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On linearly coupled systems of Schrödinger equations with critical growth

por

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João Pessoa - PB

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sob orientação do

Prof. Dr. João Marcos Bezerra do Ó

Tese apresentada ao Corpo Docente do Programa Associado de Pós-Graduação em Matemática - UFPB/UFCG, como requisito parcial para obtenção do título de Doutor em Matemática.

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Resumo

Neste trabalho estudamos a existência de ground states para a seguinte classe de sistemas acoplados envolvendo equações de Schrödinger não-lineares

$$\begin{cases} -\Delta u + V_1(x)u = f_1(x,u) + \lambda(x)v, & x \in \mathbb{R}^N, \\ -\Delta v + V_2(x)v = f_2(x,v) + \lambda(x)u, & x \in \mathbb{R}^N, \end{cases}$$

onde os potenciais $V_1 : \mathbb{R}^N \to \mathbb{R}$, $V_2 : \mathbb{R}^N \to \mathbb{R}$ são não-negativos e estão relacionados com o termo de acomplamento $\lambda : \mathbb{R}^N \to \mathbb{R}$ por $|\lambda(x)| < \delta\sqrt{V_1(x)V_2(x)}$, para algum $0 < \delta < 1$. No caso $N = 2$, as não-linearidades f_1 e f_2 possuem crescimento crítico exponencial no sentido da desigualdade de Trudinger-Moser. No caso $N \geq 3$, as não-linearidades são polinômios com expoente subcrítico e crítico no sentido de Sobolev. Estudamos ainda a seguinte classe de sistemas acoplados não-locais

$$\begin{cases} (-\Delta)^{1/2}u + V_1(x)u = f_1(u) + \lambda(x)v, & x \in \mathbb{R}, \\ (-\Delta)^{1/2}v + V_2(x)v = f_2(v) + \lambda(x)u, & x \in \mathbb{R}, \end{cases}$$

onde $(-\Delta)^{1/2}$ denota o operador raíz quadrada do laplaciano e as não-linearidades possuem crescimento crítico exponencial. Nossa abordagem é variacional e baseada na técnica de minimização sobre a variedade de Nehari.

Palavras-chave: Sistemas linearmente acoplados; Soluções de energia mínima; Variedade de Nehari; Crescimento crítico; Desigualdade de Trudinger-Moser.

Abstract

In this work we study the existence of ground states for the following class of coupled systems involving nonlinear Schrödinger equations

$$\begin{cases} -\Delta u + V_1(x)u = f_1(x,u) + \lambda(x)v, & x \in \mathbb{R}^N, \\ -\Delta v + V_2(x)v = f_2(x,v) + \lambda(x)u, & x \in \mathbb{R}^N, \end{cases}$$

where the potentials $V_1 : \mathbb{R}^N \to \mathbb{R}$, $V_2 : \mathbb{R}^N \to \mathbb{R}$ are nonnegative and related with the coupling term $\lambda : \mathbb{R}^N \to \mathbb{R}$ by $|\lambda(x)| < \delta\sqrt{V_1(x)V_2(x)}$, for some $0 < \delta < 1$. In the case $N = 2$, the nonlinearities f_1 e f_2 have critical exponential growth in the sense of Trudinger-Moser inequality. In the case $N \geq 3$, the nonlinearities are polynomials with subcritical and critical exponent in the Sobolev sense. We study also the following class of nonlocal coupled systems

$$\begin{cases} (-\Delta)^{1/2}u + V_1(x)u = f_1(u) + \lambda(x)v, & x \in \mathbb{R}, \\ (-\Delta)^{1/2}v + V_2(x)v = f_2(v) + \lambda(x)u, & x \in \mathbb{R}, \end{cases}$$

where $(-\Delta)^{1/2}$ denotes the square root of the Laplacian operator and the nonlinearities have critical exponential growth. Our approach is variational and based on minimization technique over the Nehari manifold

Keywords: Linearly couples systems; Ground state solution; Nehari manifold; Critical growth; Trudinger-Moser inequality.

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Dedicatória

À minha família...

Contents

Introduction

The present work is concerned to study the existence of ground states for the following class of coupled systems

$$\begin{cases} -\Delta u + V_1(x)u = f_1(x,u) + \lambda(x)v, & x \in \mathbb{R}^N, \\ -\Delta v + V_2(x)v = f_2(x,v) + \lambda(x)u, & x \in \mathbb{R}^N, \end{cases} \tag{1}$$

where the potentials $V_1 : \mathbb{R}^N \to \mathbb{R}$, $V_2 : \mathbb{R}^N \to \mathbb{R}$ are nonnegative and related with the coupling term $\lambda : \mathbb{R}^N \to \mathbb{R}$ by $|\lambda(x)| < \delta\sqrt{V_1(x)V_2(x)}$ for some $0 < \delta < 1$. Ground states are solutions with minimal energy among the energy of all nontrivial solutions. In the case $N = 2$, we study System (1) when the nonlinearities $f_1 : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ and $f_2 : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ have critical exponential growth motivated by classes of Trudinger-Moser inequalities introduced in [14] and [34]. The case $N \geq 3$ is studied when the nonlinearities are polynomials involving subcritical and critical exponent in the Sobolev sense. We are also concerned with the following class of coupled systems involving the nonlocal operator *square root of the Laplacian*

$$\begin{cases} (-\Delta)^{1/2}u + V_1(x)u = f_1(u) + \lambda(x)v, & x \in \mathbb{R}, \\ (-\Delta)^{1/2}v + V_2(x)v = f_2(v) + \lambda(x)u, & x \in \mathbb{R}. \end{cases} \tag{2}$$

The nonlinearities have critical exponential growth motivated by a class of Trudinger-Moser inequality introduced by T. Ozawa, see [58]. Throughout the thesis we will detail the assumptions required over the potentials and the nonlinearities.

The study of ground state solutions for coupled systems has made great progress and attracted attention of many authors for its great physical interest. Solutions of System (1) are related with standing waves of the following two-component system

$$\begin{cases} -i\dfrac{\partial \psi}{\partial t} = \Delta\psi - V_1(x)\psi + f_1(x,\psi) + \lambda(x)\phi, & x \in \mathbb{R}^N,\ t \geq 0, \\ -i\dfrac{\partial \phi}{\partial t} = \Delta\phi - V_2(x)\phi + f_2(x,\phi) + \lambda(x)\psi, & x \in \mathbb{R}^N,\ t \geq 0, \end{cases} \tag{3}$$

where i denotes the imaginary unit. Such class of systems arise in various branches of mathematical physics and nonlinear topics, and can describe different physical phenomena, such as Bose-Einstein condensates, Bose-Fermi mixture, propagation in birefringent optical fibers and Kerr-like photorefractive media in optics, see e.g. [2, 23, 48, 55, 62]. For System (3), a solution of the form

$$(\psi(x,t), \phi(x,t)) = (\exp(-iEt)u(x), \exp(-iEt)v(x)),$$

where E is some real constant is called *standing wave solution*. There are some papers involving existence of standing waves under various hypotheses on the potentials and the nonlinearities. We refer the readers to [5, 10, 11, 19–21, 44, 49, 50, 65–67, 72] and the references therein. Assuming that $f_j(x, s\xi) = f_j(x, s)\xi$, for all $s \in \mathbb{R}$, $j = 1, 2$ and $\xi \in \mathbb{C}$ with $|\xi| = 1$, it can be deduced that (ψ, ϕ) is a solution of (3) if and only if (u, v) solves the following system

$$\begin{cases} -\Delta u + (V_1(x) - E)u = f_1(x,u) + \lambda(x)v, & x \in \mathbb{R}^N, \\ -\Delta v + (V_2(x) - E)v = f_2(x,v) + \lambda(x)u, & x \in \mathbb{R}^N. \end{cases}$$

For convenience and without loss of generality, replacing $V_i(x) - E$ by $V_i(x)$, i.e., shifting E to 0, we turn to consider system (1).

Notice that if $\lambda \equiv 0$, $V_1 \equiv V_2 \equiv V$, $f_1 \equiv f_2 \equiv f$ and $u \equiv v$, then System (1) reduces to the nonlinear Schrödinger equation

$$-\Delta u + V(x)u = f(x,u), \quad x \in \mathbb{R}^N. \tag{4}$$

This class of equations has been widely studied by many researchers. In order to overcome the difficulty originated from the lack of compactness, it was introduced several classes of potentials. For instance, P. Rabinowitz, [61], considered a class of potentials bounded away from zero and coercive. He applied variational methods based on variants of Mountain Pass Theorem to get existence results for (4) when $f(x,s)$ is subcritical or superlinear. In order to improve the behavior of the potentials introduced in [61], T. Bartsch and Z.Q. Wang, [9], considered a class of uniformly positive potentials such that the level sets $\{x \in \mathbb{R}^N : V(x) \leq M\}$ have finite Lebesgue measure for all $M > 0$. Besides to weaken the previous hypothesis under $V(x)$, they also improved the existence results getting infinitely many solutions if $f(x,s)$

is odd in s, that is, $f(x,-s) = -f(x,s)$. In [64], B. Sirakov improved the class of potentials contained in [9] and preserve the compactness of the energy functional associated to (4). For more works concerning the scalar equation (4) we refer the readers to [7, 8, 10, 11, 67] and references therein. Concerning to problems defined in 2−dimensional domains and involving nonlinearities with exponential growth, we refer the readers to [3, 14, 26, 33, 35, 56, 70] and references therein.

Our work was motivated by some papers that have appeared in the recent years concerning the study of coupled systems involving nonlinear Schrödinger equations by using variational approach. In [17], Z. Chen and W. Zou studied the existence of ground states for the following class of critical coupled systems with constant potentials

$$\begin{cases} -\Delta u + \mu u = |u|^{p-2}u + \lambda v, & x \in \mathbb{R}^N, \\ -\Delta v + \nu v = |v|^{2^*-2}v + \lambda u, & x \in \mathbb{R}^N, \end{cases} \tag{5}$$

when $N \geq 3$ and $1 < p < 2^*-1$, where $2^* = 2N/(N-2)$ is the critical Sobolev exponent. They proved that there exists critical parameters $\mu_0 > 0$ and $\lambda_{\mu,\nu} \in [\sqrt{(\mu-\mu_0)\nu}, \sqrt{\mu\nu})$ such that (5) has a positive ground state when $\lambda > \lambda_{\mu,\nu}$ and has no ground state solutions when $\mu > \mu_0$ and $\lambda < \lambda_{\mu,\nu}$. Coupled systems of nonlinear Schrödinger equations of the type

$$\begin{cases} -\Delta u + \mu u = (1 + a(x))|u|^{p-1}u + \lambda v, & x \in \mathbb{R}^N, \\ -\Delta v + \nu v = (1 + b(x))|v|^{q-1}v + \lambda u, & x \in \mathbb{R}^N, \end{cases}$$

were studied by A. Ambrosetti [4] with $N = 1$ and A. Ambrosetti, G. Cerami, D. Ruiz [6] with $N \geq 2$. In [6], the authors used concentration compactness type arguments to prove existence of positive bound and ground states when $\mu = \nu = 1$, $\lambda \in (0,1)$, $1 < p = q < 2^* - 1$, $a(x)$ and $b(x)$ vanishing at infinity. In [18], Z. Chen and W. Zou extended and complemented some results introduced in [6], studying the following class of coupled systems

$$\begin{cases} -\Delta u + \mu u = f_1(u) + \lambda v, & x \in \mathbb{R}^N, \\ -\Delta v + \nu v = f_2(v) + \lambda u, & x \in \mathbb{R}^N. \end{cases}$$

The authors obtained the existence of positive radial ground states and energy estimates giving a description of the limit behavior as the parameter λ goes to zero. For more existence results concerning coupled systems we refer to [16, 43, 51, 54, 59, 73] and references therein. Note that in all of these works it was only considered nonlinearities involving polynomial growth of subcritical or critical type in terms of Sobolev embedding. On the nonlinear elliptic problems involving critical growth of Trudinger-Moser type, we refer the readers to [24,25,29,34,47,60] and references therein.

Motivated by concrete applications in many fields of physics, biology and mathematics, a great attention has been devoted to study the fractional nonlinear Schrödinger equation

$$(-\Delta)^s u + V(x)u = f(x,u), \quad x \in \mathbb{R}^N, \ 0 < s < 1,$$

under many different assumptions on the potential $V(x)$ and on the nonlinearity $f(x,u)$. In [40], it was proved the existence of positive solutions for the case when $V \equiv 1$ and $f(x,u)$ has subcritical growth in the Sobolev sense. In order to overcome the lack of compactness, the authors used a comparison argument. Another way to overcome this difficulty is requiring coercive potentials, that is, $V(x) \to +\infty$, as $|x| \to +\infty$. In this direction, the existence of ground states was studied by M. Cheng, [22], considering a polynomial nonlinearity, and S. Secchi, [63], considering a more general nonlinearity in the subcritical case. For existence results involving another types of potentials, we refer [15, 31, 41] and references therein. We point out that in all of these works it were consider dimension $N \geq 2$ and nonlinearities with polynomial growth.

It is known that when $s \to 1$, the fractional Laplacian $(-\Delta)^s$ reduces to the standard Laplace operator $-\Delta$, see [30]. In the fractional case, the critical Sobolev exponent is given by $2_s^* = 2N/(N-2s)$. If $0 < s < N/2$, then the fractional Sobolev space $H^s(\mathbb{R}^N)$ is continuously embedded into $L^q(\mathbb{R}^N)$, for any $q \in [2, 2_s^*]$. Thus, similarly the standard Laplacian case, the maximal growth on the nonlinearity $f(x,u)$ which allows to treat nonlinear fractional Schrödinger equations variationally in $H^s(\mathbb{R}^N)$, is given by $|u|^{2_s^*-1}$, when $|u| \to +\infty$. For $N = 1$ and $s \rightsquigarrow 1/2$, we have $2_s^* \rightsquigarrow +\infty$. In this case, $H^{1/2}(\mathbb{R})$ is continuously embedded into $L^q(\mathbb{R})$, for $q \in [2, +\infty)$. However, $H^{1/2}(\mathbb{R})$ is not continuously embedded into $L^\infty(\mathbb{R})$. For more details we refer the reader to [30] and the bibliographies therein. In this work, we deal with the limiting case, when $N = 1$, $s = 1/2$ and nonlinearities with the maximum growth which allows to treat System (2) variationally. For existence results considering the limiting case we refer the readers to [27, 28, 36, 37, 45] and references therein.

Motivated by the above discussion, our work is concerned to study the existence of ground states for coupled systems under several assumptions on the potentials and nonlinearities involving critical growth. Though there has been some works in this direction, not much has been done for the classes of coupled systems introduced by (1) and (2) when the nonlinear terms reached critical exponential growth. These classes of systems imposes some difficulties. The first one is the lack of compactness due to the fact that they are defined in the whole Euclidean space $\mathbb{R}^N$, which roughly speaking, originates from the invariance of $\mathbb{R}^N$ with respect to translation and dilation. Moreover, the systems involve strongly coupled Schrödinger equations because of the linear terms in the right hand side. System (2) has an additional difficulty that is the presence of the square root of the Laplacian which is a nonlocal operator, that is, it takes care of the behavior of the solution in the whole space. To overcome these difficulties, we shall use a variational approach based on Nehari manifold. The literature on the Nehari manifold is rather extensive and for a description of this subject, see for example [68]. In the following, we describe each chapter of the thesis.

In Chapter 1, we study the following class of coupled systems

$$\begin{cases} -\Delta u + V_1(x)u = \mu|u|^{p-2}u + \lambda(x)v, & x \in \mathbb{R}^N, \\ -\Delta v + V_2(x)v = |v|^{q-2}v + \lambda(x)u, & x \in \mathbb{R}^N, \end{cases} \tag{6}$$

where $N \geq 3$, $2 < p \leq q \leq 2^*$ and $2^* = 2N/(N-2)$ is the critical Sobolev exponent. Throughout the thesis, the coupling term λ will be related with the potentials V_1 and V_2 by the assumption

$$|\lambda(x)| < \delta\sqrt{V_1(x)V_2(x)}, \quad \text{for some} \quad 0 < \delta < 1. \tag{7}$$

We divided the study of System (6) into three cases:

(i) *(subcritical case)* $2 < p \leq q < 2^*$,

(ii) *(critical case)* $2 < p < q = 2^*$,

(iii) *(critical case)* $p = q = 2^*$.

The subcritical case is related with the classical paper of H. Brezis and E.H. Lieb, [12]. They proved the existence of ground states for the following class of systems

$$-\Delta u_i(x) = g^i(u(x)), \quad i = 1, 2, ..., n, \tag{8}$$

where $g^i(u) = \partial G(u)/\partial u_i$, for some $G \in C^1(\mathbb{R}^n)$, $n \geq 2$. It can be checked that when $V_1(x) \equiv \mu$, $V_2(x) \equiv \nu$ and $\lambda(x) \equiv \lambda$, with $0 < \lambda < \delta\sqrt{\mu\nu}$, System (6) becomes a particular case of System (8), satisfying all assumptions required on g^i in [12]. However, we deal with a more general coupling term $\lambda(x)$ and two classes of nonnegative potentials: periodic and asymptotically periodic. We prove the existence of positive ground state and we use a bootstrap argument to improve the regularity of the solution. In the critical case (ii), the existence of ground state will be related with the parameter μ introduced in the first equation. Indeed, we prove that if $\mu \geq \mu_0$, for some $\mu_0 > 0$, then we get ground state. Finally, in case (iii), we make use of Pohozaev identity to conclude the nonexistence of positive classical solution for System (6).

In Chapter 2, we deal with the following class of coupled systems

$$\begin{cases} -\Delta u + V_1(x)u = f_1(x,u) + \lambda(x)v, & x \in \mathbb{R}^2, \\ -\Delta v + V_2(x)v = f_2(x,v) + \lambda(x)u, & x \in \mathbb{R}^2. \end{cases} \tag{9}$$

We consider a class of potentials introduced by B. Sirakov, [64]. Since V_1 and V_2 satisfy (7), we restrict the assumptions to nonnegative potentials. However, these hypotheses involve a large class of potentials, for instance, coercive potentials. Motivated by a class of Trudinger-Moser inequalities introduced in [34] (see Lemma 2.2.1 in Section 2.2), we study System (9) when the nonlinearities have critical exponential growth in the following sense: for $i = 1,2$ and $\alpha_0^i > 0$, $f_i : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ satisfies

$$\limsup_{s\to+\infty} \frac{f_i(x,s)}{A_i(x)(e^{\alpha s^2}-1)} = \begin{cases} 0 & \text{if} \quad \alpha > \alpha_0^i, \\ \infty & \text{if} \quad \alpha < \alpha_0^i, \end{cases}$$

where $A_i(x)$ is a suitable function introduced in (V_4) (see Chapter 2). In addition to suitable assumptions, we suppose that there exists $q > 2$ such that

$$F_1(x,s) + F_2(x,t) \geq \theta(s^q + t^q), \quad \text{for all } x \in \mathbb{R}^2 \text{ and } s,t \geq 0,$$

where $F_i(x,s) := \int_0^s F_i(x,\tau)\, d\tau$, for $i = 1,2$. Using a variational approach based on Nehari manifold we prove that there exists $\theta_0 > 0$ such that System (9) possesses a positive ground state solution, for some $\theta \geq \theta_0$. Moreover, we use a bootstrap argument to get regularity and L^q-estimates to obtain an asymptotic behavior.

In Chapter 3 we study the existence of positive ground states for the following class of coupled systems

$$\begin{cases} -\Delta u + u = f_1(u) + \lambda(x)v, & x \in \mathbb{R}^2, \\ -\Delta v + v = f_2(v) + \lambda(x)u, & x \in \mathbb{R}^2, \end{cases} \tag{10}$$

when the nonlinearities $f_1(s)$ and $f_2(s)$ have critical exponential growth motivated by a class of Trudinger-Moser inequalities introduced by D.M. Cao [14] (see Theorem A in Section 3.2). For $i = 1, 2$ the function $f_i : \mathbb{R} \to \mathbb{R}$ has α_0^i-*critical growth* at $+\infty$, that is,

$$\limsup_{s\to+\infty} \frac{f_i(s)}{e^{\alpha s^2}-1} = \begin{cases} 0 & \text{if} \quad \alpha > \alpha_0^i, \\ \infty & \text{if} \quad \alpha < \alpha_0^i. \end{cases} \tag{11}$$

In order to prove the existence of ground states we assume the following hypothesis:

$$\liminf_{s\to+\infty} \frac{sf_1(s)}{e^{\alpha_0^1 s^2}} \geq \beta_0 > \frac{2e}{\alpha_0}. \tag{12}$$

The assumption (12) was introduced in [1] and refined in [26]. It has been used in many works, see e.g. [26, 35], and plays a very important role in this chapter. Indeed, (12) will be used to get a suitable upper bound for the ground state energy level associated with System (10). Thus, the ground state energy level will be in the range where we can recover the compactness of the minimizing sequence. We study also the regularity and we obtain asymptotic behavior.

Finally, in Chapter 4 we study the existence of ground states for the following class of nonlocal coupled systems

$$\begin{cases} (-\Delta)^{1/2}u + V_1(x)u = f_1(u) + \lambda(x)v, & x \in \mathbb{R}, \\ (-\Delta)^{1/2}v + V_2(x)v = f_2(v) + \lambda(x)u, & x \in \mathbb{R}, \end{cases}$$

where $(-\Delta)^{1/2}$ denotes the *square root of the Laplace operator*. Motivated by a class of Trudinger-Moser type inequalities introduced by T. Ozawa [58] (see Theorem B in Section 4.2) we consider nonlinearities with critical exponential growth (11). Our results may be considered as the extension of the main result for the scalar case in [36]. Here we improve the class of potentials and we deal with two coupled nonlocal equations.

Notation and terminology

- $C, \tilde{C}, C_0, C_1, C_2$,... denote positive (possibly different) constants;
- C_ε or $C(\varepsilon)$ denote positive constant which depends of the parameter ε;
- $B_R(x)$ denotes the open ball of radius R and center x;
- $B_R(x)^c$ denotes the complement of $B_R(x)$;
- $|A|$ denotes the Lebesgue measure of a set $A \subset \mathbb{R}^N$;
- χ_A denotes the characteristic function of a set $A \subseteq \mathbb{R}^N$, that is,

$$\chi_A(x) = \begin{cases} 1 & \text{if} \quad x \in A, \\ 0 & \text{if} \quad x \in \mathbb{R}^N \backslash A. \end{cases}$$

- For $\Omega \subseteq \mathbb{R}^N$, $u : \Omega \to \mathbb{R}$ and $c \in \mathbb{R}$, we write

$$\{u \geq c\} = \{x \in \Omega : u(x) \geq c\} \quad \text{and} \quad \{u \leq c\} = \{x \in \Omega : u(x) \leq c\};$$

- $\rightharpoonup$ denotes weak convergence in a normed space;
- $\to$ denotes strong convergence in a normed space;
- $\langle \cdot, \cdot \rangle$ denotes the duality pairing between E and the topological dual E^*;
- $o_n(1)$ denotes a sequence which converges to 0 as $n \to \infty$;
- For $1 \leq p \leq \infty$, the standard norm in $L^p(\mathbb{R}^N)$ is denoted by $\| \cdot \|_p$;
- For $1 \leq p < \infty$, $L^p(\mathbb{R}^N) \times L^p(\mathbb{R}^N)$ denotes the Lebesgue space with norm

$$\|(u, v)\|_p = \left(\|u\|_p^p + \|v\|_p^p \right)^{1/p};$$

- We denote by S the sharp constant of the embedding $D^{1,2}(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N)$

$$\int_{\mathbb{R}^N} |\nabla u|^2 \, \mathrm{d}x \geq S \left(\int_{\mathbb{R}^N} |u|^{2^*} \, \mathrm{d}x \right)^{2/2^*}$$

  where $D^{1,2}(\mathbb{R}^N) := \{u \in L^{2^*}(\mathbb{R}^N) : |\nabla u| \in L^2(\mathbb{R}^N)\}$;

- $C(\Omega)$ denotes the space of continuous real functions in $\Omega \subseteq \mathbb{R}^N$;

- $C(\overline{\Omega})$ denotes the space of continuous real functions in $\Omega \subseteq \mathbb{R}^N$, which are uniformly continuous on bounded sets of Ω;

- For an integer $k \geq 1$, $C^k(\Omega)$ denotes the space of k-times continuously differentiable real functions defined over $\Omega \subseteq \mathbb{R}^N$;

- $C^\infty(\Omega) = \bigcap_{k=0}^{\infty} C^k(\Omega)$;

- $C_0^\infty(\Omega)$ denotes the space of infinitely differentiable real functions whose support is compact in $\Omega \subseteq \mathbb{R}^N$;

- For $0 < \beta < 1$ and $\Omega \subset \mathbb{R}^N$, $C^{0,\beta}(\overline{\Omega})$ denotes the standard Hölder space, that is,

$$C^{0,\beta}(\overline{\Omega}) = \left\{ u \in C(\overline{\Omega}) : \sup_{x,y \in \Omega} \frac{|u(x) - u(y)|}{|x - y|^\beta} < \infty \right\};$$

- For an integer $k \geq 1$, $0 < \beta < 1$ and $\Omega \subset \mathbb{R}^N$, $C^{k,\beta}(\overline{\Omega})$ denotes the space of the functions in $C^k(\Omega)$ whose all derivatives up order k belongs to $C^{0,\beta}(\overline{\Omega})$;

Chapter 1

Ground states for coupled systems of Schrödinger equations on $\mathbb{R}^N$

1.1 Introduction

In this chapter, we are interested in to establish existence and nonexistence results for the following class of coupled systems involving nonlinear Schrödinger equations

$$\begin{cases} -\Delta u + V_1(x)u = \mu|u|^{p-2}u + \lambda(x)v, & x \in \mathbb{R}^N, \\ -\Delta v + V_2(x)v = |v|^{q-2}v + \lambda(x)u, & x \in \mathbb{R}^N, \end{cases} \tag{S_μ}$$

where $N \geq 3$, $2 < p \leq q \leq 2^*$ and $2^* = 2N/(N-2)$ is the critical Sobolev exponent. Our main goal here is to prove the existence of ground states for the subcritical case, that is, when $2 < p \leq q < 2^*$ and for the critical case when $2 < p < q = 2^*$. In the critical case, the existence of ground state will be related with the parameter μ introduced in the first equation. We are concerned with two classes of nonnegative potentials: periodic and asymptotically periodic. The proof of our results rely on minimization method based on the Nehari manifold. For the critical case when $p = q = 2^*$, we make use of the Pohozaev identity to prove that System (S_μ) does not admit positive solution.

1.1.1 Assumptions

In view of the presence of the potentials $V_1(x)$ and $V_2(x)$, for $i = 1, 2$ we introduce the following space

$$E_i = \left\{ u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} V_i(x)u^2 \, \mathrm{d}x < +\infty \right\},$$

endowed with the inner product

$$(u,v)_{E_i} = \int_{\mathbb{R}^N} \nabla u \nabla v \, \mathrm{d}x + \int_{\mathbb{R}^N} V_i(x) uv \, \mathrm{d}x,$$

to which corresponds the induced norm $\|u\|_{E_i}^2 = (u,u)_{E_i}$. In order to establish a variational approach to treat System (S_μ), we need to require suitable assumptions on the potentials. For each $i = 1, 2$, we assume that

(V_1) $V_i, \lambda \in C^1(\mathbb{R}^N)$ are 1-periodic in each of $x_1, x_2, ..., x_N$.

(V_2) $V_i(x) \geq 0$ for all $x \in \mathbb{R}^N$ and

$$\nu_i = \inf_{u \in E_i} \left\{ \int_{\mathbb{R}^N} |\nabla u|^2 \, \mathrm{d}x + \int_{\mathbb{R}^N} V_i(x) u^2 \, \mathrm{d}x : \int_{\mathbb{R}^N} u^2 \, \mathrm{d}x = 1 \right\} > 0.$$

(V_3) $|\lambda(x)| \leq \delta \sqrt{V_1(x) V_2(x)}$, for some $\delta \in (0,1)$, for all $x \in \mathbb{R}^N$.

(V_3') $0 < \lambda(x) \leq \delta \sqrt{V_1(x) V_2(x)}$, for some $\delta \in (0,1)$, for all $x \in \mathbb{R}^N$.

The assumption (V_2) implies that E_i is continuous embedded into $L^p(\mathbb{R}^N)$, for all $2 \leq p \leq 2^*$. We set the product space $E = E_1 \times E_2$. We have that E is a Hilbert space when endowed with the inner product

$$((u,v),(z,w))_E = \int_{\mathbb{R}^N} \left(\nabla u \nabla z + V_1(x) uz + \nabla v \nabla w + V_2(x) vw \right) \, \mathrm{d}x,$$

to which corresponds the induced norm $\|(u,v)\|_E^2 = ((u,v),(u,v))_E = \|u\|_{E_1}^2 + \|v\|_{E_2}^2$. Associated to System ($S_\mu$), we have the C^2 energy functional $I : E \to \mathbb{R}$ defined by

$$I(u,v) = \frac{1}{2} \left(\|(u,v)\|_E^2 - 2 \int_{\mathbb{R}^N} \lambda(x) uv \, \mathrm{d}x \right) - \frac{\mu}{p} \|u\|_p^p - \frac{1}{q} \|v\|_q^q,$$

which its differential is given by

$$\langle I'(u,v), (\phi, \psi) \rangle = ((u,v),(\phi,\psi)) - \int_{\mathbb{R}^N} \left(|u|^{p-2} u \phi + |v|^{q-2} v \psi + \lambda(x) \left(u \psi + v \phi \right) \right) \, \mathrm{d}x,$$

where $(\phi, \psi) \in C_0^\infty(\mathbb{R}^N) \times C_0^\infty(\mathbb{R}^N)$. Thus, critical points of I correspond to weak solutions of (S_μ) and conversely.

Definition 1.1.1. *We say that a pair $(u,v) \in E \setminus \{(0,0)\}$ is a ground state solution (least energy solution) of (S_μ), if (u,v) is a solution of (S_μ) and its energy is minimal among the energy of all nontrivial solutions of (S_μ), i.e., $I(u,v) \leq I(w,z)$ for any other nontrivial solution $(w,z) \in E$. We say that (u,v) is nonnegative (nonpositive) if $u, v \geq 0$ $(u, v \leq 0)$ and positive (negative) if $u, v > 0$ $(u, v < 0)$.*

We are also concerned with the existence of ground states for the following class of coupled systems

$$\begin{cases} -\Delta u + \tilde{V}_1(x)u = \mu|u|^{p-2}u + \tilde{\lambda}(x)v, & x \in \mathbb{R}^N, \\ -\Delta v + \tilde{V}_2(x)v = |v|^{q-2}v + \tilde{\lambda}(x)u, & x \in \mathbb{R}^N, \end{cases} \qquad (\tilde{S}_\mu)$$

when the potentials $\tilde{V}_1(x)$, $\tilde{V}_2(x)$ and $\tilde{\lambda}(x)$ are asymptotically periodic, that is, they are infinity limit of the periodic functions $V_1(x)$, $V_2(x)$ and $\lambda(x)$. In analogous way, we may define the suitable product space $\tilde{E} = \tilde{E}_1 \times \tilde{E}_2$ considering the asymptotically periodic potential $\tilde{V}_i(x)$ instead $V_i(x)$. In order to give a variational approach for our problem, for $i = 1, 2$ we assume the following hypotheses:

(V_4) $\tilde{V}_i, \tilde{\lambda} \in C^1(\mathbb{R}^N)$, $\tilde{V}_i(x) < V_i(x)$, $\lambda(x) < \tilde{\lambda}(x)$, for all $x \in \mathbb{R}^N$ and

$$\lim_{|x|\to+\infty} |V_i(x) - \tilde{V}_i(x)| = 0 \quad \text{and} \quad \lim_{|x|\to+\infty} |\tilde{\lambda}(x) - \lambda(x)| = 0.$$

(V_5) $\tilde{V}_i(x) \geq 0$ for all $x \in \mathbb{R}^N$ and

$$\tilde{\nu}_i = \inf_{u\in\tilde{E}_i} \left\{ \int_{\mathbb{R}^N} |\nabla u|^2 \, dx + \int_{\mathbb{R}^N} \tilde{V}_i(x)u^2 \, dx : \int_{\mathbb{R}^N} u^2 \, dx = 1 \right\} > 0.$$

(V_6) $|\tilde{\lambda}(x)| \leq \delta\sqrt{\tilde{V}_1(x)\tilde{V}_2(x)}$, for some $\delta \in (0, 1)$, for all $x \in \mathbb{R}^N$.

(V_6') $0 < \tilde{\lambda}(x) \leq \delta\sqrt{\tilde{V}_1(x)\tilde{V}_2(x)}$, for some $\delta \in (0, 1)$, for all $x \in \mathbb{R}^N$.

1.1.2 Statement of the main results

The main results of this chapter are the following:

Theorem 1.1.2. *Assume that (V_1)-(V_3) hold. If $2 < p \leq q < 2^*$, then there exists a nonnegative ground state solution $(u_0, v_0) \in C^{1,\beta}_{loc}(\mathbb{R}^N) \times C^{1,\beta}_{loc}(\mathbb{R}^N)$ for System (S_μ), for all $\mu \geq 0$. If (V_3') holds, then the ground state is positive.*

Theorem 1.1.3. *Assume that (V_1)-(V_3) hold. If $2 < p < q = 2^*$, then there exists $\mu_0 > 0$ such that System (S_μ) possesses a nonnegative ground state solution $(u_0, v_0) \in E$, for all $\mu \geq \mu_0$. If (V_3') holds, then the ground state is positive.*

Theorem 1.1.4. *Suppose that assumptions (V_1)-(V_6) hold. If $2 < p \leq q < 2^*$, then there exists a nonnegative ground state solution $(u_0, v_0) \in C^{1,\beta}_{loc}(\mathbb{R}^N) \times C^{1,\beta}_{loc}(\mathbb{R}^N)$ for System $(\tilde{S}_\mu)$, for all $\mu \geq 0$. Moreover, if $2 < p < q = 2^*$, then there exists $\mu_0 > 0$ such that System $(\tilde{S}_\mu)$ possesses a nonnegative ground state solution for all $\mu \geq \mu_0$. If (V_6') holds, then the ground states are positive.*

Theorem 1.1.5. *Assume $p = q = 2^*$. In addition, consider the following assumptions:*

(V_7) $0 \leq \langle \nabla V_i(x), x \rangle \leq CV_i(x)$.

(V_8) $|\langle \nabla \lambda(x), x \rangle| \leq C|\lambda(x)|$ *and* $\langle \nabla \lambda(x), x \rangle \leq 0$.

Then, System (S_μ) has no positive classical solution for all $\mu \geq 0$.

Remark 1.1.6. *A typical example of functions satisfying (V_7) and (V_8) is* $\lambda(x) = -(1/4)\|x\|^2$ *and* $V_i(x) = (1/2)\|x\|^2$.

1.1.3 Outline

The remainder of this chapter is organized as follows. In the forthcoming section we introduce and give some properties of the Nehari manifold (for a more complete description of this subject we refer the reader to [68]). In Section 1.3 we deal with System (S_μ) in the subcritical case when $2 < p \leq q < 2^*$ and the potentials are periodic. For this matter we use a minimization method based on Nehari manifold to get a positive ground state solution and a bootstrap argument to obtain regularity. In Section 1.4 we study System (S_μ) in the critical case when $2 < p < q = 2^*$ with periodic potentials. In the periodic case, the key point is to use the invariance of the energy functional under translations to recover the compactness of the minimizing sequence. In Section 1.5 we study the existence of ground states when the potentials are asymptotically periodic. For this purpose, we establish a relation between the energy levels associated to Systems (S_μ) and ($\tilde{S}_\mu$). Finally, in Section 1.6 we make use of the Pohozaev identity to prove the nonexistence of positive classical solutions for system (S_μ) in the critical case when $p = q = 2^*$.

1.2 Preliminary results

One of the features of the class of coupled systems studied in this thesis is the presence of the coupling term $\lambda(x)$ in the equations. The assumption (V_3) will be required in all chapters henceforth. The next lemma is a crucial estimate obtained by this assumption, and will be cited and used in the next chapters.

Lemma 1.2.1. *If* (V_3) *holds, then we have*

$$\|(u,v)\|_E^2 - 2\int_{\mathbb{R}^N} \lambda(x)uv \,\mathrm{d}x \geq (1-\delta)\|(u,v)\|_E^2, \quad \text{for all } (u,v) \in E. \tag{1.1}$$

Proof. For $(u,v) \in E$ we have

$$0 \leq \left(\sqrt{V_1(x)}|u| - \sqrt{V_2(x)}|v|\right)^2 = V_1(x)u^2 - 2\sqrt{V_1(x)}|u|\sqrt{V_2(x)}|v| + V_2(x)v^2,$$

which together with assumption (V_3) implies that

$$\begin{aligned}
-2\int_{\mathbb{R}^N} \lambda(x)uv \,\mathrm{d}x &\geq -2\int_{\mathbb{R}^N} |\lambda(x)||u||v| \,\mathrm{d}x \\
&\geq -2\delta \int_{\mathbb{R}^N} \sqrt{V_1(x)}|u|\sqrt{V_2(x)}|v| \,\mathrm{d}x \\
&\geq -\delta\left(\int_{\mathbb{R}^N} V_1(x)u^2 \,\mathrm{d}x + \int_{\mathbb{R}^N} V_2(x)v^2 \,\mathrm{d}x\right) \\
&\geq -\delta\|(u,v)\|_E^2,
\end{aligned}$$

which easily implies (1.1). ∎

In order to prove the existence of ground states, we introduce the Nehari manifold associated to System (S_μ)

$$\mathcal{N} = \{(u,v) \in E\backslash\{(0,0)\} : \langle I'(u,v),(u,v)\rangle = 0\}.$$

Notice that if $(u,v) \in \mathcal{N}$, then

$$\|(u,v)\|_E^2 - 2\int_{\mathbb{R}^N} \lambda(x)uv \,\mathrm{d}x = \mu\|u\|_p^p + \|v\|_q^q. \tag{1.2}$$

It is obvious that all nontrivial critical points of I belong to $\mathcal{N}$. In general, the Nehari manifold may not be a manifold. However, in our case, $\mathcal{N}$ is in fact a C^1-manifold as we can see in the following lemma:

Lemma 1.2.2. *There exists* $\alpha > 0$ *such that*

$$\|(u,v)\|_E \geq \alpha, \quad \text{for all} \quad (u,v) \in \mathcal{N}. \tag{1.3}$$

Moreover, $\mathcal{N}$ *is a* C^1*-manifold.*

Proof. Let $(u,v) \in \mathcal{N}$. By using (1.1), (1.2) and Sobolev embedding, we deduce that

$$\begin{aligned}
(1-\delta)\|(u,v)\|_E^2 &\leq \|(u,v)\|_E^2 - 2\int_{\mathbb{R}^N} \lambda(x)uv \,\mathrm{d}x \\
&= \mu\|u\|_p^p + \|v\|_q^q \\
&\leq C\left(\|(u,v)\|_E^p + \|(u,v)\|_E^q\right).
\end{aligned}$$

Hence, we have that

$$0 < \frac{1-\delta}{C} \leq \|(u,v)\|_E^{p-2} + \|(u,v)\|_E^{q-2},$$

which implies (1.3). Now, let $J : E\backslash\{(0,0)\} \to \mathbb{R}$ be the C^1-functional defined by

$$J(u,v) = \langle I'(u,v),(u,v)\rangle = \|(u,v)\|_E^2 - 2\int_{\mathbb{R}^N} \lambda(x)uv \, \mathrm{d}x - \mu\|u\|_p^p - \|v\|_q^q.$$

Notice that $\mathcal{N} = J^{-1}(0)$. If $(u,v) \in \mathcal{N}$, then it follows from (1.2) that

$$\begin{aligned} \langle J'(u,v),(u,v)\rangle &= 2\left(\|(u,v)\|^2 - 2\int_{\mathbb{R}^N} \lambda(x)uv\mathrm{d}x\right) - p\mu\|u\|_p^p - q\|v\|_q^q \\ &= (2-p)\left(\|(u,v)\|_E^2 - 2\int_{\mathbb{R}^N} \lambda(x)uv \, \mathrm{d}x\right) + (p-q)\|v\|_q^q, \end{aligned}$$

which together with (1.1), (1.3) and the fact that $2 < p \leq q$ implies that

$$\langle J'(u,v),(u,v)\rangle \leq (2-p)(1-\delta)\|(u,v)\|_E^2 \leq (2-p)(1-\delta)\alpha < 0. \tag{1.4}$$

Therefore, 0 is a regular value of J and $\mathcal{N}$ is a C^1-manifold. ■

Remark 1.2.3. *If $(u_0,v_0) \in \mathcal{N}$ is a critical point of $I\mid_{\mathcal{N}}$, then $I'(u_0,v_0) = 0$. In fact, notice that $I'(u_0,v_0) = \eta J'(u_0,v_0)$, where $\eta \in \mathbb{R}$ is the corresponding Lagrange multiplier. Taking the scalar product with (u_0,v_0) and using (1.4) we conclude that $\eta = 0$.*

We define the ground state energy associated with (S_μ) by

$$c_{\mathcal{N}} = \inf_{(u,v)\in\mathcal{N}} I(u,v).$$

We note that $c_{\mathcal{N}}$ is positive. In fact, for any $(u,v) \in \mathcal{N}$ we can deduce that

$$I(u,v) = \left(\frac{1}{2} - \frac{1}{p}\right)\left(\|(u,v)\|_E^2 - 2\int_{\mathbb{R}^N} \lambda(x)uv \, \mathrm{d}x\right) + \left(\frac{1}{p} - \frac{1}{q}\right)\|v\|_q^q.$$

Since $2 < p \leq q$, it follows from (1.1) and (1.3) that

$$I(u,v) \geq \left(\frac{1}{2} - \frac{1}{p}\right)(1-\delta)\|(u,v)\|_E^2 \geq \left(\frac{1}{2} - \frac{1}{p}\right)(1-\delta)\alpha > 0,$$

which implies that $c_{\mathcal{N}} > 0$.

The set of all nontrivial critical points of I may contain only one element, while the Nehari manifold contains infinitely many elements. Indeed, this is a consequence of the following lemma:

Lemma 1.2.4. *Assume that* (V_3) *holds. Thus, for any* $(u,v) \in E\backslash\{(0,0)\}$*, there exists a unique* $t_0 > 0$*, depending only on* (u,v)*, such that*

$$(t_0u, t_0v) \in \mathcal{N} \quad \textit{and} \quad I(t_0u, t_0v) = \max_{t\geq 0} I(tu, tv).$$

Proof. Let $(u,v) \in E\backslash\{(0,0)\}$ be fixed and consider the function $g : [0,\infty) \to \mathbb{R}$ defined by $g(t) = I(tu, tv)$. Notice that $\langle I'(tu,tv), (tu,tv)\rangle = tg'(t)$. Therefore, t_0 is a positive critical point of g if and only if $(t_0u, t_0v) \in \mathcal{N}$. It follows from assumption (V_3) that

$$\|(u,v)\|_E^2 - 2\int_{\mathbb{R}^N} \lambda(x)uv \, \mathrm{d}x \geq 0, \quad \text{for all } (u,v) \in E.$$

Since $2 < p \leq q$ and

$$g(t) = \frac{t^2}{2}\left(\|(u,v)\|_E^2 - 2\int_{\mathbb{R}^N} \lambda(x)uv \, \mathrm{d}x\right) - \frac{t^p}{p}\mu\|u\|_p^p - \frac{t^q}{q}\|v\|_q^q,$$

we conclude that $g(t) < 0$ for $t > 0$ sufficiently large. On the other hand, by using (1.1) and Sobolev embeddings, we have that

$$\begin{aligned} g(t) &\geq (1-\delta)\frac{t^2}{2}\|(u,v)\|_E^2 - C_1\frac{t^p}{p}\|u\|_{E_1}^p - C_2\frac{t^q}{q}\|v\|_{E_2}^q \\ &\geq t^2\|(u,v)\|_E^2\left(\frac{1-\delta}{2} - C_1\frac{t^{p-2}}{p}\|(u,v)\|_E^{p-2} - C_2\frac{t^{q-2}}{q}\|(u,v)\|_E^{q-2}\right) > 0, \end{aligned}$$

provided $t > 0$ is sufficiently small. Thus g has maximum points in $(0,\infty)$. In order to prove the uniqueness, let us suppose that there exists $t_1, t_2 > 0$ with $t_1 < t_2$ such that $g'(t_1) = g'(t_2) = 0$. Since every critical point of g satisfies

$$\|(u,v)\|_E^2 - 2\int_{\mathbb{R}^N} \lambda(x)uv \, \mathrm{d}x = t^{p-2}\mu\|u\|_p^p + t^{q-2}\|v\|_q^q,$$

we have that

$$0 = \left(t_1^{p-2} - t_2^{p-2}\right)\mu\|u\|_p^p + \left(t_1^{q-2} - t_2^{q-2}\right)\|v\|_q^q,$$

which contradicts the fact that $(u,v) \neq (0,0)$. ∎

1.3 Proof of Theorem 1.1.2

By Ekeland's variational principle (see [38]), there exists a sequence $(u_n, v_n)_n \subset \mathcal{N}$ such that

$$I(u_n, v_n) \to c_{\mathcal{N}} \quad \text{and} \quad I'(u_n, v_n) \to 0. \tag{1.5}$$

Notice that $(u_n, v_n)_n$ is bounded. In fact, recalling that $p \leq q$ it follows from (1.1) and (1.2) that

$$\begin{aligned} I(u_n, v_n) &= \left(\frac{1}{2} - \frac{1}{p}\right)\left(\|(u_n, v_n)\|_E^2 - 2\int_{\mathbb{R}^N} \lambda(x) u_n v_n \, dx\right) + \left(\frac{1}{p} - \frac{1}{q}\right)\|v_n\|_q^q \\ &\geq \left(\frac{1}{2} - \frac{1}{q}\right)(1-\delta)\,\|(u_n, v_n)\|_E^2. \end{aligned}$$

Since $(I(u_n, v_n))_n$ is a bounded sequence, we conclude that $(u_n, v_n)_n$ is bounded in E. Passing to a subsequence if necessary, we way assume that $(u_n, v_n) \rightharpoonup (u_0, v_0)$ weakly in E. By a standard argument, we have that $I'(u_0, v_0) = 0$. We recall the following result due to P.L. Lions [69, Lemma 1.21] (see also [52]).

Lemma 1.3.1. *Let $r > 0$ and $2 \leq s < 2^*$. If $(u_n)_n \subset H^1(\mathbb{R}^N)$ is a bounded sequence such that*

$$\lim_{n \to +\infty} \sup_{y \in \mathbb{R}^N} \int_{B_r(y)} |u_n|^s \, dx = 0,$$

then $u_n \to 0$ in $L^s(\mathbb{R}^N)$.

Proposition 1.3.2. *There exists a ground state solution for System (S_μ).*

Proof. We split the argument into two cases.
Case 1. $(u_0, v_0) \neq (0, 0)$.

In this case, (u_0, v_0) is a nontrivial critical point of the energy functional I. Thus, $(u_0, v_0) \in \mathcal{N}$. It remains to prove that $I(u_0, v_0) = c_{\mathcal{N}}$. It is clear that $c_{\mathcal{N}} \leq I(u_0, v_0)$. On the other hand, by using the semicontinuity of norm, we can deduce that

$$\begin{aligned} c_{\mathcal{N}} + o_n(1) &= I(u_n, v_n) - \frac{1}{2}\langle I'(u_n, v_n), (u_n, v_n)\rangle \\ &= \left(\frac{1}{2} - \frac{1}{p}\right)\mu\|u_n\|_p^p + \left(\frac{1}{2} - \frac{1}{q}\right)\|v_n\|_q^q \\ &\geq \left(\frac{1}{2} - \frac{1}{p}\right)\mu\|u_0\|_p^p + \left(\frac{1}{2} - \frac{1}{q}\right)\|v_0\|_q^q + o_n(1) \\ &= I(u_0, v_0) - \frac{1}{2}\langle I'(u_0, v_0), (u_0, v_0)\rangle + o_n(1) \\ &= I(u_0, v_0) + o_n(1), \end{aligned}$$

which implies that $c_{\mathcal{N}} \geq I(u_0, v_0)$. Therefore, $I(u_0, v_0) = c_{\mathcal{N}}$.
Case 2. $(u_0, v_0) = (0, 0)$.

We claim that there exists a sequence $(y_n)_n \subset \mathbb{R}^N$ and $R, \xi > 0$ such that

$$\liminf_{n \to \infty} \int_{B_R(y_n)} (u_n^2 + v_n^2) \, dx \geq \xi > 0. \tag{1.6}$$

Suppose by contradiction that (1.6) does not hold. Thus, for any $R > 0$ we have

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_R(y)} u_n^2 \, dx = 0 \quad \text{and} \quad \lim_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_R(y)} v_n^2 \, dx = 0.$$

It follows from Lemma 1.3.1 that $u_n \to 0$ strongly in $L^p(\mathbb{R}^N)$ and $v_n \to 0$ strongly $L^q(\mathbb{R}^N)$, for any $2 < p, q < 2^*$. Since $(u_n, v_n)_n \subset \mathcal{N}$, we can deduce that

$$0 < (1-\delta)\alpha \leq (1-\delta)\|(u_n, v_n)\|_E^2 \leq \mu\|u_n\|_p^p + \|v_n\|_q^q \to 0,$$

which is a contradiction. Therefore, (1.6) holds.

We may assume without loss of generality that $(y_n)_n \subset \mathbb{Z}^N$. Let us consider the shift sequence $(\tilde{u}_n(x), \tilde{v}_n(x)) = (u_n(x+y_n), v_n(x+y_n))$. Since $V_1(\cdot)$, $V_2(\cdot)$ and $\lambda(\cdot)$ are 1-periodic functions, it follows that the energy functional I is invariant under translations of the form $(u, v) \mapsto (u(\cdot - z), v(\cdot - z))$ with $z \in \mathbb{Z}^N$. By a careful computation we can deduce that

$$\|(\tilde{u}_n, \tilde{v}_n)\|_E = \|(u_n, v_n)\|_E, \quad I(\tilde{u}_n, \tilde{v}_n) = I(u_n, v_n) \to c_{\mathcal{N}} \quad \text{and} \quad I'(\tilde{u}_n, \tilde{v}_n) \to 0.$$

Moreover, arguing as before, we can conclude that $(\tilde{u}_n, \tilde{v}_n)_n$ is a bounded sequence in E. In this way, there exists a critical point $(\tilde{u}, \tilde{v})$ of I, such that, up to a subsequence, $(\tilde{u}_n, \tilde{v}_n) \rightharpoonup (\tilde{u}, \tilde{v})$ weakly in E and $(\tilde{u}_n, \tilde{v}_n) \to (\tilde{u}, \tilde{v})$ strongly in $L^2(B_R(0)) \times L^2(B_R(0))$. Thus, using (1.6) we obtain

$$\int_{B_R(0)} (\tilde{u}^2 + \tilde{v}^2) \, dx = \liminf_{n\to\infty} \int_{B_R(0)} (\tilde{u}_n^2 + \tilde{v}_n^2) \, dx = \liminf_{n\to\infty} \int_{B_R(y_n)} (u_n^2 + v_n^2) \, dx \geq \xi > 0.$$

Therefore, $(\tilde{u}, \tilde{v}) \neq (0, 0)$. The conclusion follows as in the **Case 1**. ■

Proposition 1.3.3. *There exists a nonnegative ground state solution* $(\tilde{u}, \tilde{v}) \in C^{1,\beta}_{loc}(\mathbb{R}^N) \times C^{1,\beta}_{loc}(\mathbb{R}^N)$ *for System* (S_μ).

Proof. Let $(u_0, v_0) \in \mathcal{N}$ be the ground state obtained in the proposition 1.3.2. From Lemma 1.2.4, there exists $t > 0$ such that $(t|u_0|, t|v_0|) \in \mathcal{N}$. Thus, we can deduce that

$$I(t|u_0|, t|v_0|) \leq I(tu_0, tv_0) \leq \max_{t\geq 0} I(tu_0, tv_0) = I(u_0, v_0) = c_{\mathcal{N}},$$

which implies that $(t|u_0|, t|v_0|)$ is also a minimizer of I on $\mathcal{N}$. Therefore, $(t|u_0|, t|v_0|)$ is a nonnegative ground state solution for System (S_μ).

To prove the regularity, we use the standard bootstrap argument. Let us denote $(\tilde{u}, \tilde{v}) = (t|u_0|, t|v_0|)$. First, we define

$$p_1(x) = \mu|\tilde{u}|^{p-2}\tilde{u} + \lambda(x)\tilde{v} - V_1(x)\tilde{u} \quad \text{and} \quad p_2(x) = |\tilde{v}|^{q-2}\tilde{v} + \lambda(x)\tilde{u} - V_2(x)\tilde{v}.$$

Thus, $(\tilde{u}, \tilde{v})$ is a weak solution of the restricted problem

$$\begin{cases} -\Delta\tilde{u} = p_1(x), & x \in B_1(0), \\ -\Delta\tilde{v} = p_2(x), & x \in B_1(0). \end{cases}$$

Using Sobolev embedding we have that $V_1(x)\tilde{u}, V_2(x)\tilde{v}, \lambda(x)\tilde{u}, \lambda(x)\tilde{v} \in L^{2^*}(B_1(0))$. Moreover, $|\tilde{u}|^{p-2}\tilde{u} \in L^r(B_1(0))$ for all $1 \leq r \leq 2^*/(p-1)$ and $|\tilde{v}|^{q-2}\tilde{v} \in L^s(B_1(0))$ for all $1 \leq s \leq 2^*/(q-1)$. Let us define $r_1 = 2^*/(q-1)$. Since $p \leq q$, it follows that $r_1 \leq 2^*/(p-1)$. Hence $|\tilde{u}|^{p-2}\tilde{u} \in L^{r_1}(B_1(0))$. Therefore, $p_1(x), p_2(x) \in L^{r_1}(B_1(0))$. On the other hand, for each $i = 1, 2$ let w_i be the Newtonian potential of $p_i(x)$. Thus, in light of [42, Theorem 9.9] we have $w_i \in W^{2,r_1}(B_1(0))$ and

$$\begin{cases} \Delta w_1 = p_1(x), & x \in B_1(0), \\ \Delta w_2 = p_2(x), & x \in B_1(0). \end{cases}$$

Therefore, $(\tilde{u} - w_1, \tilde{v} - w_2) \in H^1(B_1(0)) \times H^1(B_1(0))$ is a weak solution of the problem

$$\begin{cases} \Delta z_1 = 0, & \text{in } B_1(0), \\ \Delta z_2 = 0, & \text{in } B_1(0). \end{cases}$$

In light of [46, Corollary 1.2.1], we have that $(\tilde{u}-w_1, \tilde{v}-w_2) \in C^\infty(B_1(0)) \times C^\infty(B_1(0))$. Therefore, $(\tilde{u}, \tilde{v}) \in W^{2,r_1}(B_1(0)) \times W^{2,r_1}(B_1(0))$. Since $q-1 < 2^*-1$, there exists $\delta > 0$ such that $(q-1)(1+\delta) = 2^* - 1$. Thus,

$$r_1 = \frac{2^*}{q-1} = 2^* \frac{(1+\delta)}{2^*-1} = \frac{2N}{N+2}(1+\delta). \tag{1.7}$$

Recall the Sobolev embedding $W^{2,r_1}(B_1(0)) \hookrightarrow L^{s_1}(B_1(0))$, where $s_1 = Nr_1/(N-2r_1)$. We claim that there exists $r_2 \in (r_1, s_1)$ such that $(\tilde{u}, \tilde{v}) \in W^{2,r_2}(B_1(0)) \times W^{2,r_2}(B_1(0))$. Indeed, we define $r_2 = s_1/(q-1)$ and we note that $r_2 < s_1$. By using (1.7) we deduce that

$$\frac{r_2}{r_1} = \frac{Nr_1}{(q-1)(N-2r_1)r_1} = \frac{(N-2)(1+\delta)}{N-2-4\delta} > 1+\delta,$$

which implies that $r_2 \in (r_1, s_1)$. By Sobolev embedding,

$$W^{2,r_1}(B_1(0)) \hookrightarrow L^{s_1}(B_1(0)) \hookrightarrow L^{r_2}(B_1(0)).$$

Hence, $p_1(x), p_2(x) \in L^{r_2}(B_1(0))$. From the same argument used before, we can conclude that $(\tilde{u}, \tilde{v}) \in W^{2,r_2}(B_1(0)) \times W^{2,r_2}(B_1(0))$. Iterating, we obtain the sequence

$$r_{n+1} = \frac{1}{q-1}\left(\frac{Nr_n}{N-2r_n}\right).$$

Notice that $r_{n+1} \to \infty$, as $n \to \infty$. Therefore,

$$(\tilde{u}, \tilde{v}) \in W^{2,r}_{loc}(\mathbb{R}^N) \times W^{2,r}_{loc}(\mathbb{R}^N), \quad \text{for all } 2 \leq r < \infty.$$

From Sobolev embedding, we have that $(\tilde{u}, \tilde{v}) \in C^{1,\beta}(B_1(0)) \times C^{1,\beta}(B_1(0))$, for some $\beta \in (0,1)$. ■

Proposition 1.3.4. *If (V_3') holds, then the ground state is positive.*

Proof. Let $(\tilde{u}, \tilde{v}) \in E \backslash \{(0,0)\}$ be the nonnegative ground state obtained in the proposition 1.3.3. Since $(\tilde{u}, \tilde{v}) \neq (0,0)$ we may assume without loss of generality that $\tilde{u} \neq 0$. We claim that $\tilde{v} \neq 0$. In fact, arguing by contradiction, let us suppose that $\tilde{v} = 0$. Thus,

$$0 = \langle I'(\tilde{u}, 0), (0, \psi) \rangle = -\int_{\mathbb{R}^N} \lambda(x) \tilde{u} \psi \, dx, \quad \text{for all } \psi \in C_0^\infty(\mathbb{R}^N).$$

Since $\lambda(x)$ is positive, we have that $\tilde{u} = 0$ which is a contradiction. Therefore, $\tilde{v} \neq 0$.

Taking $(\varphi, 0)$ as test function one sees that

$$\int_{\mathbb{R}^N} \nabla \tilde{u} \nabla \varphi \, dx + \int_{\mathbb{R}^N} V_1(x) \tilde{u} \varphi \, dx = \int_{\mathbb{R}^N} |\tilde{u}|^{p-2} \tilde{u} \varphi \, dx + \int_{\mathbb{R}^N} \lambda(x) \tilde{v} \varphi \, dx \geq 0,$$

for all $\varphi \geq 0$, $\varphi \in C_0^\infty(\mathbb{R}^N)$. Thus, we can deduce that

$$\int_{\mathbb{R}^N} \nabla(-\tilde{u}) \nabla \varphi \, dx - \int_{\mathbb{R}^N} \left[-V_1(x)\right] (-\tilde{u}) \varphi \, dx \leq 0,$$

for all $\varphi \geq 0$, $\varphi \in C_0^\infty(\mathbb{R}^N)$. Moreover, since $V_1(x) \geq 0$ for all $x \in \mathbb{R}^N$, it follows that

$$-\int_{\mathbb{R}^N} V_1(x) \varphi \, dx \leq 0, \quad \text{for all } \varphi \geq 0, \ \varphi \in C_0^\infty(\mathbb{R}^N).$$

In order to prove that $(\tilde{u}, \tilde{v})$ is positive, we suppose by contradiction that there exists $p \in \mathbb{R}^N$ such that $\tilde{u}(p) = 0$. Thus, since $-\tilde{u} \leq 0$ in $\mathbb{R}^N$, for any $R > R_0 > 0$ we have that

$$0 = \sup_{B_{R_0}(p)} (-\tilde{u}) = \sup_{B_R(p)} (-\tilde{u}).$$

By the Strong Maximum Principle [42, Theorem 8.19] we conclude that $-\tilde{u} \equiv 0$ in $B_R(p)$, for all $R > R_0$. Therefore, $\tilde{u} \equiv 0$ in $\mathbb{R}^N$ which is a contradiction. Therefore $\tilde{u} > 0$ in $\mathbb{R}^N$. Analogously we can prove that $\tilde{v} > 0$ in $\mathbb{R}^N$. Therefore, the ground state $(\tilde{u}, \tilde{v})$ is positive. ■

Proof of Theorem 1.1.2. It follows from Propositions 1.3.2, 1.3.3 and 1.3.4. ■

1.4 Proof of Theorem 1.1.3

In this section, we deal with System (S_μ) when $2 < p < q = 2^*$. Analogously to Theorem 1.1.2, we have a sequence $(u_n, v_n)_n \subset \mathcal{N}$ satisfying (1.5). Moreover, the sequence is bounded and $(u_n, v_n) \rightharpoonup (u_0, v_0)$ weakly in E. We have also that (u_0, v_0) is a critical point of the energy functional I. In order to get a nontrivial critical point, we need the following lemma:

Lemma 1.4.1. *There exists $\mu_0 > 0$ such that $c_{\mathcal{N}} < \frac{1}{N} S^{N/2}$, for all $\mu \geq \mu_0$.*

Proof. Let us consider $(u,v) \in E$ such that $u, v \geq 0$ and $u, v \not\equiv 0$. We denote $u_\mu = \mu u$ and $v_\mu = \mu v$. It follows from Lemma 1.2.4 that for any $\mu > 0$, there exists a unique $t_\mu > 0$ such that $(t_\mu u_\mu, t_\mu v_\mu) \in \mathcal{N}$. Thus,

$$(t_\mu \mu)^2 \|(u,v)\|_E^2 = (t_\mu \mu)^p \mu \|u\|_p^p + (t_\mu \mu)^{2^*} \|v\|_{2^*}^{2^*} + 2(t_\mu \mu)^2 \int_{\mathbb{R}^N} \lambda(x) uv \, dx, \tag{1.8}$$

which implies that $\|(u,v)\|_E^2 \geq (t_\mu \mu)^{2^*-2} \|v\|_{2^*}^{2^*}$. Therefore, $(t_\mu \mu)_\mu$ is a bounded sequence. Passing to a subsequence if necessary, we may assume that $t_\mu \mu \to \tilde{t} \geq 0$, as $\mu \to +\infty$. We claim that $\tilde{t} = 0$. Indeed, arguing by contradiction we suppose that $\tilde{t} > 0$. In this case,

$$(t_\mu \mu)^p \mu \|u\|_p^p + (t_\mu \mu)^{2^*} \|v\|_{2^*}^{2^*} + 2(t_\mu \mu)^2 \int_{\mathbb{R}^N} \lambda(x) uv \, dx \to +\infty, \quad \text{as } \mu \to +\infty,$$

which contradicts (1.8). Therefore, $t_\mu \mu \to 0$ as $\mu \to +\infty$. Hence, there exists $\mu_0 > 0$ such that

$$c_{\mathcal{N}} \leq I(t_\mu u_\mu, t_\mu v_\mu) \leq \frac{(t_\mu \mu)^2}{2} \|(u,v)\|_E^2 < \frac{1}{N} S^{N/2}, \quad \text{for all } \mu \geq \mu_0. \qquad \blacksquare$$

In analogous way to the proof of Theorem 1.1.2, we split the proof into two cases.

Case 1 $(u_0, v_0) \neq (0,0)$.

This case is completely similar to the proof of the subcritical case.

Case 2 $(u_0, v_0) = (0,0)$.

Let $\mu_0 > 0$ be the parameter obtained in the preceding lemma. We claim that if $\mu \geq \mu_0$, then there exists a sequence $(y_n)_n \subset \mathbb{R}^N$ and constants $R, \xi > 0$ such that

$$\liminf_{n\to\infty} \int_{B_R(y_n)} (u_n^2 + v_n^2) \, dx \geq \xi > 0. \tag{1.9}$$

In fact, suppose that (1.9) does not hold. Thus, for any $R > 0$ we have

$$\lim_{n\to\infty} \sup_{y \in \mathbb{R}^N} \int_{B_R(y)} (u_n^2 + v_n^2) \, dx = 0.$$

It follows from Lemma 1.3.1 that $u_n \to 0$ strongly in $L^p(\mathbb{R}^N)$, for $2 < p < 2^*$. Notice that

$$I(u_n, v_n) - \frac{1}{2} \langle I'(u_n, v_n), (u_n, v_n) \rangle = \frac{p-2}{2p} \mu \|u_n\|_p^p + \frac{1}{N} \|v_n\|_{2^*}^{2^*},$$

which together with (1.5) and Lemma 1.3.1 implies that

$$N c_{\mathcal{N}} + o_n(1) = N \left(I(u_n, v_n) - \frac{1}{2} \langle I'(u_n, v_n), (u_n, v_n) \rangle - \frac{p-2}{2p} \mu \|u_n\|_p^p \right) = \|v_n\|_{2^*}^{2^*}.$$

Moreover, we can deduce that

$$Nc_{\mathcal{N}}+o_n(1) = \|v_n\|_{2^*}^{2^*}+\mu\|u_n\|_p^p+\langle I'(u_n,v_n),(u_n,v_n)\rangle = \|(u_n,v_n)\|_E^2-2\int_{\mathbb{R}^N}\lambda(x)u_nv_n\,\mathrm{d}x.$$

The preceding computations implies that

$$Nc_{\mathcal{N}}+o_n(1) = \|v_n\|_{2^*}^{2^*} \leq S^{-\frac{N}{N-2}}\|\nabla v_n\|_2^{\frac{2N}{N-2}} \leq S^{-\frac{N}{N-2}}\left(\|(u_n,v_n)\|_E^2 - 2\int_{\mathbb{R}^N}\lambda(x)u_nv_n\,\mathrm{d}x\right)^{\frac{N}{N-2}}.$$

Thus, we can conclude that

$$Nc_{\mathcal{N}} + o_n(1) \leq \left(\frac{Nc_{\mathcal{N}}}{S}\right)^{\frac{N}{N-2}} + o_n(1).$$

Therefore, $c_{\mathcal{N}} \geq \frac{1}{N}S^{N/2}$, contradicting Lemma 1.4.1.

Since (1.9) holds, we can consider the shift sequence $(\tilde{u}_n(x),\tilde{v}_n(x)) = (u_n(x+y_n),v_n(x+y_n))$ and we can repeat the same arguments used in the proof of Theorem 1.1.2 to finish the proof.

1.5 Proof of Theorem 1.1.4

In this section we are concerned with the existence of ground states for the asymptotically periodic case. We emphasize that the only difference between $V_i(x),\lambda(x)$ and $\tilde{V}_i(x),\tilde{\lambda}(x)$ is the 1-periodicity required to $V_i(x)$ and $\lambda(x)$. Thus, if $\tilde{V}_i(x)$ and $\tilde{\lambda}(x)$ are periodic potentials, we can make use of Theorems 1.1.2 and 1.1.3 to get a ground state solution for System ($\tilde{S}_\mu$). Let us suppose that they are not periodic.

Associated to System ($\tilde{S}_\mu$), we have the following energy functional

$$\tilde{I}(u,v) = \frac{1}{2}\left(\|(u,v)\|_{\tilde{E}}^2 - 2\int_{\mathbb{R}^N}\tilde{\lambda}(x)uv\,\mathrm{d}x\right) - \frac{\mu}{p}\|u\|_p^p - \frac{1}{q}\|v\|_q^q.$$

The Nehari manifold associated to System ($\tilde{S}_\mu$) is defined by

$$\tilde{\mathcal{N}} = \{(u,v)\in\tilde{E}\backslash\{(0,0)\} : \langle\tilde{I}'(u,v),(u,v)\rangle = 0\},$$

and the ground state energy is given by $c_{\tilde{\mathcal{N}}} = \inf_{\tilde{\mathcal{N}}}\tilde{I}(u,v)$. Arguing as before we deduce that

$$\tilde{I}(u,v) \geq \left(\frac{1}{2}-\frac{1}{\mu}\right)(1-\delta)\|(u,v)\|_{\tilde{E}}^2 \geq \left(\frac{1}{2}-\frac{1}{\mu}\right)(1-\delta)\rho > 0, \quad \text{for all } (u,v)\in\tilde{\mathcal{N}}.$$

Hence, $c_{\tilde{\mathcal{N}}} > 0$. The next step is to establish a relation between the energy levels $c_{\mathcal{N}}$ and $c_{\tilde{\mathcal{N}}}$.

Lemma 1.5.1. $c_{\tilde{\mathcal{N}}} < c_{\mathcal{N}}$.

Proof. Let $(u_0, v_0) \in \mathcal{N}$ be the nonnegative ground state solution for System (S_μ). It is easy to see that Lemma 1.2.4 remains true for $\tilde{I}$ and $\tilde{\mathcal{N}}$. Thus, there exists a unique $t_0 > 0$, depending only on (u_0, v_0), such that $(t_0u_0, t_0v_0) \in \tilde{\mathcal{N}}$. By using (V_4) we get

$$\int_{\mathbb{R}^N} \left[(\tilde{V}_1(x) - V_1(x))u_0^2 + (\tilde{V}_2(x) - V_2(x))v_0^2 + (\lambda(x) - \tilde{\lambda}(x))u_0v_0\right] \mathrm{d}x < 0.$$

Therefore, $\tilde{I}(t_0u_0, t_0v_0) - I(t_0u_0, t_0v_0) < 0$. Since (u_0, v_0) is a ground state for System (S_μ) we can use Lemma 1.2.4 to deduce that

$$c_{\tilde{\mathcal{N}}} \le \tilde{I}(t_0u_0, t_0v_0) < I(t_0u_0, t_0v_0) \le \max_{t\ge0} I(tu_0, tv_0) = I(u_0, v_0) = c_{\mathcal{N}},$$

which finishes the proof. ∎

Let $(u_n, v_n)_n \subset \tilde{\mathcal{N}}$ be the minimizing sequence satisfying

$$\tilde{I}(u_n, v_n) \to c_{\tilde{\mathcal{N}}} \quad \text{and} \quad \tilde{I}'(u_n, v_n) \to 0. \tag{1.10}$$

Since $(u_n, v_n)_n$ is a bounded sequence in $\tilde{E}$, we may assume up to a subsequence that $(u_n, v_n) \rightharpoonup (u_0, v_0)$ weakly in $\tilde{E}$. The main difficulty here is to prove that the weak limit is nontrivial.

Proposition 1.5.2. *The weak limit (u_0, v_0) of the minimizing sequence $(u_n, v_n)_n$ is nontrivial.*

Proof. We suppose by contradiction that $(u_0, v_0) = (0, 0)$. We may assume that

- $u_n \to 0$ and $v_n \to 0$ strongly in $L^p_{loc}(\mathbb{R}^N)$, for all $2 \le p < 2^*$;
- $u_n(x) \to 0$ and $v_n(x) \to 0$ almost everywhere in $\mathbb{R}^N$.

It follows from assumption (V_4) that for any $\varepsilon > 0$ there exists $R > 0$ such that

$$|V_1(x) - \tilde{V}_1(x)| < \varepsilon, \quad |V_2(x) - \tilde{V}_2(x)| < \varepsilon, \quad |\tilde{\lambda}(x) - \lambda(x)| < \varepsilon, \quad \text{for } |x| \ge R. \tag{1.11}$$

By using (1.11) and Sobolev embedding and local convergence there exists $n_0 \in \mathbb{N}$ such that

$$\begin{aligned}\left|\int_{\mathbb{R}^N} (V_1(x) - \tilde{V}_1(x))u_n^2 \,\mathrm{d}x\right| &\le \int_{B_R(0)} |V_1(x) - \tilde{V}_1(x)|u_n^2 \,\mathrm{d}x + C\varepsilon \int_{B_R(0)^c} u_n^2 \,\mathrm{d}x \\ &\le (\|V_1\|_{L^\infty_{loc}} + \|\tilde{V}_1\|_{L^\infty_{loc}})\|u_n\|^2_{L^2(B_R(0))} + C\varepsilon\|u_n\|^2_{\tilde{E}_1} \\ &\le (\|V_1\|_{L^\infty_{loc}} + \|\tilde{V}_1\|_{L^\infty_{loc}})\varepsilon + C\varepsilon,\end{aligned}$$

for all $n \geq n_0$. Analogously, we can deduce that

$$\left| \int_{\mathbb{R}^N} (V_2(x) - \tilde{V}_2(x)) v_n^2 \, dx \right| \leq (\|V_2\|_{L^\infty_{loc}} + \|\tilde{V}_2\|_{L^\infty_{loc}})\varepsilon + C\varepsilon.$$

We have also from (1.11) the following estimates

$$\begin{aligned} \left| \int_{\mathbb{R}^N} (\tilde{\lambda}(x) - \lambda(x)) u_n v_n \, dx \right| &\leq \int_{B_R(0)} |\tilde{\lambda}(x) - \lambda(x)||u_n||v_n| \, dx + C\varepsilon \int_{B_R(0)^c} |u_n||v_n| \, dx \\ &\leq (\|\tilde{\lambda}\|_{L^\infty_{loc}} + \|\lambda\|_{L^\infty_{loc}})\varepsilon + C\varepsilon, \end{aligned}$$

for all $n \geq \tilde{n}_0$. Therefore, we can conclude that

$$I(u_n, v_n) - \tilde{I}(u_n, v_n) = o_n(1) \quad \text{and} \quad \langle I'(u_n, v_n), (u_n, v_n)\rangle - \langle \tilde{I}'(u_n, v_n), (u_n, v_n)\rangle = o_n(1),$$

which jointly with (1.10) implies that

$$I(u_n, v_n) = c_{\tilde{\mathcal{N}}} + o_n(1) \quad \text{and} \quad \langle I'(u_n, v_n), (u_n, v_n)\rangle = o_n(1). \tag{1.12}$$

Using Lemma 1.2.4, we obtain a sequence $(t_n)_n \subset (0, +\infty)$ such that $(t_n u_n, t_n v_n)_n \subset \mathcal{N}$.

Claim 1. $\limsup_{n\to+\infty} t_n \leq 1$.

Arguing by contradiction, we suppose that there exists $\varepsilon_0 > 0$ such that, up to a subsequence, we have $t_n \geq 1 + \varepsilon_0$, for all $n \in \mathbb{N}$. Thus, using (1.12) and the fact that $(t_n u_n, t_n v_n) \subset \mathcal{N}$ we get

$$(t_n^{p-2} - 1)\mu\|u_n\|_p^p + (t_n^{q-2} - 1)\|v_n\|_q^q = o_n(1),$$

which together with $t_n \geq 1 + \varepsilon_0$ implies that

$$((1 + \varepsilon_0)^{p-2} - 1)\mu\|u_n\|_p^p + ((1 + \varepsilon_0)^{q-2} - 1)\|v_n\|_q^q \leq o_n(1). \tag{1.13}$$

Similarly to the proof of Theorems 1.1.2 and 1.1.3, there exists a sequence $(y_n)_n \subset \mathbb{R}^N$ and constants $R, \xi > 0$ such that

$$\liminf_{n\to\infty} \int_{B_R(y_n)} (u_n^2 + v_n^2) \, dx \geq \xi > 0. \tag{1.14}$$

We point out that when $q = 2^*$, (1.14) holds for parameters $\mu \geq \mu_0$, where μ_0 was introduced in Lemma 1.4.1. Let us define $(\tilde{u}_n(x), \tilde{v}_n(x)) = (u_n(x + y_n), v_n(x + y_n))$. It follows from assumption (V_4) that $\tilde{V}_1, \tilde{V}_2 \in L^\infty(\mathbb{R}^N)$. Using the continuous embedding $\tilde{E}_i \hookrightarrow H^1(\mathbb{R}^N)$ we can deduce that $(\tilde{u}_n, \tilde{v}_n)_n$ is bounded in $\tilde{E}$. Thus, up to a subsequence, we may consider $(\tilde{u}_n, \tilde{v}_n) \rightharpoonup (\tilde{u}, \tilde{v})$ weakly in $\tilde{E}$. Therefore,

$$\lim_{n\to+\infty} \int_{B_R(0)} (\tilde{u}_n^2 + \tilde{v}_n^2) \, dx = \lim_{n\to+\infty} \int_{B_R(y_n)} (u_n^2 + v_n^2) \, dx \geq \beta > 0,$$

which implies $(\tilde{u}, \tilde{v}) \neq (0,0)$. Thus, by using (1.13) and the semicontinuity of the norm, we get

$$0 < ((1+\varepsilon_0)^{p-2} - 1)\mu\|\tilde{u}\|_p^p + ((1+\varepsilon_0)^{q-2} - 1)\|\tilde{v}\|_q^q \leq o_n(1),$$

which is not possible and finishes the proof of *Claim 1.*

Claim 2. There exists $n_0 \in \mathbb{N}$ such that $t_n \geq 1$, for $n \geq n_0$.

In fact, arguing by contradiction, we suppose that up to a subsequence, $t_n < 1$. Since $(t_n u_n, t_n v_n)_n \subset \mathcal{N}$ we have that

$$c_{\mathcal{N}} \leq \frac{p-2}{2p}\mu t_n^p \|u_n\|_p^p + \frac{q-2}{2q} t_n^q \|v_n\|_q^q \leq \frac{p-2}{2p}\mu\|u_n\|_p^p + \frac{q-2}{2q}\|v_n\|_q^q = c_{\tilde{\mathcal{N}}} + o_n(1).$$

Therefore, $c_{\mathcal{N}} \leq c_{\tilde{\mathcal{N}}}$ which contradicts Lemma 1.5.1 and finishes the proof of *Claim 2.*

Combining *Claims* 1 and 2 we deduce that

$$I(t_n u_n, t_n v_n) - I(u_n, v_n) = o_n(1).$$

Thus, it follows from (1.12) that

$$c_{\mathcal{N}} \leq I(t_n u_n, t_n v_n) = I(u_n, v_n) + o_n(1) = c_{\tilde{\mathcal{N}}} + o_n(1),$$

which contradicts Lemma 1.5.1. Therefore, $(u_0, v_0) \neq (0,0)$. ∎

Proof of Theorem 1.1.4 completed. Since (u_0, v_0) is a nontrivial point of the energy functional $\tilde{I}$, it follows that $(u_0, v_0) \in \tilde{\mathcal{N}}$. Therefore, we have $c_{\tilde{\mathcal{N}}} \leq \tilde{I}(u_0, v_0)$. On the other hand, using the semicontinuity of the norm we deduce that

$$\begin{aligned} c_{\tilde{\mathcal{N}}} + o_n(1) &= \left(\frac{1}{2} - \frac{1}{p}\right)\mu\|u_n\|_p^p + \left(\frac{1}{2} - \frac{1}{q}\right)\|v_n\|_q^q \\ &\geq \left(\frac{1}{2} - \frac{1}{p}\right)\|u_0\|_p^p + \left(\frac{1}{2} - \frac{1}{q}\right)\|v_0\|_q^q + o_n(1) \\ &= \tilde{I}(u_0, v_0) + o_n(1). \end{aligned}$$

Hence, $c_{\tilde{\mathcal{N}}} \geq \tilde{I}(u_0, v_0)$. Therefore $\tilde{I}(u_0, v_0) = c_{\mathcal{N}}$. Repeating the same argument used in the proof of Theorem 1.1.2, we can deduce that there exists $t_0 > 0$ such that $(t_0|u_0|, t_0|v_0|) \in \tilde{\mathcal{N}}$ is a positive ground state solution for System ($\tilde{S}_\mu$) which finishes the proof of Theorem 1.1.4. ∎

1.6 Proof of Theorem 1.1.5

In this section we deal of the following coupled system

$$\begin{cases} -\Delta u + V_1(x)u = \mu|u|^{2^*-2}u + \lambda(x)v, & x \in \mathbb{R}^N, \\ -\Delta v + V_2(x)v = |v|^{2^*-2}v + \lambda(x)u, & x \in \mathbb{R}^N. \end{cases} \tag{1.15}$$

In order to get a nonexistence result, we prove the following Pohozaev identity.

Lemma 1.6.1. *Suppose $N \geq 3$ and (V_7),(V_8). If $(u,v) \in E$ is a classical solution of (1.15), then it satisfies the following Pohozaev identity:*

$$\int_{\mathbb{R}^N} \left(|\nabla u|^2 + |\nabla v|^2\right)\, \mathrm{d}x = \int_{\mathbb{R}^N} \left(\mu|u|^{2^*} + |v|^{2^*} + 2^*\lambda(x)uv\right)\, \mathrm{d}x + \int_{\mathbb{R}^N} \langle \nabla\lambda(x), x\rangle uv\, \mathrm{d}x$$
$$-\frac{2^*}{2}\int_{\mathbb{R}^N} \left(V_1(x)u^2 + V_2(x)v^2\right)\, \mathrm{d}x - \frac{1}{N-2}\int_{\mathbb{R}^N} \left(\langle \nabla V_1(x), x\rangle u^2 + \langle \nabla V_2(x), x\rangle v^2\right)\, \mathrm{d}x.$$

Proof. Let $(u,v) \in E$ be a classical solution of the system (1.15) and let us denote

$$f(x,u,v) = -V_1(x)u + \mu|u|^{2^*-2}u + \lambda(x)v \quad \text{and} \quad g(x,u,v) = -V_2(x)v + |v|^{2^*-2}v + \lambda(x)u.$$

We consider the cut-off function $\psi \in C_0^\infty(\mathbb{R})$ defined by

$$\psi(t) = \begin{cases} 1 & \text{if} \quad |t| \leq 1, \\ 0 & \text{if} \quad |t| \geq 2, \end{cases}$$

such that $|\psi'(t)| \leq C$, for some $C > 0$. We define $\psi_n(x) = \psi\left(|x|^2/n^2\right)$ and we note that

$$\nabla\psi_n(x) = \frac{2}{n}\psi'\left(\frac{|x|^2}{n^2}\right)x.$$

Multiplying the first equation in (1.15) by the factor $\langle \nabla u, x\rangle\psi_n$ and integrating we obtain

$$-\int_{\mathbb{R}^N} \Delta u\langle \nabla u, x\rangle\psi_n\, \mathrm{d}x = \int_{\mathbb{R}^N} f(x,u,v)\langle \nabla u, x\rangle\psi_n\, \mathrm{d}x.$$

Multiplying the second equation in (1.15) by the factor $\langle \nabla v, x\rangle\psi_n$ and integrating we get

$$-\int_{\mathbb{R}^N} \Delta v\langle \nabla v, x\rangle\psi_n\, \mathrm{d}x = \int_{\mathbb{R}^N} g(x,u,v)\langle \nabla v, x\rangle\psi_n\, \mathrm{d}x.$$

The idea is to take the limit as $n \to +\infty$ in the following equation

$$-\int_{\mathbb{R}^N} (\Delta u\langle \nabla u, x\rangle + \Delta v\langle \nabla v, x\rangle)\psi_n\, \mathrm{d}x = \int_{\mathbb{R}^N} f(x,u,v)\langle \nabla u, x\rangle\psi_n\, \mathrm{d}x + \\ + \int_{\mathbb{R}^N} g(x,u,v)\langle \nabla v, x\rangle\psi_n\, \mathrm{d}x. \tag{1.16}$$

In order to calculate the limit in the left-hand side of (1.16), we note that

$$\begin{aligned} \mathrm{div}\left(\langle \nabla u, x\rangle\psi_n\nabla u\right) &= \langle \nabla\left(\langle \nabla u, x\rangle\psi_n\right), \nabla u\rangle + \langle \nabla u, x\rangle\psi_n \mathrm{div}(\nabla u) \\ &= \psi_n\langle \nabla\left(\langle \nabla u, x\rangle\right), \nabla u\rangle + \langle \nabla u, x\rangle\langle \nabla\psi_n, \nabla u\rangle + \langle \nabla u, x\rangle\psi_n\Delta u. \end{aligned}$$

Moreover, we have

$$\begin{aligned}
\langle \nabla \left(\langle \nabla u, x \rangle \right), \nabla u \rangle &= \langle \left(\sum_{i=1}^{N} \frac{\partial}{\partial x_1} \left(\frac{\partial u}{\partial x_i} x_i \right), \ldots, \sum_{i=1}^{N} \frac{\partial}{\partial x_N} \left(\frac{\partial u}{\partial x_i} x_i \right) \right), \nabla u \rangle \\
&= \langle \left(\sum_{i=1}^{N} \frac{\partial}{\partial x_1} \left(\frac{\partial u}{\partial x_i} \right) x_i, \ldots, \sum_{i=1}^{N} \frac{\partial}{\partial x_N} \left(\frac{\partial u}{\partial x_i} \right) x_i \right) + \nabla u, \nabla u \rangle \\
&= \sum_{i=1}^{N} \sum_{j=1}^{N} \frac{\partial}{\partial x_j} \left(\frac{\partial u}{\partial x_i} \right) \frac{\partial u}{\partial x_i} x_j + |\nabla u|^2 \\
&= \langle \nabla \left(\frac{|\nabla u|^2}{2} \right), x \rangle + |\nabla u|^2.
\end{aligned}$$

Thus, we can deduce that

$$\langle \nabla u, x \rangle \psi_n \Delta u = \operatorname{div} \left(\langle \nabla u, x \rangle \psi_n \nabla u \right) - \psi_n \left(\langle \nabla \left(\frac{|\nabla u|^2}{2} \right), x \rangle - |\nabla u|^2 \right) + \langle \nabla u, x \rangle \langle \nabla \psi_n, \nabla u \rangle.$$

Since

$$\operatorname{div} \left(\psi_n \frac{|\nabla u|^2}{2} x \right) = \psi_n \langle \nabla \left(\frac{|\nabla u|^2}{2} \right), x \rangle + \frac{N}{2} \psi_n |\nabla u|^2 + \frac{|\nabla u|^2}{2} \langle \nabla \psi_n, x \rangle,$$

it follows that

$$\langle \nabla u, x \rangle \psi_n \Delta u = \operatorname{div}(\psi_n H(x,u)) + \frac{N-2}{2} \psi_n |\nabla u|^2 + \frac{|\nabla u|^2}{2} \langle \nabla \psi_n, x \rangle - \langle \nabla u, x \rangle \langle \nabla \psi_n, \nabla u \rangle, \tag{1.17}$$

where

$$H(x,u) = \langle \nabla u, x \rangle \nabla u - \frac{|\nabla u|^2}{2} x.$$

Let us denote $H^i(x,u)$ the i-coordinate of $H(x,u)$ for $1 \leq i \leq N$. Since $u \in H^2_{loc}(\mathbb{R}^N)$ and $\operatorname{supp}(\psi_n) \subset B_{2n}(0)$, we can use the definition of weak derivatives to conclude that

$$\begin{aligned}
\int_{\mathbb{R}^N} \operatorname{div}(\psi_n H(x,u)) \, \mathrm{d}x &= \sum_{i=1}^{N} \int_{B_{2n}(0)} \frac{\partial}{\partial x_i} \left(\psi_n H^i(x,u) \right) \, \mathrm{d}x \\
&= \sum_{i=1}^{N} \int_{B_{2n}(0)} \left(\frac{\partial \psi_n}{\partial x_i} H^i(x,u) + \psi_n \frac{\partial}{\partial x_i} \left(H^i(x,u) \right) \right) \, \mathrm{d}x \\
&= \sum_{i=1}^{N} \int_{B_{2n}(0)} \left(\frac{\partial \psi_n}{\partial x_i} H^i(x,u) - \frac{\partial \psi_n}{\partial x_i} H^i(x,u) \right) \, \mathrm{d}x \\
&= 0.
\end{aligned}$$

In order to use the Lebesgue dominated convergence theorem, we note that:

- $|\psi_n |\nabla u|^2| \leq |\nabla u|^2 \in L^1(\mathbb{R}^N)$ and $\psi_n |\nabla u|^2 \to |\nabla u|^2$, almost everywhere in $\mathbb{R}^N$;

- $\left|\frac{|\nabla u|^2}{2}\langle \nabla \psi_n, x\rangle\right| \le C|\nabla u|^2 \in L^1(\mathbb{R}^N)$ and $\frac{|\nabla u|^2}{2}\langle \nabla \psi_n, x\rangle \to 0$, almost everywhere in $\mathbb{R}^N$;

- $|\langle \nabla u, x\rangle\langle \nabla \psi_n, \nabla u\rangle| \le C|\nabla u|^2 \in L^1(\mathbb{R}^N)$ and $\langle \nabla u, x\rangle\langle \nabla \psi_n, \nabla u\rangle \to 0$, almost everywhere in $\mathbb{R}^N$.

Therefore, integrating (1.17) and passing the limit, we obtain

$$-\lim_{n\to\infty}\int_{\mathbb{R}^N}\langle \nabla u, x\rangle \psi_n \Delta u \,\mathrm{d}x = -\frac{N-2}{2}\int_{\mathbb{R}^N}|\nabla u|^2\,\mathrm{d}x. \tag{1.18}$$

Analogously, we can deduce the limit

$$-\lim_{n\to\infty}\int_{\mathbb{R}^N}\langle \nabla v, x\rangle \psi_n \Delta v \,\mathrm{d}x = -\frac{N-2}{2}\int_{\mathbb{R}^N}|\nabla v|^2\,\mathrm{d}x. \tag{1.19}$$

The convergences (1.18) and (1.19) implies the limit of the left-hand side in (1.16). In order to calculate the right-hand side, we note that

$$\begin{aligned}\operatorname{div}\left(\psi_n F(x,u,v)x\right) &= \langle \nabla\left(\psi_n F(x,u,v)\right), x\rangle + \psi_n F(x,u,v)\operatorname{div}(x)\\ &= \psi_n\langle \nabla F(x,u,v), x\rangle + F(x,u,v)\langle \nabla\psi_n, x\rangle + N\psi_n F(x,u,v),\end{aligned}$$

where $F(x,u,v) = -\frac{1}{2}V_1(x)u^2 + \frac{\mu}{2^*}|u|^{2^*} + \lambda(x)uv$. Moreover, we have that

$$\nabla F(x,u,v) = -\frac{1}{2}\nabla V_1(x)u^2 - (V_1(x)u + \mu|u|^{2^*-2}u + \lambda(x)v)\nabla u + \nabla\lambda(x)uv + \lambda(x)u\nabla v,$$

which implies that

$$\langle \nabla F(x,u,v), x\rangle = -\frac{1}{2}\langle \nabla V_1(x), x\rangle u^2 + f(x,u,v)\langle \nabla u, x\rangle + \langle \nabla\lambda(x), x\rangle uv + \langle \lambda(x)u\nabla v, x\rangle.$$

Therefore,

$$\begin{aligned}&\int_{\mathbb{R}^N} f(x,u,v)\langle \nabla u, x\rangle\psi_n\,\mathrm{d}x = \int_{\mathbb{R}^N}\left(\operatorname{div}(\psi_n F(x,u,v)x) - F(x,u,v)\langle\nabla\psi_n, x\rangle\psi_n\right)\,\mathrm{d}x\\ &+\int_{\mathbb{R}^N}\left(\frac{1}{2}\langle \nabla V_1(x), u\rangle u^2 - NF(x,u,v)\psi_n - \langle\nabla\lambda(x), x\rangle uv - \langle\lambda(x)u\nabla v, x\rangle\right)\psi_n\,\mathrm{d}x.\end{aligned}$$

Analogously, denoting $G(x,u,v) = -\frac{1}{2}V_2(x)v^2 + \frac{1}{2^*}|v|^{2^*} + \lambda(x)uv$, we can deduce

$$\begin{aligned}&\int_{\mathbb{R}^N} g(x,u,v)\langle \nabla v, x\rangle\psi_n\,\mathrm{d}x = \int_{\mathbb{R}^N}\left(\operatorname{div}(\psi_n G(x,u,v)x) - G(x,u,v)\langle\nabla\psi_n, x\rangle\psi_n\right)\,\mathrm{d}x\\ &+\int_{\mathbb{R}^N}\left(\frac{1}{2}\langle \nabla V_2(x), v\rangle v^2 - NG(x,u,v)\psi_n - \langle\nabla\lambda(x), x\rangle uv - \langle\lambda(x)v\nabla u, x\rangle\right)\psi_n\,\mathrm{d}x.\end{aligned}$$

We note that

$$-\int_{\mathbb{R}^N}\lambda(x)\langle u\nabla v + v\nabla u, x\rangle\psi_n\,\mathrm{d}x = -\int_{B_{2n}(0)}\lambda(x)\sum_{i=1}^N\frac{\partial(uv)}{\partial x_i}x_i\psi_n\,\mathrm{d}x.$$

Using integration by parts we have that

$$-\int_{B_{2n}(0)} \lambda(x) \sum_{i=1}^{N} \frac{\partial(uv)}{\partial x_i} x_i \psi_n = \sum_{i=1}^{N} \int_{B_{2n}(0)} \left(\frac{\partial \psi_n}{\partial x_i} x_i \lambda(x) uv + \frac{\partial \lambda}{\partial x_i} x_i uv \psi_n + N \psi_n \lambda(x) uv \right).$$

Therefore,

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} \lambda(x) \langle u\nabla v + v\nabla u, x\rangle \psi_n \, \mathrm{d}x = -\int_{\mathbb{R}^N} \langle \nabla \lambda(x), x\rangle uv \, \mathrm{d}x - N \int_{\mathbb{R}^N} \lambda(x) uv \, \mathrm{d}x.$$

Thus, using the Lebesgue dominated convergence theorem in the same way as we used when we calculate the left-hand side, we obtain

$$\begin{aligned} \lim_{n\to\infty} \int_{\mathbb{R}^N} \left(f(x,u,v)\langle \nabla u, x\rangle + g(x,u,v)\langle \nabla v, x\rangle\right) \psi_n \, \mathrm{d}x &= -N \int_{\mathbb{R}^N} F(x,u,v) \, \mathrm{d}x \\ -N \int_{\mathbb{R}^N} G(x,u,v) \, \mathrm{d}x + \frac{1}{2} \int_{\mathbb{R}^N} &\left(\langle \nabla V_1(x), x\rangle u^2 + \langle \nabla V_2(x), x\rangle v^2 \right) \, \mathrm{d}x \\ - \int_{\mathbb{R}^N} \langle \nabla \lambda(x), x\rangle uv \, \mathrm{d}x &+ N \int_{\mathbb{R}^N} \lambda(x) uv \, \mathrm{d}x. \end{aligned}$$

Replacing $F(x,u,v)$ and $G(x,u,v)$ in the equation above, we get the right-hand side of (1.16) which finishes the proof. ■

Proof of Theorem 1.1.5 completed. Let $(u,v) \in E$ be a positive classical solution of (1.15). By the definition of weak solution we obtain

$$\int_{\mathbb{R}^N} \left(|\nabla u|^2 + V_1(x)u^2 + |\nabla v|^2 + V_2(x)v^2 \right) \, \mathrm{d}x = \int_{\mathbb{R}^N} \left(|u|^{2^*} + |v|^{2^*} + 2\lambda(x)uv \right) \, \mathrm{d}x.$$

which together with the Pohozaev identity obtained in Lemma 1.6.1 implies that

$$\begin{aligned} 0 = \left(1 - \frac{2^*}{2}\right) \int_{\mathbb{R}^N} \left(V_1(x)u^2 + V_2(x)v^2 - 2\lambda(x)uv \right) \, \mathrm{d}x + \int_{\mathbb{R}^N} \langle \nabla \lambda(x), x\rangle uv \, \mathrm{d}x \\ - \frac{1}{N-2} \int_{\mathbb{R}^N} \left(\langle \nabla V_1(x), x\rangle u^2 + \langle \nabla V_2(x), x\rangle v^2 \right) \, \mathrm{d}x. \quad (1.20) \end{aligned}$$

Multiplying (1.20) by the factor $-(N-2)/2$, we get

$$\begin{aligned} \int_{\mathbb{R}^N} \left(V_1(x)u^2 + V_2(x)v^2 - 2\lambda(x)uv \right) \, \mathrm{d}x = \frac{N-2}{2} \int_{\mathbb{R}^N} \langle \nabla \lambda(x), x\rangle uv \, \mathrm{d}x \\ - \frac{1}{2} \int_{\mathbb{R}^N} \left(\langle \nabla V_1(x), x\rangle u^2 + \langle \nabla V_2(x), x\rangle v^2 \right) \, \mathrm{d}x. \end{aligned}$$

Thus, it follows from assumptions (V_7) and (V_8) that

$$\int_{\mathbb{R}^N} \left(V_1(x)u^2 + V_2(x)v^2 - 2\lambda(x)uv \right) \, \mathrm{d}x \le 0.$$

On the other hand, by assumption (V_3) we get

$$\int_{\mathbb{R}^N} \left(V_1(x)u^2 + V_2(x)v^2 - 2\lambda(x)uv \right) \, \mathrm{d}x \ge 0.$$

Thus, we conclude that

$$\int_{\mathbb{R}^N} \left(V_1(x)u^2 + V_2(x)v^2 - 2\lambda(x)uv\right)\, \mathrm{d}x = 0.$$

Therefore, we finally deduce that

$$\begin{aligned} 0 &\leq \int_{\mathbb{R}^N} \left(\sqrt{V_1(x)}u - \sqrt{V_2(x)}v\right)^2 \, \mathrm{d}x \\ &= \int_{\mathbb{R}^N} \left(V_1(x)u^2 - 2\sqrt{V_1(x)V_2(x)}uv + V_2(x)v^2\right) \, \mathrm{d}x \\ &\leq \int_{\mathbb{R}^N} \left(V_1(x)u^2 + V_2(x)v^2 - \frac{2}{\delta}\lambda(x)uv\right) \, \mathrm{d}x \\ &< \int_{\mathbb{R}^N} \left(V_1(x)u^2 + V_2(x)v^2 - 2\lambda(x)uv\right) \, \mathrm{d}x \\ &= 0, \end{aligned}$$

which is not possible and finishes the proof of Theorem 1.1.5. ■

Remark 1.6.2. *Let us set*

$$\Lambda := \{\mu > 0 : (S_\mu) \text{ has ground state}\}.$$

We proved that Λ is nonempty since for μ sufficiently large System (S_μ) possesses ground state solution. Let us define $\tilde{\mu} := \inf \Lambda$. Naturally arise the following questions:

(i) $\tilde{\mu} > 0$*?*

(ii) $\Lambda = [\tilde{\mu}, +\infty)$ *or* $\Lambda = (\tilde{\mu}, +\infty)$*?*

Moreover, let us consider the following system

$$\begin{cases} -\Delta u + V_1(x)u = |u|^{p-2}u + \lambda(x)v, & x \in \mathbb{R}^N, \\ -\Delta v + V_2(x)v = \mu|v|^{2^*-2}v + \lambda(x)u, & x \in \mathbb{R}^N. \end{cases} \tag{S_μ}$$

Does System (S_μ) possesses ground state solution for any $\mu > 0$?

Chapter 2

Ground states for coupled systems of Schrödinger equations on $\mathbb{R}^2$ involving critical exponential growth

2.1 Introduction

In this chapter we study the following class of coupled systems

$$\begin{cases} -\Delta u + V_1(x)u = f_1(x,u) + \lambda(x)v, & x \in \mathbb{R}^2, \\ -\Delta v + V_2(x)v = f_2(x,v) + \lambda(x)u, & x \in \mathbb{R}^2, \end{cases} \tag{S}$$

where the potentials V_1, V_2 are nonnegative and satisfy $|\lambda(x)| < \delta\sqrt{V_1(x)V_2(x)}$ for some $0 < \delta < 1$. Our main contribution in this chapter is to prove the existence of ground states for the class of coupled systems (S) under assumptions involving a large class of potentials that contains, for example, coercive potentials, and nonlinearities with critical exponential growth of the Trudinger-Moser type.

2.1.1 Assumptions

We will use the notation $H^1(\mathbb{R}^2)$ for the usual Sobolev space, endowed the standard scalar product and the induced norm

$$(u,v) = \int_{\mathbb{R}^2} (\nabla u \nabla v + uv) \ \mathrm{d}x, \quad \|u\|^2 = \int_{\mathbb{R}^2} \left(|\nabla u|^2 + u^2\right) \ \mathrm{d}x.$$

For each $i = 1, 2$, we consider the following weighted Sobolev space defined by

$$H_{V_i}(\mathbb{R}^2) = \left\{ u \in H^1(\mathbb{R}^2) : \int_{\mathbb{R}^2} V_i(x)u^2 \ \mathrm{d}x < \infty \right\},$$

endowed with the natural norm

$$\|u\|_{V_i} = \left(\int_{\mathbb{R}^2} |\nabla u|^2 \,\mathrm{d}x + \int_{\mathbb{R}^2} V_1(x) u^2 \,\mathrm{d}x \right)^{1/2}.$$

In order to apply variational methods based on the space $H_{V_i}(\mathbb{R}^2)$, we assume suitable conditions on the potential $V_i(x)$ for each $i = 1, 2$.

(V_1) $V_i(x) \geq 0$, for all $x \in \mathbb{R}^2$ and $V_i \in L^\infty_{loc}(\mathbb{R}^2)$.

(V_2) The infimum

$$\inf_{u \in H_{V_i}(\mathbb{R}^2)} \left\{ \int_{\mathbb{R}^2} \left(|\nabla u|^2 + V_i(x) u^2 \right) \,\mathrm{d}x : \int_{\mathbb{R}^2} u^2 \,\mathrm{d}x = 1 \right\}$$

is positive.

(V_3) Let $\Omega \subseteq \mathbb{R}^2$ be open and $2 \leq s < \infty$. There exists $s \in [2, +\infty)$ such that

$$\lim_{R \to \infty} \nu_s^i(\mathbb{R}^2 \backslash \overline{B}_R) = \infty,$$

where

$$\nu_s^i(\Omega) = \begin{cases} \inf_{u \in H_0^1(\Omega) \setminus \{0\}} \dfrac{\displaystyle\int_\Omega \left(|\nabla u|^2 + V_i(x) u^2 \right) \,\mathrm{d}x}{\left(\int_\Omega |u|^s \,\mathrm{d}x \right)^{2/s}} & \text{if } \quad \Omega \neq \emptyset, \\ \infty & \text{if } \quad \Omega = \emptyset. \end{cases}$$

(V_4) There exists functions $A_i(x) \in L^\infty_{loc}(\mathbb{R}^2)$, with $A_i(x) \geq 1$, and constants $\beta_i > 1$, $C_0, R_0 > 0$ such that

$$A_i(x) \leq C_0 \left[1 + V_i(x)^{1/\beta_i} \right], \quad \text{for all} \quad |x| \geq R_0.$$

(V_5) There exists $0 < \delta < 1$ such that $|\lambda(x)| < \delta \sqrt{V_1(x) V_2(x)}$, for all $x \in \mathbb{R}^2$.

(V_5') There exists $0 < \delta < 1$ such that $0 < \lambda(x) < \delta \sqrt{V_1(x) V_2(x)}$, for all $x \in \mathbb{R}^2$.

Motivated by a class of Trudinger-Moser type inequality proved in [34], we study a class of coupled systems when the nonlinearities have exponential critical growth. In view of this inequality, we consider nonlinearities with *maximal growth*, which allows us to treat System (S) variationally. Precisely, for $i = 1, 2$ and $\alpha_0^i > 0$, we say that $f_i : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ has α_0^i-*critical growth* at $+\infty$ if, uniformly in x, we have

$$\limsup_{s \to +\infty} \frac{f_i(x, s)}{A_i(x)(e^{\alpha s^2} - 1)} = \begin{cases} 0 & \text{if} \quad \alpha > \alpha_0^i, \\ \infty & \text{if} \quad \alpha < \alpha_0^i. \end{cases} \tag{2.1}$$

For each $i = 1, 2$, we assume the following hypotheses under the nonlinearities:

(f_1) $f_i : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ is C^1, $f_i(x,s) = 0$ for all $x \in \mathbb{R}^2$, $s \leq 0$, and

$$\lim_{s\to 0} \frac{f_i(x,s)}{A_i(x)s} = 0, \quad \text{uniformly in } x \in \mathbb{R}^2.$$

(f_2) $f_i(x,s)$ and $\partial f_i(x,s)/\partial s$ are locally bounded in s, that is, for any bounded interval $I \subset \mathbb{R}$, there exists $C > 0$ such that $|f_i(x,s)|, |\partial f_i(x,s)/\partial s| \leq C$, for all $(x,s) \in \mathbb{R}^2 \times I$.

(f_3) For each fixed $x \in \mathbb{R}^2$ the function $s \mapsto s^{-1} f_i(x,s)$ is increasing for $s > 0$;

(f_4) There exists $\mu_i > 2$ such that

$$0 < \mu_i F_i(x,s) := \mu_i \int_0^s f_i(x,\tau)\mathrm{d}\tau \leq s f_i(x,s), \quad \text{for all } x \in \mathbb{R}^2 \text{ and } s > 0.$$

We denote the product space $E = H_{V_1}(\mathbb{R}^2) \times H_{V_2}(\mathbb{R}^2)$ which is a Hilbert space when endowed with the scalar product

$$((u,v),(w,z)) = \int_{\mathbb{R}^2} (\nabla u \nabla w + V_1(x)uw + \nabla v \nabla z + V_2(x)vz)\ \mathrm{d}x,$$

to which corresponds the induced norm

$$\|(u,v)\|^2 = \|u\|_{V_1}^2 + \|v\|_{V_2}^2 = \int_{\mathbb{R}^2} \left(|\nabla u|^2 + V_1(x)u^2\right)\ \mathrm{d}x + \int_{\mathbb{R}^2} \left(|\nabla v|^2 + V_2(x)v^2\right)\ \mathrm{d}x.$$

The energy functional $I : E \to \mathbb{R}$ associated with System (S) is

$$I(u,v) = \frac{1}{2}\left(\|(u,v)\|^2 - 2\int_{\mathbb{R}^2} \lambda(x)uv\ \mathrm{d}x\right) - \int_{\mathbb{R}^2} (F_1(x,u) + F_2(x,v))\,\mathrm{d}x.$$

By standard arguments can be verified that $I \in C^2(E,\mathbb{R})$ and its derivative is given by

$$\langle I'(u,v),(\phi,\psi)\rangle = ((u,v),(\phi,\psi)) - \int_{\mathbb{R}^2} (f_1(x,u)\phi + f_2(x,v)\psi)\,\mathrm{d}x - \int_{\mathbb{R}^2} \lambda(x)\,(u\psi + v\phi)\,\mathrm{d}x,$$

where $(\phi,\psi) \in C_0^\infty(\mathbb{R}^N) \times C_0^\infty(\mathbb{R}^N)$. Thus critical points of I correspond to weak solutions of (S) and conversely.

Definition 2.1.1. *We say that a pair $(u,v) \in E \setminus \{(0,0)\}$ is a ground state solution (least energy solution) of (S), if (u,v) is a solution of (S) and its energy is minimal among the energy of all nontrivial solutions of (S), i.e., $I(u,v) \leq I(w,z)$ for any other nontrivial solution $(w,z) \in E$. We say that (u,v) is nonnegative (nonpositive) if $u,v \geq 0$ $(u,v \leq 0)$ and positive (negative) if $u,v > 0$ $(u,v < 0)$.*

2.1.2 Statement of the main result

Now we are in position to state our main result.

Theorem 2.1.2. *For each $i = 1, 2$ suppose that $f_i(x, s)$ and $\partial f_i(x, s)/\partial s$ have α_0^i-critical growth, (V_1)-(V_5) and (f_1)-(f_4) are satisfied. In addition, suppose the following hypothesis:*

(f_5) There exists $q > 2$ such that

$$F_1(x, s) + F_2(x, t) \geq \theta(s^q + t^q), \quad \text{for all } x \in \mathbb{R}^2 \text{ and } s, t \geq 0.$$

Then, there exists a constant $\theta_0 > 0$ such that System (S) possesses a nonnegative ground state solution $(u_0, v_0) \in E$, for some $\theta > \theta_0$. If (V_5') holds, then the ground state is positive. Moreover, $(u_0, v_0) \in C^{1,\alpha}_{loc}(\mathbb{R}^2) \times C^{1,\alpha}_{loc}(\mathbb{R}^2)$ for some $\alpha \in (0, 1)$ with the following asymptotic behavior

$$\|u_0\|_{C^{1,\alpha}(\overline{B_R(x_0)})} \to 0 \quad \text{and} \quad \|v_0\|_{C^{1,\alpha}(\overline{B_R(x_0)})} \to 0, \quad \text{as } |x_0| \to \infty.$$

Remark 2.1.3. *We collect some remarks on our assumptions:*

(i) A typical example of nonlinearity which satisfies the assumptions (f_1)-(f_5) is given by

$$f(s) = \begin{cases} \theta q s^{q-2}s + q s^{q-2}s(e^{\alpha_0 s^2} - 1) + 2\alpha_0 s^q s e^{\alpha_0 s^2} & \text{if } \; s > 0, \\ 0 & \text{if } \; s \leq 0, \end{cases}$$

where $q > \mu > 2$ and α_0 is the critical exponent of the definition (2.1).

(ii) There are many examples of functions $V_i(x)$ and $\lambda(x)$ satisfying (V_2)-(V_5). For instance, consider

$$V_i(x) = \begin{cases} 1 & \text{if } \; |x| \leq 1, \\ |x|^{\alpha^i} & \text{if } \; |x| > 1, \end{cases}$$

where $\alpha^i \geq 2$, and $\lambda(x) \in C_0^\infty(\mathbb{R}^2)$ such that

$$\lambda(x) = \begin{cases} 1/2 & \text{if } \; |x| \leq 1, \\ \left(\dfrac{1}{|x|^2 + 1}\right)^{1/2} & \text{if } \; |x| > 1. \end{cases}$$

(iii) We will prove the existence of ground state when the constant θ introduced in (f_5) is large enough. Accurately, we obtain the ground state for some constant satisfying

$$\theta > \theta_0 = \left(\frac{1}{1-\delta}\frac{\mu}{\mu-2}\frac{q-2}{q}\frac{\alpha_0\beta_0}{4\pi}\right)^{(q-2)/2}\frac{S_q^q}{q},$$

where $\mu = \min\{\mu_1, \mu_2\}$, $\alpha_0 = \max\{\alpha_0^1, \alpha_0^2\}$ and $\beta_0 = \max\{\frac{\beta_1}{(\beta_1-1)}, \frac{\beta_2}{(\beta_2-1)}\}$.

(iv) The assumption (f_5) can be weakened. Indeed, instead of (f_5), we can just assume for each $i = 1, 2$, the following local condition

$$\liminf_{s \to 0} \frac{F_i(x, s)}{s^q} \geq \mu > 0. \tag{2.2}$$

It can be deduced that conditions (f_4) and (2.2) imply that (f_5) holds.

2.1.3 Outline

The remainder of this chapter is organized as follows. In the forthcoming section we collect some lemmas which are crucial to give a variational approach for our work. Furthermore, we introduce and give some properties of the Nehari manifold associated with the energy functional. In Section 2.3 we make use of the Ekeland's variational principle to obtain a minimizing sequence for the ground state energy associated with System (S), and we use the growth conditions of the nonlinearities and a Trudinger-Moser type inequality to prove that the weak limit of this sequence will be a ground state solution of the problem. After that, we use the known ground state to get another ground state which will be positive. Finally, we apply a bootstrap argument and L^q-estimates to obtain regularity and asymptotic behavior.

2.2 Preliminary results

It is well known that when $N \geq 3$, it is standard to require a polynomial growth at infinity in order to define associated functionals in Sobolev spaces. However, when $N = 2$, in view of a class of Trudinger-Moser inequality, a faster growth can be allowed on the nonlinearities in order to treat System (S) variationally, see for instance [14, 32, 57, 71]. The following extension of the Trudinger-Moser inequality for the whole space $\mathbb{R}^2$ contained in [34], allow us to study System (S) when the nonlinearities have exponential growth involving the terms $A_1(x)$ and $A_2(x)$.

Lemma 2.2.1. *Suppose that (V_1)-(V_4) are satisfied and let $i = 1, 2$. Then, for any $u \in H_{V_i}(\mathbb{R}^2)$, $q \geq 2$ and $\alpha^i > 0$,*

$$\int_{\mathbb{R}^2} A_i(x)(e^{\alpha^i u^2} - 1)|u|^q \, \mathrm{d}x < \infty. \tag{2.3}$$

Furthermore, if $\alpha^i\beta_i\|u\|^2_{V_i} < 4\pi(\beta_i - 1)$*, there exists* $C > 0$ *such that*

$$\int_{\mathbb{R}^2} A_i(x)(e^{\alpha^i u^2} - 1)|u|^q \, \mathrm{d}x \le C\|u\|^q_{V_i}. \tag{2.4}$$

Lemma 2.2.2. *If* $\alpha > 0$*,* $l \ge 1$ *and* $r \ge l$*, then*

$$(e^{\alpha s^2} - 1)^l \le (e^{\alpha l s^2} - 1), \quad \textit{for all } s \in \mathbb{R}.$$

Proof. In fact, let $f : [1, +\infty) \to \mathbb{R}$ be the function defined by $f(t) = (t^l - 1) - (t-1)^l$. Notice that $f(1) = 0$ and $f'(t) \ge 0$, for all $t \ge 1$. The result follows taking $t = e^{\alpha s^2}$. ∎

For $i = 1, 2$, let us define the weighted Lebesgue space

$$L^p_{A_i}(\mathbb{R}^2) = \left\{ u : \mathbb{R}^2 \to \mathbb{R} \text{ measurable } : \int_{\mathbb{R}^2} A_i(x)|u|^p \, \mathrm{d}x < +\infty \right\},$$

endowed with the usual norm

$$\|u\|_{L^p_{A_i}} = \left(\int_{\mathbb{R}^2} A_i(x)|u|^p \, \mathrm{d}x \right)^{1/p}.$$

We set the product space $L^p_{A_i}(\mathbb{R}^2) \times L^p_{A_i}(\mathbb{R}^2)$ endowed with the norm

$$\|(u, v)\|_{L^p_{A_i}} = \left(\|u\|^p_{L^p_{A_i}} + \|v\|^p_{L^p_{A_i}} \right)^{1/p}.$$

The following embedding result has been proved by B. Sirakov in [64].

Lemma 2.2.3. *Under the assumptions* (V_2)-(V_4)*, for* $i = 1, 2$*,* $H_{V_i}(\mathbb{R}^2)$ *is compactly embedded into the Lebesgue spaces* $L^p(\mathbb{R}^2)$ *and* $L^p_{A_i}(\mathbb{R}^2)$*, for all* $2 \le q < \infty$.

Lemma 2.2.4. *Suppose that* (f_3)-(f_4) *hold. Then, for each* $i = 1, 2$ *we have*

$$s^2 \frac{\partial f_i}{\partial s}(x, s) - s f_i(x, s) > 0, \tag{2.5}$$

$$\frac{\partial f_i}{\partial s}(x, s) > 0, \tag{2.6}$$

$$s f_i(x, s) - 2F_i(x, s) > 0, \tag{2.7}$$

for all $x \in \mathbb{R}^2$ *and* $s > 0$.

Proof. For $i = 1, 2$, it follows from assumption (f_3) that

$$0 < \frac{\partial}{\partial s}\left(\frac{f_i(x, s)}{s} \right) = \frac{s^2 \frac{\partial f_i}{\partial s}(x, s) - s f_i(x, s)}{s^3}, \quad \text{for } x \in \mathbb{R}^2 \text{ and } s > 0,$$

which implies (2.5). Using the preceding estimate together with (f_4) we get

$$\frac{\partial f_i}{\partial s}(x, s) > \frac{f_i(x, s)}{s} \ge \frac{\mu_i}{s^2} F_i(x, s) > 0, \quad \text{for } x \in \mathbb{R}^2 \text{ and } s > 0,$$

which implies (2.6). Now, let $x \in \mathbb{R}^2$ be fixed and consider $0 < s < t$. Thus, using (f_3) we deduce that

$$\begin{aligned} sf_i(x,s) - 2F_i(x,s) &= s^2\frac{f_i(x,s)}{s} - 2F_i(x,t) + 2\int_s^t \frac{f_i(x,\tau)}{\tau}\tau\, \mathrm{d}\tau \\ &< s^2\frac{f_i(x,t)}{t} - 2F_i(x,t) + 2\frac{f_i(x,t)}{t}\int_s^t \tau\, \mathrm{d}\tau \\ &= s^2\frac{f_i(x,t)}{t} - 2F_i(x,t) + (t^2 - s^2)\frac{f_i(x,t)}{t} \\ &= tf_i(x,t) - 2F_i(x,t). \end{aligned}$$

Therefore, the function $sf_i(x,s) - 2F_i(x,s)$ is increasing for $s > 0$. Since $sf_i(x,0) - 2F_i(x,0) = 0$, (2.7) follows. ■

Lemma 2.2.5. *Suppose that (2.1), (V_4), (f_1), (f_2) and (f_4) are satisfied. For any $\varepsilon > 0$, $\alpha > \alpha_0^i$ and $r > 2$, there exists $C = C(\varepsilon, r) > 0$ such that*

$$f_i(x,s)s \le \varepsilon A_i(x)s^2 + CA_i(x)(e^{\alpha s^2} - 1)s^r, \tag{2.8}$$

$$F_i(x,s) \le \varepsilon A_i(x)s^2 + CA_i(x)(e^{\alpha s^2} - 1)s^r, \tag{2.9}$$

$$\frac{\partial f_i}{\partial s}(x,s) \le \varepsilon A_i(x)s^2 + CA_i(x)(e^{\alpha s^2} - 1)s^r, \tag{2.10}$$

for each $i = 1,2$ and for all $(x,s) \in \mathbb{R}^2 \times [0,+\infty)$.

Proof. Let $\varepsilon > 0$ be fixed. By using (f_1), there exists $\delta > 0$ such that

$$f_i(x,s)s \le \varepsilon A_i(x)s^2, \quad \text{for all } x \in \mathbb{R}^2 \text{ and } 0 \le s < \delta. \tag{2.11}$$

By using (2.1) for $\alpha > \alpha_0^i$, there exists $R > 0$ such that

$$f_i(x,s)s \le \varepsilon A_i(x)(e^{\alpha s^2} - 1)s \le C_1(\varepsilon,r)A_i(x)(e^{\alpha s^2} - 1)s^r, \tag{2.12}$$

for all $x \in \mathbb{R}^2$ and $s \ge R$. It follows from (V_4) and (f_2) that

$$f_i(x,s)s \le C_2(\varepsilon,r)A_i(x)(e^{\alpha s^2} - 1)s^r, \quad \text{for all } (x,s) \in \mathbb{R}^2 \times [\delta, R]. \tag{2.13}$$

From (2.12) and (2.13) we get

$$f_i(x,s)s \le C(\varepsilon,r)A_i(x)(e^{\alpha s^2} - 1)s^r, \quad \text{for all } (x,s) \in \mathbb{R}^2 \times [\delta, +\infty). \tag{2.14}$$

Combining (2.11) and (2.14) we get (2.8). By assumption (f_4) we have

$$F_i(x,s) \le \frac{1}{\mu_i} f_i(x,s)s, \quad \text{for all } (x,s) \in \mathbb{R}^2 \times [0,+\infty),$$

and (2.9) follows immediately of the estimate (2.8). Analogously we can deduce (2.10). ■

The Nehari manifold associated to System (S) is given by

$$\mathcal{N} = \{(u,v) \in E\backslash\{(0,0)\} : \langle I'(u,v),(u,v)\rangle = 0\}.$$

Notice that if $(u,v) \in \mathcal{N}$ then

$$\|(u,v)\|^2 - 2\int_{\mathbb{R}^2} \lambda(x)uv \, \mathrm{d}x = \int_{\mathbb{R}^2} f_1(x,u)u \, \mathrm{d}x + \int_{\mathbb{R}^2} f_2(x,v)v \, \mathrm{d}x. \tag{2.15}$$

Lemma 2.2.6. *If $(u,v) \in \mathcal{N}$, then $|\{u > 0\}| > 0$ or $|\{v > 0\}| > 0$.*

Proof. Arguing by contradiction, let $(u,v) \in \mathcal{N}$ be such that $|\{u > 0\}| = 0$ and $|\{v > 0\}| = 0$. We recall from Lemma 1.2.1 that

$$\|(u,v)\|^2 - 2\int_{\mathbb{R}^2} \lambda(x)uv \, \mathrm{d}x \geq (1-\delta)\|(u,v)\|^2, \quad \text{for all } (u,v) \in E. \tag{2.16}$$

By using (2.16) and the fact that $(u,v) \in \mathcal{N}$, we have that

$$\begin{aligned} 0 &< (1-\delta)\|(u,v)\|^2 \\ &\leq \|(u,v)\|^2 - 2\int_{\mathbb{R}^2} \lambda(x)uv \, \mathrm{d}x \\ &= \int_{\mathbb{R}^2} f_1(x,u)u \, \mathrm{d}x + \int_{\mathbb{R}^2} f_2(x,v) \, \mathrm{d}x = 0, \end{aligned}$$

which is not possible and finishes the proof. ■

In general, the Nehari manifold may not be a manifold. However, under our assumptions, $\mathcal{N}$ is in fact a C^1-manifold as we can see in the following lemma:

Lemma 2.2.7. *$\mathcal{N}$ is a C^1-manifold and there exists $\gamma > 0$, such that*

$$\|(u,v)\| \geq \gamma, \quad \text{for all } (u,v) \in \mathcal{N}. \tag{2.17}$$

Proof. We define the C^1-functional $\varphi : E\backslash\{(0,0)\} \to \mathbb{R}$ by $\varphi(u,v) = \langle I'(u,v),(u,v)\rangle$, that is,

$$\varphi(u,v) = \|(u,v)\|^2 - 2\int_{\mathbb{R}^2} \lambda(x)uv \, \mathrm{d}x - \int_{\mathbb{R}^2} f_1(x,u)u \, \mathrm{d}x - \int_{\mathbb{R}^2} f_2(x,v)v \, \mathrm{d}x.$$

Notice that $\mathcal{N} = \varphi^{-1}(0)$. Moreover, if $(u,v) \in \mathcal{N}$, then it follows from (2.5) and (2.15) that

$$\langle \varphi'(u,v),(u,v)\rangle = \int_{\mathbb{R}^2} \left(f_1(x,u)u - \frac{\partial f_1}{\partial u}(x,u)u^2 + f_2(x,v)v - \frac{\partial f_2}{\partial v}(x,v)v^2 \right) \mathrm{d}x < 0, \tag{2.18}$$

which implies that 0 is a regular value of φ. Therefore $\mathcal{N}$ is a C^1-manifold.

Assume by contradiction that (2.17) does not hold. Thus, there exists a sequence $(u_n, v_n)_n \subset \mathcal{N}$ such that $\|(u_n, v_n)\| \to 0$ as $n \to +\infty$. Let us consider

$$\alpha^i > \alpha_0^i \quad \text{and} \quad 0 < \gamma_1 < \gamma_2 < 4\pi(\beta_i - 1)/\alpha^i\beta_i, \quad \text{for } i = 1, 2$$

and let $n_0 \in \mathbb{N}$ be such that $\|(u_n, v_n)\|^2 \leq \gamma_1 < \gamma_2$, for $n \geq n_0$. By using Lemma 2.2.1, estimate (2.8) and Sobolev embedding, for given $r > 2$ and $\varepsilon > 0$, we deduce that

$$\begin{aligned} \int_{\mathbb{R}^2} f_1(x, u_n)u_n \, \mathrm{d}x &\leq \varepsilon\|u_n\|^2_{L^2_{A_1}} + C_2 \int_{\mathbb{R}^2} A_1(x)(e^{\alpha^1 u_n^2} - 1)|u_n|^r \, \mathrm{d}x \\ &\leq \varepsilon C_1\|u_n\|^2_{V_1} + C_2\|u_n\|^r_{V_1} \\ &\leq \varepsilon C_1\|(u_n, v_n)\|^2 + C_2\|(u_n, v_n)\|^r, \end{aligned}$$

for $n \geq n_0$. Analogously, we get

$$\int_{\mathbb{R}^2} f_2(x, v_n)v_n \, \mathrm{d}x \leq \varepsilon C_3\|(u_n, v_n)\|^2 + C_4\|(u_n, v_n)\|^r, \quad \text{for } n \geq n_0.$$

Hence,

$$\int_{\mathbb{R}^2} (f_1(x, u_n)u_n + f_2(x, v_n)v_n) \, \mathrm{d}x \leq \varepsilon\tilde{C}_1\|(u_n, v_n)\|^2 + \tilde{C}_2\|(u_n, v_n)\|^r. \tag{2.19}$$

Thus, it follows from (2.15), (2.16) and (2.19) that

$$\begin{aligned} (1 - \delta)\|(u_n, v_n)\|^2 &\leq \|(u_n, v_n)\|^2 - 2\int_{\mathbb{R}^2} \lambda(x)u_n v_n \, \mathrm{d}x \\ &= \int_{\mathbb{R}^2} (f_1(x, u_n)u_n + f_2(x, v_n)v_n) \, \mathrm{d}x \\ &\leq \varepsilon\tilde{C}_1\|(u_n, v_n)\|^2 + \tilde{C}_2\|(u_n, v_n)\|^r. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary and $\tilde{C}_1$ does not depends of ε and n, we can choose ε sufficiently small such that $1 - \delta - \varepsilon\tilde{C}_1 > 0$. Therefore,

$$0 < \gamma_3 = \left(\frac{1 - \delta - \varepsilon\tilde{C}_1}{\tilde{C}_2}\right)^{1/(r-2)} \leq \|(u_n, v_n)\|.$$

If we consider $\gamma_1 < \gamma = \min\{\gamma_2, \gamma_3\}$ we get a contradiction. Therefore (2.17) holds. ■

Let us define the ground state energy associated with System (S)

$$c_{\mathcal{N}} = \inf_{(u,v)\in\mathcal{N}} I(u, v).$$

We claim that $c_{\mathcal{N}}$ is positive. Indeed, using (f_4) and recalling that $\mu := \min\{\mu_1, \mu_2\}$, it follows that

$$\begin{aligned} I(u, v) &= \frac{1}{2}\left(\|(u, v)\|^2 - 2\int_{\mathbb{R}^2} \lambda(x)uv \, \mathrm{d}x\right) - \int_{\mathbb{R}^2} (F_1(x, u) + F_2(x, v)) \, \mathrm{d}x \\ &\geq \frac{1}{2}\left(\|(u, v)\|^2 - 2\int_{\mathbb{R}^2} \lambda(x)uv \, \mathrm{d}x\right) - \frac{1}{\mu_1}\int_{\mathbb{R}^2} f_1(x, u)u \, \mathrm{d}x - \frac{1}{\mu_2}\int_{\mathbb{R}^2} f_2(x, v)v \, \mathrm{d}x \\ &\geq \left(\frac{1}{2} - \frac{1}{\mu}\right)\left(\|(u, v)\|^2 - 2\int_{\mathbb{R}^2} \lambda(x)uv \, \mathrm{d}x\right), \end{aligned}$$

for all $(u,v) \in \mathcal{N}$, which together with Lemma 2.2.7 and (2.16) implies that

$$I(u,v) \geq \left(\frac{1}{2}-\frac{1}{\mu}\right)(1-\delta)\|(u,v)\|^2 \geq \left(\frac{1}{2}-\frac{1}{\mu}\right)(1-\delta)\gamma > 0.$$

Let us define the set

$$E_+ := \{(u,v) \in E\backslash\{(0,0)\} : |\{u>0\}| > 0 \text{ or } |\{v>0\}|>0\}.$$

Lemma 2.2.8. *Suppose that* (V_1)-(V_5)*,* (f_1)-(f_4) *and* (f_5) *holds. Then for each* $(u,v) \in E_+$*, there exists a unique* $t_0 > 0$*, depending only of* (u,v)*, such that*

$$(t_0u, t_0v) \in \mathcal{N} \quad \text{and} \quad I(t_0u,t_0v) = \max_{t\geq 0} I(tu,tv).$$

Proof. Let $(u,v) \in E_+$ be fixed and define the function $g : [0,\infty) \to \mathbb{R}$ such that $g(t) = I(tu,tv)$. Notice that

$$\langle I'(tu,tv),(tu,tv)\rangle = tg'(t).$$

Thus, it suffices to find a nontrivial positive critical point of g. From assumption (V_5) we get

$$\|(u,v)\|^2 - 2\int_{\mathbb{R}^2} \lambda(x)uv \, dx \geq 0.$$

Using assumption (V_4) we can deduce that

$$F_i(x,s) \geq C(|s|^{\mu_i}-1), \quad \text{for } s>0.$$

We may assume without loss of generality that $|\{u>0\}|>0$. Let $R>0$ be such that $|\{u>0\}\cap B_R(0)|>0$. Thus, we have that

$$\begin{aligned} g(t) &= \frac{t^2}{2}\left(\|(u,v)\|^2 - 2\int_{\mathbb{R}^2}\lambda(x)uv\,dx\right) - \int_{\mathbb{R}^2}F_1(x,tu)\,dx - \int_{\mathbb{R}^2}F_2(x,tv)\,dx \\ &\leq \frac{t^2}{2}\left(\|(u,v)\|^2 - 2\int_{\mathbb{R}^2}\lambda(x)uv\,dx\right) - \theta t^{\mu_1}\int_{\{u>0\}\cap B_R(0)}|u|^{\mu_1}\,dx. \end{aligned}$$

Since $\mu_1 > 2$, we obtain $g(t)<0$ for $t>0$ sufficiently large. On the other hand, for some $\alpha^i > \alpha_0^i$ and

$$0 < t < \min\left\{\left(\frac{4\pi(\beta_1-1)}{\alpha^1\beta_1\|u\|^2_{V_1}}\right)^{1/2}, \left(\frac{4\pi(\beta_2-1)}{\alpha^2\beta_2\|v\|^2_{V_2}}\right)^{1/2}\right\},$$

we can use (2.9) and the same ideas used to obtain the estimate (2.19), to get

$$\int_{\mathbb{R}^2}(F_1(x,tu)+F_2(x,tv))\,dx \leq C_1\varepsilon\frac{t^2}{2}\|(u,v)\|^2 + C_2t^r\|(u,v)\|^r, \tag{2.20}$$

where $r > 2$ and ε is small enough such that $1 - \delta - C_1\varepsilon > 0$. Hence, by using (2.16) and (2.20), we have

$$\begin{aligned} g(t) &= \frac{t^2}{2}\left(\|(u,v)\|^2 - 2\int_{\mathbb{R}^2} \lambda(x)uv \, \mathrm{d}x\right) - \int_{\mathbb{R}^2} F_1(x,tu)\, \mathrm{d}x - \int_{\mathbb{R}^2} F_2(x,tv)\, \mathrm{d}x \\ &\geq t^2\|(u,v)\|^2\left(\frac{1-\delta-C_1\varepsilon}{2} - C_2 t^{r-2}\|(u,v)\|^{r-2}\right). \end{aligned}$$

Therefore $g(t) > 0$ provided $t > 0$ is sufficiently small. Thus g has maximum points in $(0, \infty)$. In order to prove the uniqueness, note that every critical point of g satisfies

$$\|(u,v)\|^2 - 2\int_{\mathbb{R}^2} \lambda(x)uv \, \mathrm{d}x = \int_{\mathbb{R}^2} \frac{f_1(x,tu)u}{t}\, \mathrm{d}x + \int_{\mathbb{R}^2} \frac{f_2(x,tv)v}{t}\, \mathrm{d}x. \tag{2.21}$$

Moreover, it follows from (2.5) that

$$\frac{d}{dt}\left(\frac{f_i(x,ts)s}{t}\right) = \frac{ts^2\frac{\partial f_i}{\partial s}(x,ts) - sf_i(x,ts)}{t^2} = \frac{(ts)^2\frac{\partial f_i}{\partial s}(x,ts) - tsf_i(x,ts)}{t^3} > 0,$$

which ensure that the right-hand side of (2.21) is increasing on $t > 0$, and consequently, the critical point $t_0 \in (0, +\infty)$ is *unique*. ■

Let us define

$$S_q = \inf_{(u,v)\in E\backslash\{(0,0)\}} S_q(u,v),$$

where for any $(u,v) \in E\backslash\{(0,0)\}$, we define

$$S_q(u,v) = \frac{\left(\|(u,v)\|^2 - 2\int_{\mathbb{R}^2} \lambda(x)uv\mathrm{d}x\right)^{1/2}}{\|(u,v)\|_q}.$$

Notice that S_q is positive. In fact, by using (2.16) and Sobolev embedding we have that

$$\|(u,v)\|^2 - \int_{\mathbb{R}^2} \lambda(x)uv \, \mathrm{d}x \geq (1-\delta)\|(u,v)\|^2 \geq (1-\delta)C\|(u,v)\|_q^2,$$

for all $(u,v) \in E\backslash\{(0,0)\}$. Therefore, $S_q \geq [(1-\delta)C]^{1/2} > 0$.

Lemma 2.2.9. *For any $(u,v) \in E\backslash\{(0,0)\}$, we have*

$$\max_{t\geq 0}\left(\frac{t^2}{2}S_q(u,v)^2\|(u,v)\|_q^2 - \theta t^q\|(u,v)\|_q^q\right) = \left(\frac{1}{2} - \frac{1}{q}\right)\frac{S_q(u,v)^{2q/(q-2)}}{(q\theta)^{2/(q-2)}},$$

where θ and q are the constants introduced in (f_5).

Proof. Let $h : [0, +\infty) \to \mathbb{R}$ be defined by

$$h(t) = \frac{t^2}{2} S_q(u,v)^2 \|(u,v)\|_q^2 - \theta t^q \|(u,v)\|_q^q.$$

Thus, h is differentiable and

$$h'(t) = t S_q(u,v)^2 \|(u,v)\|_q^2 - q\theta t^{q-1} \|(u,v)\|_q^q.$$

Notice that $h'(t) \geq 0$ if and only if $t \leq \tilde{t}$, where

$$\tilde{t} = \left(\frac{S_q(u,v)^2}{q\theta \|(u,v)\|_q^{q-2}} \right)^{1/(q-2)}.$$

Therefore, $\tilde{t}$ is a maximum for h and

$$\max_{t \geq 0} h(t) = h(\tilde{t}) = \left(\frac{1}{2} - \frac{1}{q} \right) \frac{S_q(u,v)^{2q/(q-2)}}{(q\theta)^{2/(q-2)}}. \qquad \blacksquare$$

2.3 Proof of Theorem 2.1.2

By Ekeland's variational principle (see [38]), there exists a sequence $(u_n, v_n)_n \subset \mathcal{N}$ such that

$$I(u_n, v_n) \to c_{\mathcal{N}} \quad \text{and} \quad I'(u_n, v_n) \to 0. \tag{2.22}$$

Lemma 2.3.1. *We have the following facts:*

(a) The sequence $(u_n, v_n)_n$ is bounded in E.

(b) We have the following estimate

$$\limsup_{n \to \infty} \|(u_n, v_n)\|^2 \leq \frac{1}{1-\delta} \frac{\mu}{\mu - 2} \frac{q-2}{q} \frac{S_q^{2q/(q-2)}}{(q\theta)^{2/(q-2)}}. \tag{2.23}$$

Proof. By using (2.22) we get

$$c_{\mathcal{N}} + o_n(1) = \frac{1}{2} \left(\|(u_n, v_n)\|^2 - 2 \int_{\mathbb{R}^2} \lambda(x) u_n v_n \, dx \right) - \int_{\mathbb{R}^2} (F_1(x, u_n) + F_2(x, v_n)) \, dx,$$

which together with (f_4) and (2.16) implies that

$$\begin{aligned} c_{\mathcal{N}} + o_n(1) &\geq \frac{1}{2} \left(\|(u_n, v_n)\|^2 - 2 \int_{\mathbb{R}^2} \lambda(x) u_n v_n \right) - \frac{1}{\mu} \int_{\mathbb{R}^2} (f_1(x, u_n) u_n + f_2(x, v_n) v_n) \\ &\geq \left(\frac{1}{2} - \frac{1}{\mu} \right) (1 - \delta) \|(u_n, v_n)\|^2. \end{aligned}$$

Therefore $(u_n, v_n)_n$ is bounded in E.

To prove (b), using the computation in the proof of (a) we obtain

$$\limsup_{n\to\infty} \|(u_n, v_n)\|^2 \le \frac{1}{1-\delta}\frac{2\mu}{\mu-2}c_{\mathcal{N}}. \tag{2.24}$$

From Lemma 2.2.8, for any $(\psi, \phi) \in E\backslash\{(0,0)\}$ there exists $t_0 > 0$ such that $(t_0|\psi|, t_0|\phi|) \in \mathcal{N}$, which yields

$$c_{\mathcal{N}} \le I(t_0|\psi|, t_0|\phi|) = \max_{t\ge 0} I(t|\psi|, t|\phi|). \tag{2.25}$$

Thus, by using (f_5), (2.25) and the fact that $S_q(|\psi|, |\phi|)^2 \le S_q(\psi, \phi)^2$ we deduce that

$$\begin{aligned} c_{\mathcal{N}} &\le \max_{t\ge 0}\left\{\frac{t^2}{2}\left(\|(\psi,\phi)\|^2 - 2\int_{\mathbb{R}^2}\lambda(x)|\psi||\phi|\,\mathrm{d}x\right) - \theta t^q \|(\psi,\phi)\|_q^q\right\} \\ &= \max_{t\ge 0}\left\{\frac{t^2}{2}S_q(|\psi|,|\phi|)^2\|(\psi,\phi)\|_q^2 - \theta t^q\|(\psi,\phi)\|_q^q\right\} \\ &\le \max_{t\ge 0}\left\{\frac{t^2}{2}S_q(\psi,\phi)^2\|(\psi,\phi)\|_q^2 - \theta t^q\|(\psi,\phi)\|_q^q\right\}, \end{aligned}$$

which jointly with Lemma 2.2.9 ensures that

$$c_{\mathcal{N}} \le \left(\frac{1}{2} - \frac{1}{q}\right)\frac{S_q(\psi,\phi)^{2q/(q-2)}}{(q\theta)^{2/(q-2)}}. \tag{2.26}$$

Combining (2.24), (2.26) and taking the infimum over $(\psi,\phi) \in E\backslash\{(0,0)\}$ we have that

$$\limsup_{n\to\infty} \|(u_n, v_n)\|^2 \le \frac{1}{1-\delta}\frac{\mu}{\mu-2}\frac{q-2}{q}\frac{S_q^{2q/(q-2)}}{(q\theta)^{2/(q-2)}}. \qquad \blacksquare$$

Since $(u_n, v_n)_n$ is bounded in E, passing to a subsequence, we may assume that $(u_n, v_n) \rightharpoonup (u_0, v_0)$ weakly in E. Moreover, it follows from Lemma 2.2.3 that, up to a subsequence,

- $u_n \to u_0$ strongly in $L^p_{A_i}(\mathbb{R}^2)$, for all $2 \le p < \infty$;
- $v_n \to v_0$ strongly in $L^q_{A_i}(\mathbb{R}^2)$, for all $2 \le p < \infty$;
- $u_n(x) \to u_0(x)$ and $v_n(x) \to v_0(x)$ almost everywhere in $\mathbb{R}^2$.

Proposition 2.3.2. *The weak limit (u_0, v_0) is nontrivial.*

Proof. In light of (2.23), for $\alpha^i > \alpha_0^i$, there exists $\theta_0 > 0$ such that

$$\limsup_{n\to\infty} \|(u_n, v_n)\|^2 < \min\left\{4\pi(\beta_1 - 1)/(\alpha^1\beta_1), 4\pi(\beta_2-1)/(\alpha^2\beta_2)\right\}, \quad \text{for } i = 1, 2$$

provided that $\theta > \theta_0$. Let $r > 1$ sufficiently close to 1 such that

$$0 < r\|(u_n, v_n)\|^2 < 4\pi(\beta_i - 1)/(\alpha^i\beta_i), \quad \text{for some } \theta > \theta_0.$$

Now, we consider $q > 2$, $l \in (1, r)$ and $1/l + 1/l' = 1$. Thus, using Lemma 2.2.2, (2.8) and Hölder inequality we deduce that

$$\begin{aligned}\int_{\mathbb{R}^2} f_1(x,u_n)u_n\mathrm{d}x &\leq \|u_n\|^2_{L^2_{A_1}} + C_1\int_{\mathbb{R}^2} A_1(x)(e^{\alpha^1 u_n^2}-1)|u_n|^q\,\mathrm{d}x \\ &\leq \|u_n\|^2_{L^2_{A_1}} + C_1\left(\int_{\mathbb{R}^2} A_1(x)(e^{\alpha^1 u_n^2}-1)^l|u_n|^{ql}\,\mathrm{d}x\right)^{1/l}\|u_n\|^q_{L^{ql'}_{A_1}} \\ &\leq \|u_n\|^2_{L^2_{A_1}} + C_1\left(\int_{\mathbb{R}^2} A_1(x)(e^{r\alpha^1 u_n^2}-1)|u_n|^{ql'}\,\mathrm{d}x\right)^{1/l}\|u_n\|^q_{L^{ql'}_{A_1}},\end{aligned}$$

which together with Lemma 2.2.1 implies that

$$\int_{\mathbb{R}^2} f_1(x,u_n)u_n\,\mathrm{d}x \leq \|u_n\|^2_{L^2_{A_1}} + C_1\|u_n\|^q_{L^{ql'}_{A_1}}. \tag{2.27}$$

Analogously,

$$\int_{\mathbb{R}^2} f_2(x,v_n)v_n\,\mathrm{d}x \leq \|v_n\|^2_{L^2_{A_2}} + C_2\|v_n\|^q_{L^{ql'}_{A_2}}, \tag{2.28}$$

On the other hand, from Lemma 2.2.7 and (2.16) it follows that

$$0 < \gamma(1-\delta) \leq (1-\delta)\|(u_n,v_n)\|^2 \leq \|(u_n,v_n)\|^2 - 2\int_{\mathbb{R}^2}\lambda(x)u_nv_n\,\mathrm{d}x. \tag{2.29}$$

Since $(u_n, v_n)_n \subset \mathcal{N}$, one sees that

$$\|(u_n,v_n)\|^2 - 2\int_{\mathbb{R}^2}\lambda(x)u_nv_n\,\mathrm{d}x = \int_{\mathbb{R}^2} f_1(x,u_n)u_n\,\mathrm{d}x + \int_{\mathbb{R}^2} f_2(x,v_n)v_n\,\mathrm{d}x. \tag{2.30}$$

Therefore, combining (2.27), (2.28), (2.29) and (2.30), we can deduce that

$$0 < \gamma(1-\delta) \leq \|u_n\|^2_{L^2_{A_1}} + \|v_n\|^2_{L^2_{A_2}} + \tilde{C}\left(\|u_n\|^q_{L^{ql'}_{A_1}} + \|v_n\|^q_{L^{ql'}_{A_2}}\right),$$

which together with Lemma 2.2.3 implies $(u_0, v_0) \neq (0, 0)$. ■

Proposition 2.3.3. *The weak limit (u_0, v_0) is a critical point of the energy functional I.*

Proof. By the weak convergence we have that

$$((u_n,v_n),(\phi,\psi)) \to ((u_0,v_0),(\phi,\psi)), \quad \text{for all } (\phi,\psi) \in C_0^\infty(\mathbb{R}^2)\times C_0^\infty(\mathbb{R}^2).$$

Moreover, from (V_5), Lemma 2.2.3 and Hölder inequality,

$$\left|\int_{\mathbb{R}^2}\lambda(x)u_n\psi\,\mathrm{d}x - \int_{\mathbb{R}^2}\lambda(x)u_0\psi\,\mathrm{d}x\right| \leq \|\lambda(x)\|_\infty\|\psi\|_2\|u_n-u_0\|_2 \to 0,$$

and

$$\left|\int_{\mathbb{R}^2}\lambda(x)v_n\phi\,\mathrm{d}x - \int_{\mathbb{R}^2}\lambda(x)v_0\phi\,\mathrm{d}x\right| \leq \|\lambda(x)\|_\infty\|\phi\|_2\|v_n-v_0\|_2 \to 0.$$

Thus, if we prove that

$$\int_{\mathbb{R}^2} f_1(x,u_n)\phi \,\mathrm{d}x \to \int_{\mathbb{R}^2} f_1(x,u_0)\phi \,\mathrm{d}x \quad \text{and} \quad \int_{\mathbb{R}^2} f_2(x,v_n)\psi \,\mathrm{d}x \to \int_{\mathbb{R}^2} f_2(x,v_0)\psi \,\mathrm{d}x, \tag{2.31}$$

then,

$$\langle I'(u_n,v_n),(\phi,\psi)\rangle \to \langle I'(u_0,v_0),(\phi,\psi)\rangle, \quad \text{for all } (\phi,\psi) \in C_0^\infty(\mathbb{R}^2) \times C_0^\infty(\mathbb{R}^2). \tag{2.32}$$

Since the space $C_0^\infty(\mathbb{R}^2)$ is dense in $(H_{V_i}(\mathbb{R}^2), \|\cdot\|_{V_i})$, it follows from (2.32) that (u_0,v_0) is a critical point of I. Thus, it remains to prove (2.31). Notice that (2.31) holds if and only if for any compact set $K \subset \mathbb{R}^2$ we have

$$\int_K |f_1(x,u_n)| \,\mathrm{d}x \to \int_K |f_1(x,u_0)| \,\mathrm{d}x \quad \text{and} \quad \int_K |f_2(x,v_n)| \,\mathrm{d}x \to \int_K |f_2(x,v_0)| \,\mathrm{d}x. \tag{2.33}$$

Let us prove the first convergence. By using (2.27), we can deduce that

$$\int_K |f_1(x,u_n)u_n| \,\mathrm{d}x \le C_1. \tag{2.34}$$

Since $u_n, u_0 \in L^q(\mathbb{R}^2)$, we have that $u_n, u_0 \in L^1(K)$. For any $M > 0$ we can write

$$\left| \int_K |f_1(x,u_n)| \,\mathrm{d}x - \int_K |f_1(x,u_0)| \,\mathrm{d}x \right| \le I_1^n + I_2^n + I_3^n,$$

where

$$I_1^n = \int_{\{|u_n(x)| \ge M\}} |f_1(x,u_n)| \,\mathrm{d}x, \qquad I_2^n = \int_{\{|u_0(x)| \ge M\}} |f_1(x,u_0)| \,\mathrm{d}x,$$

$$I_3^n = \int_{\{|u_n(x)| < M\}} \left(|f_1(x,u_n)| - |f_1(x,u_0)|\right) \,\mathrm{d}x.$$

Let us estimate each of these integrals separately. For any $\varepsilon > 0$, we can use (2.34) and choose $M > 0$ sufficiently large such that

$$\int_{\{|u_n(x)| \ge M\}} |f_1(x,u_n)| \,\mathrm{d}x = \int_{\{|u_n(x)| \ge M\}} \frac{|f_1(x,u_n)u_n|}{|u_n|} \,\mathrm{d}x \le \frac{C_1}{M} < \frac{\varepsilon}{3}. \tag{2.35}$$

Moreover, since $f_1(x,u_0) \in L^1(K)$, we can choose $M > 0$ sufficiently large such that

$$\int_{\{|u_0(x)| \ge M\}} |f_1(x,u_0)| \,\mathrm{d}x \le \frac{\varepsilon}{3}. \tag{2.36}$$

Thus, let $M > 0$ be fixed such that (2.35) and (2.36) are satisfied. Let us denote

$$H_n(x) = \chi_{\{|u_n(x)| < M\}} |f_1(x,u_n)| - \chi_{\{|u_0| < M\}} |f_1(x,u_0)|.$$

We claim that

$$I_3^n = \int_K H_n(x) \,\mathrm{d}x \to 0. \tag{2.37}$$

In fact, notice that:

- $H_n(x) \to 0$, almost everywhere in K;
- $|H_n(x)| \leq |f_1(x, u_0)|$, if $|u_n(x)| \geq M$;
- $|H_n(x)| \leq \tilde{C} + |f_1(x, u_0)|$, if $|u_n(x)| < M$, where

$$\tilde{C} = \sup\{|f_1(x, s)| : x \in K, |s| < M\}.$$

Thus, the claim follows from Lebesgue dominated convergence theorem. Therefore, combining (2.35), (2.36) and (2.37), we obtain (2.33), and consequently (2.31). ■

Proposition 2.3.4. *The weak limit (u_0, v_0) is a ground state solution for System (S).*

Proof. Since that $(u_0, v_0) \neq (0, 0)$ and $I'(u_0, v_0) = 0$, we have that $(u_0, v_0) \in \mathcal{N}$. Therefore $c_{\mathcal{N}} \leq I(u_0, v_0)$. On the other hand, by using (2.7), it follows from Fatou's lemma that

$$\begin{aligned} c_{\mathcal{N}} + o_n(1) &= I(u_n, v_n) - \frac{1}{2}\langle I'(u_n, v_n), (u_n, v_n)\rangle \\ &= \frac{1}{2}\int_{\mathbb{R}^2} (f_1(x, u_n)u_n - 2F_1(x, u_n) + f_2(x, v_n)v_n - 2F_2(x, v_n))\, \mathrm{d}x \\ &\geq \frac{1}{2}\int_{\mathbb{R}^2} (f_1(x, u_0)u_0 - 2F_1(x, u_0) + f_2(x, v_0)v_0 - 2F_2(x, v_0))\, \mathrm{d}x + o_n(1) \\ &= I(u_0, v_0) - \frac{1}{2}\langle I'(u_0, v_0), (u_0, v_0)\rangle + o_n(1) \\ &= I(u_0, v_0) + o_n(1), \end{aligned}$$

which implies that $c_{\mathcal{N}} \geq I(u_0, v_0)$. Therefore $I(u_0, v_0) = c_{\mathcal{N}}$. ■

We have been proved that (u_0, v_0) is a ground state solution for System (S). By assumptions (f_1) and (f_4) we have for $i = 1, 2$ that

$$F_i(x, s) \leq F_i(x, |s|), \quad \text{for all } (x, s) \in \mathbb{R}^2 \times \mathbb{R}.$$

Thus, we can deduce that $I(|u_0|, |v_0|) \leq I(u_0, v_0)$.

Proposition 2.3.5. *There exists a nonnegative ground state solution $(\tilde{u}, \tilde{v}) \in C^{1,\alpha}_{loc}(\mathbb{R}^2) \times C^{1,\alpha}_{loc}(\mathbb{R}^2)$, for some $\alpha \in (0, 1)$ with the following asymptotic behavior*

$$\|\tilde{u}\|_{C^{1,\alpha}(\overline{B_R(x_0)})} \to 0 \quad \text{and} \quad \|\tilde{v}\|_{C^{1,\alpha}(\overline{B_R(x_0)})} \to 0, \quad \text{as } |x_0| \to \infty.$$

Proof. Let $(u_0, v_0) \in E$ be the ground state obtained in Proposition 2.3.4. It follows from Lemma 2.2.8 that there exists a unique $t_0 > 0$ such that $(t_0|u_0|, t_0|v_0|) \in \mathcal{N}$. Moreover, since $(u_0, v_0) \in \mathcal{N}$, we have from Lemma 2.2.6 that $(u_0, v_0) \in E_+$. Thus, it follows that $\max_{t \geq 0} I(tu_0, tv_0) = I(u_0, v_0)$. Thus, we have that

$$I(t_0|u_0|, t_0|v_0|) \leq I(t_0u_0, t_0v_0) \leq \max_{t \geq 0} I(tu_0, tv_0) = I(u_0, v_0) = c_{\mathcal{N}}.$$

Therefore, $(t_0|u_0|, t_0|v_0|) \in \mathcal{N}$ is a nonnegative ground state solution for System (S). Let us denote $(\tilde{u}, \tilde{v}) = (t_0|u_0|, t_0|v_0|)$. In order to get regularity, we use a bootstrap argument. The ground state $(\tilde{u}, \tilde{v})$ is a weak solution of the restricted problem

$$\begin{cases} -\Delta \tilde{u} = f_1(x, \tilde{u}) + \lambda(x)\tilde{v} - V_1(x)\tilde{u} = p_1(x), & B_{2R}(x_0) \\ -\Delta \tilde{v} = f_2(x, \tilde{v}) + \lambda(x)\tilde{u} - V_2(x)\tilde{v} = p_2(x), & B_{2R}(x_0) \end{cases} \tag{2.38}$$

Since $V_1(x), \lambda(x) \in L^\infty_{loc}(\mathbb{R}^2)$ and $\tilde{u} \in L^p(\mathbb{R}^2)$ for all $p \geq 2$, we have that $\lambda(x)\tilde{v}, V_1(x)\tilde{u} \in L^p(B_{2R}(x_0))$ for all $p \geq 2$. By using growth conditions of the nonlinearity, Lemmas 2.2.2 and 2.2.3 we have for $\varepsilon > 0$, $p, q \geq 2$, $r > p$ and $\alpha^1 > \alpha_0^1$ that

$$\begin{aligned} \int_{B_{2R}(x_0)} |f_1(x, \tilde{u})|^p \,\mathrm{d}x &\leq \int_{B_{2R}(x_0)} |A_1(x)\tilde{u} + CA_1(x)(e^{\alpha^1\tilde{u}^2} - 1)|\tilde{u}|^{q-1}|^p \,\mathrm{d}x \\ &\leq C\int_{B_{2R}(x_0)} A_1(x)^p|\tilde{u}|^p + C\int_{B_{2R}(x_0)} A_1(x)^p(e^{\alpha^1\tilde{u}^2} - 1)^p|\tilde{u}|^{p(q-1)} \\ &\leq C\|u_0\|^p_{L^p(B_{2R}(x_0))} + C\int_{B_{2R}(x_0)} A_1(x)(e^{r\alpha^1\tilde{u}^2} - 1)|\tilde{u}|^{p(q-1)-1}|\tilde{u}|. \end{aligned}$$

Furthermore, it follows from Lemma 2.2.1 that

$$\left(\int_{B_{2R}(x_0)} A_1(x)^2(e^{r\alpha^1\tilde{u}^2} - 1)^2|\tilde{u}|^{2(p(q-1)-1)} \,\mathrm{d}x\right)^{1/2} \|u_0\|_{L^2(B_{2R}(x_0))} \leq C\|u_0\|_{L^2(B_{2R}(x_0))}.$$

Using Hölder inequality and combining the previous estimates, we finally conclude that

$$\int_{B_{2R}(x_0)} |f_1(x, \tilde{u})|^p \,\mathrm{d}x \leq C\|u_0\|^p_{L^p(B_{2R}(x_0))} + C\|u_0\|_{L^2(B_{2R}(x_0))},$$

and since that the right-hand side is finite for all $p, q \geq 2$, we have that $p_1(x) \in L^p(B_{2R}(x_0))$ for all $p \geq 2$. Let $f_{p_1}(x)$ be the Newtonian potential of $p_1(x)$. In light of Calderon-Zygmund [42, Theorem 9.9],

$$\Delta f_{p_1} = p_1(x), \quad \text{in } B_{2R}(x_0), \tag{2.39}$$

and $f_{p_1} \in W^{2,p}(B_{2R}(x_0))$, for all $p \geq 2$. Combining (2.38) and (2.39) we deduce that

$$\int_{B_{2R}(x_0)} \nabla(\tilde{u} - f_{p_1})\varphi \,\mathrm{d}x = 0, \quad \text{for all } \varphi \in C_0^\infty(B_{2R}(x_0)),$$

which implies that $\tilde{u} - f_{p_1}$ is a weak solution of $-\Delta z = 0$ in $B_{2R}(x_0)$. Since $\tilde{u} - f_{p_1} \in W^{1,2}(B_{2R}(x_0))$, it follows from Weyl's Lemma [46, Corollary 1.2.1] that $\tilde{u} - f_{p_1} \in C^\infty(B_{2R}(x_0))$. Therefore, $\tilde{u} \in W^{2,p}(B_{2R}(x_0))$ for all $p \geq 2$. Notice that $2/p < 2$, for all $p > 2$. Thus, by Sobolev imbedding [39, Theorem 6] we conclude that $\tilde{u} \in C^{1,\alpha}(B_{2R}(x_0))$, for some $\alpha \in (0,1)$. The same argument can be used to prove that $\tilde{v} \in C^{1,\alpha}(B_{2R}(x_0))$, for some $\alpha \in (0,1)$. By interior L^p-estimates [42, Theorem 9.11], we have that

$$\|\tilde{u}\|_{W^{2,p}(B_R(x_0))} \leq C(\|\tilde{u}\|_{L^p(B_{2R}(x_0))} + \|p_1\|_{L^p(B_{2R}(x_0))}),$$

By using Sobolev Imbedding and the previous computations, we deduce that

$$\|\tilde{u}\|_{C^{1,\alpha}(\overline{B_R(x_0)})} \leq C(\|\tilde{u}\|_{L^p(B_{2R}(x_0))} + \|\tilde{u}\|_{L^2(B_{2R}(x_0))}).$$

Letting $|x_0| \to \infty$, we get $\|\tilde{u}\|_{C^{1,\alpha}(\overline{B_R(x_0)})} \to 0$. The same idea can be applied to $\tilde{v}$. ■

Proposition 2.3.6. *If (V_5') holds then the ground state is positive.*

Proof. The idea is similar to the proof of Lemma 1.3.4 and for convenience we give a short version here. Let $(\tilde{u}, \tilde{v}) \in E \backslash \{(0,0)\}$ be the nonnegative ground state obtained in the preceding proposition. Using (V_5'), we can conclude that $\tilde{u} \neq 0$ and $\tilde{v} \neq 0$. Taking $(\varphi, 0)$ as test function one sees that

$$\int_{\mathbb{R}^2} \nabla\tilde{u}\nabla\varphi \, \mathrm{d}x + \int_{\mathbb{R}^2} V_1(x)\tilde{u}\varphi \, \mathrm{d}x = \int_{\mathbb{R}^2} f_1(x,\tilde{u})\varphi \, \mathrm{d}x + \int_{\mathbb{R}^2} \lambda(x)\tilde{v}\varphi \, \mathrm{d}x \geq 0,$$

for all $\varphi \geq 0$, $\varphi \in C_0^\infty(\mathbb{R}^2)$. We suppose by contradiction that there exists $p \in \mathbb{R}^2$ such that $\tilde{u}(p) = 0$. Thus, since $-\tilde{u} \leq 0$ in $\mathbb{R}^2$, for any $R > R_0 > 0$ we have that

$$0 = \sup_{B_{R_0}(p)} (-\tilde{u}) = \sup_{B_R(p)} (-\tilde{u}).$$

By the Strong Maximum Principle [42, Theorem 8.19] we conclude that $-\tilde{u} \equiv 0$ in $B_R(p)$, for all $R > R_0$. Therefore $-\tilde{u} \equiv 0$ in $\mathbb{R}^2$, which is a contradiction. Therefore $\tilde{u} > 0$ in $\mathbb{R}^2$. Analogously, we can prove that $\tilde{v} > 0$ in $\mathbb{R}^2$. Therefore, the ground state $(\tilde{u}, \tilde{v})$ is positive. ■

Proof of Theorem 2.1.2. It follows from Propositions 2.3.2, 2.3.3, 2.3.4, 2.3.5 and 2.3.6. ■

Remark 2.3.7. *We stress that Theorem 2.1.2 holds for some $\theta > \theta_0$ sufficiently large, see Remark 2.1.3 (iii). Notice that by estimate (2.23) the norm of the minimizing sequence is so small as we want, and it is controlled by the choice of θ_0. However, in the lemma 2.2.7, we proved that the norm of any element that belongs to Nehari manifold is greater or equal to a positive constant γ, which is strictly less than $4\pi(\beta_i - 1)/\alpha^i\beta_i$, for $i = 1, 2$. Thus, our proof holds for any θ contained in a bounded interval of the real line. Let us consider, for instance,*

$$\vartheta^* := \sup\{\vartheta \in \mathbb{R} : (S) \text{ has ground states}\}.$$

Naturally, it arises the following questions: ϑ^* is finite? If ϑ^* is finite, then there exists ground states at $\vartheta = \vartheta^*$?

Chapter 3

On coupled systems of nonlinear equations with critical exponential growth

3.1 Introduction

This chapter is devoted to study the following class of coupled systems involving nonlinear Schrödinger equations

$$\begin{cases} -\Delta u + u = f_1(u) + \lambda(x)v, & x \in \mathbb{R}^2, \\ -\Delta v + v = f_2(v) + \lambda(x)u, & x \in \mathbb{R}^2. \end{cases} \tag{S}$$

Our main contribution in this work is to prove the existence of positive ground state solutions for (S) when the nonlinearities $f_1(s)$, $f_2(s)$ have critical exponential growth motivated by a class of Trudinger-Moser inequalities introduced by D.M. Cao [14] (see Theorem A in Section 3.2).

3.1.1 Assumptions.

For $i = 1, 2$ we assume the following assumptions on f_i:

(H_1) The function f_i belongs to $C^1(\mathbb{R})$, $f_i(s) = 0$ for all $s \leq 0$ and

$$\lim_{s \to 0} \frac{f_i(s)}{s} = 0.$$

(H_2) The function $s \mapsto s^{-1} f_i(s)$ is increasing for $s > 0$.

(H_3) There exists $\mu_i > 2$ such that

$$0 < \mu_i F_i(s) := \mu_i \int_0^s f_i(\tau)\, \mathrm{d}\tau \leq f_i(s)s, \quad \text{for all } s > 0.$$

(H_4) There exists $M > 0$ such that

$$0 < F_i(s) \leq M f_i(s), \quad \text{for all } s > 0.$$

(H_λ) There exists $\delta > 0$ such that $0 < \lambda(x) \leq \delta < 1$ for all $x \in \mathbb{R}^2$. Moreover, $\lambda(x)$ is 1-periodic, that is, $\lambda(x) = \lambda(x+z)$ for all $x \in \mathbb{R}^2$ and $z \in \mathbb{Z}^2$.

(CG) The function $f_i : \mathbb{R} \to \mathbb{R}$ has α_0^i-*critical growth* at $+\infty$, that is, there exists $\alpha_0^i > 0$ such that

$$\limsup_{s \to +\infty} \frac{f_i(s)}{e^{\alpha s^2} - 1} = \begin{cases} 0 & \text{if } \alpha > \alpha_0^i, \\ \infty & \text{if } \alpha < \alpha_0^i. \end{cases}$$

Let us consider $E = H^1(\mathbb{R}^2) \times H^1(\mathbb{R}^2)$ endowed with the natural scalar product

$$((u,v),(w,z)) = \int_{\mathbb{R}^2} (\nabla u \nabla w + uw + \nabla v \nabla z + vz)\, \mathrm{d}x,$$

and the induced norm $\|(u,v)\|^2 = ((u,v),(u,v))$. Associated to System (S) we have the energy functional $I : E \to \mathbb{R}$ defined by

$$I(u,v) = \frac{1}{2}\left(\|(u,v)\|^2 - 2\int_{\mathbb{R}^2} \lambda(x) uv\, \mathrm{d}x \right) - \int_{\mathbb{R}^2} (F_1(u) + F_2(v))\, \mathrm{d}x.$$

By using the assumptions on $f_i(s)$ and $\lambda(x)$ and Trudinger-Moser inequality, we can easily see that I is well defined. Moreover, it's standard to check that I is $C^2(E,\mathbb{R})$ and

$$\langle I'(u,v),(\phi,\psi)\rangle = ((u,v),(\phi,\psi)) - \int_{\mathbb{R}^2} (f_1(u)\phi + f_2(v)\psi)\, \mathrm{d}x - \int_{\mathbb{R}^2} \lambda(x)\,(u\psi + v\phi)\, \mathrm{d}x.$$

The critical points of I are precisely the solutions (in the weak sense) of System (S).

Definition 3.1.1. *We say that a pair $(u,v) \in E \setminus \{(0,0)\}$ is a ground state solution (least energy solution) of (S), if (u,v) is a solution of (S) and its energy is minimal among the energy of all nontrivial solutions of (S), i.e., $I(u,v) \leq I(w,z)$ for any other nontrivial solution $(w,z) \in E$. We say that (u,v) is nonnegative (nonpositive) if $u,v \geq 0$ $(u,v \leq 0)$ and positive (negative) if $u,v > 0$ $(u,v < 0)$.*

3.1.2 Statement of the main result.

We are now in position to formulate our main result and we also give some remarks on our assumptions.

Theorem 3.1.2. *Suppose that* (H_λ) *holds and assume that for each* $i = 1, 2$ $f_i(s)$, $f_i'(s)$ *have* α_0^i*-critical growth at* $+\infty$ (CG) *and satisfies* (H_1)-(H_4). *In addition, we consider the following assumption:*

$$(H_5) \qquad \liminf_{s\to+\infty} \frac{sf_i(s)}{e^{\alpha_0^i s^2}} \geq \beta_0 > \frac{2e}{\alpha_0},$$

where $\alpha_0 = \max\{\alpha_0^1, \alpha_0^2\}$. *Then System* (S) *possesses a positive ground state solution* $(u_0, v_0) \in C^{1,\alpha}_{loc}(\mathbb{R}^2) \times C^{1,\alpha}_{loc}(\mathbb{R}^2)$, *for some* $0 < \alpha < 1$ *with the following asymptotic behavior*

$$\|u_0\|_{C^{1,\alpha}(\overline{B_R(x_0)})} \to 0 \quad \text{and} \quad \|v_0\|_{C^{1,\alpha}(\overline{B_R(x_0)})} \to 0, \quad \text{as } |x_0| \to \infty.$$

Furthermore, the set $\mathcal{K}$ *of all ground state solutions of System* (S) *is a compact subset of* E.

Remark 3.1.3. *A typical example of nonlinear term satisfying conditions* (H_1)-(H_4) *and* (CG) *is given by* $f(s) = e^{\alpha_0 s^2}(qs^{q-1} + 2\alpha_0 s^{q+1})$ *if* $s \geq 0$ *and* $f(s) = 0$ *if* $s < 0$, *where* α_0 *is the critical constant introduced in* (CG).

Remark 3.1.4. *A typical example of coupling term satisfying* (H_λ) *is given by* $\lambda(x) = \lambda \in (0, \delta)$, *for all* $x \in \mathbb{R}^2$, *for some* $\delta < 1$. *The assumption* (H_λ) *will be crucial through the paper. It will be used to guarantee that the Nehari manifold is bounded away from* $(0, 0)$ *(see Lemma 3.2.3).*

Remark 3.1.5. *As we comment in the introduction, assumption* (H_5) *was introduced in [1] and refined in [26]. It plays a very important role in the proof of Theorem 3.1.2, because it will ensure that the ground state energy associated to System* (S) *is strictly less than* $2\pi/\alpha_0$ *(Proposition 3.2.5). This fact will allow the use of Theorem A in the minimizing sequence obtained by Ekeland's variational principle (see* (3.15) *and* (3.23)*).*

3.1.3 Outline

The remainder of this chapter is organized as follows: In the forthcoming Section we collect some results which are crucial to study our problem by a variational approach. Moreover, we introduce and give some properties of the Nehari manifold. In Section 3.3, we prove Theorem 3.1.2. We make use of the Ekeland's variational principle to get a minimizing sequence for the energy functional associated to the problem. We will use the invariance of the energy functional by translation to recover the compactness of the minimizing sequence and a Trudinger-Moser type inequality to prove that the weak limit of that sequence will be a ground state solution of the problem. After that, we obtain a nonnegative ground state solution. Finally, we get regularity and asymptotic behavior of the ground state using a bootstrap argument and L^q-estimates. The positivity will be a consequence of the strong maximum principle.

3.2 Preliminary results

In this section, we provide preliminary results which will be used throughout the chapter. The notion of *criticality* used in this work is motivated by the following result which was first considered by D.M. Cao [14] (see also [32]).

Theorem A. *If $\alpha > 0$ and $u \in H^1(\mathbb{R}^2)$, then*

$$\int_{\mathbb{R}^2} (e^{\alpha u^2} - 1)\, \mathrm{d}x < \infty.$$

Moreover, if $0 < \alpha < 4\pi$, $\|\nabla u\|_2 \leq 1$, $\|u\|_2 \leq \tilde{C}$, then there exists a constant $C = C(\alpha, \tilde{C}) > 0$, depending only on α and $\tilde{C}$, such that

$$\int_{\mathbb{R}^2} (e^{\alpha u^2} - 1)\, \mathrm{d}x \leq C.$$

Lemma 3.2.1. *Let assumptions (H_1)-(H_3) hold. Then*

$$f_i'(s)s^2 - f_i(s)s > 0, \tag{3.1}$$

$$f_i'(s) > 0, \tag{3.2}$$

$$\mathcal{H}_i(s) = f_i(s)s - 2F_i(s) > 0, \tag{3.3}$$

for $i = 1, 2$ and for all $s > 0$.

Proof. The proof is quite similar to Lemma 2.2.4 and will be omitted here. ∎

We introduce the Nehari manifold associated to (S) define by

$$\mathcal{N} = \{(u, v) \in E \backslash \{(0, 0)\} : \langle I'(u, v), (u, v) \rangle = 0\}.$$

Notice that if $(u, v) \in \mathcal{N}$ then

$$\|(u, v)\|^2 - 2\int_{\mathbb{R}^2} \lambda(x) uv\, \mathrm{d}x = \int_{\mathbb{R}^2} f_1(u)u\, \mathrm{d}x + \int_{\mathbb{R}^2} f_2(v)v\, \mathrm{d}x. \tag{3.4}$$

Remark 3.2.2. *We can prove analogously Lemma 2.2.6 to the Nehari manifold defined under assumptions of this chapter.*

Lemma 3.2.3. *Suppose that (CG), (H_1) and (H_2) hold. Then*

(a) $\mathcal{N}$ *is a* C^1*-manifold.*

(b) There exists $\rho > 0$*, such that*

$$\|(u, v)\| \geq \rho, \quad \text{for all } (u, v) \in \mathcal{N}. \tag{3.5}$$

(c) $(u, v) \in E\backslash\{(0, 0)\}$ *is a critical point of* I *if and only if* (u, v) *is a critical point of* $I\,|_{\mathcal{N}}$.

Proof. Let $\varphi : E\backslash\{(0, 0)\} \to \mathbb{R}$ be the C^1-functional defined by

$$\varphi(u, v) = \langle I'(u, v), (u, v)\rangle = \|(u, v)\|^2 - 2\int_{\mathbb{R}^2} \lambda(x)uv \, dx - \int_{\mathbb{R}^2} f_1(u)u \, dx - \int_{\mathbb{R}^2} f_2(v)v \, dx.$$

Notice that $\mathcal{N} = \varphi^{-1}(0)$. If $(u, v) \in \mathcal{N}$, it follows from (3.1) and (3.4) that

$$\langle \varphi'(u, v), (u, v)\rangle = \int_{\mathbb{R}^2} \left(f_1(u)u - f_1'(u)u^2\right) \, dx + \int_{\mathbb{R}^2} \left(f_2(v)v - f_2'(v)v^2\right) \, dx < 0.$$

Therefore, 0 is a regular value of φ which implies that $\mathcal{N}$ is a C^1-manifold.

Arguing by contradiction, we suppose that (3.5) does not hold. Thus, we have a sequence

$$(u_n, v_n)_n \subset \mathcal{N}, \text{ such that } \|(u_n, v_n)\| \to 0 \text{ as } n \to +\infty. \tag{3.6}$$

Consider $\alpha > \alpha_0$ and $\rho_0 > 0$ such that $\alpha\rho_0^2 < 4\pi$. As consequence of (3.6), there exists $n_0 \in \mathbb{N}$ such that $\|(u_n, v_n)\| \leq \rho_1 < \rho_0$, for $n \geq n_0$. By using the growth conditions (H_1) and (CG), for any $\varepsilon > 0$ and $p > 2$, there exists a constant $C = C(\varepsilon, p) > 0$ such that

$$f_i(s) \leq \varepsilon|s| + C(\varepsilon, p)(e^{\alpha s^2} - 1)|s|^p, \quad \text{for all } s \in \mathbb{R} \text{ and } i = 1, 2. \tag{3.7}$$

We recall from Lemma 2.2.2 that for $\alpha > 0$, $l \geq 1$ and $r \geq l$ we have

$$(e^{\alpha s^2} - 1)^l \leq (e^{\alpha l s^2} - 1), \quad \text{for all } s \in \mathbb{R}. \tag{3.8}$$

Let us consider $l > 1$ close enough to 1 such that $l\alpha\rho_0^2 < 4\pi$. Thus, it follows from Theorem A, (3.8) and Hölder inequality that

$$\int_{\mathbb{R}^2} (e^{\alpha u_n^2} - 1)|u_n|^p \, dx \leq \left(\int_{\mathbb{R}^2} (e^{l\alpha\|u_n\|^2\left(\frac{u_n}{\|u_n\|}\right)^2} - 1) \, dx\right)^{1/l} \|u_n\|_{pl'}^p \leq C\|u_n\|_{pl'}^p. \tag{3.9}$$

Combining (3.7), (3.9) and using Sobolev embedding one sees that

$$\int_{\mathbb{R}^2} f_1(u_n)u_n \, dx \leq \varepsilon C_1\|(u_n, v_n)\|^2 + C_2\|(u_n, v_n)\|^p, \quad \text{for } n \geq n_0.$$

Analogously, we deduce that

$$\int_{\mathbb{R}^2} f_2(v_n)v_n \, \mathrm{d}x \le \varepsilon C_3\|(u_n,v_n)\|^2 + C_4\|(u_n,v_n)\|^p, \quad \text{for } n \ge n_0.$$

Combining theses estimates we get

$$\int_{\mathbb{R}^2} (f_1(u_n)u_n + f_2(v_n)v_n) \, \mathrm{d}x \le \varepsilon \tilde{C}_1\|(u_n,v_n)\|^2 + \tilde{C}_2\|(u_n,v_n)\|^p. \tag{3.10}$$

Since $\varepsilon > 0$ is arbitrary and $\tilde{C}_1$ does not depend on ε and n, we can choose ε sufficiently small such that $1-\delta-\varepsilon\tilde{C}_1 > 0$. We recall from Lemma 1.2.1 that

$$\|(u,v)\|^2 - 2\int_{\mathbb{R}^2} \lambda(x)uv \, \mathrm{d}x \ge (1-\delta)\|(u,v)\|^2, \quad \text{for all } (u,v) \in E. \tag{3.11}$$

Thus, it follows from (3.10) and (3.11) that

$$(1-\delta)\|(u_n,v_n)\|^2 \le \int_{\mathbb{R}^2} (f_1(u_n)u_n + f_2(v_n)v_n) \, \mathrm{d}x \le \varepsilon\tilde{C}_1\|(u_n,v_n)\|^2 + \tilde{C}_2\|(u_n,v_n)\|^p,$$

which yields

$$0 < (1-\delta-\varepsilon\tilde{C}_1)\|(u_n,v_n)\|^2 \le \tilde{C}_2\|(u_n,v_n)\|^p.$$

Therefore, denoting $\rho_2 = (1-\delta-\varepsilon\tilde{C}_1)/\tilde{C}_2$ we obtain

$$0 < \rho_2^{1/(p-2)} \le \|(u_n,v_n)\|,$$

If we choose $\rho_1 < \rho = \min\{\rho_0, \rho_2^{1/(p-2)}\}$ we get a contradiction. Therefore, $\|(u,v)\|^2 \ge \rho$ for all $(u,v) \in \mathcal{N}$.

Finally, if $(u,v) \ne (0,0)$ is a critical point of I, we have $I'(u,v) = 0$ and obviously $(u,v) \in \mathcal{N}$. Conversely, if (u,v) is a critical point of I on $\mathcal{N}$, we have that $\lambda\varphi'(u,v) = I'(u,v)$, where $\lambda \in \mathbb{R}$ is the Lagrange multiplier. Taking the scalar product with (u,v) and recalling the previous results we conclude that $\lambda = 0$, and the lemma is proved. ■

Let us define the set

$$E_+ := \{(u,v) \in E\backslash\{(0,0)\} : |\{u>0\}| > 0 \text{ or } |\{v>0\}| > 0\}.$$

Lemma 3.2.4. *Suppose that* (H_1)-(H_3) *and* (H_λ) *hold. For any* $(u,v) \in E_+$*, there exists a unique* $t_0 > 0$*, depending of* (u,v)*, such that*

$$(t_0u, t_0v) \in \mathcal{N} \quad \text{and} \quad I(t_0u,t_0v) = \max_{t\ge0} I(tu,tv).$$

Proof. Let $(u,v) \in E_+$ be fixed and define the function $h : [0,\infty) \to \mathbb{R}$ such that $h(t) = I(tu,tv)$. Notice that $\langle I'(tu,tv),(tu,tv)\rangle = th'(t)$. Thus, it suffices to find a nontrivial positive critical point of h. After integrating (H_3), we get

$$F_i(s) \geq C_0(|s|^{\mu_i} - 1), \quad \text{for all } s > 0.$$

We may assume without loss of generality that $|\{u > 0\}| > 0$. Let $R > 0$ be such that $|\{u > 0\} \cap B_R(0)| > 0$. Thus, we have that

$$h(t) \leq \frac{t^2}{2}\left(\|(u,v)\|^2 - 2\int_{\mathbb{R}^2} \lambda(x)uv \, dx\right) - C_0 \int_{|\{u>0\}\cap B_R(0)|} t^{\mu_1}|u|^{\mu_1} \, dx - \tilde{C}|B_R(0)|.$$

Since $\mu_1 > 2$, we conclude that $h(t) < 0$ for $t > 0$ sufficiently large. On the other hand, by using growth conditions we have that for any $\varepsilon > 0$ and $p > 2$, there exists $C = C(\varepsilon,p) > 0$ such that

$$F_i(s) \leq \varepsilon|s| + C(e^{\alpha s^2} - 1)|s|^{p-1}, \quad \text{for all } s \in \mathbb{R}.$$

By similar arguments used to get (3.10) we can deduce that

$$\int_{\mathbb{R}^2} (F_1(tu) + F_2(tv)) \, dx \leq C_1\varepsilon\frac{t^2}{2}\|(u,v)\|^2 + C_2 t^p\|(u,v)\|^p.$$

Thus, we have

$$\begin{aligned} g(t) &= \frac{t^2}{2}\left(\|(u,v)\|^2 - 2\int_{\mathbb{R}^2}\lambda(x)uv \, dx\right) - \int_{\mathbb{R}^2} F_1(tu)\, dx - \int_{\mathbb{R}^2} F_2(tv)\, dx \\ &\geq t^2\|(u,v)\|^2\left(\frac{1-\delta-\varepsilon C}{2} - Ct^{p-2}\|(u,v)\|^{p-2}\right). \end{aligned}$$

Choosing $\varepsilon > 0$ small enough such that $1 - \delta - \varepsilon C > 0$, we conclude that $h(t) > 0$ for $t > 0$ sufficiently small. Therefore h has maximum points in the interval $(0,+\infty)$. Finally, notice that every critical point of h satisfies

$$\|(u,v)\| - 2\int_{\mathbb{R}^2} \lambda(x)uv \, dx = \int_{\mathbb{R}^2} \frac{f_1(tu)u}{t} \, dx + \int_{\mathbb{R}^2} \frac{f_2(tv)v}{t} \, dx. \tag{3.12}$$

It is easy to see that (3.1) implies that the right-hand side of (3.12) is strictly increasing on $t > 0$. Thus, the critical point $t_0 \in (0,+\infty)$ is *unique*. ■

We define the ground state energy associated with System (S) by

$$c_{\mathcal{N}} = \inf_{(u,v)\in\mathcal{N}} I(u,v).$$

The next Proposition plays a very important role and will be proved in Section 3.4.

Proposition 3.2.5. *The energy level $c_{\mathcal{N}}$ satisfies*

$$0 < c_{\mathcal{N}} < \frac{2\pi}{\alpha_0} \tag{3.13}$$

Remark 3.2.6. *We can use Lemma 3.2.4 to get the following minimax characterization:*

$$c_{\mathcal{N}} = \inf_{(u,v)\in\mathcal{N}} I(u,v) \leq \inf_{(u,v)\in E_+} \max_{t\geq 0} I(tu,tv). \tag{3.14}$$

3.3 Proof of Theorem 3.1.2

By Ekeland's variational principle (see [38]), there exists a minimizing sequence $(u_n, v_n)_n \subset \mathcal{N}$ such that

$$I(u_n, v_n) \to c_{\mathcal{N}} \quad \text{and} \quad I'(u_n, v_n) \to 0. \tag{3.15}$$

Proposition 3.3.1. *The minimizing sequence $(u_n, v_n)_n$ is bounded in E and*

$$\int_{\mathbb{R}^2} f_1(u_n)u_n \, \mathrm{d}x \le C, \quad \textit{and} \quad \int_{\mathbb{R}^2} F_1(u_n) \, \mathrm{d}x \le C,$$
$$\int_{\mathbb{R}^2} f_2(v_n)v_n \, \mathrm{d}x \le C, \quad \textit{and} \quad \int_{\mathbb{R}^2} F_2(v_n) \, \mathrm{d}x \le C.$$

Proof. We obtain from (3.15) that

$$c_{\mathcal{N}}+o_n(1) = I(u_n, v_n) = \frac{1}{2}\left(\|(u_n, v_n)\|^2 - 2\int_{\mathbb{R}^2} \lambda(x)u_n v_n \, \mathrm{d}x\right) - \int_{\mathbb{R}^2} (F_1(u_n)+F_2(v_n)) \, \mathrm{d}x.$$

Thus, by using (H_3), (3.11) and the fact that $(u_n, v_n)_n \subset \mathcal{N}$, we deduce that

$$c_{\mathcal{N}} + o_n(1) \ge \left(\frac{1}{2} - \frac{1}{\mu}\right)(1-\delta)\|(u_n, v_n)\|^2, \tag{3.16}$$

where $\mu = \min\{\mu_1, \mu_2\}$. Therefore, $(u_n, v_n)_n$ is bounded in E. It follows from (3.15) that

$$\left|\|(u_n, v_n)\|^2 - 2\int_{\mathbb{R}^2} \lambda(x)u_n v_n \, \mathrm{d}x - \int_{\mathbb{R}^2} f_1(u_n)u_n \, \mathrm{d}x - \int_{\mathbb{R}^2} f_2(v_n)v_n \, \mathrm{d}x\right| \le o_n(1)\|(u_n, v_n)\|.$$

Combining these estimates together with (H_3) we get

$$\begin{aligned}
\int_{\mathbb{R}^2} (f_1(u_n)u_n + f_2(v_n)v_n) &\le 2c_{\mathcal{N}} + 2o_n(1) + o_n(1)\|(u_n, v_n)\| + 2\int_{\mathbb{R}^2} (F_1(u_n) + F_2(v_n)) \\
&\le 2c_{\mathcal{N}} + 2o_n(1) + o_n(1)\|(u_n, v_n)\| + \frac{2}{\mu}\int_{\mathbb{R}^2} (f_1(u_n)u_n + f_2(v_n)v_n),
\end{aligned}$$

which implies that

$$\int_{\mathbb{R}^2} (f_1(u_n)u_n + f_2(v_n)v_n) \, \mathrm{d}x \le \frac{\mu}{\mu-2}(2c_{\mathcal{N}} + 2o_n(1) + \varepsilon_n\|(u_n, v_n)\|). \tag{3.17}$$

Since $\|(u_n, v_n)\| \le C$ for some $C > 0$, using (H_3) and (3.17) we conclude the proof. ■

By the preceding proposition, we may assume, up to a subsequence, that

- $(u_n, v_n) \rightharpoonup (u_0, v_0)$ weakly in E;
- $u_n \to u_0$ and $v_n \to v_0$ strongly in $L^p_{loc}(\mathbb{R}^2)$, for all $2 \le p < \infty$;

- $u(x) \to u_0(x)$ and $v(x) \to v_0(x)$, almost everywhere in $\mathbb{R}^2$.

Proposition 3.3.2. *Let $(u_n, v_n)_n \subset \mathcal{N}$ be a minimizing sequence satisfying (3.15). Then $(u_n, v_n)_n$ satisfies exactly one of the following conditions:*

(i) $(u_n, v_n) \to (0,0)$ strongly in E;

(ii) There exists a sequence $(y_n)_n \subset \mathbb{R}^2$ and constants $R, \xi > 0$ such that

$$\liminf_{n\to\infty} \int_{B_R(y_n)} (u_n^2 + v_n^2)\, \mathrm{d}x \geq \xi > 0.$$

Proof. Suppose that (ii) does not hold. Thus, for any $R > 0$ we have

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^2} \int_{B_R(y)} (u_n^2 + v_n^2)\, \mathrm{d}x = 0. \tag{3.18}$$

Assertion 3.3.3. *If (3.18) holds, then*

$$\lim_{n\to\infty} \int_{\mathbb{R}^2} \lambda(x) u_n v_n\, \mathrm{d}x = \lim_{n\to\infty} \int_{\mathbb{R}^2} F_1(u_n)\, \mathrm{d}x = \lim_{n\to\infty} \int_{\mathbb{R}^2} F_2(v_n)\, \mathrm{d}x = 0. \tag{3.19}$$

In fact, if (3.18) holds it follows from Lemma 1.3.1 that $u_n \to 0$ and $v_n \to 0$ in $L^p(\mathbb{R}^2)$ for any $p > 2$. Thus, up to a subsequence $u_n(x) \to 0$ and $v_n(x) \to 0$ almost everywhere in $\mathbb{R}^2$. By using assumption (H_λ), Sobolev imbedding, Hölder inequality and the fact that the minimizing sequence is bounded, we have for $p > 2$ that

$$\left| \int_{\mathbb{R}^2} \lambda(x) u_n v_n\, \mathrm{d}x \right| \leq \delta \|u_n\|_p \|v_n\|_{p'} \leq C \|v_n\| \|u_n\|_p \leq \tilde{C} \|u_n\|_p \to 0.$$

By a similar argument used to get (2.33) (see also [26, Lemma 2.1]), we can deduce that

$$\int_{B_R(0)} f_1(u_n)\, \mathrm{d}x \to 0 \quad \text{and} \quad \int_{B_R(0)} f_2(v_n)\, \mathrm{d}x \to 0, \quad \text{as } n \to \infty,$$

for any $R > 0$. Therefore, by using assumption (H_4) and generalized Lebesgue dominated convergence theorem we conclude that

$$\int_{B_R(0)} F_1(u_n)\, \mathrm{d}x \to 0 \quad \text{and} \quad \int_{B_R(0)} F_2(v_n)\, \mathrm{d}x \to 0, \quad \text{as } n \to \infty. \tag{3.20}$$

Thus, it remains to prove that for given $\varepsilon > 0$ there exists $R = R(\varepsilon) > 0$ such that

$$\int_{\mathbb{R}^2\setminus B_R(0)} F_1(u_n)\, \mathrm{d}x \leq \varepsilon \quad \text{and} \quad \int_{\mathbb{R}^2\setminus B_R(0)} F_2(v_n)\, \mathrm{d}x \leq \varepsilon.$$

Let $B > 0$ be such that $2MCB^{-1} < \varepsilon$ and let us define the set

$$\Omega_B = \{x \in \mathbb{R}^2 \setminus B_R(0) : |u_n(x)| \geq B\}.$$

It follows from assumption (H_4) and Proposition 3.3.1 that

$$\int_{\Omega_B} F_1(u_n)\,\mathrm{d}x \le M \int_{\Omega_B} f_1(u_n)\,\mathrm{d}x \le \frac{M}{B}\int_{\Omega_B} f_1(u_n)u_n\,\mathrm{d}x \le \frac{MC}{B} \le \frac{\varepsilon}{2}. \tag{3.21}$$

By assumption (H_1), for any $\tilde{\varepsilon} > 0$, there exists $C = C(\tilde{\varepsilon}, B) > 0$ such that

$$F_1(s) \le \tilde{\varepsilon} s^2 + C s^4, \quad \text{for } |s| \le B.$$

Denoting $\Omega_B^c = \{x \in \mathbb{R}^2 \backslash B_R(0) : |u_n(x)| \le B\}$ and using the fact that $u_n \to 0$ in $L^4(\mathbb{R}^2)$, we have that

$$\limsup_{n\to\infty} \int_{\Omega_B^c} F_1(u_n)\,\mathrm{d}x \le \tilde{\varepsilon} \sup_n \|u_n\|_2^2 \le \tilde{\varepsilon} \sup_n \|(u_n, v_n)\|^2 \le \frac{\varepsilon}{2}, \tag{3.22}$$

since $(u_n, v_n)_n$ is bounded in E and $\tilde{\varepsilon}$ is arbitrary. Combining (3.20), (3.21) and (3.22) we get the second limit in (3.19). The same idea can be used to get the third limit in (3.19), and Assertion 3.3.3 is proved.

By using (3.15) and Assertion 3.3.3 we deduce that

$$c_{\mathcal{N}} + o_n(1) = \frac{1}{2}\|(u_n, v_n)\|^2 - \int_{\mathbb{R}^2} \left(F_1(u_n) + F_2(v_n) + \lambda(x) u_n v_n\right)\,\mathrm{d}x = \frac{1}{2}\|(u_n, v_n)\|^2 + o_n(1),$$

which together with Proposition 3.2.5 implies that

$$\limsup_{n\to\infty} \|(u_n, v_n)\|^2 = 2c_{\mathcal{N}} < \frac{4\pi}{\alpha_0}. \tag{3.23}$$

It follows from (3.23) that we can consider $\alpha > \alpha_0$ and $r > l > 1$ sufficiently close to 1 such that $r\alpha\|(u_n, v_n)\|^2 < 4\pi$. Thus, by using (3.7) and (3.9) we have that

$$\int_{\mathbb{R}^2} (f_1(u_n)u_n + f_2(v_n)v_n)\,\mathrm{d}x \le \varepsilon C_1 \|(u_n, v_n)\|^2 + C_2 \|(u_n, v_n)\|_{ql'}^q, \tag{3.24}$$

for $q > 2$. By choosing $\varepsilon > 0$ such that $1 - \varepsilon C_1 > 0$, it follows from (3.15), (3.19) and (3.24) that

$$(1 - \varepsilon C_1)\|(u_n, v_n)\|^2 \le \varepsilon C_2 \|(u_n, v_n)\|_{ql'}^q + o_n(1),$$

which jointly with Lemma 1.3.1 implies that $\|(u_n, v_n)\| \to 0$ and the lemma follows. ■

Proposition 3.3.4. *The weak limit (u_0, v_0) is a critical point of I.*

Proof. For any $(\phi, \psi) \in C_0^\infty(\mathbb{R}^2) \times C_0^\infty(\mathbb{R}^2)$ we have by the weak convergence that

$$((u_n, v_n), (\phi, \psi)) \to ((u_0, v_0), (\phi, \psi)).$$

Moreover, also by weak convergence we have the following convergences

$$\int_{\mathbb{R}^2} \lambda(x)\psi u_n\,\mathrm{d}x \to \int_{\mathbb{R}^2} \lambda(x)\psi u_0\,\mathrm{d}x, \quad \text{and} \quad \int_{\mathbb{R}^2} \lambda(x)\phi v_n\,\mathrm{d}x \to \int_{\mathbb{R}^2} \lambda(x)\phi v_0\,\mathrm{d}x.$$

Thus, if we get the convergences

$$\int_{\mathbb{R}^2} f_1(u_n)\phi \, dx \to \int_{\mathbb{R}^2} f_1(u_0)\phi \, dx, \quad \text{and} \quad \int_{\mathbb{R}^2} f_2(v_n)\psi \, dx \to \int_{\mathbb{R}^2} f_2(v_0)\psi \, dx, \tag{3.25}$$

we conclude that

$$\langle I'(u_n, v_n), (\phi, \psi)\rangle \to \langle I'(u_0, v_0), (\phi, \psi)\rangle,$$

for any $(\phi, \psi) \in C_0^\infty(\mathbb{R}^2) \times C_0^\infty(\mathbb{R}^2)$, which together with (3.15) and the fact that $C_0^\infty(\mathbb{R}^2)$ is dense in $H^1(\mathbb{R}^2)$, implies that (u_0, v_0) is a critical point of I. In order to prove (3.25), notice that the convergences holds if and only if

$$\int_K |f_1(u_n)| \, dx \to \int_K |f_1(u_0)| \, dx, \quad \text{and} \quad \int_K |f_2(v_n)| \, dx \to \int_K |f_2(v_0)| \, dx, \tag{3.26}$$

for any compact set $K \subset \mathbb{R}^2$. It follows by Theorem A that $f_1(u_n), f_2(v_n) \in L^1(\mathbb{R}^2)$, for any $n \in \mathbb{N}$. Thus, by a quiet similar argument to used to obtain (2.33), we get (3.26). ■

Let us now complete the proof of the existence of ground state for System (S). We split the argument into two cases.

Case 1 $u_0 \not\equiv 0$ and $v_0 \not\equiv 0$.

In this case (u_0, v_0) is a nontrivial critical point of I, thus $(u_0, v_0) \in \mathcal{N}$. We only need to prove that $I(u_0, v_0) = c_{\mathcal{N}}$. Since $(u_0, v_0) \in \mathcal{N}$ we have $c_{\mathcal{N}} \leq I(u_0, v_0)$. On the other hand,

$$c_{\mathcal{N}} = I(u_n, v_n) - \frac{1}{2}\langle I'(u_n, v_n), (u_n, v_n)\rangle + o_n(1) = \frac{1}{2}\int_{\mathbb{R}^2} (\mathcal{H}_1(u_n) + \mathcal{H}_2(v_n)) \, dx + o_n(1),$$

which together with (3.3) and Fatou's lemma implies that

$$c_{\mathcal{N}} = \frac{1}{2}\int_{\mathbb{R}^2} (\mathcal{H}_1(u_n) + \mathcal{H}_2(v_n)) \, dx + o_n(1) \geq \frac{1}{2}\int_{\mathbb{R}^2} (\mathcal{H}_1(u_0) + \mathcal{H}_2(v_0)) \, dx = I(u_0, v_0).$$

Therefore $c_{\mathcal{N}} \geq I(u_0, v_0)$ and (u_0, v_0) is a ground state for System (S).

Case 2 $u_0 \equiv 0$ or $v_0 \equiv 0$.

Recalling that $I(u_n, v_n) \to c_{\mathcal{N}} > 0$ and I is continuous, we conclude that $(u_n, v_n)_n$ can not converge to zero strongly in E. Thus, it follows from Proposition 3.3.2 that the sequence is *non-vanishing*, that is, there exists a sequence $(y_n)_n \subset \mathbb{R}^2$ and constants $R, \xi > 0$ such that

$$\liminf_{n\to\infty} \int_{B_R(y_n)} (u_n^2 + v_n^2) \, dx \geq \xi > 0.$$

We may assume, without loss of generality, that $(y_n)_n \subset \mathbb{Z}^2$. Let us consider the shift sequence $(\tilde{u}_n(x), \tilde{v}_n(x)) = (u_n(x+y_n), v_n(x+y_n))$. By the invariance of I under translations of the form $(u,v) \mapsto (u(\cdot - z), v(\cdot - z))$ with $z \in \mathbb{Z}^2$, we conclude that

$$\|(\tilde{u}_n, \tilde{v}_n)\| = \|(u_n, v_n)\|, \quad I(\tilde{u}_n, \tilde{v}_n) = I(u_n, v_n) \to c_{\mathcal{N}} \quad \text{and} \quad I'(\tilde{u}_n, \tilde{v}_n) \to 0.$$

We may assume that $(y_n)_n$ is bounded in $\mathbb{Z}^2$. Repeating the same arguments used in Propositions 3.3.1 and 3.3.4, we can deduce that $(\tilde{u}_n, \tilde{v}_n)_n$ is a bounded sequence in E, which implies that $(\tilde{u}_n, \tilde{v}_n) \rightharpoonup (\tilde{u}, \tilde{v})$ and $I'(\tilde{u}, \tilde{v}) = 0$. Thus,

$$\liminf_{n\to\infty} \int_{B_R(0)} (\tilde{u}_n^2 + \tilde{v}_n^2)\, dx = \liminf_{n\to\infty} \int_{B_R(y_n)} (u_n^2 + v_n^2)\, dx \geq \xi > 0.$$

Therefore, $\tilde{u} \not\equiv 0$ or $\tilde{v} \not\equiv 0$. Let us consider without loss of generality that $\tilde{v} \not\equiv 0$. If we suppose that $\tilde{u} \equiv 0$, then using the fact that $I'(\tilde{u}, \tilde{v}) = 0$, we get

$$0 = \langle I'(\tilde{u}, \tilde{v}), (\tilde{v}, 0)\rangle = -\int_{\mathbb{R}^2} \lambda(x)\tilde{v}^2\, dx.$$

Since $\lambda(x) > 0$ we must have $\tilde{v} \equiv 0$. This contradiction implies that $\tilde{u} \not\equiv 0$ and $\tilde{v} \not\equiv 0$. The conclusion follows from the same idea used in the **Case 1**.

Remark 3.3.5. *If $(\tilde{u}, \tilde{v}) \in E$ is a ground state for System (S), then there exists $C = C(\delta, \mu) > 0$ such that $C\|(\tilde{u}, \tilde{v})\|^2 \leq c_{\mathcal{N}}$. In fact, by a similar argument used to get (3.16) we can deduce that*

$$c_{\mathcal{N}} = I(\tilde{u}, \tilde{v}) \geq \left(\frac{1}{2} - \frac{1}{\mu}\right)(1-\delta)\|(\tilde{u}, \tilde{v})\|^2.$$

We note by assumptions (H_1) and (H_3) that for each $i = 1, 2$ we have

$$F_i(s) \leq F_i(|s|) \quad \text{for all } s \in \mathbb{R}.$$

Thus, $I(|\tilde{u}|, |\tilde{v}|) \leq I(\tilde{u}, \tilde{v})$. Since $(|\tilde{u}|, |\tilde{v}|) \in E\backslash\{(0,0)\}$, it follows from Lemma 3.2.4 that there exists a unique $t_0 > 0$ such that $(t_0|\tilde{u}|, t_0|\tilde{v}|) \in \mathcal{N}$. Moreover, since $(\tilde{u}, \tilde{v}) \in \mathcal{N}$, we point out that $\max_{t\geq 0} I(t\tilde{u}, t\tilde{v}) = I(\tilde{u}, \tilde{v})$. Thus,

$$I(t_0|\tilde{u}|, t_0|\tilde{v}|) \leq I(t_0\tilde{u}, t_0\tilde{v}) \leq \max_{t\geq 0} I(t\tilde{u}, t\tilde{v}) = I(\tilde{u}, \tilde{v}) = c_{\mathcal{N}}.$$

Therefore, $(t_0|\tilde{u}|, t_0|\tilde{v}|) \in \mathcal{N}$ is a nonnegative ground state for System (S). The positivity and regularity are obtained by a similar argument used to get in the proof of Theorem 2.1.2 (see Chapter 2).

Let $\mathcal{K}$ be the set of all ground state solutions for System (S)

$$\mathcal{K} := \{(u,v) \in E : (u,v) \in \mathcal{N},\ I(u,v) = c_{\mathcal{N}} \text{ and } I'(u,v) = 0\}.$$

Let $(u_n, v_n)_n \subset \mathcal{K}$ be a bounded sequence. Thus, $I(u_n, v_n) = c_{\mathcal{N}}$ and $I'(u_n, v_n) = 0$. Passing to a subsequence we have $(u_n, v_n) \rightharpoonup (u,v)$ weakly in E. By a similar argument used before, we can prove that there exists a sequence $(y_n)_n \subset \mathbb{R}^2$ and constants $R, \xi > 0$ such that

$$\liminf_{n\to\infty} \int_{B_R(y_n)} (u_n^2 + v_n^2)\, \mathrm{d}x \geq \xi > 0.$$

By the invariance of I, we may assume that $(y_n)_n$ is bounded in $\mathbb{Z}^2$. Therefore, $(u,v) \neq 0$. Repeating the same argument used in Proposition 3.3.4, we conclude that $I'(u,v) = 0$. As before, we have also that $I(u,v) = c_{\mathcal{N}}$. By using (H_3), the weakly lower semi-continuity of the norm and Fatou's lemma, we can deduce that

$$\begin{aligned} c_{\mathcal{N}} + o_n(1) &= I(u_n, v_n) - \frac{1}{\mu}\langle I'(u_n, v_n), (u_n, v_n)\rangle \\ &\geq I(u,v) - \frac{1}{\mu}\langle I'(u,v), (u,v)\rangle + o_n(1) \\ &= c_{\mathcal{N}} + o_n(1). \end{aligned}$$

Therefore $\|(u_n, v_n)\| \to \|(u,v)\|$, which implies that $(u_n, v_n) \to (u,v)$ strongly in E.

3.4 Proof of Proposition 3.2.5

First, it follows from (H_3), (3.11) and Lemma 3.2.3 that for any $(u,v) \in \mathcal{N}$ we have

$$I(u,v) \geq \left(\frac{1}{2} - \frac{1}{\mu}\right)\left(\|(u,v)\|^2 - 2\int_{\mathbb{R}^2} \lambda(x) uv\, \mathrm{d}x\right) \geq \left(\frac{1}{2} - \frac{1}{\mu}\right)(1-\delta)\rho > 0,$$

which implies that $c_{\mathcal{N}} > 0$. Now, let $f : (0,\infty) \to \mathbb{R}$ defined by $f(r) = 4e^{r^2/2}/r^2$. Thus

$$f'(r) = \frac{4e^{r^2/2}(r^2-1)}{r^3}.$$

Hence, $r = \sqrt{2}$ is the unique critical point and it is a minimum for f. Therefore, $\min_{r>0} 4e^{r^2/2}/r^2 = 2e$. Thus, it follows from assumption (H_5) that there exists $r > 0$ such that

$$\beta_0 > \frac{4e^{r^2/2}}{\alpha_0 r^2}. \tag{3.27}$$

We consider the following Moser's sequence of functions (see [26, 35, 57])

$$\omega_n(x) = \frac{1}{\sqrt{2\pi}} \begin{cases} \sqrt{\log(n)} & \text{if} \quad |x| \leq \dfrac{r}{n}, \\ \dfrac{\log(r/|x|)}{\sqrt{\log(n)}} & \text{if} \quad \dfrac{r}{n} \leq |x| \leq r, \\ 0 & \text{if} \quad |x| \geq r. \end{cases}$$

It is well known that $\|\nabla \omega_n\|_2^2 = 1$ and $\|\omega_n\|_2^2 = r^2/4\log(n) + o_n(r^2/\log(n))$. Thus,

$$\|\omega_n\|^2 = 1 + \frac{d_n(r)}{\log(n)}, \quad \text{where} \quad d_n(r) = \frac{r^2}{4} + o_n(1).$$

Let us define $\overline{\omega}_n = \omega_n/\|\omega_n\|$. Notice that $\|\overline{\omega}_n\| = 1$ and for $|x| \leq r/n$ we have

$$(\overline{\omega}_n)^2(x) = \frac{1}{2\pi}\log(n)\frac{\log(n)}{\log(n)+d_n(r)} = \frac{1}{2\pi}\left(\log(n) - d_n(r)\frac{\log(n)}{\log(n)+d_n(r)}\right).$$

Therefore, for n sufficiently large we deduce that

$$(\overline{\omega}_n)^2(x) \geq \frac{1}{2\pi}\left(\log(n) - d_n(r)\right), \quad \text{for } |x| \leq \frac{r}{n}. \tag{3.28}$$

In light of the minimax characterization (3.14), we note that to prove (3.13) it suffices to get $(w,z) \in E_+$ such that $\max_{t\geq 0} I(tw,tz) < 2\pi/\alpha_0$. The idea is to prove that there exists $n_0 \in \mathbb{N}$ such that

$$\max_{t\geq 0} I(t\overline{\omega}_{n_0}, 0) < \frac{2\pi}{\alpha_0}. \tag{3.29}$$

Arguing by contradiction, we suppose that (3.29) does not hold, that is

$$\max_{t\geq 0} I(t\overline{\omega}_n, 0) \geq \frac{2\pi}{\alpha_0}, \quad \text{for all } n \in \mathbb{N}. \tag{3.30}$$

By using Lemma 3.2.4 for each $n \in \mathbb{N}$, there exists $t_n > 0$ such that

$$(t_n\overline{\omega}_n, 0) \in \mathcal{N} \quad \text{and} \quad I(t_n\overline{\omega}_n, 0) = \max_{t\geq 0} I(t\overline{\omega}_n, 0).$$

We claim that the sequence $(t_n)_n \subset (0, +\infty)$ is bounded. In fact, it follows from (H_5) that for any $\varepsilon > 0$, there exists $R = R(\varepsilon) > 0$, such that

$$sf_1(s) \geq (\beta_0 - \varepsilon)e^{\alpha_0 s^2}, \quad \text{for all } s \geq R.$$

Moreover, since $(t_n\overline{\omega}_n, 0) \in \mathcal{N}$ we have that

$$t_n^2 = \|t_n\overline{\omega}\|^2 = \int_{\mathbb{R}^2} f_1(t_n\overline{\omega}_n)t_n\overline{\omega}_n \, dx.$$

Thus, we can conclude that

$$t_n^2 \geq (\beta_0 - \varepsilon) \int_{\mathbb{R}^2} e^{\alpha_0 t_n^2 \overline{\omega}_n^2} \, dx, \quad \text{for all } \; t_n \overline{\omega}_n \geq R. \tag{3.31}$$

Notice that if $x \in B_{\frac{r}{n}}(0)$, then

$$t_n \overline{\omega}_n = \frac{t_n}{\|\omega_n\|} \frac{\sqrt{\log(n)}}{\sqrt{2\pi}} \to +\infty, \quad \text{as } \; n \to +\infty.$$

Thus, for given $\varepsilon > 0$, we can consider n sufficiently large such that $t_n \overline{\omega}_n \geq R$. Therefore, by using (3.28) and (3.31) we get

$$t_n^2 \geq (\beta_0 - \varepsilon) \int_{B_{\frac{r}{n}}(0)} e^{\alpha_0 t_n^2 \overline{\omega}_n^2} \, dx \geq \pi r^2 (\beta_0 - \varepsilon) e^{\alpha_0/(2\pi) t_n^2 (\log(n) - r^2/4 - o_n(1)) - 2\log(n)}, \tag{3.32}$$

which implies that the sequence $(t_n^2)_n$ is bounded. It follows from (3.30) that

$$\frac{t_n^2}{2} = I(t_n \overline{w}_n, 0) + \int_{\mathbb{R}^2} F_1(t_n \overline{\omega}_n) \, dx \geq I(t_n \overline{\omega}_n, 0) = \max_{t \geq 0} I(t \overline{\omega}_n, 0) \geq \frac{1}{2} \frac{4\pi}{\alpha_0},$$

which implies that $t_n^2 \geq 4\pi/\alpha_0$. Thus, up to a subsequence, $t_n^2 \to t_0 \in [4\pi/\alpha_0, +\infty)$. We claim that $t_0 = 4\pi/\alpha_0$. In fact, suppose by contradiction that $t_0 = 4\pi/\alpha_0 + \gamma$, for some $\gamma > 0$. For $n \in \mathbb{N}$ large enough such that $t_n^2 > 4\pi/\alpha_0 + \varepsilon$ we have

$$\frac{\alpha_0}{2\pi} t_n^2 (\log(n) - r^2/4 - o_n(1)) - 2\log(n) > \frac{\alpha_0 \varepsilon}{2\pi} \log(n) - \left(2 + \frac{\alpha_0 \varepsilon}{2\pi}\right) \left(\frac{r^2}{4} + o_n(1)\right) \to +\infty,$$

as $n \to +\infty$, which contradicts (3.32). Since $t_n^2 \geq 4\pi/\alpha_0$ and $t_n^2 \to 4\pi/\alpha_0$, we get

$$\liminf_{n \to +\infty} \int_{B_{\frac{r}{n}}(0)} e^{\alpha_0 t_n^2 \overline{\omega}_n^2} \, dx \geq \liminf_{n \to +\infty} \int_{B_{\frac{r}{n}}(0)} e^{\alpha_0 t_n^2 \log(n) - r^2/4 + o_n(1)} \, dx \geq \pi r^2 e^{-r^2/2},$$

which together with (3.31) implies that

$$\frac{4\pi}{\alpha_0} = \lim_{n \to +\infty} t_n^2 \geq (\beta_0 - \varepsilon) \pi r^2 e^{-r^2/2}.$$

Since ε is arbitrary, we conclude that $\beta_0 \leq 4e^{r^2/2}/(\alpha_0 r^2)$, which contradicts (3.27). Therefore, there exists $n_0 \in \mathbb{N}$ such that (3.29) holds. Thus,

$$c_{\mathcal{N}} = \inf_{(u,v) \in \mathcal{N}} I(u, v) \leq \inf_{(u,v) \in E_+} \max_{t \geq 0} I(tu, tv) \leq \max_{t \geq 0} I(t \overline{\omega}_{n_0}, 0) < \frac{2\pi}{\alpha_0},$$

which finishes the proof of Proposition 3.2.5.

Chapter 4

Coupled systems involving the square root of the Laplacian and critical exponential growth

4.1 Introduction

In the last few years, a great attention has been focused on the study of problems involving fractional Sobolev spaces and corresponding nonlocal equations, both from a pure mathematical point of view and their concrete applications, since they naturally arise in many different contexts, such as, among the others, obstacle problems, flame propagation, minimal surfaces, conservation laws, financial market, optimization, crystal dislocation, phase transition and water waves, see for instance [13, 30] and references therein. This chapter deals with the existence of ground states to the following class of coupled systems involving fractional nonlinear Schrödinger equations

$$\begin{cases} (-\Delta)^{1/2}u + V_1(x)u = f_1(u) + \lambda(x)v, & x \in \mathbb{R}, \\ (-\Delta)^{1/2}v + V_2(x)v = f_2(v) + \lambda(x)u, & x \in \mathbb{R}, \end{cases} \tag{S}$$

where $(-\Delta)^{1/2}$ denotes the *square root of the Laplace operator*, the potentials $V_1(x)$, $V_2(x)$ are nonnegative and satisfy $|\lambda(x)| \leq \delta\sqrt{V_1(x)V_2(x)}$, for some $\delta \in (0,1)$ and for all $x \in \mathbb{R}$. Here we consider the case when $V_1(x)$, $V_2(x)$ and $\lambda(x)$ are periodic, and also when these functions are asymptotically periodic, that is, the limits of $V_1(x)$, $V_2(x)$ and $\lambda(x)$ are periodic functions when $|x| \to +\infty$. Our main goal here is to study the existence of ground states for (S), when the nonlinearities $f_1(u)$, $f_2(v)$ have critical exponential growth motivated by a class of Trudinger-Moser inequality introduced by

T. Ozawa (see Theorem B in the Section 4.2).

4.1.1 Assumptions

We start this subsection recalling some preliminary concepts about the fractional operator, for a more complete discussion we refer the readers to [30]. For $s \in (0,1)$, the *fractional Laplace operator* of a measurable function $u : \mathbb{R} \to \mathbb{R}$ is defined by

$$(-\Delta)^s u(x) = -\frac{1}{2}C(s)\int_{\mathbb{R}} \frac{u(x+y)+u(x-y)-2u(x)}{|y|^{1+2s}}\,\mathrm{d}y,$$

where

$$C(s) = \left(\int_{\mathbb{R}} \frac{1-\cos(\xi)}{|\xi|^{1+2s}}\,\mathrm{d}\xi\right)^{-1}.$$

The particular case when $s = 1/2$ its called the *square root of the Laplacian*. We recall the definition of the fractional Sobolev space

$$H^{1/2}(\mathbb{R}) = \left\{u \in L^2(\mathbb{R}) : \int_{\mathbb{R}^2} \frac{|u(x)-u(y)|^2}{|x-y|^2}\,\mathrm{d}x\,\mathrm{d}y < \infty\right\},$$

endowed with the natural norm

$$\|u\|_{1/2} = \left([u]_{1/2}^2 + \int_{\mathbb{R}} u^2\,\mathrm{d}x\right)^{1/2}, \quad [u]_{1/2} = \left(\int_{\mathbb{R}^2} \frac{|u(x)-u(y)|^2}{|x-y|^2}\,\mathrm{d}x\,\mathrm{d}y\right)^{1/2}$$

where the term $[u]_{1/2}$ is the so-called *Gagliardo semi-norm* of the function u. In light of [30, Proposition 3.6] we have that

$$\|(-\Delta)^{1/4}u\|_2^2 = \frac{1}{2\pi}\int_{\mathbb{R}^2} \frac{|u(x)-u(y)|^2}{|x-y|^2}\,\mathrm{d}x\,\mathrm{d}y, \quad \text{for all } u \in H^{1/2}(\mathbb{R}).$$

In view of the potentials $V_1(x)$ and $V_2(x)$, we define the following subspace of $H^{1/2}(\mathbb{R})$

$$E_i = \left\{u \in H^{1/2}(\mathbb{R}) : \int_{\mathbb{R}} V_i(x)u^2\,\mathrm{d}x < +\infty\right\}, \quad \text{for } i = 1,2,$$

endowed with the inner product

$$(u,v)_{E_i} = \int_{\mathbb{R}} (-\Delta)^{1/4}u(-\Delta)^{1/4}v\,\mathrm{d}x + \int_{\mathbb{R}} V_i(x)u^2\,\mathrm{d}x,$$

to which corresponds the induced norm $\|u\|_{E_i}^2 = (u,u)_{E_i}$. In order to establish a variational approach to treat System (S), we need to require suitable assumptions on the potentials. For each $i = 1,2$, we assume that

(V_1) $V_i(x)$, $\lambda(x)$ are periodic, that is, $V_i(x) = V_i(x+z)$, $\lambda(x) = \lambda(x+z)$, for all $x \in \mathbb{R}$, $z \in \mathbb{Z}$.

(V_2) $V_i(x) \in L^\infty_{loc}(\mathbb{R})$, $V_i(x) \geq 0$ for all $x \in \mathbb{R}$ and

$$\nu_i = \inf_{u \in E_i} \left\{ \frac{1}{2\pi}[u]^2_{1/2} + \int_{\mathbb{R}} V_i(x) u^2 \, dx : \int_{\mathbb{R}} u^2 \, dx = 1 \right\} > 0.$$

(V_3) $|\lambda(x)| \leq \delta\sqrt{V_1(x)V_2(x)}$, for some $\delta \in (0,1)$, for all $x \in \mathbb{R}$.

We set the product space $E = E_1 \times E_2$ endowed with the scalar product

$$((u,v),(w,z))_E = \int_{\mathbb{R}} \left((-\Delta)^{1/4}u(-\Delta)^{1/4}w + V_1(x)uw + (-\Delta)^{1/4}v(-\Delta)^{1/4}z + V_2(x)vz \right),$$

to which corresponds the induced norm $\|(u,v)\|^2_E = ((u,v),(u,v))_E = \|u\|^2_{E_1} + \|v\|^2_{E_2}$. It follows from assumption (V_2) that E is a Hilbert space.

We are also concerned with the existence of ground states for the following class of coupled systems

$$\begin{cases} (-\Delta)^{1/2}u + \tilde{V}_1(x)u = f_1(u) + \tilde{\lambda}(x)v, & x \in \mathbb{R}, \\ (-\Delta)^{1/2}v + \tilde{V}_2(x)v = f_2(v) + \tilde{\lambda}(x)u, & x \in \mathbb{R}, \end{cases} \tag{$\tilde{S}$}$$

when the potentials $\tilde{V}_1(x)$, $\tilde{V}_2(x)$ and $\tilde{\lambda}(x)$ are asymptotically periodic. In analogous way, we may define the suitable space $\tilde{E} = \tilde{E}_1 \times \tilde{E}_2$ considering $\tilde{V}_i(x)$ instead $V_i(x)$. In order to establish an existence theorem for $(\tilde{S})$, for $i = 1,2$ we introduce the following assumptions:

(V_4) $\tilde{V}_i(x) < V_i(x)$, $\lambda(x) < \tilde{\lambda}(x)$ and

$$\lim_{|x| \to +\infty} |V_i(x) - \tilde{V}_i(x)| = 0 \quad \text{and} \lim_{|x| \to +\infty} |\tilde{\lambda}(x) - \lambda(x)| = 0.$$

(V_5) $\tilde{V}_i(x) \in L^\infty_{loc}(\mathbb{R})$, $\tilde{V}_i(x) \geq 0$ for all $x \in \mathbb{R}$ and

$$\tilde{\nu}_i = \inf_{u \in \tilde{E}_i} \left\{ \frac{1}{2\pi}[u]^2_{1/2} + \int_{\mathbb{R}} \tilde{V}_i(x) u^2 \, dx : \int_{\mathbb{R}} u^2 \, dx = 1 \right\} > 0.$$

(V_6) $|\tilde{\lambda}(x)| \leq \delta\sqrt{\tilde{V}_1(x)\tilde{V}_2(x)}$, for some $\delta \in (0,1)$, for all $x \in \mathbb{R}$.

We suppose here that the nonlinearities $f_1(s)$ and $f_2(s)$ have *critical exponential growth*. Precisely, we say that $f_i : \mathbb{R} \to \mathbb{R}$ for $i = 1,2$ has α_0^i-*critical growth* at $\pm\infty$ if there exists $\alpha_0^i > 0$ such that

$$\limsup_{s \to \pm\infty} \frac{f_i(s)}{e^{\alpha s^2} - 1} = \begin{cases} 0 & \text{if} \quad \alpha > \alpha_0^i, \\ \pm\infty & \text{if} \quad \alpha < \alpha_0^i. \end{cases} \tag{CG}$$

This notion of *criticality* is motivated by a class of Trudinger-Moser type inequality introduced by T. Ozawa (see Section 4.2). Furthermore, for $i = 1, 2$ we make the following assumptions on the nonlinearities:

(H_1) The function f_i belongs to $C^1(\mathbb{R})$, convex function on $\mathbb{R}^+$, $f_i(-s) = -f_i(s)$ for $s \in \mathbb{R}$, and

$$\lim_{s \to 0} \frac{f_i(s)}{s} = 0.$$

(H_2) The function $s \mapsto s^{-1} f_i(s)$ is increasing for $s > 0$.

(H_3) There exists $\mu_i > 2$ such that

$$0 < \mu_i F_i(s) := \mu_i \int_0^s f_i(\tau)\, \mathrm{d}\tau \le f_i(s)s, \quad \text{for all } s \in \mathbb{R} \backslash \{0\}.$$

(H_4) There exist $q > 2$ and $\vartheta > 0$ such that

$$F_i(s) \ge \vartheta |s|^q, \quad \text{for all } s \in \mathbb{R}.$$

4.1.2 Statement of the main results

We are in condition to state our existence theorem for the case when the potentials are periodic.

Theorem 4.1.1. *Suppose that assumptions (V_1)-(V_3) hold. Assume that for each $i = 1, 2$ $f_i(s)$ and $f_i'(s)s$ have α_0^i-critical growth (CG) and satisfy (H_1)-(H_4). Then, System (S) possesses a nonnegative ground state solution provided ϑ in (H_4) is large enough.*

Theorem 4.1.2. *Suppose that assumptions (V_1)-(V_6) hold and for each $i = 1, 2$ assume that $f_i(s)$ has α_0^i-critical growth (CG), satisfies (H_1)-(H_4) and $f_i'(s)s$ has α_0^i-critical growth (CG). Then, System $(\tilde{S})$ possesses a nonnegative ground state solution provided ϑ in (H_4) is large enough.*

Remark 4.1.3. *We collect the following remarks on our assumptions:*

(i) A typical example of nonlinearity which satisfies the assumptions (H_1)-(H_4) is

$$f(s) = \vartheta q |s|^{q-2} s + q|s|^{q-2} s (e^{\alpha_0 s^2} - 1) + 2\alpha_0 |s|^q s e^{\alpha_0 s^2}, \quad \text{for } 2 < \mu < q \text{ and } s \in \mathbb{R},$$

where α_0 is the critical exponent introduced in (CG).

(ii) The assumption (H_4) could be replaced by the following local condition: there exists $q > 2$ and $\tilde{\vartheta}$ such that

$$\liminf_{s\to 0} \frac{F_i(s)}{|s|^q} \geq \tilde{\vartheta} > 0. \tag{4.1}$$

In fact, we can use the critical exponential growth of the nonlinearities, Ambrosetti-Rabinowitz condition (H_3) and assumption (4.1) to deduce (H_4). In order to make ease the presentation of this paper and avoid certain technicalities, we simply assume (H_4).

(iii) The assumption (H_4) plays a very important role in the proof of Theorems 4.1.1 and 4.1.2. We will prove the existence of ground states when ϑ is large enough. Precisely, if

$$\vartheta > \vartheta_0 = \frac{S_q^q}{q}\left(\frac{1}{1-\delta}\frac{\mu}{\mu-2}\frac{q-2}{q}\frac{\alpha_0\kappa^{-1}}{\omega}\right)^{(q-2)/2}, \tag{4.2}$$

where $\alpha_0 = \max\{\alpha_0^1, \alpha_0^2\}$, $\mu = \min\{\mu_1, \mu_2\}$, ω is introduced in Theorem B, $\kappa^{-1} = \max\{\kappa_1^{-1}, \kappa_2^{-1}\}$ where κ_i is introduced in Lemma 4.2.3 and S_q is introduced in Section 4.5. The estimate (4.2) will allow us to apply the Trudinger-Moser inequality (see Section 4.2, Theorem B) in the minimizing sequence obtained by Ekeland's variational principle (see Lemma 4.5.2) in order to prove that the weak limit of this sequence belongs to Nehari manifold.

(iv) Theorems 4.1.1 and 4.1.2 may be considered as the extension of the main result for the scalar case in [36], because we consider a class of potentials and the nonlinear term different from them. If we take $u = v$ and $\lambda = 0$ in System (S) then we solve the single equation found in that paper but under our hypotheses.

4.1.3 Outline

The remainder of this chapter is organized as follows. In Sections 4.2 and 4.3, we collect some results which are crucial to give a variational approach for our problem. In Section 4.4, we introduce and give some properties of the Nehari manifold. In Section 4.5, we study the periodic case. For this purpose, we make use of the Ekeland's variational principle to obtain a minimizing sequence for the energy functional on the Nehari manifold. We shall use a fractional version of a lemma introduced by P.L. Lions, a Brezis-Lieb type lemma and a Trudinger-Moser type inequality to prove that the weak limit of this sequence will be a ground state solution for the problem. In the periodic case, the key point is to use the invariance of the energy functional under translations to recover the compactness of the minimizing sequence. Finally, in Section 4.6 we study the asymptotically periodic case. For this matter, the key point is a relation obtained between the ground state energy associated with Systems (S) and ($\tilde{S}$) (see Lemma 4.6.1).

4.2 Preliminary results

In this section we provide preliminary results which will be used throughout the chapter. One of the features of the class of the systems (S) and ($\tilde{S}$) is the presence of the nonlocal operator, square root of the Laplacian. Another feature is the critical exponential behavior of the nonlinearities in the sense of Trudinger-Moser. We are motivated by the following Trudinger-Moser type inequality which was introduced by T. Ozawa, see [58].

Theorem B. *There exists $\omega \in (0, \pi)$ such that, for all $\alpha \in (0, \omega]$, there exists $H_\alpha > 0$ with*

$$\int_{\mathbb{R}} (e^{\alpha u^2} - 1) \, \mathrm{d}x \leq H_\alpha \|u\|_2^2, \tag{4.3}$$

for all $u \in H^{1/2}(\mathbb{R})$ such that $\|(-\Delta)^{1/4} u\|_2^2 \leq 1$.

The following result is a consequence of Theorem B, more details can be found in [36, Lemma 2.2].

Lemma 4.2.1. *Let $u \in H^{1/2}(\mathbb{R})$ and $\rho_0 > 0$ be such that $\|u\|_{1/2} \leq \rho_0$. Then, there exists $C = C(\alpha, \rho_0) > 0$ such that*

$$\int_{\mathbb{R}} (e^{\alpha u^2} - 1) \, \mathrm{d}x \leq C, \quad \textit{for every } 0 < \alpha \rho_0^2 < \omega.$$

Remark 4.2.2. *In light of [53, Theorem 8.5], for any $p \geq 2$, there exists $C = C(p)$, such that*

$$\|u\|_p \leq C \|u\|_{1/2}, \quad \textit{for all } u \in H^{1/2}(\mathbb{R}). \tag{4.4}$$

Lemma 4.2.3. *Assume that (V_2) holds. Then for each $i = 1, 2$ there exists $\kappa_i > 0$ such that*

$$\kappa_i \|u\|_{1/2}^2 \leq \frac{1}{2\pi} [u]_{1/2}^2 + \int_{\mathbb{R}} V_i(x) u^2 \, \mathrm{d}x, \quad \textit{for all } u \in E_i. \tag{4.5}$$

Proof. The proof can be found in [27] and for the sake of convenience, we sketch the proof here. Suppose that (4.5) does not holds. Thus, there exists a sequence $(u_n)_n \subset E_i$ such that $\|u_n\|_{1/2} = 1$ and

$$\frac{1}{2\pi}[u_n]_{1/2}^2 + \int_{\mathbb{R}} V_i(x)u_n^2 \, \mathrm{d}x < \frac{1}{n}.$$

By using (V_2), we have that

$$0 < \lambda_i \leq \frac{1}{\|u_n\|_2^2}\left(\frac{1}{2\pi}[u_n]_{1/2}^2 + \int_{\mathbb{R}} V_i(x)u_n^2 \, \mathrm{d}x\right) < \frac{1}{n}\frac{1}{\|u_n\|_2^2},$$

which implies that $\|u_n\|_2^2 \to 0$ and $[u_n]_{1/2}^2 \to 1$. Therefore, since $V_i \geq 0$, we conclude that

$$o_n(1) = -\|u_n\|_2^2 \leq \int_{\mathbb{R}} V_i(x)u_n^2 \, \mathrm{d}x < \frac{1}{n} - \frac{1}{2\pi}[u_n]_{1/2}^2 \to -\frac{1}{2\pi},$$

which is not possible and finishes the proof. ∎

Notice that combining Remark 4.2.2 and Lemma 4.2.3 we have that E_i is continuously embedded into $L^p(\mathbb{R})$, for any $p \geq 2$. The next lemma is a very important tool to overcome the lack of compactness. The *vanishing lemma* was proved originally by P.L. Lions [52, Lemma I.1] and here we use the following version to fractional Sobolev spaces.

Lemma 4.2.4. *Assume that $(u_n)_n$ is a bounded sequence in $H^{1/2}(\mathbb{R})$ satisfying*

$$\lim_{n\to+\infty} \sup_{y\in\mathbb{R}} \int_{y-R}^{y+R} |u_n|^2 \, \mathrm{d}x = 0, \tag{4.6}$$

for some $R > 0$. Then, $u_n \to 0$ strongly in $L^p(\mathbb{R})$, for $2 < p < \infty$.

Proof. Given $r > p$, $R > 0$ and $y \in \mathbb{R}$ it follows by standard interpolation that

$$\|u_n\|_{L^p(B_R(y))} \leq \|u_n\|_{L^2(B_R(y))}^{1-\theta}\|u_n\|_{L^r(B_R(y))}^{\theta},$$

for some $\theta \in (0,1)$ such that

$$\frac{1-\theta}{2} + \frac{\theta}{r} = \frac{1}{q}.$$

Using a locally finite covering of $\mathbb{R}$ consisting of open balls of radius R, the continuous embedding $H^{1/2}(\mathbb{R}) \hookrightarrow L^r(\mathbb{R})$, the fact that $\|u_n\|_{1/2} \leq C$ and assumption (4.6), we can conclude that

$$\lim_{n\to+\infty} \|u_n\|_p \leq C \lim_{n\to+\infty} \sup_{y\in\mathbb{R}} \left(\int_{y-R}^{y+R} |u_n|^2 \, \mathrm{d}x\right)^{(1-\theta)/2} = 0.$$

∎

4.3 The Variational Setting

The energy functional $I : E \to \mathbb{R}$ associated to System (S) is defined by

$$I(u,v) = \frac{1}{2}\left(\|(u,v)\|_E^2 - 2\int_{\mathbb{R}} \lambda(x)uv \,\mathrm{d}x\right) - \int_{\mathbb{R}} \left(F_1(u) + F_2(v)\right) \,\mathrm{d}x.$$

Under our assumptions on $f_i(s)$, $V_i(x)$ and $\lambda(x)$, its standard to check that I is well defined. Moreover, $I \in C^2(E)$ and its differential is given by

$$\langle I'(u,v),(\phi,\psi)\rangle = ((u,v),(\phi,\psi)) - \int_{\mathbb{R}} \left(f_1(u)\phi + f_2(v)\psi\right) \,\mathrm{d}x - \int_{\mathbb{R}} \lambda(x)\left(u\psi + v\phi\right) \,\mathrm{d}x.$$

The critical points of I are precisely solutions (in the weak sense) to (S).

Definition 4.3.1. *We say that a pair $(u,v) \in E \setminus \{(0,0)\}$ is a ground state solution (least energy solution) of (S), if (u,v) is a solution of (S) and its energy is minimal among the energy of all nontrivial solutions of (S), i.e., $I(u,v) \le I(w,z)$ for any other nontrivial solution $(w,z) \in E$.*

Lemma 4.3.2. *If (H_1)-(H_3) hold, then we have the following facts:*

$$f_i'(s)s^2 - f_i(s)s > 0, \tag{4.7}$$

$$f_i'(s) > 0, \tag{4.8}$$

$$\phi_i(s) = f_i(s)s - 2F_i(s) > 0, \tag{4.9}$$

$$\phi_i(s) > \phi_i(ts), \text{ for all } t \in (0,1), \tag{4.10}$$

for each $i = 1, 2$ and for all $s \in \mathbb{R}\backslash\{0\}$.

Proof. The proof is quite similar to Lemma 2.2.4 and we omitted here. ∎

Lemma 4.3.3. *Suppose that (H_1) and (H_3) hold. If $f_i(s)$ and $f_i'(s)s$ have α_0^i-critical growth, then for each $i = 1, 2$, for any $\varepsilon > 0$, $\alpha > \alpha_0^i$ and $p > 2$, there exists $C = C(\varepsilon, p) > 0$ such that*

$$f_i(s) \le \varepsilon|s| + C(e^{\alpha s^2} - 1)|s|^{p-1}, \tag{4.11}$$

$$f_i'(s)s \le \varepsilon|s| + C(e^{\alpha s^2} - 1)|s|^{p-1}, \tag{4.12}$$

$$F_i(s) \le \varepsilon s^2 + C(e^{\alpha s^2} - 1)|s|^p., \tag{4.13}$$

for each $i = 1, 2$ and for all $s \in \mathbb{R}\backslash\{0\}$.

Proof. The proof is similar to Lemma 2.2.5 and we omitted here. ∎

4.4 The Nehari manifold

We introduce the Nehari manifold associated to System (S)

$$\mathcal{N} = \{(u,v) \in E\backslash\{(0,0)\} : \langle I'(u,v),(u,v)\rangle = 0\}.$$

Notice that if $(u,v) \in \mathcal{N}$ then

$$\|(u,v)\|_E^2 - 2\int_{\mathbb{R}} \lambda(x)uv \, \mathrm{d}x = \int_{\mathbb{R}} f_1(u)u \, \mathrm{d}x + \int_{\mathbb{R}} f_2(v)v \, \mathrm{d}x. \tag{4.14}$$

Lemma 4.4.1. *$\mathcal{N}$ is a C^1-manifold and there exists $\rho > 0$, such that*

$$\|(u,v)\|_E \geq \rho, \quad \textit{for all } (u,v) \in \mathcal{N}. \tag{4.15}$$

Proof. The proof of the lemma is similar to Lemma 3.2.3, but for the sake of convenience we give the proof here. Let $J : E\backslash\{(0,0)\} \to \mathbb{R}$ be the C^1-functional defined by

$$J(u,v) = \langle I'(u,v),(u,v)\rangle = \|(u,v)\|_E^2 - 2\int_{\mathbb{R}} \lambda(x)uv \, \mathrm{d}x - \int_{\mathbb{R}} f_1(u)u \, \mathrm{d}x - \int_{\mathbb{R}} f_2(v)v \, \mathrm{d}x.$$

Notice that $\mathcal{N} = J^{-1}(0)$. If $(u,v) \in \mathcal{N}$, it follows from (4.7) and (4.14) that

$$\langle J'(u,v),(u,v)\rangle = \int_{\mathbb{R}} \left(f_1(u)u - f_1'(u)u^2\right) \, \mathrm{d}x + \int_{\mathbb{R}} \left(f_2(v)v - f_2'(v)v^2\right) \, \mathrm{d}x < 0. \tag{4.16}$$

Therefore, 0 is a regular value of J which implies that $\mathcal{N}$ is a C^1-manifold.

To prove the second part, we suppose by contradiction that (4.15) does not hold. Thus, we have a sequence

$$(u_n,v_n)_n \subset \mathcal{N}, \text{ such that } \|(u_n,v_n)\|_E \to 0 \text{ as } n \to +\infty. \tag{4.17}$$

Let us consider $\rho_0 > 0$ such that $\alpha\rho_0^2 < \omega$. As consequence of (4.17), there exists $n_0 \in \mathbb{N}$ such that $\kappa^{-1}\|(u_n,v_n)\|_E^2 \leq \rho_1^2 < \rho_0^2$, for $n \geq n_0$, where $\kappa^{-1} = \max\{\kappa_1^{-1}, \kappa_2^{-1}\}$. For given $p > 2$ and $\varepsilon > 0$, it follows from estimate (4.11) that

$$\int_{\mathbb{R}} f_1(u_n)u_n \, \mathrm{d}x \leq \varepsilon\|u_n\|_2^2 + C_2\int_{\mathbb{R}} (e^{\alpha u_n^2} - 1)|u_n|^p \, \mathrm{d}x. \tag{4.18}$$

We recall from Lemma 2.2.2 that for $\alpha > 0$, $l \geq 1$ and $r \geq l$ we have

$$(e^{\alpha s^2} - 1)^l \leq (e^{\alpha l s^2} - 1), \quad \text{for all } s \in \mathbb{R}. \tag{4.19}$$

Let $r > l > 1$ be sufficiently close to 1 such that $r\alpha\rho_0^2 < \omega$. Thus, it follows from Lemma 4.2.1, (4.19) and Hölder inequality that

$$\int_{\mathbb{R}} (e^{\alpha u_n^2} - 1)|u_n|^p \, \mathrm{d}x \leq \left(\int_{\mathbb{R}} (e^{r\alpha u_n^2} - 1) \, \mathrm{d}x\right)^{1/l} \|u_n\|_{pl'}^p \leq C\|u_n\|_{pl'}^p,$$

which together with (4.18) and Sobolev embedding implies that

$$\int_{\mathbb{R}} f_1(u_n)u_n \, dx \leq \varepsilon C_1 \|u_n\|_{E_1}^2 + C_2\|u_n\|_{E_1}^p \leq \varepsilon C_1 \|(u_n, v_n)\|_E^2 + C_2\|(u_n, v_n)\|_E^p.$$

Analogously, we deduce that

$$\int_{\mathbb{R}} f_2(v_n)v_n \, dx \leq \varepsilon C_3 \|(u_n, v_n)\|_E^2 + C_4\|(u_n, v_n)\|_E^p.$$

Combining theses estimates we get,

$$\int_{\mathbb{R}} (f_1(u_n)u_n + f_2(v_n)v_n) \, dx \leq \varepsilon \tilde{C}_1 \|(u_n, v_n)\|_E^2 + \tilde{C}_2\|(u_n, v_n)\|_E^p. \tag{4.20}$$

Since $\varepsilon > 0$ is arbitrary and $\tilde{C}_1$ does not depend of ε and n, we can choose ε sufficiently small such that $1 - \delta - \varepsilon\tilde{C}_1 > 0$. We recall from Lemma 1.2.1 that

$$\|(u, v)\|_E^2 - 2\int_{\mathbb{R}^2} \lambda(x)uv \, dx \geq (1 - \delta)\|(u, v)\|_E^2, \quad \text{for all } (u, v) \in E. \tag{4.21}$$

Thus, combining (4.20), (4.21) and the fact that $(u_n, v_n)_n \subset \mathcal{N}$ we get

$$(1 - \delta)\|(u_n, v_n)\|_E^2 \leq \int_{\mathbb{R}} (f_1(u_n)u_n + f_2(v_n)v_n) \, dx \leq \varepsilon\tilde{C}_1\|(u_n, v_n)\|_E^2 + \tilde{C}_2\|(u_n, v_n)\|_E^p,$$

which yields

$$0 < (1 - \delta - \varepsilon\tilde{C}_1)\|(u_n, v_n)\|_E^2 \leq \tilde{C}_2\|(u_n, v_n)\|_E^p.$$

Hence, denoting $\rho_2 = (1 - \delta - \varepsilon\tilde{C}_1)/\tilde{C}_2$ we obtain

$$0 < \rho_2^{1/(p-2)} \leq \|(u_n, v_n)\|_E.$$

Choosing $\rho_1 < \rho = \min\{\rho_0, \rho_2^{1/(p-2)}\}$ we get a contradiction and we conclude that (4.15) holds. ∎

Remark 4.4.2. *If $(u_0, v_0) \in \mathcal{N}$ is a critical point of $I \mid_{\mathcal{N}}$, then $I'(u_0, v_0) = 0$. In fact, recall the notation $J(u_0, v_0) = \langle I'(u_0, v_0), (u_0, v_0)\rangle$ and notice that $I'(u_0, v_0) = \eta J'(u_0, v_0)$, where $\eta \in \mathbb{R}$ is the corresponding Lagrange multiplier. Taking the scalar product with (u_0, v_0) and using (4.16) we conclude that $\eta = 0$.*

Let us define the ground state energy associated with System (S)

$$c_{\mathcal{N}} = \inf_{(u,v)\in\mathcal{N}} I(u, v).$$

We point out that $c_{\mathcal{N}}$ is positive. In fact, if $(u, v) \in \mathcal{N}$ it follows from (H_3) that

$$\begin{aligned} I(u, v) &\geq \frac{1}{2}\left(\|(u, v)\|_E^2 - 2\int_{\mathbb{R}} \lambda(x)uv \, dx\right) - \frac{1}{\mu_1}\int_{\mathbb{R}} f_1(u)u \, dx - \frac{1}{\mu_2}\int_{\mathbb{R}} f_2(v)v \, dx \\ &\geq \left(\frac{1}{2} - \frac{1}{\mu}\right)\left(\|(u, v)\|_E^2 - 2\int_{\mathbb{R}} \lambda(x)uv \, dx\right), \end{aligned}$$

Thus, combining with (4.21) we conclude that

$$I(u, v) \geq \left(\frac{1}{2} - \frac{1}{\mu}\right)(1 - \delta)\|(u, v)\|_E^2 \geq \left(\frac{1}{2} - \frac{1}{\mu}\right)(1 - \delta)\rho > 0, \quad \text{for all } (u, v) \in \mathcal{N}.$$

Lemma 4.4.3. *Suppose that* (V_3) *and* (H_1)-(H_4) *hold. For any* $(u,v) \in E\backslash\{(0,0)\}$, *there exists a unique* $t_0 > 0$, *depending only of* (u,v), *such that*

$$(t_0 u, t_0 v) \in \mathcal{N} \quad \textit{and} \quad I(t_0 u, t_0 v) = \max_{t \geq 0} I(tu, tv).$$

Moreover, if $\langle I'(u,v),(u,v)\rangle < 0$, *then* $t_0 \in (0,1)$.

Proof. Let $(u,v) \in E\backslash\{(0,0)\}$ be fixed and consider the function $g : [0,\infty) \to \mathbb{R}$ defined by $g(t) = I(tu,tv)$. Notice that

$$\langle I'(tu,tv),(tu,tv)\rangle = tg'(t).$$

The result follows if we find a positive critical point of g. After integrating (H_3), we deduce that

$$F_i(s) \geq C_0(|s|^{\mu_i} - 1), \quad \text{for all } s \neq 0,$$

which implies that

$$g(t) \leq \frac{t^2}{2}\left(\|(u,v)\|_E^2 - 2\int_{\mathbb{R}} \lambda(x)uv \,\mathrm{d}x\right) - C_0\int_{-R}^{R}(t^{\mu_1}|u|^{\mu_1} + t^{\mu_2}|v|^{\mu_2}) \,\mathrm{d}x - \tilde{C}.$$

Since $\mu_1, \mu_2 > 2$, we obtain $g(t) < 0$ for $t > 0$ sufficiently large. On the other hand, for some $\alpha > \alpha_0$ and $\rho_0 > 0$ satisfying $\alpha\rho_0^2 < \omega$, we consider $t > 0$ sufficiently small such that $t\kappa^{-1}\|(u,v)\|_E^2 < \rho_0^2$. Thus, for $\varepsilon > 0$ and $p > 2$, we can use (4.13) and the same ideas used to obtain (4.20) to get

$$\int_{\mathbb{R}}(F_1(tu) + F_2(tv)) \,\mathrm{d}x \leq \varepsilon C_1\frac{t^2}{2}\|(u,v)\|_E^2 + C_2 t^p\|(u,v)\|_E^p. \tag{4.22}$$

Since C_1 does not depends of ε which is arbitrary, we can take it small enough such that $1 - \delta - C_1\varepsilon > 0$. Hence, by using (4.21) and (4.22) we have

$$g(t) \geq t^2\|(u,v)\|_E^2\left(\frac{1-\delta-C_1}{2} - C_2 t^{p-2}\|(u,v)\|_E^{p-2}\right).$$

Thus, $g(t) > 0$ provided $t > 0$ is sufficiently small. Therefore, g has maximum points in $(0,\infty)$. In order to prove the uniqueness, we note that every critical point of g satisfies

$$\|(u,v)\|_E^2 - 2\int_{\mathbb{R}}\lambda(x)uv \,\mathrm{d}x = \int_{\mathbb{R}}\frac{f_1(tu)u}{t}\,\mathrm{d}x + \int_{\mathbb{R}}\frac{f_2(tv)v}{t}\,\mathrm{d}x. \tag{4.23}$$

Furthermore, by using (4.7) we get

$$\frac{d}{dt}\left(\frac{f_i(ts)s}{t}\right) = \frac{f_i'(ts)ts^2 - f_i(ts)s}{t^2} = \frac{f_i'(ts)t^2s^2 - f_i(ts)ts}{t^3} > 0, \tag{4.24}$$

which implies that the right-hand side of (4.23) is strictly increasing on $t > 0$, and consequently, the critical point $t_0 \in (0,+\infty)$ is *unique*.

Finally, we assume that $\langle I'(u,v),(u,v)\rangle < 0$ and we suppose by contradiction that $t_0 \geq 1$. Since t_0 is a critical point of g, we have

$$0 = g'(t_0) = \|(u,v)\|_E^2 - 2\int_{\mathbb{R}} \lambda(x)uv\, \mathrm{d}x - \int_{\mathbb{R}} \frac{f_1(t_0 u)u}{t_0}\, \mathrm{d}x + \int_{\mathbb{R}} \frac{f_2(t_0 v)v}{t_0}\, \mathrm{d}x.$$

Therefore, by using the monotonicity obtained above, we conclude that

$$0 \leq \|(u,v)\|_E^2 - 2\int_{\mathbb{R}} \lambda(x)uv\, \mathrm{d}x - \int_{\mathbb{R}} f_1(u)u\, \mathrm{d}x + \int_{\mathbb{R}} f_2(v)v\, \mathrm{d}x = \langle I'(u,v),(u,v)\rangle < 0,$$

which is a contradiction and the lemma is proved. ∎

4.5 Proof of Theorem 4.1.1

For $q > 2$ considered in (H_4), we define the constant

$$S_q = \inf_{(u,v)\in E\setminus\{(0,0)\}} S_q(u,v),$$

where

$$S_q(u,v) = \frac{\left(\|(u,v)\|_E^2 - 2\int_{\mathbb{R}} \lambda(x)uv\, \mathrm{d}x\right)^{1/2}}{\|(u,v)\|_q}, \quad \text{for } (u,v) \in E\setminus\{(0,0)\}.$$

Lemma 4.5.1. *Let ϑ and q be the constants introduced in (H_4).*

(a) The constant S_q is positive.

(b) For any $(u,v) \in E\setminus\{(0,0)\}$, we have

$$\max_{t\geq 0}\left(\frac{t^2}{2}S_q(u,v)^2\|(u,v)\|_q^2 - \vartheta t^q\|(u,v)\|_q^q\right) = \left(\frac{1}{2}-\frac{1}{q}\right)\frac{S_q(u,v)^{2q/(q-2)}}{(q\vartheta)^{2/(q-2)}}.$$

Proof. The proof is the same of Lemma 2.2.9 and we omitted here. ∎

By Ekeland's variational principle (see [38]), there exists a sequence $(u_n,v_n)_n \subset \mathcal{N}$ such that

$$I(u_n,v_n) \to c_{\mathcal{N}} \quad \text{and} \quad I'(u_n,v_n) \to 0. \tag{4.25}$$

Now we summarize some properties of $(u_n,v_n)_n$ which are useful to study our problem.

Lemma 4.5.2. *The minimizing sequence $(u_n,v_n)_n$ satisfies the following properties:*

(a) $(u_n,v_n)_n$ is bounded in E.

(b) $\limsup_{n\to+\infty} \|(u_n, v_n)\|_E^2 \leq \dfrac{1}{1-\delta}\dfrac{\mu}{\mu-2}\dfrac{q-2}{q}\dfrac{S_q^{2q/(q-2)}}{(q\vartheta)^{2/(q-2)}}$.

(c) $(u_n, v_n)_n$ *does not converge strongly to zero in* $L^m(\mathbb{R}) \times L^m(\mathbb{R})$*, for some* $m > 2$.

(d) There exists a sequence $(y_n)_n \subset \mathbb{R}$ *and constants* $\beta, R > 0$ *such that*

$$\liminf_{n\to+\infty} \int_{y_n-R}^{y_n+R} (u_n^2 + v_n^2)\, \mathrm{d}x \geq \beta > 0. \tag{4.26}$$

Proof. It follows from assumption (4.25) that

$$c_{\mathcal{N}} + o_n(1) = I(u_n, v_n) = \frac{1}{2}\left(\|(u_n, v_n)\|_E^2 - 2\int_{\mathbb{R}} \lambda(x) u_n v_n \, \mathrm{d}x\right) - \int_{\mathbb{R}} (F_1(u_n) + F_2(v_n))\, \mathrm{d}x.$$

Thus, by using (H_3), (4.21) and the fact that $(u_n, v_n)_n \subset \mathcal{N}$, we deduce that

$$c_{\mathcal{N}} + o_n(1) \geq \left(\frac{1}{2} - \frac{1}{\mu}\right)(1-\delta)\|(u_n, v_n)\|_E^2.$$

Therefore, $(u_n, v_n)_n$ is bounded in E. Moreover, the preceding estimate implies that

$$\limsup_{n\to\infty} \|(u_n, v_n)\|_E^2 \leq \frac{1}{1-\delta}\frac{2\mu}{\mu-2} c_{\mathcal{N}}. \tag{4.27}$$

To prove item (b), we have from (H_4) that

$$F_1(s) + F_2(t) \geq \vartheta(|s|^q + |t|^q), \quad \text{for all } \ s, t \in \mathbb{R}. \tag{4.28}$$

By using Lemma 4.4.3, for any $(w, z) \in E\backslash\{(0,0)\}$ there exists a unique $t_0 > 0$ such that $(t_0 w, t_0 z) \in \mathcal{N}$. Thus, since that $c_{\mathcal{N}} \leq I(t_0 w, t_0 z) \leq \max_{t\geq 0} I(tw, tz)$, we can use (4.28) to get

$$c_{\mathcal{N}} \leq \max_{t\geq 0}\left\{\frac{t^2}{2}\left(\|(w,z)\|^2 - 2\int_{\mathbb{R}} \lambda(x) wz\, \mathrm{d}x\right) - \vartheta t^q \|(w,z)\|_q^q\right\}.$$

Recalling the definition of $S_q(w, z)$ and using Lemma 4.5.1 (b), we conclude that

$$c_{\mathcal{N}} \leq \max_{t\geq 0}\left\{\frac{t^2}{2} S_q(\psi, \phi)^2 \|(w,z)\|_q^2 - \vartheta t^q \|(w,z)\|_q^q\right\} = \left(\frac{1}{2} - \frac{1}{q}\right)\frac{S_q(w,z)^{2q/(q-2)}}{(q\vartheta)^{2/(q-2)}}. \tag{4.29}$$

Combining (4.27), (4.29) and taking the infimum over $(w, z) \in E\backslash\{(0,0)\}$ we have that

$$\limsup_{n\to\infty} \|(u_n, v_n)\|_E^2 \leq \frac{1}{1-\delta}\frac{\mu}{\mu-2}\frac{q-2}{q}\frac{S_q^{2q/(q-2)}}{(q\vartheta)^{2/(q-2)}}.$$

Concerning (c), let $\alpha, \rho_0 > 0$ be such that $\alpha > \alpha_0$ and $0 < \alpha\rho_0^2 < \omega$. By using item (b), there exists $\vartheta_0 > 0$ such that

$$\kappa^{-1} \limsup_{n\to+\infty} \|(u_n, v_n)\|_E^2 \leq \rho_0^2, \quad \text{for } \vartheta > \vartheta_0.$$

By similar arguments used in the proof of Lemma 4.4.1, for given $p > 2$, $r > l > 1$, sufficiently close to 1, such that $r\alpha\rho_0^2 < \omega$ and a suitable $\varepsilon > 0$, we can deduce that

$$0 < (1-\delta-\varepsilon C_1)\rho^2 \leq (1-\delta-\varepsilon C_1)\|(u_n,v_n)\|_E^2 \leq C_2\|(u_n,v_n)\|_{pl'}^p,$$

where $1/l + 1/l' = 1$. Therefore, $(u_n, v_n)_n$ cannot converge to zero in $L^{pl'}(\mathbb{R})$.

Finally to prove item (d), we suppose by contradiction that (4.26) does not holds. Thus, for any $R > 0$, we have

$$\lim_{n\to+\infty} \sup_{y\in\mathbb{R}} \int_{y-R}^{y+R} (u_n^2 + v_n^2)\,\mathrm{d}x = 0.$$

By using Lemma 4.2.4, it follows that $(u_n, v_n) \to 0$ strongly in $L^p(\mathbb{R}) \times L^p(\mathbb{R})$ for any $p > 2$. In particular, for $pl' > 2$ contradicting item (c). ■

Proposition 4.5.3. *There exists a minimizing sequence which converges to a nontrivial weak limit.*

Proof. Let $(u_n, v_n)_n \subset E$ be the minimizing sequence satisfying (4.25). By Lemma 4.5.2 (a), $(u_n, v_n)_n$ is bounded in E. Thus, passing to a subsequence, we may assume that $(u_n, v_n) \rightharpoonup (u_0, v_0)$ weakly in E. Let us define the shift sequence $(\tilde{u}_n(x), \tilde{v}_n(x)) = (u_n(x+y_n), v_n(x+y_n))$. Notice that the sequence $(\tilde{u}_n, \tilde{v}_n)_n$ is also bounded in E which implies that, up to a subsequence, $(\tilde{u}_n, \tilde{v}_n) \rightharpoonup (\tilde{u}, \tilde{v})$ weakly in E. By using assumption (V_1), we can note that the energy functional is invariant by translations of the form $(u, v) \mapsto (u(\cdot - z), v(\cdot - z))$, with $z \in \mathbb{Z}$. Thus, by a careful computation we can deduce that

$$\|(\tilde{u}_n,\tilde{v}_n)\| = \|(u_n,v_n)\|, \quad I(\tilde{u}_n,\tilde{v}_n) = I(u_n,v_n) \to c_{\mathcal{N}} \quad \text{and} \quad I'(\tilde{u}_n,\tilde{v}_n) \to 0.$$

Therefore,

$$\lim_{n\to+\infty} \int_{-R}^{R} (\tilde{u}_n^2 + \tilde{v}_n^2)\,\mathrm{d}x = \lim_{n\to+\infty} \int_{y_n-R}^{y_n+R} (u_n^2 + v_n^2)\,\mathrm{d}x \geq \beta > 0,$$

which implies $(\tilde{u}, \tilde{v}) \neq (0,0)$. ■

For the sake of simplicity, we will keep the notation $(u_n, v_n)_n$ and (u_0, v_0). In order to prove that $(u_0, v_0) \in \mathcal{N}$, we will use the following Brezis-Lieb type lemma which has been proved by J.M. do Ó et al. [36, Lemma 2.6].

Lemma 4.5.4. *Let $(u_n)_n \subset H^{1/2}(\mathbb{R})$ be a sequence such that $u_n \rightharpoonup u$ weakly in $H^{1/2}(\mathbb{R})$ and $\|u_n\|_{1/2} < \rho_0$ with $\rho_0 > 0$ small. Then, as $n \to \infty$, we have*

$$\int_{\mathbb{R}} f(u_n)u_n\,\mathrm{d}x = \int_{\mathbb{R}} f(u_n-u)(u_n-u)\,\mathrm{d}x + \int_{\mathbb{R}} f(u)u\,\mathrm{d}x + o_n(1),$$

$$\int_{\mathbb{R}} F(u_n)\,\mathrm{d}x = \int_{\mathbb{R}} F(u_n-u)\,\mathrm{d}x + \int_{\mathbb{R}} F(u)\,\mathrm{d}x + o_n(1).$$

As consequence of Lemma 4.5.4, we have the following lemma:

Lemma 4.5.5. *If $w_n = u_n - u_0$ and $z_n = v_n - v_0$, then*

$$\langle I'(u_0, v_0), (u_0, v_0)\rangle + \liminf_{n\to+\infty} \langle I'(w_n, z_n), (w_n, z_n)\rangle = 0. \tag{4.30}$$

Therefore, either $\langle I'(u_0, v_0), (u_0, v_0)\rangle \le 0$ or $\liminf_{n\to+\infty}\langle I'(w_n, z_n), (w_n, z_n)\rangle < 0$.

Proof. By easy computations we can deduce that

$$\|u_n\|^2_{E_1} = \|w_n\|^2_{E_1} + \|u_0\|^2_{E_1} + 2\left(\int_{\mathbb{R}} (-\Delta)^{1/4} w_n (-\Delta)^{1/4} u_0 \, \mathrm{d}x + \int_{\mathbb{R}} V_1(x) w_n u_0 \, \mathrm{d}x\right),$$

$$\|v_n\|^2_{E_2} = \|z_n\|^2_{E_2} + \|v_0\|^2_{E_2} + 2\left(\int_{\mathbb{R}} (-\Delta)^{1/4} z_n (-\Delta)^{1/4} v_0 \, \mathrm{d}x + \int_{\mathbb{R}} V_2(x) z_n v_0 \, \mathrm{d}x\right).$$

Thus, since $(w_n, z_n) \rightharpoonup 0$ weakly in E, we have

$$\begin{aligned} \|(u_n, v_n)\|^2_E &= \|(w_n, z_n)\|^2_E + \|(u_0, v_0)\|^2_E + 2((w_n, z_n), (u_0, v_0))_E \\ &= \|(w_n, z_n)\|^2_E + \|(u_0, v_0)\|^2_E + o_n(1). \end{aligned} \tag{4.31}$$

Moreover, we have also that

$$\int_{\mathbb{R}} \lambda(x) w_n z_n \, \mathrm{d}x = \int_{\mathbb{R}} \lambda(x) u_n v_n \, \mathrm{d}x + \int_{\mathbb{R}} \lambda(x) u_0 v_0 \, \mathrm{d}x - \int_{\mathbb{R}} \lambda(x) u_n v_0 \, \mathrm{d}x - \int_{\mathbb{R}} \lambda(x) v_n u_0 \, \mathrm{d}x.$$

By the weak convergence we have the following convergences

$$\int_{\mathbb{R}} \lambda(x) v_0 u_n \, \mathrm{d}x \to \int_{\mathbb{R}} \lambda(x) v_0 u_0 \, \mathrm{d}x \quad \text{and} \quad \int_{\mathbb{R}} \lambda(x) u_0 v_n \, \mathrm{d}x \to \int_{\mathbb{R}} \lambda(x) v_0 u_0 \, \mathrm{d}x,$$

which yields

$$\int_{\mathbb{R}} \lambda(x) w_n z_n \, \mathrm{d}x = \int_{\mathbb{R}} \lambda(x) u_n v_n \, \mathrm{d}x - \int_{\mathbb{R}} \lambda(x) u_0 v_0 \, \mathrm{d}x + o_n(1). \tag{4.32}$$

By using Lemma 4.5.4, (4.31), (4.32) and the fact that $(u_n, v_n)_n \subset \mathcal{N}$, we conclude that

$$\liminf_{n\to+\infty} \langle I'(w_n, z_n), (w_n, z_n)\rangle = -\langle I'(u_0, v_0), (u_0, v_0)\rangle,$$

which completes the proof. ■

Proposition 4.5.6. *The weak limit (u_0, v_0) satisfies $\langle I'(u_0, v_0), (u_0, v_0)\rangle = 0$.*

Proof. We have divided the proof into two steps.

Step 1. $\langle I'(u_0, v_0), (u_0, v_0)\rangle \ge 0$.

Suppose by contradiction that $\langle I'(u_0, v_0), (u_0, v_0)\rangle < 0$. Thus, from Lemma 4.4.3, there exists $t_0 \in (0, 1)$ such that $(t_0 u_0, t_0 v_0) \in \mathcal{N}$. By using (4.9) and Fatou's lemma, we obtain

$$c_{\mathcal{N}} + o_n(1) = \frac{1}{2}\int_{\mathbb{R}} (\phi_1(u_n) + \phi_2(v_n)) \, \mathrm{d}x \ge \frac{1}{2}\int_{\mathbb{R}} (\phi_1(u_0) + \phi_2(v_0)) \, \mathrm{d}x + o_n(1).$$

Since $t_0 \in (0,1)$, it follows from (4.10) that

$$\frac{1}{2}\int_{\mathbb{R}}(\phi_1(u_0)+\phi_2(v_0))\ \mathrm{d}x + o_n(1) > \frac{1}{2}\int_{\mathbb{R}}(\phi_1(t_0u_0)+\phi_2(t_0v_0))\ \mathrm{d}x + o_n(1).$$

Combining these estimates and using the fact that $(t_0u_0, t_0v_0) \in \mathcal{N}$, we conclude that

$$c_{\mathcal{N}} + o_n(1) > I(t_0u_0,t_0v_0) - \frac{1}{2}\langle I'(t_0u_0,t_0v_0),(t_0u_0,t_0v_0)\rangle + o_n(1) = I(t_0u_0,t_0v_0) + o_n(1).$$

Hence, $I(t_0u_0,t_0v_0) < c_{\mathcal{N}}$, which is a contradiction. Therefore, $\langle I'(u_0,v_0),(u_0,v_0)\rangle \geq 0$.

Step 2. $\langle I'(u_0,v_0),(u_0,v_0)\rangle \leq 0$.

Suppose by contradiction, that $\langle I'(u_0,v_0),(u_0,v_0)\rangle > 0$. By Lemma 4.5.5, we have that

$$\liminf_{n\to+\infty}\langle I'(w_n,z_n),(w_n,z_n)\rangle < 0. \tag{4.33}$$

Thus, passing to a subsequence, we have $\langle I'(w_n,z_n),(w_n,z_n)\rangle < 0$, for $n \in \mathbb{N}$ sufficiently large. By Lemma 4.4.3, there exists a sequence $(t_n)_n \subset (0,1)$ such that $(t_nw_n, t_nz_n)_n \subset \mathcal{N}$. Passing to a subsequence, we may assume that $t_n \to t_0 \in (0,1]$. Arguing by contradiction, we suppose that $t_0 = 1$. Thus, it follows that

$$\|(w_n,z_n)\|_E^2 - 2\int_{\mathbb{R}}\lambda(x)w_nz_n\ \mathrm{d}x = \|(t_nw_n,t_nz_n)\|_E^2 - 2\int_{\mathbb{R}}\lambda(x)t_nw_nt_nz_n\ \mathrm{d}x + o_n(1). \tag{4.34}$$

If we prove the following convergences

$$\int_{\mathbb{R}} f_1(w_n)w_n\ \mathrm{d}x = \int_{\mathbb{R}} f_1(t_nw_n)t_nw_n\ \mathrm{d}x + o_n(1), \tag{4.35}$$

$$\int_{\mathbb{R}} f_2(z_n)z_n\ \mathrm{d}x = \int_{\mathbb{R}} f_2(t_nz_n)t_nz_n\ \mathrm{d}x + o_n(1), \tag{4.36}$$

then combining with (4.34) and the fact that $(t_nw_n,t_nz_n)_n \subset \mathcal{N}$ we conclude that

$$\langle I'(w_n,z_n),(w_n,z_n)\rangle = \langle I'(t_nw_n,t_nz_n),(t_nw_n,t_nz_n)\rangle + o_n(1) = o_n(1),$$

which contradicts (4.33). This contradiction implies that $t_0 \in (0,1)$. It remains to prove (4.35) and (4.36). For this purpose, for each $i = 1,2$ we apply the mean value theorem to the function $g_i(t) = f_i(t)t$. Thus, we get a sequence of functions $(\tau_n^i)_n \subset (0,1)$ such that

$$f_1(w_n)w_n - f_1(t_nw_n)t_nw_n = (f_1'(\sigma_n^i)\sigma_n^i + f_1(\sigma_n^i))w_n(1-t_n), \tag{4.37}$$

$$f_2(z_n)z_n - f_2(t_nz_n)t_nz_n = (f_2'(\sigma_n^i)\sigma_n^i + f_2(\sigma_n^i))z_n(1-t_n), \tag{4.38}$$

where $\sigma_n^1 = w_n + \tau_n^1 w_n(t_n-1)$ and $\sigma_n^2 = z_n + \tau_n^2 z_n(t_n-1)$. By using Lemma 4.5.2 (b), there exists $\vartheta_0 > 0$ such that $\kappa^{-1}\|(u_n,v_n)\|_E^2 \leq \rho_0^2$, for some $\alpha > \alpha_0$, $0 < \alpha\rho_0^2 < \omega$ and $\vartheta > \vartheta_0$. Since we have

$$\|u_n\|_{E_1}^2 = \|w_n\|_{E_1}^2 + \|u_0\|_{E_1}^2 + o_n(1),$$

it follows that $\kappa^{-1}\limsup_{n\to+\infty}\|w_n\|_{E_1}^2 \le \rho_0^2$. Thus, up to a subsequence, we get

$$\|\sigma_n^1\|_{E_1} = \|w_n + \tau_n^1 w_n(t_n-1)\|_{E_1} = |1-(1-t_n)\tau_n^1|\|w_n\|_{E_1} \le \kappa\rho_0,$$

for $n\in\mathbb{N}$ sufficiently large. We claim that

$$\sup_n \int_{\mathbb{R}} f_1(\sigma_n^1)w_n \,\mathrm{d}x < \infty \quad \text{and} \quad \sup_n \int_{\mathbb{R}} f_1'(\sigma_n^1)\sigma_n^1 w_n \,\mathrm{d}x < \infty, \tag{4.39}$$

$$\sup_n \int_{\mathbb{R}} f_2(\sigma_n^2)z_n \,\mathrm{d}x < \infty \quad \text{and} \quad \sup_n \int_{\mathbb{R}} f_2'(\sigma_n^2)\sigma_n^2 z_n \,\mathrm{d}x < \infty. \tag{4.40}$$

In fact, for $p>2$ it follows from (4.4), (4.11) and Hölder inequality that

$$\int_{\mathbb{R}} f_1(\sigma_n^1)w_n \,\mathrm{d}x \le C\|\sigma_n^1\|_{E_1}\|w_n\|_{E_1} + C\int_{\mathbb{R}} (e^{\alpha(\sigma_n^1)^2}-1)|\sigma_n^1|^{p-1}|w_n| \,\mathrm{d}x.$$

Consider $r>l>1$, sufficiently close to 1, such that $0<r\alpha\rho_0^2<\omega$. By using Sobolev embedding, Lemma 4.2.1, (4.19) and Hölder inequality we get

$$\begin{aligned}\int_{\mathbb{R}} (e^{\alpha(\sigma_n^1)^2}-1)|\sigma_n^1|^{p-1}|w_n| \,\mathrm{d}x &\le \left(\int_{\mathbb{R}} (e^{r\alpha(\sigma_n^1)^2}-1)\,\mathrm{d}x\right)^{1/l}\left(\int_{\mathbb{R}} |\sigma_n^1|^{l'(p-1)}|w_n|^{l'}\,\mathrm{d}x\right)^{1/l'}\\ &\le C\left(\int_{\mathbb{R}} |\sigma_n^1|^{2l'(p-1)}\,\mathrm{d}x\right)^{1/2l'}\left(\int_{\mathbb{R}} |w_n|^{2l'}\,\mathrm{d}x\right)^{1/2l'}\\ &\le C\|\sigma_n^1\|_{E_1}^{p-1}\|w_n\|_{E_1},\end{aligned}$$

where $1/l+1/l'=1$ and we have used the fact that $2l'(p-1)>2$. Therefore,

$$\int_{\mathbb{R}} f_1(\sigma_n^1)w_n \,\mathrm{d}x \le C\|\sigma_n^1\|_{E_1}\|w_n\|_{E_1} + C\|\sigma_n^1\|_{E_1}^{p-1}\|w_n\|_{E_1} \le C\rho_0^2 + C\rho_0^{p-1}\rho_0 < \infty.$$

By using (4.12) and similar computations we obtain

$$\int_{\mathbb{R}} f_1'(\sigma_n^1)\sigma_n^1 w_n \,\mathrm{d}x \le C\|\sigma_n^1\|_{E_1}\|w_n\|_{E_1} + C\|\sigma_n^1\|_{E_1}^{p-1}\|w_n\|_{E_1} < \infty.$$

Analogously we obtain (4.40) and the claim is proved. From (4.39) and (4.40) we conclude that

$$\sup_n \int_{\mathbb{R}} |f_1(\sigma_n^1)\sigma_n^1 + f_1(\sigma_n^1)||w_n| \,\mathrm{d}x < \infty \quad \text{and} \quad \sup_n \int_{\mathbb{R}} |f_2(\sigma_n^2)\sigma_n^2 + f_2(\sigma_n^2)||z_n| \,\mathrm{d}x < \infty. \tag{4.41}$$

Finally, combining (4.37), (4.38), (4.41) and $t_n\to 1$, we get (4.35) and (4.36).

The preceding arguments concluded that, up to a subsequence, $t_n\to t_0\in(0,1)$. By a similar argument used in the *Step 1*, we can deduce that

$$c_{\mathcal{N}} + o_n(1) = \frac{1}{2}\int_{\mathbb{R}} (\phi_1(u_n)+\phi_2(v_n))\,\mathrm{d}x \ge \frac{1}{2}\int_{\mathbb{R}} (\phi_1(t_nu_n)+\phi_2(t_nv_n))\,\mathrm{d}x. \tag{4.42}$$

Notice that $t_nu_n \rightharpoonup t_0u_0$ and $\kappa^{-1}\|t_nu_n\|_{E_1}^2 \le \rho_0^2$. Thus, by using Lemma 4.5.4 we have

$$\int_{\mathbb{R}} \phi_1(t_nu_n)\,\mathrm{d}x = \int_{\mathbb{R}} \phi_1(t_nu_n - t_0u_0)\,\mathrm{d}x + \int_{\mathbb{R}} \phi_1(t_0u_0)\,\mathrm{d}x. \tag{4.43}$$

Let us denote $\hat{t}_n = t_n - t_0 \to 0$. By the mean value theorem, there exists a sequence of functions $(\gamma_n)_n \subset (0,1)$ such that

$$\phi_1(t_n u_n - t_0 u_0) - \phi_1(t_n w_n) = \phi_1'((1-\gamma_n)(t_n u_n - t_0 u_0) + \gamma t_n w_n)\hat{t}_n u_0.$$

Notice that $t_n u_n - t_0 u_0 = t_n w_n + \hat{t}_n u_0$. Thus, it follows that

$$\phi_1(t_n u_n - t_0 u_0) - \phi_1(t_n w_n) = \phi_1'(\zeta_n)\hat{t}_n u_0, \tag{4.44}$$

where $\zeta_n = (1-\gamma_n)\hat{t}_n u_0 + t_n w_n$. Recalling that $\kappa^{-1}\|w_n\|_{E_1}^2 \le \rho_0^2$ we have

$$\|\zeta_n\|_{E_1} = \|(1-\gamma_n)\hat{t}_n u_0 + t_n w_n\|_{E_1} \le \hat{t}_n\|u_0\|_{E_1} + t_n\|w_n\|_{E_1} \le \rho_0,$$

for n sufficiently large. Repeating the same argument used to deduce (4.41), we get

$$\sup_n \int_{\mathbb{R}} |\phi_1'(\zeta_n)||u_0| \,\mathrm{d}x \le \sup_n \int_{\mathbb{R}} |f_1'(\zeta_n)\zeta_n + f_1(\zeta_n)||w_n| \,\mathrm{d}x < \infty. \tag{4.45}$$

By using (4.44), (4.45) and the fact that $\hat{t}_n \to 0$, we conclude that

$$\int_{\mathbb{R}} \phi_1(t_n u_n - t_0 u_0) \,\mathrm{d}x = \int_{\mathbb{R}} \phi_1(t_n w_n) \,\mathrm{d}x + o_n(1). \tag{4.46}$$

Since $t_n v_n \rightharpoonup t_0 v_0$ and $\kappa^{-1}\|t_n v_n\|_{E_2}^2 \le \rho_0^2$, we can check analogously that

$$\int_{\mathbb{R}} \phi_2(t_n v_n) \,\mathrm{d}x = \int_{\mathbb{R}} \phi_2(t_n v_n - t_0 v_0) \,\mathrm{d}x + \int_{\mathbb{R}} \phi_2(t_0 v_0) \,\mathrm{d}x, \tag{4.47}$$

$$\int_{\mathbb{R}} \phi_2(t_n v_n - t_0 v_0) \,\mathrm{d}x = \int_{\mathbb{R}} \phi_2(t_n z_n) \,\mathrm{d}x + o_n(1). \tag{4.48}$$

Therefore, by using (4.42), (4.43), (4.46), (4.47), (4.48) and the fact that $(t_n w_n, t_n z_n) \in \mathcal{N}$, we have that

$$\begin{aligned}
c_{\mathcal{N}} + o_n(1) &\ge \frac{1}{2}\int_{\mathbb{R}} (\phi_1(t_n u_n) + \phi_2(t_n v_n)) \,\mathrm{d}x \\
&= \frac{1}{2}\int_{\mathbb{R}} (\phi_1(t_n u_n - t_0 u_0) + \phi_2(t_n v_n - t_0 v_0)) \,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}} (\phi_1(t_0 u_0) + \phi_2(t_0 v_0)) \,\mathrm{d}x \\
&= \frac{1}{2}\int_{\mathbb{R}} (\phi_1(t_n w_n) + \phi_2(t_n z_n)) \,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}} (\phi_1(t_0 u_0) + \phi_2(t_0 v_0)) \,\mathrm{d}x + o_n(1),
\end{aligned}$$

which implies that

$$c_{\mathcal{N}} + o_n(1) \ge I(t_n w_n, t_n z_n) + \frac{1}{2}\int_{\mathbb{R}} (\phi_1(t_0 u_0) + \phi_2(t_0 v_0)) \,\mathrm{d}x + o_n(1).$$

Since $(u_0, v_0) \neq (0,0)$, it follows from (4.9) that

$$\frac{1}{2}\int_{\mathbb{R}} (\phi_1(t_0 u_0) + \phi_2(t_0 v_0)) \,\mathrm{d}x > 0,$$

which jointly with (4.49) implies that $I(t_n w_n, t_n z_n) < c_{\mathcal{N}}$ for n large, contradicting the definition of $c_{\mathcal{N}}$. Therefore, $\langle I'(u_0, v_0), (u_0, v_0)\rangle = 0$ and the proof is complete. ■

Proof of Theorem 4.1.1 completed. Finally, we will conclude that (u_0, v_0) is in fact a ground state solution for System (S), even though we do not know if (u_n, v_n) converges strongly in E. By Propositions 4.5.3 and 4.5.6, we have that $(u_0, v_0) \in \mathcal{N}$. Thus, $c_{\mathcal{N}} \leq I(u_0, v_0)$. On the other hand, by using (4.9) and similar arguments as used before, we deduce that

$$c_{\mathcal{N}}+o_n(1) = \frac{1}{2}\int_{\mathbb{R}}(\phi_1(u_n)+\phi_2(v_n))\,\mathrm{d}x \geq \frac{1}{2}\int_{\mathbb{R}}(\phi_1(u_0)+\phi_2(v_0))\,\mathrm{d}x+o_n(1) = I(u_0,v_0)+o_n(1),$$

which implies that $c_{\mathcal{N}} \geq I(u_0, v_0)$. Therefore $I(u_0, v_0) = c_{\mathcal{N}}$ and jointly with Remark 4.4.2 implies that (u_0, v_0) is a ground state solution for System (S).

In order to get a nonnegative ground state, we note that $I(|u_0|, |v_0|) \leq I(u_0, v_0)$. Moreover, by using Lemma 4.4.3, there exists $t_0 > 0$, depending on $(|u_0|, |v_0|)$, such that $(t_0|u_0|, t_0|v_0|) \in \mathcal{N}$. Since $(u_0, v_0) \in \mathcal{N}$, we have also from Lemma 4.4.3 that $\max_{t\geq 0} I(tu_0, tv_0) = I(u_0, v_0)$. Hence,

$$I(t_0|u_0|, t_0|v_0|) \leq I(t_0u_0, t_0v_0) \leq \max_{t\geq 0} I(tu_0, tv_0) = I(u_0, v_0) = c_{\mathcal{N}}.$$

Therefore, $(t_0|u_0|, t_0|v_0|) \in \mathcal{N}$ is a nonnegative ground state solution for System (S) which finishes the proof of Theorem 4.1.1. ■

Remark 4.5.7. *Let $\mathcal{K}$ be the set of all ground state solutions for System (S), that is,*

$$\mathcal{K} := \{(u, v) \in E : (u, v) \in \mathcal{N},\ I(u, v) = c_{\mathcal{N}} \text{ and } I'(u, v) = 0\}.$$

Let $(u_n, v_n)_n \subset \mathcal{K}$ be a bounded sequence. Thus, up to a subsequence, we may assume $(u_n, v_n) \rightharpoonup (u, v)$ weakly in E. Proceeding analogously to the proof of Proposition 4.5.3, we can conclude that there exists a sequence $(y_n)_n \subset \mathbb{Z}$ and constants $R, \xi > 0$ such that

$$\liminf_{n\to\infty} \int_{B_R(y_n)} (u_n^2 + v_n^2)\,\mathrm{d}x \geq \xi > 0.$$

Using the invariance of I, we may conclude that $(u, v) \neq 0$. Repeating the same arguments used in the proof of Proposition 4.5.6, we deduce that $(u, v) \in \mathcal{N}$. As before, we see also that $I(u, v) = c_{\mathcal{N}}$. Thus, using ($H_3$), the weakly lower semi-continuity of the norm and Fatou's lemma, we have

$$\begin{aligned} c_{\mathcal{N}} + o_n(1) &= I(u_n, v_n) - \frac{1}{\mu}\langle I'(u_n, v_n), (u_n, v_n)\rangle \\ &\geq I(u, v) - \frac{1}{\mu}\langle I'(u, v), (u, v)\rangle + o_n(1) \\ &= c_{\mathcal{N}} + o_n(1). \end{aligned}$$

Thus, $\|(u_n, v_n)\| \to \|(u, v)\|$, which implies that $(u_n, v_n) \to (u, v)$ strongly in E. Therefore, $\mathcal{K}$ is a compact set in E.

4.6 Proof of Theorem 4.1.2

In this section we will be concerned with the existence of ground states for the asymptotically periodic case. The idea is the same used in the proof of Theorem 1.5. We emphasize that the only difference between the potentials $V_i(x)$, $\lambda(x)$ and $\tilde{V}_i(x)$, $\tilde{\lambda}(x)$ is the periodicity by translations required to $V_i(x)$ and $\lambda(x)$. Thus, if $\tilde{V}_i(x)$ and $\tilde{\lambda}(x)$ are periodic potentials, we can make use of Theorem 4.1.1 to get a ground state solution for System ($\tilde{S}$). Let us suppose that they are not periodic.

Associated to System ($\tilde{S}$), we have the following energy functional

$$\tilde{I}(u,v) = \frac{1}{2}\left(\|(u,v)\|_{\tilde{E}}^2 - 2\int_{\mathbb{R}} \tilde{\lambda}(x)uv \, \mathrm{d}x\right) - \int_{\mathbb{R}} \left(F_1(u) + F_2(v)\right) \, \mathrm{d}x.$$

The Nehari manifold for System ($\tilde{S}$) is defined by

$$\tilde{\mathcal{N}} = \{(u,v) \in \tilde{E}\backslash\{(0,0)\} : \langle \tilde{I}(u,v), (u,v)\rangle\},$$

and the ground state energy associated $c_{\tilde{\mathcal{N}}} = \inf_{\tilde{\mathcal{N}}} \tilde{I}(u,v)$. Similarly to Section 4.4, for any $(u,v) \in \tilde{\mathcal{N}}$, we can deduce that

$$\tilde{I}(u,v) \geq \left(\frac{1}{2} - \frac{1}{\mu}\right)(1-\delta)\|(u,v)\|_{\tilde{E}}^2 \geq \left(\frac{1}{2} - \frac{1}{\mu}\right)(1-\delta)\rho > 0.$$

Hence, $c_{\tilde{\mathcal{N}}} > 0$. The next step is to establish a relation between the levels $c_{\mathcal{N}}$ and $c_{\tilde{\mathcal{N}}}$.

Lemma 4.6.1. *Assume the hypotheses of Theorem 4.1.2. Then* $c_{\tilde{\mathcal{N}}} < c_{\mathcal{N}}$.

Proof. The proof is quite similar to Lemma 1.5.1 and we omitted here. ■

As in the proof of Theorem 4.1.1, there exists a sequence $(u_n, v_n)_n \subset \mathcal{N}$ such that

$$\tilde{I}(u_n, v_n) \to c_{\tilde{\mathcal{N}}} \quad \text{and} \quad \tilde{I}'(u_n, v_n) \to 0. \tag{4.49}$$

Notice that in the proof of Theorem 4.1.1 the only step we used the periodicity of the potentials was to guarantee that a minimizing sequence converges to a nontrivial limit (see Proposition 4.5.3). Thus, Lemma 4.5.2 remains true for the minimizing sequence obtained above to the asymptotically periodic case. Since $(u_n, v_n)_n$ is a bounded sequence in $\tilde{E}$, we may assume up to a subsequence that $(u_n, v_n) \rightharpoonup (u_0, v_0)$ weakly in $\tilde{E}$. The main difficulty is to prove that the weak limit is nontrivial.

Proposition 4.6.2. *The weak limit* (u_0, v_0) *of the minimizing sequence* $(u_n, v_n)_n$ *is nontrivial.*

Proof. Arguing by contradiction, we suppose that $(u_0, v_0) = (0, 0)$. We may assume that

- $u_n \to 0$ and $v_n \to 0$ strongly in $L^p_{loc}(\mathbb{R})$, for all $2 \le p < \infty$;
- $u_n(x) \to 0$ and $v_n(x) \to 0$ almost everywhere in $\mathbb{R}$.

It follows from (V_4) that for any $\varepsilon > 0$ there exists $R > 0$ such that

$$|V_1(x) - \tilde{V}_1(x)| < \varepsilon, \quad |V_2(x) - \tilde{V}_2(x)| < \varepsilon, \quad |\tilde{\lambda}(x) - \lambda(x)| < \varepsilon, \quad \text{for } |x| \ge R. \tag{4.50}$$

By the same idea used to get (1.12) we can deduce that

$$I(u_n, v_n) = c_{\tilde{\mathcal{N}}} + o_n(1) \quad \text{and} \quad \langle I'(u_n, v_n), (u_n, v_n)\rangle = o_n(1). \tag{4.51}$$

Using Lemma 4.4.3, there exists $(t_n)_n \subset (0, +\infty)$ such that $(t_n u_n, t_n v_n)_n \subset \mathcal{N}$.

Claim 1. $\limsup_{n\to+\infty} t_n \le 1$.

In fact, we suppose by contradiction that there exists $\varepsilon_0 > 0$ such that, up to a subsequence, we have $t_n \ge 1 + \varepsilon_0$, for all $n \in \mathbb{N}$. Combining (4.51) and the fact that $(t_n u_n, t_n v_n) \subset \mathcal{N}$, we can deduce that

$$\int_{\mathbb{R}} \left(\frac{f_1(t_n u_n)u_n}{t_n} - f_1(u_n)u_n \right) \mathrm{d}x + \int_{\mathbb{R}} \left(\frac{f_2(t_n v_n)v_n}{t_n} - f_2(v_n)v_n \right) \mathrm{d}x = o_n(1).$$

By using (4.7) (see (4.24)) and the fact that $t_n \ge 1 + \varepsilon_0$, we have that

$$\int_{\mathbb{R}} \left(\frac{f_1((1+\varepsilon_0)u_n)u_n}{1+\varepsilon_0} - f_1(u_n)u_n \right) \mathrm{d}x + \int_{\mathbb{R}} \left(\frac{f_2((1+\varepsilon_0)v_n)v_n}{1+\varepsilon_0} - f_2(v_n)v_n \right) \mathrm{d}x = o_n(1). \tag{4.52}$$

Arguing similar to the proof of Proposition 4.5.3 we consider the shift sequence $(\tilde{u}_n(x), \tilde{v}_n(x)) = (u_n(x + y_n), v_n(x + y_n))$. The sequence $(\tilde{u}_n(x), \tilde{v}_n(x))$ is bounded in $\tilde{E}$ and, up to a subsequence, $(\tilde{u}_n(x), \tilde{v}_n(x)) \rightharpoonup (\tilde{u}, \tilde{v})$. Therefore,

$$\lim_{n\to+\infty} \int_{-R}^{R} (\tilde{u}_n^2 + \tilde{v}_n^2)\, \mathrm{d}x = \lim_{n\to+\infty} \int_{y_n-R}^{y_n+R} (u_n^2 + v_n^2)\, \mathrm{d}x \ge \beta > 0,$$

which implies $(\tilde{u}, \tilde{v}) \ne (0, 0)$. Thus, by using (4.7), (4.52) and Fatou's lemma, we conclude that

$$0 < \int_{\mathbb{R}} \left(\frac{f_1((1+\varepsilon_0)\tilde{u})\tilde{u}}{1+\varepsilon_0} - f_1(\tilde{u})\tilde{u} \right) \mathrm{d}x + \int_{\mathbb{R}} \left(\frac{f_2((1+\varepsilon_0)\tilde{v})\tilde{v}}{1+\varepsilon_0} - f_2(\tilde{v})\tilde{v} \right) \mathrm{d}x = o_n(1),$$

which is not possible and finishes the proof of *Claim 1.*

Claim 2. There exists $n_0 \in \mathbb{N}$ such that $t_n \ge 1$, for $n \ge n_0$.

In fact, arguing by contradiction, we suppose that up to a subsequence, $t_n < 1$. By using (4.10) and the fact that $(t_n u_n, t_n v_n)_n \subset \mathcal{N}$ we have

$$c_{\mathcal{N}} \leq \frac{1}{2}\int_{\mathbb{R}} (\phi_1(t_n u_n) + \phi_2(t_n v_n))\,\mathrm{d}x \leq \frac{1}{2}\int_{\mathbb{R}} (\phi_1(u_n) + \phi_2(v_n))\,\mathrm{d}x = c_{\tilde{\mathcal{N}}} + o_n(1).$$

Therefore, $c_{\mathcal{N}} \leq c_{\tilde{\mathcal{N}}}$ which contradicts Lemma 4.6.1 and finishes the proof of *Claim 2*.

Combining *Claims* 1 and 2, we can deduce that

$$\int_{\mathbb{R}} (F_1(t_n u_n) - F_1(u_n) + F_2(t_n v_n) - F_2(v_n))\,\mathrm{d}x = \int_1^{t_n}\int_{\mathbb{R}} (f_1(\tau u_n)u_n + f_2(\tau v_n)v_n)\,\mathrm{d}x\mathrm{d}\tau = o_n(1).$$

Moreover, we have that

$$\frac{t_n^2 - 1}{2}\left(\|(u_n, v_n)\|_E^2 - 2\int_{\mathbb{R}} \lambda(x) u_n v_n\,\mathrm{d}x\right) = o_n(1).$$

These convergences imply that $I(t_n u_n, t_n v_n) - I(u_n, v_n) = o_n(1)$. Thus, it follows from (4.51) that

$$c_{\mathcal{N}} \leq I(t_n u_n, t_n v_n) = I(u_n, v_n) + o_n(1) = c_{\tilde{\mathcal{N}}} + o_n(1),$$

which contradicts Lemma 4.6.1. Therefore, $(u_0, v_0) \neq (0,0)$ and the proposition is proved. ■

Proof of Theorem 4.1.2 completed. We point out that we did not use the periodicity on the potentials $V_i(x)$ and $\lambda(x)$ to prove Proposition 4.5.6. Thus, since $(u_0, v_0) \neq (0,0)$, we can repeat the same proof to conclude that $(u_0, v_0) \in \tilde{\mathcal{N}}$. Therefore, we have $c_{\tilde{\mathcal{N}}} \leq I(u_0, v_0)$. On the other hand, by using (4.9) and similar arguments as used before, we deduce that

$$\begin{aligned} c_{\tilde{\mathcal{N}}} + o_n(1) &= \frac{1}{2}\int_{\mathbb{R}} (\phi_1(u_n) + \phi_2(v_n))\,\mathrm{d}x \\ &\geq \frac{1}{2}\int_{\mathbb{R}} (\phi_1(u_0) + \phi_2(v_0))\,\mathrm{d}x + o_n(1) \\ &= \tilde{I}(u_0, v_0) + o_n(1), \end{aligned}$$

which implies that $c_{\tilde{\mathcal{N}}} \geq I(u_0, v_0)$. Therefore $\tilde{I}(u_0, v_0) = c_{\mathcal{N}}$. Repeating the same argument used in the proof of Theorem 4.1.1, we can deduce that there exists $t_0 > 0$ such that $(t_0|u_0|, t_0|v_0|) \in \tilde{\mathcal{N}}$ is a ground state solution for System ($\tilde{S}$) which finishes the proof of Theorem 4.1.2. ■

Remark 4.6.3. *Let $\tilde{\mathcal{K}}$ be the set of all ground state solutions for System ($\tilde{S}$), that is,*

$$\tilde{\mathcal{K}} := \{(u,v) \in \tilde{E} : (u,v) \in \tilde{\mathcal{N}},\ \tilde{I}(u,v) = c_{\tilde{\mathcal{N}}} \text{ and } \tilde{I}'(u,v) = 0\}.$$

Using Proposition 4.40 instead Proposition 4.5.3, we can apply a similar argument used in Remark 4.6.2 , with I replaced by $\tilde{I}$, to conclude that $\tilde{\mathcal{K}}$ is a compact set in $\tilde{E}$.

Remark 4.6.4. *The main goal of this chapter was to prove the existence of ground states for Systems* (S) *and* ($\tilde{S}$), *when the constant* ϑ *introduced in* (H_4) *is large enough. In the lemma 4.4.1, we proved that the norm of any element that belongs to the Nehari manifold is greater or equal to a positive constant* ρ, *which is strictly less than* $\kappa\omega/\alpha_0$. *However, we note by Lemma 4.5.2 (b) that the norm of the minimizing sequence is so small as we want, and it is controlled by the choice of* ϑ. *Thus, our proof holds for any* ϑ *contained in a bounded interval of the real line. Let us consider, for instance,*

$$\vartheta^* := \sup\{\vartheta \in \mathbb{R} : (S) \text{ has ground states}\}.$$

Naturally, it arises the following questions: ϑ^* *is finite? If* ϑ^* *is finite, then there exists ground states at* $\vartheta = \vartheta^*$*?*

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