{2k} * f^*.$$

By Proposition 5.3 we have that v satisfies $u^* \leq v$. Note that, when $m = 2k+1$ by Lemma 2.1 we have that $v \in W^{m,n/m}(\mathbb{R}^n)$. Furthermore, for $m = 2k$ we have

$$\|u\|_{m,n,\frac{n}{m}} = \|(I-\Delta)^k u\|_{n/m} = \|((I-\Delta)^k u)^*\|_{n/m} = \|v\|_{m,n,\frac{n}{m}},$$

and for $m = 2k+1$, by Pólya-Szegö inequality, we have

$$\begin{aligned}\|u\|^{n/m}_{m,n,\frac{n}{m}} &= \|\nabla(-\Delta+I)^k u\|^{n/m}_{n/m} + \|(-\Delta+I)^k u\|^{n/m}_{n/m} \\ &\geq \|\nabla((I-\Delta)^k u)^*\|^{n/m}_{n/m} + \|((I-\Delta)^k u)^*\|^{n/m}_{n/m} = \|v\|^{n/m}_{m,n,\frac{n}{m}},\end{aligned}$$

■

Corollary 1.5 follows immediately form the Theorem 1.4.

Remark 5.3. *The use of comparison results related to the technique of rearrangement was intensively used in the study of solutions to partial differential equations. Some references related to this subject can be fond in [36, 38, 40, 12, 7].*

6. Proof of the nonexistence of extremals

In this sections we will prove Theorem 1.3. By Corollary 1.5 we only need to look at the radially symmetric functions. So let $n > 2$ and $m \geq 2$ integers such that $n/m = 2$. Given $\beta \in (0, (4\pi)^m m!)$ and $u \in W^{m,2}_{rad}(\mathbb{R}^n)$, we define the following functional

$$\mathscr{J}_\beta(u) := \int_{\mathbb{R}^n} \left(e^{\beta |u|^2} - 1 \right) \mathrm{d}x,$$

and $\mathscr{M} := \{u \in W^{m,2}_{rad}(\mathbb{R}^n) : \|u\|_{m,n,2} = 1\}$, for $\|u\|_{m,n,2}$ given as in (1.11). Thus, in this notation we have

$$\eta_{\beta,n,m} = \sup_{u \in \mathscr{M}} \mathscr{J}_\beta(u).$$

Now, given $u \in \mathscr{M}$ we also define $f_{u,\beta} : (0,\infty) \to \mathbb{R}$ given by

$$f_{u,\beta}(t) := \mathscr{J}_\beta \left(\frac{I_t(u)}{\|I_t(u)\|_{m,n,2}} \right),$$

where I_t was defined in (3.1). If $u \in \mathscr{M}$ is an function that attains $\eta_{\beta,n,m}$ then $t = 1$ will be a critical point of $f_{u,\beta}$, i.e., $f'_{u,\beta}(1) = 0$. Therefore we only need to prove that for β sufficiently small $f'_{u,\beta}(1) \neq 0$ for all $u \in \mathscr{M}$. In order to derive $f_{u,\beta}$ we look at the norms of $I_t(u)$ and of their derivatives

$$\|I_t(u)\|^{2j}_{2j} = \int_{\mathbb{R}^n} \left| t^{1/2} u(t^{1/n} x) \right|^{2j} \mathrm{d}x = t^{j-1} \|u\|^{2j}_{2j};$$

$$\|\nabla^r I_t(u)\|^2_2 = \int_{\mathbb{R}^n} \left| t^{(n+2r)/2n} (\nabla^r u)(t^{1/n} x) \right|^2 \mathrm{d}x = t^{2r/n} \|\nabla^r u\|^2_2.$$

Hence we can write $f_{u,\beta}(t)$ as follows:

$$\begin{aligned}
f_{u,\beta}(t) = \mathscr{J}_\beta\left(\frac{I_t(u)}{\|I_t(u)\|_{m,n,2}}\right) &= \sum_{j=1}^{\infty} \frac{\beta^j}{j!}\frac{\|I_t(u)\|_{2j}^{2j}}{\|I_t(u)\|_{m,n,2}^{2j}} \\
&= \sum_{j=1}^{\infty} \frac{\beta^j}{j!}\frac{t^{j-1}\|u\|_{2j}^{2j}}{\left(\sum_{r=0}^{m}\binom{m}{r}\|\nabla^r I_t(u)\|_2^2\right)^j} \\
&= \sum_{j=1}^{\infty} \frac{\beta^j}{j!}\frac{t^{j-1}\|u\|_{2j}^{2j}}{\left(\sum_{r=0}^{m}\binom{m}{r}t^{2r/n}\|\nabla^r u\|_2^2\right)^j},
\end{aligned}$$

where we have consider the form (2.1). Then we will calculate and estimate $f'_{u,\beta}(1)$, that is,

$$f'_{u,\beta}(1) = \sum_{j=1}^{\infty}\frac{\beta^j}{j!}\|u\|_{2j}^{2j}\left(\sum_{r=0}^{m}\binom{m}{r}\|\nabla^r u\|_2^2\right)^{j-1}\frac{(j-1)\left(\sum_{r=0}^{m}\binom{m}{r}\|\nabla^r u\|_2^2\right) - j\left(\sum_{r=1}^{m}\binom{m}{r}\frac{2r}{n}\|\nabla^r u\|_2^2\right)}{\left(\sum_{r=0}^{m}\binom{m}{r}\|\nabla^r u\|_2^2\right)^{2j}}$$

and, since

$$\sum_{r=0}^{m}\binom{m}{r}\|\nabla^r u\|_2^2 = \|u\|_{m,n,2}^2 = 1 \quad \text{and} \quad n = 2m,$$

we have

$$\begin{aligned}
f'_{u,\beta}(1) &= \sum_{j=1}^{\infty}\frac{\beta^j}{j!}\|u\|_{2j}^{2j}\left((j-1) - j\left(\sum_{r=1}^{m}\binom{m}{r}\frac{2r}{n}\|\nabla^r u\|_2^2\right)\right) \\
&\le \sum_{j=1}^{\infty}\frac{\beta^j}{j!}\|u\|_{2j}^{2j}(j-1-j\|\nabla^m u\|_2^2) \\
&\le \sum_{j=1}^{\infty}\frac{\beta^j}{j!}(j-1)\|u\|_{2j}^{2j} - \beta\|\nabla^m u\|_2^2\|u\|_2^2.
\end{aligned}$$

Now, using that

$$\sum_{j=1}^{\infty}\frac{\beta^j}{j!}(j-1)\|u\|_{2j}^{2j} = \sum_{j=2}^{\infty}\frac{\beta^j}{j!}(j-1)\|u\|_{2j}^{2j} \le \sum_{j=2}^{\infty}\frac{\beta^j}{(j-1)!}\|u\|_{2j}^{2j},$$

we obtain that

$$f'_{u,\beta}(1) \le \beta\|\nabla^m u\|_2^2\|u\|_2^2\left(\sum_{j=2}^{\infty}\frac{\beta^{j-1}}{(j-1)!}\frac{\|u\|_{2j}^{2j}}{\|\nabla^m u\|_2^2\|u\|_2^2} - 1\right). \tag{6.1}$$

Now to conclude the proof we will use the Proposition 1.1 and the fact that $\|\nabla^m u\|_2^2 \le 1$ to estimate the positive term of the right hand side of inequality (6.1). Applying Proposition 1.1 to some fixed $\bar{\beta} < (4\pi)^m m!$, we have

$$\frac{\bar{\beta}^j}{j!}\frac{\|u\|_{2j}^{2j}}{\|\nabla^m u\|_2^{2j}} \le \int_{\mathbb{R}^n}\left(e^{\bar{\beta}\left(\frac{|u|}{\|\nabla^m u\|_2}\right)^2} - 1\right)\,\mathrm{d}x \le C_{\bar{\beta},m,n}\frac{\|u\|_2^2}{\|\nabla^m u\|_2^2}, \quad \forall\, u \in W_{rad}^{m,2}(\mathbb{R}^n).$$

for all $j \in \mathbb{N}$. Thus, by inequality (6.1) and $\|\nabla^m u\|_2^2 \leq 1$, we have

$$\frac{\|u\|_{2j}^{2j}}{\|\nabla^m u\|_2^2 \|u\|_2^2} \leq \frac{\|u\|_{2j}^{2j}}{\|\nabla^m u\|_2^{2(j-1)} \|u\|_2^2} \leq \frac{j!}{\bar{\beta}^j} C_{\bar{\beta},m,n}, \quad \forall\, u \in W_{rad}^{m,2}(\mathbb{R}^n) \quad \text{and} \quad j = 2,3,\dots.$$

Then, for all $\beta > 0$ such that $\beta/\bar{\beta} < 1/2$ we have

$$f'_{u,\beta}(1) \leq \beta \|\nabla^m u\|_2^2 \|u\|_2^2 \left(\beta \frac{1}{\bar{\beta}^2} \sum_{j=2}^{\infty} j \left(\frac{1}{2}\right)^{j-1} C_{\bar{\beta},m,n} - 1 \right).$$

Therefore for $\beta < \min\left\{ \left(1/\bar{\beta}^2 \sum_{j=2}^{\infty} j\,(1/2)^{j-1} C_{\bar{\beta},m,n} \right)^{-1}, \bar{\beta}/2 \right\}$, we obtain

$$f'_{u,\beta}(1) < 0, \quad \forall\, u \in \mathscr{M},$$

which, as we said before, guarantees that no $u \in W_{rad}^{m,2}(\mathbb{R}^n)$ such that $\|u\|_{m,n,2} = 1$ can be an extremal function to $\eta_{\beta,n,m}$.

7. Proof of the Theorem 1.6

The argument used here to prove the Theorem 1.6 follows the ideas used to prove a similar result for a bounded domain in [16]. To proceed with the proof we enunciate the Lemma 2 proved in [16] that describe a relation between weak convergence sequences and spherically symmetric decreasing rearrangement.

Lemma 7.1. *Let $f_i, f \in L^q(\mathbb{R}^n)$ such that $f_i \rightharpoonup f$ weakly in $L^q(\mathbb{R}^n)$, $q > 1$. Then, up to a subsequence, $f_i^* \to g$ almost everywhere for some $g \in L^q(\mathbb{R}^n)$ such that $\|g\|_q \geq \|f^*\|_q$.*

In what follows we prove Theorem 1.6. Let $u_i, u \in W_0^{m,n/m}(\Omega)$ such that $\|u_i\|_{m,n,\frac{n}{m}} \leq 1$, $u \neq 0$ and $u_i \rightharpoonup u$ in $W_0^{m,n/m}(\Omega)$. Let $k \in \mathbb{N}$ such that $m = 2k+1$ or $m = 2k$. We take $\tilde{u}_i, \tilde{u} \in W^{m,n/m}(\mathbb{R}^n)$ given by

$$\tilde{u}_i(x) = \begin{cases} u_i(x), & \text{for } x \in \Omega \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad \tilde{u}(x) = \begin{cases} u(x), & \text{for } x \in \Omega \\ 0, & \text{otherwise} \end{cases}$$

and f_i, f given by $f_i(x) = (I-\Delta)^k \tilde{u}_i(x)$ and $f(x) = (I-\Delta)^k \tilde{u}$. Note that, when m is odd $f_i, f \in W^{1,n/m}(\mathbb{R}^n)$. By Lemma 2.5, since $u_i \rightharpoonup u$ in $W_0^{m,n/m}(\Omega)$, we have that $f_i \rightharpoonup f$ in $L^{n/m}(\mathbb{R}^n)$, for m even, and in $W^{1,n/m}(\mathbb{R}^n)$, for m odd.

Now we take $v_i \in W_{rad}^{m,n/m}(\mathbb{R}^n)$, given by

$$v_i = G_{2k} * f_i^*,$$

where f_i^* is the spherically symmetric decreasing rearrangement of f_i. Thus, from Proposition 5.2,

$$\int_\Omega e^{\beta_0 \gamma |u_i|^{n/(n-m)}}\, \mathrm{d}x = \int_{\mathbb{R}^n} e^{\beta_0 \gamma |\tilde{u}_i|^{n/(n-m)}}\, \mathrm{d}x = \int_{\mathbb{R}^n} e^{\beta_0 \gamma (\tilde{u}_i^*)^{n/(n-m)}}\, \mathrm{d}x \leq \int_{\mathbb{R}^n} e^{\beta_0 \gamma v_i^{n/(n-m)}}\, \mathrm{d}x.$$

Hence, by Lemma 7.1, $f_i^* \to g$ almost everywhere in $\mathbb{R}^n$ and $\|g\|_{n/m} \geq \|f^*\|_{n/m} = \|(I-\Delta)^k u\|_{n/m}$. Then, if we consider $\|\cdot\|$ the norm in $L^{n/m}(\mathbb{R}^n)$ for m even and in $W^{1,n/m}(\mathbb{R}^n)$ for m odd, by Brezis-Lieb's Lemma,

$$\|f_i^* - g\| \to c \leq 1 - \|g\|.$$

Hence, as γ satisfies

$$\gamma < \left(1 - \left\|(I-\Delta)^k u\right\|_{n/m}^{n/m}\right)^{-m/(n-m)} \leq (1 - \|g\|^{n/m})^{-m/(n-m)},$$

for $v \in W^{m,n/m}(\mathbb{R}^n)$ given by $v = G_{2k} * g$ we have that

$$\begin{aligned}
&\int_{\mathbb{R}^n} \Phi\left(\beta_0 \gamma v_i^{n/(n-m)}\right) \, \mathrm{d}x \\
&\quad \leq \int_{\mathbb{R}^n} \Phi\left(\beta_0 \gamma (1+\delta)^{n/(n-m)} (v_i - v)^{n/(n-m)} + \beta_0 \gamma (1+1/\delta)^{n/(n-m)} v^{n/(n-m)}\right) \, \mathrm{d}x \\
&\quad \leq \frac{1}{q} \left(\int_{\mathbb{R}^n} \Phi\left(\beta_0 \, \frac{(v_i - v)^{n/(n-m)}}{\|v_i - v\|_{m,n,\frac{n}{m}}^{n/(n-m)}} \right) \, \mathrm{d}x \right)^{1/q} + \frac{q-1}{q} \left(\int_{\mathbb{R}^n} \Phi\left(\beta_0 \gamma \tilde{q} v^{n/(n-m)}\right) \, \mathrm{d}x \right)^{1-1/q},
\end{aligned}$$

for i sufficiently large and some $\delta > 0$, $q > 1$, with $\tilde{q} = q'(1+1/\delta)^{n/(n-m)} n/m$. Then applying Theorem C and Theorem D the result follows.

Remark 7.2. *Here we indicated some important open questions related with the Adams type inequality on unbounded domains:*

i) Does the nonexistence of extremal result hold for inequalities (1.7) *and* (1.8) *when $p \neq 2$ and $\Omega = \mathbb{R}^n$? Should be mentioned that if $p \neq 2$ inequality* (1.3) *has always an extremal function, including the subcritical case.*

ii) Based in Theorem 1.6 the inequalities (1.7) *and* (1.8) *can be improved in sense of the result due to Adimurthi and O. Druet [4]? (see also [13] and [26])*

References

[1] Adachi, S.; Tanaka, K.: *Trudinger type inequalities in $\mathbb{R}^N$ and their best exponents*. Proc. Amer. Math. Soc. **128**, 2051–2057 (2000).

[2] Adams, D. R.: *A sharp inequality of J. Moser for higher order derivatives*. Ann. Math. **128**, 385–398 (1988).

[3] Adams, D. R.; Fournier, J. F.: *Sobolev spaces*. Second edition. Pure and Applied Mathematics, 140. Elsevier/Academic Press, Amsterdam, (2003).

[4] Adimurthi, Druet, O.: *Blow-up analysis in dimension 2 and a sharp form of Trudinger-Moser inequality*. Comm. Partial Differential Equations **29**, 295–322 (2004).

[5] Adimurthi; Struwe, M.: *Global Compactness properties of semilinear elliptic equations with critical exponential growth*. J. Functional Analysis **175**, 125–167 (2000).

[6] Berestycki, H.; Lions, P.-L.: *Nonlinear scalar field equations. I. Existence of a ground state.* Arch. Ration. Mech. Anal. **82**, 313–345 (1983).

[7] Bonheure, D.; Moreira dos Santos, E.; Ramos, M.: *Ground state and non-ground state solutions of some strongly coupled elliptic systems.* Trans. Amer. Math. Soc. **364**, 447–491 (2012).

[8] Cao, D. M.: *nontrivial solution of semilinear elliptic equation with critical exponent in* $\mathbb{R}^2$. Comm. Partial Differential Equations **17**, 407–435 (1992).

[9] Carleson, L., Chang, A.: *On the existence of an extremal function for an inequality of J. Moser*. Bull. sci. Math. **110**, 113–127 (1986).

[10] de Figueiredo, D. G.; do Ó, J. M.; Ruf, B.: *On an inequality by N. Trudinger and J. Moser and related elliptic equations*. Comm. Pure Appl. Math. **55**, 135–152 (2002).

[11] de Figueiredo, D. G.; do Ó, J. M.; Ruf, B.: *Elliptic Equations and Systems with critical Trudinger-Moser nonlinearities*. Discrete Contin. Dyn. Syst. **30**, 455–476 (2011).

[12] de Figueiredo, D. G.; Moreira, E.; Miyagaki, O. H.: *Sobolev spaces of symmetric functions and applications*. J. Functional Analysis **261** , no. 12, 3735–3770 (2011).

[13] de Souza, M.; do Ó, J. M.: *A sharp Trudinger-Moser type inequality in* $\mathbb{R}^2$, to appear in Transactions of the American Mathematical Society.

[14] do Ó, J. M.: *Semilinear Dirichlet problems for the N-Laplacian in* $\mathbb{R}^N$ *with nonlinearities in the critical growth range*. Differential Integral Equations **9**, 967–979 (1996).

[15] do Ó, J. M.: *n-Laplacian equations in* $\mathbb{R}^n$ *with critical growth*. Abstr. Appl. Anal. **2**, 301–315 (1997).

[16] do Ó, J. M.; Macedo, A. C.: *Concentration-compactness principle for an inequality by D. Adams*. Calc. Var. Partial Differential Equations (2013). doi: 10.1007/s00526-013-0671-z.

[17] Flucher, M.: *Extremal functions for the Trudinger-Moser inequality in 2 dimensions*. Comment. Math. Helv. **67**, 471–497 (1992).

[18] Ishiwata, M.: *Existence and nonexistence of maximizers for variational problems associated with Trudinger-Moser type inequalities in* $\mathbb{R}^n$. Math. Ann. **351**, 781–804 (2011).

[19] Ishiwata, M.; Nakamura, M.; Wadade, H.: *On the sharp constant for the weighted Trudinger-Moser type inequality of the scaling invariant form*. Ann. Inst. H. Poincaré Anal. Non Linéaire (2013).

[20] Kavian, O.: *Introduction à la Théorie des Points Critiques et Applications aux Problèmes elliptiques*. Mathèmatiques et Applications (Berlin), Vol. 13, Springer–Verlag, Paris, 1993. MR1276944 (95e:58036).

[21] Lam, N.; Lu, G.: *Sharp Adams type inequalities in Sobolev spaces $W^{m,n/m}(\mathbb{R}^n)$ for arbitrary integer m* . J. Differential Equations **253**, 1143–1171 (2012).

[22] Lam, N.; Lu, G.: *A new approach to sharp Moser-Trudinger and Adams type inequalities: A rearrangement-free argument*. J. Differential Equations **255**, 298–325 (2013).

[23] Li, Y.; Ruf, B.: *A sharp Trudinger-Moser type inequality for unbounded domains in $\mathbb{R}^n$*. Indiana Univ. Math. J. **57**, 451–480 (2008).

[24] Lin, K.: *Extremal functions for Moser's inequality*. Trans. Amer. Math. Soc. **348**, 2663–2671 (1996).

[25] Lions, P.-L.: *The concentration compactness principle in the calculus of variation, the limit case, part I*. Rev. Mat. Iberoam. **1**, 145–201 (1985).

[26] Lu, G.; Yang Y.: *Adams' inequalities for bi-Laplacian and extremal functions in dimension four*. Adv. Math. **220**, 1135–1170 (2009).

[27] Panda, R.: *On semilinear Neumann problems with critical growth for the $N-$Laplacian*. Nonlinear Anal. **26**, 1347–1366 (1996).

[28] Peetre, J.: *Espaces d'interpolation et théorème de Sobolev*. Ann. Inst. Fourier (Grenoble) **16**, 279–317 (1966).

[29] Pohozaev, S. I.: *The Sobolev embedding in the case $pl = n$*. Proceedings of the Technical Scientific Conference on Advances of Scientific Research 1964–1965. Mathematics Section, Moscov. Energet. Inst., 158–170 (1965).

[30] Moser, J.: *A sharp form of an inequality by N. Trudinger*. Indiana Univ. Math. J. **20**, 1077–1092 (1971).

[31] Ruf, B: *A sharp Trudinger-Moser type inequality for unbounded in $\mathbb{R}^2$*. J. Functional Analysis **219**, 340–367 (2005).

[32] Ruf, B; Sani, F.: *sharp Adams-Type Inequalities in $\mathbb{R}^n$*. Trans. Amer. Math. Soc. 3**65**, 645–670 (2013).

[33] Stein, E. M.: *Singular Integrals and Differentiability Properties of Functions*. Princeton Math. Ser., **30**, Princeton University Press (1970).

[34] Struwe, M.: *Critical points of embeddings of $H_0^{1,n}$ into Orlicz spaces*. Ann. Inst. H. Poincaré Anal. Non Linéaire **5**, 425–464 (1988).

[35] Struwe, M.: *Positive solution of critical semilinear elliptic equations on non-contractible planar domain*. J. Eur. Math. Soc. **2**, 100–126 (2000).

[36] Talenti, G.: *Elliptic equations and rearrangements*. Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **3**, 697–718 (1976).

[37] Tarsi, C.: *Adams' inequality and limiting Sobolev embeddings into Zygmund spaces*. Potential Anal. **37**, 353–385 (2012).

[38] Trombetti, G.; Vázquez, J. L.: Ann. Fac. Sci. Toulouse Math. (5) **7** . 2, 137–150 (1985).

[39] Trudinger, N. S.: *On imbeddings into Orlicz spaces and some applications*. J. Math. Mech. **17**, 473–483 (1967).

[40] Weinberger, H.: *Symmetrization in uniformly elliptic problems*. Studies in Math. Anal, Stanford Univ. Press, 424–428 (1962).

[41] Yudovich, V. I.: *Some estimates connected with integral operators and with solutions of elliptic equations*. Dok. Akad. Nauk SSSR **138**, 804–808 (1961) [English translation in Soviet Math. Doklady 2, 746–749 (1961)].