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RODRIGO GENUINO CLEMENTE

Some classes of elliptic problems with singular nonlinearities

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Tese apresentada ao Programa de Pós-graduação em Matemática da Universidade Federal de Pernambuco como requisito parcial para obtenção do título de Doutor em Matemática.

Orientador: Prof. Dr. **JOÃO MARCOS BEZERRA DO Ó**

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SOME CLASSES OF ELLIPTIC PROBLEMS WITH SINGULAR NONLINEARITIES

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À minha família.

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Abstract

Singular elliptic problems has been extensively studied and it has attracted the attention of many research in various contexts and applications. The purpose of this thesis is to study singular elliptic problems in riemannian manifolds. We investigate a semilinear elliptic problem involving singular nonlinearities and advection and we prove the existence of a parameter $\lambda^* > 0$ such that for $\lambda \in (0, \lambda^*)$ there exists a minimal classical solution which is semi-stable and for $\lambda > \lambda^*$ there are no solutions of any kind. Futhermore we obtain L^p estimates for minimal solutions uniformly in λ and determine the critical dimension for this class of problems. As an application, we prove that the extremal solution is classical whenever the dimension of the riemannian manifold is below the critical dimension. We analyse the branch of minimal solutions and we prove multiplicity of solutions close to extremal parameter. We also prove symmetry and monotonicity properties for the class of semi-stable solutions and we obtain L^∞ estimates for the extremal solution. Moreover, we study a class of problems involving the $p-$Laplace Beltrami operator in a geodesic ball of a riemannian model and we establish L^∞ and $W^{1,p}$ estimates for semi-stable, radially symmetric and decreasing solutions. As an application we prove regularity results for extremal solution of a quasilinear elliptic problem with Dirichlet boundary conditions. In the last chapter we study an elliptic system and we prove the existence of a curve which splits the positive quadrant of the plane into two disjoint sets, where there is classical solution while in the other there is no solution. We establish upper and lower estimates for the critical curve and regularity results for solutions on this curve.

Keywords: Nonlinear elliptic problems. Quasilinear elliptic problems. Nonlinear elliptic systems. Gelfand-Liouville problems. MEMS problems. Extremal solution. Singular solution. Stability. Advection. Semi-stable solutions. Regularity.

Resumo

Problemas elípticos singulares têm sido extensivamente estudados nas últimas décadas. Nesta tese, abordamos classes de problemas não lineares modelados em variedades riemannianas. Investigamos inicialmente um problema elíptico semilinear envolvendo não linearidades singulares e advecção e provamos resultados de existência do parâmetro extremal $\lambda^* > 0$ tal que para $\lambda \in (0, \lambda^*)$ existe uma solução minimal clássica a qual é semiestável e para $\lambda > \lambda^*$ não existem soluções de nenhum tipo. Além disso, obtivemos estimativas L^p para as soluções minimais que são uniformes em λ e determinamos as dimensões críticas para esta classe de problemas. Como uma aplicação, provamos a regularidade da solução extremal quando a dimensão da variedade riemanniana está abaixo da dimensão crítica. Analisamos o ramo das soluções minimais e provamos multiplicidade de soluções próximo do λ^*. Provamos também simetria e monotonicidade para a classe das soluções semiestáveis e provamos estimativas L^∞ para a solução extremal. Estudamos também uma classe de equações envolvendo o operador $p-$Laplace Beltrami em uma bola geodésica de uma variedade Riemanniana modelo e estabelecemos estimativas L^∞ e $W^{1,p}$ para soluções semiestáveis, radialmente simétricas e decrescentes. Como aplicação, provamos resultados de regularidade para soluções extremais para um problema quasilinear com condição de fronteira de Dirichlet. No último capítulo, estudamos um sistema elíptico e provamos a existência de uma curva que divide o primeiro quadrante do plano em dois conjuntos disjuntos, um dos quais existe solução clássica enquanto que no outro não existe solução. Estabelecemos estimativas superiores e inferiores para tal curva e resultados de regularidade para soluções sobre a curva.

Palavras-chave: Problemas elípticos não lineares. Problemas elípticos quasilineares. Sistemas elípticos não lineares. Problemas tipo Gelfand-Liouville. Problemas tipo MEMS. Solução extremal. Solução singular. Estabilidade. Advecção. Soluções semiestáveis. Regularidade.

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1 Introduction

Singular elliptic problems of the form

$$-\Delta u = \lambda g(u) \tag{1}$$

defined on smooth bounded domain Ω of $\mathbb{R}^N$ with zero Dirichlet boundary condition where g is smooth, increasing, convex with $g(0) = 1$ and satisfying

$$\lim_{u\to+\infty} \frac{g(u)}{u} = +\infty \tag{H_1}$$

or

$$\lim_{u\to+\infty} g(u) = +\infty \tag{H_2}$$

has been extensively studied and it has attracted the attention of many research in various contexts and applications (1, 2, 3, 4, 5, 6, 7). For example, Liouville (8) consider this equation when $g(u) = \mathrm{e}^u$, or more commonly Gelfand's problem (9), in connection to surfaces with constant Gauss curvature. In physics, the exponencial nonlinearity appears in connection with the structure of stars and equilibrium of gas spheres (10, 11, 12).

A general objective concerning equation (1) with zero Dirichlet boundary condition for nonlinearities of the form (H_1) and (H_2) is to study the existence and qualitative properties of singular solutions. A basic result shared by these problems is that there exist a critical parameter $\lambda^* > 0$ such that (1) admits positive solutions for $0 < \lambda < \lambda^*$, while no positive solutions exist for $\lambda > \lambda^*$. We call λ^* the extremal parameter. Later, H. Brezis and J. L. Vazquez (3) treated the delicate issue of regularity of solutions of (1) at $\lambda = \lambda^*$. In particular, for the nonlinearities mentioned above, the solutions at the critical parameter λ^* are uniformly bounded in lower dimensions, while in higher dimensions they are unbounded.

For more general nonlinearities $g(s)$ satisfying (H_1), regularity of solutions of (1) at $\lambda = \lambda^*$ has been established by G. Nedev (13). Precisely, he proved that the extremal solution belongs to $L^\infty(\Omega)$ if $N \leq 3$, while it belongs to $H_0^1(\Omega)$ if $N \leq 5$, for every bounded domain Ω. After that X. Cabré (14) proved regularity when $N \leq 4$ assuming conditions contained in (H_1) for convex and bounded domain Ω. Under the same assumptions, X. Cabré and M. Sanchón (15) completed the analysis of regularity for dimensions $5 \leq N \leq 9$. We have to call attention that in (14, 15) they make no assumption of convexity on the nonlinearity, but in contrast with Nedev's result, they assume Ω to be convex.

Note that in MEMS case, i.e. when $g(u) = 1/(1-u^2)$, it is sufficient in view of standard elliptic regularity theory to show that $\sup u^* < 1$. Some results related to this question in a bounded domain of $\mathbb{R}^N$ was obtain by N. Ghoussoub and Y. Guo (16), where they proved regularity for the extremal solution when $N \leq 7$. Here we give some historical background on the MEMs problems, for a complete study on this subject we refer the reader to (17, 18). Micro-electromechanical systems (MEMS) are often used to combine electronics with microsize mechanical devices in the design of various types of microscopic machinery. MEMS devices have therefore become key components of many commercial systems, including accelerometers for airbag deployment in automobiles, ink jet printer heads, optical switches and chemical sensors.

Consider now the quasilinear elliptic problem with $p > 1$

$$\begin{cases} -\Delta_p u = \lambda f(u) & \text{in} \quad \Omega, \\ \quad u = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{2}$$

In (19, 20), I. Peral, J. Puel and J. García-Azorero obtained optimal bounds for the extremal solution for $g(u) = \mathrm{e}^u$ when $p > 1$. In (21), using certain assumptions on f, the authors proved that every semi-stable solution is bounded for a explicit exponent which is optimal for the boundedness of semi-stable solutions and, in particular, it is bigger than the critical Sobolev exponent $p^* - 1$. For general f and $p > 1$ the interested reader can consult (4, 22, 23, 24, 25) for more regularity results about the extremal solution. In (22), X. Cabré, A. Capella and M. Sanchón treated the delicate issue about regularity of extremal solutions u^* of (2) at $\lambda = \lambda^*$ when Ω is the unit ball of $\mathbb{R}^N$. Among other results, they established pointwise, L^q and $W^{1,q}$ estimates which are optimal and do not depend on the nonlinearity f namely,

(i) If $N < p + 4p/(p-1)$ then $u^* \in L^\infty$.

(ii) If $N = p + 4p/(p-1)$ then $u^* \in L^q$ for all $q < +\infty$.

(iii) If $q < Np/(N - 2\sqrt{(N-1)/(p-1)})$ then $u^* \in W^{1,q}(B_1)$.

When Ω is a smooth domain of a Riemannian manifold $(\mathcal{M}, g)$, a recent work due D. Castorina and M. Sanchón (26) proved qualitative properties for semi-stable solutions and they established L^∞, L^q and $W^{1,q}$ estimates which do not depends on the nonlinearity g. Futhermore, the authors established regularity results for the extremal solution for exponencial and power nonlinearities. A similar setting has been considered, in the case $p = 2$, by E. Berchio, A. Ferrero and G. Grillo (27) in order to study qualitative properties of radial solutions when $g(u) = |u|^{m-1}u$ with $m > 1$ on certain classes of Cartan–Hadamard manifolds and by (28) that studied existence and uniqueness of positive radial solutions of

$$\begin{cases} \Delta_g u + \lambda u + u^p = 0 & \text{in} \quad \mathcal{A}, \\ u = 0 & \text{on} \quad \partial\mathcal{A}, \end{cases}$$

with $\lambda < 0$, $\mathcal{A}$ being an annular domain in a certain class of Riemannian manifold $\mathcal{M}$ and Δ_g denotes the Laplace-Beltrami operator.

A natural generalization of problem (1) is to consider the presence of advection term. In engineering and physics, advection is a conserved property by a fluid due to the fluid's motion or a transport mechanism of a substance. The fluid's motion is described mathematically as a vector field and the transported material is described by a scalar field giving its distribution over space. For more informations about advection and applications to partial differential equations the interested reader can consult (29). In Chapter 3 we study the following class of semilinear elliptic problems with singular nonlinearities and advection

$$\begin{cases} \mathcal{L}(u) := -\Delta_g u + A(x) \cdot \nabla_g u = \lambda f(u) & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{P_λ}$$

on a smooth bounded domain Ω of a complete Riemannian manifold $(\mathcal{M}, g)$ with zero Dirichlet boundary condition, where A is a smooth vector field. We prove the existence of $\lambda^* = \lambda^*(N, \Omega, A) > 0$ such that for $\lambda \in (0, \lambda^*)$ there exists a minimal classical solution $\underline{u}_\lambda$ which is semi-stable, and for $\lambda > \lambda^*$ there are no solutions of any kind. Furthermore, we obtain L^p-estimates for $\underline{u}_\lambda$ uniformly in λ, and we determine the critical dimensions for this class of problems for singular nonlinearities of type MEMS, Gelfand and power case. As an application, we prove that the extremal solution $u^* := \lim_{\lambda \nearrow \lambda^*} \underline{u}_\lambda$ is classical whenever the dimension of $\mathcal{M}$ is below the critical dimension. Moreover, we analyze the branch of minimal solutions and we prove multiplicity of solutions when $\lambda \in (\lambda^* - \delta, \lambda^*)$ for some $\delta > 0$ and uniqueness at λ^*. In particular, when $\mathcal{M}$ is a Riemannian model and Ω is a geodesic ball of $\mathcal{M}$, we establish symmetry and monotonicity for the class of semi-stable solutions and we also prove L^∞-estimates for u^*.

The Chapter 4 is devoted to study the reaction-diffusion equation involving the p-

Laplace Beltrami operator on Riemannian manifolds of the form

$$-\operatorname{div}(|\nabla_g u|^{p-2}\nabla_g u) = h(u) \quad \text{in} \quad \mathcal{B}_1 \setminus \{\mathcal{O}\} \tag{3}$$

where $1 < p < +\infty$, $\mathcal{B}_1$ is a geodesic ball of a Riemannian model $\mathcal{M}$ with radius 1, $\mathcal{O}$ is a pole and h is a locally Lipschitz positive nonlinearity. In this way, we establish L^∞ and $W^{1,p}$ estimates for semi-stable, radially symmetric, and decreasing solutions of (3). Our result do not depend on the specific form of the nonlinearity, precisely, our L^∞ and $W^{1,p}$ estimates hold for every locally Lipschitz nonlinearity h. This may regarded as a result on removable singularities because u may be unbounded at the pole $\mathcal{O}$.

As an application of our estimates, we prove regularity results for the following quasilinear elliptic problem with Dirichlet boundary condition

$$\begin{cases} -\operatorname{div}(|\nabla_g u|^{p-2}\nabla_g u) = \lambda f(u) & \text{in} \quad \mathcal{B}_1, \\ u > 0 & \text{in} \quad \mathcal{B}_1, \\ u = 0 & \text{on} \quad \partial\mathcal{B}_1, \end{cases}$$

where $\lambda > 0$ and f is an increasing C^1 function with $f(0) > 0$ and

$$\lim_{t\to+\infty} \frac{f(t)}{t^{p-1}} = +\infty.$$

In Chapter 5, we analyse the Lane-Emden system

$$\begin{cases} -\Delta u = \dfrac{\lambda f(x)}{(1-v)^2} & \text{in} \quad \Omega \\ -\Delta v = \dfrac{\mu g(x)}{(1-u)^2} & \text{in} \quad \Omega \\ 0 \le u, v < 1 & \text{in} \quad \Omega \\ u = v = 0 & \text{on} \quad \partial\Omega \end{cases} \tag{$S_{\lambda,\mu}$}$$

where λ and μ are positive parameters and Ω is a smooth bounded domain of $\mathbb{R}^N$ $(N \ge 1)$. We prove the existence of a critical curve Γ which splits the positive quadrant of the (λ, μ)-plane into two disjoint sets $\mathcal{O}_1$ and $\mathcal{O}_2$ such that the problem $(S_{\lambda,\mu})$ has a smooth minimal stable solution (u_λ, v_μ) in $\mathcal{O}_1$, while for $(\lambda, \mu) \in \mathcal{O}_2$ there are no solutions of any kind. We also establish upper and lower estimates for the critical curve Γ and regularity results on this curve if $N \le 5$. Our proof is based on a delicate combination involving L^p estimates for semi-stable solutions of $(S_{\lambda,\mu})$.

2 Preliminaries

This section is devoted to present some basic facts used along the text. We start introducing the notions of smooth and Riemannian manifolds and Riemannian measure. For more details the interested reader can consult (30). Here after we use Einstein summation convention that implies summation over a set of index terms in a formula.

2.1 Riemannian manifolds

Definition 2.1. Let $\mathcal{M}$ a topological space. A N-dimensional chart on $\mathcal{M}$ is any couple (U, φ) where U is an open subset of $\mathcal{M}$ and φ is a homeomorphism of U onto an open subset of $\mathbb{R}^N$.

Definition 2.2. A manifold of dimension N is a Hausdorff topological space $\mathcal{M}$ with a countable base such that any point of $\mathcal{M}$ belongs to a N-dimensional chart.

Let $\mathcal{M}$ be a manifold of dimension N. For any chart (U, φ) on $\mathcal{M}$, the local coordinate system $x^1, ..., x^N$ is defined in U by taking the φ-pullback of the cartesian coordinate system in $\mathbb{R}^N$.

A family $\mathcal{A}$ of charts on a manifold is called a C^k-atlas if the charts from $\mathcal{A}$ covers all $\mathcal{M}$ and the change of coordinates in the intersection of any two charts from $\mathcal{A}$ is given by C^k-functions. Two C^k-atlases are said to be compatible if their union is again a C^k-atlas. The union of all compatible C^k-atlases determines a C^k-structure on $\mathcal{M}$.

Definition 2.3. A smooth manifold is a manifold endowed with a C^∞-structure.

Hereafter we always consider a manifold $\mathcal{M}$ as a smooth manifold.

Definition 2.4. A mapping $\xi : C^\infty(\mathcal{M}) \to \mathbb{R}$ is called an $\mathbb{R}$-differentiation at a point $x_0 \in \mathcal{M}$ if ξ is linear and

$$\xi(fg) = \xi(f)g(x_0) + \xi(g)f(x_0),$$

for all $f, g \in C^\infty(\mathcal{M})$.

The set of all $\mathbb{R}$-differentiations at x_0 is denoted by $T_{x_0}\mathcal{M}$. The linear space $T_{x_0}\mathcal{M}$ is called the tangent space of $\mathcal{M}$ at x_0.

Theorem 2.1. If $\mathcal{M}$ is a smooth manifold of dimension N then the tangent space $T_{x_0}\mathcal{M}$ is a linear space of the same dimension N.

A vector field on a smooth manifold $\mathcal{M}$ is a family $\{v(x)\}_{x\in\mathcal{M}}$ of tangent vectors such that $v(x) \in T_x\mathcal{M}$ for any $x \in \mathcal{M}$. In local coordinates, $x^1, ..., x^N$ it can be represented in the form

$$v(x) = v^i(x)\frac{\partial}{\partial x^i}.$$

The vector field $v(x)$ is called smooth if all the functions $v^i(x)$ are smooth in any chart.

Let $\mathcal{M}$ be a smooth N-dimensional manifold. A Riemannian metric on $\mathcal{M}$ is a family $g = \{g(x)\}_{x\in\mathcal{M}}$ such that $g(x)$ is a symmetric, positive definite, bilinear form on the tangent space $T_x\mathcal{M}$ smoothly depending on $x \in \mathcal{M}$.

Using the metric tensor, ones defines an inner product $\langle\cdot,\cdot\rangle_g$ in any tangent space $T_x\mathcal{M}$ by

$$\langle\xi,\eta\rangle_g \equiv g(x)(\xi,\eta),$$

for all tangent vectors $\xi, \eta \in T_x\mathcal{M}$. In the local coordinates $x^1, ..., x^N$ the inner product in $T_x\mathcal{M}$ has the form

$$\langle\xi,\eta\rangle_g = g_{i,j}(x)\xi^i\eta^j, \tag{4}$$

where $(g_{i,j}(x))_{i,j=1}^N$ is a symmetric positive definite $N \times N$ matrix. It follows from (4) that

$$g_{i,j} = \langle\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\rangle_g.$$

Definition 2.5. A Riemannian manifold is a couple $(\mathcal{M}, g)$ where g is a Riemannian metric on a smooth manifold $\mathcal{M}$.

Let $(\mathcal{M}, g)$ be a Riemannian manifold. The metric tensor g provides a canonical way of identifying the tangent space $T_x\mathcal{M}$ with the cotangent space $T_x^*\mathcal{M}$. Observe that if $\xi \neq 0$ then $g(x)\xi$ is also non-zero as covector. Therefore, the mapping

$$g(x) : T_x\mathcal{M} \to T_x^*\mathcal{M}$$

is bijective. Consequently, it has the inverse mapping

$$g^{-1}(x) : T_x^*\mathcal{M} \to T_x\mathcal{M},$$

whose components are denoted by $(g^{i,j})$ so that

$$\left(g^{i,j}\right) = (g_{i,j})^{-1}.$$

Hence, for any covector $w \in T_x^*\mathcal{M}$, $g^{-1}(x)w$ is a vector whose components are given by

$$w^i := \left(g^{-1}(x)w\right)^i = g^{i,j}w_j.$$

Obviously, g^{-1} can be considered as an inner product in $T_x^*\mathcal{M}$. It follows that, in the local coordinates

$$\langle v, w\rangle_{g^{-1}} = g^{i,j}v_iw_j$$

for all $v, w \in T_x^*\mathcal{M}$.

Definition 2.6. For any smooth function $f \in C^\infty(\mathcal{M})$, define its gradient $\nabla_g f(x)$ at a point $x \in \mathcal{M}$ by

$$\nabla_g f(x) = g^{-1}(x)\mathrm{d}f(x)$$

that is, $\nabla_g f(x)$ is a covector version of $\mathrm{d}f(x)$.

Remark 2.1. In the local coordinates $x^1, ..., x^N$ we obtain

$$(\nabla_g f)^i = g^{i,j}\frac{\partial f}{\partial x^j}.$$

Let $\mathcal{M}$ be a smooth manifold of dimension N. Let $\mathcal{B}(\mathcal{M})$ be the smallest σ-algebra containing all open sets in $\mathcal{M}$. The elements of $\mathcal{B}(\mathcal{M})$ are called Borel sets. We say that a set $E \subset \mathcal{M}$ is measurable if for any chart U, the set $\varphi(E \cap U)$ is a Lebesgue measurable set in $\varphi(U)$. Obviously, the family of all measurable sets in $\mathcal{M}$ forms a σ-algebra and we will denote by $\Lambda(\mathcal{M})$. We now show that any Riemannian manifold $(\mathcal{M}, g)$ features a canonical measure ν defined on $\Lambda(\mathcal{M})$ which is called Riemannian measure.

Theorem 2.2. For any Riemannian manifold $(\mathcal{M}, g)$ there exists a unique measure ν on $\Lambda(\mathcal{M})$ such that in any chart U,

$$\mathrm{d}\nu = \sqrt{\det g}\,\mathrm{d}\lambda$$

where $g = (g_{i,j})$ is the matrix of the Riemannian metric g in U and λ is the Lebesgue measure in U. Furthermore, the measure ν is complete, $\nu(K) < \infty$ for any compact set $K \subset \mathcal{M}$, $\nu(\Omega) > 0$ for any non-empty open set $\Omega \subset \mathcal{M}$ and ν is regular in the following sense: for any set $A \in \Lambda(\mathcal{M})$

$$\nu(A) = \sup\left\{\nu(K) : K \subset A, K \text{ compact}\right\}$$

and

$$\nu(A) = \inf\left\{\nu(U) : A \subset U, U \text{ open}\right\}.$$

For any smooth vector field $v(x)$ on a Riemannian manifold $(\mathcal{M}, g)$, its divergence $\mathrm{div}v(x)$ is a smooth function on $\mathcal{M}$ defined by the following statement.

Definition 2.7. For any C^∞-vector field $v(x)$ on a Riemannian manifold $\mathcal{M}$, there exists a unique smooth function on $\mathcal{M}$ denoted by $\mathrm{div}v$ such that the following identity holds

$$\int_{\mathcal{M}} (\mathrm{div}v)u \, \mathrm{d}\nu = -\int_{\mathcal{M}} \langle v, \nabla_g u\rangle_g \, \mathrm{d}\nu$$

for all $u \in C_0^\infty(\mathcal{M})$.

We can see that the divergence in local coordinates can be defined by

$$\mathrm{div}v = \frac{1}{\sqrt{\det g}} \frac{\partial}{\partial x^k} \left(\sqrt{\det g} v^k\right).$$

Having defined gradient and divergence, we can define the Laplace-Beltrami operator on any Riemannian manifold $(\mathcal{M}, g)$ as follows

$$\Delta_g = \mathrm{div} \circ \nabla_g.$$

In local coordinates we have

$$\Delta_g = \frac{1}{\sqrt{\det g}} \frac{\partial}{\partial x^i} \left(\sqrt{\det g} g^{i,j} \frac{\partial}{\partial x^j}\right),$$

where $g = (g_{i,j})$.

Theorem 2.3 (Green formula)**.** If u and v are smooth functions on a Riemannian manifold $\mathcal{M}$ and one of then has a compact support then

$$\int_{\mathcal{M}} u\Delta_g v \, \mathrm{d}\nu = -\int_{\mathcal{M}} \nabla_g u \cdot \nabla_g v \, \mathrm{d}\nu = \int_{\mathcal{M}} v\Delta_g u \, \mathrm{d}\nu.$$

2.2 Polar coordinates and Model manifolds

In $\mathbb{R}^N$, $N \geq 2$, every point $x \neq 0$ can be represented in the polar coordinates as a couple (r, ϑ) where $r := |x| > 0$ is the polar radius and $\vartheta = \frac{x}{|x|} \in \mathbb{S}^N$ is the polar angle. We already know that the canonical Euclidean metric $g_{\mathbb{R}^N}$ has the following representation in polar coordinates:

$$g_{\mathbb{R}^N} = \mathrm{d}r^2 + r^2 g_{\mathbb{S}^{N-1}},$$

where $g_{\mathbb{S}^{N-1}}$ is the canonical spherical metric.

For sphere, we consider now the polar coordinates on $\mathbb{S}^N$. Let p be the north pole and q be the south pole of $\mathbb{S}^N$. We can use the stereographic projection to prove that the canonical spherical metric $g_{\mathbb{S}^N}$ has the following representation in polar coordinates:

$$g_{\mathbb{S}^N} = \mathrm{d}r^2 + \sin^2 r g_{\mathbb{S}^{N-1}}.$$

The hyperbolic space $\mathbb{H}^N$, $N \geq 2$, is defined as follows. Consider in $\mathbb{R}^{N+1}$ a hyperboloid H given by the equation $(x^{N+1})^2 - (x')^2 = 1$, where $x' = (x^1, ..., x^N) \in \mathbb{R}^N$ and $x^{N+1} > 0$. This implies that $\mathbb{H}^N$ is a submanifold of $\mathbb{R}^{N+1}$ of dimension N. Consider in $\mathbb{R}^{N+1}$ the Minkowski metric

$$g_{Mink} = (\mathrm{d}x^1)^2 + ... + (\mathrm{d}x^N)^2 - (\mathrm{d}x^{N+1})^2$$

which is a bilinear symmetric form but not positive definite. Let $g_{\mathbb{H}^N}$ be the restriction of the tensor g_{Mink} to $\mathbb{H}$. With this, $(\mathbb{H}, g_{\mathbb{H}^N})$ is a Riemannian manifold and $g_{\mathbb{H}^N}$ is called the canonical hyperbolic metric, which has the following representation in the polar coordinates:

$$g_{\mathbb{H}^N} = \mathrm{d}r^2 + \sinh^2 r g_{\mathbb{S}^{N-1}}.$$

Definition 2.8. An N-dimensional Riemannian manifold $(\mathcal{M}, g)$ is called a Riemannian model if:

(i) There is a chart on $\mathcal{M}$ that covers all $\mathcal{M}$, and the image of this chart in $\mathbb{R}^N$ is a ball $B_{r_0} := \left\{x \in \mathbb{R}^N : |x| < r_0\right\}$ of radius $r_0 \in (0, +\infty]$.

(ii) The metric g in the polar coordinates (r, ϑ) in the above chart has the form

$$g = \mathrm{d}r^2 + \psi^2(r) g_{\mathbb{S}^{N-1}},$$

where $\psi(r)$ is a smooth positive funtion on $(0, r_0)$.

Remark 2.2. The number r_0 is called the radius of the model $\mathcal{M}$.

Lemma 2.1. On a model manifold $(\mathcal{M}, g)$, the Riemannian measure ν is given in the polar coordinates by

$$\mathrm{d}\nu = \psi^{N-1} \mathrm{d}r \mathrm{d}\vartheta,$$

where $\mathrm{d}\vartheta$ stands for the Riemannian measure on $\mathbb{S}^{N-1}$, and the Laplace operator on $(\mathcal{M}, g)$ has a form

$$\Delta_g = \frac{\partial^2}{\partial r^2} + \left(\frac{\mathrm{d}}{\mathrm{d}r} \log \psi^{N-1}\right) \frac{\mathrm{d}}{\mathrm{d}r} + \frac{1}{\psi^2(r)} \Delta_{\mathbb{S}^{N-1}}.$$

2.3 Geodesic distance

Definition 2.9. Let $\mathcal{M}$ be a smooth manifold. A path on $\mathcal{M}$ is any continuous mapping $\gamma : (a, b) \to \mathcal{M}$ where $-\infty \leq a < b \leq +\infty$.

Definition 2.10. For any smooth path $\gamma(t)$, its velocity $\gamma'(t)$ is an $\mathbb{R}$-differentiation at the point $\gamma(t)$ defined by

$$\gamma'(t)(f) = (f \circ \gamma)'(t) \text{ for all } f \in C^\infty(\mathcal{M}).$$

Recall that the the lenght of a tangent vector $\xi \in T_x\mathcal{M}$ is defined by $|\xi| = \sqrt{\langle \xi, \xi \rangle_g}$. For any smooth path $\gamma : (a, b) \to \mathcal{M}$, its lenght $l(\gamma)$ is defined by

$$l(\gamma) = \int_a^b |\gamma'(t)| \mathrm{d}t.$$

We use the paths to define a distance function on the manifold $(\mathcal{M}, g)$. We say that a path $\gamma : [a, b] \to \mathcal{M}$ connects points x and y if $\gamma(a) = x$ and $\gamma(b) = y$. The geodesic distance $\mathrm{d}(x, y)$ between points $x, y \in \mathcal{M}$ is defined by

$$\mathrm{d}(x, y) = \inf_\gamma l(\gamma),$$

where the infimum is taken over all smooth paths connecting x and y. If the infimum is attained on a path γ then γ is called a shortest (or a minimizing) geodesic between x and y.

Clain: The geodesic distance satisfies the following properties.

(i) $\mathrm{d}(x, y) \in [0, +\infty]$ and $\mathrm{d}(x, x) = 0$.

(ii) $\mathrm{d}(x, y) = \mathrm{d}(y, x)$.

(iii) $\mathrm{d}(x, y) \leq \mathrm{d}(x, z) + \mathrm{d}(y, z)$.

This implies that the geometric distance is a metric on $\mathcal{M}$. Futhermore, the topology of the metric space $(\mathcal{M}, \mathrm{d})$ coincides with the original topology of the smooth manifold $\mathcal{M}$.

2.4 Sobolev spaces on manifolds

For any manifold $\mathcal{M}$, define the space of test functions $\mathcal{D}(\mathcal{M})$ as $C_0^\infty(\mathcal{M})$ with the following convergence: $\phi_k \to \phi$ if

(i) In any chart U and for any multiindex α, $\partial^\alpha \phi_k$ converges uniformly to $\partial^\alpha \phi$ in U.

(ii) All supports $\operatorname{supp} \phi_k$ are contained in a compact subset of $\mathcal{M}$.

A distribution is a continuous linear functional on $\mathcal{D}(\mathcal{M})$. The set of all distributions $\mathcal{D}'(\mathcal{M})$ is a linear space. The convergence in $\mathcal{D}'(\mathcal{M})$ is defined as follows: $u_k \to u$ if $(u_k, \phi) \to (u, \phi)$ for all $\phi \in \mathcal{D}(\mathcal{M})$.

We would like to identify a function on $\mathcal{M}$ as a distribution and for this we need a measure on $\mathcal{M}$. Assume that $(\mathcal{M}, g, \mu)$ is a Riemannian manifold endowed with the measure μ. The notion of measurable and integrable functions are defined as well as the Lebesgue function space $L^p(\mathcal{M}) = L^p(\mathcal{M}, \mu)$, $1 \leq p \leq +\infty$. Note that $L^p(\mathcal{M})$ are Banach spaces and $L^2(\mathcal{M})$ is a Hilbert space. We also know that if $1 \leq p < +\infty$ then $L^p(\mathcal{M})$ is separable and $\mathcal{D}(\mathcal{M})$ is dense in $L^p(\mathcal{M})$.

Let $\overrightarrow{\mathcal{D}}(\mathcal{M})$ be the space of all smooth vector fields on $\mathcal{M}$ with compact supports endowed with the convergence similar to that in $\mathcal{D}(\mathcal{M})$. The elements of the dual space $\overrightarrow{\mathcal{D}'}(\mathcal{M})$ are called distributional vector fields. The convergence in $\overrightarrow{\mathcal{D}'}(\mathcal{M})$ is defined in the same way as in $\mathcal{D}'(\mathcal{M})$.

A vector field v on $\mathcal{M}$ is called measurable if all its components in any chart are measurable functions. By definition, the space $\overrightarrow{L^p}(\mathcal{M})$ consists of (the equivalence class of) measurable vector fields v such that $|v| \in L^p(\mathcal{M})$ (where $|v| = \langle v, v \rangle_g^{1/2}$ is the lenght of v). The norm in $\overrightarrow{L^p}(\mathcal{M})$ is defined by $\|v\|_{\overrightarrow{L^p}} := \||v|\|_{L^p}$. The space $\overrightarrow{L^p}$ are also complete.

For any distribution $u \in \mathcal{D}'(\mathcal{M})$, define its distributional gradient $\nabla_g u \in \overrightarrow{\mathcal{D}}(\mathcal{M})$ by means of the identity

$$(\nabla u, \psi) = -(u, \operatorname{div}_\mu \psi) \text{ for all } \psi \in \overrightarrow{\mathcal{D}}(\mathcal{M}).$$

Define the following Sobolev space

$$W^{1,p}(\mathcal{M}) = W^{1,p}(\mathcal{M}, g, \mu) := \left\{ u \in L^p(\mathcal{M}) : \nabla u \in \overrightarrow{L^p}(\mathcal{M}) \right\}.$$

It is easy to see that $W^{1,p}$ is a linear normed space endowed with the norm

$$\|u\|_{W^{1,p}} = \|u\|_{L^p} + \|\nabla u\|_{\overrightarrow{L^p}}.$$

We are going to define Sobolev spaces of integer order on a Riemannian manifold. In general, higher order Sobolev Spaces W^k cannot be defined in the same way as in $\mathbb{R}^N$ because the partial derivatives of higher order are not well-defined on $\mathcal{M}$. For Sobolev spaces on the open sets of Euclidean space $\mathbb{R}^N$ we recommend the book (31).

Definition 2.11. Let $(\mathcal{M}, g)$ be a smooth Riemannian manifold of dimension N. For a real function ϕ belonging to $C^k(\mathcal{M})$, k a non-negative integer and $\alpha = (\alpha_1, ..., \alpha_k)$ a multi-index with order less than or equal to k. We define

$$|\nabla^k \phi|^2 = \nabla^{\alpha_1}\nabla^{\alpha_2}\cdots\nabla^{\alpha_k}\phi\nabla_{\alpha_1}\nabla_{\alpha_2}\cdots\nabla_{\alpha_k}\phi.$$

In particular, $|\nabla^0\phi| = |\phi|$, $|\nabla^1\phi|^2 = |\nabla\phi|^2$. The notation $\nabla^k\phi$ will mean any k-th covariant derivative of ϕ.

Let us consider the vector space $\mathcal{C}^{k,p}$ of C^∞ functions ϕ such that $|\nabla^l\phi| \in L^p(\mathcal{M})$ for all $l \in \mathbb{Z}$ with $0 \le l \le k$ and $p \ge 1$ is a real number.

Definition 2.12. The Sobolev space $W^{k,p}(\mathcal{M})$ is the completion of $\mathcal{C}^{k,p}$ with respect to the norm

$$\|\phi\|_{W^{k,p}} = \sum_{l=0}^{k} \|\nabla^l\phi\|_p.$$

It is possible to consider some other norms which are equivalent, for instance, we could use

$$\left[\sum_{l=0}^{k} \|\nabla^l\phi\|_p^p\right]^{\frac{1}{p}}.$$

Using partition of unity we can obtain all results about Sobolev embeddings for a compact manifold $\mathcal{M}$. The interested reader should consult (32, 33).

Theorem 2.4. Let $\mathcal{M}$ be a smooth compact manifold of dimension N.

(i) If $\frac{1}{r} \ge \frac{1}{p} - \frac{k}{N}$ then $W^{k,p}(\mathcal{M})$ is continuous embedded in $L^r(\mathcal{M})$.

(ii) The above immersion is compact if $\frac{1}{r} > \frac{1}{p} - \frac{k}{N}$.

(iii) If $\alpha \in (0,1)$ and $\frac{1}{p} \le \frac{k-\alpha}{N}$ then $W^{k,p}(\mathcal{M})$ is continuous embedded in $C^\alpha(\overline{\mathcal{M}})$.

(iv) If $\frac{1}{s} \ge \frac{1}{N-1}\left(\frac{N}{p} - k\right)$ then $W^{k,p}(\mathcal{M})$ is continuous embedded in $L^s(\partial\mathcal{M})$.

(v) The above immersion is compact if $\frac{1}{s} > \frac{1}{N-1}\left(\frac{N}{p} - k\right)$.

2.5 Maximum principle and Regularity results

We use a Comparison Principle for weak solutions of quasilinear elliptic differential equation in divergence form on complete Riemannian manifold. We need a simple version of Theorem 3.3 found in (34).

Proposition 2.1 (Maximum Principle)**.** Let A a smooth vector field over $\overline{\Omega}$ and w be a weak supersolution of the problem $-\Delta_g u + A \cdot \nabla u = 0$, that is,

$$-\Delta w + A \cdot \nabla w \geq 0.$$

If $w \geq 0$ on $\partial\Omega$, then $w \geq 0$ in Ω.

For the sake of completeness, we prove the Sub and Supersolution result in Proposition 2.2 using the Monotone Iteration Method. In this way, T. Kura (35) has proved many results about the existence of a solution between sub and supersolutions for quasilinear problems.

Proposition 2.2 (The sub- and super-solution method)**.** Let $\underline{u}$ and $\overline{u}$ weak subsolution and weak supersolution respectively of

$$\begin{cases} -\Delta_g u + A(x) \cdot \nabla_g u = \lambda f(u) & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{P_λ}$$

that satisfies $\underline{u} \leq \overline{u}$ a.e. in Ω. Then problem (P_λ) has a weak solution u such that $\underline{u} \leq u \leq \overline{u}$ a.e. in Ω.

Proof. Denote by $u_0 = \underline{u}$. We define a sequence (u_n) inductively where each u_n is the unique weak solution of the problem

$$\begin{cases} -\Delta_g u_n + A \cdot \nabla_g u_n + c u_n = \lambda f(u_{n-1}) + c u_{n-1} & \text{in} \quad \Omega, \\ u_n = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{5}$$

This sequence satisfies $\underline{u} \leq u_{n-1} \leq u_n \leq \overline{u}$. In fact, consider (5) where $n = 1$. We have $u_1 \in W_0^{1,2}(\Omega)$ and by Maximum Principle follows $\underline{u} \leq u_1 \leq \overline{u}$. In the same way $u_2 \in W_0^{1,2}(\Omega)$ and satisfies $\underline{u} \leq u_1 \leq u_2 \leq \overline{u}$. By induction we have the result i.e., $\underline{u} \leq u_{n-1} \leq u_n \leq \overline{u}$. Now, observe that (u_n) is bounded in $W_0^{1,2}(\Omega)$ and has a subsequence that converges weakly to $u \in W_0^{1,2}(\Omega)$. Taking the limit in the equation follows that u is a weak solution of the problem

$$\begin{cases} -\Delta_g u + A \cdot \nabla_g u = \lambda f(u) & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

$\square$

We also have a version of Hodge-Helmholtz decomposition in order to deal with general vector fields A. This decomposition of vector fields is one of the fundamental theorem in fluid dynamics. It describes a vector field in terms of its divergence-free and rotation-free components. For more results in this subject we refer the reader to (36).

Lemma 2.2. Any vector field $A \in C^\infty(\overline{\Omega}, TM)$ can be decomposed as $A = -\nabla_g a + C$ where a is a smooth scalar function and C is a smooth bounded vector field such that $\mathrm{div}(\mathrm{e}^a C) = 0$.

Proof. Let ν the unit outer normal on $\partial\Omega$. Using Krein-Rutman theorem, we can find a positive solution w of

$$\begin{cases} \Delta_g w + \mathrm{div}(wA) = \mu w & \text{in} \quad \Omega, \\ (\nabla_g w + wA) \cdot \nu = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{6}$$

for a constant $\mu \in \mathbb{R}$. Integrating the equation over Ω one sees that $\mu = 0$. By the maximum principle (Theorem 3.3 in (34)), w is positive up to the boundary. Now define $a := \log(w)$ and $C := A + \nabla_g a$. Observe that

$$\begin{aligned} \mathrm{div}\,(e^a C) &= e^a \nabla_g a \cdot C + e^a \mathrm{div} C \\ &= e^a \nabla_g a \cdot (A + \nabla_g a) + e^a \mathrm{div}\,(A + \nabla_g a) \\ &= e^a \nabla_g a \cdot A + e^a |\nabla_g a|^2 + e^a \mathrm{div} A + e^a \mathrm{div}\,(\nabla_g a) \\ &= e^a \nabla_g \cdot A + e^a |\nabla_g|^2 + e^a \mathrm{div} A + e^a \nabla_g a. \end{aligned}$$

Since $e^a = e^{\ln w} = w$, $\nabla_g a = 1/w \nabla_g w$ and $\Delta_g a = -1/w^2 |\nabla_g w|^2 + 1/w \Delta_g w$ we can use the above calculations to obtain

$$\begin{aligned} \mathrm{div}\,(e^a C) &= \nabla_g w \cdot A + 1/w |\nabla_g w|^2 + w \mathrm{div} A + w\left(-1/w |\nabla_g w|^2 + 1/w \Delta_g w\right) \\ &= \Delta_g + w \mathrm{div} A + \nabla_g w \cdot A \\ &= \Delta_g w + \mathrm{div}\,(wA) = 0. \end{aligned}$$

□

Regularity results on manifolds are similar to the more traditional ones expressed in the Euclidean case, because regularity is a local notion. The regularity result we will mostly use is the following: if $f \in W^{k,p}(\mathcal{M})$ for some $k \in \mathbb{N}$ and $p > 1$, then a weak solution u to the equation

$$L(u) = f$$

is in $W^{k+2,p}(\mathcal{M})$. In particular, it follows from Sobolev embedding theorem that when f is smooth then u is also smooth. About regularity theory, a classical reference is Appendix B in (37) and also Gilbarg-Trudinger (38).

We recall some general facts for the p-Laplace Beltrami operator that are extensions, to a quasilinear setting, of some results of spectral theory. The reader may consult (39, 40, 41, 42, 43, 57, 58, 59). The basic technical material that is necessary for our purpose about regularity is summarized in the following.

Theorem 2.5. Let $\Omega \subset \mathcal{M}$ be a relatively compact open domain with $C^{1,\alpha}$ boundary for some $0 < \alpha < 1$. Let $1 < p < +\infty$, $h \in L^\infty(\Omega)$, $\xi \in C^{1,\alpha}(\partial\Omega)$ and suppose that $u \in W^{1,p}(\Omega)$ is a solution of

$$\begin{cases} -\Delta_p u = h & \text{in} \quad \Omega, \\ \quad\; u = \xi & \text{on} \quad \partial\Omega. \end{cases} \tag{7}$$

Then

(i) [Boundedness] We have $u \in L^\infty_{\mathrm{loc}}(\Omega)$ and for any relatively compact open domains $\Omega' \Subset \Omega'' \Subset \Omega$ there exists a positive constant $C = C(p, h, N, \xi, \Omega, \|u\|_{L^p(\Omega'')})$ such that

$$\|u\|_{L^\infty(\Omega')} \leq C. \tag{8}$$

If $\xi \in C^{2,\alpha}(\partial\Omega)$, C can be chosen globally on Ω, and thus $u \in L^\infty(\Omega)$.

(ii) [$C^{1,\beta}$-regularity] When $u \in L^\infty(\Omega)$ there exists $\beta \in (0,1)$ depending on p, N, h, α and on upper bounds for $\|u\|_{L^\infty}$, $\|h\|_{L^\infty}$, $\|\xi\|_{C^{1,\alpha}}$ on Ω such that

$$\|u\|_{C^{1,\beta}(\overline{\Omega})} \leq C \tag{9}$$

for some constant C depending on α, p, the geometry of Ω and upper bounds for $\|u\|_{L^\infty}$, $\|h\|_{L^\infty}$, $\|\xi\|_{C^{1,\alpha}}$ on Ω.

(iii) [Harnack inequality] For any relatively compact open sets $\Omega' \Subset \Omega'' \Subset \Omega$ there exists $C = C(p, N, \Omega', \Omega'') > 0$ such that $u \in W^{1,p}(\Omega)$ nonnegative solution of $-\Delta_p u = 0$ on Ω,

$$\sup_{\Omega'} u \leq C \inf_{\Omega''} u. \tag{10}$$

In particular, either $u > 0$ on Ω or $u \equiv 0$ on Ω.

(iv) [Hopf lemma] Suppose that $\xi \geq 0$, $g \geq 0$ and let $u \in C^1(\overline{\Omega})$ be a nonnegative solution of (7) which is not identically zero. If $x \in \partial\Omega$ is such that $u(x) = \xi(x) = 0$ then we have $\langle \nabla u, \nu \rangle > 0$, where ν is the inward unit normal vector to $\partial\Omega$ at x.

The interested reader can verify the next remarks:

(a) The local boundedness of u (8) is a particular case of Serrin's theorem (44) and does not need the boundary condition. The global boundedness of (8) can be reached via reflection technique described at (39) and (45, 46) when $\xi \in C^{2,\alpha}(\partial\Omega)$.

(b) The $C^{1,\beta}$-regularity (9) is a global version (60), of a local regularity result in (46) and (47).

(c) The Harnack inequality (10) is due to J. Serrin (44).

(d) The Hopf lemma can be found in (48).

2.6 Krein-Rutman theorem

The Krein-Rutman theorem is a important tool in nonlinear partial differential equations, as it provides the abstract basis for the proof of the existence of various principal eigenvalues, which are crucial in topological degree calculations and bifurcation theory. The interested reader can consult (49).

Definition 2.13. Let X be a Banach space. By a cone $K \subset X$ we mean a closed convex set such that $\lambda K \subset K$ for all $\lambda \geq 0$ and $K \cap (-K) = \{0\}$.

It is clear that a cone K in X induces a partial ordering $\leq$ by the rule: $u \leq v$ if and only if $v - u \in K$. A Banach space with such an ordering is usually called a partially ordered Banach space and the cone generating the partial ordering is called the positive cone of the space X. We write $u > v$ if $u - v \in K \setminus \{0\}$ and $u \gg v$ if $u - v \in \mathrm{int}(K)$.

Definition 2.14. Let $K \subset X$ be a cone in a Banach space X. If $\overline{K - K} = X$ then K is called a total cone. If $K - K = X$ then K is called a reproducing cone. If a cone K has nonempty interior then it is called a solid cone.

Let X^* denote the dual space of X. The set $K^* := \{l \in X^* : l(x) \geq 0 \, \forall x \in K\}$ is called the dual cone of K. It is easy to check that K^* is closed and convex and $\lambda K^* \subset K^*$ for any $\lambda \geq 0$.

Theorem 2.6 (Krein-Rutman Theorem)**.** Let X be a Banach space, $K \subset X$ a total cone and $T : X \to X$ a compact linear operator that is positive, i.e., $T(K) \subset K$ with positive spectral radius $r(T)$. Then $r(T)$ is an eigenvalue with an eigenvector $u \in K \setminus \{0\}$. Moreover, $r(T^*) = r(T)$ is an eigenvalue of T^* with an eigenvector $u^* \in K$.

Using the Krein-Rutman Theorem 2.6 we can obtain the following useful result.

Theorem 2.7. Let X be a Banach space, $K \subset X$ a solid cone and $T : X \to X$ a compact linear operator which is strongly positive, i.e., $Tu \gg 0$ if $u > 0$. Then

(i) $r(T) > 0$ and $r(T)$ is a simple eigenvalue with an eigenvector $v \in \operatorname{int}(K)$. Moreover, there is no other eigenvalue with a positive eigenvector.

(ii) $|\lambda| < r(T)$ for all eigenvalues $\lambda \neq r(T)$.

3 Singular elliptic problems on manifolds

In this chapter we investigate the following class of nonlinear elliptic differential equations involving singular nonlinearities and advection

$$\begin{cases} -\Delta_g u + A(x)\cdot\nabla_g u = \lambda f(u) & \text{in} \quad \Omega, \\ u > 0 & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{P_λ}$$

where A is a smooth vector field over $\overline{\Omega}$ and Ω is a smooth bounded domain of a complete Riemannian manifold $(\mathcal{M}, g)$ with dimension N. We analyse (P_λ) for the following types of nonlinearities and we are assuming the following values for $s_0 \in (0, +\infty]$

Nonlinearity	s_0	Type
$f(u) = 1/(1-u)^2$	1	MEMS
$f(u) = \mathrm{e}^u$	$+\infty$	Gelfand
$f(u) = (1+u)^m$	$+\infty$	Power

where $m > 1$. Along this chapter we deal with many types of solutions. To avoid confusion, we prefer to state all this definitions here.

Definition 3.1.

Classical solution: $u \in C^2(\Omega)\cap C(\overline{\Omega})$ is a classical solution of (P_λ) if it solves (P_λ) in the classical sense (i.e. using the classical notion of derivative).

Weak solution: $u \in W_0^{1,2}(\Omega)$ is a weak solution of (P_λ) if $0 \le u < s_0$ almost everywhere in Ω and $u = s_0$ in a subset with measure zero such that $f(u) \in L^2(\Omega)$ and

$$\int_\Omega (\nabla_g u \cdot \nabla_g \phi + \phi A\cdot\nabla_g u)\, \mathrm{d}v_g = \lambda \int_\Omega f(u)\phi\, \mathrm{d}v_g, \quad \forall \phi \in W_0^{1,2}(\Omega).$$

Weak subsolution: $u \in W_0^{1,2}(\Omega)$ is a weak subsolution of (P_λ) if $0 \le u < s_0$ almost everywhere in Ω and $u = s_0$ in a subset with measure zero such that $f(u) \in L^2(\Omega)$ and

$$\int_\Omega (\nabla_g u \cdot \nabla_g \phi + \phi A\cdot\nabla_g u)\, \mathrm{d}v_g \le \lambda \int_\Omega f(u)\phi\, \mathrm{d}v_g, \quad \forall \phi \in W_0^{1,2}(\Omega) \quad \text{with} \quad \phi \ge 0.$$

Weak supersolution: Analogously one defines weak supersolution of (P_λ) by reversing the above inequality.

Minimal solution: For problem (P_λ), we say that a weak solution $u \in W_0^{1,2}(\Omega)$ is a minimal solution if $u \leq v$, almost everywhere, for all v supersolution. We denote minimal solution of (P_λ) by $\underline{u}_\lambda$.

Regular solution: We say that a weak solution u of (P_λ) is a regular solution if $\sup_{x\in\Omega} u(x) < s_0$.

Semi-stable solution: We say that a classical solution u of (P_λ) is semi-stable solution when

$$\int_\Omega \left(|\nabla_g \xi|^2 + \xi A(x) \cdot \nabla_g \xi\right) \mathrm{d}v_g \geq \int_\Omega \lambda f'(u)\xi^2 \mathrm{d}v_g, \quad \forall \xi \in C_0^1(\Omega). \tag{11}$$

Analogously one defines stable solution when we have the strict inequality in (11). We say that a classical solution u of (P_λ) is unstable when u is not semi-stable.

We will study the minimal branch and regularity properties for minimal solutions of (P_λ). We first prove that there exists some positive finite critical paramater λ^* such that for all $0 < \lambda < \lambda^*$ the problem (P_λ) has a smooth minimal stable solution $\underline{u}_\lambda$ while for $\lambda > \lambda^*$ there are no solutions of (P_λ) in any sense (cf. Theorems 3.1). We determine the critical dimension N^* for this class of problems, precisely we prove that the extremal solution (see section 3.3) of (P_λ) is regular for $N < N^*$ and it is singular for $N \geq N^*$. We see that the critical dimension depends only on the nonlinearity f and does not depend of the Manifold $\mathcal{M}$ (cf. Theorem 3.2 and Table 1). Furthermore, we establish L^∞ estimates, which are crucial to obtain our regularity results on the extremal solutions. We also prove multiplicity of solutions near the extremal parameter λ^* and uniqueness on it (cf. Theorem 3.5 and Theorem 3.7).

Remark 3.1. If we cover $\mathcal{M}$ by coordinate neighborhoods and consider a partition of unity subordinate to this cover, we can use elliptic estimates to see that any regular solution u of (P_λ) belongs to $C^{1,\alpha}(\overline{\Omega})$. Moreover, using Schauder estimates it is easy to prove that $u \in C^{2,\alpha}(\overline{\Omega})$ and consequently any regular solution of (P_λ) is a classical solution.

Remark 3.2. The class of semi-stable solutions includes local minimizers, minimal solutions, extremal solutions and certain class of solutions found between a sub and a supersolution.

3.1 Existence Results

We can construct a supersolution for the problem (P_λ) when λ is sufficient small.

Lemma 3.1. Let $w \in W_0^{1,2}(\Omega)$ be a weak solution of the problem

$$\begin{cases} -\Delta_g w + A \cdot \nabla_g w = 1 & \text{in} \quad \Omega, \\ w = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{12}$$

There exists $\beta > 0$ such that βw is a supersolution of (P_λ) for λ sufficient small.

Proof. For a large $c > 0$, let $\tilde{\mathcal{L}}w = -\Delta_g w + A \cdot \nabla_g w + cw$ and consider the problem

$$\begin{cases} \tilde{\mathcal{L}}w = f & \text{in} \quad \Omega, \\ w = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{13}$$

If we write $w = \tilde{\mathcal{L}}^{-1}(1 + cw) = \mathcal{N}(w)$ we can use Schauder Fixed Point Theorem to find a solution w of (12). By elliptic estimates $w \in C^1(\overline{\Omega})$ so we can take $\beta > 0$ such that $\beta \max_{\overline{\Omega}} w < s_0$. If $\lambda \le \beta / f(\beta \max_{\overline{\Omega}} w)$ we have

$$\int_\Omega (\nabla_g(\beta w)\nabla_g \phi + \phi A \cdot \nabla_g(\beta w))\, \mathrm{d}v_g = \beta \int_\Omega \phi \ge \int_\Omega \lambda \phi f(\beta w)\, \mathrm{d}v_g,$$

for all $\phi \in W_0^{1,2}(\Omega)$ with $\phi \ge 0$,i.e., βw is a supersolution of (P_λ). $\square$

Let us define

$$\Lambda := \{\lambda \ge 0 : \ (P_\lambda) \text{ has a classical solution}\}.$$

Remark 3.3. Using Sub and Supersolution Method (Proposition 2.2) we can find a regular solution between 0 and βw. With this, $\sup \Lambda > 0$.

Lemma 3.2. The interval Λ is bounded.

Proof. Suppose that exists a classical solution u of (P_λ), for λ sufficiently large. We can suppose that $\lambda > \lambda_1$, where λ_1 is the first eigenvalue associate to the operator $L = -\Delta_g + A \cdot \nabla_g$. Let v_1 the eigenfunction associated to λ_1, i.e.,

$$\begin{cases} -\Delta_g v_1 + A \cdot \nabla_g v_1 = \lambda_1 v_1 & \text{in} \quad \Omega, \\ v_1 = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

By regularity theory, it follows that $v_1 \in C^{1,\alpha}(\overline{\Omega})$. By homogeneity, we can suppose $\|v_1\|_\infty < 1$. So v_1 and u satisfies

$$-\Delta_g v_1 + A \cdot \nabla_g v_1 = \lambda_1 v_1 < \lambda f(u) = -\Delta_g u + A \cdot \nabla_g u.$$

By Maximum Principle follows that $v_1 \leq u$. Now, given $\varepsilon > 0$, we take v_2 a solution of

$$\begin{cases} -\Delta_g v_2 + A \cdot \nabla_g v_2 = (\lambda_1 + \varepsilon) v_1 & \text{in} \quad \Omega, \\ v_2 = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

As above, $v_1 \leq v_2 \leq u$. By induction, we have solutions v_n such that

$$\begin{cases} -\Delta_g v_n + A \cdot \nabla_g v_n = (\lambda_1 + \varepsilon) v_{n-1} & \text{in} \quad \Omega, \\ v_n = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

with $v_1 \leq ... \leq v_{n-1} \leq v_n \leq u$ in $C^{1,\alpha}(\overline{\Omega})$. Thus $v_n \rightharpoonup v$ in $W_0^{1,2}(\Omega)$. It follows that v satisfies

$$\begin{cases} -\Delta_g v + A \cdot \nabla_g v = (\lambda_1 + \varepsilon) v & \text{in} \quad \Omega, \\ v = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

This is impossible since the first eigenvalue is isolated. □

Since Λ is bounded, we can define the extremal parameter

$$\lambda^* = \sup \Lambda.$$

Remark 3.4. Clearly, there are no classical solution of (P_λ) for $\lambda > \lambda^*$.

Lemma 3.3. The set Λ is an interval.

Proof. Initially, we prove that Λ does not consist of just $\lambda = 0$. Let u be a classical solution for problem (P_λ) with $\lambda < \lambda^*$. Observe that $u_0 = 0$ and u are sub and supersolution, respectively, for the problem (P_λ). Using the Sub and Supersolution Method (see Proposition 2.2), there exist a weak solution $v \in W_0^{1,2}(\Omega)$ such that $u_0 \leq v < u < s_0$. By Remark 3.1, v is a classical solution. This solution is a supersolution for (P_μ) when $\mu \in (0, \lambda)$. Again, there exist a classical solution for the problem (P_μ). Thus, Λ is an interval. □

3.2 Minimal solutions

Lemma 3.4. For each $\lambda < \lambda^*$, there exists a unique minimal solution $\underline{u}_\lambda$ for the problem (P_λ). Therefore, for all $x \in \Omega$, the map $\lambda \to \underline{u}_\lambda$ is strictly increasing.

Proof. Consider the weak solution u given by Proposition 2.2. By the maximum principle (see Proposition 2.1), all supersolutions v of (P_λ) satisfies $u \leq v$. Thus u is minimal. The uniqueness follows by minimality of u. In this way, we define $u := \underline{u}_\lambda$. Therefore, if $\lambda < \mu$, we have that $\underline{u}_\mu$ is a supersolution of (P_λ). Thus, $\underline{u}_\lambda < \underline{u}_\mu$. □

Let u be a semi-stable solution of (P_λ), and let us consider the following eigenvalue problem involving the linearized operator $-\Delta_g + A \cdot \nabla_g - \lambda f'(u)$ at u,

$$\begin{cases} L_{u,\lambda}\phi = \mu\phi & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

It is well known that there exists a smallest positive eigenvalue μ, which we denote by $\mu_{1,\lambda}$, and an associated eigenfunction $\phi_{1,\lambda} > 0$ in Ω, and $\mu_{1,\lambda}$ is a simple eigenvalue and has the following variational characterization

$$\mu_{1,\lambda} = \inf \left\{ \langle L_{u,\lambda}\phi, \phi \rangle_{L^2(\Omega)} : \phi \in W_0^{1,2}(\Omega), \int_\Omega \phi^2 \, \mathrm{d}v_g = 1 \right\}.$$

Lemma 3.5. If $0 \le \lambda < \lambda^*$, the minimal solutions are semi-stable.

Proof. Let $\underline{u}_\lambda$ be the minimal solution of (P_λ). Suppose that $\underline{u}_\lambda$ is not semi-stable i.e., the first eigenvalue $\mu_{1,\lambda}$ of operator $L_{u,\lambda}$ is negative. Consider the function $\psi_\varepsilon = \underline{u}_\lambda - \varepsilon\psi \in W_0^{1,2}(\Omega)$, where $\psi \in W_0^{1,2}(\Omega)$ is the first positive eigenvector of $-\Delta_g + A \cdot \nabla_g - \lambda f'(\underline{u}_\lambda)$. Using Taylor's formula, for $\|\xi\|_{W_0^{1,2}(\Omega)}$ sufficiently small we have

$$\begin{aligned} -\Delta_g \psi_\varepsilon + A \cdot \nabla_g \psi_\varepsilon - \lambda f(\psi_\varepsilon) &= -\varepsilon\lambda f(\psi) + \lambda f(\underline{u}_\lambda) - \varepsilon\kappa\psi - \varepsilon\lambda f'(\underline{u}_\lambda)\psi \\ &= -\varepsilon\kappa\psi - \lambda\varepsilon^2 f''(\xi)\psi^2 \ge 0, \end{aligned}$$

for ε sufficiently small, because $\kappa < 0$. Thus ψ_ε is a supersolution of (P_λ) and, by minimality of $\underline{u}_\lambda$ we have a contradiction. □

Theorem 3.1. There exists a critical parameter $\lambda^* \in \mathbb{R},\ \lambda^* > 0$ such that

1. For all $\lambda \in (0, \lambda^*)$ problem (P_λ) possesses an unique minimal classical solution $\underline{u}_\lambda$ which is positive and semi-stable, and the map $\lambda \to \underline{u}_\lambda$ is increasing on $(0, \lambda^*)$ for each $x \in \Omega$.

2. The following estimates hold

$$\beta(1 - \beta \max_{\overline{\Omega}} w)^2 \le \lambda^* \le \lambda_1,$$

where w and β are given in Lemma 3.1 and λ_1 is the first eigenvalue of $-\Delta_g + A \cdot \nabla_g$ with zero Dirichlet boundary condition.

3. For $\lambda > \lambda^*$ there are no solutions, even in weak sense.

4. semi-stable solutions of (P_λ) are necessarily minimal solutions.

Proof of Theorem 3.1. (1) The existence of λ^* follows from Lemma 3.2. By Lemmas 3.4 and 3.5, there exists $\underline{u}_\lambda$ minimal solution of (P_λ) which is semi-stable and the function $\lambda \to \underline{u}_\lambda(x)$ is strictly increasing.
(2) Note that since $u_0 = 0$ is a subsolution of (P_λ), $\underline{u}_\lambda$ is non negative. In the same way, since a classical solution of (P_λ) is also a supersolution, it follows that $\underline{u}_\lambda$ is a classical solution. The estimate is a consequence of Lemma 3.1 and Lemma 3.2.
(3) Let u_μ be a weak solution of (P_μ) with $\lambda^* < \mu$. Observe that $w = (1-\varepsilon)u_\mu$ is a weak solution of $-\Delta_g w + A \cdot \nabla_g w = (1-\varepsilon)\mu f(u_\mu)$, that is,

$$\int_\Omega (\nabla_g w \cdot \nabla_g \phi + \phi A \cdot \nabla_g w)\, \mathrm{d}v_g = (1-\varepsilon)\mu \int_\Omega \phi f(u_\mu)\, \mathrm{d}v_g.$$

An easy calculation shows that w is a supersolution for $(P_{(1-\varepsilon)\mu})$. Thus there exist a weak solution $v \le w$. Since $v \le w < u_\mu$, it follows that v is a classical solution of $(P_{(1-\varepsilon)\mu})$. If ε is sufficiently small, $\lambda^* < (1-\varepsilon)\mu$. Furthermore, since u^* is a monotone limit of measurable functions, it is also measurable.
(4) Now, to prove that a semi-stable solution of (P_λ) is minimal, let u and v a semi-stable solution and a supersolution of (P_λ) respectively. For $\vartheta \in [0,1]$ and $0 \le \phi \in W_0^{1,2}(\Omega)$, we have

$$I_{\vartheta,\phi} := \int_\Omega (\nabla_g(\vartheta u + (1-\vartheta)v) \cdot \nabla_g \phi + \phi A \cdot \nabla_g(\vartheta u + (1-\vartheta)v))\, \mathrm{d}v_g \tag{14}$$

$$-\lambda \int_\Omega \phi f(\vartheta u + (1-\vartheta)v)\, \mathrm{d}v_g \ge 0, \tag{15}$$

due to the convexity of function $s \to f(s)$. Since $I_{1,\phi} = 0$, the derivative of $I_{\vartheta,\phi}$ at $\vartheta = 1$ is non positive, that is

$$\int_\Omega (\nabla_g(u-v) \cdot \nabla_g \phi + \phi A \cdot \nabla_g(u-v))\, \mathrm{d}v_g - \int_\Omega \lambda(u-v)\phi f'(u)\, \mathrm{d}v_g \le 0, \quad \forall \phi \ge 0.$$

Testing $\phi = (u-v)^+$ and using that u is semi-stable we get that

$$\int_\Omega \left(|\nabla_g(u-v)^+|^2 + (u-v)^+ A \cdot \nabla_g(u-v)\right)\, \mathrm{d}v_g - \int_\Omega \lambda(u-v)(u-v)^+ f'(u)\, \mathrm{d}v_g = 0,$$

for all $\phi \ge 0$. Since $I_{\vartheta,(u-v)^+} \ge 0$ for any $\vartheta \in [0,1]$ and $I_{1,(u-v)^+} = \partial_\vartheta I_{1,(u-v)^+} = 0$, we have

$$\partial^2_{\vartheta\vartheta} I_{1,(u-v)^+} = -\int_\Omega \lambda(u-v)^2(u-v)^+ f''(u)\, \mathrm{d}v_g \ge 0.$$

Clearly we have $(u-v)^+ = 0$ a.e. in Ω and therefore $\int_\Omega |\nabla_g(u-v)^+|^2\, \mathrm{d}v_g = 0$, from which we conclude that $u \le v$ a.e. in Ω. □

3.3 Determining the Critical Dimension

In view of item 1 of Theorem 3.1, we can define

$$u^*(x) := \lim_{\lambda \nearrow \lambda^*} \underline{u}_\lambda(x),$$

which is a measurable function. If u^* is a weak solution of (P_λ) with $\lambda = \lambda^*$ it will be called extremal solution.

An important question which has attracted a lot of attention is whether the extremal function solution u^* is a classical solution. Here we are going to prove regularity of the extremal solution u^* when the dimension of $\mathcal{M}$ is below the critical dimension N^*. C. Cowan and N. Ghoussoub in (50) proved regularity results for the extremal solution of (P_λ) when $f(u) = 1/(1-u)^2$ or $f(u) = e^u$ and Ω is a bounded domain of $\mathbb{R}^N$. Their result hold for a general class of advection term, which it is not necessarily incompressible. For that they used an argument based on a class of Hardy inequality (51). We could not use a similar argument here because our problem is defined in a domain of a Riemannian manifold. It is known that the existence of Hardy or Gagliardo-Nirenberg or Caffarelli-Kohn-Nirenberg inequality implies qualitative properties on the Riemannian manifold. Precisely, it was shown that if $(\mathcal{M}, g)$ is a complete Riemannian manifold with nonnegative Ricci curvature in which a Hardy or Gagliardo-Nirenberg or Caffarelli-Kohn-Nirenberg type inequalities holds then $\mathcal{M}$ is close to Euclidean space in some suitable sense (52). In this section we want to prove the following theorem:

Theorem 3.2. If $1 \leq N < N^*$ then u^* is a classical solution of (P_{λ^*}).

For the MEMS case, the next lemma is the principal estimate, which was already behind the proof of the regularity of semi-stable solutions in dimensions lower than 7 (see Lemma 3.6).

When $A = -\nabla_g a + C$ the problem (P_λ) can be rewritten as

$$-\mathrm{div}_g(\mathrm{e}^a \nabla_g u) + \mathrm{e}^a C \cdot \nabla_g u = \frac{\lambda \mathrm{e}^a}{(1-u)^2}.$$

Thus the semi-stability and weak solution conditions becomes, respectively

$$\int_\Omega \left(\mathrm{e}^a |\nabla_g \eta|^2 \ + \mathrm{e}^a \eta C \cdot \nabla_g \eta\right) \, \mathrm{d}v_g \geq \int_\Omega \frac{2\lambda \mathrm{e}^a}{(1-u)^3} \eta^2 \, \mathrm{d}v_g, \quad \forall \eta \in C_0^1(\Omega).$$

and

$$\int_\Omega (\mathrm{e}^a \nabla_g u \cdot \nabla_g \phi + \mathrm{e}^a \phi C \cdot \nabla_g u) \, \mathrm{d}v_g = \int_\Omega \frac{\lambda \mathrm{e}^a \phi}{(1-u)^2} \, \mathrm{d}v_g, \quad \forall \phi \in W_0^{1,2}(\Omega).$$

Lemma 3.6. If u is a semi-stable solution of (P_λ) with $0 < \lambda < \lambda^*$, $f(u) = 1/(1-u)^2$ and $0 < t < 2+\sqrt{6}$, holds the following estimate

$$\|\mathrm{e}^{2a/(2t+3)}(1-u)^{-2}\|_{L^{t+3/2}} \leq \left[\frac{4(2t+1)}{2+4t-t^2}\right]^{2/t} C_1\|\Omega\|^{2/(2t+3)}.$$

Proof. Let $0 < t < 2+\sqrt{6}$ and u semi-stable solution of (P_λ). Define $\eta := (1-u)^{-t} - 1$ and $\phi := (1-u)^{-2t-1} - 1$. Testing η in the semistability condition we have

$$\begin{aligned}
0 &\leq \int_\Omega \mathrm{e}^a\{t^2(1-u)^{-2t-2}|\nabla_g u|^2 - 2\lambda(1-u)^{-3}\left[(1-u)^{-t}-1\right]^2\}\mathrm{d}v_g \\
&= \int_\Omega \mathrm{e}^a\{t^2(1-u)^{-2t-2}|\nabla_g u|^2 - 2\lambda(1-u)^{-2t-3} + 4\lambda(1-u)^{-t-3} - 2\lambda(1-u)^{-3}\}\mathrm{d}v_g \\
&\leq \int_\Omega \left\{\mathrm{e}^a t^2(1-u)^{-2t-2}|\nabla_g u|^2 - 2\lambda\mathrm{e}^a(1-u)^{-2t-3} + 4\lambda\mathrm{e}^a(1-u)^{-t-3}\right\}\mathrm{d}v_g.
\end{aligned}$$

Due to this choice of η we have $\int_\Omega \mathrm{e}^a\eta C\cdot\nabla_g\eta\,\mathrm{d}v_g = 0$. It follows that

$$-\int_\Omega \mathrm{e}^a(1-u)^{-2t-2}|\nabla_g u|^2\mathrm{d}v_g \leq -\frac{2\lambda}{t^2}\int_\Omega \mathrm{e}^a(1-u)^{-2t-3}\mathrm{d}v_g + \frac{4\lambda}{t^2}\int_\Omega \mathrm{e}^a(1-u)^{-t-3}\mathrm{d}v_g. \quad (16)$$

Testing ϕ in the weak solution condition we obtain

$$\begin{aligned}
\int_\Omega \mathrm{e}^a|\nabla_g u|^2(2t+1)(1-u)^{-2t-2}\mathrm{d}v_g &= \int_\Omega \left\{\lambda\mathrm{e}^a(1-u)^{-2t-3} - \lambda\mathrm{e}^a(1-u)^{-2}\right\}\mathrm{d}v_g \\
&\leq \int_\Omega \lambda\mathrm{e}^a(1-u)^{-2t-3}\mathrm{d}v_g.
\end{aligned} \quad (17)$$

With this choice of ϕ we can check that $\int_\Omega \mathrm{e}^a\phi C\cdot\nabla_g u\,\mathrm{d}v_g = 0$. Using (17) and (16) we have

$$-\frac{1}{2t+1}\int_\Omega \mathrm{e}^a(1-u)^{-2t-3}\mathrm{d}v_g \leq -\frac{2}{t^2}\int_\Omega \mathrm{e}^a(1-u)^{-2t-3}\mathrm{d}v_g + \frac{4}{t^2}\int_\Omega \mathrm{e}^a(1-u)^{-t-3}\mathrm{d}v_g$$

and it follows that

$$\left(\frac{2}{t^2} - \frac{1}{2t+1}\right)\int_\Omega \mathrm{e}^a(1-u)^{-2t-3}\mathrm{d}v_g \leq \frac{4}{t^2}\int_\Omega \mathrm{e}^a(1-u)^{-t-3}\mathrm{d}v_g.$$

Using Hölder inequality with conjugate exponents $(2t+3)/(t+3)$ and $(2t+3)/t$, we have

$$\begin{aligned}
&\left(\frac{2}{t^2} - \frac{1}{2t+1}\right)\int_\Omega \mathrm{e}^a(1-u)^{-2t-3}\mathrm{d}v_g \\
&\qquad\leq \frac{4}{t^2}\|\Omega\|^{t/(2t+3)}\left[\int_\Omega \mathrm{e}^{a(2t+3)/(t+3)}(1-u)^{-2t-3}\mathrm{d}v_g\right]^{(t+3)/(2t+3)} \\
&\qquad\leq \frac{4C_1}{t^2}\|\Omega\|^{t/(2t+3)}\left[\int_\Omega \mathrm{e}^a(1-u)^{-2t-3}\mathrm{d}v_g\right]^{(t+3)/(2t+3)},
\end{aligned}$$

where $C_1 = \left[\sup_{\overline{\Omega}} \mathrm{e}^{at/(t+3)}\right]^{(t+3)/(2t+3)}$. Thus,

$$\left(\frac{2}{t^2} - \frac{1}{2t+1}\right)\left[\int_\Omega \mathrm{e}^a (1-u)^{-2t-3} \mathrm{d}v_g\right]^{t/(2t+3)} \leq \frac{4C_1}{t^2}\|\Omega\|^{t/(2t+3)}$$

and therefore

$$\| \mathrm{e}^{2a/(2t+3)}(1-u)^{-2} \|_{L^{t+3/2}} \leq \left[\frac{4(2t+1)}{2+4t-t^2}\right]^{2/t} C_1\|\Omega\|^{2/(2t+3)},$$

this is the desired estimate. □

Remark 3.5. In the above estimate we used that $2(2t+1) > t^2$ which is an immediately consequence of our assumption $0 < t < 2+\sqrt{6}$.

Remark 3.6. By the above estimate, $\mathrm{e}^a(1-\underline{u}_\lambda)^{-2}$ is bounded uniformly in λ over $L^p(\Omega)$ for all $p < p_0 := 7/2+\sqrt{6}$. By elliptic estimates, $\underline{u}_\lambda$ is bounded in $W_0^{1,p}(\Omega)$ uniformly in λ. Thus u^* is a weak solution of (P_{λ^*}). If we take the limit $\lambda \nearrow \lambda^*$ in the inequality given by Lemma 3.6, we obtain the same L^p estimate to extremal solution u^*.

Proposition 3.1. If $1 \leq N \leq 7$ and $f(u) = 1/(1-u)^2$ then u^* is a classical solution of (P_{λ^*}).

Proof. Note that $\mathrm{e}^a(1-u^*)^{-2} \in L^{3N/4}(\Omega)$. By elliptic regularity we have $u^* \in W_0^{2,3N/4}(\Omega)$ and by Sobolev immersion $u^* \in C^{0,2/3}(\overline{\Omega})$. If we suppose that $\|u^*\|_\infty = 1$, there exist a element $x_0 \in \Omega$ such that $u^*(x_0) = 1$. Since $|1-u^*(x)| \leq C\,\mathrm{dist}(x,x_0)^{2/3}$ we have,

$$\frac{\mathrm{e}^{a/2}}{1-u^*(x)} \geq \frac{\mathrm{e}^{a/2}}{C\,\mathrm{dist}(x,x_0)^{2/3}},$$

and hence

$$\infty > \int_\Omega \frac{\mathrm{e}^{3Na/4}}{((1-u^*)^2)^{3N/4}}\mathrm{d}v_g \geq C \inf_{x\in\overline{\Omega}}\{\mathrm{e}^{a/2}\}\int_\Omega \frac{1}{\mathrm{dist}(x,x_0)^N}\mathrm{d}v_g = \infty.$$

This is a contradiction. Thus $\mathrm{e}^a(1-u^*)^{-2} \in L^\infty(\Omega)$ and u^* is a classical solution of (P_{λ^*}). □

With a slight variation of the above arguments, the same approach works on the Gelfand and Power-type cases.

Lemma 3.7. If u is a semi-stable solution of (P_λ) with $0 < \lambda < \lambda^*$, $f(u) = \mathrm{e}^u$ and $0 < t < 2$, holds the following estimate

$$\| \mathrm{e}^{1/(2t+1)a+u} \|_{L^{2t+1}} \leq \left[\frac{2t}{2t-t^2}\right]^{1/t} C_1\|\Omega\|^{1/(2t+1)}.$$

Proof. Let $0 < t < 2$ and u semi-stable solution of (P_λ). Define $\eta := \mathrm{e}^{tu} - 1$ and $\phi := \mathrm{e}^{2tu} - 1$. Testing η in the semistability condition we have

$$0 \leq \int_\Omega t^2 \mathrm{e}^{a+2tu}|\nabla_g u|^2 - \lambda \mathrm{e}^{a+(2t+1)u} + 2\lambda \mathrm{e}^{a+(t+1)u} \mathrm{d}v_g.$$

With this choice of η we have $\int_\Omega \mathrm{e}^a \eta C \nabla_g \eta \, \mathrm{d}v_g = 0$. It follows that

$$-\int_\Omega \mathrm{e}^{a+2tu}|\nabla_g u|^2 \mathrm{d}v_g \leq -\frac{\lambda}{t^2}\int_\Omega \mathrm{e}^{a+(2t+1)u}\mathrm{d}v_g + \frac{2\lambda}{t^2}\int_\Omega \mathrm{e}^{a+(t+1)u}\mathrm{d}v_g. \tag{18}$$

Testing ϕ in the weak solution condition we obtain

$$2t\int_\Omega \mathrm{e}^{a+2tu}|\nabla_g u|^2 \mathrm{d}v_g \leq \lambda \int_\Omega \mathrm{e}^{a+(2t+1)u}\mathrm{d}v_g. \tag{19}$$

With this choice of ϕ we can check that $\int_\Omega \mathrm{e}^a \phi C \cdot \nabla_g u \, \mathrm{d}v_g = 0$. Using (18) and (19) we have

$$\left(\frac{1}{t^2} - \frac{1}{2t}\right)\int_\Omega \mathrm{e}^{a+(2t+1)u}\mathrm{d}v_g \leq \frac{2}{t^2}\int_\Omega \mathrm{e}^{a+(t+1)u}\mathrm{d}v_g.$$

We remark that we can apply Hölder inequality with conjugate exponents $(2t+1)/(t+1)$ and $(2t+1)/t$, we have

$$\left(\frac{1}{t^2} - \frac{1}{2t}\right)\int_\Omega \mathrm{e}^{a+(2t+1)u}\mathrm{d}v_g \leq \frac{2C_1}{t^2}\left[\int_\Omega \mathrm{e}^{a+(2t+1)u}\right]^{(t+1)/(2t+1)} \|\Omega\|^{t/(2t+1)}\mathrm{d}v_g,$$

where $C_1 = \left[\sup_{\overline{\Omega}} \mathrm{e}^{t/(t+1)a}\right]^{(t+1)/(2t+1)}$. Therefore

$$\| \mathrm{e}^{1/(2t+1)a+u} \|_{L^{2t+1}} \leq \left[\frac{2t}{2t-t^2}\right]^{1/t} C_1\|\Omega\|^{1/(2t+1)}.$$

□

Remark 3.7. In the above estimate we used that $1/t^2 - 1/(2t) > 0$ which is an immediately consequence of our assumption $0 < t < 2$.

Remark 3.8. The above estimate says that e^{a+u} is bounded uniformly in λ over $L^p(\Omega)$ for all $p < p_0 := 4 + 1 = 5$. By elliptic estimates, $\underline{u}_\lambda$ is uniformly bounded in $W_0^{1,p}(\Omega)$. Thus u^* is a weak solution of (P_{λ^*}). Taking the limit in λ, we obtain the same L^p estimate above to extremal solution u^*.

Proposition 3.2. If $1 \leq N \leq 9$ and $f(u) = e^u$ then u^* is a classical solution of (P_{λ^*}).

Proof. Note that $\mathrm{e}^{a+u} \in L^p(\Omega)$ with $p < 5$. By elliptic regularity we have $u^* \in W_0^{2,p}(\Omega)$ and by Sobolev immersion $u^* \in C^{0,\alpha}(\overline{\Omega})$ when $N < 10$. Thus u^* is a classical solution of (P_{λ^*}). □

Lemma 3.8. If u is a semi-stable solution of (P_λ) with $0 < \lambda < \lambda^*$, $f(u) = (1+u)^m$, $b > 0$ and $\frac{m}{b} - \frac{\sqrt{m(m-1)}}{b} < t < \frac{m}{b} + \frac{\sqrt{m(m-1)}}{b}$, holds the following estimate

$$\| \,\mathrm{e}^{am/[2bt+m-1]}(1+u)^m \,\|_{L^{[2bt+m-1]/m}} \leq \left(1 - \frac{b^2t^2}{m[2bt-1]}\right)^{-1/[tb]} C_1\|\Omega\|^{1/[(2t+1)b]}.$$

Proof. Define $\eta := (1+u)^{bt} - 1$ and $\phi := (1+u)^{2bt-1} - 1$ where $b > 0$. Testing η in the semistability condition we have

$$\int_\Omega b^2t^2\mathrm{e}^a(1+u)^{2bt-2}|\nabla_g u|^2 \mathrm{d}v_g \geq \lambda m \int_\Omega \mathrm{e}^a(1+u)^{(m-1)}\left[(1+u)^{2bt} - 2(1+u)^{bt}\right]\mathrm{d}v_g.$$

With this choice of η we have $\int_\Omega \mathrm{e}^a\eta C\nabla_g\eta\,\mathrm{d}v_g = 0$. Testing ϕ in the weak solution condition we obtain

$$(2bt-1)\int_\Omega \mathrm{e}^a(1+u)^{2bt-2}|\nabla_g u|^2\mathrm{d}v_g \leq \lambda\int_\Omega \mathrm{e}^a(1+u)^{2bt+m-1}\mathrm{d}v_g.$$

With this choice of ϕ we can check that $\int_\Omega \mathrm{e}^a\phi C\nabla_g u\,\mathrm{d}v_g = 0$. It follows that

$$\left(1 - \frac{b^2t^2}{m[2bt-1]}\right)\int_\Omega \mathrm{e}^a(1+u)^{2bt+m-1}\mathrm{d}v_g \leq 2\int_\Omega \mathrm{e}^a(1+u)^{bt+m-1}\mathrm{d}v_g.$$

Using Hölder inequality with conjugate exponents $\frac{2bt+m-1}{bt+m-1}$ and $\frac{2bt+m-1}{bt}$, we have

$$\left(1 - \frac{b^2t^2}{m[2bt-1]}\right)\int_\Omega \mathrm{e}^a(1+u)^{2bt+m-1}\mathrm{d}v_g \leq 2C_1\left[\int_\Omega \mathrm{e}^a(1+u)^{2bt+m-1}\right]^{\frac{bt+m-1}{2bt+m-1}}\mathrm{d}v_g,$$

where $C_1 = \left[\sup_{\overline{\Omega}} \mathrm{e}^{at/(t+1)}\right]^{t/(2t+1)} \cdot \|\Omega\|^{\frac{bt}{2bt+m-1}}$. Therefore

$$\| \,\mathrm{e}^{am/[2bt+m-1]}(1+u)^m \,\|_{L^{[2bt+m-1]/m}} \leq \left(1 - \frac{b^2t^2}{m[2bt-1]}\right)^{-1/[tb]} C_1\|\Omega\|^{1/[(2t+1)b]}.$$

□

Remark 3.9. The above estimate said that $\mathrm{e}^a(1+u)^m$ is bounded uniformly in λ over $L^p(\Omega)$ for all $p < 3 - \frac{1}{m} + \frac{2}{m}\sqrt{m(m-1)}$. By elliptic estimates, $\underline{u}_\lambda$ is uniformly bounded in $W_0^{1,p}(\Omega)$. Thus u^* is a weak solution of (P_{λ^*}). Taking the limit in λ, we obtain the same L^p estimate above to extremal solution u^*.

Proposition 3.3. If $1 \leq N \leq 10$ and $f(u) = (1+u)^m$ with $m > 1$ then u^* is a classical solution of (P_{λ^*}).

Proof. Since $\mathrm{e}^a(1+u^*)^m \in L^p(\Omega)$ with $p < 3 - \frac{1}{m} + \frac{2}{m}\sqrt{m(m-1)}$, we can use elliptic regularity to obtain $u^* \in W_0^{2,p}(\Omega)$ and by Sobolev immersion u^* is a classical solution when $N < 6 + \frac{4}{m-1}\left(\sqrt{m(m-1)} + 1\right)$. Observe that $\frac{\sqrt{m(m-1)}+1}{m-1} > 1$ An immediate consequence is that when $m > 1$ and $N \leq 10$, u^* is a classical solution of (P_{λ^*}). □

Proof of Theorem 3.2. The proof follows immediately from Propositions 3.1, 3.2, 3.3 □

Here we stress the fact that the critical dimension depends only on the nonlinearity f and does not depend of the manifold $\mathcal{M}$, which is given precisely in next table

Nonlinearity	Critical dimension N^*
$f(u) = 1/(1-u)^2$	8
$f(u) = \mathrm{e}^u$	10
$f(u) = (1+u)^m$	11

Table 1: Critical dimensions

3.4 Symmetry and Monotonicity

We prove radial symmetry and monotonicity for semi-stable solutions of (P_λ) when $\Omega = \mathcal{B}_R$ is a geodesic ball of a Riemannian model $\mathcal{M}$ (cf. Theorem 3.3). The class of Riemannian model $(\mathcal{M}, g)$ includes the classical space forms. Precisely, a manifold $\mathcal{M}$ of dimension $N \geq 2$ admitting a pole $\mathcal{O}$ and whose metric g is given, in polar coordinates around $\mathcal{O}$, by

$$\mathrm{d}s^2 = \mathrm{d}r^2 + \psi(r)^2 \mathrm{d}\vartheta^2 \quad \text{for } r \in (0, R) \text{ and } \vartheta \in \mathbb{S}^{N-1}, \tag{20}$$

where r is by construction the Riemannian distance between the point $P = (r, \vartheta)$ to the pole $\mathcal{O}$, ψ is a smooth positive function in $(0, R)$ and $\mathrm{d}\vartheta^2$ is the canonical metric on the unit sphere $\mathbb{S}^{N-1}$. Note that our results apply to the important case of space forms, i.e., the unique complete and simply connected Riemannian manifold of constant sectional curvature K_ψ corresponding to the choice of ψ namely,

Space form	$\psi(r)$	K_ψ
Hyperbolic space $\mathbb{H}^N$	$\sinh r$	-1
Euclidean space $\mathbb{R}^N$	r	0
Elliptic space $\mathbb{S}^N$	$\sin r$	1

In the proof of Theorem 3.3 the radial symmetry relies on the fact that any angular derivative of u would be either a sign changing first eigenfunction of linearized operator at u or identically zero, thanks to the semistability condition. The monotonicity is due the positivity of the nonlinearity.

Theorem 3.3. If $u \in C^2(\mathcal{B}_R)$ is a classical stable solution of (P_λ) with a radial vector field A, then u is radially symmetric and decreasing.

Proof of Theorem 3.3. Let $u \in C^2(\mathcal{B}_R)$ a stable solution of (P_λ). The stability condition (3.1) is equivalent to the positivity of the first eigenvalue of $L_{u,\lambda}$ in $\mathcal{B}_R$, i.e.,

$$\mu_{1,\lambda} = \inf_{\xi \in W_0^{1,2}(\mathcal{B}_R)\setminus\{0\}} \frac{\int_{\mathcal{B}_R}\{|\nabla_g \xi|^2 + \xi A \cdot \nabla_g \xi - \lambda f'(u)\xi^2\}\mathrm{d}v_g}{\int_{\mathcal{B}_R}\xi^2 \mathrm{d}v_g} > 0.$$

Now, consider $u_\vartheta = \frac{\partial u}{\partial \vartheta}$ any angular derivative of u. By the fact $u \in C^2(\mathcal{B}_R)$, we have

$$\int_{\mathcal{B}_R} |\nabla_g u_\vartheta|^2 \mathrm{d}v_g < \infty.$$

Moreover, the regularity up the boundary of u and the fact that $u = 0$ on $\partial\mathcal{B}_R$ give that $u_\vartheta = 0$ on $\partial\mathcal{B}_R$. Hence, $u_\vartheta \in W_0^{1,2}(\mathcal{B}_R)$. Differentiate the problem (P_λ) we obtain that u_ϑ weakly satisfies

$$\begin{cases} -\Delta_g u_\vartheta + A \cdot \nabla_g u_\vartheta = \lambda f'(u) u_\vartheta & \text{in} \quad \mathcal{B}_R, \\ \qquad\qquad u_\vartheta = 0 & \text{on} \quad \partial\mathcal{B}_R. \end{cases}$$

Multiplying the above equation by u_ϑ and integrating by parts we have

$$\int_{\mathcal{B}_R} \{|\nabla_g u_\vartheta|^2 + u_\vartheta A \cdot \nabla_g u_\vartheta - \lambda f'(u) u_\vartheta^2\}\mathrm{d}v_g = 0.$$

It follows that either $|u_\vartheta|$ is the first eigenvalue of linearized operator at u or $u_\vartheta = 0$. But the first alternative can not occur because $\mu_{1,\lambda} > 0$. it follows that $u_\vartheta = 0$ for all $\vartheta \in \mathbb{S}^{N-1}$. Thus u is radial. On the other hand, in spherical coordinates given by (20), the Laplacian operator of $u = u(r, \vartheta_1, ..., \vartheta_{N-1})$ is given by

$$\Delta_g u = \frac{1}{\psi^{N-1}}(\psi^{N-1}u_r)_r + \frac{1}{\psi^2}\Delta_{\mathbb{S}^{N-1}} u,$$

where $\Delta_{\mathbb{S}^{N-1}}$ is the Laplacian on the unit sphere $\mathbb{S}^{N-1}$. To prove the monotonicity, note that since $u = u(r)$ and $A = A(r)$, the equation (P_λ) becomes

$$\int_0^s \int_0^{2\pi} \mathrm{e}^a (\psi^{N-1}u_r)_r \, \mathrm{d}r \, \mathrm{d}\vartheta = \int_0^s \int_0^{2\pi} -\mathrm{e}^a \psi^{N-1} f(u) \, \mathrm{d}r \, \mathrm{d}\vartheta.$$

Therefore, $u_r < 0$. □

3.5 Regularity in the Radial Case

In view of the previous section, we can write the problem (P_λ) for radial solutions $u \in C^2(\mathcal{B}_R)$ and for a radial vector field A as

$$\begin{cases} -(\mathrm{e}^a\psi^{N-1}u_r)_r + \mathrm{e}^a\psi^{N-1}C(r)u_r = \lambda \mathrm{e}^a\psi^{N-1}f(u) & \text{em} \quad (0,R), \\ u > 0 & \text{em} \quad (0,R), \\ u_r(0) = u(R) = 0. \end{cases} \tag{R_λ}$$

In the same way, the semistability and weak solution condition becomes, respectively

$$\int_0^R \mathrm{e}^a\psi^{N-1}\xi_r^2 + \mathrm{e}^a\psi^{N-1}C(r)\xi\xi_r\,\mathrm{d}r \geq \int_0^R \lambda\mathrm{e}^a\psi^{N-1}f'(u)\xi^2\,\mathrm{d}r$$

and

$$\int_0^R \mathrm{e}^a\psi^{N-1}u_r\phi_r + \mathrm{e}^a\psi^{N-1}C(r)u_r\phi\,\mathrm{d}r = \int_0^R \lambda\mathrm{e}^a\psi^{N-1}f(u)\phi\,\mathrm{d}r.$$

In radial case, we obtain a more precise information about the L^∞ norm of the extremal solution. Again, we will start with MEMS case.

Lemma 3.9. If u is a classical semi-stable solution of (R_λ) with $f(u) = 1/(1-u)^2$, then for all $0 < t < 2+\sqrt{6}$ we have

$$\|\mathrm{e}^{2a/(2t+3)}\psi^{2(N-1)/(2t+3)}(1-u)^{-2}\|_{L^{t+3/2}} \leq \left[\frac{4(2t+1)}{2+4t-t^2}\right]^{2/t} C_2R^{2/(2t+3)}.$$

Proof. We follow the proof of Lemma 3.6. Let $0 < t < 2+\sqrt{6}$ and u semi-stable classical solution of (R_λ). Define $\eta := (1-u)^{-t} - 1$ and $\phi := (1-u)^{-2t-1} - 1$. Applying η in the semistablity condition we have

$$-\int_0^R \mathrm{e}^a\psi^{N-1}(1-u)^{-2t-2}u_r^2\mathrm{d}r \leq \frac{2}{t^2}\int_0^R \mathrm{e}^a\psi^{N-1}(1-u)^{-2t-3}\mathrm{d}r + \frac{4}{t^2}\int_0^R \mathrm{e}^a\psi^{N-1}(1-u)^{-t-3}\mathrm{d}r. \tag{21}$$

Applying ϕ in the weak solution condition it follows that

$$\int_0^R \mathrm{e}^a\psi^{N-1}u_r^2(2t+1)(1-u)^{-2t-2}\mathrm{d}r \leq \int_0^R \mathrm{e}^a\psi^{N-1}(1-u)^{-2t-3}\mathrm{d}r. \tag{22}$$

Using (21) and (22) we obtain

$$\left(\frac{2}{t^2} - \frac{1}{2t+1}\right)\int_0^R \mathrm{e}^a\psi^{N-1}(1-u)^{-2t-3}\mathrm{d}r \leq \frac{4}{t^2}\int_0^R \mathrm{e}^a\psi^{N-1}(1-u)^{-t-3}\mathrm{d}r.$$

Using Hölder inequality with conjugate exponents $(2t+3)/(t+3)$ and $(2t+3)/t$,

$$\begin{aligned}\left(\frac{2}{t^2} - \frac{1}{2t+1}\right)&\int_0^R \mathrm{e}^a\psi^{N-1}(1-u)^{-2t-3}\mathrm{d}r \\ &\leq \frac{4}{t^2}R^{t/(2t+3)}C_2(t,\psi)\left[\int_0^R \mathrm{e}^a\psi^{N-1}(1-u)^{-2t-3}\,\mathrm{d}r\right]^{(t+3)/(2t+3)},\end{aligned}$$

where $C_2 := \left[\sup_{[0,R]} \mathrm{e}^{at/(t+3)}\psi^{(N-1)t/(t+3)}\right]^{(t+3)/(2t+3)}$. Thus,

$$\left(\frac{2}{t^2} - \frac{1}{2t+1}\right)\left[\int_0^R \mathrm{e}^a\psi^{N-1}(1-u)^{-2t-3}\right]^{t/(2t+3)} \leq \frac{4}{t^2}C_2R^{t/(2t+3)}$$

and therefore

$$\| \mathrm{e}^{2a/(2t+3)}\psi^{2(N-1)/(2t+3)}(1-u)^{-2} \|_{L^{t+3/2}} \leq \left[\frac{4(2t+1)}{2+4t-t^2}\right]^{2/t} C_2R^{2/(2t+3)}.$$

□

Lemma 3.10. Let u be a radially decreasing and semi-stable classical solution of (R_λ) with $f(u) = 1/(1-u)^2$. If $1 < p < \infty$, we have the estimate

$$u(0) \geq u(r) \geq u(0) - C_3\|\mathrm{e}^{a/p}\psi^{(N-1)/p}(1-u)^{-2}\|_p r.$$

Proof. By the Mean value theorem, there exists $c \in (0,r)$ such that

$$-u(r) + u(0) = -u'(c)r \tag{23}$$

Integrating the equation (R_λ) from 0 to c we obtain

$$\begin{aligned} -\mathrm{e}^{a(c)}\psi^{N-1}(c)u'(c) &= \int_0^c \mathrm{e}^a\psi^{N-1}(1-u)^{-2} \\ &\leq \left[\int_0^R \mathrm{e}^a\psi^{(N-1)}(1-u)^{-2p}\right]^{1/p}\left[\int_0^R \mathrm{e}^{a(1-1/p)}\psi^{(N-1)(1-1/p)}\mathrm{d}r\right]^{p/(p-1)}. \end{aligned}$$

Using (23) we conclude the proof because

$$-u(r) + u(0) \leq C_3 \| \mathrm{e}^{a/p}\psi^{(N-1)/p}(1-u)^{-2} \|_{L^p}\, r$$

where $C_3 = \mathrm{e}^{-a(c)}\psi^{1-N}(c)\left[\int_0^R \mathrm{e}^{a(1-1/p)}\psi^{(N-1)(1-1/p)}\mathrm{d}r\right]^{p/(p-1)}$. □

Lemma 3.11. Let u a radially decreasing and semi-stable classical solution of (R_λ) with $1 \leq N \leq 7$. Then, for all $0 < t < 2+\sqrt{6}$, we have

$$\int_0^r \frac{\mathrm{e}^a\psi^{N-1}}{D_2(r)^{2t+3}}\mathrm{d}r \leq \left(\frac{4(2t+1)}{4t+2-t^2}\right)^{(2t+3)/t},$$

where $D_2(r) := 1 - \|u\|_\infty + C_4\left(4(2t+1)/(2+4t-t^2)\right)^{2/t} R^{1/p}r$.

Proof. Take $p = t + 3/2$. By Lemma 3.10,

$$1 - u(r) \leq 1 - u(0) + C_3 \| \mathrm{e}^{a/p}\psi^{(N-1)/p}(1-u)^{-2} \|_{L^p}\, r.$$

Multiplying some positive terms and using Lemma 3.9, it follows that

$$\mathrm{e}^{-a}\psi^{-(N-1)}(1-u(r))^{2t+3} \leq \mathrm{e}^{-a}\psi^{-(N-1)}(1-u(0)+C_1C_3\left(\frac{4(2t+1)}{2+4t-t^2}\right)^{2/t} R^{1/p}r)^{2t+3}.$$

We have

$$\int_0^R \frac{\mathrm{e}^a\psi^{N-1}dr}{D_2(r)^{2t+3}} \leq \int_0^R \frac{\mathrm{e}^a\psi^{N-1}dr}{(1-u(r))^{2t+3}}.$$

where $C_4 := C_1C_3$. Thus,

$$\int_0^R \frac{\mathrm{e}^a\psi^{N-1}dr}{D_2(r)^{2t+3}} \leq \left(\frac{4(2t+1)}{2+4t-t^2}\right)^{(2t+3)/t} R^{1/p}.$$

□

We can improve the result of Theorem 3.2 giving an estimate for the radial case.

Theorem 3.4. Let u be the extremal solution of (P_{λ^*}) on a geodesic ball $\mathcal{B}_r$ of a Riemannian model with $2 \leq N < N^*$. Then u^* is a classical solution and

$$\|u^*\|_\infty \leq c,$$

where $c > 0$ is a constant which does not depends of λ. We emphasize that, for the case $f(u) = 1/(1-u)^2$ we have $c < 1$.

We split the proof of Theorem 3.4 in three cases, namely, MEMS, Gelfand and Power cases.

Proof of Theorem 3.4 (MEMS case). Using the Lemma 3.11, we have

$$\int_0^R \frac{\mathrm{e}^a\psi^{N-1}}{D_2(r)^{2t+3}}\mathrm{d}r \leq \left(\frac{4(2t+1)}{4t+2-t^2}\right)^{(2t+3)/t} R^{1/p}. \tag{24}$$

Calculating the integral in the left-hand side, we have

$$\begin{aligned} \int_0^R \frac{\mathrm{e}^a\psi^{N-1}}{D_2(r)^{2t+3}}\mathrm{d}r &= \frac{\int_0^R \mathrm{e}^a\psi^{N-1}\,\mathrm{d}r}{D_2(R)^{2t+3}} + (2t+4)\int_0^R \frac{D_2'\int_0^R \mathrm{e}^a\psi^{N-1}\,\mathrm{d}r}{D_2^{2t+4}}\,\mathrm{d}r \\ &\geq \frac{\int_0^R \mathrm{e}^a\psi^{N-1}\,\mathrm{d}r}{D_2(R)^{2t+3}}. \end{aligned} \tag{25}$$

Applying (25) in (24), it follows that

$$\frac{\int_0^R \mathrm{e}^a\psi^{N-1}\,\mathrm{d}r}{D_2(R)^{2t+3}} \leq \left(\frac{4(2t+1)}{4t+2-t^2}\right)^{(2t+3)/t} R^{1/p}. \tag{26}$$

Calculating the equation (26) and taking $\lambda \nearrow \lambda^*$ we have

$$\|u^*\|_\infty \leq 1-C,$$

where
$$C := \left[\int_0^R \mathrm{e}^a \psi^{N-1}\,\mathrm{d}r\right]^{1/(2t+3)} \big(4(2t+1)/(4t+2-t^2)\big)^{-1/t} R^{1/p^2}$$
$$- C_4\left(4(2t+1)/(2+4t-t^2)\right)^{2/t} R^{1+1/p}.$$

□

With a slight variation of the above arguments, the same approach works for the Gelfand problem with advection.

Lemma 3.12. Let u a radially decreasing and semi-stable classical solution of (R_λ) with $f(u) = \mathrm{e}^u$. If $1 < p < \infty$, we have the estimate
$$u(0) \geq u(r) \geq u(0) - C_3\|\mathrm{e}^{a/p+u}\psi^{(N-1)/p}\|_p r.$$

Proof. There exists $c \in (0, r)$ such that
$$-u(r) + u(0) = -u'(c)r. \tag{27}$$

Integrating the equation (R_λ) from 0 to c we obtain
$$-\mathrm{e}^{a(c)}\psi^{N-1}(c)u'(c) = \int_0^c \mathrm{e}^{a+u}\psi^{N-1}$$
$$\leq \left[\int_0^R \mathrm{e}^{a+up}\psi^{N-1}\right]^{1/p}\left[\int_0^R \mathrm{e}^{a(1-1/p)}\psi^{(N-1)(1-1/p)}\mathrm{d}r\right]^{p/(p-1)}.$$

Using (27) we obtain
$$-u(r) + u(0) \leq C_3 \parallel \mathrm{e}^{a/p+u}\psi^{(N-1)/p} \parallel_{L^p} r$$

where $C_3 = e^{-a(c)}\psi^{1-N}(c)\left[\int_0^R \mathrm{e}^{a(1-1/p)}\psi^{(N-1)(1-1/p)}\mathrm{d}r\right]^{p/(p-1)}$. □

Proof of Theorem 3.4 (Gelfand case). Take $p = 2t+1$. By Lemma 3.12 and using Lemma 3.7 it follows that
$$-u(r) \leq -u(0) + C_1C_3\left((2t)/(2t-t^2)\right)^{1/t} R^{1/p}r.$$

Multiplying some positive terms we have
$$\mathrm{e}^a\psi^{(N-1)}\mathrm{e}^{-u(2t+1)} \leq \mathrm{e}^a\psi^{(N-1)}\mathrm{e}^{(-u(0)+C_1C_3\left(4(2t)/(2t-t^2)\right)^{1/t}R^{1/p}r)(2t+1)}.$$

Thus,
$$\int_0^R \mathrm{e}^a\psi^{(N-1)}\mathrm{e}^{(u(0)-C_4\left(4(2t)/(2t-t^2)\right)^{1/t}R^{1/p}r)(2t+1)} \leq C_1\left(\frac{2t}{2t-t^2}\right)^{(2t+1)/t} R^{1/p},$$

where $C_4 := C_1C_3$. Calculating the left-hand side above, we have

$$\mathrm{e}^{\left(u(0)-C_4\left(4(2t)/(2t-t^2)\right)^{1/t}R^{1/p}r\right)(2t+1)}\int_0^R \mathrm{e}^a\psi^{(N-1)}\,\mathrm{d}r \le C_4\left(\frac{2t}{2t-t^2}\right)^{(2t+1)/t} R^{1/p}.$$

Taking the limit $\lambda \nearrow \lambda^*$ we have

$$\| u^* \|_\infty \le \frac{\ln\left(C_1R^{1/p}\left(\frac{2t}{2t-t^2}\right)^{(2t+1)/t}\right)}{(2t+1)} + C_4\left(4(2t)/(2t-t^2)\right)^{1/t} R^{1+1/p}.$$

□

Lemma 3.13. Let u a radially decreasing and semi-stable classical solution of (R_λ) with $f(u) = (1+u)^m$. If $1 \le p < \infty$, we have the estimate

$$u(0) \ge u(r) \ge u(0) - C_3 \parallel \mathrm{e}^{a/p}\psi^{(N-1)/p}(1+u)^m \parallel_{L^p} r.$$

Proof. There exists $c \in (0, r)$ such that

$$-u(r) + u(0) = -u'(c)r. \tag{28}$$

Integrating the equation (R_λ) from 0 to c we obtain

$$\begin{aligned} -\mathrm{e}^{a(c)}\psi^{N-1}(c)u'(c) &= \int_0^c \mathrm{e}^a\psi^{N-1}(1+u)^m \\ &\le \left[\int_0^R \mathrm{e}^a\psi^{(N-1)}(1+u)^{mp}\,\mathrm{d}r\right]^{1/p}\left[\int_0^R \mathrm{e}^{a(1-1/p)p'}\psi^{(N-1)(1-1/p)p'}\mathrm{d}r\right]^{1/p'}. \end{aligned}$$

Using (28) we obtain

$$-u(r) + u(0) \le C_3 \parallel \mathrm{e}^{a/p}\psi^{(N-1)/p}(1+u)^m \parallel_{L^p} r.$$

where $C_3 = e^{-a(c)}\psi^{1-N}(c)\left[\int_0^r e^{a(1-1/p)}\psi^{(N-1)(1-1/p)}\mathrm{d}r\right]^{p/(p-1)}$. □

Proof of Theorem 3.4 (Power case). Take $p = (2bt + m - 1)/m$. By Lemma 3.13 and using Lemma 3.8 it follows that

$$\begin{aligned} &\int_0^R \mathrm{e}^{a/p}\psi^{(N-1)/p}(1+u(0))^m\mathrm{d}r \le \\ &\qquad \int_0^R \mathrm{e}^{a/p}\psi^{(N-1)/p}\left(1+u(r)+C\left(1-\frac{b^2t^2}{m[2bt-1]}\right)^{-1/[tb]}R^{1/(2bt+b)}r\right)^m \mathrm{d}r. \end{aligned}$$

Thus, we have

$$\|u\|^m \le 2^mC\left(1-\frac{b^2t^2}{m[2bt-1]}\right)^{-\frac{1}{tb}}R^{\frac{1}{2bt+b}} + \frac{2^mC^m}{m+1}\left(1-\frac{b^2t^2}{m[2bt-1]}\right)^{-\frac{m}{tb}}R^{\frac{m}{2bt+b}}r^{m+1}.$$

Using the above inequality and taking the limit $\lambda \nearrow \lambda^*$, it follows that

$$\|u^*\|^m \le 2^mC\left(1-\frac{b^2t^2}{m[2bt-1]}\right)^{-\frac{1}{tb}}R^{\frac{1}{2bt+b}} + \frac{2^mC^m}{m+1}\left(1-\frac{b^2t^2}{m[2bt-1]}\right)^{-\frac{m}{tb}}R^{\frac{m}{2bt+b}}r^{m+1}.$$

□

3.6 Branch of minimal solutions

In this section we prove multiplicity of solutions near the extremal parameter and uniqueness on it for the problem

$$\begin{cases} -\Delta_g u + A(x)\cdot\nabla_g u = \lambda f(u) & \text{in} \quad \Omega, \\ u > 0 & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{P_λ}$$

Let u be a semi-stable solution of (P_λ), and let us consider the following eigenvalue problem involving the linearized operator $-\Delta_g + A\cdot\nabla_g - \lambda f'(u)$ at u,

$$\begin{cases} L_{u,\lambda}\phi = \mu\phi & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

It is well known that there exists a smallest positive eigenvalue μ, which we denote by $\mu_{1,\lambda}$, and an associated eigenfunction $\phi_{1,\lambda} > 0$ in Ω, and $\mu_{1,\lambda}$ is a simple eigenvalue and has the following variational characterization

$$\mu_{1,\lambda} = \inf\left\{ \langle L_{u,\lambda}\phi, \phi\rangle_{L^2(\Omega)} : \phi \in W_0^{1,2}(\Omega), \int_\Omega \phi^2 \, \mathrm{d}v_g = 1 \right\}.$$

We start with the following lemma:

Lemma 3.14. Let u and v be a weak solution and a weak supersolution, respectively, of (P_λ).

(i) If $\mu_1(\lambda, u) > 0$, then $u \le v$ a.e. in Ω.

(ii) If u is a regular solution of (P_λ) and $\mu_1(\lambda, u) = 0$, then $u = v$ a.e. in Ω.

Proof. Let $\vartheta \in [0,1]$ and $0 \le \phi \in W_0^{1,2}(\Omega)$. By convexity of $s \to f(s)$ we have

$$\begin{aligned} I_{\vartheta,\phi} := & \int_\Omega \left(\nabla_g(\vartheta u + (1-\vartheta)v)\cdot\nabla_g\phi + \phi A\cdot\nabla_g(\vartheta u + (1-\vartheta)v)\right) \mathrm{d}v_g \\ & - \int_\Omega \lambda f(\vartheta u + (1-\vartheta)v)\phi \, \mathrm{d}v_g \\ & \ge \lambda \int_\Omega \left(\vartheta f(u) + (1-\vartheta)f(v) - f(\vartheta u - (1-\vartheta)v)\right)\phi \, \mathrm{d}v_g \ge 0. \end{aligned}$$

Since $I_{1,\phi} = 0$, the derivative of $I_{\vartheta,\phi}$ at $\vartheta = 1$ is nonpositive. If $\mu_1(\lambda, u) > 0$ using the maximum principle (Theorem 3.3 in (34)) clearly $u \le v$. We shall prove that this holds true if $\mu_1(\lambda, u) \ge 0$. Indeed, we have

$$\int_\Omega \left(\nabla_g(u - v)\cdot\nabla_g\phi + \phi A\cdot\nabla_g(u-v)\right) \mathrm{d}v_g - \int_\Omega \lambda f'(u)(u-v)\phi \, \mathrm{d}v_g = 0. \tag{29}$$

Since $I_{\vartheta,\phi} \geq 0$ for any $\vartheta \in [0,1]$ and $I_{1,\phi} = \partial_\vartheta I_{1,\phi} = 0$, we have

$$\partial^2_{\vartheta\vartheta} I_{1,\phi} = -\int_\Omega \lambda f''(u)(u-v)^2 \phi \, \mathrm{d}v_g \geq 0.$$

Take $\phi = (u-v)^+$. We have $(u-v)^+ = 0$ in Ω and we get $\int_\Omega |\nabla_g (u-v)^+|^2 \, \mathrm{d}v_g = 0$. It follows that $u \leq v$ a.e. in Ω as claimed. Now, if $\mu_{1,\lambda}(u) = 0$ let $\psi_{1,\lambda}$ the first eigenfunction of $L_{u,\lambda}$. Observe that $\psi_{1,\lambda}$ is in the kernel of the linearized operator $L_{u,\lambda}$, and (29) is valid when we replace $u - v$ with $u - v - t\psi_{1,\lambda}$. We have

$$\begin{aligned}\int_\Omega \left(|\nabla_g (u-v-t\psi_{1,\lambda})^+|^2 + (u-v-t\psi_{1,\lambda})^+ A \cdot \nabla_g (u-v-t\psi_{1,\lambda})^+ \right) \mathrm{d}v_g \\ - \int_\Omega \lambda f(u)((u-v-t\psi_{1,\lambda})^+)^2 \, \mathrm{d}v_g = 0.\end{aligned}$$

We claim that if $u < v - \bar{t}\psi_{1,\lambda}$ on a set Ω' of positive measure, then there exists $\varepsilon > 0$ such that $u < v - t\psi_{1,\lambda}$ a.e. in Ω for any $\bar{t} \leq t < \bar{t} + \varepsilon$. Since we have a variational characterization of $\psi_{1,\lambda}$ we get that $(u-v-t\psi_{1,\lambda})^+ = \beta\psi_{1,\lambda}$ a.e. in Ω for some $\beta \in \mathbb{R}$. We can find, by assumption, a set $\Omega' \subset \Omega$ of positive measure such that $u < v - \bar{t}\psi_{1,\lambda} - \delta$ for $\delta > 0$ and consequently, for some $\varepsilon > 0$ sufficient small that $u < v - t\psi_{1,\lambda}$ in Ω' for any $\bar{t} \leq t \leq \bar{t} + \varepsilon$. Hence $\beta\psi_{1,\lambda} = 0$ a.e. in Ω'. Since $\psi_{1,\lambda} > 0$ in Ω we have $\beta = 0$ and $u < v + t\psi_{1,\lambda}$ a.e. in Ω for any $\bar{t} \leq t \leq \bar{t} + \varepsilon$ and this finishes the proof of claim. Now, by contradiction, assume that u is not equal to v a.e. in Ω. Since $u \leq v$, we find a set Ω' of positive measure so that $u < v$ in Ω'. Applying the above claim with $\bar{t} = 0$ we get some $\varepsilon > 0$, $u < v - t\psi_{1,\lambda}$ a.e. in Ω for any $0 \leq t < \varepsilon$. Set now $t_0 = \sup\{t > 0 : u < v - t\psi_{1,\lambda} \text{ a.e. in } \Omega\}$. Clearly, $u \leq v - t_0\psi_{1,\lambda}$ a.e. in Ω. The claim and maximal property of t_0 imply that necessarily $u = v - t_0\psi_{1,\lambda}$ a.e. in Ω since (29) holds for any $0 \leq \phi \in W_0^{1,2}(\Omega)$. Taking $\phi = v - u$ and arguing as before we have $\int_\Omega |\nabla_g (u-v)|^2 \, \mathrm{d}v_g = 0$ contradicting the assumption that $u < v$ on a set of positive measure. $\square$

Theorem 3.5. For dimension $1 \leq N < N^*$, u^* is the unique classical solution of (P_{λ^*}) among all weak solutions.

Proof of Theorem 3.5. Using Theorem 3.2, we have that u^* exists as a classical solution. On the other hand, we have that $\mu_{1,\lambda^*} \geq 0$. If we suppose that $\mu_{1,\lambda^*} > 0$, then the Implicit Function Theorem could be applied to the operator L_{u^*,λ^*} to allow for the continuation of the minimal branch $\lambda \nearrow \underline{u}_\lambda$ beyond λ^*, which is a contradiction. Therefore $\mu_{1,\lambda^*} = 0$. The uniqueness of u^* in the class of weak solutions follows from the Lemma 3.14. $\square$

Proposition 3.4. If $0 < \lambda < \lambda^*$, the minimal solutions are stable.

Proof. Define
$$\lambda^{**} = \sup\{\lambda > 0 : \underline{u}_\lambda \text{ is a stable solution for } (P_\lambda)\}.$$
Obviously λ^{**} satisfies $\lambda^{**} \leq \lambda^*$. If $\lambda^{**} < \lambda^*$, then $\underline{u}_{\lambda^{**}}$ is a minimal solution of $(P_{\lambda^{**}})$. For $\lambda \leq \lambda^{**}$, we have that $\lim_{\lambda \nearrow \lambda^{**}} \underline{u}_\lambda \leq \underline{u}_{\lambda^{**}}$. Since u^{**} is solution of $(P_{\lambda^{**}})$ and by minimality follows that $\lim_{\lambda \nearrow \lambda^{**}} \underline{u}_\lambda = \underline{u}_{\lambda^{**}}$ and $\mu_{1,\lambda^{**}} \geq 0$. If we suppose that $\mu_{1,\lambda^{**}} = 0$, we get that $\underline{u}_{\lambda^{**}} = \underline{u}_\lambda$ for any $\lambda^{**} < \lambda < \lambda^*$. But this is a contradiction, which proves that $\lambda^{**} = \lambda^*$. □

Proposition 3.5. For each $x \in \Omega$, the function $\lambda \to \underline{u}_\lambda(x)$ is differentiable and strictly increasing on $(0, \lambda^*)$.

Proof. Since $\underline{u}_\lambda$ is stable, the linearized operator $L_{\underline{u}_\lambda,\lambda}$ at u_λ is invertible for any $0 < \lambda < \lambda^*$. By the Implicit Function Theorem $\lambda \to \underline{u}_\lambda(x)$ is differentiable in λ. By monotonicity, $\frac{\mathrm{d}\underline{u}_\lambda}{\mathrm{d}\lambda}(x) \geq 0$ for all $x \in \Omega$. Finally, by differentiating (P_λ) with respect to λ we get that $\frac{\mathrm{d}\underline{u}_\lambda}{\mathrm{d}\lambda}(x) > 0$, for all $x \in \Omega$. □

It is standard to show the existence of a second branch of solutions near λ^* (18). We make use of Mountain Pass Theorem to provide a variational characterization for this solutions. To apply the Mountain Pass Theorem we will need to truncate the singular nonlinearity into a subcritical case, that is, we consider a regularized C^1 nonlinearity $g_\varepsilon(u)$, $0 < \varepsilon < 1$ of the following form for MEMS case
$$g_\varepsilon(u) = \begin{cases} \dfrac{1}{(1-u)^2} & \text{if } u < 1-\varepsilon \\ \dfrac{1}{\varepsilon^2} - \dfrac{2(1-\varepsilon)}{p\varepsilon^3} + \dfrac{2u^p}{p\varepsilon^3(1-\varepsilon)^{p-1}} & \text{if } u \geq 1-\varepsilon \end{cases} \tag{30}$$
and for Gelfand or Power-type
$$g_\varepsilon(u) = \begin{cases} f(u) & \text{if } u < t_0-\varepsilon \\ f(s_0-\varepsilon) - \dfrac{f'(s_0-\varepsilon)(s_0-\varepsilon)}{p} + \dfrac{f'(s_0-\varepsilon)u^p}{p(s_0-\varepsilon)^{p-1}} & \text{if } u \geq t_0-\varepsilon \end{cases} \tag{31}$$
where $p > 1$ if $N = 1, 2$ and $1 < p < (N+2)/(N-2)$ if $3 \leq N \leq N^*$. For $\lambda \in (0, \lambda^*)$ and $A = \nabla_g a$, we associate the elliptic problem
$$\begin{cases} -\mathrm{div}\left(\mathrm{e}^{-a}\nabla_g u\right) = \lambda \mathrm{e}^{-a} g_\varepsilon(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{S_λ}$$

We can define a energy functional on $W_0^{1,2}(\Omega)$ associated to (S_λ) given by

$$J_{\varepsilon,\lambda}(u) = \frac{1}{2}\int_\Omega \mathrm{e}^{-a}|\nabla_g u|^2 \,\mathrm{d}v_g - \lambda \int_\Omega \mathrm{e}^{-a} G_\varepsilon(u)\,\mathrm{d}v_g,$$

where $G_\varepsilon(u) = \int_{-\infty}^{u} g_\varepsilon(s)\,\mathrm{d}s$. We can fix $0 < \varepsilon < \frac{1-\|u^*\|_\infty}{2}$ for MEMS case or $0 < \varepsilon < \frac{t_0-\|u^*\|_\infty}{2}$ for Gelfand and Power-type, and observe that for λ close enough to λ^*, the minimal solution $\underline{u}_\lambda$ of (P_λ) is also a solution of (S_λ) that satisfies $\mu_{1,\lambda}(-\mathrm{div}(\mathrm{e}^{-a}\nabla_g) - \lambda g'_\varepsilon(\underline{u}_\lambda)) > 0$.

Lemma 3.15. If $1 \le N < N^*$ and if λ is close enough to λ^*, then the minimal solution $\underline{u}_\lambda$ of (S_λ) is a strict local minimum of $J_{\varepsilon,\lambda}$ on $W_0^{1,2}(\Omega)$.

Proof. We first show that the minimal solution $\underline{u}_\lambda$ is a local minimum in $C^1(\overline{\Omega})$. Indeed, since $\mu_{1,\lambda}((-\mathrm{div}(\mathrm{e}^{-a}\nabla_g) - \lambda g'_\varepsilon(\underline{u}_\lambda)) > 0$ and $\underline{u}_\lambda < 1-\varepsilon$, we have the inequality

$$\int_\Omega \mathrm{e}^{-a}|\nabla_g \phi|^2\,\mathrm{d}v_g - 2\lambda \int_\Omega \frac{\mathrm{e}^{-a}\phi^2}{(1-\underline{u}_\lambda)^3}\,\mathrm{d}v_g \ge \mu_{1,\lambda}\int_\Omega \phi^2\,\mathrm{d}v_g,$$

for any $\phi \in W_0^{1,2}(\Omega)$. Now take $\phi \in W_0^{1,2}(\Omega) \cap C^1(\overline{\Omega})$ satisfying $\underline{u}_\lambda + \phi \le 1-\varepsilon$ and such that $\|\phi\|_{C^1} \le \delta_\lambda$. Thus we have

$$\begin{aligned}
&J_{\varepsilon,\lambda}(\underline{u}_\lambda+\phi) - J_{\varepsilon,\lambda}(\underline{u}_\lambda) = \\
&\quad \frac{1}{2}\int_\Omega \mathrm{e}^{-a}|\nabla_g\phi|^2\,\mathrm{d}v_g + \int_\Omega \mathrm{e}^{-a}\nabla_g\underline{u}_\lambda\cdot\nabla_g\phi\,\mathrm{d}v_g - \lambda\int_\Omega \mathrm{e}^{-a}\left(\frac{1}{1-\underline{u}_\lambda-\phi} - \frac{1}{1-\underline{u}_\lambda}\right)\mathrm{d}v_g \\
&\ge \frac{\mu_{1,\lambda}}{2}\int_\Omega \phi^2\,\mathrm{d}v_g - \lambda\|\mathrm{e}^{-a}\|_\infty \int_\Omega \left(\frac{1}{1-\underline{u}_\lambda-\phi} - \frac{1}{1-\underline{u}_\lambda} - \frac{\phi}{(1-\underline{u}_\lambda)^2} - \frac{\phi^2}{(1-\underline{u}_\lambda)^3}\right)\mathrm{d}v_g.
\end{aligned}$$

For some $C > 0$ we have

$$\left|\frac{1}{1-\underline{u}_\lambda-\phi} - \frac{1}{1-\underline{u}_\lambda} - \frac{\phi}{(1-\underline{u}_\lambda)^2} - \frac{\phi^2}{(1-\underline{u}_\lambda)^3}\right| \le C|\phi|^3$$

and this implies

$$J_{\varepsilon,\lambda}(u_\lambda+\phi) - J_{\varepsilon,\lambda}(u_\lambda) \ge \left(\frac{\mu_{1,\lambda}}{2} - C\lambda\|\mathrm{e}^{-a}\|_\infty\delta_\lambda\right)\int_\Omega \phi^2\,\mathrm{d}v_g > 0$$

provided δ_λ is small enough. This proves that $\underline{u}_\lambda$ is a local minimum of $J_{\varepsilon,\lambda}$ in the C^1 topology. We can apply Theorem 2.1 of (53) and get that u_λ is a local minimum of $J_{\varepsilon,\lambda}$ in $W_0^{1,2}(\Omega)$. For Gelfand and Power cases we take $\phi \in W_0^{1,2}(\Omega) \cap C^1(\overline{\Omega})$ such that $\underline{u}_\lambda + \phi \le t_0 - \varepsilon$ and $\|\phi\|_{C^1} \le \delta_\lambda$. With similar arguments we conclude that u_λ is a local minimum of $J_{\varepsilon,\lambda}$ in $W_0^{1,2}(\Omega)$. ∎

Now we proof the existence of a second solution for (S_λ). We need a version of mountain pass theorem (54).

Theorem 3.6 (Critical point of Mountain pass type)**.** Let J be a C^1 functional defined on a Banach space E that satisfies the Palais-Smale condition, that is, any sequence in E such that $(J(u_n))_n$ is bounded and $J'(u_n) \to 0$ in E^* is relatively compact in E. Assume the following conditions:

(i) There exists a neighborhood B of some u in E and a constant $\sigma > 0$ such that

$$J(v) \geq J(u) + \sigma \quad \text{for all } v \in \partial B.$$

(ii) There exists $w \notin B$ such that $J(w) \leq J(u)$.

Defining

$$\Gamma = \{y \in C([0,1], E) : \gamma(0) = u, \gamma(1) = w\}$$

then there exists $u \in E$ such that $J'(u) = 0$ and $J(u) = c$, where

$$c = \inf_{\gamma \in \Gamma} \max_{0 \leq t \leq 1} \{J(\gamma(t)) : t \in (0,1)\}.$$

Lemma 3.16. Assume that $\{w_n\} \subset W_0^{1,2}(\Omega)$ satisfies

$$J_{\varepsilon,\lambda_n}(w_n) \leq C, \quad J'_{\varepsilon,\lambda_n} \to 0 \text{ in } W_0^{-1,2}(\Omega),$$

for $\lambda_n \to \lambda > 0$. The sequence (w_n) then admits a convergent subsequence in $W_0^{1,2}(\Omega)$.

Proof. By (3.16) we have as $n \to +\infty$

$$\int_\Omega \mathrm{e}^{-a} |\nabla_g w_n|^2 \, \mathrm{d}v_g - \lambda_n \int_\Omega \mathrm{e}^{-a} g_\varepsilon(w_n) w_n \, \mathrm{d}v_g = o(\|w_n\|_{W_0^{1,2}}).$$

We have the inequality

$$\vartheta G_\varepsilon(u) \leq u g_\varepsilon(u) \quad \text{for } u \geq M_\varepsilon$$

for some $M_\varepsilon > 0$ large and $\vartheta > 2$. We obtain

$$\begin{aligned}
C &\geq \frac{1}{2} \int_\Omega \mathrm{e}^{-a} |\nabla_g w_n|^2 \, \mathrm{d}v_g - \lambda_n \int_\Omega \mathrm{e}^{-a} G_\varepsilon(w_n) \, \mathrm{d}v_g \\
&= \left(\frac{1}{2} - \frac{1}{\vartheta}\right) \int_\Omega \mathrm{e}^{-a} |\nabla_g w_n|^2 \, \mathrm{d}v_g + \lambda_n \int_\Omega \mathrm{e}^{-a} \left(\frac{1}{\vartheta} w_n g_\varepsilon(w_n) - G_\varepsilon(w_n)\right) \mathrm{d}v_g \\
&+ o\left(\|w_n\|_{W_0^{1,2}(\Omega)}\right) \\
&\geq \left(\frac{1}{2} - \frac{1}{\vartheta}\right) \int_\Omega \mathrm{e}^{-a} |\nabla_g w_n|^2 \, \mathrm{d}v_g + o\left(\|w_n\|_{W_0^{1,2}(\Omega)}\right) - C_\varepsilon.
\end{aligned}$$

It follows that $\sup_{n\in\mathbb{N}} \|w_n\|_{W_0^{1,2}(\Omega)} < +\infty$. We have the compactness of embedding $W_0^{1,2}(\Omega) \hookrightarrow L^{p+1}(\Omega)$ and thus, up to a subsequence, $w_n \rightharpoonup w$ weakly in $W_0^{1,2}(\Omega)$ and strongly in $L^{p+1}(\Omega)$ for some $w \in W_0^{1,2}(\Omega)$. It follows that

$$\int_\Omega \mathrm{e}^{-a}|\nabla_g w|^2 \, \mathrm{d}v_g = \lambda \int_\Omega g_\varepsilon(w) w \, \mathrm{d}v_g$$

and we deduce that

$$\begin{aligned}\int_\Omega \mathrm{e}^{-a}|\nabla_g (w_n - w)|^2 \, \mathrm{d}v_g &= \int_\Omega \mathrm{e}^{-a}|\nabla_g w_n|^2 \, \mathrm{d}v_g - \int_\Omega \mathrm{e}^{-a}|\nabla_g w|^2 \, \mathrm{d}v_g + o(1) \\ &= \lambda_n \int_\Omega g_\varepsilon(w_n) w_n \, \mathrm{d}v_g - \lambda \int_\Omega g_\varepsilon(w) w \, \mathrm{d}v_g + o(1) \to 0.\end{aligned}$$

as $n \to +\infty$, and the lemma is proved. □

Theorem 3.7. Let $1 \leq N < N^*$ and $A = \nabla_g a$. There exists $\delta > 0$ such that for any $\lambda \in (\lambda^* - \delta, \lambda^*)$ we have a second branch of solutions U_λ given by mountain pass for $J_{\varepsilon,\lambda}$ on $W_0^{1,2}(\Omega)$.

Proof of Theorem 3.7. We first show that $J_{\varepsilon,\lambda}$ has a mountain pass geometry in $W_0^{1,2}(\Omega)$. Since $\underline{u}_\lambda$ is a local minimum for $J_{\varepsilon,\lambda}$ for $\lambda \nearrow \lambda^*$, condition (i) of Theorem 3.6 is satisfied. Consider $r > 0$ such that $B_{2r} \subset \Omega$ and a cutoff function χ so that $\chi = 1$ on B_r and $\chi = 0$ outside B_{2r}. Let $w_\varepsilon = (1-\varepsilon)\chi \in W_0^{1,2}(\Omega)$. In MEMS case, we have

$$J_{\varepsilon,\lambda}(w_\varepsilon) \leq \frac{(1-\varepsilon)^2}{2} \int_\Omega \mathrm{e}^{-a}|\nabla\chi|^2 \, \mathrm{d}v_g - \frac{\lambda}{\varepsilon^2} \int_{B_r} \mathrm{e}^{-a} \, \mathrm{d}v_g \to -\infty$$

as $\varepsilon \to 0$ and uniformly for λ bounded away from 0. With a similar argument we can prove the same result for Gelfand and Power cases. Thus we have

$$J_{\varepsilon,\lambda}(\underline{u}_\lambda) \to J_{\varepsilon,\lambda^*}(u_{\lambda^*}) \text{ as } \lambda \to \lambda^*.$$

We get for $\varepsilon > 0$ sufficiently small that

$$J_{\varepsilon,\lambda}(w_\varepsilon) < J_{\varepsilon,\lambda}(\underline{u}_\lambda)$$

holds for λ close to λ^*. It follows by Lemma 3.16 that the functional $J_{\varepsilon,\lambda}$ satisfies the Palais-Smale condition on $W_0^{1,2}(\Omega)$. We fix $\varepsilon > 0$ small enough and for λ close to λ^* we define

$$c_{\varepsilon,\lambda} = \inf_{\gamma\in\Gamma} \max_{u\in\gamma} J_{\varepsilon,\lambda}(u).$$

We can use the mountain pass theorem to get a solution $U_{\varepsilon,\lambda}$ of (S_λ) for λ close to λ^*. A similar proof as in Lemma 3.14 shows that the convexity of g_ε ensures that problem

(S_λ) has a unique solution at $\lambda = \lambda^*$, which is u^*. By elliptic regularity theory we get that $U_{\varepsilon,\lambda} \to u^*$ uniformly in $C(\overline{\Omega})$. Thus $U_{\varepsilon,\lambda} \leq t_0 - \varepsilon$ for λ close to λ^*. Therefore, $U_{\varepsilon,\lambda}$ is a second solution for (P_λ) bifurcating from u^*, that we denote by U_λ. Since U_λ is a mountain pass solution, U_λ is not a minimal solution. Thus U_λ is unstable solution of (P_λ). □

4 Quasilinear problems on manifolds

In this chapter we consider the following reaction-diffusion equation involving the p-Laplace Beltrami operator on Riemannian manifolds,

$$-\operatorname{div}(|\nabla_g u|^{p-2}\nabla_g u) = h(u) \quad \text{in} \quad \mathcal{B}_1 \setminus \{\mathcal{O}\} \tag{32}$$

where $1 < p < +\infty$, $\mathcal{B}_1$ is a geodesic ball of a Riemannian model $\mathcal{M}$ with radius 1 and h is a locally Lipschitz positive nonlinearity. In this way, we establish L^∞ and $W^{1,p}$ estimates for semi-stable, radially symmetric, and decreasing solutions of (32). Our results do not depend on the specific form of the nonlinearity, precisely, our L^∞ and $W^{1,p}$ estimates hold for every positive locally Lipschitz nonlinearity h. This may be regarded as a result on removable singularities because u may be unbounded at the pole $\mathcal{O}$.

As an application of our estimates, we prove regularity results for the following quasilinear elliptic problem with Dirichlet boundary condition

$$\begin{cases} -\operatorname{div}(|\nabla_g u|^{p-2}\nabla_g u) = \lambda f(u) & \text{in} \quad \mathcal{B}_1, \\ u > 0 & \text{in} \quad \mathcal{B}_1, \\ u = 0 & \text{on} \quad \partial\mathcal{B}_1, \end{cases} \tag{Q_λ}$$

where $\lambda > 0$ and f is an increasing C^1 function with $f(0) > 0$ and

$$\lim_{t\to+\infty} \frac{f(t)}{t^{p-1}} = +\infty. \tag{33}$$

The study of the above nonlinear problem (Q_λ) requires to extend the classical results of Crandall, Rabinowitz, Brezis, et. al. (2, 5) for the Euclidean setting to the general case of Riemannian manifolds, precisely, since we have a comparison principle for $-\Delta_p$ (because it is uniformly elliptic) and the first eigenvalue (as well the corresponding eigenfunction) of $-\Delta_p$ on Ω is positive, it is standard to prove that there exists a parameter $\lambda^* \in (0, +\infty)$ such that if $0 < \lambda < \lambda^*$ then (Q_λ) admits a minimal smooth solution u_λ and for $\lambda > \lambda^*$ problem (Q_λ) admits no solution. Moreover, for every $0 < \lambda < \lambda^*$ the minimal solution u_λ is semi-stable and we can define the limit

$$u^* = \lim_{\lambda \nearrow \lambda^*} u_\lambda.$$

When we can establish that u^* is a weak solution of (P_{λ^*}), it is called the extremal solution.

Hereafter we will consider the problem (32) posed in $\mathcal{B}_1$ with radius 1 of a Riemannian model, precisely, a manifold $\mathcal{M}$ of dimension $N \geq 2$ admitting a pole $\mathcal{O}$ and whose metric g is given, in polar coordinates around $\mathcal{O}$, by

$$\mathrm{d}s^2 = \mathrm{d}r^2 + \psi(r)^2 \mathrm{d}\vartheta^2 \quad \text{for } r \in (0, R) \text{ and } \vartheta \in \mathbb{S}^{N-1},$$

where r is by construction the Riemannian distance between the point $P = (r, \vartheta)$ to the pole $\mathcal{O}$, ψ is a smooth positive function in $(0, R)$ and $\mathrm{d}\vartheta^2$ is the canonical metric on the unit sphere $\mathbb{S}^{N-1}$. We need to assume some hypotheses on the Riemannian metric g to obtain some results. Suppose that ψ satisfies

$$0 < \tau := \inf_{[0,1]} \frac{-\psi''\psi + (\psi')^2}{(\psi')^2}. \tag{H_1}$$

and

$$\text{there exists } \overline{\psi} \in L^\infty(0,1) \text{ such that } \psi(r) \leq r\overline{\psi}(r) \text{ a.e. in } (0,1). \tag{H_2}$$

Observe that hypotheses (H_1) and (H_2) holds for the space forms (see Remark 4.1).

As discussed in (55, 56), we need to assume

$$\psi(0) = \phi''(0) = 0 \text{ and } \psi'(0) = 1. \tag{H_3}$$

These assumptions are sufficient to extend in a C^2 manner the metric $\mathrm{d}s^2$ to the whole $\mathbb{R}^N$. An important consequence of (H_3) is that on geodesic balls of $\mathcal{M}$ the p-Laplace Beltrani operator Δ_p is uniformly elliptic.

Remark 4.1. Note that our results apply to the important case of space forms, i.e., the unique complete and simply connected Riemannian manifold of constant sectional curvature K_ψ corresponding to the choice of ψ namely,

Space form	$\psi(r)$	K_ψ	τ
Hyperbolic space $\mathbb{H}^N$	$\sinh r$	-1	$\inf_{[0,1]} \frac{1}{\cosh^2 r}$
Euclidean space $\mathbb{R}^N$	r	0	1
Elliptic space $\mathbb{S}^N$	$\sin r$	1	$\inf_{[0,1]} \frac{1}{\cos^2 r}$

4.1 Key-ingredients

We recall some general facts for the p-Laplace Beltrami operator that are extensions, to a quasilinear setting, of some results of spectral theory. The reader may consult (39, 40, 41, 42, 43, 57, 58, 59). The basic technical material that is necessary for our purpose about regularity is summarized in the following:

Theorem 4.1. Let $\Omega \subset \mathcal{M}$ be a relatively compact open domain with $C^{1,\alpha}$ boundary for some $0 < \alpha < 1$. Let $1 < p < +\infty$, $g \in L^\infty(\Omega)$, $\xi \in C^{1,\alpha}(\partial\Omega)$ and suppose that $u \in W^{1,p}(\Omega)$ is a solution of

$$\begin{cases} -\Delta_p u = g & \text{in} \quad \Omega, \\ u = \xi & \text{on} \quad \partial\Omega. \end{cases} \tag{34}$$

Then

Boundedness: We have $u \in L^\infty_{\mathrm{loc}}(\Omega)$ and for any relatively compact open domains $\Omega' \Subset \Omega'' \Subset \Omega$ there exists a positive constant $C = C(p, g, N, \xi, \Omega, \|u\|_{L^p(\Omega'')})$ such that

$$\|u\|_{L^\infty(\Omega')} \leq C.$$

If $\xi \in C^{2,\alpha}(\partial\Omega)$, C can be chosen globally on Ω, and thus $u \in L^\infty(\Omega)$.

$C^{1,\beta}$-regularity: When $u \in L^\infty(\Omega)$ there exists $\beta \in (0,1)$ depending on p, N, g, α and on upper bounds for $\|u\|_{L^\infty}$, $\|g\|_{L^\infty}$, $\|\xi\|_{C^{1,\alpha}}$ on Ω such that

$$\|u\|_{C^{1,\beta}(\overline{\Omega})} \leq C$$

for some constant C depending on α, p, the geometry of Ω and upper bounds for $\|u\|_{L^\infty}$, $\|g\|_{L^\infty}$, $\|\xi\|_{C^{1,\alpha}}$ on Ω.

Harnack inequality: For any relatively compact open sets $\Omega' \Subset \Omega'' \Subset \Omega$ there exists $C = C(p, N, \Omega', \Omega'') > 0$ such that $u \in W^{1,p}(\Omega)$ nonnegative solution of $-\Delta_p u = 0$ on Ω,

$$\sup_{\Omega'} u \leq C \inf_{\Omega''} u.$$

In particular, either $u > 0$ on Ω or $u \equiv 0$ on Ω.

Hopf lemma: Suppose that $\xi \geq 0$, $g \geq 0$ and let $u \in C^1(\overline{\Omega})$ be a solution of (34) with $u \geq 0$, $u \neq 0$. If $x \in \partial\Omega$ is such that $u(x) = \xi(x) = 0$ then we have $\langle \nabla_g u, \nu \rangle > 0$, where ν is the inward unit normal vector to $\partial\Omega$ at x.

The local boundedness of u is a particular case of Serrin's theorem (44) and does not need the boundary condition. The global boundedness can be reached via reflection technique described at (39), check also (45, 46), when $\xi \in C^{2,\alpha}(\partial\Omega)$. The $C^{1,\beta}$-regularity is a global version (60), of a local regularity result in (46) and (47). The Harnack inequality is due to J. Serrin (44). The Hopf lemma can be found in (48).

Remark 4.2. We do not assume the Dirichlet boundary condition or any other boundary condition to obtain our L^∞ and $W^{1,p}$ estimates for semi-stable, radially symmetric, and decreasing solutions of (32). Since $u \in W^{1,p}$ is radially decreasing, we can use the Sobolev embedding to obtain $u \in L^\infty_{loc}(B_1 \setminus \mathcal{O})$. Using know regularity results for degenerate elliptic equations (60), we have that $u \in C^{1,\alpha}_{loc}(\overline{B_1} \setminus \mathcal{O})$ for some $\alpha \in (0,1)$.

4.2 Estimates for semi-stable solutions

In this section we prove the principal estimate (see Lemma 4.1), which was already behind of the regularity of the semi-stable solutions in Theorem 4.3. Before we state our main results we recall some standard notation and definitions related with problem (32).

Definition 4.1. We say that $u \in W^{1,p}(\mathcal{B}_1)$ is radially symmetric and decreasing when $u(x) = u(r)$ and $u_r(r) < 0$ for all $r \in (0,1)$ where $r = \mathrm{dist}(x, \mathcal{O})$ and u_r denotes the radial derivative.

Consider the energy functional

$$J(u) := \frac{1}{p}\int_{\mathcal{B}_1} |\nabla_g u|^p \, \mathrm{d}v_g - \int_{\mathcal{B}_1} F(u)\, \mathrm{d}v_g.$$

Definition 4.2. Let $u \in W^{1,p}(\mathcal{B}_1)$ be a radially symmetric solution in $\mathcal{B}_1$ of (32) with $u_r(r) < 0$ for all $r \in (0,1)$. We say that u is semi-stable when the second variation of energy functional at u satisfies

$$Q(\xi) := \int_{\mathcal{B}_1} (p-1)|u_r|^{p-2}|\xi_r|^2 - h'(u)\xi^2 \, \mathrm{d}v_g \geq 0 \tag{35}$$

for every radially symmetric function $\xi \in C^1_c(\mathcal{B}_1 \setminus \mathcal{O})$.

We can write the problem (32) for radial solutions $u \in W^{1,p}(\mathcal{B}_1)$ as

$$-\frac{1}{\psi^{N-1}}\frac{\partial}{\partial r}\left(\psi^{N-1}|u_r|^{p-2}u_r\right) = h(u) \quad \text{ for } r \in (0,1).$$

The above expression is equivalent to

$$-(p-1)|u_r|^{p-2}u_{rr}-\frac{(N-1)\psi'}{\psi}|u_r|^{p-2}u_r=h(u)\quad \text{with } r\in(0,1). \tag{36}$$

In the next lemma, we prove that for a suitable choice of test functions we can rewrite the semistability condition without the term $h'(u)$. This is the reason why our main theorems do not depend on nonlinearity f.

Lemma 4.1. Let $u\in W^{1,p}(\mathcal{B}_1)$ be a radial solution in $\mathcal{B}_1\setminus\mathcal{O}$ of (32) with $u_r(r)<0$ for all $r\in(0,1)$ and Q be the quadratic form defined in (35). Then, taking $\xi=u_r\eta$ in (35) we have

$$Q(u_r\eta)=\int_{\mathcal{B}_1}|u_r|^p\left[(p-1)|\eta_r|^2+\frac{\partial}{\partial r}\left(\frac{(N-1)\psi'}{\psi}\right)\eta^2\right]\mathrm{d}v_g.$$

Proof. Let $\eta\in C_c^1(\mathcal{B}_1\setminus\mathcal{O})$ be a radial function with compact support in $\mathcal{B}_1\setminus\mathcal{O}$ and $\zeta\in C^1(\mathcal{B}_1\setminus\mathcal{O})$ be a radial function. We can take $\xi=\zeta\eta\in C_c^1(\mathcal{B}_1\setminus\mathcal{O})$ in (35) to obtain

$$\begin{aligned}Q(\zeta\eta)&=\int_{\mathcal{B}_1}(p-1)|u_r|^{p-2}|\nabla_g(\zeta\eta)|^2-h'(u)\zeta^2\eta^2\,\mathrm{d}v_g\\&=\int_{\mathcal{B}_1}(p-1)|u_r|^{p-2}\zeta^2|\nabla_g\eta|^2+(p-1)|u_r|^{p-2}\nabla_g(\eta^2\zeta)\nabla_g\zeta-h'(u)\zeta^2\eta^2\,\mathrm{d}v_g.\end{aligned}\tag{37}$$

Now we multiplying (36) by $(\eta^2u_r\psi^{N-1})_r$, integrating and using integration by parts, it follows that

$$\begin{aligned}0&=\int_0^1(p-1)|u_r|^{p-2}u_{rr}\left(\eta^2u_r\psi^{N-1}\right)_r\,\mathrm{d}v_g\\&+\int_0^1\left[\frac{(N-1)\psi'}{\psi}|u_r|^{p-2}u_r+h(u)\right]\left(\eta^2u_r\psi^{N-1}\right)_r\,\mathrm{d}v_g\\&=\int_0^1(p-1)|u_r|^{p-2}u_{rr}(\eta^2u_r\psi^{N-1})_r\,\mathrm{d}v_g\\&-\int_0^1\left[\frac{(N-1)\psi'}{\psi}|u_r|^{p-2}u_r+h(u)\right]_r\eta^2u_r\psi^{N-1}\,\mathrm{d}v_g\end{aligned}$$

Observe that $\partial_r(|u_r|^{p-2}u_r)=(p-1)|u_r|^{p-2}u_{rr}$. Using this fact

$$\begin{aligned}0=&\int_0^1(p-1)|u_r|^{p-2}u_{rr}\partial_r(\eta^2u_r\psi^{N-1})\,\mathrm{d}r-\int_0^1\partial_r\left(\frac{(N-1)\psi'}{\psi}\right)|u_r|^{p-2}u_r\eta^2u_r\psi^{N-1}\,\mathrm{d}r\\&-\int_0^1\frac{(N-1)\psi'}{\psi}(p-1)|u_r|^{p-2}u_{rr}\eta^2u_r\psi^{N-1}\,\mathrm{d}r-\int_0^1h'(u)u_r\eta^2u_r\psi^{N-1}.\end{aligned}$$

Thus,

$$\begin{aligned}0=&\int_0^1(p-1)|u_r|^{p-2}u_{rr}\partial_r\left(\eta^2u_r\right)\psi^{N-1}\,\mathrm{d}r-\int_0^1\partial_r\left(\frac{(N-1)\psi'}{\psi}\right)|u_r|^{p-2}u_r^2\eta^2\psi^{N-1}\,\mathrm{d}r\\&-\int_0^1h'(u)\eta^2u_r^2\psi^{N-1}\,\mathrm{d}r.\end{aligned}$$

It follows that

$$\int_{\mathcal{B}_1} \partial_r \left(\frac{(N-1)\psi'}{\psi} \right) |u_r|^p \eta^2 \, \mathrm{d}v_g = \int_{\mathcal{B}_1} (p-1)|u_r|^{p-2} u_{rr} \partial_r(\eta^2 u_r) \, \mathrm{d}v_g - \int_{\mathcal{B}_1} h'(u) \eta^2 u_r^2 \, \mathrm{d}v_g. \tag{38}$$

Take $\zeta = u_r$ in (37) and using (38) we have

$$\begin{aligned} Q(u_r\eta) &= \int_{\mathcal{B}_1} (p-1)|u_r|^{p-2} u_{rr} u_r^2 |\nabla_g \eta|^2 \, \mathrm{d}v_g \\ &+ \int_{\mathcal{B}_1} (p-1)|u_r|^{p-2} u_{rr} \nabla_g(\eta^2 u_r) \nabla_g u_r - h'(u) u_r^2 \eta^2 \, \mathrm{d}v_g \\ &= \int_{\mathcal{B}_1} (p-1)|u_r|^p u_{rr} |\nabla_g \eta|^2 \, \mathrm{d}v_g + \int_{\mathcal{B}_1} \frac{\partial}{\partial r} \left(\frac{(N-1)\psi'}{\psi} \right) |u_r|^p \eta^2 \, \mathrm{d}v_g \\ &= \int_{\mathcal{B}_1} |u_r|^p \left[(p-1)|\eta_r|^2 + \frac{\partial}{\partial r} \left(\frac{(N-1)\psi'}{\psi} \eta^2 \right) \right] \mathrm{d}v_g. \end{aligned}$$

□

Using Lemma 4.1 and the semistability assumption, we can establish the following result. It is an L^p estimate for u_r with a weight in $\mathcal{B}_1$, for a certain positive exponent α in terms of $W^{1,p}$ norm of u. As said before, this is the key estimate behind the proof of Theorem 4.2. Note that when $N < p$ we have $W^{1,p}(\mathcal{B}_1) \subset L^\infty(\mathcal{B}_1)$ and hence solutions are bounded.

Lemma 4.2. Assume that $(\mathcal{M}, g)$ satisfies (H_1), $N \geq p$ and let $u \in W^{1,p}(\mathcal{B}_1)$ be a semi-stable radial solution in $\mathcal{B}_1 \setminus \mathcal{O}$ of (32) satisfying $u_r(r) < 0$ for $r \in (0,1)$. Let α such that

$$1 \leq \alpha < 1 + \sqrt{\frac{\tau(N-1)}{p-1}}. \tag{39}$$

Then

$$\int_{\mathcal{B}_1} \left[(N-1)[-\psi''\psi + (\psi')^2] - (p-1)(\psi')^2(1-\alpha)^2 \right] |u_r|^p \psi^{-2\alpha} \, \mathrm{d}v_g \leq C_{N,p,\psi} \|\nabla_g u\|^p_{L^p(\mathcal{B}_1)}$$

where $C_{N,p,\psi}$ is a constant depending only on N and p.

Proof. Using the semistability of u and Lemma 4.1 applied with η replaced by $\psi\eta$ we have that

$$-\int_{\mathcal{B}_1} \frac{\partial}{\partial r} \left(\frac{(N-1)\psi'}{\psi} \right) \psi^2 \eta^2 |u_r|^p \, \mathrm{d}v_g \leq \int_{\mathcal{B}_1} |u_r|^p (p-1) |(\psi\eta_r)|^2 \, \mathrm{d}v_g.$$

Thus,

$$(N-1) \int_{\mathcal{B}_1} \left[-\psi''\psi + (\psi')^2 \right] |u_r|^p \eta^2 \, \mathrm{d}v_g \leq (p-1) \int_\Omega |u_r|^p |(\psi\eta)_r|^2 \, \mathrm{d}v_g \tag{40}$$

holds for all $\eta \in C_c^1(\mathcal{B}_1 \setminus \{\mathcal{O}\})$. By an approximation argument (21), the inequality (40) holds for all Lipschitz function η that vanishes on $\partial\mathcal{B}_1$ and also in a neighborhood of the pole $\mathcal{O}$. In fact, this estimate holds for ever radial Lipschitz function vanishing on $\partial\mathcal{B}_1$ but not necessarily vanishing in around the pole $\mathcal{O}$. Now, take α satisfying (39), $\varepsilon \in (0,1)$ sufficiently small and

$$\eta_\varepsilon(r) = \begin{cases} \psi^{-\alpha}(\varepsilon) - \psi(1) \text{ for } 0 \leq r \leq \varepsilon \\ \psi^{-\alpha}(r) - \psi(1) \text{ for } \varepsilon < r \leq 1 \end{cases} \tag{41}$$

a Lipschitz function which vanishes on $\partial\mathcal{B}_1$. Taking $\eta = \eta_\varepsilon$ we have

$$\begin{aligned}(N-1)\int_{\mathcal{B}_1\setminus\mathcal{B}_\varepsilon}\left[-\psi''\psi+(\psi')^2\right]\left(\psi^{-\alpha}-\psi(1)\right)^2|u_r|^p\,\mathrm{d}v_g+\\ (N-1)\left(\psi^{-\alpha}(\varepsilon)-\psi(1)\right)^2\int_{\mathcal{B}_\varepsilon}\left[-\psi''\psi+(\psi')^2\right]|u_r|^p\,\mathrm{d}v_g\ \leq\\ (p-1)\int_{\mathcal{B}_1\setminus\mathcal{B}_\varepsilon}|u_r|^p(\psi')^2\left[(1-\alpha)\psi^{-\alpha}-\psi(1)\right]^2\mathrm{d}v_g+\\ (p-1)\left[\psi^{-\alpha}(\varepsilon)-\psi(1)\right]^2\int_{\mathcal{B}_\varepsilon}|u_r|^p(\psi')^2\,\mathrm{d}v_g\end{aligned}$$

For ε sufficiently small, it follows that

$$\begin{aligned}(N-1)\int_{\mathcal{B}_1\setminus\mathcal{B}_\varepsilon}\left[-\psi''\psi+(\psi')^2\right]\left(\psi^{-\alpha}-\psi(1)\right)^2|u_r|^p\,\mathrm{d}v_g\leq\\ (p-1)\int_{\mathcal{B}_1\setminus\mathcal{B}_\varepsilon}|u_r|^p(\psi')^2\left[(1-\alpha)\psi^{-\alpha}-\psi(1)\right]^2\mathrm{d}v_g\end{aligned}$$

Developing the squares we have

$$\begin{aligned}\int_{\mathcal{B}_1\setminus\mathcal{B}_\varepsilon}\left[(N-1)[-\psi''\psi+(\psi')^2]-(p-1)(\psi')^2(1-\alpha)^2\right]|u_r|^p\psi^{-2\alpha}\,\mathrm{d}v_g\leq\\ 2\psi(1)\int_{\mathcal{B}_1\setminus\mathcal{B}_\varepsilon}[(N-1)[-\psi''\psi+(\psi')^2]-(p-1)(\psi')^2(1-\alpha)]\psi^{-\alpha}|u_r|^p\,\mathrm{d}v_g\\ +\psi^2(1)\int_{\mathcal{B}_1\setminus\mathcal{B}_\varepsilon}[-(N-1)[-\psi''\psi+(\psi')^2]+(p-1)(\psi')^2]|u_r|^p\,\mathrm{d}v_g\end{aligned}$$

Using Young inequality and some calculations, it follows that

$$\begin{aligned}\frac{1}{2}\int_{\mathcal{B}_1\setminus\mathcal{B}_\varepsilon}\left[(N-1)[-\psi''\psi+(\psi')^2]-(p-1)(\psi')^2(1-\alpha)^2\right]|u_r|^p\psi^{-2\alpha}\,\mathrm{d}v_g\leq\\ \psi^2(1)\int_{\mathcal{B}_1\setminus\mathcal{B}_\varepsilon}[(N-1)[-\psi''\psi+(\psi')^2]+3(p-1)(\psi')^2]|u_r|^p\,\mathrm{d}v_g\end{aligned}$$

Taking $\varepsilon \to 0$, it follows that

$$\begin{aligned}\frac{1}{2}\int_{\mathcal{B}_1}\left[(N-1)[-\psi''\psi+(\psi')^2]-(p-1)(\psi')^2(1-\alpha)^2\right]|u_r|^p\psi^{-2\alpha}\,\mathrm{d}v_g\leq\\ \psi^2(1)\int_{\mathcal{B}_1}[(N-1)[-\psi''\psi+(\psi')^2]+3(p-1)(\psi')^2]|u_r|^p\,\mathrm{d}v_g\end{aligned}$$

From this, we can conclude

$$\int_{\mathcal{B}_1} \left[(N-1)[-\psi''\psi + (\psi')^2] - (p-1)(\psi')^2(1-\alpha)^2\right] |u_r|^p \psi^{-2\alpha} \, \mathrm{d}v_g \leq$$

$$C_{N,p,\psi} \|\nabla_g u\|^p_{L^p(\mathcal{B}_1)}$$

□

4.3 Regularity for semi-stable solutions

Our first main result establishes a priori estimates for radial semi-stable classical solutions of (32). Since the extremal solutions can be obtained as the limit of classical minimal solutions, this result is useful in order to obtain regularity results about u^*.

Theorem 4.2. Let $\mathcal{M}$ is a Riemannian model such that (H_1) and (H_2) hold, f be a locally Lipschitz function and $u \in W^{1,p}(\mathcal{B}_1)$ be a semi-stable radial solution in $\mathcal{B}_1 \setminus \mathcal{O}$ of (32) satisfying $u_r(r) < 0$ for all $r \in (0,1)$.

(i) If $N < p + 2 + \frac{2\tau}{p-1} + \frac{2}{p-1}\sqrt{(p^2-1)\tau + \tau^2}$ then $u \in L^\infty(\mathcal{B}_1)$. Moreover,

$$\|u\|_{L^\infty(\mathcal{B}_1)} \leq C_{N,p,\psi} \|u\|_{W^{1,p}(\mathcal{B}_1)}.$$

(ii) Assume f is nonnegative. Then

$$\|\nabla_g u\|_{L^p(\mathcal{B}_1)} \leq C_{N,p,\psi} \left\{ \| \left(u - u(1)\right)^{p-1} \|_{L^1(\mathcal{B}_1)} \|f(u)\|^{\frac{1}{p-1}}_{L^1(\mathcal{B}_1)} \right\}$$

The proof of Theorem 4.2 relies essentially on the following key estimate

$$\int_{\mathcal{B}_1} \left[(N-1)[-\psi''\psi + (\psi')^2] - (p-1)(\psi')^2(1-\alpha)^2\right] |u_r|^p \psi^{-2\alpha} \, \mathrm{d}v_g \leq$$

$$C_{N,p,\psi} \|\nabla_g u\|^p_{L^p(\mathcal{B}_1)}$$

for some range of explicit α (see Lemma 4.2 below). We obtained this estimate using the radial symmetry of the solution and by choosing a suitable test function in the semistability condition. With this choice, we have to be careful in the computations due to the appearance of the Riemannian metric. Remember that in the Euclidean case, ψ has first and second derivatives identically 1 and 0, respectively.

Proof of Theorem 4.2. Every radial function $u \in W^{1,p}(\mathcal{B}_1)$ also belongs to the Sobolev space $W^{1,p}(\delta, 1)$ in one dimension for a given $\delta \in (0,1)$. Using the Sobolev embedding in one dimension, u becomes a continuous function of $r = \mathrm{dist}(x, \mathcal{O}) \in [\delta, 1]$ and

$$|u(1)| \leq C_{N,p} \|u\|_{W^{1,p}(\mathcal{B}_1)}$$

In view of this estimate, we can assume that

$$u > 0 = u(1) \text{ in } \mathcal{B}_1.$$

Note that the case $N < p$ is very simple and it follows immediately by Sobolev embeddings. Let α satisfying (39) and $0 < t < 1$. For the sake of simplicity, set

$$\gamma(r) := \left[(N-1)[-\psi''\psi + (\psi')^2] - (p-1)(\psi')^2(1-\alpha)^2\right].$$

Using Hölder inequality, it follows that

$$\begin{aligned} u(t) &= \int_t^1 -u_r \psi^{\frac{-2\alpha+N-1}{p}} \psi^{\frac{2\alpha-N+1}{p}} \,\mathrm{d}r \\ &\le C_{N,p} \left[\int_{\mathcal{B}_1} \gamma(r)|u_r|^p \psi^{-2\alpha} \mathrm{d}v_g\right]^{\frac{1}{p}} \left[\int_t^1 \frac{\psi^{\frac{p'(2\alpha-N+1)}{p}}}{[\gamma(r)]^{1/p-1}} \mathrm{d}r\right]^{\frac{1}{p'}} \end{aligned}$$

where $p' = p/(p-1)$. By Lemma 4.2 and (H_2), we have

$$\begin{aligned} u(t) &\le C_{N,p,\psi} \|u\|_{W^{1,p}(\mathcal{B}_1)} \left[\int_t^1 \frac{\psi^{\frac{p'(2\alpha-N+1)}{p}}}{[\gamma(r)]^{1/p-1}} \mathrm{d}r\right]^{\frac{1}{p'}} \\ &\le C_{N,p,\overline{\psi}} \|u\|_{W^{1,p}(\mathcal{B}_1)} \left[\int_t^1 \frac{r^{\frac{p'(2\alpha-N+1)}{p}}}{[\gamma(r)]^{1/p-1}} \,\mathrm{d}r\right]^{\frac{1}{p'}}. \end{aligned} \tag{42}$$

(i) In the case $p \le N < p + 2 + \frac{2\tau}{p-1} + \frac{2}{p-1}\sqrt{(p^2-1)\tau + \tau^2}$ we can choose α satisfying

$$\frac{N-p}{2} < \alpha < 1 + \sqrt{\frac{\tau(N-1)}{p-1}}.$$

In addition, we can assume $\alpha \ge 1$. With this, $\gamma(r) > 0$ for all $r \in [0,1]$ and

$$\int_t^1 r^{\frac{p'(2\alpha-N+1)}{p}} < +\infty.$$

Thus, we have the desired L^∞ estimate from (42).

(ii) Using a similar argument as in (22), we can prove that

$$\|\psi^{N-1}|u_r|^{p-1}\|_{L^\infty(\mathcal{B}_1)} \le C_{N,p}\|f(u)\|_{L^1(\mathcal{B}_1)}.$$

To control $\|\nabla_g u\|_{L^p(\mathcal{B}_1)}$, we assume first $N < p$. Then, using (H_2) it follows that

$$\begin{aligned} \int_0^1 \psi^{N-1}|u_r|^p \,\mathrm{d}r &= \int_0^1 \left(\psi^{N-1}|u_r|^{p-1}\right)^{\frac{p}{p-1}} \psi^{-\frac{N-1}{p-1}} \,\mathrm{d}r \\ &\le \|\psi^{N-1}|u_r|^{p-1}\|_{L^\infty(\mathcal{B}_1)}^{\frac{p}{p-1}} \int_0^1 \psi^{-\frac{N-1}{p-1}} \,\mathrm{d}r \\ &\le C_{N,p,\overline{\psi}}\|f(u)\|_{L^1(\mathcal{B}_1)}. \end{aligned}$$

This implies

$$\|\nabla_g u\|_{L^p(\mathcal{B}_1)} \le C_{N,p,\overline{\psi}} \|f(u)\|_{L^1(\mathcal{B}_1)}.$$

In the case $p \le N$, take α satisfying (39) and using Lemma 39

$$\begin{aligned}
\int_{\mathcal{B}_1} &\left[(N-1)[-\psi''\psi + (\psi')^2] - (p-1)(\psi')^2(1-\alpha)^2\right] |u_r|^p \psi^{-2\alpha} \,\mathrm{d}v_g \\
&\le C_{N,p,\psi} \int_{\mathcal{B}_1} |u_r|^p \,\mathrm{d}x \\
&= C_{N,p,\psi} \int_{\mathcal{B}_{r_0}} |u_r|^p \,\mathrm{d}x + C_{N,p,\psi} \int_{\mathcal{B}_1 \setminus \overline{\mathcal{B}_{r_0}}} |u_r|^p \,\mathrm{d}x.
\end{aligned}$$

Now, choose r_0 such that $2C_{N,p,\psi} \le \psi^{-2\alpha}$ in $r_0 \in (0, r_0)$ to obtain

$$C_{N,p,\psi} \int_{\mathcal{B}_{r_0}} |u_r|^p \,\mathrm{d}x \le \frac{1}{2} \int_{\mathcal{B}_1} \psi^{-2\alpha} |u_r|^p \,\mathrm{d}x.$$

With this, we deduce that

$$C_\psi \int_{\mathcal{B}_1} |u_r|^p \,\mathrm{d}x \le \int_{\mathcal{B}_1} \psi^{-2\alpha} |u_r|^p \,\mathrm{d}x \le C_{N,p,\psi} \int_{\mathcal{B}_1 \setminus \overline{\mathcal{B}_{r_0}}} |u_r|^p \,\mathrm{d}x.$$

Since u is decreasing, we have that

$$u(r_0)^{p-1} \le C_{N,p} \|u^{p-1}\|_{L^1(\mathcal{B}_{r_0})}.$$

Thus,

$$\begin{aligned}
\int_{\mathcal{B}_1 \setminus \overline{\mathcal{B}_{r_0}}} |u_r|^p \,\mathrm{d}x &= C_{N,\psi} \int_{r_0}^1 |u_r|^p \psi^{N-1} \,\mathrm{d}r \\
&\le C_{N,\psi} \|\psi^{N-1} |u_r|^{p-1}\|_{L^\infty(\mathcal{B}_1)} \int_{r_0}^1 -u_r \,\mathrm{d}r \\
&\le C_{N,p,\psi} \|f(u)\|_{L^1(\mathcal{B}_1)} \|u^{p-1}\|_{L^1(\mathcal{B}_1)}^{\frac{1}{p-1}}.
\end{aligned}$$

We can conclude that

$$\begin{aligned}
\int_{\mathcal{B}_1} |u_r|^p \,\mathrm{d}x &\le C_{N,p,\psi} \int_{\mathcal{B}_1 \setminus \overline{\mathcal{B}_{r_0}}} |u_r|^p \,\mathrm{d}x \\
&\le C_{N,p,\psi} \|f(u)\|_{L^1(\mathcal{B}_1)} \|u^{p-1}\|_{L^1(\mathcal{B}_1)}^{\frac{1}{p-1}}.
\end{aligned}$$

□

4.4 Regularity for extremal solutions

Applying the Theorem 4.2 to minimal solutions u_λ of (Q_λ) and letting $\lambda \nearrow \lambda^*$ we can prove that u^* has the same regularity properties as the ones stated in Theorem 4.2. For this, we need to bound u^{p-1} and $f(u)$ in $L^1(\mathcal{B}_1)$ uniformly in λ. This is possible because we have the growth condition (33) on f and the radially decreasing property of the minimal solutions u_λ.

Theorem 4.3. Let $\mathcal{M}$ is a Riemannian model such that (H_1) and (H_2) hold, f be a positive C^1 function in $[0,+\infty)$ satisfying (33). For $\lambda \in (0,\lambda^*)$ let u_λ be the minimal solution of (Q_λ). Then $u^* \in W^{1,p}(\mathcal{B}_1)$ and

$$\|u_\lambda^{p-1}\|_{L^1(B_1)} + \|f(u_\lambda)\|_{L^1(B_1)} \leq C$$

Moreover, u^* is a semi-stable radially symmetric decreasing weak solution of (Q_λ). As a consequence, u^* has the regularity stated in Theorem 4.2. In particular,

$$u^* \in L^\infty(\mathcal{B}_1) \text{ if } N < p + 2 + \frac{2\tau}{p-1} + \frac{2}{p-1}\sqrt{(p^2-1)\tau + \tau^2}.$$

Proof of Theorem 4.3. We can obtain a minimal solution using monotone iteration starting by $u_0 = 0$ by solving $-\Delta_{g,p} u_{k+1} = \lambda f(u_k)$. With this, the limit u_λ, when $k \to +\infty$, is radial. Observe that u_λ is decreasing. Indeed, integrating the equation (Q_λ) we get that $\frac{\partial u_\lambda}{\partial r}(r) < 0$. Now, for $\lambda \in (0,\lambda^*)$ let $\rho_\lambda \in (1/2,1)$ such that mean value property holds, more precisely,

$$\frac{\partial u_\lambda}{\partial r}(\rho_\lambda) = \frac{u_\lambda(1/2) - u_\lambda(1)}{1/2}.$$

With this,

$$\left[\frac{\partial u_\lambda}{\partial r}(\rho_\lambda)\right]^{p-1} = [2u_\lambda(1/2)]^{p-1} \leq C_{N,p,\psi}\|u_\lambda^{p-1}\|_{L^1(B_{1/2})}.$$

By monotonicity, we have

$$\|\psi^{N-1}|\frac{\partial u_\lambda}{\partial r}|^{p-1}\|_{L^\infty(B_{1/2})} \leq C_{N,p,\psi}\|u_\lambda^{p-1}\|_{L^1(B_{1/2})}. \tag{43}$$

If we use a radial test function $\phi(r) = \min 1, (2-4r)^+$ and using (43), we obtain

$$\|\lambda f(u_\lambda)\|_{L^1(B_{1/4})} \leq C_{N,p,\psi}\int_{1/4}^{1/2} \psi^{N-1}|\frac{\partial u_\lambda}{\partial r}|^{p-1}\,\mathrm{d}r \leq C_{N,p,\psi}\|u_\lambda^{p-1}\|_{L^1(B_{1/2})}. \tag{44}$$

Using the assumption (33), given $\delta > 0$, we have

$$\lambda f(t) \geq \frac{1}{\delta}t^{p-1} - C_\delta$$

holds for all $t > 0$ and $\lambda \in (\lambda^*/2, \lambda^*)$, where C_δ does not depend on λ. With this

$$\|u_\lambda^{p-1}\|_{L^1(B_{1/4})} \leq C_{N,p,\psi}\delta\|u_\lambda^{p-1}\|_{L^1(B_{1/2})} + C_\delta. \tag{45}$$

Since u_λ is decreasing, it follows that

$$\|u_\lambda^{p-1}\|_{L^1(B_{1/2}\setminus\overline{B}_{1/4})} \leq C_{N,p,\psi}u_\lambda^{p-1}(1/4) \leq C_{N,p,\psi}\|u_\lambda^{p-1}\|_{B_{1/4}}. \tag{46}$$

Now, take δ sufficiently small and combine (45) with (46) to obtain

$$\|u_\lambda^{p-1}\|_{L^1(B_{1/4})} \leq C,$$

where C is a constant independent of λ. Repeating the argument in (46), we are able to obtain an estimate uniform in λ for $\|u_\lambda^{p-1}\|_{L^1(B_1)}$. Using this in (44) we obtain an estimate for $\|f(u_\lambda)\|_{L^1(B_{1/4})}$. Again by monotonicity, we can apply the same arguments used above to control $\|f(u_\lambda)\|_{L^1(B_1)}$ uniformly in λ. Thus,

$$\|u_\lambda^{p-1}\|_{L^1(B_1)} + \|f(u_\lambda)\|_{L^1(B_1)} \leq C$$

where C is a constant independent of λ. By Theorem 4.2, we deduce a bound for $\|u_\lambda\|_{W^{1,p}(B_1)}$. Using the compactness and since $u_\lambda \to u^*$ as $\lambda \to \lambda^*$, it follows that $u^* \in W_0^{1,p}(B_1)$. We can pass to the limit and conclude that u^* is a weak solution of (Q_λ) for $\lambda = \lambda^*$. It is clear that u^* is radially symmetric and nonincreasing. Using the same argument at the beginning of this proof we can obtain that u^* is decreasing. By Fatou's Lemma we obtain that u^* is semi-stable. Finally, the regularity statement follows as a consequence of Theorem 4.2. $\square$

5 Systems with singular nonlinearities and applications to MEMS

In this chapter we deal with Hamiltonian systems of coupled singular elliptic equations of second-order of the form

$$\begin{cases} -\Delta u = \dfrac{\lambda f(x)}{(1-v)^2} & \text{in } \quad \Omega, \\ -\Delta v = \dfrac{\mu g(x)}{(1-u)^2} & \text{in } \quad \Omega, \\ 0 \leq u, v < 1 & \text{in } \quad \Omega, \\ u = v = 0 & \text{on } \quad \partial\Omega, \end{cases} \tag{$S_{\lambda,\mu}$}$$

where λ and μ are positive parameters, Ω is a smooth bounded domain of $\mathbb{R}^N$ $(N \geq 1)$ and f and g satisfy the following conditions:

$$\begin{aligned} &f, g \in C^\alpha(\overline{\Omega}) \text{ for some } \alpha \in (0,1],\ 0 \leq f, g \leq 1 \\ &f, g > 0 \text{ on a subset of } \Omega \text{ of positive measure.} \end{aligned} \tag{H}$$

5.1 Motivation and related results

System $(S_{\lambda,\mu})$ can be seen as a Lane-Emden type system with nonlinearities with negative exponents (11, 61, 62, 63, 64). A lot of work has been devoted to existence and nonexistence of solutions to elliptic systems with continuous power like nonlinearities, among which we recall (65, 66, 67, 68, 69, 70, 71, 72) and the survey (73), just recently Lane-Emden type singular nonlinearities have been considered in (74). Here we address the problem of studying existence, non-existence and regularity results by means of the nonlinear eigenvalue problem $(S_{\lambda,\mu})$, in which for the sake of clarity we consider a Coulomb nonlinear source though most results extend to more general situations. Related results for systems with continuous nonlinearities have been obtained in (75, 76).

Another important motivation to consider $(S_{\lambda,\mu})$ comes from recent works on the study of the equations that models MEMS

$$\begin{cases} -\Delta v = \lambda \frac{g(x)}{(1-v)^2} & \text{in } \Omega, \\ 0 \leq v < 1 & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{P_λ}$$

Micro-electromechanical systems (MEMS) are often used to combine electronics with microsize mechanical devices in the design of various types of microscopic machinery. MEMS devices have therefore become key components of many commercial systems, including accelerometers for airbag deployment in automobiles, ink jet printer heads, optical switches and chemical sensors.

Nonlinear interaction described in terms of coupling of semilinear elliptic equations has revealed through the last decades a fundamental tool in studying nonlinear phenomena (67, 68, 77, 78, 79, 80, 81, 82) and references therein. In all the above contexts the nonlinearity is fairly represented by a continuous function. More recently, a rigorous mathematical approach in modeling and designing Micro Electro Mechanical Systems has demanded the need to consider also nonlinearities which develop singularities. In a nutshell, one may think of MEMS' actuation as governed by the dynamic of a micro plate which deflects towards a fixed plate, under the effect of a Coulomb force, once that a drop voltage is applied.

In the stationary case, the naive model which describes this device cast into the following second order elliptic PDE (P_λ), where Ω is a bounded smooth domain in $\mathbb{R}^N$ and the positive function g is bounded and related to dielectric properties of the material, check the survey (18) and also (16, 83, 84) for more technical aspects. The key feature of the equation in (P_λ) is retained by the discontinuity of the nonlinearity which blows up as $v \to 1^-$ and this corresponds in applications to a snap through of the device.

The difficulties arising in studies of such equations are well know. The role of the positive parameter λ is that of tuning the drop voltage, whence from the PDE point of view, yields the threshold between existence and non-existence of solutions which exist up to a maximal value λ^* in which the following alternative occurs: the extremal solution does not reach the maximal height (set to be one in (P_λ)) and this is the case in which the pointwise solution (classical) is smooth or the touch down has occurred and a weaker notion of solution, which reflect the conservation of energy, shows up; this is referred in literature as the regularity issue for extremal solutions (18, 85, 86, 87).

The higher order version of (P_λ) has been considered in (88), results further extended in (89). We mention that singular nonlinearities appear in different contexts also in (90, 91).

Here we mention some recent papers on semilinear elliptic system of cooperative type which are close related with our work. M. Montenegro in (76) studied elliptic systems of the form $\Delta u = \lambda f(x,u,v)$, $\Delta v = \mu g(x,u,v)$ defined in Ω a smooth bounded domain under homogeneous Dirichlet boundary conditions. Under some suitable assumptions, which include in particular that the systems are cooperative, it was proved that there exists a monotone continuous curve Υ in the positive quadrant $\mathcal{Q}$ separating this set into two connected components: U "below" Υ, where there are $C^1(\overline{\Omega})$ minimal positive solutions, and V "above" Υ, where there is no such solution. For points on Υ there are weak solutions (in the sense of the weighted Lebesgue space $L^1_d(\Omega)$, where $d(x)$ is the distance to the boundary $\partial\Omega$. Linearized stability of solutions in U is also proved. The existence proof uses sub- and supersolutions, and the existence of weak solutions is shown by a limiting argument involving a priori estimates in $L^1_d(\Omega)$ for classical solutions.

A question that attracted a lot of attention is the regularity of the extremal solution. C. Cowan (92) considered the particular case of nonlinearities of Gelfand type, that is, when $f(x,u,v) = \mathrm{e}^v$ and $g(x,u,v) = \mathrm{e}^u$. He studied the regularity of the extremal solutions on the critical curve, precisely, he proved that if $3 \leq N \leq 9$ and $(N-2)/8 < \mu^*/\lambda^* < 8/(N-2)$ then the associated extremal solutions are smooth. This implies that $N = 10$ is the critical dimension for the Gelfand systems, because the scalar equation related with this class of problems may be singular if $N \geq 10$. Later, C. Cowan and M. Fazly in (93) examined the elliptic systems given by

$$-\Delta u = \lambda f'(u)g(v), \quad -\Delta v = \mu f(u)g'(v) \text{ in } \Omega, \tag{H_1}$$

and

$$-\Delta u = \lambda f(u)g'(v), \quad -\Delta v = \mu f'(u)g(v) \text{ in } \Omega \tag{H_2}$$

with zero Dirichlet boundary condition in a bounded convex domain Ω. They proved that for a general nonlinearities f and g, the extremal solution associated with (H_1) are bounded when $N \leq 3$. For a radial domain, they proved the extremal solution are bounded provided $N < 10$. The extremal solution associated with (H_2) are bounded in the case where f is a general nonlinearity and $g(v) = (1+v)^q$ for $1 < q < +\infty$ and $N \leq 3$. For the explicit nonlinearities of the form $f(u) = (1+u)^p$ and $g(v) = (1+v)^q$ certain regularity results are also obtained in higher dimensions for (H_1) and (H_2).

The main goal of this chapter is to complement the study of the works (76, 92, 93) proving similar results for singular nonlinearity in problems of MEMS type. In the next section we bring some auxiliary results used in the text. Moreover, we study the existence of a critical curve, extremal parameter and minimal solutions. We also establish upper and lower bounds for the critical curve Γ and monotonicity results for the extremal parameter. In Sect. 5.6 we obtain some estimates and properties for the branch of minimal solutions that allow us to prove the regularity result about the extremal solution.

5.2 A critical curve: existence of classical solutions

The main goal of this section is to prove Theorems 5.1, 5.2 and 5.3. Precisely, by the method of sub-super solutions we prove that there exists a non-increasing continuous function Γ of the parameter λ such that $(S_{\lambda,\mu})$ has at least one solution for $0 < \mu < \Gamma(\lambda)$ whereas $(S_{\lambda,\mu})$ has no solutions for $\mu > \Gamma(\lambda)$. In what follows unless otherwise stated, by solution we mean a classical solution of class $\mathcal{C}^2(\Omega)$.

Lemma 5.1. Let λ and μ be positive parameters such that there exists a classical super solution (U, V) for $(S_{\lambda,\mu})$, namely, $U, V \in C^2(\overline{\Omega})$ satisfying pointwisely the following system of inequalities

$$\begin{cases} -\Delta U \geq \dfrac{\lambda f(x)}{(1-V)^2} & \text{in} \quad \Omega, \\ -\Delta V \geq \dfrac{\mu g(x)}{(1-U)^2} & \text{in} \quad \Omega, \\ 0 \leq U, V < 1 & \text{in} \quad \Omega, \\ U = V = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

Then there exists a classical solution (u, v) of $(S_{\lambda,\mu})$ such that $u \leq U$ and $v \leq V$.

Proof. Setting $(u_0, v_0) = (U, V)$ let (u_1, v_1) be the unique solution of the linear problem

$$\begin{cases} -\Delta u_1 = \dfrac{\lambda f(x)}{(1-v_0)^2} & \text{in} \quad \Omega, \\ -\Delta v_1 = \dfrac{\mu g(x)}{(1-u_0)^2} & \text{in} \quad \Omega, \\ 0 \leq u_1, v_1 < 1 & \text{in} \quad \Omega, \\ u_1 = v_1 = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

By the classical maximum principle we deduce that $u_1 \leq u_0$ and $v_1 \leq v_0$. Defining (u_n, v_n)

inductively as follows

$$\begin{cases} -\Delta u_n = \dfrac{\lambda f(x)}{(1-v_{n-1})^2} & \text{in} \quad \Omega, \\ -\Delta v_n = \dfrac{\mu g(x)}{(1-u_{n-1})^2} & \text{in} \quad \Omega, \\ \ \ 0 \leq u_n, v_n < 1 & \text{in} \quad \Omega, \\ \ u_n = v_n = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

by the maximum principle, we have

$$0 < u_n \leq u_{n-1} \leq \ldots u_1 \leq u_0 \text{ and } 0 < v_n \leq v_{n-1} \leq \ldots v_1 \leq u_0.$$

Thus, there exists (u, v) such that

$$0 \leq u = \lim_{n\to\infty} u_n \leq U < 1 \text{ and } 0 \leq v = \lim_{n\to\infty} v_n \leq V < 1$$

and by a standard compactness argument we have that the above convergence holds in $C^{2,\alpha}(\overline{\Omega})$ to a solution (u, v) of $(S_{\lambda,\mu})$ and in particular different from zero. □

We now state and prove a monotonicity result on the coordinates of a solution of $(S_{\lambda,\mu})$, precisely,

Lemma 5.2. Suppose that (u, v) is a smooth solution of $(S_{\lambda,\mu})$ where $0 < \mu \leq \lambda$ and $f \equiv g \equiv 1$. Then

$$\frac{\mu}{\lambda}u \leq v \leq u \qquad \text{a.e in } \Omega.$$

Proof. Take the difference of the equations in $(S_{\lambda,\mu})$ to obtain

$$-\Delta(u - v) = \frac{\lambda}{(1-v)^2} - \frac{\mu}{(1-u)^2}$$

in Ω and multiplying this equation by $(u - v)_-$ and integrating by parts we have

$$\int_\Omega |\nabla(u - v)_-|^2 \, dx = \int_\Omega \left(\frac{\lambda}{(1-v)^2} - \frac{\mu}{(1-u)^2} \right) (u - v)_- \, dx.$$

Since the right hand side is nonpositive and the left hand side is nonnegative, we see that $(u - v)_- = 0$ a.e. in Ω and so $u \geq v$ a.e. in Ω. Now, since $u \geq v$,

$$-\Delta\left(v - \frac{\mu}{\lambda}u\right) = \mu\left(\frac{1}{(1-u)^2} - \frac{1}{(1-v)^2}\right) \geq 0.$$

Thus, $\frac{\mu}{\lambda}u \leq v$ and we finish the proof. □

We are going to prove that $(S_{\lambda,\mu})$ has a classical solution for λ and μ sufficiently small. Let $\mathcal{Q}$ be the positive quadrant of the (λ,μ)-plane. The set

$$\Lambda_{(\Omega,f,g)} = \Lambda := \{(\lambda,\mu) \in \mathcal{Q} : \ (S_{\lambda,\mu}) \text{ has at least a classical solution}\}$$

has nonempty interior.

Lemma 5.3. There exists $\lambda_1 > 0$ such that $(0,\lambda_1] \times (0,\lambda_1] \subset \Lambda$.

Proof. Let B_R be a ball of radius R such that $\Omega \subset B_R$ and let $\mu_{1,R}$ be the first eigenvalue of the Dirichlet boundary value problem

$$\begin{cases} -\Delta\varphi = \nu\varphi & \text{in} \quad B_R \\ \varphi = 0 & \text{in} \quad \partial\Omega \end{cases}$$

and denote the corresponding eigenfunction by $\psi_{1,R}$ which we may assume to be positive and also that $\sup_{B_R} \psi_{1,R} = 1$. Now we show that there exists $\vartheta > 0$ such that $\psi = \vartheta\psi_{1,R}$ is a super-solution to $(S_{\lambda,\lambda})$ provided $\lambda > 0$ is sufficiently small. Notice that we can choose $\vartheta \in (0,1)$ such that $0 < 1 - \vartheta\psi_{1,R} < 1$ in B. Thus

$$\begin{cases} -\Delta\psi = \mu_{1,R}\vartheta\psi_{1,R} \geq \dfrac{\lambda f(x)}{(1-\psi)^2} = \dfrac{\lambda f(x)}{(1-\vartheta\psi_{1,R})^2} & \text{in} \quad \Omega \\ -\Delta\psi = \mu_{1,R}\vartheta\psi_{1,R} \geq \dfrac{\lambda g(x)}{(1-\psi)^2} = \dfrac{\lambda g(x)}{(1-\vartheta\psi_{1,R})^2} & \text{in} \quad \Omega \end{cases}$$

provided

$$\mu_{1,R}\vartheta\psi_{1,R}(1-\vartheta\psi_{1,R})^2 \geq \lambda \max\{f(x), g(x)\}, \quad x \in \Omega.$$

Notice that $s_1 := \inf_{x\in\Omega} \vartheta\psi_{1,R} < s_2 := \sup_{x\in\Omega} \vartheta\psi_{1,R} < 1$, and $s_1, \ s_2$ depend of R. Setting $Z(s) := s(1-s)^2$, it is easy to see that we can choose $\lambda > 0$ sufficiently small such that

$$\mu_{1,R} \inf_{x\in\Omega} Z(\vartheta\psi_{1,R}(x)) \geq \lambda \max\{\sup_{\Omega} g(x), \ \sup_{\Omega} f(x)\}.$$

Thus, using Lemma 5.1 we conclude that $(\lambda,\mu) \in \Lambda$, for all $\lambda,\mu \in (0,\lambda_1]$. □

Lemma 5.4. Λ is bounded.

Proof. Let $(\lambda,\mu) \in \Lambda$ and (u,v) the corresponding solution of $(S_{\lambda,\mu})$. Multiplying the first equation in $(S_{\lambda,\mu})$ by $\psi_{1,R}$ and integrating by parts we get

$$\begin{aligned} |B_R|\mu_{1,R} \geq \mu_{1,R} \int_{B_R} u\psi_{1,R}\,\mathrm{d}x &= -\int_{B_R} u\Delta\psi_{1,R}\,\mathrm{d}x \\ &= -\int_{B_R} \Delta u\psi_{1,R}\,\mathrm{d}x \\ &= \lambda \int_{B_R} \frac{f(x)\psi_{1,R}}{(1-v)^2}\,\mathrm{d}x \end{aligned}$$

which implies that

$$|B_R|\mu_{1,R} \geq \lambda \int_{B_R} f(x)\psi_{1,R}\,\mathrm{d}x$$

Analogously we obtain

$$|B_R|\mu_{1,R} \geq \mu \int_{B_R} g(x)\psi_{1,R}\,\mathrm{d}x$$

and therefore Λ is bounded. $\square$

Now we state that Λ is a convex set, precisely,

Lemma 5.5. If $(\lambda', \mu') \in \mathcal{Q}$ and $\lambda' \leq \lambda$ and $\mu' \leq \mu$ for some $(\lambda, \mu) \in \Lambda$ then $(\lambda', \mu') \in \Lambda$.

Proof. It follows from Lemma 5.1. Indeed, the solution associated to the pair $(\lambda, \mu) \in \Lambda$ turns out to be a super-solution to $(S_{\lambda',\mu'})$. $\square$

5.3 Critical curve

For each fixed $\vartheta > 0$ considere the line L_ϑ start from $(0,0)$

$$L_\vartheta = \{\lambda > 0 : (\lambda, \vartheta\lambda) \in \Lambda\}.$$

Observe that Lemma 5.3 and Lemma 5.4 implies that for each ϑ fixed, the line L_ϑ is nonempty and bounded. This allow us to define the curve $\Gamma : (0, +\infty) \to \mathcal{Q}$ by

$$\Gamma(\vartheta) := (\lambda^*(\vartheta), \mu^*(\vartheta))$$

where

$$\lambda^*(\vartheta) := \sup L_\vartheta \quad \text{and} \quad \mu^*(\vartheta) = \vartheta\lambda^*(\vartheta)$$

Our first result deals with the existence of a curve that split the positive quadrant into two connected components.

Theorem 5.1. Suppose that condition (H) holds. Then, there exists a curve Γ that separates the positive quadrant $\mathcal{Q}$ of the (λ, μ)-plane into two connected components $\mathcal{O}_1$ and $\mathcal{O}_2$. For $(\lambda, \mu) \in \mathcal{O}_1$, problem $(S_{\lambda,\mu})$ has a positive classical minimal solution (u_λ, v_λ). Otherwise, if $(\lambda, \mu) \in \mathcal{O}_2$, there are no solutions.

Proof of Theorem 5.1. Define $\mathcal{O}_1 = \Lambda \setminus \Gamma$. Given $(\lambda_1, \mu_1), (\lambda_2, \mu_2) \in \mathcal{O}_1$, there exist $\vartheta_1, \vartheta_2 > 0$ such that $\mu_1 = \vartheta_1\lambda_1$ and $\mu_2 = \vartheta_2\lambda_2$. We can define, using the Lemma 5.5, a path linking (λ_1, μ_1) to $(0, 0)$ and another path linking $(0, 0)$ to (λ_2, μ_2). Follows that

$\mathcal{O}_1$ is connected. The Lemma 5.1 implies that for each $(\lambda, \mu) \in \mathcal{O}_1$ there exists a positive minimal classical solution (u_λ, v_μ) for problem $(S_{\lambda,\mu})$. Now, define $\mathcal{O}_2 = \mathcal{Q} \setminus \{\Lambda \cup \Gamma\}$. Let $(\lambda_1, \mu_1), (\lambda_2, \mu_2) \in \mathcal{O}_2$. Take $(\lambda_{\max}, \mu_{\max}) \in \mathcal{O}_2$, where $\lambda_{\max} = \max\{\lambda_1, \lambda_2\}$ and $\mu_{\max} = \max\{\mu_1, \mu_2\}$. We can take a path linking (λ_1, μ_1) to $(\lambda_{\max}, \mu_{\max})$ and another path linking $(\lambda_{\max}, \mu_{\max})$ to (λ_2, μ_2). Follows that $\mathcal{O}_2$ is connected. □

5.4 Upper and lower bounds for the critical curve

The next lemma will be the main tool to obtain the estimates contained in Theorem 5.2.

Lemma 5.6. Assume that $\Omega = B = B_R$ and f, g are radial, that is, $f(x) = f(|x|)$ and $g(x) = g(|x|)$, for all $x \in B$. Then

$$(0, a_{(f,R,N)}] \times (0, a_{(g,R,N)}] \subset \Lambda$$

where

$$a_{(f,R,N)} := C_N \frac{1}{\sup_B f(x)} \frac{1}{R^2}, \quad a_{(g,R,N)} := C_N \frac{1}{\sup_B g(x)} \frac{1}{R^2}.$$

and

$$C_N = \max\left\{\frac{8N}{27}, \frac{6N-8}{9}\right\}.$$

Proof. Notice that the function

$$w(x) := \frac{1}{3}\left(1 - \frac{|x|^2}{R^2}\right)$$

satisfies

$$\begin{aligned} -\Delta w = \frac{2N}{3R^2} &= \frac{2N\left(1-\frac{1}{3}\right)^2}{3R^2} \frac{1}{\left(1-\frac{1}{3}\right)^2} \\ &\geq \frac{8N}{27R^2 \sup_B f} \frac{f(x)}{\left[1 - \frac{1}{3}\left(1 - \frac{|x|^2}{R^2}\right)\right]^2} \\ &= \frac{8N}{27R^2 \sup_B f} \frac{f(x)}{(1-w)^2}. \end{aligned}$$

Similarly,

$$-\Delta w \geq \frac{8N}{27R^2 \sup_B g} \frac{g(x)}{(1-w)^2}.$$

Thus, for

$$\lambda \leq \frac{8N}{27R^2 \sup_B f} \text{ and } \mu \leq \frac{8N}{27R^2 \sup_B g}$$

we have that (w, w) is a super-solution of $(S_{\lambda,\mu})$ in B. Similarly, we can see that, taking

$$v(x) := 1 - \left(\frac{|x|}{R}\right)^{2/3},$$

the pair (v, v) is a super-solution for $(S_{\lambda,\mu})$ in B provided that

$$\lambda \leq \frac{6N-8}{9R^2 \sup_B f} \text{ and } \mu \leq \frac{6N-8}{9R^2 \sup_B g},$$

which completes the proof. $\square$

The following results contain upper and lower estimates for the critical curve in Theorems 5.2 and 5.3 respectively. These estimates depend only on $f, g, |\Omega|$ and the dimension N, namely,

Theorem 5.2. Suppose f, g satisfy (H). Then the region $\mathcal{O}_1$ is nonempty, more precisely, there exist a positive constant C_N which depends only of the dimension N such that

$$(0, a_{(f,|\Omega|,N)}] \times (0, a_{(g,|\Omega|,N)}] \subset \mathcal{O}_1,$$

where

$$a_{(f,|\Omega|,N)} := C_N \frac{1}{\sup_\Omega f(x)} \left(\frac{\omega_N}{|\Omega|}\right)^{2/N}, \; a_{(g,R,N)} := C_N \frac{1}{\sup_\Omega g(x)} \left(\frac{\omega_N}{|\Omega|}\right)^{2/N}$$

and

$$C_N = \max\left\{\frac{8N}{27}, \frac{6N-8}{9}\right\}.$$

Proof of Theorem 5.2. Since $\sup_{B_R} f^\sharp = \sup_\Omega f$ and $\sup_{B_R} g^\sharp = \sup_\Omega g$, setting

$$R = \left(\frac{|\Omega|}{\omega_N}\right)^{1/N}$$

the proof follows as an applications of Proposition 5.5 and Lemma 5.6. $\square$

Proposition 5.1. Assume that $\Omega = B = B_R$ and $f(x) = |x|^\alpha, \quad g(x) = |x|^\beta$ with $\alpha,\ \beta \geq 0$, then

$$(0, a_{(\alpha,R,N)}] \times (0, b_{(\beta,R,N)}] \subset \Lambda,$$

where

$$a_{(\alpha,R,N)} := \max\left\{\frac{4(2+\alpha)(N+\alpha)}{27}, \frac{(2+\alpha)(3N+\alpha-4)}{9}\right\} \frac{1}{R^{2+\alpha}}$$

and

$$b_{(\beta,R,N)} := \max\left\{\frac{4(2+\beta)(N+\beta)}{27}, \frac{(2+\beta)(3N+\beta-4)}{9}\right\} \frac{1}{R^{2+\beta}}.$$

Proof. Consider the function

$$w_{(\alpha,R)}(x) = \frac{1}{3}\left(1 - \frac{|x|^{2+\alpha}}{R^{2+\alpha}}\right).$$

Using a similar computation as we have done in the previous lemma we can prove that the pair $(w_{(\alpha,R)}, w_{(\beta,R)})$ is a super-solution of $(S_{\lambda,\mu})$ in B provided that

$$\lambda \leq \frac{4(2+\alpha)(N+\alpha)}{27R^{2+\alpha}} \text{ and } \mu \leq \frac{4(2+\beta)(N+\beta)}{27R^{2+\beta}}.$$

The same holds for the function

$$w(x) = 1 - \left(\frac{|x|}{R}\right)^{(2+\alpha)/3}$$

if

$$\lambda \leq \frac{(2+\alpha)(3N+\alpha-4)}{9R^{2+\alpha}} \text{ and } \mu \leq \frac{(2+\beta)(3N+\beta-4)}{9R^{2+\beta}}.$$

□

Theorem 5.3. Assume that $\inf_\Omega f(x) > 0$ (respectively $\inf_\Omega g(x) > 0$), then

$$\lambda^* \leq \frac{4\mu_1}{27}\frac{1}{\inf_\Omega f(x)} \quad \left(\text{respectively} \quad \mu^* \leq \frac{4\mu_1}{27}\frac{1}{\inf_\Omega g(x)}\right),$$

where μ_1 is the first eigenvalue of $(-\Delta, H_0^1(\Omega))$. Therefore, if $\inf_\Omega f(x) > 0$ and $\inf_\Omega g(x) > 0$ the region $\mathcal{O}_1$ is bounded, precisely,

$$\mathcal{O}_1 \subset \left(0, \frac{4\mu_1}{27}\frac{1}{\inf_\Omega f(x)}\right) \times \left(0, \frac{4\mu_1}{27}\frac{1}{\inf_\Omega g(x)}\right).$$

Proof of Theorem 5.3. Let $(\lambda, \mu) \in \Lambda$ and (u, v) the corresponding solution of $(S_{\lambda,\mu})$. Let μ_1 be the first eigenvalue for the Dirichlet boundary conditions

$$\begin{cases} -\Delta\varphi = \nu\varphi \text{ in } \Omega, \\ \varphi = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{47}$$

and the corresponding positive eigenfunction we denote by ψ_1. Taking ψ_1 as a test function in the first equation of $(S_{\lambda,\mu})$ and using integration by parts we obtain

$$\int_\Omega \left(-\mu_1 u + \frac{\lambda f(x)}{(1-v)^2}\right)\psi_1 \,\mathrm{d}x = 0$$

which implies that $\lambda > \lambda^*$ when

$$-\mu_1 u + \frac{\lambda f(x)}{(1-v)^2} > 0 \text{ in } \Omega. \tag{48}$$

After a simple calculation we find that (48) holds when

$$\lambda > \frac{4\mu_1}{27}\frac{1}{\inf_\Omega f(x)}.$$

Using the same approach in the second equation we finish the proof. □

5.5 Monotonicity results for the extremal parameter

Let $G_\Omega(x,\xi) = G(x,\xi)$ be the Green's function of the Laplace operator for the region Ω, with $G(x,\xi) = 0$ if $x \in \partial\Omega$. We shall write $(u_{n,\Omega}(x), v_{n,\Omega}(x)) = (u_n(x), v_n(x))$ for the sequence obtained by the interaction process as follows: $(u_0, v_0) = (0,0)$ in Ω and

$$\begin{cases} u_n(x) = \displaystyle\int_\Omega \frac{\lambda f(x) G(x,\xi)}{(1-v_{n-1})^2}\, d\xi \text{ in } \Omega, \\ v_n(x) = \displaystyle\int_\Omega \frac{\mu g(x) G(x,\xi)}{(1-u_{n-1})^2}\, d\xi \text{ in } \Omega. \end{cases} \tag{49}$$

It is easy to see that the sequence above converges uniformly for a minimal solution of $(S_{\lambda,\mu})$ provided that $0 < \lambda < \lambda^*$ and $0 < \mu < \Gamma(\lambda)$. This construction will help us to prove the monotonicity result for λ^* stated in Theorem 5.4.

We also establish some monotonicity properties for the critical parameter λ^*.

Theorem 5.4. If $(S_{\lambda,\mu})$ has a solution in Ω, then it also has a solution for any subdomain $\Omega' \subset \Omega$ for which the Green's function exists. Furthermore, $\lambda^*(\Omega') \geq \lambda^*(\Omega)$ and for the corresponding minimal solutions, we have

$$(u_{\Omega'}(x), v_{\Omega'}(x)) \leq (u_\Omega(x), v_\Omega(x)) \text{ in } \Omega.$$

Proof of Theorem 5.4. Let $(u_{n,\Omega'}, v_{n,\Omega'})$ be defined as in (49) with Ω replaced by Ω'. Using the corresponding Green's functions for the subdomains $\Omega' \subset \Omega$ satisfy the inequality

$$G_{\Omega'}(x,\xi) \leq G_\Omega(x,\xi)$$

we have

$$u_{1,\Omega'}(x) = \int_{\Omega'} \lambda f(x) G_{\Omega'}(x,\xi) d\xi \leq \int_\Omega \lambda f(x) G_\Omega(x,\xi)\, d\xi \text{ in } \Omega',$$
$$v_{1,\Omega'}(x) = \int_{\Omega'} \mu g(x) G_{\Omega'}(x,\xi)\, d\xi \leq \int_\Omega \mu g(x) G_\Omega(x,\xi)\, d\xi \text{ in } \Omega'.$$

By induction we conclude that

$$u_{n,\Omega'}(x) \leq u_{n,\Omega}(x) \text{ in } \Omega',$$
$$v_{n,\Omega'}(x) \leq v_{n,\Omega}(x) \text{ in } \Omega'.$$

On the other hand, since

$$u_{n,\Omega}(x) \leq u_{n+1,\Omega}(x) \text{ in } \Omega,$$
$$v_{n,\Omega}(x) \leq v_{n+1,\Omega}(x) \text{ in } \Omega,$$

for each n, we get that

$$u_{n,\Omega'}(x) \le u_\Omega(x) \text{ in } \Omega',$$
$$v_{n,\Omega'}(x) \le v_\Omega(x) \text{ in } \Omega',$$

and we are done. □

Corollary 5.1. Suppose $f_1, f_2, g_1, g_2 : \overline{\Omega} \to \mathbb{R}$ satisfy condition (H) and

$$f_1(x) \le f_2(x) \text{ and } g_1(x) \le g_2(x) \text{ for all } x \in \Omega,$$

then $\lambda^*(f_1, g_1) \ge \lambda^*(f_2, g_2)$ and for each $\lambda \in (0, \lambda^*(f_2, g_2))$. Furthermore

$$(u_1(x), v_1(x)) \le (u_2(x), v_2(x)) \text{ for all } x \in \Omega$$

for the corresponding minimal solutions. If

$$f_1(x) < f_2(x) \text{ or } g_1(x) < g_2(x)$$

on a subset of positive measure, then

$$(u_1(x), v_1(x)) < (u_2(x), v_2(x)) \text{ for all } x \in \Omega.$$

We shall use Schwarz symmetrization method (94). Let $B_R = B_R(0)$ the Euclidian ball in $\mathbb{R}^N$ with radius $R > 0$ centered at origin such that $|B_R|=|\Omega|$, and let $u^\sharp$ be the symmetrization of u, then it is well known that $u^\sharp$ depends only on $|x|$ and $u^\sharp$ is a decreasing function of $|x|$.

Theorem 5.5. Let f, g satisfying (H) and $f^\sharp, g^\sharp$ the Schwarz symmetrization of f and g respectively. Then $\lambda^*(\Omega, f, g) \ge \lambda^*(B_R, f^\sharp, g^\sharp)$ and for each $\lambda \in (0, \lambda^*(B_R, f, g))$ we have $\Gamma_{(\Omega,f,g)}(\lambda) \ge \Gamma_{(B_R,f^\sharp,g^\sharp)}(\lambda)$.

Proof of Theorem 5.5. For each $\lambda \in (0, \lambda^*(B_R, f, g))$ and $\mu \in (0, \Gamma_{(B_R,f^\sharp,g^\sharp)}(\lambda))$ we consider the minimal sequence (u_n, v_n) for $(S_{\lambda,\mu})$ as defined in (51), and let $(\widehat{u}_n, \widehat{v}_n)$ be the minimal sequence for the corresponding Schwarz symmetrized problem:

$$\begin{cases} -\Delta u = \dfrac{\lambda f^\sharp(x)}{(1-v)^2} & \text{in } B_R, \\ -\Delta v = \dfrac{\mu g^\sharp(x)}{(1-u)^2} & \text{in } B_R, \\ 0 < u, v < 1 & \text{in } B_R, \\ u = v = 0 & \text{on } \partial B_R. \end{cases} \tag{50}$$

Since $\lambda \in (0, \lambda^*(B_R, f, g))$ and $\mu \in (0, \Gamma_{(B_R, f^\sharp, g^\sharp)}(\lambda))$ we can consider the corresponding minimal solution $(\widehat{u}, \widehat{v})$ of (50). As in the proof of Lemma 5.7 we have $0 < \widehat{u}_n \le \widehat{u} < 1$ and $0 < \widehat{v}_n \le \widehat{v} < 1$ on B_R for all n. We shall prove for the sequence (u_n, v_n) we also have $0 < u_n^\sharp \le \widehat{u} < 1$ and $0 < v_n^\sharp \le \widehat{v} < 1$ on B_R for all n. Therefore, the minimal sequence (u_n, v_n) for $(S_{\lambda,\mu})$ satisfies

$$u_n(x) \le \max_{x \in B_R} \widehat{u} \quad \text{and} \quad v_n(x) \le \max_{x \in B_R} \widehat{v}$$

and again as in the proof of Lemma 5.1, there exists a minimal solution (u, v) for $(S_{\lambda,\mu})$. $\square$

5.6 The branch of minimal solutions

Next, assuming the existence of solutions for System $(S_{\lambda,\mu})$, we obtain also existence and uniqueness of minimal solution.

Lemma 5.7. For any $0 < \lambda < \lambda^*$ and $0 < \mu < \Gamma(\lambda)$, there exists a unique minimal solution (u, v) of $(S_{\lambda,\mu})$.

Proof. This minimal solution is obtained as the limit of the sequence of pair of functions (u_n, v_n) constructed recursively as follows: $(u_0, v_0) = (0, 0)$ in Ω and for each $n = 1, 2, \ldots$, (u_n, v_n) is the unique solution of the boundary value problem:

$$\begin{cases} -\Delta u_n = \dfrac{\lambda f(x)}{(1 - v_{n-1})^2} & \text{in } \Omega, \\ -\Delta v_n = \dfrac{\mu g(x)}{(1 - u_{n-1})^2} & \text{in } \Omega, \\ 0 < u_n, v_n < 1 & \text{in } \Omega, \\ u_n = v_n = 0 & \text{on } \partial\Omega. \end{cases} \tag{51}$$

Let (U, V) be any solution for problem $(S_{\lambda,\mu})$. First, it is clear that $1 \ge U > u_0 \equiv 0$ and $1 \ge V > v_0 \equiv 0$ in Ω. Now, assume that $U \ge u_{n-1}$ and $V \ge v_{n-1}$ in Ω. Thus,

$$\begin{cases} -\Delta(U - u_n) = \lambda f(x) \left[\dfrac{1}{(1 - V)^2} - \dfrac{1}{(1 - u_{n-1})^2} \right] \ge 0 & \text{in } \Omega, \\ -\Delta(V - v_n) = \mu g(x) \left[\dfrac{1}{(1 - U)^2} - \dfrac{1}{(1 - v_{n-1})^2} \right] \ge 0 & \text{in } \Omega, \\ U - u_n = V - v_n = 0 & \text{on } \partial\Omega. \end{cases} \tag{52}$$

By the maximum principle we conclude that

$$1 > U \ge u_n > 0 \text{ and } 1 > V \ge v_n > 0 \text{ in } \Omega.$$

It is clear that this kind of argument implies that (u_n, v_n) is a monotone increasing sequence. Therefore, (u_n, v_n) converges uniformly to a solution (u, v) of $(S_{\lambda,\mu})$, which by construction is unique in this class of minimal solutions. $\square$

We can introduce for any solution u of (P_λ), the linearized operator at u defined by $L_{u,\lambda} = -\Delta - \frac{2\lambda f(x)}{(1-u)^3}$ and its eigenvalues $\{\mu_{k,\lambda}(u); k = 1, 2, ...\}$. The first eigenvalue is then simple and can be characterized variationally by

$$\mu_{1,\lambda}(u) = \inf \left\{ \langle L_{u,\lambda}\phi, \phi \rangle_{H_0^1(\Omega)} \,;\, \phi \in C_0^\infty(\Omega), \int_\Omega |\phi(x)|^2 dx = 1 \right\}.$$

Stable solutions (resp., *semi-stable solutions*) of $(S)_\lambda$ are those solutions u such that $\mu_{1,\lambda}(u) > 0$ (resp., $\mu_{1,\lambda}(u) \geq 0$).

In the case that (u, v) is a solution of $(S_{\lambda,\mu})$ we consider the first eigenvalue $\nu_1 = \nu_1((\lambda, \mu), (u, v))$ of the linearization $\mathfrak{L} := -\overrightarrow{\Delta} - A(x)$ around (u, v) under Dirichlet boundary conditions, where

$$\overrightarrow{\Delta}\Phi = \begin{pmatrix} \Delta\phi_1 \\ \Delta\phi_2 \end{pmatrix} \text{ and } A(x) := \begin{pmatrix} 0 & a_{12}(x) \\ a_{21}(x) & 0 \end{pmatrix} = \begin{pmatrix} 0 & \frac{2\lambda f(x)}{(1-v(x))^3} \\ \frac{2\mu g(x)}{(1-u(x))^3} & 0 \end{pmatrix}$$

that is, the eigenvalue problem

$$\mathfrak{L}\Phi = \nu\Phi, \quad \Phi \in W_0^{1,2}(\Omega) \times W_0^{1,2}(\Omega),$$

namely, ν_1 is the first eigenvalue of the problem

$$\begin{cases} -\Delta\phi_1 - \dfrac{2\lambda f(x)}{(1-v)^3}\phi_2 = \nu\phi_1 \text{ in } \Omega, \\ -\Delta\phi_2 - \dfrac{2\mu g(x)}{(1-u)^3}\phi_1 = \nu\phi_2 \text{ in } \Omega, \\ \qquad \phi_1 = \phi_2 = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{$E_{(\lambda,\mu)}$}$$

We recall that in (95, Proposition 3.1) was proved that there exists a unique eigenvalue ν_1 with strictly positive eigenfunction $\phi = (\phi_1, \phi_2)$ of $(E_{(\lambda,\mu)})$, that is, $\phi_i > 0$ in Ω for $i = 1, 2$.

Remark 5.1. The first eigenvalue of the linearized single equation has a variational characterization; no such analogous formulation is available for our system.

Definition 5.1 (Stable and Semi-stable Solution)**.** A solution of problem $(S_{\lambda,\mu})$ is said to be stable (resp. semi-stable) if $\nu_1 > 0$ (resp., $\nu_1 \geq 0$).

Proposition 5.2. Suppose that $(\lambda, \mu) \in \Lambda$ with $0 < \mu \leq \lambda$ and we let (u, v) denote the minimal solution of $(S_{\lambda,\mu})$. Let ν, ϕ_1, ϕ_2 as in $(E_{(\lambda,\mu)})$. Then

$$\frac{\phi_2}{\phi_1} \geq \frac{\mu}{\lambda} \quad \text{in } \Omega.$$

Proof. Take the difference equation in $(E_{(\lambda,\mu)})$ and use Lemma 5.2 to obtain

$$\begin{aligned} -\Delta(\phi_2 - \phi_1) &= \frac{2\mu\phi_1}{(1-u)^3} - \frac{2\lambda\phi_2}{(1-v)^3} + \nu(\phi_2 - \phi_1) \\ &\geq \frac{\mu(\phi_1 - \phi_2)}{(1-v)^3} + \frac{(\mu-\lambda)\phi_2}{(1-v)^3} + \nu(\phi_2 - \phi_1). \end{aligned}$$

Rewriting this we have

$$-\Delta(\phi_2 - \phi_1) - \nu(\phi_2 - \phi_1) + \frac{\mu(\phi_2 - \phi_1)}{(1-v)^3} \geq \frac{(\mu-\lambda)\phi_2}{(1-v)^3} \quad \text{in } \Omega.$$

Now, define a elliptic operator $L := -\Delta - \nu$. We have that

$$\begin{aligned} L\left(\psi_2 - \psi_1 + \frac{\lambda-\mu}{\lambda}\psi_1\right) &+ \frac{\mu}{(1-v)^3}\left(\psi_2 - \psi_1 + \frac{\lambda-\mu}{\lambda}\right) \\ &\geq L\left(\psi_2 - \psi_1 + \frac{\lambda-\mu}{\lambda}\psi_1\right) + \frac{\mu}{(1-v)^3}(\psi_2 - \psi_1) \\ &\geq \frac{(\mu-\lambda)\phi_2}{(1-v)^3} + \frac{\lambda-\mu}{\lambda}L(\phi_1) = 0 \end{aligned}$$

Using the maximum principle, we have

$$\phi_2 - \phi_1 + \frac{\lambda-\mu}{\lambda}\phi_1 \geq 0 \text{ in } \Omega.$$

Re-arranging the above equation follows

$$\frac{\phi_2}{\phi_1} \geq \frac{\mu}{\lambda}$$

and this finish the proof. □

5.7 Estimates for minimal solutions

The next lemmas are crucial to obtain the estimates which are already behind the proof of the regularity of semi-stable solutions. We start with a lemma that can be found in (96).

Lemma 5.8. Suppose (u,v) is a smooth minimal solution of $-\Delta u = \gamma G(u,v)$, $-\Delta v = \lambda F(u,v)$ in Ω with $u = v = 0$ on $\partial\Omega$. Here, $F(u,v)$ and $G(u,v)$ are positive nonlinearities which are increasing in u and v. Then

$$\int \left(\gamma G_u \alpha^2 + \lambda F_v \beta^2\right) \mathrm{d}x + 2\int \sqrt{F_u G_v}\alpha\beta \, \mathrm{d}x \le \int |\nabla \alpha|^2 \, \mathrm{d}x + \int |\nabla \beta|^2 \, \mathrm{d}x$$

for all $\alpha, \beta \in H_0^1(\Omega)$.

Lemma 5.9. Suppose that $(u_\lambda, v_{\sigma\lambda})$ denotes a sequence of smooth minimal solutions of $(S_{\lambda,\sigma\lambda})$ where $0 < \sigma \le 1$ and let $0 < t < \sqrt{\sigma} + \sqrt{\sigma + \sqrt{\sigma}}$. Then $(u_\lambda, v_{\sigma\lambda})$ is uniformly bounded in $L^p(\Omega) \times L^p(\Omega)$ for all $p \le t + 3/2$.

Proof. Taking $\alpha = \beta = (1-u)^{-t} - 1$ in Lemma 5.8 we have that

$$\sqrt{\lambda\mu} \int_\Omega \sqrt{\frac{1}{(1-u)^3}\frac{1}{(1-v)^3}} \left((1-u)^{-t} - 1\right)^2 \mathrm{d}x \le t^2 \int_\Omega (1-u)^{-2t-2} |\nabla u|^2 \, \mathrm{d}x.$$

Multiplying $-\Delta u = \frac{\lambda}{(1-v)^2}$ by $(1-u)^{-2t-1} - 1$ and integrating by parts we have

$$(2t+1) \int_\Omega |\nabla u|^2 (1-u)^{-2t-2} \, \mathrm{d}x = \int_\Omega \frac{\lambda\left((1-u)^{-2t-1} - 1\right)}{(1-v)^2} \, \mathrm{d}x.$$

Follows that

$$\sqrt{\sigma} \int_\Omega \sqrt{\frac{1}{(1-u)^3}\frac{1}{(1-v)^3}} \left((1-u)^{-t} - 1\right)^2 \mathrm{d}x \le \frac{t^2}{2t+1} \int_\Omega \frac{\left((1-u)^{-2t-1} - 1\right)}{(1-v)^2} \, \mathrm{d}x$$

By monotonicity, follows that

$$\left[\sqrt{\sigma} - \frac{t^2}{2t+1}\right] \int_\Omega \frac{1}{(1-v)^2} \frac{1}{(1-u)^{2t+1}} \, \mathrm{d}x \le 2 \int_\Omega \frac{1}{(1-v)^2} \frac{1}{(1-u)^{t+1}} \, \mathrm{d}x$$

Using Hölder inequality with conjugate exponents $p_1 = \frac{2t+1}{t+1}$ and $q_1 = \frac{2t+1}{t}$ we have

$$\begin{aligned} \left[\sqrt{\sigma} - \frac{t^2}{2t+1}\right] &\int_\Omega \frac{1}{(1-v)^2} \frac{1}{(1-u)^{2t+1}} \, \mathrm{d}x \\ &\le 2 \left[\int_\Omega \frac{1}{(1-v)^2} \frac{1}{(1-u)^{2t+1}} \, \mathrm{d}x\right]^{1/p_1} \left[\int_\Omega \frac{1}{(1-v)^2} \, \mathrm{d}x\right]^{1/q_1}. \end{aligned}$$

Thus

$$\left[\sqrt{\sigma} - \frac{t^2}{2t+1}\right] \int_\Omega \frac{1}{(1-v)^2} \frac{1}{(1-u)^{2t+1}} \, \mathrm{d}x \le 2 \int_\Omega \frac{1}{(1-v)^2} \, \mathrm{d}x.$$

Using the monotonicity and Hölder inequality with conjugate exponents $p_2 = \frac{2t+3}{2}$ and $q_2 = \frac{2t+3}{2t+1}$, we have

$$\int_\Omega \frac{1}{(1-v)^{2t+3}} \, \mathrm{d}x \le 2 \int_\Omega \frac{1}{(1-v)^2} \, \mathrm{d}x \le C \left[\int_\Omega \frac{1}{(1-v)^{2t+3}} \, \mathrm{d}x\right]^{1/p_2}.$$

Follows that

$$\left[\int_\Omega \frac{1}{(1-v)^{2t+3}}\,\mathrm{d}x\right]^{1/q_2} \leq C.$$

□

Analogously to scalar case as in (16), we can define the notion of extremal solution of $(S_{\lambda,\mu})$ for points on the critical curve. Precisely, for (λ^*,μ^*) a point on the critical curve Γ, we can write $\mu^* = \sigma\lambda^*$ for some $\sigma > 0$. Let us consider (λ_n) an increasing sequence converging to λ^*, and consequently $\sigma\lambda_n^* \to \sigma\lambda^* = \mu^*$. In view of Theorem 5.1, we can consider the minimal solution $(u_{\lambda_n}, v_{\sigma\lambda_n})$ of System $(S_{\lambda_n,\sigma\lambda_n})$. Now, we can define the extremal solution (u^*,v^*) at (λ^*,μ^*) by passing to the limit when $\lambda_n \nearrow \lambda^*$ along the ray given by $\mu = \sigma\lambda$, namely,

$$(u^*,v^*) = \lim_{\lambda\nearrow\lambda^*}(u_{\lambda_n}, v_{\sigma\lambda_n}).$$

The following theorem deals with regularity properties for solutions of $(S_{\lambda,\mu})$. Our method are based on systematic use of maximum principle in combination with energy estimates to conclude that extremal solutions of $(S_{\lambda,\mu})$ are smooth in lower dimensions provided one stays close to the diagonal of (λ,μ)-plane, precisely

Theorem 5.6. Assume that f, g are bounded nontrivial functions and let $0 < \sigma \leq 1$. Then the extremal solution (u^*,v^*) of System $(S_{\lambda^*,\sigma\lambda^*})$ is smooth when $N \leq 5$.

Proof of Theorem 5.6. The above estimate said that $(1-u)^3$ is bounded uniformly in λ over $L^p(\Omega)$ for all $p \leq 2,6$. By elliptic estimates, u is uniformly bounded in $W_0^{2,p}(\Omega)$. Taking the limit in λ, we obtain that u^* is a classical solution when $N \leq 5$. □

Remark 5.2. It remains an interesting and open question to determine the critical dimension for this class of Lane-Emden systems, precisely determine the dimension N^* such that the extremal solution is smooth when $N < N^*$ and singular when $N \geq N^*$. In Theorem 5.6 we prove that the extremal solution $(u_{\lambda^*}, v_{\sigma\lambda^*})$ is smooth when $N \leq 5$. Now, if Ω is the unit ball, we take $u = v$ and the system turns into a scalar equation and the function $u^*(x) = 1 - |x|^{2/3}$ is a singular solution for $(S_{\lambda^*,\sigma\lambda^*})$ if $N \geq 8$. In view of this facts we believe that $N^* = 8$ is the critical dimension for $(S_{\lambda,\mu})$.

Remark 5.3. Using a result due to W. Troy (97, Theorem 1), we can see that any smooth solution of $(S_{\lambda,\mu})$ is radially symmetric and decreasing when Ω is a ball of $\mathbb{R}^N$.

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