Project Proposal

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Title of proposal: Semi-classical States of Nonlinear Elliptic Equations and Systems

Abstract

We are concerned with the nonlinear Schrodinger equation

$$(1) \qquad -\varepsilon^2\Delta u + V(x)u = f(u), \quad u \in H^1(R^N)$$

and the coupled Schrodinger system

$$(2) \qquad \begin{cases} -\varepsilon^2\Delta u + a(x)u = \mu_1 u^3 + \beta u v^2, x \in R^3, \\ -\varepsilon^2\Delta v + b(x)v = \mu_1 v^3 + \beta v u^2, x \in R^3. \end{cases}$$

By using variational methods, we shall consider the semi-classical states of problem (1) and (2). More precisely, the object is to establish the existence of solutions highly concentrated near a certain number of points.

Research Background

Equation (1) and (2) appear in the theory of Bose-Einstein condensates, nonlinear optics, etc. In recent years, considerable attentions have been paid to problem (1) and (2). From the physical point of view, one of the most interesting solutions of problem (1) and (2) should be of finite energy. Such solutions are called bound states. Furthermore, to describe the transition from quantum mechanics to classical mechanics, ones consider bound states of (1) and (2) for small ε, which are referred to as semi-classical states. In this art, we mainly consider the existence of spikes of problem (1) and (2) around the non-minimal critical points of the potential.

Floer and Weinstein [1] firstly considered the existence of single peak solutions to (1) for $N=1, f(s)=s^3$. They constructed a single peak solution which concentrates around any given non-degenerate critical point of V. Subsequently, Oh [2] generalized this result to the higher dimension. Their arguments are based on a Lyapunov-Schmidt reduction, where it is essential to require a non-degeneracy condition (ND) on a ground state solution for a limiting problem. However, the non-degeneracy condition is, in general, rather difficult to prove. So far, only a very restricted class of f had been founded to have such property. In the last decade, many people have made efforts to relax or remove the non-degeneracy condition in the singularly perturbed problems. Without (ND), Rabinowitz [3] proved the existence of positive solutions to (1) for small ε. Later, del Pino and Felmer [4] developed the variational approach initiated by Rabinowitz and proved, by using a penalization approach, the existence of single-peak solutions which concentrate around the local minimum points of V. But in the works above, the following assumptions were required:

$$(\mathrm{AR}): 0 < \mu F(s) \le s f(s),\ s>0,\ \mu>2 \text{ and } (\mathrm{H}): \frac{f(s)}{s} \text{is non-decreasing on } (0,\infty).$$

Recently, some people have tried to weaken or eliminate the assumption (AR) or (H). In this direction, Byeon and Jeanjean [5] developed a new variational approach. By using a localized deformation argument, the authors proved that problem (1) has a spike solution around the local minimum points of V if f satisfies the Berestycki-Lions conditions. In [4-5], the condition (ND) was not needed. With the assumption (AR), P. d'Avenia, A. Pomponio and D. Ruiz [6] developed a min-max argument to establish the existence of spikes around the saddle points of the potential. Through all these works above, the assumption $V_0 = \inf_{x\in R^N} V(x) > 0$ was imposed. It is easy to see that there exists no positive solution of (1) for small ε if $V_0 < 0$. The case $V_0 = 0$ is called the critical frequency. In the case $V_0 = 0$, the semi-classical states have attracted many mathematician's attention, such as Ambrosetti, Malchiodi, Byeon, Van Schaftingen, Moroz, Wang, Cao, Peng, Yan, etc. Many interesting results have been obtained, but the nonlinear term f usually satisfies (AR) or (H). For more general nonlinearity f, Bae and Byeon [7] obtained the existence of spikes of (1) around the positive local minima of V. In [7] f only satisfies Berestycki-Lions conditions and other suitable conditions. In [8], J. Di Cosmo and J. Van Schaftingen obtained solutions of (1) which concentrate at local maxima of the potential V in the critical frequency case. But in [8] only the nonlinearities like $f(s)=s^p$ were considered. A natural open problem which has not been settled so far is that in the critical frequency case, whether there exist spikes around the local maxima of V for more general nonlinearity f.

System (2) has been studied by many experts, such as Wei, Pomponio, Montefusco, Pellacci, Squassina, Tanaka, Wang, Peng, Zou, etc. In [9], with the suitable asymptotic behaviours of a,b at infinity, Lin and Wei established the existence of a least energy nontrivial vector solution of (2) for $\beta>0$ and small ε. In [10], Montefusco, Pellacci, and Squassina showed that, if a,b are strictly positive and satisfy some local potential well conditions, system (2) has a non-zero solution for $\beta>0$ and small ε. In [11], Ikoma and Tanaka constructed a family of solutions of (2) which concentrates around the local minima of m, where for $P\in R^N$, $m(p)$ denotes the least energy level for non-trivial vector solutions of the rescaled "limit" problem

$$(3)\qquad \begin{cases}-\varepsilon^2\Delta u+a(P)u=\mu_1u^3+\beta uv^2, x\in R^3,\\ -\varepsilon^2\Delta v+b(P)v=\mu_1v^3+\beta vu^2, x\in R^3.\end{cases}$$

Recently, Chen and Zou [12] extended the result in [11] to the critical frequency case. A natural question arises that whether there exist spikes of system (2) around the local maxima of m.

Research Objectives

1. In the critical frequency case, i.e., $V_0=0$, we study the semi-classical states of (1) around the local maxima of V. The nonlinearity f is more general, i.e., (AR) and (H) are not needed. Moreover, we also consider the semi-classical states of (1) involving critical growth.
2. We consider the existence of the non-trivial vector solutions to (2) for ε small and study the concentration phenomenon around the local maxima of m.

Expected Results

- To obtain the existence of spikes of (1) around the local maxima of V in the case $V_0=0$.
- To establish the concentration phenomenon of the non-trivial vector solutions to (2) around the local maxima of m.

Research Plan

In this art, we will use variational approach to study the existence of the semi-classical states of (1) and (2) around the non-minima of V.

For Objective 1, we consider the existence of spikes of (1) around the local maxima of V. In [6], the authors established the concentration phenomenon around the saddle points of V, but the restriction: $V_0=\inf_{x\in R^N}V(x)>0$ is essential in the proof. In [8], the authors obtained the existence of spikes (1) around the local maxima of V in the case : $V_0=0$, but the proof heavy relies on the special nonlinearity $f(s)=s^p$. So for the more general nonlinear term and the critical frequency case, the methods in [6,8] are not applicable. The diffculty is third-fold: firstly, since the potential V may vanish or have a compact support, the energy functional does not belong to C^1 and the usual variational methods can not be used directly. This diffculty can be overcome by the variational penalization method devised by del Pino and Felmer in [4]. Then we consider the truncated problem of (1). By Hardy's inequality, we can show that the modified energy functional is well defined and of C^1. Secondly, the assumption (AR) is usually to guarantee the boundedness of (PS)-sequence. Without (AR), the (PS)-condition is not easy to verify. We will use the local deformation argument introduced in [5] to seek the critical point of the modified energy function in the small neighbourhood of approximate solutions. Thirdly, the lower estimates on the gradient of the energy functional will play a crucial role in the deformation argument. In contrast with the case: the local minima of V , the lower estimates become rather more complicated in the case: the local maxima . We will use an almost center of mass to construct some neighbourhood of approximate solutions, where the lower estimate of the gradient can be obtained. A similar argument also can be founded in [13]. Finally, by the elliptic estimates, we can show that the solutions of the truncated problem exactly solve the original problem (1).

For Objective 2, by using a mini-max approach, we consider the semi-classical states of system (2) around the local maxima of m . Firstly, in [11,12], the characteristic of local minima of m makes it rather easy to get a lower estimate of a mini-max value. But in the case: local maxima of m , the problem will become more thorny and tough. To bypass this obstacle, we use a barycenter type map to define a set of mountain paths , whose their centers always locate at the origin. Then we can define another mini-max value and obtain the lower estimate desired. By comparing the two mini-max values, the desired lower estimate is obtained. Similar arguments also can be founded in [6] to deal with problem (1). Secondly, to guarantee the concentration around the local maxima of m , we also use the variational penalization method devised by del Pino and Felmer in [4]. Finally, by the elliptic estimates, the solutions obtained actually solve the original problem (2).

Impact

By starting this proposal, we can explore what are the essential features which guarantee the existence of spikes around the local maxima. In this proposal, the curent methods can not be used directly in our problems. For system (2), due to the couple term, there is a fundamental difference between the single equation and the system. Thus, to complete this proposal, some original ideas are needed. This will promote the ability of research.

References

[1] A. Floer and A. Weinstein, Nonspreading wave packets for the cubic Schrodinger equations with a bounded potential, J. Funct. Anal., 69(1986), 397-408.

[2] Y. G. Oh, Existence of semiclassical bound states of nonlinear Schrodinger equations with potentials of the class V, Comm.Partial Differential Equations, 13(1988), 1499-1519.

[3] P. H. Rabinowitz, On a class of nonlinear Schrodinger equations, Z. Angew. Math. Phys., 43(1992), 270-291.

[4] M. del Pino and P. Felmer, Local mountain passes for semilinear elliptic problems in unbounded domains, Calc. Var. Partial Differential Equations, 4(1996), 121-137.

[5] J. Byeon and L. Jeanjean, Standing waves for nonlinear Schrodinger equations with a general nonlinearity, Arch. Ration. Mech. Anal., 185(2007),185-200.

[6] P. d'Avenia, A. Pomponio and D. Ruiz, Semiclassical states for the nonlinear Schrodinger equation on saddle points of the potential via variational methods, J. Funct. Anal., 262(2012), 4600-4633.

[7] S. Bae and J. Byeon, Standing waves of nonlinear Schrodinger equations with optimal conditions for potential and nonlinearity, Comm. Pure Appl.Anal., 12(2013), 831-850.

[8] J. Di Cosmo and J. Van Schaftingen, Stationary solutions of the nonlinear Schrodinger equation with fast-decay potentials concentrating around local maxima, Calc. Var. Partial Differential Equations, 47(2013), 121-137.

[9] T. Lin and J. Wei, Spikes in two-component systems of nonlinear Schrodinger equations with trapping potentials, J. Differ. Equations 229, 538–569 (2006).

[10]E. Montefusco, B. Pellacci, and M. Squassina, Semiclassical states for weakly coupled nonlinear Schrodinger systems, J. Eur. Math. Soc. 10, 47–71 (2007).

[11] N. Ikoma and K. Tanaka, A local mountain pass type result for a system of nonlinear Schrodinger equations, Calc. Var. PDE. 40(2011), 449–480 .

[12] Z. Chen and W. Zou, Standing waves for coupled nonlinear Schrodinger equations with decaying potentials, Journal of Mathematical Physics 54(2013), 111505 .

[13] J. Byeon and K. Tanaka, Semi-classical standing waves for nonlinear Schrodinger equations at structurally stable critical points of the potential, J. Eur. Math. Soc., 15(2013), 1859-1899.