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EUDES MENDES BARBOZA

Hénon type equations with nonlinearities in the critical growth range

RECIFE

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Tese apresentada ao Programa de Pós-graduação em Matemática da Universidade Federal de Pernambuco como requisito parcial para obtenção do título de Doutor em Matemática.

Orientador: Prof. Dr. **JOÃO MARCOS BEZERRA DO Ó**

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Muito obrigado.

Abstract

In this work, using variational methods we have investigated the existence of solutions for some Hénon type equations, which are characterized by the presence of the weight $|x|^\alpha$ in the nonlinearity with $\alpha > 0$. When we are working in the radial context, this characteristic modifies the critical growth of the nonlinearities in some senses. This fact allows us to study some well-known problems under new perspectives. For this purpose, we have considered three different classes of problems with critical nonlinearity which presents the weight of Hénon. Firstly, we have studied the class of problem with a Trudinger-Moser nonlinearity in critical range in $\mathbb{R}^2$. In the subcritical case, there was no difference if we have looked for weak solutions in $H_0^1(B_1)$ or in $H_{0,\text{rad}}^1(B_1)$. Nevertheless, in the critical case we have needed to adapt some hypotheses when we have changed the space where we were seeking the solutions. For the second problem, we have kept working with exponential nonlinearity in $\mathbb{R}^2$, but we were treating an Ambrosseti-Prodi problem for which we have searched two weak solutions. In the subcritical case, analogously to first problem, the radially symmetric solutions were obtained as the solutions in $H_0^1(B_1)$, what have not happened in the critical case. Thus, again some assumptions have had to depend on the context where we were searching for the solutions. Lastly, we have studied a natural version of the second problem with the nonlinearity involving critical Sobolev growth in $\mathbb{R}^N (N \geq 3)$. In this last problem, we have searched the existence of solutions only in the radial critical case because the others cases were almost identical to problems with nonlinearities without the weight of Hénon.

Keywords: Partial differential equations. Hénon type equations. Nonlinearity in critical growth range. Variational methods.

Resumo

Neste trabalho, utilizando métodos variacionais investigamos a existência de soluções para algumas equações do tipo Hénon, que são caracterizadas pela presença do peso $|x|^\alpha$ na não-linearidade com $\alpha > 0$. Quando estamos trabalhando no contexto radial, essa característica modifica o crescimento crítico das não-linearidades em alguns sentidos. Este fato nos permite estudar problemas bem conhecidos sob novas perspectivas. Com este propósito, consideramos três classes diferentes de problemas com uma não-linearidade que apresenta o peso de Hénon. Em primeiro lugar, estudamos a classe de problema envolvendo uma não-linearidade do tipo Trudinger-Moser com imagem crítica em $\mathbb{R}^2$. No caso subcrítico, não houve diferença se procuramos soluções fracas em $H_0^1(B_1)$ ou em $H^1_{0,\mathrm{rad}}(B_1)$. No entanto, no caso crítico, precisamos adaptar algumas hipóteses quando mudamos o espaço onde buscávamos as soluções. Para o segundo problema, continuamos trabalhando com uma não-linearidade exponencial em $\mathbb{R}^2$, mas desta vez tratando de um problema do tipo Ambrosseti-Prodi para o qual buscamos duas soluções fracas. No caso subcrítico, analogamente ao primeiro problema, as soluções radialmente simétricas foram obtidas do mesmo modo das soluções em $H_0^1(B_1)$, o que não aconteceu no caso crítico. Assim, algumas hipótese novamente tiveram que depender do contexto em que buscávamos as soluções. Por fim, estudamos uma versão natural do segundo problema com a não-linearidade envolvendo o crescimento crítico do tipo Sobolev em $\mathbb{R}^N (N \geq 3)$. Neste último problema, pesquisamos a existência apenas de soluções radiais no caso crítico porque os outros casos eram quase idênticos a problemas com não-linearidades sem o peso de Hénon.

Palavras-chave: Equações diferenciais parciais. Equações tipo Hénon. Não-linearidades com imagem crítica. Métodos variacionais.

ILLUSTRATIONS LIST

SYMBOL LIST

$C, C_1, C_2, \ldots$	Positive constants.
$\mathbb{R}^N$	N-dimensional Euclidian space, where $x = (x_1, ..., x_n)$, $x_i \in \mathbb{R}, i = 1, ..., n$.
$\|x\|$	$\sqrt{x_1^2 + ... + x_N^2}$.
$B_R(x) = B(x, R)$	Open ball of Euclidian space $\mathbb{R}^N$ centered at x with radius $R > 0$, i.e. , $B_R(x) = \{y \in \mathbb{R}^n; \|x - y\| < R\}$.
B_R	Open ball of Euclidian space $\mathbb{R}^N$ centered at origin.
$\overline{B_R(x)} = \overline{B(x, R)}$	Closet ball of Euclidian space $\mathbb{R}^N$ centered at x with radius $R > 0$, i.e. , $B_R(x) = \{y \in \mathbb{R}^n; \|x - y\| \leq R\}$.
$\overline{B_R}$	Closet ball of Euclidian space $\mathbb{R}^N$ centered at origin.
$\partial B_R(x)$	Circumference of $\mathbb{R}^n$ centered at x with radius $R > 0$, i.e., $\partial B_R(x) = \{y \in \mathbb{R}^n; \|y - x\| = R\}$.
$u_+(x)$	$\max\{u(x), 0\}$.
$\mathrm{supp} u$	The closure of $\{x$ in domain of $u; u(x) \neq 0\}$.
$\frac{\partial u}{\partial x_i} = u_{x_i}$	i^{th} partial derivative of function u.
∇u	The vector $\left(\frac{\partial u}{\partial x_1}, ..., \frac{\partial u}{\partial x_N}\right)$.
Δu	$\sum_{i=1}^{N} \frac{\partial^2 u}{\partial x_i^2}$.

$u(x) = o(g(x))$	$\lim\limits_{x \to x_0} \dfrac{\lvert u(x)\rvert}{\lvert g(x)\rvert} = 0.$
$u(x) = O(g(x))$	$\lim\limits_{x \to x_0} \dfrac{\lvert u(x)\rvert}{\lvert g(x)\rvert} \leq C$ for $C > 0$.
$C_c^\infty(\Omega)$	The set of C^∞-function with compact support.
$L^q(\Omega)$	$\{u : \Omega \to \mathbb{R}; \int_\Omega \lvert u\rvert^q < \infty\}.$
$L^1_{loc}(\Omega)$	The set of functions which are integrables on every compact subset of its domain of definition.
$L^\infty(\Omega)$	$\{u : \Omega \to \mathbb{R}; \mathrm{supess}\lvert u\rvert < \infty\}.$
$C^0(\Omega)$	The set of continuous functions in Ω.
$C^k(\Omega)$	The set of functions in Ω which has continuos partial derivative in Ω of order less than or equal to k .
$C^\infty(\Omega)$	The set of functions in Ω which has continuos partial derivative in Ω of all orders $k \in \mathbb{N}$.
$W^{k,p}(\Omega)$	Space of functiosn weakly k-differenciable and p-integrable.
$W_0^{k,p}$	The closure of $C_c^\infty(\Omega)$ in $W^{k,p}(\Omega)$.
$H_0^1(\Omega)$	$W_0^{1,2}(\Omega)$.
$H^1_{0,\mathrm{rad}}(\Omega)$	The set of radially symmetric functions which belong to $H_0^1(\Omega)$.

$H^\perp$	The orthogonal subspace to space H.
$H \oplus H^\perp$	Direct sum of subspaces H and $H^\perp$.
$\dim E$	Dimention of vectorial espace E.
$u_n \to u$	The sequence (u_n) converges to u.
$u_n \rightharpoonup u$	The sequence (u_n) converges weakly to u.
$e^t = \exp(t)$	Exponential function.
$O(N)$	Space of rotations in $\mathbb{R}^N$.

Contents

1 Introduction

This work has as main theme the study of Hénon type Problems with nonlinearity in critical range. The Hénon equations arose in 1973, when M. Hénon [33], using the concentric shell model in order to investigate numerically the stability of spherical stellar system steady state with respect to spherical disturbances, studied the following problem

$$\begin{cases} \Delta u = & |x|^{\alpha} u^{p-1}, \quad \text{and} \quad u > 0 \quad \text{in } B_1; \\ u = & 0 \qquad\qquad\qquad\qquad \text{on } \partial B_1, \end{cases} \tag{1}$$

where B_1 is the unit ball centered at the origin in $\mathbb{R}^N (N \geq 3)$, $p > 1$ and the power α is positive (HÉNON, 1973).

Since $H_0^1(B_1)$ is compactly embedded in $L^q(B_1)$ for $1 \leq q < 2^* = 2N/(N-2)$, we can use standard arguments of Nonlinear Analysis in the search of solution for (1). However it is known that the presence of the weight $|x|^{\alpha}$ modifies, in some sense, the global homogeneity properties of the equation. Indeed, for instance, we notice that the usual arguments based on the Pohozaev identity yield that Problem (1) has no solution if $p \geq 2_{\alpha}^* = 2(N+\alpha)/(N-2)$ (see [27, 30](FIGUEIREDO; LION, 1982; FIGUEIREDO et al, 2008)).

Still on these topics, W. Ni [41] used the power-like decay away from the origin of radially symmetric functions of $H_0^1(B_1)$ (Strauss Lemma [3, 54] AMBROSETTI; MALCHIODI, 2007; STRAUSS, 1977) together with the weight $|x|^{\alpha}$ in order to prove the compactness of the embedding $H_{0,\text{rad}}^1(B_1)$ in $L^q(B_1, |x|^{\alpha})$ for any $1 \leq q < 2_{\alpha}^*$. Due to this fact, he obtained via the Mountain–Pass argument that (1) admits at least one radially symmetric solution for every $2 < p < 2_{\alpha}^*$ (NI, 1982).

Although the initial study of this kind of equation is only numerical, the paper of W. Ni motivated subsequent researches which showed that the above problem exhibits very rich features from the functional-analytic point of view, see for example [7, 8, 13, 17, 18, 19, 30, 35, 36, 37, 43, 50, 51, 53, 56](BADIALE; SERRA, 2004;

BARUELLO; SECCHI, 2008; BONHEURE et al, 2008; CALANCHI; TERRANEO, 2005; CARRIÃO et al, 2009; CHEN, 2000; FIGUEIREDO et al, 2008; LONG; YANG, 2012; LONG; YANG, 2010; LI; PENG, 2009; PENG, 2006; SECCHI; SERRA, 2006; SERRA, 2005; SMETS et al, 2002; TARSI; 2008).

In particular, various questions that arise quite naturally concerning existence, multiplicity and qualitative properties of solutions have given the Hénon type equation the role of a very interesting item in nonlinear analysis and critical point theory.

Considering, more specifically, the existence of solution for Hénon equations, we can emphasize that D. Smets, J. Su and M. Willem [53] proved that for any $p \in (2, 2^*)$ if $N \geq 3$ and for all $p \in (2, \infty)$ if $N = 2$, there exists at least a non-radial solution of the Hénon problem provided that α is large enough. From this fact together with the result of W. Ni mentioned above, we can conclude that (1) admits the coexistence of radial and non-radial positive solutions for $N \geq 3$ and larger α (SMETS et al, 2002). In [51], E. Serra showed that (1) also possesses a non-radial solution for $N \geq 3$ and p critical, that is, $p = 2^*$ (SERRA, 2006). In this direction, considering the critical Sobolev exponent, W. Long and J. Yang [36] showed the existence of a positive solution for the following Hénon type equation involving Sobolev critical growth

$$\begin{cases} \Delta u = & \lambda u + |x|^\alpha |u|^{2^*-1} \quad \text{in} \quad \Omega; \\ \quad u = & 0 \quad\quad\quad\quad\quad\quad \text{on} \quad \partial\Omega, \end{cases} \tag{2}$$

for $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain, $N \geq 4, \lambda < \lambda_1$ and $\alpha > 0$ is small enough (LONG; YANG, 2010). In [35], for the same nonlinearity they proved the identical result for $\lambda > \lambda_1$ and $N \geq 7$ (LONG; YANG, 2012).

If we focus our attention to the plane, due to the well-known Trudinger-Moser inequality, we can say, with relevant differences, that the growth $e^{4\pi t^2}$ in $\mathbb{R}^2$ corresponds to the critical growth $|t|^{2^*}$ in $\mathbb{R}^N$ for $N \geq 3$. This can be made precise by introducing the class of Orlicz spaces, see [40, 57] (MOSER, 1971; TRUDINGER, 1967). This inequality can be summarized as follows

$$\sup_{\substack{u \in H_0^1(B_1) \\ \|\nabla u\|_2 = 1}} \int_{B_1} e^{\beta u^2} \, \mathrm{d}x \begin{cases} < +\infty & \text{if} \quad \beta \leq 4\pi; \\ = +\infty & \text{if} \quad \beta > 4\pi. \end{cases} \tag{3}$$

The embedding in the sense of the Trudinger-Moser inequality for the Sobolev space of radial functions $H^1_{0,\text{rad}}(B_1)$ was considered by (CALANCHI; TERRANEO, 2005) [17] (see also [13] BONHEURE et al, 2008). In this case the behavior of the weight $|x|^\alpha$ at zero produces a higher embedding than the usual case (3), precisely,

$$\sup_{\substack{u\in H^1_{0,\mathrm{rad}}(B_1)\\ \|\nabla u\|_2=1}} \int_{B_1} |x|^\alpha e^{\beta u^2}\,\mathrm{d}x \begin{cases} <+\infty & \text{if} \quad \beta \le 2\pi(2+\alpha);\\ =+\infty & \text{if} \quad \beta > 2\pi(2+\alpha). \end{cases} \tag{4}$$

Based on the fact, M. Calanchi and E. Terraneo, proved the existence of a non-radial solution, analogously to [53] (SMETS et al, 2002), for the following Hénon type equation in $\mathbb{R}^2$ with exponential growth nonlinearity

$$\begin{cases} \Delta u = & |x|^\alpha (e^{p|u|^\gamma} - 1 - p|u|^\gamma) \quad \text{in } B_1(0) \subset \mathbb{R}^2;\\ \quad u = & 0 \qquad\qquad\qquad\qquad\qquad \text{on } \partial B_1(0), \end{cases}$$

where $\alpha > 0$ and $p > 0, 1 < \gamma < 2$ or $0 < p < 4\pi, \gamma = 2$. After this work, some attention has been given to the study of this type of equation with Trudinger-Moser nonlinearity, as we can see in [13, 56] (BONHEURE et al, 2008; TARSI, 2008).

Motivated by the results above, we questioned: How can the presence of the weight $|x|^\alpha$ influence a search for solutions to relevant problems of Nonlinear Analysis? Can we improve some assumptions of classical problems if we seek radial solutions when we add this weight on the nonlinearity?

In order to answer some points of these questions, we studied three problems with nonlinearity in the critical range with the weight $|x|^\alpha$: the first two problems with nonlinearities in the critical Trudinger-Moser growth range and the last problem with critical Sobolev growth. For the matter, we use these well-known critical point results (the Mountain–Pass Theorem or the Linking Theorem). Thus, we need to prove some geometric conditions satisfied by the functionals associated with each problems. In addition, for the critical case, we have to estimate the minimax levels which have to lie below some appropriate constants.

In order to present these studies, we organized this work in four chapters. Firstly, in Chapter , we present some basic facts and previous results used along the text and, in the others, we trate our problems.

In Chapter 1, inspired by studies of the Hénon type equations with the Trudinger-Moser growth nonlinearity in $\mathbb{R}^2$, we focus our attention on the natural two-dimensional extension for problem (2). More specifically, we study the Dirichlet problem in the unit ball B_1 of $\mathbb{R}^2$ for the Hénon type equation of the form

$$\begin{cases} -\Delta u &= \lambda u + |x|^\alpha f(u) \quad \text{in} \quad B_1, \\ \quad u &= 0 \qquad\qquad\qquad \text{on} \quad \partial B_1, \end{cases}$$

where $f(t)$ is a C^1-function in the critical growth range motivated by the Trudinger-Moser inequality.

In this chapter, we search for a weak solutions as critical points of functional associated to problem in two spaces: $H_0^1(B_1)$ and $H_{0,\text{rad}}^1(B_1)$. In the subcritical case, our arguments in the proof of boundedness of minimax levels are the same for the both spaces. However, in the critical case, we must consider the Trudinger-Moser inequality as (3), when we work in $H_0^1(B_1)$ (see [57] TRUDINGER, 1967), or as (4), when we look for a solution in $H_{0,\text{rad}}^1(B_1)$ (see [13] BONHEURE, 2008). Then, in the last case, the minimax levels increase its boundedness from above. Because of this fact, to prove the existence of radially symmetric solutions, we modify in some points our arguments to work in Sobolev space $H_{0,\text{rad}}^1(B_1)$.

In Chapters 2 and 3, we intend to establish a link between Ambrosetti-Prodi problems and Hénon type equations. Ambrosetti-Prodi problems have been studied, explored and extended by an enormous variety of authors. The resulting literature is nowadays rather rich (see, for instance, [24, 29, 46] FIGUEIREDO, 1980; FIGUEIREDO; YANG, 1999; RAMOS, 1988 and references therein). In this direction, one searches for solution of nonhomogeneous problems such that

$$\begin{cases} -\Delta u &= \tilde{g}(x,u) + f(x) \quad \text{in} \quad \Omega, \\ \quad u &= 0 \qquad\qquad\qquad \text{on} \quad \partial\Omega, \end{cases} \tag{5}$$

where "$\tilde{g}$ jumps eigenvalues", meaning that the limits $\lim_{s\to-\infty} \tilde{g}(s)/s < \lim_{s\to+\infty} \tilde{g}(s)/s$ form an interval that contains at least one eigenvalue of $(-\Delta, H_0^1(B_1))$. This class of problems are also called problems with jumping nonlinearities in a terminology introduced by S. Fučik (see [31] FUČIK; 1980).

Talking specifically about multiplicity of solutions for this kind of problem, we note that, in almost all Ambrosetti-Prodi problems since the work of J. Kazdan and F. Warner in [34] (KAZDAN; WARNER, 1975), it is extensively considered the heavy dependence of an usual hypothesis regarding a suitable parametrization of the forcing term f. More precisely, we assume that f satisfies

$$f(x) = h(x) t\phi_1(x),$$

where $h \in L^{\mu}(B_1)$, ϕ_1 is the eigenfunction associated to the first eigenvalue of the Laplacian operator in B_1, $\mu > N$ and

$$\int_{B_1} h\phi_1 \, \mathrm{d}x = 0.$$

Then, one should have that

$$|\tilde{g}(x,s)| \leq C|s|^{p-1} + C \quad \text{with} \quad 2 < p \leq 2^*.$$

The subcritical case ($p < 2^*$) of (5) was studied by B. Ruf and P. N. Srikanth [49] (RUF; SRIKANTH, 1986) considered $\tilde{g}(x,s) = s_+^p + \lambda s$ and $\lambda > \lambda_1$ (see also [23] FIGUEIREDO, 1988). The first paper that addressed Ambrosetti-Prodi problem (5) involving critical nonlinearity of the type $\tilde{g}(s) = |s|^{2_*-1} + \tilde{K}(s)$ was the work of D. YinBin, [58](YINBIN; 1991), where $\tilde{K}(s)$ is a lower perturbation of the expression with the critical exponent. Thus, we can say that the author considered a function that was superlinear both in $+\infty$ and $-\infty$ and, so, asymptotically jumping all the eigenvalues of $(-\Delta, H_0^1(\Omega))$.

In [29], D. Figueiredo and J. Yang dealt with one-sided critical growth, which allowed to explore more jumping possibilities. Due to natural limitations on the techniques they used, the existence of multiple solutions was investigated in dimensions $N \geq 7$ only (FIGUEIREDO; YANG, 1999). In [15], M. Calanchi and B. Ruf extended their results and proved that the same problem has at least two solutions provided $N \geq 6$. They also added a lower order growth term to the nonlinearity, which guaranteed the existence of solutions in lower dimensions as well (CALANCHI; RUF, 2002).

If we turn our attention to problems on domais in $\mathbb{R}^2$, where the critical growth is well-known to be exponential, we find the paper of M. Calanchi et al. [16], which considered the following problem:

$$\begin{cases} -\Delta u &= \lambda u + g(u_+) + f(x) & \text{in} \quad \Omega, \\ u &= 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{6}$$

where Ω is bounded and smooth in $\mathbb{R}^2$ and g has a subcritical or critical Trudinger-Moser growth. They proved the existence of two solutions for some forcing terms f, depending on the usual parametrization $f(x) = h(x) + t\phi_1(x)$ (CALANCHI et al, 2004). This result was a natural extension to the ones found in [15, 29] (CALANCHI; RUF, 2002; FIGUEIREDO; YANG, 1999).

In Chapter 2, we consider a similar problem to (6) with the weight $|x|^\alpha$ in the nonlinearity. Thus, we deal with the following class of problems

$$\begin{cases} -\Delta u = \lambda u + |x|^\alpha g(u_+) + f(x) & \text{in} \quad B_1, \\ \quad u = 0 & \text{on} \quad \partial B_1, \end{cases} \tag{7}$$

where B_1 is the unit ball in $\mathbb{R}^2$, g is a C^1 function in $[0, +\infty)$ which is assumed to be in the subcritical or critical growth range of Trudinger-Moser type and $f \in L^\mu(B_1)$ for some $\mu > 2$. Under suitable hypotheses on the constant λ, we prove existence of at least two solutions for this problem using variational methods. If we consider that f is radially symmetric, the two solutions will be radially symmetric as well.

We remark that existence of a first solution for Ambrosseti-Prodi problems does not require any growth of the nonlinearity. Only when we find out a second solution, we need to consider the growth of the nonlinearity at infinite. For this, we get a first solution ψ using the Fredholm Alternative in a related linear equation. A second solution is obtained via the Mountain–Pass Theorem, if $\lambda < \lambda_1$, or the Linking Theorem, if $\lambda > \lambda_1$.

In Chapter 3, we studied a natural extension for problem (7) for $N \geq 3$. But we concentrated on the radial critical case, once the critical case in $H_0^1(B_1)$ is very near to the result of M. Calanch and B.Ruf in [15](CALANCHI; RUF, 2002). Thus, we search the existence of radially symmetric solutions for problems with jumping nonlinearities in the presence of a typical weight of a Hénon equation. More specifically, our goal is to study the following class of Hénon type problems

$$\begin{cases} -\Delta u = \lambda u + |x|^\alpha k(u_+) + f(x) & \text{in} \quad B_1, \\ \quad u = 0 & \text{on} \quad \partial B_1, \end{cases}$$

where B_1 is the unit ball in $\mathbb{R}^N$, $k(t)$ is a C^1 function in $[0, +\infty)$ which is assumed to be in the critical level with subcritical perturbation, f is radially symmetric and belongs to $L^\mu(B_1)$ for suitable μ depending on $N \geq 3$. Under appropriate hypotheses on the constant λ, we prove the existence of at least two radial solutions. Analogously to Chapter 3, we get a first solution ψ using the Fredholm Alternative in a related linear equation. A second solution is obtained via Critical Points Theorems, consequently we need to show the geometric condition and estimate the minimax level. For this matter, we consider $k(t) = t^{2^*_\alpha} + g(t)$, where $g(t)$ is an appropriated subcritical perturbation, which can vanish in high dimensions. However, the assumptions on $g(t)$ are crucial in order to obtain the estimate of minimax level in lower dimensions. Because of this, we add this lower order growth term to the nonlinearity for all $N \geq 3$ to guarantee a second radial solution of this last problem.

Throughout this work, we realize that a nonlinearity with this new characteristic, the weight $|x|^\alpha$, allows us to deal with a well-studied problem under new perspectives. For instance, if we work with a critical growth nonlinearity, we need to assume suitable hypotheses when we are seeking radial solutions for the three problems. Due to this fact, we must adapt some technique to obtain radial versions for classic problems with Hénon weight.

It is important we emphasize that there are two technical differences between Chapter 1 and Chapters 2 and 3, when we search for a solution in the critical cases using Critical Points Theorems. In Chapter 2, we show that the associated functional of the problem satisfies the $(PS)_c$ condition for levels c in an interval bounded from above for a suitable constant. In Chapters 2 and 3, we only prove the boundedness of the minimax levels by appropriate constants, consequently, we also need to show that the solution obtained this away is not trivial.

In these last chapters, we use a technique based on the paper of B. Ruf and F. Gazzola [32] (RUF; GAZZOLA, 1997), which is not used in Chapter 1. Precisely, in Chapters 2 and 3 when we work in Linking case, we separate the supports of the functions used to prove the estimates of the minimax levels (functions of Moser and functions of Talenti, respectly) and the supports of functions spanned by the first k^{th}- eigenfunctions. Thus, we make some estimates easier to handle, due to lack of interference between these modified functions.

2 Preliminaries

This chapter is devoted to present some important results used along the text. Throughout this chapter, Ω is an open set in $\mathbb{R}^N$. We start introducing the notions of Sobolev spaces.

0.1 Sobolev spaces

Before we define Sobolev space, we need to know what is a weak derivative.

Definition 0.1.1. Assume $u, v \in L^1_{loc}(\Omega)$ and $\sigma = (\sigma_1, ..., \sigma_N)$ is a multiindex of order $|\sigma| = \sigma_1 + ... + \sigma_k$. We say that v is the σ^{th}**-weak partial derivative** of u, written

$$D^\sigma u = v$$

provided

$$\int_\Omega u D^\sigma \phi \mathrm{d}x = (-1)^{|\sigma|} \int_\Omega v\phi \mathrm{d}x$$

for all test functions $\phi \in C_c^\infty(\Omega)$.

Definition 0.1.2. Fix $1 \leq p \leq \infty$ and let k a nonnegative integer. The **Sobolev space**

$$W^{k,p}(\Omega)$$

consists of all locally summable functions $u : \Omega \to \mathbb{R}$ such that for each multiindex σ with $|\sigma| \leq k$, $D^\sigma u$ exists in the weak sense and belongs to $L^p(\Omega)$.

Definition 0.1.3. If $u \in W^{k,p}(\Omega)$, we define its **norm** to be

$$\|u\|_{W^{k,p}(\Omega)} := \begin{cases} \left(\sum_{|\sigma| \leq k} \int_\Omega |D^\sigma u|^p \mathrm{d}x \right)^{1/p} & \text{if} \quad 1 \leq p < \infty; \\ \sum_{|\sigma| \leq k} \operatorname*{ess\,sup}_\Omega |D^\sigma u| & \text{if} \quad p = \infty. \end{cases}$$

Example 0.1.1. If $p = 2$, we usually write

$$H^k(\Omega) = W^{k,2}(\Omega) \quad (k = 0, 1, ...).$$

We can prove that $H^k(\Omega)$ with the norm above is a Hilbert space. We still notice that $H^0(\Omega) = L^2(\Omega)$.

Definition 0.1.4. We denote by

$$W_0^{k,p}(\Omega)$$

the closure of $C_c^\infty(\Omega)$ in $W^{k,p}(\Omega)$.

It is customary to write

$$H_0^k(\Omega) = W_0^{k,2}(\Omega).$$

0.2 Elliptic Differential Partial Equation

Initially, we consider the following problem

$$\begin{cases} Lu = f & \text{in} \quad \Omega \\ \;\; u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{8}$$

where $u : \bar{\Omega} \to \mathbb{R}$ is the unknown, $u = u(x)$, $f : U \to \mathbb{R}$ is given, and L denotes a second-order partial differential operator having the form

$$Lu = -\sum_{i,j=1}^{n} (a^{ij}(x) u_{x_i})_{x_j} + \sum_{i=1}^{n} b^i(x) u_{x_i} + c(x)u$$

for given coefficient functions $a^{ij} = a^{ji}, b^j, c (i, j = 1, ..., n)$.

Definition 0.2.1. The partial differential operator L is **elliptic** if there exists a constant $\vartheta > 0$ such that

$$\sum_{i,j=1}^{n} a^{ij}(x)\xi_i\xi_j \geq \vartheta|\xi|^2$$

for *almost every* $x \in \Omega$ *and all* $\xi \in \mathbb{R}^N$.

Thus, we can say that ellipticity means that for each point $x \in \Omega$, *the symmetric* $N \times N$ *matrix* (a_{ij}) *is positive definite, with smallest eigenvalue greater than or equal to* ϑ.

Example 0.2.1. Let $\Omega \subset \mathbb{R}^N$ and a $C^2(\Omega)$-function. The **Laplace operator** of u, denoted by Δu is given by

$$\Delta u = \sum_{j=1}^{n} u_{x_i x_i} = \text{div} Du.$$

The Laplace operator is a second-order elliptic operator.

Definition 0.2.2. The **bilinear form** $B[\cdot,\cdot]$ **associated with the elliptic operator** L is

$$B[u,v] := \int_\Omega \sum_{i,j=1}^{N} a^{ij} u_{x_i} v_{x_j} + \sum_{i=1}^{N} b^j u_{x_i} v + cuv \mathrm{d}x$$

for $u, v \in H_0^1(\Omega)$.

Definition 0.2.3. We say that $u \in H_0^1(\Omega)$ is a **weak solution** of the boundary-value problem (8) if

$$B[u,v] = (f,v)$$

for all $v \in H_0^1(\Omega)$, where $(\cdot,\cdot)$ denotes the inner product in $L^2(\Omega)$. This identity is called thevariational form of problem (8).

0.3 Fredholm alternative

The Fredholm alternative will be used with more emphasis in last two chapters, but this important result is applied to the Laplace Operator. For this purpose we present its short version as following.

Theorem 0.3.1 (Fredholm alternative)**.** Precisely one of the following statements holds:

either

for each $f \in L^2(\Omega)$ there exists a unique weak solution u of the problem

$$(a) \quad \begin{cases} Lu = f & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

or else

there exists a weak solution $u \not\equiv 0$ of the homogeneous problem

$$(b) \quad \begin{cases} Lu = 0 & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

See [9, 21].

0.4 The Principle of Symmetric Criticality of Palais

Another important result which will be used in last chapters is the Principle of Symmetric Criticality of Palais. This celebrated result is also used to study some characteristics of eigenfunctions when we are working in radial contexts.

Definition 0.4.1. We consider an action of a group G on a normed vectorial space H.

(i) The **invariant space under** G is the closet subspace of H defined by

$$\text{Fix}(G) = \{u \in H; g(u) = u \quad \text{for all} \quad g \in G\}.$$

(ii) A **set** $A \subset H$ is called G-**invariant** if $g(A) = A$ for all $g \in G$.

(iii) A **functional** $J : H \to \mathbb{R}$ is called G-**invariant** if $J \circ g = J$ for all $g \in G$.

(iv) A **function** $f : H \to H$ is called G-**invariant** if $f \circ g = g \circ f$ for all $g \in G$.

Theorem 0.4.1 (The Principle of Symmetric Criticality of Palais)**.** Consider an isometric action of a group G under a Hilbert Space $(H, \langle \cdot, \cdot \rangle)$. If the functional $J : H \to \mathbb{R}$ is G-invariant and u is a critical point of J on $\text{Fix}(G)$, then u is a critical point of J in H.

See [42].

0.5 Spectrum of a operator

This section is based on results and definitions that can be found in [9, 21]. We will use them some times throughout this work.

Theorem 0.5.1. **(i)** There exists an at most countable set $\sum \subset \mathbb{R}$ such that the problem

$$\begin{cases} Lu = \lambda u + f & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega \end{cases} \tag{9}$$

has a unique weak solution for each $f \in L^2(\Omega)$ if and only if $\lambda \notin \sum$.

(ii) If $\sum$ is infinite, then $\sum = \{\lambda_k\}_{k=1}^{\infty}$, the values of a nondecreasing sequence with

$$\lambda_k \to +\infty.$$

Definition 0.5.1. We call $\sum$ the **spectrum of the operator** L. We notice that Problem (9) with $f \equiv 0$ has a nontrivial solution ϕ if and only if $\lambda \in \sum$, in which case λ is called **eigenvalue** of L and ϕ a corresponding **eigenfunction**.

From now on, we will focus our attention to (Δ, B_1), where B_1 is the unity ball of $\mathbb{R}^N$ for $N \geq 2$. This is because we search solutions for problems with this specific operator.

0.5.1 Spectrum of $(-\Delta, H_0^1(B_1))$

We denote by $0 < \lambda_1 < \lambda_2 \leq \lambda_3 ...$ the eigenvalues of $(-\Delta, H_0^1(B_1))$. The next result gives us a basis for $H_0^1(B_1)$ associated to these eigenvalues.

Theorem 0.5.2. There exists an orthonormal basis $\{\phi_k\}_{k=1}^{\infty}$ of $L^2(B_1)$ and $H_0^1(B_1)$, where ϕ_k is an eigenfunctions corresponding to λ_k:

$$\begin{cases} -\Delta\phi_k & = & \lambda_k\phi_k & \text{in} & B_1; \\ \phi_k & = & 0 & \text{on} & \partial B_1 \end{cases}$$

for $k = 1, 2, ...$

Since ∂B_1 is smooth, from regularity theory, we can conclude that $\phi_k \in C^{\infty}(\bar{B}_1)$ for $k = 1, 2, ...$

Definition 0.5.2. We call $\lambda_1 > 0$ the **principal eigenvalue** of $(-\Delta, H_0^1(B_1))$.

Theorem 0.5.3 (Variational principle of the principal eigenvalue)**.** We have

$$\lambda_1 = \inf_{\|u\|_2^2=1} \|\nabla u\|_2^2 \quad \text{for all} \quad u \in H_0^1(B_1) \setminus \{0\}.$$

The number λ_1 is attained for a function ϕ_1, which is radially symmetric and positive within B_1. We can write this characterization as

$$\lambda_1 = \inf_{u \in H_0^1(B_1)\setminus\{0\}} \frac{\|\nabla u\|_2^2}{\|u\|_2^2}.$$

Definition 0.5.3. If $k \geq 2$ we say that λ_k is a **high order eigenvalue**. We define H_k as the finite dimensional subspace of $H_0^1(B_1)$ spanned by $\phi_1, \phi_2, ...\phi_k$. Notice that we can write

$$H_0^1(B_1) = H_k \oplus H_k^{\perp} = \bigcup_{j=1}^{\infty} H_j,$$

which are called **orthonormal decompositions of** $H_0^1(B_1)$.

Similar to λ_1, we have a variational characterization to λ_k and λ_{k+1}, respectively, given by

$$\lambda_k = \sup_{u \in H_k\setminus\{0\}} \frac{\|\nabla u\|_2^2}{\|u\|_2^2} \quad \text{and} \quad \lambda_{k+1} = \inf_{u \in H_k^{\perp}\setminus\{0\}} \frac{\|\nabla u\|_2^2}{\|u\|_2^2}.$$

0.5.2 Spectrum of $(-\Delta, H^1_{0,\mathrm{rad}}(B_1))$

Since we also seek radially symmetric solutions for our problems, in these moments, we need to consider a suitable subspace of $H^1_0(B_1)$. More specifically, some times we work in

$$H^1_{0,\mathrm{rad}}(B_1) := \{u \in H^1_0(B_1); u(Sx) = u(x) \text{ for all } S \in O(N)\},$$

where $O(N)$ is space of rotations in $\mathbb{R}^N$.

First, we observe that $H^1_{0,\mathrm{rad}}(B_1)$ is an infinite dimensional subspace of $H^1_0(B_1)$. Indeed, let us take $\{e^{n|x|^2} - e^n \text{ with } n \in \mathbb{N}\} \subset H^1_{0,rad}(B_1)$, which is linearly independent. Furthermore, $H^1_{0,\mathrm{rad}}(B_1)$ is a Hilbert space with the norm of $H^1_0(B_1)$.

And analogously to $H^1_0(B_1)$, we can write $H^1_{0,\mathrm{rad}}(B_1)$ as an infinite union of finite dimensional orthogonal subspaces. From the Principle of Symmetric Criticality of Palais, we can see that ϕ_1 is radially symmetric. So we have $H_1 = H_1 \cap H^1_{0,\mathrm{rad}}(B_1)$. Now, we set

$$H^*_k = H_k \cap H^1_{0,\mathrm{rad}}(B_1) \text{ for all } k \in \mathbb{N}$$

and we have

$$H^1_{0,\mathrm{rad}}(B_1) = \bigcup_{k=1}^{\infty} H^*_k. \tag{10}$$

Moreover, it is straightforward to prove that the spectrum of $(-\Delta, H^1_{0,\mathrm{rad}}(B_1))$ is a subsequence of (λ_k) that we will denote by $\lambda^*_1 = \lambda_1 < \lambda^*_2 \leq \lambda^*_3 \leq ... \leq \lambda^*_k \leq ...$ where $\lambda^*_j \geq \lambda_j$ for all $j = 1, 2, 3....$ This fact guarantees that the variational characterizations of eigenvalues in $H^1_{0,\mathrm{rad}}(B_1)$ do not change.

0.6 Critical Point Theorems

In this section, we introduce the theorems which are used to guarantee our existence results. We need the following concept.

Definition 0.6.1. We consider real C^1-functional Φ defined on a Banach space E. When looking for critical points of Φ it has become standard to assume the following compactness condition

$(PS)_c$: any sequence (u_j) in E such that

$$\Phi(u_j) \to c \quad \text{and} \quad \Phi'(u_j) \to 0 \text{ in } E^* \tag{11}$$

has a convergence subsequence. A sequence that satisfies (11) is called a (PS) **sequence (at level** c**)**.

We shall use two well-known critical-point theorems, namely, the Mountain–Pass and the Linking Theorems, to $\lambda < \lambda_1$ and $\lambda_j < \lambda < \lambda_{j+1}$, respectively. Let us state them here, for the sake of completeness. Initially, we have the Mountain–Pass and the Linking Theorems in the versions which will be used in the Chapter 1. For the proofs we refer the reader to [5, 12, 44, 52].

Theorem A (Mountain–Pass)**.** *Let Φ be a C^1 functional on a Banach space E satisfying*

(Φ_0) *There exist $\varrho > 0$ such that Φ satisfies (PS) condition for levels in the interval $(0, \varrho)$.*

(Φ_1) *There exist constants ρ, $\delta > 0$ such that $\Phi(u) \geq \delta$ for all $u \in E$ such that $\|u\| = \rho$,*

(Φ_2) *$\Phi(0) < \delta$ and $\Phi(v) < \delta$ for some $v \in E \setminus \{0\}$ and $\|v\| \neq \rho$.*

Considere $\Gamma := \{\eta \in C([0,1], E) : \eta(0) = 0$ and $\eta(1) = v\}$ and set

$$c = \inf_{\eta\in\Gamma} \max_{t\in[0,1]} \Phi(\eta(t)). \geq \delta.$$

Then $c \in (0, \varrho)$ and it is a critical value of Φ.

Moreover, if $f(t)$ is a odd function, one has many infinitely solutions.

Theorem B (Linking Theorem)**.** *Let Φ be a C^1 functional on a Banach space $E = E_1 \oplus E_2$ such that $\dim E_1$ is finite. If Φ satisfies (Φ_0) and*

(Φ_3) *There exist constants δ, $\rho > 0$ such that $\Phi(u) \geq \delta$ for all $u \in E_2$ such that $\|u\| = \rho$,*

(Φ_4) *There exist an $z \notin E_1$ with $\|z\| = 1$ and $R > \rho$ such that if*

$$Q := \{v + sz : v \in E_1, \|v\| \leq R \text{ and } 0 \leq s \leq R\},$$

then

$$\Phi(u) \leq 0 \text{ for all } u \in \partial Q.$$

Consider $\Gamma := \{\eta \in C(\bar{Q}, E) : \eta(u) = u$ if $u \in \partial Q\}$ and set

$$c = \inf_{\eta\in\Gamma} \max_{u\in Q} \Phi(\eta(u)) \geq \delta.$$

Then $c \in (0, \varrho)$ and it is a critical value of Φ.

Moreover, if $f(t)$ is a odd function, one has many infinitely solutions.

In Chapters 3 and 4, we also shall use these well-known critical-point theorems, but now both without the (PS) condition. For the proofs we refer the reader to [11, 12, 25, 39, 44].

Theorem C (Mountain-Pass without the (PS) condition)**.** *Let* Φ *be a* C^1 *functional on a Banach space* E *and satisfying*

$(\Phi 1')$ *There exist constants* ρ, $\delta > 0$ *such that* $\Phi(u) \geq \delta$ *if* $u \in E$ *and* $\|u\| = \rho$,

$(\Phi 2')$ $\Phi(0) < \delta$ *and* $\Phi(v) < \delta$ *for some* $v \in E$ *if* $\|u\| > \rho$.

Consider $\Gamma := \{\eta \in C([0,1], E) : \eta(0) = 0$ *and* $\eta(1) = v\}$ *and set*

$$c = \inf_{\eta \in \Gamma} \max_{t \in [0,1]} \Phi(\eta(t)) \geq \delta.$$

Then there exists a sequence (u_j) *in* E *satisfying* (11)*. Moreover, if* Φ *satisfies* $(PS)_c$, *then* c *is a critical value for* Φ.

Theorem D (Linking Theorem without the (PS) condition)**.** *Let* Φ *be a* C^1 *functional on a Banach space* $E = E_1 \oplus E_2$ *such that* $\dim E_1$ *is finite and satisfying*

$(\Phi 3')$ *There exist constants* $\delta, \rho > 0$ *such that* $\Phi(u) \geq \delta$ *if* $u \in E_2$ *and* $\|u\| = \rho$,

$(\Phi 4')$ *There exists* $z \in E_2$ *with* $\|z\| = 1$ *and there exists* $R > \rho$ *such that*

$$\Phi(u) \leq 0 \text{ for all } u \in \partial Q,$$

where

$$Q := \{v + sz : v \in E_1, \|v\| \leq R \text{ and } 0 \leq s \leq R\}.$$

Consider $\Gamma := \{\eta \in C(\bar{Q}, E) : \eta(u) = u$ *if* $u \in \partial Q\}$ *and set*

$$c = \inf_{\eta \in \Gamma} \max_{u \in Q} \Phi(\eta(u)) \geq \delta.$$

Then there exists a sequence (u_j) *in* E *satisfying* (11)*. Moreover, if* Φ *satisfies* $(PS)_c$, *then* c *is a critical value for* Φ.

0.7 Other results

Here, we will introduce results used in this work, which are proved in our references.

Lemma 0.7.1. Let (u_n) be a sequence of functions in $L^1(\Omega)$ converging to u in $L^1(\Omega)$. Assume that $f(x,u_n(x))$ and $f(x,u(x))$ are also L^1 functions. If

$$\int_\Omega |f(x,u_n(x))u_n(x)|\mathrm{d}x \leq C_1$$

then $f(x,u_n)$ converges in L^1 to $f(x,u)$.

See [28, Lemma 2.1].

Now we consider the so-called Moser functions

$$\bar{z}_n(x) = \frac{1}{\sqrt{2\pi}} \begin{cases} (\log n)^{1/2}, & 0 \leq |x| \leq \frac{1}{n}; \\ \dfrac{\log \frac{1}{|x|}}{(\log n)^{1/2}} & \frac{1}{n} \leq |x| \leq 1; \\ 0 & |x| > 1. \end{cases} \tag{12}$$

It is known that $\bar{z}_n \in H_0^1(B_1)$, where $B_1 \subset \mathbb{R}^2$ and $\|\bar{z}_n\| = 1$ for all $n \in \mathbb{N}$ and $\|\bar{z}_n\|_2 = O(1/\log n)^{1/2}$, for details see [40]. Now, we have the following lemma, which gives us some estimates used when we work with the orthogonal projection on $(H_k)^\perp$.

Lemma 0.7.2. Let $T_k : H_0^1(B_1) \to (H_k)^\perp$ be the orthogonal projection. We consider (12) and define

$$\bar{w}_n(x) = T_k \bar{z}_n(x). \tag{13}$$

The following estimates hold

$$1 - \frac{A_k}{\log n} \leq \|\bar{w}_n\|^2 \leq 1. \tag{14}$$

$$\bar{w}_n(x) \geq \begin{cases} -\dfrac{B_k}{(\log n)^{1/2}} \text{ for all } x \in B_1; \\ \dfrac{(\log n)^{1/2}}{\sqrt{2\pi}} - \dfrac{B_k}{(\log n)^{1/2}} \text{ for all } x \in B_{1/n}, \end{cases} \tag{15}$$

where $A_k, B_k > 0$ are such that

$$\|u\| \leq A_k \|u\|_2 \text{ and } \|u\|_\infty \leq (B_k/B)\|u\|_2 \text{ for all } u \in H_k,$$

and B satisfies $\|\bar{z}_n\|_2 \leq B/(\log n)^{1/2}$ for all $n \in \mathbb{N}$.

See [48, Lemma 2].

Theorem 0.7.1 (Dominated convergence theorem)**.** Let $\{f_n\}$ be a sequence of measurable functions on a measure space (Ω, Σ, μ). Suppose that the sequence converges pointwise to a function f and is dominated by some integrable function g in the sense that

$$|f_n(x)| \leq g(x)$$

for all numbers n in the index set of the sequence and all points $x \in \Omega$. Then f is integrable and

$$\lim \int_\Omega f_n \quad \mathrm{d}\mu = \int_\Omega f \quad \mathrm{d}\mu.$$

See [22].

Theorem 0.7.2. Let $(u_n) \subset W_0^{1,N}$ satisfy: $\|\nabla u_n\|_N \leq 1$. Without loss of generality we may assume that $u_n \to u, |\nabla u_n|^2 \to \nu$ weakly. Then either $\nu = \delta_{x_0}$(the Dirac measure of mass 1 concentrated at some $x_0 \in \bar{\Omega}$) for some $x_0 \in \bar{\Omega}$ and $u_n \to 0$, $e^{\beta_N \|u_n\|^{N/N-1}} \to c\delta_{x_0}$ for some $c \geq 0$, or there exists $\beta > 0$ such that $e^{\beta_N \|u_n\|^{N/N-1}}$ is bounded in $L^1(\Omega)$ and thus

$$e^{\beta_N \|u_n\|^{N/N-1}} \to e^{\beta_N \|u\|^{N/N-1}} \quad \text{in} \quad L^1(\Omega).$$

In particular this is the case if $u \not\equiv 0$.

See [38, Theorem I.6].

Now we consider the following sequence

$$y_n(s) = \begin{cases} \dfrac{s}{n^{1/2}}(1-\delta_n)^{1/2}, & 0 \leq s \leq n; \\ \dfrac{1}{(n(1-\delta_n))^{1/2}} \log\left(\dfrac{A_n + 1}{A_n + e^{-(s-n)}}\right) + (n(1-\delta_n)^{1/2}, & n \leq s, \end{cases} \tag{16}$$

where $\delta_n = 2\log n/n$ and $A_n = 1/(en) + O(1/n^4)$.

Theorem 0.7.3. Let $F(t)$ be a $C^1(\mathbb{R})$-function with Trudinger-Moser critical growth, then

$$\lim \int_{B_1(0)} F(u_n) \quad \mathrm{d}x \in [0, e^{1+\frac{1}{2}+\ldots+\frac{1}{N-1}}|B_1(0)|]$$

for any normalized concentrating sequence $\{u_n\}$. In particular, there exists an explicit normalized concentrating sequence $\{y_n\}$ with

$$\lim \int_{B_1(0)} F(y_n) \quad \mathrm{d}x = e^{1+\frac{1}{2}+\ldots+\frac{1}{N-1}}|B_1(0)|.$$

See [26, Theorem 1.4].

Chapter 1

Hénon elliptic equations in $\mathbb{R}^2$ with subcritical and critical exponential growth

In this chapter, using variational methods, we prove the existence of a nontrivial solution for the following class of Hénon type equations

$$\begin{cases} -\Delta u &= \lambda u + |x|^{\alpha} f(u) \quad \text{in} \quad B_1, \\ \quad u &= 0 \qquad\qquad\qquad \text{on} \quad \partial B_1, \end{cases} \tag{P1}$$

where $\lambda > 0, \alpha \geq 0$ (for Hénon type equations we should have $\alpha > 0$, but in this work we also consider α as possibly equal to 0) and B_1 is a unity ball centred at origin of $\mathbb{R}^2$, which is a natural two dimensional extension of (2).

For this purpose, we assume that $f(t)$ has the maximum growth which allows us to study (P1) variationally in Sobolev Spaces, motivated by the well-known Trudinger-Moser inequality, given by (3), when we seek a solution in $H_0^1(B_1)$, and by (4), if we search a radial solution. Therefore, from (3)-(4), we have naturally associated notions of subcriticality and criticality: we say that

(SG) $f(t)$ has subcritical growth at $+\infty$ if for all β one has

$$\lim_{t \to +\infty} \frac{|f(t)|}{e^{\beta t^2}} = 0;$$

(CG) $f(t)$ has critical growth at $+\infty$ if there exists $\beta_0 > 0$ such that

$$\lim_{t \to +\infty} \frac{|f(t)|}{e^{\beta t^2}} = 0, \quad \forall \beta > \beta_0; \quad \lim_{t \to +\infty} \frac{|f(t)|}{e^{\beta t^2}} = +\infty, \quad \forall \beta < \beta_0.$$

Similarly we define subcritical and critical growth at $-\infty$. In this chapter we will consider problems where the nonlinearities have critical or subcritical growth.

1.1 Hypotheses

Before stating our main results, we shall introduce the following assumptions on the nonlinearity $f(t)$:

(H_1) The function $f(t)$ is continuous and $f(0) = 0$.

(H_2) There exist t_0 and $M > 0$ such that

$$0 < F(t) =: \int_0^t f(s)\,\mathrm{d}s \leq M|f(t)| \quad \text{for all} \quad |t| > t_0.$$

(H_3) $0 < 2F(t) \leq f(t)t$ for all $t \in \mathbb{R} \setminus \{0\}$.

Remark 1.1.1. **(a)** *We can see that from conditions (H_1) and (H_2), it follows that there are constants $C, t_0 > 0$ such that*

$$F(t) > Ce^{|t|/M} \quad \textit{for all} \quad |t| > t_0. \tag{1.1}$$

Hence we can see that $F(t)$ and $f(t)$ have exponential growth for $|t|$ large enough.

(b) *Also by (H_1) and (H_2), we notice that $f(t)$ satisfies the well-known Ambrosetti–Rabinowitz condition, in exact terms, there exist $\vartheta > 2$ and $t_\vartheta > 0$ such that*

$$\vartheta F(t) \leq f(t)t \quad \textit{for all} \quad |t| \geq t_\vartheta. \tag{1.2}$$

The Ambrosetti–Rabinowitz condition has been used in most of the studies for superlinear problems and plays an important role in studying the existence of nontrivial solutions of many nonlinear elliptic boundary value problems. Here, we use (1.2) *in order to ensure that the functional associated with our problem has the geometric condition of the Mountain–Pass Theorem if $\lambda < \lambda_1$ or the Linking Theorem if $\lambda_k < \lambda < \lambda_{k+1}$ and also guarantee the boundedness of the Palais–Smale sequence. Thus, we obtain the weak convergence of the Palais–Smale sequence to the solution. Then condition (H_2) guarantees the nontriviality of our weak solutions.*

(c) *In the subcritical case, we can replace condition (H_3) by the following weak hypothesis*

(H_3') *$F(t) > 0$ and $f(t)t > 0$ for all $t \in \mathbb{R} \setminus \{0\}$.*

(d) *We mention that we do not use condition (H_3) to prove the existence of solutions attaining the minimax level in the subcritical case. For the critical case, we can show the existence of weak solution of* (P1) *without condition (H_3). Nevertheless, we use (H_3), in our arguments, to prove the existence of a critical point achieving the minimax level. In some cases, these solutions will be ground state solutions.*

1.2 Statement of main results

Here, we divide our results in six theorems, which are considered depending on the growth conditions of the nonlinearity and also on the interval where λ lies. In the critical case, since the weight $|x|^\alpha$ has an important role on the estimate of the minimax levels, the variational setting and methods used in $H_0^1(B_1)$ and $H_{0,\text{rad}}^1(B_1)$ are different and, therefore, are given in separate theorems. Another aspect, which we should consider in this case, is an asymptotic hypothesis on $f(t)t/e^{\beta_0 t^2}$ at infinity. We shall suppose a certain kind of asymptotic behavior for $f(t)t/e^{\beta_0 t^2}$ at infinity, if $\lambda < \lambda_1$ and $\alpha \geq 0$, in order to find a solution in $H_0^1(B_1)$ or in $H_{0,\text{rad}}^1(B_1)$. If $\lambda > \lambda_1$ and $\alpha > 0$, an analogous hypothesis can be assumed, ensuring, thus, the existence of a solution in $H_0^1(B_1)$. However, if we search a solution for (P1) in $H_{0,\text{rad}}^1(B_1)$, when we are supposing $\lambda > \lambda_1$ and $\alpha \geq 0$, we demand a stronger hypothesis on the behavior of $f(t)t/e^{\beta_0 t^2}$. Then, we have the following results.

Theorem 1.2.1. (The subcritical case, local minimum at 0). Assume $(H_1) - (H_3)$ and that $f(t)$ has subcritical growth (SG) at both $+\infty$ and $-\infty$. Furthermore, suppose that $0 < \lambda < \lambda_1$ and

(H_4) $\displaystyle\limsup_{t\to 0} \frac{2F(t)}{t^2} \leq \lambda_1 - \lambda.$

Then, problem (P1) has a nontrivial (radially symmetric) solution. Moreover, if $f(t)$ is an odd function, then (P1) has infinitely many solutions.

Theorem 1.2.2. (The subcritical case, saddle point at 0). Assume $(H_1) - (H_3)$ and that $f(t)$ has subcritical growth (SG) at both $+\infty$ and $-\infty$. Furthermore, suppose that $\lambda_k < \lambda < \lambda_{k+1}$ and

(H_5) There exist $\delta > 0$ and $0 < \mu < \lambda_{k+1} - \lambda$, such that

$$F(t) \leq \frac{\mu}{2}t^2, \quad \text{for all} \quad |t| \leq \delta.$$

Then, problem (P1) has a nontrivial (radially symmetric) solution. Moreover, if we assume that $f(t)$ is an odd function, then (P1) has infinitely many solutions.

For the next two theorems, we will use the following notations: for $0 \leq \varepsilon < 1$ let us consider

$$M_\varepsilon = \lim_{n\to\infty} \int_0^1 n e^{n((1-\varepsilon)t^2 - t)}\, \mathrm{d}t \quad \text{and} \quad \widehat{M} = \lim_{\varepsilon\to 0} M_\varepsilon.$$

We use the notation M_0 for M_ε with $\varepsilon = 0$, and using calculus, one can see that $M_0 = 2$ and $\widehat{M} = 1$. We also consider constants $0 < r, d < 1$ such that there exists a ball $B_r \subset B_1$ so that $|x| > d$ for all $x \in B_r$.

Theorem 1.2.3. (The critical case, local minimum at 0). Assume $(H_1) - (H_4)$ and that $f(t)$ has critical growth (CG) at both $+\infty$ and $-\infty$. Furthermore, suppose that $0 < \lambda < \lambda_1$ and

$$(H_6) \quad \lim_{t\to+\infty} \frac{f(t)t}{e^{\beta_0 t^2}} \geq \xi, \quad \xi > \frac{4}{\beta_0 M_0 d^\alpha r^2} \text{ with } d > 0.$$

Then, problem (P1) has a nontrivial solution.

Theorem 1.2.4. (The critical case, saddle point at 0 with $\alpha > 0$). Assume $(H_1) - (H_3)$, (H_5) and that $f(t)$ has critical growth (CG) at both $+\infty$ and $-\infty$. Furthermore, suppose that $\lambda_k < \lambda < \lambda_{k+1}$, $\alpha > 0$ and

$$(H_7) \lim_{t\to+\infty} \frac{f(t)t}{e^{\beta_0 t^2}} \geq \xi, \text{ with } \xi > \frac{4}{\beta_0 r^{2+\alpha}} \left[\left(\frac{d}{r}\right)^\alpha - \frac{2}{2+\alpha} \right]^{-1},$$

where $0 < r \leq d < 1$ are such that

$$\left(\frac{d}{r}\right)^\alpha > \frac{2}{2+\alpha}.$$

Then problem (P1) has a nontrivial solution.

Theorem 1.2.5. (The radial critical case, local minimum at 0). Assume $(H_1) - (H_4)$ and that $f(t)$ has critical growth (CG) at both $+\infty$ and $-\infty$. Furthermore, suppose that $0 < \lambda < \lambda_1$ and

$$(H_{6,\mathrm{rad}}) \quad \lim_{t\to+\infty} \frac{f(t)t}{e^{\beta_0 t^2}} \geq \xi, \quad \xi > \frac{8}{(2+\alpha)\beta_0 e}.$$

Then, problem (P1) has a nontrivial radially symmetric solution.

Theorem 1.2.6. (The radial critical case, saddle point at 0). Assume $(H_1) - (H_3)$, (H_5) and that $f(t)$ has critical growth (CG) at both $+\infty$ and $-\infty$. Furthermore, suppose that $\lambda_k < \lambda < \lambda_{k+1}$ and for all $\gamma \geq 0$ there exists $c_\gamma \geq 0$ such that

(H_8) $\dfrac{f(t)t}{e^{\beta_0 t^2}} \geq \gamma h(t)$ for all $t > c_\gamma$, where $h : \mathbb{R} \to \mathbb{R}^+$ is a Carathéodory function satisfying $\liminf\limits_{t\to+\infty} \dfrac{\log(h(t))}{t} > 0$.

Then problem (P1) has a nontrivial radially symmetric solution.

Remark 1.2.1. **(a)** *Note that an example for nonlinearity with subcritical growth which satisfies* $(H_1) - (H_3)$ *is given by*

$$f(t) = e^{|t|^q}|t|^{p+q-2}t + \frac{p}{q}|t|^{p-2}te^{|t|^q} + ct,$$

where $1 \leq q < 2 < p$.

Moreover, if $\lambda < \lambda_1$ *and* $0 < c \leq \lambda_1 - \lambda$, *assumption* (H_4) *is satisfied. Thus, applying Theorem 1.2.1, we obtain a nontrivial (radial) solution of* (P1). *Similarly, in view of Theorem 1.2.2, we can obtain a nontrivial (radial) solution of* (P1) *provided that* $\lambda_k < \lambda < \lambda_{k+1}$.

(b) *Our Theorem 1.2.1 is a version of Theorem 1.1 of [28], for* $\alpha = 0$ *and the autonomous nonlinearity and Theorem 1.2.2 in this chapter is analogous to Theorem 1.2 in [28] (FIGUEIREDO et al, 1995).*

(c) *For examples of nonlinearities with critical growth satisfying the assumptions of Theorems 1.2.3, 1.2.4 or 1.2.5, one can consider*

$$f(t) = e^{\beta_0 t^2}|t|^p t + \frac{p}{2\beta_0}|t|^{p-2}te^{\beta_0 t^2} + ct,$$

where $p > 1$ *and* c *can be chosen so that* $f(t)$ *satisfies* (H_4) *or* (H_5).

(d) *We also realize that Theorem 1.2.3 is a version of Theorem 1.3 in [28] for the autonomous case. Moreover, Theorem 1.2.5 is a radial version of Theorem 1.2.3 and, consequently, of Theorem 1.3 in [28] with autonomous linearity when* $\alpha = 0$. *We also mention that for the radial case we obtain an improvement in assumption of behavior of* $f(t)/t$ *at infinity. Indeed, in radial context, the estimate from bellow in* $(H_{6,\text{rad}})$ *is smaller than the one in* (H_6) *(FIGUEIREDO et al, 1995).*

(e) *Our Theorem 1.2.4 is not a similar version of Theorem 1.4 in [28], because here we are demanding that* $\alpha > 0$, *which was crucial to guarantee that the minimax level is in the range of compactness of the associated functional (FIGUEIREDO et al,* *1995). In order to obtain a similar result to Theorem 1.2.4 with α equal to 0 we need to replace (H_7) with (H_8), which would make this result weaker.*

(f) *For $p > 1$ and appropriated choice of constants $c > 0$, we can see that the following nonlinearity*

$$f(t) = e^{\beta_0 t^2 + t}|t|^p t + \frac{p}{2\beta_0}|t|^{p-2} t e^{\beta_0 t^2 + t} + ct$$

satisfies the hypotheses of Theorem 1.2.6.

(g) *In Theorem 1.2.6, although α can be taken equal to 0, we do not obtain a similar result to the autonomous case for Theorem 1.4 in [28], because we need to assume (H_8) instead of (H_7) (FIGUEIREDO et al, 1995).*

(h) *Following a similar argument as the one due to P. H. Rabinowitz [45, Proposition 3.11] (RABINOWITZ, 1992), we can prove that the Mountain–Pass solution obtained in Theorems 1.2.1, 1.2.3 and 1.2.5 are indeed ground state solution provided that $f(t) = o(|t|)$ at 0 and $f(t)/t$ is increasing.*

Since we prove the existence of a solution for the Hénon type equations using variational methods, we considere the nonlinearity with subcritical or critical exponential growth. We do so in the same spirit of D. Figueiredo, O. Miyagaki and B. Ruf [28] (FIGUEIREDO et al, 1995), following the Brezis-Nirenberg argument [11] (BREZIS; NIRENBERG, 1983).

As we can see in the proofs of Theorems 1.2.3, 1.2.4, 1.2.5 and 1.2.6, the minimax levels of the functional depend heavily on the asymptotic behavior of $f(t)t/e^{\beta_0 t^2}$ at infinity. Moreover, the parameters α, λ and the Sobolev spaces ($H_0^1(B_1)$ or $H_{0,\mathrm{rad}}^1(B_1)$) exert a significant influence on the choice of our assumptions $(H_6), (H_{6,\mathrm{rad}}), (H_7)$ and (H_8). For example, in $H_0^1(B_1)$, we were inspired by Adimurthi, in [2] (ADIMURTHI, 1990) , and D. Figueiredo, O. Miyagaki and B. Ruf, in [28] (FIGUEIREDO, 1995), who showed the existence of solution for similar problems to (P1), in a bounded domain $\Omega \subset \mathbb{R}^2$, provided that $\tilde{f}$ satisfies (among other conditions) the asymptotic hypothesis

(h_0) $\displaystyle\lim_{t\to\infty} \frac{\tilde{f}(x,t)t}{e^{\beta_0 t^2}} > C(r)$, where r is the radius of the largest open ball in the domain.

To study the existence of solutions for (P1) in $H_0^1(B_1)$ for nonlinearity with the vanishing weight at origin of the type $\tilde{f}(x,t) = |x|^\alpha f(t)$, we assume that an analogous condition to

(h_0) holds uniformly on a small ball $B(x_0, \delta) \subset B_1 \setminus \{0\}$ in order to avoid the influence of the weight $|x|^\alpha$.

Nevertheless, we can prove the existence of solution for (P1) in $H_0^1(B_1)$ in the critical case supposing $\lambda_k < \lambda < \lambda_{k+1}$ and (H_7) for $\alpha > 0$. For the saddle point at 0 case, we cannot see how it is possible to find a solution for (P1) in $H_0^1(B_1)$ with $\alpha = 0$ or in $H^1_{0,\text{rad}}(B_1)$ for $\alpha \geq 0$, using a hypothesis similar to (h_0). Hence we have to assume a stronger hypothesis than (H_7), which will be used in Theorem 1.2.6.

In Theorem 1.2.5, although we search for a radial solution for (P1), since $\lambda < \lambda_1$, we can use a technique introduced by D. Figueiredo, J. do Ó and B. Ruf, in [26] (FIGUEIREDO et al, 2002), and assume a hypothesis for $f(t)$ as in (h_0). Moreover, this technique together with the weight $|x|^\alpha$ in $H^1_{0,\text{rad}}(B_1)$ minimize the constant which bounded from below the asymptotic behavior of $f(t)$. Consequently, we achieve better estimatives for the last case. Indeed, in [28] (FIGUEIREDO, 1995), it is used the hypothesis as (h_0) with

$$C_1(r) = \frac{4}{M_0 \beta_0 r^2}$$

in order to prove a similar result to Theorem 1.2.5, in which we use the analogous hypothesis $(H_{6,\text{rad}})$ with

$$C_2(r) = \frac{8}{(2+\alpha)\beta_0 e r^2}.$$

Once $C_1(r) > C_2(r)$, we can notice that, even with α equal to zero, we obtain a significant improvement in this kind of estimate.

1.2.1 Outline

This Chapter is organized as follows. In Sect. 1.3, we introduce the variational framework and prove the boundedness of Palais-Smale sequences of the functional associated to Problem (P1). After that, we consider the subcritical case and prove that the associated functional of the problem satisfies the $(PS)_c$ condition for all $c > 0$. For critical case, we show a similar result for levels c in an interval bounded from above for a suitable constant, which depends only on β_0 in $H_0^1(B_1)$ and on β_0 and α in $H^1_{0,\text{rad}}(B_1)$. In Sect. 1.4, we consider a nonlinearity with subcritical growth and we obtain the geometric conditions for the functional in order to prove the existence of a solution to both Sobolev Spaces, $H_0^1(B_1)$ and $H^1_{0,\text{rad}}(B_1)$. In Sect. 1.5 and 1.6, we study Problem (P1) in critical growth range and guarantee the existence of a solution in $H_0^1(B_1)$ and $H^1_{0,\text{rad}}(B_1)$, respectively. In these cases, we also prove the geometric conditions and the boundedness of the minimax levels.

1.3 The variational formulation

We can notice that by (SC) or (CG) there exist positive constants C and β such that

$$|f(t)| \leq Ce^{\beta t^2} \text{ for all } t \in \mathbb{R}. \tag{1.3}$$

We denote $H = H^1_{0,\text{rad}}(B_1)$ in the radial case or $H = H^1_0(B_1)$ in the another case, with the norm

$$\|u\| = \left(\int_{B_1} |\nabla u|^2 \, \mathrm{d}x\right)^{1/2} \quad \text{for all } u \in H.$$

Recalling that our efforts are searching for a nontrivial weak solution to Problem (P1), we define the functional $\Phi_\lambda : H \to \mathbb{R}$ by

$$\Phi_\lambda(u) = \frac{1}{2}\int_{B_1} |\nabla u|^2 \, \mathrm{d}x - \frac{\lambda}{2}\int_{B_1} |u|^2 \, \mathrm{d}x - \int_{B_1} |x|^\alpha F(u) \, \mathrm{d}x,$$

where $\alpha \geq 0$ (only in Theorem 1.2.4, we should have α strictly positive) and $\lambda > 0$. By $(H_1) - (H_2)$ and (1.3), we see that the functional Φ_λ is C^1 and its derivative is given by

$$\langle \Phi'_\lambda(u), v\rangle = \int_{B_1} \nabla u \nabla v \, \mathrm{d}x - \lambda \int_{B_1} uv \, \mathrm{d}x - \int_{B_1} |x|^\alpha f(u)v \, \mathrm{d}x \quad \text{for all} \quad v \in H$$

and the critical points of Φ_λ are a (weak) solution to (P1). Hence to find a solution of (P1) one can look for critical points of the functional Φ_λ with critical value $c > 0$.

1.3.1 The boundedness of a $(PS)_c$ sequence

We will use (1.2) to prove the following lemma that shows that a $(PS)_c$ sequence of the functional Φ_λ is bounded. Let us notice that the proof of the Palais-Smale condition for the functional associated to the problem follows traditional methods. Then we will present a brief proof for this condition.

Lemma 1.3.1. Assume $(H_1) - (H_3)$. Let $(u_n) \subset H$ be a $(PS)_c$ sequence of Φ_λ. Then (u_n) is bounded in H.

Proof: Let $(u_n) \subset H$ be a $(PS)_c$ sequence, i.e.

$$\left|\frac{1}{2}\|u_n\|^2 - \frac{\lambda}{2}\|u_n\|_2^2 - \int_{B_1} |x|^\alpha F(u_n) \, \mathrm{d}x - c\right| \to 0 \tag{1.4}$$

and

$$\left|\int_{B_1} \nabla u_n \nabla v \, \mathrm{d}x - \lambda \int_{B_1} u_n v \, \mathrm{d}x - \int_{B_1} |x|^\alpha f(u_n) v \, \mathrm{d}x\right| \leq \varepsilon_n \|v\| \text{ for all } v \in H, \tag{1.5}$$

where $\varepsilon_n \to 0$. It follows from (1.2) that, for $\vartheta = 4$, we obtain $t_4 > 0$ such that

$$F(t) \leq \frac{1}{4} f(t)t \quad \text{for all} \quad |t| \geq t_4.$$

Using (1.4) and (1.5), we have

$$\begin{aligned} c + \frac{\varepsilon_n \|u_n\|}{2} &\geq \frac{1}{2} \int_{B_1} |x|^\alpha f(u_n) u_n - \int_{B_1} |x|^\alpha F(u_n) \\ &= \frac{1}{2} \int_{B_1} |x|^\alpha f(u_n) u_n - \int_{\{u_n \geq t_4\}} |x|^\alpha F(u_n) - \int_{\{u_n \leq t_4\}} |x|^\alpha F(u_n) \\ &\geq \frac{1}{2} \int_{B_1} |x|^\alpha f(u_n) u_n - \frac{1}{4} \int_{\{u_n \geq t_4\}} |x|^\alpha f(u_n) u_n - C_0 \\ &\geq \frac{1}{4} \int_{B_1} |x|^\alpha f(u_n) u_n - C_0. \end{aligned}$$

So we have

$$\int_{B_1} |x|^\alpha f(u_n) u_n \leq C + \varepsilon_n \|u_n\| \text{ with } C > 0, \tag{1.6}$$

and, consequently,

$$\int_{B_1} |x|^\alpha |f(u_n)| \, \mathrm{d}x \leq C + \varepsilon_n \|u_n\|. \tag{1.7}$$

Indeed, given $M_0 > 0$, we have

$$\begin{aligned} \int_{B_1} |x|^\alpha f(u_n) &= \int_{\{u_n \leq M_0\}} |x|^\alpha f(u_n) + \int_{\{u_n \geq M_0\}} |x|^\alpha f(u_n) \\ &\leq C_1 + \int_{\{u_n \geq M_0\}} |x|^\alpha f(u_n) \frac{u_n}{u_n} \\ &\leq C_1 + \frac{1}{M_0} \int_{\{u_n \geq M_0\}} |x|^\alpha f(u_n) u_n \\ &\leq C_1 + \frac{1}{M_0} \int_{B_1} |x|^\alpha f(u_n) u_n \\ &\leq C + \varepsilon_n \|u_n\|. \end{aligned}$$

Firstly, we consider $0 < \lambda < \lambda_1$. Using (1.5) and (1.6) and the variational characterization of λ_1, we have

$$\varepsilon_n\|u_n\| \geq \left(1 - \frac{\lambda}{\lambda_1}\right)\|u_n\|^2 - (C + \varepsilon_n\|u_n\|).$$

Therefore, (u_n) is a bounded sequence.

Now we consider $\lambda_k < \lambda < \lambda_{k+1}$. It is convenient to decompose H into appropriate subspaces:

$$H = H_k \oplus H_k^{\perp}, \tag{1.8}$$

where H_k is finite dimensional and defined by

$$H_k = [\phi_1, ..., \phi_k], \tag{1.9}$$

where ϕ_j is the eigenfunction associated to eigenvalue λ_k.

This notation is standard when dealing with this framework of high order eigenvalues, and we will use it throughout this chapter.

For all u in H, let us take $u = u^k + u^{\perp}$, where $u^k \in H_k$ and $u^{\perp} \in H_k^{\perp}$. We notice that

$$\int_{B_1} \nabla u \nabla u^k \, \mathrm{d}x - \lambda \int_{B_1} u u^k \, \mathrm{d}x = \|u^k\|^2 - \lambda\|u^k\|_2^2 \tag{1.10}$$

and

$$\int_{B_1} \nabla u \nabla u^{\perp} \, \mathrm{d}x - \lambda \int_{B_1} u u^{\perp} \, \mathrm{d}x = \|u^{\perp}\|^2 - \lambda\|u^{\perp}\|_2^2. \tag{1.11}$$

By (1.5), (1.10) and the variational characterization of λ_k, we can see that

$$\begin{aligned} -\varepsilon_n\|u_n^k\| &\leq \int_{B_1} \nabla u_n \nabla u_n^k \, \mathrm{d}x - \lambda \int_{B_1} u_n u_n^k \, \mathrm{d}x - \int_{B_1} |x|^{\alpha} f(u_n) u_n^k \, \mathrm{d}x \\ &\leq \left(1 - \frac{\lambda}{\lambda_k}\right)\|u_n^k\|^2 - \int_{B_1} |x|^{\alpha} f(u_n) u_n^k \, \mathrm{d}x. \end{aligned}$$

Therefore,

$$\begin{aligned} C\|u_n^k\|^2 \leq & \ \varepsilon_n\|u_n^k\| - \int_{B_1} |x|^{\alpha} f(u_n) u_n^k \, \mathrm{d}x \\ \leq & \ \varepsilon_n\|u_n^k\| + \|u_n^k\|_{\infty} \int_{B_1} |x|^{\alpha}|f(u_n)|. \end{aligned} \tag{1.12}$$

Then, since H_k is a finite dimensional subspace, by (1.7) and (1.12)

$$\begin{aligned} C\|u_n^k\|^2 &\leq \varepsilon_n\|u_n^k\| + \|u_n^k\|_\infty \int_{B_1} |x|^\alpha f(u_n) \\ &\leq \varepsilon_n\|u_n^k\| + C\|u_n^k\|(C + \varepsilon_n\|u_n\|) \\ &\leq C + C\|u_n\| + C\varepsilon_n\|u_n\|^2. \end{aligned} \tag{1.13}$$

Now we notice that, by (1.4), (1.11) and the characterization of λ_{k+1}, we obtain

$$\begin{aligned} -\varepsilon_n\|u_n^\perp\| &\leq -\int_{B_1} \nabla u_n \nabla u_n^\perp + \lambda \int_{B_1} u_n u_n^\perp + \int_{B_1} |x|^\alpha f(u_n)u_n^\perp \\ &\leq \left(\frac{\lambda}{\lambda_{k+1}} - 1\right)\|u_n\|^2 + \int_{B_1} |x|^\alpha f(u_n)u_n^\perp. \end{aligned}$$

Therefore,

$$C\|u_n^\perp\|^2 \leq \varepsilon_n\|u_n^\perp\| + \int_{B_1} |x|^\alpha f(u_n)u_n^\perp. \tag{1.14}$$

By (1.7), (1.6) and (1.14), we get

$$\begin{aligned} C\|u_n^\perp\|^2 &\leq \varepsilon_n\|u_n^\perp\| + \int_{B_1} |x|^\alpha f(u_n)u_n + \|u_n^k\|_\infty(C + \varepsilon_n\|u_n\|) \\ &\leq \varepsilon_n\|u_n^{\perp\|} + C + \varepsilon_n\|u_n\| + C\|u_n\|(C + \varepsilon_n\|u_n\|) \\ &\leq C + C\|u_n\| + C\varepsilon_n\|u_n\|^2. \end{aligned} \tag{1.15}$$

By summing the inequalities in (1.13) and (1.15), we can see that

$$\|u_n\|^2 \leq C + C\|u_n\| + C\varepsilon_n\|u_n\|^2, \tag{1.16}$$

proving the boundedness of the sequence (u_n) as desired. ■

Remark 1.3.1. (a) *In the proof of Lemma 1.3.1 there is no difference between assuming subcritical or critical growth or considering the radial case or not. Then we can conclude that even in the case of critical growth, every Palais-Smale sequence is bounded.*

(b) *Lemma 1.3.1 still implies that*

$$\|u_n\| \leq C, \quad \|u_n\|_2 \leq C, \quad \int_{B_1} f(u_n)u_n \,\mathrm{d}x \leq C, \quad \int_{B_1} F(u_n)\,\mathrm{d}x \leq C \tag{1.17}$$

and if we assume (H_3), then the two integrals above are non-negative.

1.3.2 The $(PS)_c$ condition

Now we show that $(PS)_c$ condition is satisfied by functional Φ_λ. However, we can observe that there is an important difference in assuming subcritical or critical growth of nonlinearity in order to guarantee the $(PS)_c$ condition. Indeed, if we consider subcritical case, we shall obtain the $(PS)_c$ condition for all levels in $\mathbb{R}$. In critical case, we only obtain this condition for certain levels, which are bounded from above by suitable constants. Working in $H_0^1(B_1)$, we achieve a constant that depends on β_0, given in (CG). In $H^1_{0,\mathrm{rad}}(B_1)$, this constant also depends on α. First, we show this condition for subcritical case.

Lemma 1.3.2. Assume $(H_1)-(H_2)$ and (SG). Then the functional Φ_λ satisfies the $(PS)_c$ condition for all $c \in \mathbb{R}$.

Proof: We consider (u_n) a $(PS)_c$ sequence. By Lemma 1.3.1, we have a subsequence denoted again by (u_n) such that, for some $u \in H$, we have

$$u_n \rightharpoonup u \text{ in } H_0^1(B_1),$$

$$u_n \to u \text{ in } L^q(B_1) \text{ or on } L^q_{\mathrm{rad}}(|x|^\alpha, B_1) \text{ for all } q \geq 1,$$

$$u_n(x) \to u(x) \text{ almost everywhere in } B_1(0).$$

Notice that there is nothing else to prove in the case when $\|u_n\| \to 0$. Thus, one may suppose that $\|u_n\| \geq k > 0$ for n sufficiently large. It follows from (H_2) and [28, Lemma 2.1] that

$$\int_{B_1} |x|^\alpha F(u_n)\,\mathrm{d}x \to \int_{B_1} |x|^\alpha F(u)\,\mathrm{d}x.$$

We will prove that

$$\int_{B_1} |x|^\alpha f(u_n)u_n\,\mathrm{d}x \to \int_{B_1} |x|^\alpha f(u)u\,\mathrm{d}x. \tag{1.18}$$

In fact, we have

$$\left|\int_{B_1} |x|^\alpha f(u_n)u_n\,\mathrm{d}x - \int_{B_1} |x|^\alpha f(u)u\,\mathrm{d}x\right|$$

$$\leq \left|\int_{B_1} [f(u_n) - f(u)]u\,\mathrm{d}x\right| + \left|\int_{B_1} f(u_n)(u_n - u)\,\mathrm{d}x\right|.$$

Initially, let us focus on the second integral in the left side of this last estimate. By (SG), we get

$$\left|\int_{B_1} |x|^\alpha f(u_n)(u_n - u)\ \mathrm{d}x\right| \leq C \int_{B_1} e^{\beta u_n^2}|u_n - u|\ \mathrm{d}x \text{ for all } \beta > 0.$$

Using Höder inequality, we obtain

$$\int_{B_1} e^{\beta u_n^2}|u_n - u|\ \mathrm{d}x \leq \left(\int_{B_1} e^{q\beta\left(\frac{u_n}{\|u_n\|}\right)^2\|u_n\|^2}\ \mathrm{d}x\right)^{\frac{1}{q}} \|u_n - u\|_{q'},$$

where $1/q + 1/q' = 1$. We take $q > 1$ and by (SG) and Lemma 1.3.1, we can choose β sufficiently small such that $q\beta\|u_n\|^2 \leq 4\pi$. Thus by the Trudinger-Moser inequality, we have

$$\int_{B_1} e^{\beta u_n^2}|u_n - u|\ \mathrm{d}x \leq C_1\|u_n - u\|_{q'}. \tag{1.19}$$

Since $u_n \to u$ strongly in $L^{q'}$, one has

$$\left|\int_{B_1} |x|^\alpha f(u_n)(u_n - u)\ \mathrm{d}x\right| \to 0.$$

It remains to show that

$$\left|\int_{B_1} |x|^\alpha [f(u_n) - f(u)]u\ \mathrm{d}x\right| \to 0$$

as well. Indeed, let $\varepsilon > 0$ be given. By similarly arguments used to prove (1.19), there exists C_2 such that

$$|f(u_n) - f(u)\|_{2,|x|^\alpha} \leq C_2.$$

Consider $\xi \in C_0^\infty(B_1)$ such that $\|\xi - u\|_2 < \varepsilon/2C_2$. Now, since

$$\int_{B_1} |x|^\alpha |f(u_n) - f(u)|\mathrm{d}x \to 0,$$

for this ε, there exists n_ε such that

$$\int_{B_1} |x|^\alpha |f(u_n) - f(u)|\mathrm{d}x < \frac{\varepsilon}{2\|\xi\|_\infty}$$

for all $n \geq n_\varepsilon$. Therefore

$$\begin{aligned}&\left|\int_{B_1} |x|^\alpha [f(u_n) - f(u)]u\ \mathrm{d}x\right|\\ &\leq \int_{B_1} |x|^\alpha |f(u_n) - f(u)||\xi|\ \mathrm{d}x\\ &+ \int_{B_1} |x|^\alpha |f(u_n) - f(u)||\xi - u|\ \mathrm{d}x < \varepsilon\end{aligned}$$

for all $n \geq n_\varepsilon$, as desired. Consequently, we conclude that (1.18) holds.

Now taking $v = u$ and $n \to \infty$ in (1.5), we have

$$\|u\|^2 = \lambda \|u\|_2^2 + \int_{B_1} |x|^\alpha f(u) u \, \mathrm{d}x.$$

On the other hand, if $n \to \infty$ in (1.5) with $v = u_n$, since (1.18) is true, we have

$$\|u_n\|^2 \to \lambda \|u\|_2^2 + \int_{B_1} |x|^\alpha f(u) u \, \mathrm{d}x.$$

Hence, $\|u_n\| \to \|u\|$ and so $u_n \to u$ in H. ■

Remark 1.3.2. *Notice that* (1.19) *holds because, since we are supposing* (SG)*, we can choose* β *small enough. This fact is not true if we assume* (CG)*. Thus, only in the subcritical case, we can conclude the* $(PS)_c$ *condition is satisfied for all* $c \in \mathbb{R}$*.*

Now we consider the critical case, which will dramatically change the arguments in order to show the $(PS)_c$ condition for Φ_λ. Initially, we are working in $H_0^1(B_1)$, so we will prove the $(PS)_c$ condition for c below an appropriate constant depending on β_0.

Lemma 1.3.3. Assume $(H_1) - (H_3)$, (CG) and $H = H_0^1(B_1)$. Then the functional Φ_λ satisfies the $(PS)_c$ condition for $< 0 \leq < 2\pi/\beta_0$.

Proof: First, we consider (u_n) a $(PS)_c$ sequence. By Lemma 1.3.1, we obtain a subsequence denoted again by (u_n) such that, for some $u \in H$, we have

$$u_n \rightharpoonup u \text{ in } H_0^1(B_1),$$

$$u_n \to u \text{ in } L^q(B_1) \text{ or on } L^q_{\text{rad}}(|x|^\alpha, B_1) \text{ for all } q \geq 1,$$

$$u_n(x) \to u(x) \text{ almost everywhere in } B_1(0).$$

It follows from (H_2) and [28, Lemma 2.1], that $|x|^\alpha F(u_n) \to |x|^\alpha F(u)$ in $L^1(B_1(0))$. Using (1.4), we obtain

$$\lim \|u_n\|^2 = 2c + \lambda \|u\|_2^2 + 2 \int_{B_1} |x|^\alpha F(u) \, \mathrm{d}x \tag{1.20}$$

and, consequently, by (1.5) it follows

$$\lim \int_{B_1} |x|^\alpha f(u_n) u_n \, \mathrm{d}x = 2c + 2 \int_{B_1} |x|^\alpha F(u) \, \mathrm{d}x. \tag{1.21}$$

By (H_3) and (1.21), we conclude that $c \geq 0$. Then any $(PS)_c$ sequence approaches a non-negative level. It follows from [28, Lemma 2.1] and (1.5) that

$$\int_{B_1} \nabla u \nabla \psi \, \mathrm{d}x = \lambda \int_{B_1} u\psi \, \mathrm{d}x + \int_{B_1} |x|^\alpha f(u)\psi \, \mathrm{d}x, \quad \text{for all} \quad \psi \in C_0^\infty(B_1(0)).$$

Using elliptic regularity one can see that u is a classical solution for (P1). We still have that

$$\|u\|^2 - \lambda\|u\|_2^2 = \int_{B_1} |x|^\alpha f(u)u \, \mathrm{d}x \geq 2\int_{B_1} |x|^\alpha F(u) \, \mathrm{d}x.$$

Consequently, we can see that $\Phi_\lambda(u) \geq 0$. Now we need to show that $u_n \to u \in H$, in order to have this convergence, we separate the proof into three cases.

Case 1 $c = 0$. In this case, using (1.20) and the convergence of u_n in L^q for all $q > 1$, we have

$$0 \quad \leq \Phi_\lambda(u) \leq \liminf \Phi_\lambda(u_n) = \liminf \frac{1}{2}\|u_n\|^2 - \left(\frac{\lambda}{2}\|u\|_2^2 + \int_{B_1} |x|^\alpha F(u) \, \mathrm{d}x\right) = 0.$$

Then $\|u_n\| \to \|u\|$ and $u_n \to u$ in H, as we desire.

Case 2 $c \neq 0$ and $u = 0$. We show that this case cannot happen for a $(PS)_c$ sequence.

For some $q > 1$ and a $(PS)_c$ sequence $(u_n) \subset H$, such that $u_n \to 0$, we claim that

$$\int_{B_1} |f(u_n)|^q \, \mathrm{d}x \leq C_0. \tag{1.22}$$

Since $u = 0$, it follows from (1.20) that, given $\varepsilon > 0$, for n large enough we have

$$\|u_n\| \leq 2c + \varepsilon. \tag{1.23}$$

In order to estimate the integral (1.22) using (CG), we notice that

$$\int_{B_1} |f(u_n)|^q \, \mathrm{d}x \leq C\int_{B_1} e^{q\beta_0 u_n^2} \, \mathrm{d}x \leq C\int_{B_1} e^{q\beta_0\|(u_n)\|^2\left(\frac{u_n}{\|(u_n)\|}\right)^2} \, \mathrm{d}x. \tag{1.24}$$

By the Trudinger-Moser inequality, the integral in (1.24) is bounded, independently from n, if $q\beta_0\|u_n\|^2 \leq 4\pi$. Because of (1.23), this will be indeed the case $c \leq 2\pi/\beta_0$, if we choose $q > 1$ sufficiently close to 1 and ε is sufficiently small.

Now using (1.5) with $v = u_n$, we have

$$\left| \|u_n\|^2 - \lambda\|u_n\|_2^2 - \int_{B_1} |x|^\alpha f(u_n)u_n \, \mathrm{d}x \right| \leq \varepsilon_n\|u_n\| \leq (c + \varepsilon)\varepsilon_n. \tag{1.25}$$

We estimate the last integral above using the Hölder inequality. Initially, we consider q' such that $(1/q) + (1/q') = 1$. Then by (1.24), we have

$$\left| \int_{B_1} |x|^\alpha f(u_n) u_n \, \mathrm{d}x \right| \leq C \left(\int_{B_1} |u_n|^{q'} \, \mathrm{d}x \right)^{1/q'}.$$

Then, from (1.22), since $u_n \to 0$ in $L^{q'}$, we conclude that $\|u_n\| \to 0$. This contradicts (1.20), which says, $\|u_n\|^2 \to 2c \neq 0$.

Case 3 $c \neq 0$ and $u \neq 0$. We shall prove that

$$\Phi_\lambda(u) = c.$$

Assume by contraction that $\Phi_\lambda < c$. Thus,

$$\|u\|^2 < 2 \left(c + \lambda/2 \|u\|_2^2 + \int_{B_1} |x|^\alpha F(u) \, \mathrm{d}x \right). \tag{1.26}$$

Let $v_n = u_n / \|u_n\|$ and

$$v = \frac{u}{\sqrt{2(c + \lambda/2\|u\|_2^2 + \int_{B_1} |x|^\alpha F(u) \, \mathrm{d}x)}}.$$

Since $v_n \rightharpoonup v$, $\|v_n\| = 1$, and $\|v\| < 1$, it follows by [38, Theorem I.6] that

$$\sup \int_{B_1} e^{4\pi p v_n^2} \, \mathrm{d}x < \infty, \quad \text{for all} \quad p < \frac{1}{1 - \|v\|^2}.$$

Using (CG), we can obtain the following estimate of $f(u_n)$ in L^q

$$\int_{B_1} |f(u_n)|^q \, \mathrm{d}x \leq C \int_{B_1} e^{q\beta_0 u_n} \, \mathrm{d}x \leq C \int_{B_1} e^{q\beta_0 u_n^2} \, \mathrm{d}x \leq C \int_{B_1} e^{q\beta_0 \|u_n\|^2 v_n^2} \, \mathrm{d}x$$

and this will be bounded if q and p can be chosen such that

$$q\beta_0 \|u_n\|^2 \leq 4\pi p \leq 4\pi \frac{c + \lambda/2\|u\|_2^2 + \int_{B_1} |x|^\alpha F(u) \, \mathrm{d}x}{c - \Phi_\lambda(u)} = \frac{4\pi}{1 - \|v\|^2}.$$

For large n, this will happen if

$$\frac{\beta_0}{2\pi} < \frac{1}{c - \Phi_\lambda(u)},$$

which is actually so, since $\Phi_\lambda(u) \geq 0$ and $c < 2\pi/\beta_0$.

Then, from (1.22), since $u_n \to u$ in $L^{q'}$ and $f(u_n)$ is bounded in L^q for some q, we conclude that $\|u_n\| \to \|u\|$. Consequently, $u_n \to u$ in H. This is impossible in view of (1.20) and (1.26) and we conclude that $\Phi_\lambda(u) = c$. Thus, the proof of Lemma 1.3.3 is complete. ■

If we consider $H = H^1_{0,\text{rad}}(B_1)$ and $f(t)$ having a critical growth, we should also prove that the $(PS)_c$ condition is satisfied by minimax level, which are bounded from above by an appropriate constant. But in this context, this constant, in addition to relying on β_0, it also depends on α, which allows an increasing in the range of the minimax levels.

Lemma 1.3.4. Assume $(H_1) - (H_3)$, (CG) and $H = H^1_{0,\text{rad}}(B_1)$. Then the functional Φ_λ satisfies the $(PS)_c$ condition for $c < (2+\alpha)\pi/\beta_0$.

Proof: Since we work in $H = H^1_{0,\text{rad}}(B_1)$ and the wieght $|x|^\alpha$ is in the nonlinearity, we can change (3) into (4), so the proof follows exactly the same steps of Lemma 1.3.3. ■

1.4 Proof of Theorems 1.2.1 and 1.2.2: Subcritical Case

This section is devoted to the proof of Theorems 1.2.1 and 1.2.2. Here, we consider $H = H^1_0(B_1)$, although, in the case that $H = H^1_{0,\text{rad}}(B_1)$, the proof uses the same arguments. We begin proving the geometric conditions of the Mountain–Pass Theorem and the Linking Theorem are satisfied for associated functional Φ_λ in Theorems 1.2.1 and 1.2.2, respectively.

1.4.1 Geometric conditions

Initially, we will show that $\Phi_\lambda(u) \leq 0$, when u belongs to a finite dimensional subspace of H and norm of u is large enough. This fact helps us to prove the geometric condition of the Mountain–Pass Theorem to $\lambda < \lambda_1$ and of the Linking Theorem to $\lambda_k < \lambda < \lambda_{k+1}$.

Proposition 1.4.1. *Assume (H_1) and (H_2). Let V be a finite dimensional subspace of H. Then Φ_λ is bounded from above in V, and moreover, given $M > 0$ there is an $R > 0$ such that*

$$\Phi_\lambda(u) \leq -M \text{ for all } \|u\| \geq R \text{ and } u \in V.$$

Proof: Given $u_0 \in V$, let us define

$$\eta(t) = \Phi_\lambda(tu_0) = \frac{t^2}{2}\|u_0\|^2 - \frac{\lambda}{2}\|u_0\|_2^2 - \int_{B_1} |x|^\alpha F(tu_0)\,\mathrm{d}x \text{ for all } t \in \mathbb{R}.$$

It follows from (H_2) (see (1.1)) that for $\sigma > 2$ there is a constant $C > 0$ such that

$$F(t) \geq C|t|^\sigma - C \text{ for all } t \in \mathbb{R}. \tag{1.27}$$

Thus,

$$\eta(t) \leq \frac{t^2}{2}\|u_0\|^2 - \frac{t^2\lambda}{2}\|u_0\|_2^2 - C\int_{B_1} |x|^\alpha |t|^\sigma |u_0|^\sigma \, \mathrm{d}x + C \text{ for all } t.$$

By the equivalence of norms in V, we obtain

$$\eta(t) \leq \frac{t^2}{2}\|u_0\|^2 - C\frac{2\pi}{2+\alpha}|t|^\sigma \|u_0\|^\sigma + C \text{ for all } t,$$

which implies that $\eta(t) \to -\infty$ as $t \to \infty$. The result follows by compactness. ■

Proposition 1.4.2. *Assume* $0 < \lambda < \lambda_1$, (H_1), (H_2), (H_4) *and* (1.3). *Then there exists* $a > 0$ *and* $\rho > 0$ *such that*

$$\Phi_\lambda(u) \geq a \text{ if } \|u\| = \rho.$$

Proof: By (H_4) we know that there exist $\mu < \lambda_1 - \lambda$ and $\delta > 0$ such that

$$F(t) \leq \frac{\mu}{2}t^2, \text{ if } |t| \leq \delta.$$

On the other hand, from (1.3) we obtain

$$F(t) \leq Ce^{\beta t^2}|t|^3 \text{ for all } t \in \mathbb{R} \text{ for } \beta \text{ large enough .}$$

Hence we obtain

$$F(t) \leq \frac{\mu}{2}t^2 + Ce^{\beta t^2}|t|^3 \text{ for all } t \in \mathbb{R},$$

which implies

$$\begin{aligned}
\Phi_\lambda(u) \ &\geq \frac{1}{2}\|u\|^2 - \frac{1}{2}\left(\frac{\mu+\lambda}{\lambda_1}\right)\|u\|_2^2 - C\int e^{\beta u^2}|u|^3 \, \mathrm{d}x \\
&\geq \frac{1}{2}\left[1 - \left(\frac{\mu+\lambda}{\lambda_1}\right)\right]\|u\|^2 - C\left(\int_{B_1} e^{2\beta u^2} \, \mathrm{d}x\right)^{1/2} \|u\|_6^3.
\end{aligned}$$

Using the Trudinger-Moser inequality we have that

$$\int_{B_1} e^{2\beta u^2} \, \mathrm{d}x = \int_{B_1} e^{2\beta\|u\|^2\left(\frac{u}{\|u\|}\right)^2} \, \mathrm{d}x < C_0$$

if $\|u\|^2 \leq \tau$ and $2\beta\tau < 4\pi$ if $u \in H_0^1(B_1)$. So

$$\Phi_\lambda(u) \geq \frac{1}{2}\left[1 - \left(\frac{\mu+\lambda}{\lambda_1}\right)\right]\|u\|^2 - C\|u\|^3.$$

Now we choose $\rho > 0$ as the point where the function $g(s) = \frac{1}{2}\left[1 - \left(\frac{\mu+\lambda}{\lambda_1}\right)\right] s^2 - Cs^3$ assumes its maximum and take $a = g(\rho)$. The result follows. ■

Here, we consider ϕ_j to be the eigenfunctions of $(-\Delta, H_0^1(B_1))$ corresponding to the eigenvalues λ_j for all $j \in \mathbb{N}$ and let us denote the subspace H_k as in (1.9).

Proposition 1.4.3. *Assume $\lambda_k < \lambda < \lambda_{k+1}$, (H_1), (H_2), (1.3) and (H_5). Then, there exist $a, \rho > 0$ such that*

$$\Phi_\lambda(u) \geq a \text{ if } \|u\| = \rho \text{ and } u \in H_k^\perp.$$

Using the variational characterization of λ_{k+1}, the proof follows the same way of Proposition 1.4.2. See below. From (H_5), we notice that there exist $\mu < \lambda_{k+1} - \lambda$ and $\delta > 0$ such that

$$F(t) \leq \frac{\mu}{2} t^2, \text{ if } |t| \leq \delta.$$

Moreover, from (1.3) we get

$$F(t) \leq Ce^{\beta t^2} |t|^3 \text{ for all } t \in \mathbb{R}.$$

Consequently, we obtain

$$F(t) \leq \frac{\mu}{2} t^2 + Ce^{\beta t^2} |t|^3 \text{ for all } t \in \mathbb{R},$$

which implies, together with the variational characterization of λ_{k+1}, that

$$\begin{aligned} \Phi_\lambda(u) &\geq \frac{1}{2}\|u\|^2 - \frac{1}{2}\left(\frac{\mu+\lambda}{\lambda_{\lambda_{k+1}}}\right)\|u\|_2^2 - C\int_{B_1} e^{\beta u^2}|u|^3 \,\mathrm{d}x \\ &\geq \frac{1}{2}\left[1 - \left(\frac{\mu+\lambda}{\lambda_{k+1}}\right)\right]\|u\|^2 - C\left(\int_{B_1} e^{2\beta u^2}\,\mathrm{d}x\right)^{1/2}\|u\|_6^3. \end{aligned}$$

By the Trudinger-Moser inequality, we know that

$$\int_{B_1} e^{2\beta u^2}\,\mathrm{d}x = \int_{B_1} e^{2\beta\|u\|^2\left(\frac{u}{\|u\|}\right)^2}\,\mathrm{d}x < C_0$$

if $\|u\| \leq \delta^{1/2}$, where $\beta\delta < 2\pi$. So

$$\Phi_\lambda(u) \geq C\|u\|^2 - C\|u\|^3 > 0.$$

for all $\|u\| < \min\{1, \delta^{1/2}\}$. ■

Remark 1.4.1. *Propositions 1.4.1, 1.4.2 and 1.4.3 are available for both cases (subcritical and critical).*

Proposition 1.4.4. *Suppose that* $(H_1)-(H_3)$, (1.3) *and* (H_5) *hold and* $\lambda_k < \lambda < \lambda_{k+1}$. *Let* Q *be as*

$$Q := \{v + s\phi_{k+1} : v \in H_k, \|v\| \leq R \text{ and } 0 \leq s \leq R\},$$

where ϕ_{k+1} *is eigenfunctions corresponding to eigenvalue* λ_{k+1}. *Then*

$$\Phi_\lambda(u) \leq 0 \text{ for all } u \in \partial Q.$$

Proof: We set $R_0 > \rho$ and $R \geq R_0$. Now we consider a usual split $\partial Q = Q_1 \cup Q_2 \cup Q_3$, where

$$\begin{aligned} Q_1 &= \{v \in H_k : \|v\| \leq R\}; \\ Q_2 &= \{v + s\phi_{k+1} : \|v\| = R \text{ and } 0 \leq s \leq R\}; \\ Q_3 &= \{v + R\phi_{k+1} : \|v\| \leq R\}. \end{aligned}$$

Let v be on Q_1, then by (H_3) and the variational characterization of λ_k, it follows that

$$\begin{aligned} \Phi_\lambda(v) &\leq \frac{1}{2}\|u\|^2 - \frac{\lambda}{2}\|u\|_2^2 \\ &\leq \frac{1}{2}\left(1 - \frac{\lambda}{\lambda_k}\right)\|v\|^2 \leq 0, \end{aligned} \tag{1.28}$$

independently of $R > 0$.

On Q_2, we have

$$\|v + s\phi_{k+1}\|^2 = \|v\| + s\|^2\phi_{k+1}\|^2 \geq \|v\|^2 = R^2$$

and on Q_3, we can see that

$$\|v + R\phi_{k+1}\|^2 = R^2\|\phi_{k+1}\|^2 + \|v\|^2 \geq R^2.$$

Let us take $R \geq R_0$ large enough, such that by Proposition 1.4.1 one has $\Phi_\lambda(u) \leq 0$ for $u \in \partial Q$. ■

Remark 1.4.2. *Propositions 1.4.1 and 1.4.2 are proved exactly the same way when we work in* $H^1_{0,\mathrm{rad}}(B_1)$. *However, in order to show that Propositions 1.4.3 and 1.4.4 are also available in* $H^1_{0,\mathrm{rad}}(B_1)$, *we need to split it into two orthogonal subspaces as we usually do with* $H^1_0(B_1)$. *Let us recall the notation introduced in* (1.9) *and that*

$$H^1_{0,\mathrm{rad}}(B_1) = \bigcup_{k=1}^{\infty} H^*_k. \tag{1.29}$$

Then, in order to prove Lemma 1.3.1, when one considers the radial case, we should use the decomposition given by (1.29).

1.4.2 Proof of Theorems 1.2.1 and 1.2.2 completed

We know that from the hypotheses in both theorems that Φ_λ satisfies $(PS)_c$ for all $c \in \mathbb{R}$. Using the Propositions 1.4.1 and 1.4.2 and applying the Mountain–Pass Theorem, we prove Theorem 1.2.1. From Propositions 1.4.3 and 1.4.4, Theorem 1.2.2 follows, applying the Linking Theorem. If we consider $H = H_{0,\mathrm{rad}}(B_1)$ in the proofs of these last Theorems, we have that solutions are radially symmetric. ■

1.5 Proof of Theorems 1.2.3 and 1.2.4: Critical Case in $H_0^1(B_1)$

This section is dedicated to the proof of Theorems 1.2.3 and 1.2.4. Since we cannot obtain a Palais-Smale condition for the functional Φ_λ as we have done in the subcritical case, we need to show that the minimax levels are below an appropriate constant in order to recover a similar compactness property for the functional. Before that, we show the geometric conditions of the Mountain–Pass Theorem when $\lambda < \lambda_1$ and in the Linking Theorem if $\lambda_k < \lambda < \lambda_{k+1}$.

1.5.1 Geometric conditions

Initially, we need to introduce the so-called Moser functions

$$\bar{z}_n(x) = \frac{1}{\sqrt{2\pi}} \begin{cases} (\log n)^{1/2}, & 0 \leq |x| \leq \frac{1}{n}; \\ \dfrac{\log \frac{1}{|x|}}{(\log n)^{1/2}} & \frac{1}{n} \leq |x| \leq 1; \\ 0 & |x| > 1. \end{cases} \tag{1.30}$$

This kind of sequence is usually used to guarantee that minimax levels lies under an appropriate constant, which will allow us to recover the compactness properties for Φ_λ that are lost when dealing with critical growth ranges. Let us consider a suitable translation of Moser's functions in a region of B_1 far from the origin where the presence of $|x|^\alpha$ can be neglected. First, let us set $x_0 \in B_1$ and $0 < r \leq d < 1$ such that $B_r(x_0) \subset B_1$ and $|x| > d$ in $B_r(x_0)$ and define the functions

$$z_n(x) = \bar{z}_n\left(\frac{x - x_0}{r}\right), \tag{1.31}$$

which are in $H_0^1(B_1)$, with $\|z_n\| = 1$ and $\mathrm{supp} z_n \subset B_r(x_0)$.

Remark 1.5.1. *If $\alpha = 0$ and $0 < \lambda < \lambda_1$, we do not need to have $|x| > d$, thus it is sufficient to consider x_0 equal to origin and $r = 1$.*

Fist of all, we will see that for large n we still have the same geometric condition proven in the subcritical case. For $0 < \lambda < \lambda_1$, we see that Φ_λ has a local minimum at 0. Indeed, by Proposition 1.4.2, we have that there exist $a, \rho > 0$ such that

$$\Phi_\lambda(u) \geq a \text{ if } \|u\| = \rho \text{ for all } u \in H$$

and from Proposition 1.4.1, we see that, for n large enough, there exist $R_n > \rho$ such that

$$\Phi_\lambda(R_n z_n) \leq 0.$$

For $\lambda_k < \lambda < \lambda_{k+1}$, the proof of Theorem 1.2.4 is accomplished by the use of the Linking Theorem. As in Proposition 1.4.3, we have that there exist $a, \rho > 0$

$$\Phi_\lambda(u) \geq a \text{ if } \|u\| = \rho \text{ for all } u \in H_k^\perp.$$

Now, we prove the following proposition in order to conclude that geometric conditions of the saddle point case are satisfied.

Proposition 1.5.1. *Suppose that $\lambda_k < \lambda < \lambda_{k+1}$, $(H_1) - (H_3)$, (CG) and (H_5) hold. We consider $z \notin H_k$. Let Q be as follows*

$$Q := \{v + sz : v \in H_k, \|v\| \leq R \text{ and } 0 \leq s \leq R\},$$

$$\Phi_\lambda(v) \leq 0 \text{ for all } v \in \partial Q.$$

Proof: First, we split ∂Q as usual

$$\partial Q = Q_1 \cup Q_2 \cup Q_3$$

where

$$\begin{aligned} Q_1 &= \{v \in H_k : \|v\| \leq R\};\\ Q_2 &= \{v + sz : \|v\| = R, 0 \leq s \leq R\};\\ Q_3 &= \{v + Rz : \|v\| \leq R\}. \end{aligned}$$

If $v \in Q_1$, using (H_3) and the variational characterization of λ_k, we have

$$\Phi_\lambda(v) \leq \frac{1}{2}\|v\|^2 - \frac{\lambda}{2}\|v\|_2^2 \leq \frac{1}{2}\left(1 - \frac{\lambda}{\lambda_k}\right)\|v\|^2 \leq 0,$$

independently of R.

On Q_2, let us take $R_1 > \rho$, with $\|v\| = R_1$ and $0 \leq s \leq R_1$, such that there exists R_2 as in Proposition 1.4.1 satisfying

$$\|v + sz\| > R_2 > \rho,$$

and

$$\Phi_\lambda(v + sz) \leq 0.$$

Analogously, on Q_3 let us take $R_3 > \rho$, so there exists R_4 such that

$$\|v + R_3 z\| \geq R_4 > \rho,$$

and we also obtain by Proposition 1.4.1

$$\Phi_\lambda(v + R_3 z) \leq 0.$$

For $R \geq \max\{R_1, R_3, R_4\}$, the result follows. ■

Remark 1.5.2. *Notice that z_n, given in* (1.31)*is not in H_k for all $n \in \mathbb{N}$, so we will use Proposition 1.5.1 with $z = z_n$.*

1.5.2 Estimate of minimax levels

For the Mountain–Pass case, we define the minimax level of Φ_λ by

$$\tilde{c} = \tilde{c}(n) = \inf_{\upsilon \in \Gamma} \max_{w \in \upsilon([0,1])} \Phi_\lambda(w) \tag{1.32}$$

where

$$\Gamma = \{\upsilon \in C([0,1], H) : \upsilon(0) = 0 \text{ and } \upsilon(1) = R_n z_n\},$$

R_n being such that $\Phi_\lambda(R_n z_n) \leq 0$ and z_n is given in (1.31).

Proposition 1.5.2. *Let $\tilde{c}(n)$ be given in* (1.32)*. Then there exists n large enough such that*

$$\tilde{c}(n) < \frac{2\pi}{\beta_0}.$$

Proof: We claim that there exists n such that

$$\max_{t \geq 0} \Phi_\lambda(t z_n) < \frac{2\pi}{\beta_0}.$$

Suppose by contradiction that is not case. So, for all n, this maximum is larger than or equal to $2\pi/\beta_0$ (it is indeed a maximum, in view of Proposition 1.4.1). Let $t_n > 0$ be such that

$$\Phi_\lambda(t_n z_n) = \max_{t\geq 0} \Phi_\lambda(t z_n). \tag{1.33}$$

Then

$$\Phi_\lambda(t_n z_n) \geq \frac{2\pi}{\beta_0} \text{ for all } n \in \mathbb{N},$$

and, consequently,

$$t_n^2 \geq \frac{4\pi}{\beta_0} \text{ for all } n \in \mathbb{N}. \tag{1.34}$$

Let us prove that $t_n^2 \to 4\pi/\beta_0$. From (1.33) we obtain

$$\frac{d}{dt}(\Phi_\lambda(t z_n))\bigg|_{t=t_n} = 0.$$

Multiplying this last equation by t_n and noticing that $\|z_n\| = 1$, $\lambda, f(t) \geq 0$, we have, for n large enough, that

$$t_n^2 \geq \int_{B_r(x_0)} |x|^\alpha f(t_n z_n) t_n z_n \,\mathrm{d}x,$$

By (1.31), one has $|x| \geq d$, for all $x \in B_r(x_0)$. Thus,

$$t_n^2 \geq d^\alpha \int_{B_r(x_0)} f(t_n z_n) t_n z_n \,\mathrm{d}x. \tag{1.35}$$

By (H_6), it follows that given $\varepsilon > 0$ there is $t_\varepsilon > 0$ such that

$$f(t)t \geq (\xi - \varepsilon) e^{\beta_0 t^2}, \text{ for all } t > t_\varepsilon.$$

Then from (1.35), for n large enough, we obtain

$$\begin{aligned} t_n^2 &\geq (\xi - \varepsilon) d^\alpha \int_{B_r(x_0)} e^{\beta_0 t_n^2 z_n^2} \,\mathrm{d}x \\ &= (\xi - \varepsilon)\pi d^\alpha r^2 e^{2\log n \left(\frac{\beta_0 t_n^2}{4\pi} - 1\right)}, \end{aligned} \tag{1.36}$$

which implies that t_n is bounded. And moreover, (1.34) together with (1.36) gives us that $t_n^2 \to 4\pi/\beta_0$.

Let us estimate (1.35) more precisely. Let

$$A_n = \{x \in B_r(x_0) : t_n z_n(x) \geq t_\varepsilon\}, \quad B_n = B_r(x_0) \setminus A_n,$$

and split the integral in (1.35) into a sum of the integrals over A_n and B_n. Using (H_6), we have

$$t_n^2 \geq d^\alpha \left[(\xi - \varepsilon) \int_{B_r(x_0)} e^{\beta_0 t_n^2 z_n^2} \,\mathrm{d}x + \int_{B_n} f(t_n z_n) t_n z_n \,\mathrm{d}x - (\xi - \varepsilon) \int_{B_n} e^{\beta_0 t_n^2 z_n} \,\mathrm{d}x \right]. \tag{1.37}$$

Since $t_n z_n < t_\varepsilon$ for $x \in B_n$, one has $\chi_{B_n} \to 1$ almost everywhere in $B_r(x_0)$ as $n \to \infty$. Thus, by Lebesgue Dominated Convergence Theorem, we conclude that

$$\int_{B_n} f(t_n z_n) t_n z_n \, \mathrm{d}x \to 0 \quad \text{and} \quad \int_{B_n} e^{\beta_0 t_n^2 z_n} \, \mathrm{d}x \to r^2 \pi.$$

Passing to the limit in (1.37), we obtain

$$\frac{4\pi}{\beta_0} \geq d^\alpha(\xi - \varepsilon) \left[\lim \int_{B_r(x_0)} e^{4\pi 2 z_n^2} \, \mathrm{d}x - r^2 \pi \right]. \tag{1.38}$$

The last integral in (1.38), denoted by J_n, is evaluated as follows

$$J_n = r^2 \int_{B_1} e^{4\pi \bar{z}_n^2} \, \mathrm{d}x = r^2 \left\{ \frac{\pi}{n^2} e^{4\pi \frac{1}{2\pi} \log n} + 2\pi \int_{1/n}^{1} e^{4\pi \frac{1}{2\pi} \frac{\left(\log \frac{1}{r}\right)^2}{\log n}} r \, \mathrm{d}r \right\}. \tag{1.39}$$

Changing variables in the integral in (1.39), $s = \log(\frac{1}{r}) / \log n$, we have

$$J_n = r^2 \left\{ \pi + 2\pi \log n \int_0^1 e^{s^2 \log n - 2s \log n} \, \mathrm{d}s \right\}.$$

Thus from (1.38), we obtain

$$\frac{4\pi}{\beta_0} \geq (\xi - \varepsilon) d^\alpha r^2 \pi M_0 \text{ for all } \varepsilon > 0,$$

which implies

$$\xi \leq \frac{4}{\beta_0 M_0 d^\alpha r^2},$$

a contradiction to (H_6). ■

In the Linking case with $\alpha > 0$, we define the minimax level of Φ_λ by

$$\hat{c} = \hat{c}(n) = \inf_{\upsilon \in \Gamma} \sup_{w \in \upsilon(Q)} \Phi_\lambda(w) \tag{1.40}$$

where

$$\Gamma = \{\upsilon \in C(Q, H) : \upsilon(\omega) = \omega \text{ if } \omega \in \partial Q\}.$$

First, in order to prove Theorem 1.2.4, let us prove the following proposition.

Proposition 1.5.3. *Let $\hat{c}(n)$ be given in (1.40). Assume $\alpha > 0$ and (H_7), then there exists n large enough such that*

$$\hat{c}(n) < \frac{2\pi}{\beta_0}.$$

Proof: In the same spirit of [2], we demand to show that there is $n > \rho$ such that

$$\max\{\Phi_\lambda(v + sz_n) : \|v\| \leq R_n, 0 \leq s \leq R_n\} < \frac{2\pi}{\beta_0} \tag{1.41}$$

where we set z_n as in (1.31). First, we need to take d and r, given in (1.31), according to hypothesis (H_7). Since, $\alpha > 0$, we can choose $0 < r \leq d < 1$, previously fixed, satisfying

$$\left(\frac{d}{r}\right)^\alpha \geq 1 > \frac{2}{2+\alpha}. \tag{1.42}$$

Now, we assume, by contradiction, that (1.41) is not true. Then for all n

$$\max\{\Phi_\lambda(v + sz_n) : \|v\| \leq R_n, 0 \leq s \leq R_n\} \geq \frac{2\pi}{\beta_0}.$$

Let $u_n = v_n + t_n z_n$ be the point where this maximum is achieved. So

$$\Phi_\lambda(u_n) \geq \frac{2\pi}{\beta_0} \tag{1.43}$$

and using the fact that the derivative of Φ_λ, restricted to $H_k \oplus \mathbb{R}z_n$, is zero at u_n, we obtain

$$\|u_n\|^2 - \lambda\|u_n\|_2^2 - \int_{B_1} |x|^\alpha f(u_n)u_n \,\mathrm{d}x = 0. \tag{1.44}$$

Initially, we accept the following claims, which will be proved later.

Claim 1. *$(v_n), (t_n)$ are bounded sequences.*

We can assume that the subsequences also denoted by (v_n) and (t_n) such that $v_n \to v_0$ and $t_n \to t_0$.

Claim 2. *$v_0 = 0$ and $t_0^2 = \frac{4\pi}{\beta_0}$.*

From (1.44), we notice that

$$\lim \|u_n\|^2 \geq \lim \int_{B_r(x_0)} |x|^\alpha f(u_n) \,\mathrm{d}x \tag{1.45}$$

and consider

$$\tilde{A}_n = \{x \in B_r(x_0) : u_n \geq t_\varepsilon\}, \quad \tilde{B}_n = B_r(x_0) \setminus \bar{A}_n,$$

and split the integral in (1.45) into a sum of the integrals over $\tilde{A}_n$ and $\tilde{B}_n$. Using (H_7) we can see that

$$\begin{aligned} \int_{B_1} |x|^\alpha f(u_n)u_n \,\mathrm{d}x \geq \quad & (\xi - \varepsilon)\int_{B_r(x_0)} |x|^\alpha e^{\beta_0 u_n^2} \,\mathrm{d}x \\ & + \int_{\tilde{B}_n} |x|^\alpha f(u_n)u_n \,\mathrm{d}x - (\xi - \varepsilon)\int_{\tilde{B}_n} |x|^\alpha e^{\beta_0 u_n^2} \,\mathrm{d}x. \end{aligned} \tag{1.46}$$

Since $\chi_{\tilde{B}_n} \to 1$, from Claim 2 and Dominated Convergence Theorem, we have

$$\lim \int_{\bar{B}_n} |x|^\alpha f(u_n)u_n \, \mathrm{d}x = 0 \quad \text{and} \quad \lim \int_{\bar{B}_n} |x|^\alpha e^{\beta_0 u_n^2} \, \mathrm{d}x = \frac{2\pi r^{2+\alpha}}{2+\alpha}.$$

Now we notice that, for n large enough, given $\varepsilon > 0$, there is C_ε such that

$$u_n^2 \geq (1-\varepsilon)t_n^2 z_n^2 - C_\varepsilon v_n^2.$$

Since $t_n^2 \to 4\pi/\beta_0$, we obtain for n large enough

$$u_n^2 \geq (1-\varepsilon)^2 \frac{4\pi}{\beta_0} z_n^2 - C_\varepsilon \|v\|_\infty. \tag{1.47}$$

Thus, using (1.42) and (1.47), the first integral on the right side in (1.46) is estimated from below by

$$\int_{B_r(x_0)} |x|^\alpha e^{\beta_0 u_n^2} \, \mathrm{d}x \geq d^\alpha e^{-C_\varepsilon \|v\|_\infty} \int_{B_r(x_0)} e^{(1-\varepsilon)^2 4\pi \bar{z}_n^2} \, \mathrm{d}x. \tag{1.48}$$

We denote the integral over $B_r(x_0)$ in (1.48) by J_n and we can evaluate it as follows

$$\begin{aligned} J_n = &\ r^2 \int_{B_1} e^{(1-\varepsilon)^2 4\pi z_n^2} \, \mathrm{d}x \\ = &\ r^2 \left\{ \tfrac{\pi}{n^2} e^{(1-\varepsilon)^2 2\log n} + 2\pi \int_{1/n}^1 e^{(1-\varepsilon)^2 (\log \frac{1}{r})^2 (\log n)^{-1}} r \, \mathrm{d}r \right\}. \end{aligned} \tag{1.49}$$

Changing variables in the integral, $s = \log\left(\frac{1}{r}\right) / \log n$, we can see that the second integral in (1.49) converges to $\pi \widehat{M}$. Hence, $J_n \to r^2 \pi \widehat{M}$. Hence we obtain

$$\|u_n\|^2 \geq (\xi - \varepsilon)\pi r^2 \left(d^\alpha \widehat{M} - \frac{2r^\alpha}{2+\alpha} \right) = (\xi-\varepsilon)\pi r^{2+\alpha} \left[\left(\frac{d}{r}\right)^\alpha - \frac{2}{2+\alpha} \right]. \tag{1.50}$$

On the hand, since $v_n \to 0$, for any ε, we have

$$\lim \|u_n\|^2 = \lim t_n^2 \|z_n\|^2 + \lim 2t_n \int_{B_1} z_n v_n \, \mathrm{d}x + \lim \|v_n\|^2 \leq (1+\varepsilon) \lim t_n^2 = (1+\varepsilon)\frac{4\pi}{\beta_0}.$$

Then it follows from (1.48) and (1.50) that

$$(1+\varepsilon)\frac{4\pi}{\beta_0} \geq (\xi - \varepsilon)\pi r^{2+\alpha} \left[\left(\frac{d}{r}\right)^\alpha - \frac{2}{2+\alpha} \right], \text{ for all } \varepsilon > 0,$$

which implies

$$\xi \leq \frac{4}{\beta_0 r^{2+\alpha}} \left[\left(\frac{d}{r}\right)^\alpha - \frac{2}{2+\alpha} \right]^{-1},$$

a contradiction to (H_7). ■

Remark 1.5.3. *We notice that the last contradiction only happens because $\alpha > 0$ and r and d satisfy* (1.42)*. If $\alpha = 0$, we should assume (H_8), so we can obtain a result as Proposition 1.63 in the next section.*

Proof of Claim 1: Given (t_n) and (v_n) as above, one of the following two possibilities has to hold:

(i) either there exists a constant $C_0 > 0$ such that $t_n/\|v_n\| \geq C_0$, or

(ii) there are subsequences, denoted also by (t_n) and (v_n), such that $t_n/\|v_n\| \to 0$.

If **(i)** holds, then there is a constant $C > 0$ such that

$$\|u_n\| \leq \|v_n\| + t_n \leq Ct_n,$$

which applied to (1.44) gives

$$C^2 t_n^2 \geq \|u_n\|^2 \geq \int_{B_r(x_0)} |x|^\alpha f(u_n) u_n \, \mathrm{d}x \geq d^\alpha (\xi - \varepsilon) \int_{B_{r/n}} e^{\beta_0 u_n^2} \, \mathrm{d}x, \tag{1.51}$$

where (H_7) was used. We can write

$$u_n(x) = \frac{t_n}{\sqrt{2\pi}} (\log n)^{1/2} \left(\frac{v_n(x)}{t_n} \frac{\sqrt{2\pi}}{(\log n)^{1/2}} + 1 \right) \text{ for all } x \in B_{r/n}(x_0).$$

Hence, given $\varepsilon \in (0,1)$, we have $u_n(x) \geq (1-\varepsilon) t_n (\log n)^{1/2}/\sqrt{2\pi}$ for large n and $x \in B_{r/n}(x_0)$. Together with (1.51), we obtain

$$C^2 t_n^2 \geq d^\alpha (\xi - \varepsilon) \frac{\pi r^2}{n^2} e^{\beta_0/2\pi t_n^2 (1-\varepsilon)^2 \log n},$$

which can be written as

$$1 \quad \geq d^\alpha (\xi - \varepsilon) \frac{\pi r^2}{C^2 t_n^2 n^2} e^{\beta_0/2\pi t_n^2 (1-\varepsilon)^2 \log n} \tag{1.52}$$

It follows that t_n is bounded. Consequently, $\|v_n\|$ is also bounded in this case.

Now, we assume that **(ii)** is true. Then , for n sufficiently large, we have $t_n \leq \|v_n\|$, which implies $\|u_n\| \leq 2\|v_n\|$. We suppose, by contradiction, that $\|v_n\| \to \infty$. As before, let $s_\varepsilon > 0$ be such that

$$f(t)t \geq (\xi - \varepsilon) e^{\beta_0 s^2}, \text{ for all } s \geq s_\varepsilon.$$

Thus by (1.44), it follows that

$$1 \geq d^\alpha \int_{u_n \geq s_\varepsilon} \frac{f(u_n) u_n}{\|u_n\|^2} \, \mathrm{d}x \geq (\xi - \varepsilon) \frac{d^\alpha}{4} \int_{u_n \geq s_\varepsilon} \frac{e^{\beta_0 u_n^2}}{\|v_n\|^2} \, \mathrm{d}x. \tag{1.53}$$

On the other hand, we notice that

$$\frac{u_n}{\|v_n\|}\chi_{u_n\geq s_\varepsilon} = \frac{v_n}{\|v_n\|} + \frac{t_n}{\|v_n\|}z_n - \frac{u_n}{\|v_n\|}\chi_{u_n<s_\varepsilon}.$$

Hence, we can see that

$$\frac{u_n(x)}{\|v_n(x)\|}\chi_{u_n\geq s_\varepsilon}(x) \to \widehat{v} \text{ a. e. in } H_0^1(B_1),$$

where $\widehat{v} \in H_k$, with $v_n/\|v_n\| \to \widehat{v}$ and $\|\widehat{v}\| = 1$. Then using Fatou's Lemma in (1.53) and since we have assumed that $\|v_n\| \to \infty$, we come to a contradiction. Thus $\|v_n\|$ is bounded and, consequently, t_n is also bounded. ■

Proof of Claim 2: First, we will show that

$$t_0^2 \geq 4\pi/\beta_0. \tag{1.54}$$

Once z_n converges weakly to zero in H, we have

$$\|u_n\|^2 \to t_0^2 + \|v_0\|^2.$$

On the other hand, we notice that $u_n \to v_0$ in $L^1(B_1)$ and by (1.44) we have

$$\int_{B_1} |x|^\alpha f(u_n)u_n \,\mathrm{d}x \leq \|u_n\|^2 \leq C$$

and using (H_2), we ensure that the other hypotheses of [28, Lemma 2.1] are satisfied. Therefore, we obtain

$$\int_{B_1} |x|^\alpha F(u_n) \,\mathrm{d}x \to \int_{B_1} |x|^\alpha F(v_0) \,\mathrm{d}x \tag{1.55}$$

Using this information in (1.43), we can see that

$$\frac{2\pi}{\beta_0} \leq \frac{1}{2}(t_0^2 + \|v_0\|^2 - \lambda\|v_0\|_2^2) - \int_{B_1} |x|^\alpha F(v_0) \,\mathrm{d}x.$$

From the variational characterization of λ_k and since $\lambda > \lambda_k$, we obtain

$$\frac{2\pi}{\beta_0} \leq \frac{1}{2}t_0^2 + \left(1 - \frac{\lambda}{\lambda_k}\right)\|v_0\|^2 \leq \frac{1}{2}t_0^2,$$

which implies that $t_0^2 \geq 4\pi/\beta_0$.

We will work again with the alternative set in the proof of Claim 1. We first observe that by (1.54) the alternative **(ii)** cannot hold. Hence let us assume **(i)**. We conclude from (1.52) that

$$t_0^2 \leq \frac{4\pi}{\beta_0(1-\varepsilon)^2} \Rightarrow t_0^2 \leq \frac{4\pi}{\beta_0}.$$

This fact together with (1.54) implies $t_0^2 = 4\pi/\beta_0$.

Now we must have $v_0 = 0$. First, observe that $v_n \to v_0$, $t_n \to t_0$, $\|z_n\| = 1$ and $\|z_n\|_2^2 \to 0$, from (1.43) and (1.55), we can see that

$$\Phi_\lambda(v_0) + \frac{t_0^2}{2} \geq \frac{4\pi}{\beta_0}.$$

Then we conclude that $\Phi_\lambda(v_0) \geq 0$. But, we have that

$$\Phi_\lambda(v_0) \leq \frac{1}{2}\left(\|v_n\|^2 - \lambda\|v_0\|^2\right) \leq \frac{1}{2}\left(1 - \frac{\lambda}{\lambda_k}\right)\|v_0\|^2.$$

Hence, we obtain $\Phi_\lambda(v_0) = 0$.

Now we must show that if $v_0 \in H_k$ and $\Phi_\lambda(v_0) = 0$, then $v_0 = 0$ and we finish the proof of Claim 2.

Consider $v_0 \in H_k$, then

$$\begin{aligned} 0 &= \Phi_\lambda(v_0) = \frac{1}{2}\|v_0\|^2 - \frac{\lambda}{2}\|v_0\|_2^2 - \int_{B_1} |x|^\alpha F(v_0)\,\mathrm{d}x \\ &\leq \frac{1}{2}\left(1 - \frac{\lambda}{\lambda_k}\right)\|v_0\|^2 - \int_{B_1} |x|^\alpha F(v_0)\,\mathrm{d}x \\ &\leq -\int_{B_1} |x|^\alpha F(v_0)\,\mathrm{d}x. \end{aligned}$$

Since $F(t) \geq 0$, we can see that

$$\int_{B_1} |x|^\alpha F(v_0)\,\mathrm{d}x = 0.$$

Thus

$$\begin{aligned} 0 = \Phi_\lambda(v_0) = &\ \frac{1}{2}\|v_0\|^2 - \frac{\lambda}{2}\|v_0\|_2^2 \\ \leq &\ \frac{1}{2}\left(1 - \frac{\lambda}{\lambda_k}\right)\|v_0\|^2. \end{aligned}$$

Thus we can see that

$$\frac{1}{2}\left(1 - \frac{\lambda}{\lambda_k}\right)\|v_0\|^2 \geq 0,$$

which give us $v_0 = 0$. ■

1.5.3 Proof of Theorems 1.2.3 and 1.2.4 completed

Proof: We know that from the hypotheses in these three theorems that Φ_λ satisfies $(PS)_c$ for $c < 2\pi/\beta_0$. From the Propositions 1.4.1, 1.4.2 and 1.5.2 and applying the Mountain–Pass Theorem, we prove Theorem 1.2.3. By Propositions 1.4.3, 1.5.1 and 1.5.3, Theorem 1.2.4 satisfies the hypotheses of the Linking Theorem. ■

1.6 Proof of Theorems 1.2.5 and 1.2.6: Critical case in $H^1_{0,\mathrm{rad}}(B_1)$

In this section, we treat the radial case with $f(t)$ having critical growth. In this case, a solution u for Problem (P1) should be $H = H^1_{0,\mathrm{rad}}(B_1)$, which will force us to change some calculations.

1.6.1 The geometric conditions

Analogously to Subsection 1.5.1, we need to introduce sequences of functions, which helps us to guarantee the minimax levels lie under an appropriate constant, in order to achieve the compactness properties for Φ_λ as before. If $0 < \lambda < \lambda_1$, we consider the following sequence (y_n), given by

$$y_n(s) = \begin{cases} \dfrac{s}{n^{1/2}}(1-\delta_n)^{1/2}, & 0 \le s \le n; \\ \dfrac{1}{(n(1-\delta_n))^{1/2}} \log\left(\dfrac{A_n + 1}{A_n + e^{-(s-n)}}\right) + (n(1-\delta_n))^{1/2}, & n \le s, \end{cases} \tag{1.56}$$

where $\delta_n = 2\log n/n$ and $A_n = 1/(en) + O(1/n^4)$. Since $f(t)$ satisfies (CG), by [26, Theorem 1.4], we have

$$\int_0^\infty |y_n'(t)|^2 \, \mathrm{d}t = 1 \quad \text{and} \quad \lim_{n\to\infty} \int_{B_1} F(y_n) \, \mathrm{d}t = \pi e. \tag{1.57}$$

Using this sequence together with Proposition 1.4.1 and Proposition 1.4.2, we see that Φ_λ has a local minimum at 0.

If $\lambda_k < \lambda < \lambda_{k+1}$, we have to consider the split $H = H_l^* \oplus \big((H_l^*)^\perp \cap H_{0,\mathrm{rad}}(B_1)\big)$, as in (1.29) and similarly to Proposition 1.4.3, we have that there exist constants $a, \rho > 0$ such that $\Phi_\lambda(u) \ge a$ for all $u \in \big((H_l^*)^\perp \cap H_{0,\mathrm{rad}}(B_1)\big)$ with $\|u\| = \rho$. Taking $\bar{z}_n$ as in (1.30), we set $\tilde{T}_l : H^1_{0,\mathrm{rad}}(B_1) \to \big((H_l^*)^\perp \cap H_{0,\mathrm{rad}}(B_1)\big)$ as the orthogonal projection. More explicitly, we have

$$\tilde{w}_n(x) = \tilde{T}_l \bar{z}_n(x). \tag{1.58}$$

Since $H_l^* \subset H_k$, we have that $\tilde{w}_n$ satisfies (14) and (15) for all $n \in \mathbb{N}$. By the same away of Proposition 1.4.4, we conclude that, if n is large enough, there exists $R_n = R(n) > \rho$ such that for

$$Q := \{v + s\tilde{w}_n : v \in H_l^*; \|v\| \leq R_n \text{ and } 0 \leq s \leq R_n\},$$

we have $\Phi_\lambda(u) \leq 0$ for all $u \in \partial Q$.

Remark 1.6.1. *Sequences (y_n), given in (1.56), and $(\tilde{w}_n)$, given in (1.58), will be used in the proofs of Theorems 1.2.5 and 1.2.6, respectively.*

Remark 1.6.2. *By definition, $\tilde{w}_n$ is not in H_k for all n. From (15), we realize that $\tilde{w}_n \not\equiv 0$, given in (1.58), for all $n \in \mathbb{N}$. Since $\tilde{w}_n$ is the projection of z_n on $H_{0,\mathrm{rad}}(B_1)$), we have that z_n, given in (1.31), is also not in H_k for all $n \in \mathbb{N}$, so we will use Proposition 1.5.1 with $z = z_n$ and with $z = \tilde{w}_n$ in order to prove Theorems 1.2.4 and 1.2.6, respectively. Geometrically, the next figures illustrate this fact.*

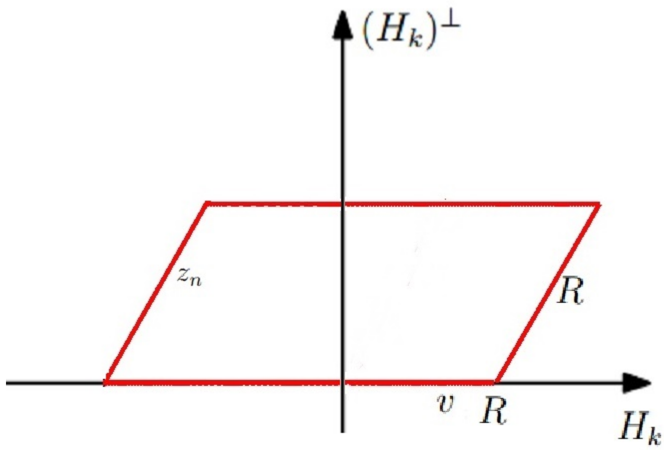

Figure 1.1: $Q := \{v + sz_n : v \in H_k, \|v\| \leq R \text{ and } 0 \leq s \leq R\}$.

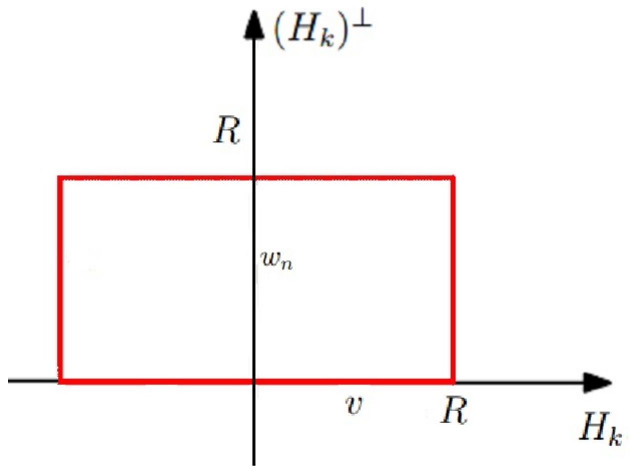

Figure 1.2: $Q := \{v + sw_n : v \in H_k, \|v\| \leq R \text{ and } 0 \leq s \leq R\}$.

1.6.2 Estimate of minimax level

First, we consider $\lambda < \lambda_1$. From Lemma 1.3.4, we know that Φ_λ satisfies $(PS)_c$ for $c < (2+\alpha)\pi/\beta_0$. To conclude via the Mountain–Pass Theorem that (P1) has a radial solution, we need to show that there is a $z \in H^1_{0,\text{rad}}(B_1)$ with $\|z\| = 1$ such that $\max\{\Phi_\lambda(tz) : t \geq 0\} < (2+\alpha)\pi/\beta_0$. It is common to use the Moser sequence, given in (1.30), in order to show this kind of result, but here we will use sequence (y_n) given in (1.56) and, thus, we can obtain better estimates.

Initially, we notice that, since we are working in the radial case, we can rewrite the functional J_λ in radial coordinates:

$$2\pi \int_0^1 \left(\frac{1}{2}|u_r|^2 - \frac{\lambda}{2}|u|^2 - r^\alpha F(u) \right) r \, \text{d}r,$$

where u_r denotes the derivative of u in relation to the variable r. Cancelling the factor 2π, we see that

$$\int_0^1 \left(\frac{1}{2}|u_r|^2 - \frac{\lambda}{2}|u|^2 - r^\alpha F(u) \right) r \, \text{d}r$$

satisfies $(PS)_c$ for $c < (2+\alpha)/2\beta_0$. Next, we perform a change of variables to transform the interval $(0,1)$ into the interval $(0,+\infty)$. Let

$$r = e^{-s/2}; \quad dr = -\frac{1}{2}e^{-s/2}ds \quad u_s = u_r\frac{dr}{ds} = -\frac{1}{2}u_r e^{-s/2},$$

and hence we obtain

$$\int_0^\infty \left(\frac{1}{2}|2u_s e^{s/2}|^2 - \frac{\lambda}{2}|u|^2 - e^{-s\alpha/2}F(u) \right) \frac{1}{2}e^{-s} \, \text{d}s.$$

Multiplying by 2, we see that the functional

$$\int_0^\infty \left(2|u_s|^2 - \frac{\lambda}{2}|u|^2 e^{-s} - F(u)e^{-s(1+\alpha/2)}\right) \mathrm{d}s$$

satisfies $(PS)_c$ for $(2+\alpha)/\beta_0$. Finally, substitute $y = 2\sqrt{\pi}u$ and multiply by π to achieve

$$\Phi_\lambda(y) = \int_0^\infty \left(\frac{1}{2}|y_s|^2 - \frac{\lambda}{8}|y|^2 e^{-s} - \pi F\left(\frac{1}{2\sqrt{\pi}}y\right) e^{-s(1+\alpha/2)}\right) \mathrm{d}s, \tag{1.59}$$

which again satisfies $(PS)_c$ for $c < (2+\alpha)\pi/\beta_0$.

Now we can define the minimax level

$$\dot{c} = \dot{c}(n) = \inf_{\nu\in\Gamma} \max_{w\in\nu([0,1])} \Phi_\lambda(\nu(w)) \tag{1.60}$$

where $\Gamma = \{\nu \in C([0,1], H); \nu(0) = 0 \text{ and } \nu(1) = R_n y_n\}$.

Proposition 1.6.1. *Let $\dot{c}(n)$ be given as in* (1.60). *Then there exists n large enough such that*

$$\dot{c}(n) < \frac{(2+\alpha)\pi}{\beta_0}.$$

Proof: We claim that there exists n such that

$$\max_{t\geq 0} \Phi_\lambda(ty_n) < \frac{(2+\alpha)\pi}{\beta_0}.$$

Suppose by contradiction that this is not the case. So, for all n, this maximum is larger than or equal to $(2+\alpha)\pi/\beta_0$ (it is indeed a maximum, in view of Proposition 1.4.2).

Let $t_n > 0$ be such that

$$\Phi_\lambda(t_n y_n) = \max_{t\geq 0} \Phi_\lambda(ty_n). \tag{1.61}$$

Then

$$\Phi_\lambda(t_n y_n) \geq \frac{(2+\alpha)\pi}{\beta_0} \text{ for all } n \in \mathbb{N},$$

and, consequently,

$$t_n^2 \geq \frac{2(2+\alpha)\pi}{\beta_0} \text{ for all } n \in \mathbb{N}.$$

In order to prove that $t_n^2 \to 2(2+\alpha)\pi/\beta_0$, first, we observe that, from (1.61), we have

$$\frac{d}{dt}(\Phi_\lambda(ty_n))\bigg|_{t=t_n} = 0.$$

So, using condition $(H_{6,\mathrm{rad}})$, we conclude that

$$\begin{aligned} t_n^2 \geq \;& \pi \int_0^\infty f\left(\frac{t_n}{2\sqrt{\pi}}y_n\right) \frac{t_n}{2\sqrt{\pi}} y_n e^{-s\left(1+\frac{\alpha}{2}\right)} \mathrm{d}s \\ \geq \;& (\xi - \varepsilon)\pi \int_n^\infty e^{\beta_0 t_n^2/4\pi(n-2\log n) - s\left(1+\frac{\alpha}{2}\right)} \mathrm{d}s. \end{aligned}$$

We suppose that

$$\lim_{n\to\infty} t_n^2 > \frac{2(2+\alpha)\pi}{\beta_0}.$$

Consequently, there exists a subsequence of t_n with $t_n^2 \geq \delta + 2(2+\alpha)\pi/\beta_0$ for some $\delta > 0$. Then we have

$$\begin{aligned} t_n^2 \geq &\ (\xi - \varepsilon)\pi \int_n^\infty e^{\left(\frac{2(2+\alpha)\pi}{\beta_0}+\delta\right)\frac{\beta_0}{4\pi}(n-2\log n) - t\left(1+\frac{\alpha}{2}\right)}\, \mathrm{d}s \\ = &\ (\xi - \varepsilon)\pi e^{\delta\frac{\beta_0}{4\pi}n - \left(\frac{2(2+\alpha)\pi}{\beta_0}+\delta\right)\frac{\beta_0}{2\pi}\log n}. \end{aligned}$$

Thus, we conclude that t_n^2 is bounded and converges to $2(2+\alpha)\pi/\beta_0$.

In order to estimate more precisely, we set $A > 0$ and set

$$[0, b_n) = \{s \in [0,\infty) : t_n y_n(s) < A\}.$$

Since $y_n(s) = s((1-\delta_n)/n)^{1/2} \to 0$ for every fixed $s \geq 0$, we can see that $b_n \to \infty$. Then we have

$$\begin{aligned} t_n^2 \geq &\ (\xi - \varepsilon)\pi \int_0^\infty e^{\beta_0/4\pi t_n^2 y_n^2 - s\left(1+\frac{\alpha}{2}\right)}\, \mathrm{d}s + \pi \int_0^{b_n} f\left(\frac{t_n}{2\sqrt{\pi}} y_n\right) \frac{t_n}{2\sqrt{\pi}} y_n e^{-s\frac{\alpha}{2}}\, \mathrm{d}s \\ &-(\xi - \varepsilon)\pi \int_0^{b_n} e^{\beta_0/4\pi t_n^2 y_n^2 - s\left(1+\frac{\alpha}{2}\right)}\, \mathrm{d}s. \end{aligned} \tag{1.62}$$

The last integral in (1.62) goes to 1. In fact, we have

$$\int_0^{b_n} e^{-s\left(1+\frac{\alpha}{2}\right)}\, \mathrm{d}s \leq \int_0^{b_n} e^{s_n^2 y_n^2 - t\left(1+\frac{\alpha}{2}\right)}\, \mathrm{d}s = \int_0^{b_\varepsilon} e^{s_n^2 y_n^2 - t\left(1+\frac{\alpha}{2}\right)}\, \mathrm{d}s + \int_{b_\varepsilon}^{b_n} e^{s_n^2 y_n^2 - t\left(1+\frac{\alpha}{2}\right)}\, \mathrm{d}s,$$

where we choose for given $\varepsilon > 0$ the number $b_\varepsilon > 0$ such that

$$\int_{b_\varepsilon}^{b_n} e^{\beta_0/4\pi t_n^2 y_n^2 - s\left(1+\frac{\alpha}{2}\right)}\, \mathrm{d}s \leq e^{\beta_0/4\pi A^2} \int_{b_\varepsilon}^{b_n} e^{-s\left(1+\frac{\alpha}{2}\right)}\, \mathrm{d}s \leq \frac{\varepsilon}{2} \text{ for all } n.$$

Now, using that $y_n(s) \leq \varepsilon_n \to 0$ on $[0, b_\varepsilon]$, we choose N_ε sufficiently large such that

$$\int_0^{b_\varepsilon} e^{\beta_0/4\pi t_n^2 y_n^2 - t\left(1+\frac{\alpha}{2}\right)}\, \mathrm{d}s \leq e^{\beta_0/4\pi\varepsilon_n^2} \int_0^{b_\varepsilon} e^{-t\left(1+\frac{\alpha}{2}\right)}\, \mathrm{d}s \leq 1 + \frac{\varepsilon}{2} \text{ for all } n > N_\varepsilon.$$

The second integral in (1.62) is positive, and in fact goes to zero (as can be seen using a similar argument). Hence we have in the limit

$$\begin{aligned} \lim_{n\to\infty} t_n^2 = &\ (\xi - \varepsilon)\pi \left(\lim_{n\to\infty} \int_0^\infty e^{\beta_0/4\pi t_n^2 y_n^2 - s\left(1+\frac{\alpha}{2}\right)}\, \mathrm{d}s - 1 \right) \\ = &\ (\xi - \varepsilon)\pi \left(\lim_{n\to\infty} \int_0^\infty e^{\left(1+\frac{\alpha}{2}\right) y_n^2 - s\left(1+\frac{\alpha}{2}\right)}\, \mathrm{d}s - 1 \right) \end{aligned}$$

First consider the change of variable $s = (1+\alpha/2)t$. After, changing $y_n(s)$ into $(1+\alpha/2)^{-1}x_n(s)$. From (1.57), it follows that

$$\begin{aligned}\frac{2(2+\alpha)}{\beta_0}\pi \geq & \ (\xi-\varepsilon)\left(1+\frac{\alpha}{2}\right)^2\pi\left(\lim_{n\to\infty}\int_0^\infty e^{x_n^2-t}\,\mathrm{d}t-1\right)\\ = & \ (\xi-\varepsilon)\left(1+\frac{\alpha}{2}\right)^2\pi e,\end{aligned}$$

for ε small enough, which implies

$$\xi \leq \frac{8}{(2+\alpha)\beta_0 e},$$

a contradiction to $(H_{6,\mathrm{rad}})$. ■

Now, we consider the Linking case and define the minimax level

$$\bar{c} = \bar{c}(n) = \inf_{\nu\in\Gamma}\max_{w\in\nu(\partial Q)}\Phi_\lambda(h(w)) \tag{1.63}$$

where $\Gamma = \{\nu \in C(Q,H); \nu(u) = u \text{ if } u \in \partial Q\}$.

Proposition 1.6.2. *Let $\bar{c}(n)$ be given as in* (1.63)*. Then there exists n large enough such that*

$$\bar{c}(n) < \frac{(2+\alpha)\pi}{\beta_0}.$$

Proof: Suppose by contradiction that for all n we have $\bar{c}(n) \geq (2+\alpha)\pi/\beta_0$. Thus we obtain

$$\bar{c}(n) \leq \max\{\Phi_\lambda(v+t\tilde{w}_n) : v \in H_l^*, \|v\| \leq R_n \text{ and } t \geq 0\}.$$

It follows that for each n there exist $v_n \in H_l^*$ and $t_n > 0$ such that

$$\Phi_\lambda(v_n+t_n\bar{w}_n) = \max\{\Phi_\lambda(v+t\bar{w}_n); v \in H_l^* \cap \overline{B}_{R_n}, t \geq 0\} \tag{1.64}$$

and we have

$$\Phi_\lambda(v_n+t_n\bar{w}_n) \geq \frac{(2+\alpha)\pi}{\beta_0} \text{ for all } n \in \mathbb{N}. \tag{1.65}$$

So, since $\tilde{w}_n \in (H_l^*)^\perp$, from (1.28) we obtain

$$t_n^2 \geq \frac{2(2+\alpha)\pi}{\beta_0} \text{ for all } n \in \mathbb{N}. \tag{1.66}$$

Analogously to Proposition 1.5.2, let us assume the following claims, whose proofs we will give later.

Claim 3. *(v_n) and (t_n) are bounded sequences.*

Claim 4. $t_n^2 \to \dfrac{2(2+\alpha)\pi}{\beta_0}$ *in* $\mathbb{R}$ *and* $v_n \to 0$ *in* H.

Since $v_n \in H_l^*$, in view of Claim 4, we also obtain $\|v\|_\infty \to 0$. However, we have $v_n + t_n\tilde{w}_n \to \infty$ uniformly in $B_{1/n}$.

We can suppose, without loss of generality, that

$$K_0 \leq \frac{\log(h(t))}{t} \leq C_0 \tag{1.67}$$

for all t large enough for some $C_0, K_0 > 0$.

Indeed, note that if $h(s)$ satisfies (H_8), there exists $\tilde{h}(t)$ such that $h(t) \geq \tilde{h}(t)$ for all t large enough and

$$0 < \liminf_{t\to+\infty} \frac{\log(\tilde{h}(t))}{t} \leq \limsup_{t\to+\infty} \frac{\log(\tilde{h}(t))}{t} < +\infty.$$

Finally, consider γ such that

$$\gamma > \frac{(2+\alpha)^2}{\beta_0} \exp\left(\frac{4(2+\alpha)\pi B_k}{\sqrt{2\pi}} + \frac{C_0^2}{4\beta_0}\right). \tag{1.68}$$

From (1.64), using the fact that derivative of Φ_λ, restricted to $H_k \oplus \mathbb{R}w_n$ is zero at $v_n + t_n\tilde{w}_n$, we obtain

$$\|v_n + t_n\tilde{w}_n\|^2 - \lambda\|v_n + t_n\tilde{w}_n\|_2^2 - \int |x|^\alpha f(v_n + t_n\tilde{w}_n)(v_n + t_n\tilde{w}_n)\, \mathrm{d}x = 0 \tag{1.69}$$

and we can see, for n large enough, we have

$$t_n^2 \geq \int_{B_{1/n}} |x|^\alpha f(v_n + t_n\tilde{w}_n)(v_n + t_n\tilde{w}_n)\, \mathrm{d}x.$$

Since $t_n \geq \sqrt{2(2+\alpha)\pi/\beta_0} > 0$ and $v_n \to 0$, by (15) we can take n large enough that

$$t_n\tilde{w}_n - \|v_n\|_\infty \geq c_\lambda,$$

where c_λ is given in (H_8). Thus, we obtain

$$t_n^2 \geq \gamma \int_{B_{1/n}} |x|^\alpha h\left(t_n\tilde{w}_n - \|v_n\|_\infty\right) \exp\left(\beta_0\left(t_n\tilde{w}_n - \|v_n\|_\infty\right)^2\right)\, \mathrm{d}x.$$

We have

$$\begin{aligned} t_n^2 \;&\geq \gamma \int_{B_{1/n}} |x|^\alpha \exp\left[-\left(\frac{\log[h(t_n\tilde{w}_n - \|v_n\|_\infty)]}{2\sqrt{\beta_0}(t_n\tilde{w}_n - \|v_n\|_\infty)}\right)^2\right. \\ &\left. +\beta_0\left(t_n\tilde{w}_n - \|v_n\|_\infty + \frac{\log[h(t_n\tilde{w}_n - \|v_n\|_\infty)]}{2\beta_0(t_n\tilde{w}_n - \|v_n\|_\infty)}\right)^2\right]\, \mathrm{d}x. \end{aligned}$$

But if n is large, (1.67) shows that

$$-\left(\frac{\log[h(t_n\tilde{w}_n - \|v_n\|_\infty)]}{2\sqrt{\beta_0}(t_n\tilde{w}_n - \|v_n\|_\infty)}\right)^2 \geq -\frac{C_0^2}{4\beta_0}$$

and

$$\frac{\log[h(t_n\tilde{w}_n - \|v_n\|_\infty)]}{2\beta_0(t_n\tilde{w}_n - \|v_n\|_\infty)} \geq \frac{K_0}{2\beta_0},$$

from which we obtain

$$\begin{aligned} t_n^2 &\geq \gamma \exp\left(-\frac{C_0^2}{4\beta_0}\right)\int_{B_{1/n}} |x|^\alpha \exp\left(\beta_0\left(t_n\tilde{w}_n - \|v_n\|_\infty + \frac{K_0}{2\beta_0}\right)^2\right) \mathrm{d}x \\ &\geq \gamma \exp\left(-\frac{C_0^2}{4}\right)\int_{B_{1/n}} |x|^\alpha \exp(\beta_0 t_n^2\tilde{w}_n^2)\,\mathrm{d}x. \end{aligned}$$

Consequently, from (15) and (1.66) we can see that

$$\begin{aligned} t_n^2 &\geq \gamma \exp\left(-\frac{C_0^2}{4}\right)\int_{B_{1/n}} |x|^\alpha \exp[2(2+\alpha)\pi\tilde{w}_n^2]\,\mathrm{d}x \\ &= \gamma \exp\left(-\frac{C_0^2}{4}\right)\frac{2\pi}{(2+\alpha)n^{2+\alpha}}\exp\left[2(2+\alpha)\pi\left(\frac{(\log n)^{1/2}}{\sqrt{2\pi}} - \frac{B_k}{(\log n)^{1/2}}\right)^2\right] \\ &= \gamma e^{\left(-\frac{C_0^2}{4}\right)\frac{2\pi}{(2+\alpha)n^{2+\alpha}}+[(2+\alpha)\log n]+(-4(2+\alpha)\pi B_k/\sqrt{2\pi})+[2(2+\alpha)\pi B_k^2/\log n]}. \end{aligned}$$

Now, letting $n \to \infty$ in the inequality above, one gets

$$\gamma \leq \frac{(2+\alpha)^2}{\beta_0}\exp\left(\frac{4(2+\alpha)\pi B_k}{\sqrt{2\pi}} + \frac{C_0^2}{4\beta_0}\right),$$

which is contrary to the choice of γ in (1.68). This contradiction follows the assumption $\check{c}(n) \geq (2+\alpha)\pi/\beta_0$ for all $\in \mathbb{N}$, which concludes the proof. ■

Although Claims 3 and 4 are similar to Claims 1 and 2, respectively, we must use different arguments in order to prove them, because we assume (H_8) instead of (H_7). Now we will prove them briefly.

Proof of Claim 3: It is sufficient to prove that all subsequence of (t_n) and (v_n) have bounded subsequences. Let us suppose that this is not true. So, we can find subsequences, which by convenience we still denote by (t_n) and (v_n), respectively, such that all of their subsequences are unbounded. That means we can assume the following

$$t_{n_k} + \|v_{n_k}\| \to \infty \text{ for all subsequences } (n_k). \tag{1.70}$$

Therefore, one of the following two possibilities has that hold:

(i) either there exists a constant $C_0 > 0$ such that $t_n/\|v_n\| \geq C_0$, or

(ii) there exist subsequences such that $t_n/\|v_n\| \to 0$.

Assume that **(i)** holds and using (1.70), we have that $t_n \to \infty$. Now we can see from (H_8) and (1.64) that

$$\begin{aligned} t_n^2 \geq & \int_{B_{r/n}(x_0)} |x|^\alpha f(v_n + t_n\tilde{w}_n)(v_n + t_n\tilde{w}_n)\,\mathrm{d}x \\ \geq & \;\gamma \int_{B_{r/n}(x_0)} |x|^\alpha h(v_n + t_n\tilde{w}_n)\exp\left(\beta_0(v_n + t_n\tilde{w}_n)^2\right)\,\mathrm{d}x \end{aligned}$$

for n large enough from (H_8) we have

$$h(s) \geq \tilde{C} \tag{1.71}$$

when s is large. Thus we have

$$t_n^2 \geq \gamma\tilde{C}\int_{B_{r/n}(x_0)} |x|^\alpha \exp\left(\beta_0(v_n + t_n\tilde{w}_n)^2\right)\,\mathrm{d}x. \tag{1.72}$$

We notice that since H_k has finite dimension, we have that $\|v_n\|_\infty/t_n$ is bounded for all $x \in B_{r/n}(x_0)$. This fact, together with (15), give us

$$\begin{aligned} v_n(x) + t_n\tilde{w}_n(x) &= t_n\tilde{w}_n(x)\left(1 + \frac{v_n(x)}{t_n}\frac{1}{\tilde{w}_n}\right) \\ &\geq \frac{t_n}{\sqrt{2\pi}}\left((\log n)^{1/2} - \frac{\sqrt{2\pi}B_k}{(\log n)^{1/2}}\right)\left(1 - C\left(\frac{(\log n)^{1/2}}{\sqrt{2\pi}} - \frac{B_k}{(\log n)^{1/2}}\right)^{-1}\right) \\ &\geq \frac{t_n}{2}\frac{1}{\sqrt{2\pi}}\left((\log n)^{1/2} - \frac{\sqrt{2\pi}B_k}{(\log n)^{1/2}}\right) \end{aligned}$$

and taking n such that $(\log n)^{1/2} - \sqrt{2\pi}B_k/(\log n)^{1/2} \geq (1/2)(\log n)^{1/2}$, we obtain

$$v_n(x) + t_n\tilde{w}_n(x) \geq \frac{t_n}{4\sqrt{2\pi}}(\log n)^{1/2}$$

and by (1.72) it follows that

$$\begin{aligned} t_n^2 \geq & \;\gamma\tilde{C}\int_{B_{r/n}(x_0)} |x|^\alpha \exp\left(\beta_0\left(\frac{t_n}{4\sqrt{2\pi}}(\log n)^{1/2}\right)^2\right)\,\mathrm{d}x \\ & = \gamma\tilde{C}\pi\frac{r^{2+\alpha}}{2+\alpha}\exp\left(\left(\beta_0\frac{t_n^2}{32\pi} - 2\right)\log n\right). \end{aligned}$$

Consequently, t_n must be bounded in case **(i)**, which contradicts $t_n \to \infty$.

Then **(ii)** occurs. Since $\lim_{n\to\infty} t_n/\|v_n\| = 0$, by (1.70) we conclude that $v_n \to \infty$. From (1.69), we obtain

$$\|u_n\|^2 \geq \int_{B_1} |x|^\alpha f(t_n\tilde{w}_n + v_n)(t_n\tilde{w}_n + v_n)\,\mathrm{d}x. \tag{1.73}$$

Using (H_8) and (1.71), for n large enough, we have

$$\|u_n\|^2 \geq \gamma\tilde{C}\int_{\{u_n\geq c_\gamma\}} |x|^\alpha e^{\beta_0(t_n\tilde{w}_n+v_n)^2}\,\mathrm{d}x.$$

Since we are supposing **(ii)**, it follows that

$$1 \geq \tilde{C}\gamma\int_{\{u_n\geq c_\gamma\}} |x|^\alpha \frac{e^{\beta_0(t_n\tilde{w}_n+v_n)^2}}{\|u_n\|^2}\,\mathrm{d}x \geq \tilde{C}\frac{\gamma}{2}\int_{\{u_n\geq c_\gamma\}} |x|^\alpha \frac{e^{\beta_0(t_n\tilde{w}_n+v_n)^2}}{\|v_n\|^2}\,\mathrm{d}x \tag{1.74}$$

and just like with Claim 2, using Fatou's Lemma, we come to a contradiction. Then $\|v_n\|$ is bounded and, consequently, t_n is also bounded. ■

Proof of Claim 3: First, we notice for some appropriated subsequences we have $v_n \to v_0$ in H and $t_n \to t_0$. Since $z_n \rightharpoonup 0$, we have $\tilde{w}_n \rightharpoonup 0$ and $\tilde{w}_n \to 0$ for all $x \in B_1$. Then, it follows that

$$v_n + t_n\tilde{w}_n \to v_0 \text{ almost everywhere in } B_1. \tag{1.75}$$

Moreover, in view of (1.69) we see that

$$\int_{B_1} |x|^\alpha f(v_n + t_n\tilde{w}_n)(v_n + t_n\tilde{w}_n)\,\mathrm{d}x \leq \|v_n + t_n\tilde{w}_n\|^2 \leq C. \tag{1.76}$$

However, using [28, Lemma 2.1], once (H_1), (1.75) and (1.76) hold we have

$$\int_{B_1} |x|^\alpha F(v_n + t_n\tilde{w}_n)\,\mathrm{d}x \to \int_{B_1} |x|^\alpha F(v_0)\,\mathrm{d}x.$$

From (1.65) and (1.76), we can see that

$$\Phi_\lambda(v_0) + \frac{t_0^2}{2} \geq \frac{(2+\alpha)\pi}{\beta_0}. \tag{1.77}$$

and since $v_0 \in H_k$, in view of $\Phi_\lambda(v_0) \leq 0$, we have

$$t_0^2 \geq \frac{2(2+\alpha)\pi}{\beta_0}.$$

Let us prove that $t_0^2 = 2(2+\alpha)\pi/\beta_0$. Then, suppose that it is not true. We have $t_0^2 > 2(2+\alpha)\pi/\beta_0$. Thus we can take small enough $\varepsilon > 0$ so that

$$t_n^2 > (1+\varepsilon)\frac{2(2+\alpha)\pi}{\beta_0}$$

for all large n. We consider

$$\varepsilon_n = \sup_{B_{1/n}} \frac{|v_n(x)|}{t_n \tilde{w}_n},$$

clearly we see that $\varepsilon_n \to 0$, which yields

$$\begin{aligned} C &\geq \gamma \tilde{C} \int_{B_{r/n}(x_0)} |x|^\alpha \exp[\beta_0 (v_n + t_n \tilde{w}_n)^2] \, \mathrm{d}x \\ &\geq \gamma \tilde{C} \int_{B_{r/n}(x_0)} |x|^\alpha \exp[\beta_0 (-\varepsilon_n t_n \tilde{w}_n + t_n \tilde{w}_n)^2] \, \mathrm{d}x. \end{aligned}$$

Now, using (H_8) and (15) and large n, we see that

$$\begin{aligned} C &\geq \gamma \tilde{C} \frac{\pi r^{2+\alpha}}{(2+\alpha) n^{2+\alpha}} \exp\left(\beta_0 (1-\varepsilon_n)^2 t_n^2 \left[\frac{(\log n)^{1/2}}{\sqrt{2\pi}} - \frac{B_k}{(\log n)^{1/2}} \right]^2 \right) \\ &= \gamma \tilde{C} \frac{\pi r^{2+\alpha}}{(2+\alpha) e^{(2+\alpha)\log n}} \exp\left(\beta_0 (1-\varepsilon_n)^2 t_n^2 \left[\frac{B_k^2}{\log n} - 2\frac{B_k}{\sqrt{2\pi}} \right] \right) \exp\left(\beta_0 \frac{(1-\varepsilon_n)^2 t_n^2 \log n}{2\pi} \right). \end{aligned}$$

Because $t_n^2 \geq 2(2+\alpha)\pi/\beta_0$ and $\varepsilon_n \to 0$, we notice that

$$\exp\left(\beta_0 (1-\varepsilon_n)^2 t_n^2 \left[\frac{B_k^2}{\log n} - 2\frac{B_k}{\sqrt{2\pi}} \right] \right) > C_1$$

for n large enough and some $C_1 > 0$. Using $t_n^2 \geq (1+\varepsilon)4\pi/\beta_0$, we have

$$\begin{aligned} C &\geq C_1 \gamma \tilde{C} \pi \frac{r^{2+\alpha}}{2+\alpha} \exp\left(\beta_0 (1-\varepsilon_n)^2 t_n^2 \frac{\log n}{2\pi} - (2+\alpha) \right) \\ &\geq C_1 \gamma \tilde{C} \pi \frac{r^{2+\alpha}}{2+\alpha} \exp\left((2+\alpha) \log n [(1-\varepsilon_n)^2 (1+\varepsilon) - 1] \right) \to \infty, \end{aligned}$$

which is a contradiction. Consequently, we must have $t_0^2 = 2(2+\alpha)\pi/\beta_0$ as desired.

Thus, by (1.77), we obtain $\Phi_\lambda(v_0) \geq 0$. But we know that $v_0 \in H_k$, so by (1.28), we have $\Phi_\lambda(v_0) = 0$. As with Claim 2, we can conclude that if $v_0 \in H_k$ and $\Phi_\lambda(v_0) = 0$, then $v_0 = 0$. This finishes the proof of Claim 4. ■

1.6.3 Proof of Theorems 1.2.5 and 1.2.6 completed

Proof: We know that from the hypotheses in both Theorems that Φ_λ satisfies $(PS)_c$ for $c < (2+\alpha)\pi/\beta_0$. For $\lambda < \lambda_1$, we see that Φ_λ has a local minimum point at 0, thus, applying the Mountain–Pass Theorem, we prove Theorem 1.2.5. If $\lambda_k < \lambda < \lambda_{k+1}$, we have a saddle point at 0 and Theorem 1.2.6 follows from the Linking Theorem. ■

Chapter 2

Hénon type equations with one-sided exponential growth

In this chapter, we establish a link between Ambrosetti–Prodi problems and Hénon equations. First of all, we remember that the study these problem starts with the celebrate paper by A. Ambrosetti and G. Prodi [4] in 1975. Ever since, this kind of problems has been studied, explored and extended by an enormous variety of authors. For a brief review, we refer the reader to [24, 29, 46]. In short, it deals with non-homogeneous problems such as

$$\begin{cases} -\Delta u = g(x,u) + f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where at least one eigenvalue of $(-\Delta, H_0^1(B_1))$ is in the interval, whose the extremes are the limits $\lim_{s\to-\infty} g(s)/s < \lim_{s\to+\infty} g(s)/s$. Moreover, the existence of multiple solutions depends heavily on an usual hypothesis regarding a suitable parametrization of the forcing term f, which is extensively considered in almost all Ambrosetti-Prodi problems since the work of J. Kazdan and F. Warner in [34].

Since our work deals with nonlinearities in critical growth range, let us focus our discussion to more specifically related studies. As far as we know, the first paper that addressed an Ambrosetti-Prodi problem involving critical nonlinearities was the work of D. YinBin, [58]. There, the author considered a function that was superlinear both in $+\infty$ and $-\infty$ and, so, asymptotically jumping all the eigenvalues of $(-\Delta, H_0^1(\Omega))$. In [29], D. Figueiredo and J. Yang explored more jumping possibilities with one-sided critical growth equation. However, their techniques have natural limitations, which allowed to investigate the existence of multiple solutions in dimensions $N \geq 7$ only. In [15], M. Calanchi and B. Ruf improved these results, once they proved that the same problem has at least two solutions provided $N \geq 6$. They also guarantee the existence of solutions in lower dimensions, when they added a lower order growth term to the nonlinearity.

Considering similar problems on domains in $\mathbb{R}^2$ with the exponential growth, we have the work of CALANCHI et al, 2004 [16], which studied the following problem:

$$\begin{cases} -\Delta u &= \lambda u + g(u_+) + f(x) \quad \text{in} \quad \Omega, \\ u &= 0 \qquad\qquad\qquad\quad \text{on} \quad \partial\Omega, \end{cases}$$

where Ω is bounded and smooth in $\mathbb{R}^2$ and g has a subcritical or critical Trudinger-Moser growth. They proved the existence of two solutions for some forcing terms f, depending on the usual parametrization $f(x) = h(x) + t\phi_1(x)$.

Here we consider a similar problem with the weight $|x|^\alpha$ on the nonlinearity, which is proper of Hénon equations. More specifically, we we study the solvability of problems of the type

$$\begin{cases} -\Delta u &= \lambda u + |x|^\alpha g(u_+) + f(x) \quad \text{in} \quad B_1, \\ u &= 0 \qquad\qquad\qquad\qquad\quad \text{on} \quad \partial B_1, \end{cases} \tag{P2}$$

where $\lambda, \alpha \geq 0$ and $B_1 = \{x \in \mathbb{R}^2 : |x| < 1\}$. We assume that g has the maximum growth which allows us to treat problem (P2) variationally in suitable Sobolev Spaces, due to the well-known Trudinger-Moser inequality (see [40, 57]), which is given by (3) and (4) (this last in radial context with the typical Hénon weight). Motivated by these inequalities, similarity to we did in previous chapter, we say that g has subcritical growth at $+\infty$ if

$$\lim_{t\to+\infty} \frac{g(t)}{e^{\beta t^2}} = 0, \quad \forall\beta \tag{2.1}$$

and g has critical growth at $+\infty$ if there exists $\beta_0 > 0$ such that

$$\lim_{t\to+\infty} \frac{g(t)}{e^{\beta t^2}} = 0, \quad \forall\beta > \beta_0; \quad \lim_{t\to+\infty} \frac{g(t)}{e^{\beta t^2}} = +\infty, \quad \forall\beta < \beta_0. \tag{2.2}$$

2.1 Hypotheses

Before stating our main results, we shall introduce the following assumptions on the nonlinearity g:

(g_0) $g \in C(\mathbb{R}, \mathbb{R}^+)$, $g(s) = 0$ for all $s \leq 0$.

(g_1) There exist s_0 and $M > 0$ such that

$$0 < G(s) = \int_0^s g(t)\,\mathrm{d}t \leq Mg(s) \text{ for all } s > s_0.$$

(g_2) $|g(s)| = o(|s|)$ when $|s| \to 0$.

We recall that we consider $\lambda_1 < \lambda_2 \leq \lambda_3 \leq ... \leq \lambda_j \leq ...$ the sequence of eigenvalues of $(-\Delta, H_0^1(B_1))$ and ϕ_j is a j^{th} eigenfunction of $(-\Delta, H_0^1(B_1))$.

We observe that, using assumption (g_0), one can see that ψ is a nonpositive solution of (P2) if and only if it is a nonpositive solution for the linear problem

$$\begin{cases} -\Delta\psi &= \lambda\psi + f(x) \quad \text{in} \quad B_1, \\ \psi &= 0 \qquad\qquad\ \text{on} \quad \partial B_1. \end{cases} \tag{2.3}$$

In order to get such solutions for (2.3), let us assume that

(f_1) $f(x) = h(x) + t\phi_1(x)$,

where $h \in L^\mu(B_1)$, $\mu > 2$ and

$$\int_{B_1} h\phi_1\,\mathrm{d}x = 0.$$

For that matter, the parameter t plays a crucial role. We shall use this hypothesis in the first Theorem of this chapter.

2.2 Statement of main results

We divide our results in four theorems. The first one deals with the first solution of the problem, which is nonpositive and is obtained by a simple remark about a linear problem related to our equation. The other theorems concern the second solution and are considered depending on the growth conditions of the nonlinearity. In the critical case, since the weight $|x|^\alpha$ has an important role on the estimate of the minimax levels, the variational setting and methods used in $H_0^1(B_1)$ and $H_{0,\mathrm{rad}}^1(B_1)$ are different and, therefore, are given in separate theorems.

Theorem 2.2.1 (The linear Problem)**.** Assume that $\lambda \neq \lambda_j$ for all $j \in \mathbb{N}$ and (f_1) holds, then there exists a constant $T = T(h) > 0$ such that:

(i) If $\lambda < \lambda_1$, there exists $\psi_t < 0$, a solution for (2.3) and, consequently, for (P2), for all $t < -T$.

(ii) If $\lambda > \lambda_1$, there exists $\psi_t < 0$, a solution for (2.3) and, consequently, for (P2), for all $t > T$.

Furthermore, if f is radially symmetric, then ψ_t is radially symmetric as well.

Theorem 2.2.2 (The subcritical case)**.** Let $f \in L^{\mu}(B_1)$ with $\mu > 2$ such that there exists a nonpositive solution ψ for (P2). Assume $(g_0) - (g_2)$, (2.1) and $\lambda \neq \lambda_j$ for all $j \in \mathbb{N}$. Then, problem (P2) has a second solution. Furthermore, if ψ is radially symmetric, the second solution is also radially symmetric.

Theorem 2.2.3 (The critical case)**.** Let $f \in L^{\mu}(B_1)$ with $\mu > 2$ such that there exists a nonpositive solution ψ for (P2). Assume $(g_0) - (g_2)$, (2.2), $\lambda \neq \lambda_j$ for all $j \in \mathbb{N}$. Furthermore suppose that for all $\gamma \geq 0$ there exists $c_\gamma \geq 0$ such that

$$g(s)s \geq \gamma e^{(\beta_0 s^2)} h(s) \quad \text{for all} \quad s > c_\gamma, \tag{2.4}$$

where $h : \mathbb{R} \to \mathbb{R}^+$ is a continuous function satisfying

$$\liminf_{s \to +\infty} \frac{\log(h(s))}{s} > 0. \tag{2.5}$$

Then, problem (P2) has a second solution.

Theorem 2.2.4 (The radial critical case)**.** Let $f \in L^{\mu}(B_1)$ with $\mu > 2$ such that there exists a radial and nonpositive solution ψ for (P2). Assume $(g_0) - (g_2)$, (2.2), (2.4) with $h : \mathbb{R} \to \mathbb{R}^+$ being a continuous function satisfying

$$\liminf_{s \to +\infty} \frac{\log(h(s))}{s} \geq 4\beta_0 \|\psi\|_\infty, \tag{2.6}$$

where β_0 is given in (2.2). If $\lambda \neq \lambda_j$ for all $j \in \mathbb{N}$, problem (P2) has a second solution which is radially symmetric.

Example 2.2.1. For examples of non-linearities with critical growth satisfying the assumptions of Theorem 2.2.3, one can consider

$$g(t) = \begin{cases} e^{\beta_0 s^2 + K_0 s}(2\beta_0 s^3 + K_0 s^2 + 2s) & \text{if} \quad s \geq 0 \\ 0 & \text{if} \quad s < 0 \end{cases},$$

with $h(s) = e^{K_0 s}(2\beta_0 s^3 + K_0 s^2 + 2s)$ and $K_0 > 0$. If $K_0 \geq 4\beta_0 \|\psi\|_\infty$, then $g(s)$ satisfies the hypotheses of Theorem 2.2.4.

Remark 2.2.1. *We notice that if $\lambda < \lambda_1$, we shall use the Mountain–Pass Theorem in the proofs of Theorems 2.2.2, 2.2.3 and 2.2.4. On the other hand, if $\lambda > \lambda_1$, we need to use the Linking Theorem.*

Remark 2.2.2. *We also must point out an important fact: the hypothesis* (2.5) *and* (2.6) *are definitely stricter than the one used in [16], since they do not require this additional function h. What happens is that, as far as we are concerned, it is not proven, nor we were able to prove that the estimate given in Lemma 3.5 of that article is uniform in $m = m(n)$. It concerned the minimax level $c(k)$ and, although it is not clear there, this level actually depends also on the parameter n, which surely changes with the considered terms of the Moser functions $z_{k,r(n)}$. So, an uniform estimate in this level must be proven in order to conclude that the weak limit of the associated (PS) sequence is actually nontrivial, as established on Proposition 3.4 of that paper.*

In Theorems 2.2.2, 2.2.3 and 2.2.4, we assume that f is such that (P2) admits a nonpositive solution ψ. Then, a second solution will be given by $u + \psi$, where u is a nontrivial solution of the following problem:

$$\begin{cases} -\Delta u = \lambda u + |x|^\alpha g(u+\psi)_+ & \text{in} \quad B_1; \\ u = 0 & \text{on} \quad \partial B_1. \end{cases} \tag{2.7}$$

This means that we will focus our attention in looking for a nontrivial solution for Problem (2.7), considering each of the cases given by the Theorems above. This translation has the advantage of working with a homogeneous problem, without a forcing term that could hinder the desired estimates on the minimax levels. On the other hand, the function ψ plays this role in complicating the estimates, but we could handle them by performing some delicate arguments involving the additional hypothesis (2.4), with h satisfying (2.5) or (2.6).

Under certain condition, we can see that the first solution we obtained is radially symmetric, due to a celebrated result of R. Palais, namely: the Principle of Symmetric Criticality (see [42]). After that, we get a second solution that is radially symmetric as well. In the subcritical case, we obtain this second solution with almost the same arguments regardless the space we consider, $H_0^1(B_1)$ or $H_{0,\text{rad}}^1(B_1)$. However, in the critical case, we must consider the Trudinger-Moser inequality as given in (3), when we work in $H_0^1(B_1)$, or as in (4), in $H_{0,\text{rad}}^1(B_1)$, which will dramatically change the arguments. In the last case, the minimax levels have a higher upper boundedness, that should indicate an easier task, but, since we can only work with radial functions, it is impossible to follow the steps we give in the $H_0^1(B_1)$ framework.

2.2.1 Outline

This chapter is organized as follows. In Sect. 2.3, we study a linear problem related to our equation. This is an important step to guarantee a first solution for the problem, which will be denoted by ψ and is nonpositive. This case is almost identical to the previous results in [15, 16, 29, 47], but we give a little remark concerning existence of a radially symmetric solution when the forcing term is also radial. In Sect. 2.4, we introduce the variational framework and prove the boundedness of Palais-Smale sequences of the functional associated to Problem (2.7). We also show that this functional satisfies the (PS) condition, in the subcritical case. In Sect. 2.5, we obtain the geometric conditions for the functional in order to prove the existence of a second solution to the problem, considering the subcritical growth both in $H_0^1(B_1)$ and $H_{0,\text{rad}}^1(B_1)$. In Sect. 2.6 and 2.7, we consider the problem in the critical growth range and guarantee the existence of a second solution in $H_0^1(B_1)$ and $H_{0,\text{rad}}^1(B_1)$, respectively. In these cases, we also show that the geometric conditions are satisfied and we prove the boundedness of the minimax levels by appropriate constants depending on β_0 when we consider the functional defined in $H_0^1(B_1)$ and on β_0 and α when the functional is considered in $H_{0,\text{rad}}^1(B_1)$.

2.3 The Linear Problem

In this section, we prove Theorem 2.2.1, for that we consider the linear problem

$$\begin{cases} -\Delta\psi = \lambda\psi + f(x) & \text{in} \quad B_1, \\ \psi = 0 & \text{on} \quad \partial B_1. \end{cases} \tag{2.8}$$

It is easy to see that if f is such that this linear problem admits a nonpositive solution, then it will also be a solution for Problem (P2). Considering f decomposed as in (f_1), we will see that the sign of the (unique) solution of (2.8) can be established depending on t. Moreover, we give an idea on how to obtain radial solutions as well.

Proof of Theorem 1: Up to the point where we discuss the radial case, this proof follows exactly the same arguments found in [15, 16, 29, 47], but we bring it here for the sake of completeness.

Since $\lambda \neq \lambda_j$ for all $j \in \mathbb{N}$, we obtain, by the Fredholm Alternative, a unique solution ψ of (2.8) in $H_0^1(B_1)$. Using (f_1), we can write $\psi = \psi_t = \omega + s_t\phi_1$, with

$$\int_{B_1} \omega\phi_1 \,\mathrm{d}x = 0 \quad \text{and} \quad s_t = \frac{t}{\lambda_1 - \lambda}.$$

We recall that $f \in L^\mu(B_1)$, with $\mu > 2$. Thus, by elliptic regularity, we have $\omega \in C^{1,\nu}$ for some $0 < \nu < 1$. Then

$$\left\| \frac{\lambda_1 - \lambda}{t}\psi - \phi_1 \right\|_{C^1} = \left| \frac{\lambda_1 - \lambda}{t} \right| \|\omega\|_{C^1}.$$

Let $\varepsilon > 0$ be such that $v > 0$ for all $v \in C^{1,\nu}$ such that $\|v - \phi_1\|_{C^1} < \varepsilon$. Since we want $\psi < 0$, we must have

$$\frac{\lambda - \lambda_1}{t} > 0 \quad \text{and} \quad \frac{\lambda_1 - \lambda}{t} \|\omega\|_{C^1} < \varepsilon$$

So there exists $T > 0$ such that for $\lambda < \lambda_1$ and $t < -T$ or for $\lambda > \lambda_1$ and $t > T$, we have $\psi < 0$.

Now we notice that ψ is a solution for (2.8) if only if it is a critical point of the following functional $I : H_0^1(B_1) \to \mathbb{R}$, given by

$$I(\psi) := \frac{1}{2}\int_{B_1} |\nabla\psi|^2 \,\mathrm{d}x - \lambda\frac{1}{2}\int_{B_1} \psi^2 \,\mathrm{d}x - \int_{B_1} f(x)\psi \,\mathrm{d}x.$$

When we restrict I to $H^1_{0,\mathrm{rad}}(B_1)$, we also obtain a critical point of this functional on this subspace. If f is radially symmetric, by the Principle of Symmetric Criticality of Palais (see [42]), we can see that all critical points on $H^1_{0,\mathrm{rad}}(B_1)$ are also critical points on $H_0^1(B_1)$. So, due to the fact that I admits only one critical point in the whole space, we get that ψ is also radially symmetric. This completes the proof of Theorem 2.2.1. ■

2.4 The variational formulation

Again, we will denote $H = H_0^1(B_1)$ or $H = H^1_{0,\mathrm{rad}}(B_1)$, depending on which theorem we are considering, with the Dirichlet norm

$$\|u\| = \left(\int_{B_1} |\nabla u|^2 \,\mathrm{d}x\right)^{1/2} \quad \text{for all} \quad u \in H.$$

Recalling that our efforts are searching for a nontrivial weak solution to Problem (2.7), we define the functional $J_\lambda : H \to \mathbb{R}$ as

$$J_\lambda(u) = \frac{1}{2}\int_{B_1} |\nabla u|^2 \,\mathrm{d}x - \frac{\lambda}{2}\int_{B_1} |u|^2 \,\mathrm{d}x - \int_{B_1} |x|^\alpha G(u+\psi)_+ \,\mathrm{d}x, \tag{2.9}$$

where $\alpha, \lambda \geq 0$ and $\lambda \neq \lambda_j$ for all $j \in \mathbb{N}$. By $(g_0), (g_1)$ and (2.1) or (2.2), we have that the functional J_λ is C^1 and we can see that its derivative is given by

$$\langle J_\lambda'(u), v\rangle = \int_{B_1} \nabla u \nabla v \,\mathrm{d}x - \lambda \int_{B_1} uv \,\mathrm{d}x - \int_{B_1} |x|^\alpha g(u+\psi)_+ v \,\mathrm{d}x \quad \text{for all} \quad v \in H \tag{2.10}$$

and the critical points of J_λ are (weak) solution of (2.7). We observe the $u = 0$ satisfies $J_\lambda'(0) = 0$, which corresponds to the negative solution ψ for (2.3). To find a second solution of (P2) we shall look for critical points of the functional J_λ with critical values $c > 0$.

2.4.1 Palais-Smale condition

Initially, from (g_1), we can see that for every $\sigma > 0$ there exists $s_\sigma > 0$ such that

$$0 < G(s) \leq \frac{1}{\sigma} g(s)s \text{ for all } s \geq s_\sigma. \tag{2.11}$$

The proof of Palais–Smale condition for the subcritical case is essentially standard. For the reader's convenience, we sketch the proof in the next lemma. We must point out that it is not necessary to suppose that $g(s)$ is $O(s^2)$ at $s = 0$ as required in [16]. The well-known techniques we used here can also be handled in case $\alpha = 0$ and for that it is sufficient to admit that $g(s) = o(s)$ at $s = 0$, following usual assumptions.

Lemma 2.4.1. Suppose $(g_0) - (g_2)$. Let $(u_n) \subset H$ be a $(PS)_c$ sequence of J_λ. Then (u_n) is bounded in H.

Proof: Let $(u_n) \subset H$ be a $(PS)_c$ sequence of J_λ, that is,

$$\left|\frac{1}{2}\|u_n\|^2 - \frac{\lambda}{2}\|u_n\|_2^2 - \int_{B_1} |x|^\alpha G(u_n+\psi)_+ \,\mathrm{d}x - c\right| \to 0 \tag{2.12}$$

and

$$\left|\int_{B_1} \nabla u_n \nabla v \,\mathrm{d}x - \lambda \int_{B_1} u_n v \,\mathrm{d}x - \int_{B_1} |x|^\alpha g(u_n+\psi)_+ v \,\mathrm{d}x\right| \leq \varepsilon_n \|v\| \text{ for all } v \in H, \tag{2.13}$$

where $\varepsilon_n \to 0$. By (2.11), let us take $s_0 > 0$ such that

$$G(s) \leq \frac{1}{4} g(s)s \text{ for all } s \geq s_0. \tag{2.14}$$

Using (2.12), (2.13) and (2.14), we get

$$\begin{aligned} c + \frac{\varepsilon_n \|u_n\|}{2} &\geq \frac{1}{2}\int_{B_1} |x|^\alpha g(u_n+\psi)_+(u_n) - \int_{B_1} |x|^\alpha G(u_n+\psi)_+ \\ &= \frac{1}{2}\int_{B_1} |x|^\alpha g(u_n+\psi)_+(u_n) - \int_{\{(u_n+\psi)_+ \geq s_0\}} |x|^\alpha G(u_n+\psi)_+ \\ &\quad - \int_{\{(u_n+\psi)_+ \leq s_0\}} |x|^\alpha G(u_n+\psi)_+ \\ &\geq \frac{1}{2}\int_{B_1} |x|^\alpha g(u_n+\psi)_+(u_n) - \frac{1}{4}\int_{\{(u_n+\psi)_+ \geq s_0\}} |x|^\alpha g(u_n+\psi)_+(u_n+\psi)_+ - C_0 \\ &\geq \frac{1}{4}\int_{B_1} |x|^\alpha g(u_n+\psi)_+(u_n+\psi) - C_0. \end{aligned}$$

We notice that the last integral above is obtained because $\psi \leq 0$. Thus, we have

$$\int_{B_1} |x|^\alpha g(u_n+\psi)_+(u_n+\psi)_+ \,\mathrm{d}x \leq C + \varepsilon_n \|u_n\| \text{ with } C > 0 \tag{2.15}$$

and, similarly to (1.7) in previous chapter, we obtain

$$\int_{B_1} |x|^\alpha g(u_n+\psi)_+ \,\mathrm{d}x \leq C + \varepsilon_n \|u_n\|. \tag{2.16}$$

Initially, we consider the case: $0 \leq \lambda < \lambda_1$. Using (2.13) and (2.15), we have

$$\varepsilon_n \|u_n\| \geq \left(1 - \frac{\lambda}{\lambda_1}\right) \|u_n\|^2 - (C + \varepsilon_n \|u_n\|).$$

Thus, (u_n) is a bounded sequence.

Now we consider the case: $\lambda_k < \lambda < \lambda_{k+1}$. It is convenient to decompose H into appropriate subspaces:

$$H = H_k \oplus H_k^\perp, \tag{2.17}$$

where H_k is finite dimensional and defined by

$$H_k = [\phi_1, ..., \phi_k]. \tag{2.18}$$

This notation is standard when dealing with this framework of high order eigenvalues, and we will use it throughout this work.

For all u in H, let us take $u = u^k + u^\perp$, where $u^k \in H_k$ and $u^\perp \in H_k^\perp$. We notice that

$$\int_{B_1} \nabla u \nabla u^k \, \mathrm{d}x - \lambda \int_{B_1} u u^k \, \mathrm{d}x = \|u^k\|^2 - \lambda \|u^k\|_2^2 \tag{2.19}$$

and

$$\int_{B_1} \nabla u \nabla u^\perp \, \mathrm{d}x - \lambda \int_{B_1} u u^\perp \, \mathrm{d}x = \|u^\perp\|^2 - \lambda \|u^\perp\|_2^2.$$

By (2.13), (2.19) and the characterization of λ_k, we can see that

$$\begin{aligned} -\varepsilon_n \|u_n^k\| &\leq \int_{B_1} \nabla u_n \nabla u_n^k \, \mathrm{d}x - \lambda \int_{B_1} u_n u_n^k \, \mathrm{d}x - \int_{B_1} |x|^\alpha g(u_n + \psi)_+ u_n^k \, \mathrm{d}x \\ &\leq \left(1 - \frac{\lambda}{\lambda_k}\right) \|u_n^k\|^2 - \int_{B_1} |x|^\alpha g(u_n + \psi)_+ u_n^k \, \mathrm{d}x. \end{aligned}$$

Therefore,

$$C\|u_n^k\|^2 \leq \varepsilon_n \|u_n^k\| - \int_{B_1} |x|^\alpha g(u_n + \psi)_+ u_n^k \, \mathrm{d}x \text{ with } C > 0. \tag{2.20}$$

Similarly, we get

$$C\|u_n^\perp\|^2 \leq \varepsilon_n \|u_n^\perp\| + \int_{B_1} |x|^\alpha g(u_n + \psi)_+ u_n^\perp \, \mathrm{d}x. \tag{2.21}$$

Then, since H_k is a finite dimensional subspace, by (2.16) and (2.20), we obtain

$$\begin{aligned} C\|u_n^k\|^2 &\leq \varepsilon_n \|u_n^k\| + \|u_n^k\|_\infty \int_{B_1} |x|^\alpha g(u_n + \psi)_+ \, \mathrm{d}x \\ &\leq \varepsilon_n \|u_n^k\| + C\|u_n^k\|(C + \varepsilon_n \|u_n\|) \\ &\leq C + C\|u_n\| + C\varepsilon_n \|u_n\|^2. \end{aligned} \tag{2.22}$$

Using (2.15), (2.16) and (2.21), we have

$$\begin{aligned} C\|u_n^\perp\|^2 \leq \; & \varepsilon_n \|u_n^\perp\| + \int_{B_1} |x|^\alpha g(u_n + \psi)_+ u_n \, \mathrm{d}x + \|u_n^k\|_\infty (C + \varepsilon_n \|u_n\|) \\ \leq \; & \varepsilon_n \|u_n^\perp\| + \int_{B_1} |x|^\alpha g(u_n + \psi)_+ (u_n + \psi) \, \mathrm{d}x \\ & + \|\psi\|_\infty \int_{B_1} |x|^\alpha g(u_n + \psi)_+ \, \mathrm{d}x + \|u_n^k\|_\infty (C + \varepsilon_n \|u_n\|) \\ \leq \; & \varepsilon_n \|u_n^\perp\| + C + \varepsilon_n \|u_n\| + \|\psi\|_\infty \int_{B_1} g(u_n + \psi)_+ \, \mathrm{d}x + C\|u_n\|(C + \varepsilon_n \|u_n\|) \\ \leq \; & C + C\|u_n\| + C\varepsilon_n \|u_n\|^2. \end{aligned} \tag{2.23}$$

By summing the inequalities in (2.23) and (2.22), we reach

$$\|u_n\|^2 \leq C + C\|u_n\| + C\varepsilon_n\|u_n\|^2,$$

proving the boundedness of the sequence (u_n) as desired. ■

Remark 2.4.1. *In the proof of Lemma 2.4.1 there is no difference between assuming subcritical or critical growth or considering the radial case or not. So we can conclude that even in the case of critical growth, every Palais-Smale sequence is bounded.*

In case of subcritical growth, we can obtain the $(PS)_c$ condition for all levels in $\mathbb{R}$.

Lemma 2.4.2. Assume $(g_0) - (g_2)$ and (2.1). Then the functional J_λ satisfies the $(PS)_c$ condition for all $c \in \mathbb{R}$.

Proof: Let (u_n) be a $(PS)_c$ sequence. By Lemma 2.4.1, we know that (u_n) is bounded. So we consider a subsequence denoted again by (u_n) such that, for some $u \in H$, we have

$$\begin{aligned}
&u_n \rightharpoonup u \quad \text{weakly in} \quad H,\\
&u_n(x) \to u(x) \quad \text{almost everywhere in} \quad B_1,\\
&u_n \to u \quad \text{strongly in} \quad L^q(B_1) \text{ or } L^q_{\text{rad}}(|x|^\alpha B_1) \quad \text{for all} \quad q \geq 1.
\end{aligned}$$

Notice that there is nothing else to prove if $\|u_n\| \to 0$. Thus, one may suppose that $\|u_n\| \geq k > 0$ for n sufficiently large.

It follows from (g_1) and [28, Lemma 2.1] that

$$\int_{B_1} |x|^\alpha G(u_n + \psi)_+ \, \mathrm{d}x \to \int_{B_1} |x|^\alpha G(u + \psi)_+ \, \mathrm{d}x.$$

We will prove that

$$\int_{B_1} |x|^\alpha g(u_n + \psi)_+ u_n \, \mathrm{d}x \to \int_{B_1} |x|^\alpha g(u + \psi)_+ u \, \mathrm{d}x. \tag{2.24}$$

In fact, we have

$$\left| \int_{B_1} |x|^\alpha g(u_n + \psi)_+ u_n \, \mathrm{d}x - \int_{B_1} |x|^\alpha g(u + \psi)_+ u \, \mathrm{d}x \right|$$

$$\leq \left| \int_{B_1} |x|^\alpha [g(u_n + \psi)_+ - g(u + \psi)_+] u \, \mathrm{d}x \right| + \left| \int_{B_1} |x|^\alpha g(u_n + \psi)_+ (u_n - u) \, \mathrm{d}x \right|.$$

First, let us focus on the second integral in the left side of this last estimate. By (2.1), we get

$$\left|\int_{B_1} |x|^\alpha g(u_n+\psi)_+(u_n-u)\ \mathrm{d}x\right| \le C\int_{B_1} e^{\beta u_n^2}|u_n-u|\ \mathrm{d}x \text{ for all } \beta>0.$$

Using the Höder inequality, we obtain

$$\int_{B_1} e^{\beta u_n^2}|u_n-u|\ \mathrm{d}x \le \left(\int_{B_1} e^{q\beta\left(\frac{u_n}{\|u_n\|}\right)^2\|u_n\|^2}\ \mathrm{d}x\right)^{\frac{1}{q}}\|u_n-u\|_{q'},$$

where $1/q+1/q'=1$. We take $q>1$ and by (2.1) and Lemma 2.4.1, we can choose β sufficiently small such that $q\beta\|u_n\|^2\le 4\pi$. Thus by the Trudinger-Moser inequality, we have

$$\int_{B_1} e^{\beta u_n^2}|u_n-u|\ \mathrm{d}x \le C_1\|u_n-u\|_{q'}. \tag{2.25}$$

Since $u_n\to u$ strongly in $L^{q'}$, one has

$$\left|\int_{B_1} |x|^\alpha g(u_n+\psi)_+(u_n-u)\ \mathrm{d}x\right| \to 0.$$

It remains to prove that

$$\left|\int_{B_1} |x|^\alpha [g(u_n+\psi)_+ - g(u+\psi)_+]u\ \mathrm{d}x\right| \to 0$$

as well. Indeed. Let $\varepsilon>0$ be given. By analogous arguments used to prove (2.25), there exists C_2 such that

$$\|g(u_n+\psi)_+ - g(u+\psi)_+\|_{2,|x|^\alpha} \le C_2.$$

Consider $\xi\in C_0^\infty(B_1)$ such that $\|\xi-u\|_2<\varepsilon/2C_2$. Now, since

$$\int_{B_1} |x|^\alpha|g(u_n+\psi)_+ - g(u+\psi)_+|\mathrm{d}x \to 0,$$

for this ε, there exists n_ε such that

$$\int_{B_1} |x|^\alpha|g(u_n+\psi)_+ - g(u+\psi)_+|\mathrm{d}x < \frac{\varepsilon}{2\|\xi\|_\infty}$$

for all $n\ge n_\varepsilon$. Therefore

$$\begin{aligned}&\left|\int_{B_1} |x|^\alpha [g(u_n+\psi)_+ - g(u+\psi)_+]u\ \mathrm{d}x\right|\\ &\le \int_{B_1} |x|^\alpha|g(u_n+\psi)_+ - g(u+\psi)_+||\xi|\ \mathrm{d}x\\ &+\int_{B_1} |x|^\alpha|g(u_n+\psi)_+ - g(u+\psi)_+||\xi-u|\ \mathrm{d}x<\varepsilon\end{aligned}$$

for all $n \geq n_\varepsilon$, as desired. Consequently, we conclude that (2.24) holds.

Now taking $v = u$ and $n \to \infty$ in (2.13), we have

$$\|u\|^2 = \lambda \|u\|_2^2 + \int_{B_1} |x|^\alpha g(u+\psi)_+ u \, \mathrm{d}x.$$

On the other hand, if $n \to \infty$ in (2.13) with $v = u_n$,

$$\|u_n\|^2 \to \lambda \|u\|_2^2 + \int_{B_1} |x|^\alpha g(u+\psi)_+ u \, \mathrm{d}x$$

again by (2.24). Consequently, $\|u_n\| \to \|u\|$ and so $u_n \to u$ in H. ■

2.5 Proof of Theorem 2.2.2: The subcritical case

This section is devoted to the proof of Theorem 2.2.2. Here, we consider ψ radially symmetric and $H = H^1_{0,\mathrm{rad}}(B_1)$. In the case that ψ is not necessarily radial and $H = H^1_0(B_1)$, the proof uses the same arguments.

2.5.1 The geometric condition

First of all, we will prove the following lemma, which will be used in the proofs of geometric conditions for the associated functionals.

Lemma 2.5.1. Assume $(g_0) - (g_2)$. Then, for all $\varepsilon > 0$ and $\beta > 0$ if g satisfies (2.1) or $\beta > \beta_0$ if g satisfies (2.2), there exists K_ε such that

$$G(u) \leq \varepsilon u^2 + K_\varepsilon u^q e^{\beta u^2}, \text{ for all } q \geq 1. \tag{2.26}$$

Proof: Given ε, by (g_2) there exists $C_\varepsilon > 0$ such that if $|u| \leq C_\varepsilon$, we have

$$G(u) - G(0) = \int_0^u g(t) \, \mathrm{d}t \leq \varepsilon \tilde{C} \int_0^u |t| \, \mathrm{d}t \leq \varepsilon C \left(\frac{1}{2} u^2\right).$$

Since $G(0) = 0$, we see that

$$G(u) \leq \varepsilon C u^2 \text{ if } |u| \leq C_\varepsilon.$$

By (2.1) or (2.2), there exists C_1 such that if $|u| > C_1$, then

$$g(u)u \leq \varepsilon C e^{\beta u^2} u \leq \varepsilon C e^{\beta u^2} u^q$$

for ssuitable $\beta > 0$.

From (2.11) we obtain

$$G(u) \le \varepsilon C e^{\beta u^2} u^q \text{ for } |u| > C_1.$$

Once the set where $C_\varepsilon \le |u| \le C_1$ is compact, using (g_0) we complete the proof. ■

Now, we consider $\lambda < \lambda_1$. We will show that the hypotheses of the Mountain–Pass Theorem hold for the functional J_λ.

Proposition 2.5.1. *Suppose that $\lambda < \lambda_1$, $(g_0) - (g_2)$ and (2.1). Then there exist $\rho, a > 0$ such that $J_\lambda(u) \ge a$ if $\|u\| = \rho$.*

Proof: Since $\lambda < \lambda_1$, the variational characterization of the eigenvalues gives us

$$J_\lambda(u) \ge \left(1 - \frac{\lambda}{\lambda_1}\right) \|u\|^2 - \int_{B_1} |x|^\alpha G(u+\psi)_+ \, \mathrm{d}x$$

and the Lemma 2.26 with $q = 3$ give us

$$J_\lambda(u) \ge C\|u\|^2 - \varepsilon \int_{B_1} |x|^\alpha (u+\psi)_+^2 \, \mathrm{d}x - K_\varepsilon \int_{B_1} |x|^\alpha e^{\beta(u+\psi)_+^2} (u+\psi)_+^3 \, \mathrm{d}x.$$

From the monotonicity and Hölder's inequality, we have

$$\begin{aligned} \int_{B_1} |x|^\alpha (u+\psi)_+^2 \, \mathrm{d}x \le & \left(\int_{B_1} |x|^{2\alpha} \, \mathrm{d}x\right)^{1/2} \left(\int_{B_1} (u+\psi)_+^4 \, \mathrm{d}x\right)^{1/2} \\ \le & \frac{1}{(2\alpha+2)^{1/2}} \left(\int_{B_1} |u|^4 \, \mathrm{d}x\right)^{1/2} = \frac{1}{(2\alpha+2)^{1/2}} \|u\|_4^2 \\ \le & \frac{1}{(2\alpha+2)^{1/2}} \|u\|^2 \le C_\alpha \|u\|^2. \end{aligned}$$

and

$$\begin{aligned} \int_{B_1} |x|^\alpha e^{\beta(u+\psi)_+^2} (u+\psi)_+^3 \, \mathrm{d}x \le & \left(\int_{B_1} e^{2\beta(u+\psi)_+^2} \, \mathrm{d}x\right)^{1/2} \left(\int_{B_1} |x|^{4\alpha} \, \mathrm{d}x\right)^{1/4} \left(\int_{B_1} (u+\psi)_+^{12} \, \mathrm{d}x\right)^{1/4} \\ \le & \left(\int_{B_1} e^{2\beta u_+^2} \, \mathrm{d}x\right)^{1/2} \left(\int_{B_1} |x|^{4\alpha} \, \mathrm{d}x\right)^{1/4} \left(\int_{B_1} |u|^{12} \, \mathrm{d}x\right)^{1/4} \\ \le & \tilde{C}_\alpha \left(\int_{B_1} e^{2\beta u^2} \, \mathrm{d}x\right)^{1/2} \|u\|^3. \end{aligned}$$

Since we are supposing (2.1), β can be chosen such that $2\beta \le 4\pi$. So by Trudinger-Moser inequality, we obtain

$$J_\lambda(u) \ge C\|u\|^2 - C\|u\|^3 > 0$$

for all $u \in H$ with $\|u\| < \min\{1, \delta^{1/2}\}$. The second constant comes from the Trudinger-Moser inequality. Therefore, take $\rho = \|u\|$ and $a = C\rho^2 - C\rho^3$. ■

Proposition 2.5.2. *Suppose* $(g_0) - (g_2)$ *and* (2.1) *hold. Then, there exists* $R > \rho$ *such that*

$$J_\lambda(R\phi_1) \leq 0,$$

where ϕ_1 *is a first eigenfunction of* $(-\Delta, H)$ *(with* $\phi_1 > 0$ *and* $\|\phi_1\| = 1$*) and* ρ *is given in Proposition 2.5.1.*

Proof: We fix $R_0 > \rho$ and $0 < r < 1$ such that

$$\phi_1(x) \geq \frac{2\|\psi\|_\infty}{R_0} \text{ almost everywhere in } B_r.$$

We observe that (2.11) gives us

$$G(t) \geq C_\sigma t^\sigma - D_\sigma \tag{2.27}$$

for $\sigma > 2$, and $C_\sigma, D_\sigma \geq 0$. Thus, we obtain

$$J_\lambda(R\phi_1) \leq \frac{R^2}{2} - R^\sigma C_\sigma \int_{B_r} |x|^\alpha \left(\phi_1 + \frac{\psi}{R}\right)_+^\sigma \mathrm{d}x + \pi r^2 D_\sigma.$$

Let $R \geq R_0$. We estimate the last integral

$$\int_{B_r} |x|^\alpha \left(\phi_1 + \frac{\psi}{R}\right)_+^\sigma \mathrm{d}x \geq \left(\frac{\|\psi\|_\infty}{R_0}\right)^\sigma \frac{2r^{2+\alpha}\pi}{2+\alpha} = \tau > 0.$$

It follows that

$$J_\lambda(R\phi_1) \leq \frac{R^2}{2} - C_\sigma R^\sigma \tau + \pi r^2 D_\sigma.$$

Since $\sigma > 2$, we can choose $R > \rho$ such that

$$J_\lambda(R\phi_1) \leq 0,$$

which is the desired conclusion. ■

Next, we consider $\lambda_k < \lambda < \lambda_{k+1}$. Before the proof of the geometric conditions of the Linking Theorem for J_λ, we need to split $H^1_{0,\mathrm{rad}}(B_1)$ into two orthogonal subspaces as we have done to $H^1_0(B_1)$ in (2.17). Recalling the notation

$$H^*_k = H_k \cap H^1_{0,\mathrm{rad}}(B_1) \text{ for all } k \in \mathbb{N}.$$

We have that

$$H^1_{0,\mathrm{rad}}(B_1) = \bigcup_{k=1}^{\infty} H^*_k. \tag{2.28}$$

Remark 2.5.1. *In the proof of Lemma 2.4.1, when we consider the radial case we use the decomposition given by* (2.28).

For $\lambda_k < \lambda$, we consider the corresponding subspace H_k and we write $H^1_{0,\text{rad}}(B_1)$ as

$$H = H^1_{0,\text{rad}}(B_1) = H^*_l \oplus \big((H^*_l)^\perp \cap H^1_{0,\text{rad}}(B_1)\big)$$

where $H^*_l \subset H_k$ with $l = \max\{j : H^*_j \subset H_k\}$. Since $H^*_l \subset H_k$, we notice that $\big((H^*_l)^\perp \cap H^1_{0,\text{rad}}(B_1)\big) \subset H_k^\perp$. This decomposition will allow us to use the same variational inequalities that characterize λ_k and λ_{k+1} in the $H^1_0(B_1)$ environment.

Proposition 2.5.3. *Suppose that $\lambda_k < \lambda < \lambda_{k+1}$, $(g_0) - (g_2)$ and* (2.1) *hold. Then there exist ρ, $a > 0$ such that $J_\lambda(u) \geq a$ for all $u \in \big((H^*_l)^\perp \cap H^1_{0,\text{rad}}(B_1)\big)$ with $\|u\| = \rho$.*

Proof: Since $\lambda < \lambda_{k+1}$, the variational characterization of the eigenvalues gives us

$$J_\lambda(u) \geq \left(1 - \frac{\lambda}{\lambda_{k+1}}\right) \|u\|^2 - \int_{B_1} |x|^\alpha G((u+\psi)_+)\, \mathrm{d}x$$

and by Lemma 2.5.1 with $q = 3$, we have

$$J_\lambda(u) \geq C\|u\|^2 + C_\varepsilon \int_{B_1} |x|^\alpha (u+\psi)_+^2\, \mathrm{d}x - C\int_{B_1} |x|^\alpha e^{\beta(u+\psi)_+^2}(u+\psi)_+^3\, \mathrm{d}x.$$

The Hölder's inequality and the monotonicity give us

$$\int_{B_1} |x|^\alpha (u+\psi)_+^2\, \mathrm{d}x \leq C_\alpha \|u\|^2.$$

and

$$\int_{B_1} |x|^\alpha e^{\beta(u+\psi)_+^2}(u+\psi)_+^3\, \mathrm{d}x \leq C_\alpha \left(\int_{B_1} e^{2\beta u^2\, \mathrm{d}x}\right)^{1/2} \|u\|^3.$$

Choosing $\beta \leq 2\pi$, by Trudinger-Moser inequality, we obtain

$$J_\lambda(u) \geq C\|u\|^2 - C\|u\|^3 > 0,$$

for all $u \in \big((H^*_l)^\perp \cap H^1_{0,\text{rad}}(B_1)\big)$ with $\|u\| < \min\{1, \delta^{1/2}\}$. The second constant comes from the Trudinger-Moser inequality. Therefore, take $\rho = \|u\|$ and $a = C\rho^2 - C\rho^3$.

■

Proposition 2.5.4. *Suppose that $\lambda_k < \lambda < \lambda_{k+1}$, $(g_0) - (g_2)$ and* (2.1) *hold. There exists $z \in \big((H^*_l)^\perp \cap H^1_{0,\text{rad}}(B_1)\big) \subset H_k^\perp$ and $R > 0$ such that $R\|z\| > \rho$ and*

$$J_\lambda(u) \leq 0 \text{ for all } u \in \partial Q.$$

where $Q := \{v \in H^*_l : \|v\| \leq R\} \oplus \{sz : 0 \leq s \leq R\}$.

Proof: We fix $R_0 > \rho$ and choose $z \in \big((H_l^*)^\perp \cap H^1_{0,\mathrm{rad}}(B_1)\big) \subset H_k^\perp$ and $x_0 \in B_1$ such that

(a) $\|z\|^2 < \dfrac{\lambda}{\lambda_k} - 1$;

(b) $B_r(x_0) \subset B_1$ and

$$z(x) \geq \left(K + \frac{2\|\psi\|_\infty}{R_0}\right) \text{ almost everywhere in } B_r(x_0)$$

with $K > 0$ satisfying $\|v\|_\infty \leq K\|v\|$ for all $v \in H_k$. This choice is possible because $\big((H_l^*)^\perp \cap H^1_{0,\mathrm{rad}}(B_1)\big)$ possesses unbounded functions and H_k has finite dimension.

We consider a usual split $\partial Q = Q_1 \cup Q_2 \cup Q_3$, where

$$\begin{aligned} Q_1 &= \{v \in H_l^* : \|v\| \leq R\}, \\ Q_2 &= \{v + sz : v \in H_l^*, \|v\| = R \text{ and } 0 \leq s \leq R\}, \\ Q_3 &= \{v + Rz : v \in H_l^* \text{ and } \|v\| \leq R\}. \end{aligned}$$

Let u be on Q_1, by (g_2) and characterization of λ_k, it follows

$$\begin{aligned} J_\lambda(u) &\leq \frac{1}{2}\|u\|^2 - \frac{\lambda}{2}\|u\|_2^2 \\ &\leq \frac{1}{2}\left(1 - \frac{\lambda}{\lambda_k}\right)\|u\|^2 \leq 0 \end{aligned}$$

independently of $R > 0$.

For Q_2, we get

$$\begin{aligned} J_\lambda(v + sz) &\leq \frac{1}{2}\|v\|^2 + \frac{1}{2}s^2\|z\|^2 - \frac{\lambda}{2}\|v\|_2^2 \\ &\leq \frac{1}{2}R^2 + \frac{1}{2}R^2\|z\|^2 - \frac{\lambda}{2\lambda_k}R^2 \\ &\leq \frac{1}{2}R^2\left(1 - \frac{\lambda}{\lambda_k} + \|z\|^2\right) < 0, \end{aligned}$$

independently of $R > 0$.

Now for Q_3, using (2.27) and obtain

$$\begin{aligned} J_\lambda(v + Rz) &\leq \frac{R^2}{2}\|z\|^2 - \int_{B_1} |x|^\alpha G((v + Rz + \psi)_+)\,\mathrm{d}x \\ &\leq \frac{R^2}{2}\|z\|^2 - R^\sigma C_\sigma \int_{B_1} |x|^\alpha \left(z + \frac{\psi + v}{R}\right)_+^\sigma \mathrm{d}x - D_\sigma. \end{aligned}$$

Let $R \geq R_0$ be, we estimate the last integral

$$\int_{B_1} |x|^\alpha \left(z + \frac{\psi + v}{R}\right)_+^\sigma \mathrm{d}x \geq \int_{B_r(x_0)} |x|^\alpha \left(z - \frac{\|\psi\|_\infty + \|v\|_\infty}{R}\right)^\sigma \mathrm{d}x$$

$$\geq \left(\frac{\|\psi\|_\infty}{R}\right)^\sigma \int_{B_r(x_0)} |x|^\alpha \geq \frac{\pi r^{2+\alpha}}{2+\alpha} \left(\frac{\|\psi\|_\infty}{R}\right)^\sigma = \tau > 0.$$

Thus, we obtain

$$J_\lambda(v + Rz) \leq \frac{R^2}{2}\|z\|^2 - C_\sigma R^\sigma \tau - D_\sigma.$$

Since $\sigma > 2$, we finish the proof. ■

2.5.2 Proof of Theorem 2.2.2 completed

We have proved that J_λ satisfies the geometric and the compactness conditions required in the Mountain–Pass Theorem when $\lambda < \lambda_1$ and in the Linking Theorem if $\lambda_k < \lambda < \lambda_{k+1}$. Thus there exists a nontrivial critical point for J_λ and thus a solution of (2.7). ■

2.6 Proof of Theorem 2.2.3: Critical case in $H_0^1(B_1)$

It is well known that for non-linear elliptic problems involving critical growth some concentration phenomena may occur, due to the action of the non-compact group of dilations. For problems (3) and (4) there are loss of compactness at the limiting exponent $\beta = 4\pi$ and $\beta = 2\pi(2+\alpha)$ respectively. Thus, the energy functional J_λ fails to satisfy the $(PS)_c$ condition for certain levels c. Such a failure makes it difficulty to apply standard variational approach to this class of problems. Our proofs here relies on a Brezis-Nirenberg type argument: We begin by proving the geometric condition of the Mountain–Pass Theorem when $\lambda < \lambda_1$ and the geometric condition in the Linking Theorem if $\lambda_k < \lambda < \lambda_{k+1}$. In the second step we show that the minimax levels belong to the intervals where the $(PS)-$condition holds for the functional J_λ.

2.6.1 The geometric conditions

Initially we consider $\lambda < \lambda_1$. We will prove the geometric condition of the Mountain–Pass Theorem.

Proposition 2.6.1. *Assume* $(g_0) - (g_2)$, $\lambda < \lambda_1$ *and* (2.2). *Then, there exist* $a, \rho > 0$ *such that*

$$J_\lambda(u) \geq a \text{ if } \|u\| = \rho.$$

Proof: We can now proceed analogously to the proof of Proposition 2.5.1. ■

Now, let us recall the so-called Moser sequence. That is, for each n, we define

$$\bar{z}_n(x) = \frac{1}{\sqrt{2\pi}} \begin{cases} (\log n)^{1/2}, & 0 \leq |x| \leq \frac{1}{n}; \\ \\ \dfrac{\log \frac{1}{|x|}}{(\log n)^{1/2}} & \frac{1}{n} \leq |x| \leq 1. \end{cases} \tag{2.29}$$

It is known that $z_n \in H$, $\|z_n\| = 1$ for all n and $\|z_n\|_2 = O\left(1/(\log n)^{1/2}\right)$. For details see [40].

In order to apply our techniques, we shall consider a suitable translation of Moser's functions in a region of B_1 far from the origin where the presence of $|x|^\alpha$ can be neglected. We begin by noticing that, from (2.5), we obtain ε_0 and s_0 such that

$$\frac{\log(h(s))}{s} \geq \varepsilon_0, \tag{2.30}$$

for all $s > s_0$.

Since $\psi \equiv 0$ on ∂B_1, we can fix $r > 0$, small enough, and x_0, sufficiently close to ∂B_1, such that

$$\|\psi\|_{\infty,r} := \|\psi|_{B_{r^2}(x_0)}\|_\infty \leq \frac{\varepsilon_0}{2\beta_0} \quad \text{and} \quad |x| \geq \frac{1}{2} \text{ in } B_{r^2}(x_0), \tag{2.31}$$

with $B_r(x_0) \subset B_1$.

Let us take the following family of functions:

$$z_n^r(x) = \frac{1}{\sqrt{2\pi}} \begin{cases} (\log n)^{1/2}, & 0 \leq |x - x_0| \leq \frac{r^2}{n}; \\ \\ \dfrac{\log \frac{r^2}{|x-x_0|}}{(\log n)^{1/2}}, & \frac{r^2}{n} \leq |x - x_0| \leq r^2; \\ \\ 0, & |x - x_0| \geq r^2. \end{cases} \tag{2.32}$$

We notice that $\text{supp} z_n^r = \overline{B_{r^2}(x_0)}$, as we can see in Figure 3.1, and, for all $n \in \mathbb{N}$, one has

$$\|z_n^r\|^2 = \int_{B_1} \sum_{i=1}^{2} \left(\frac{\partial z_n^r}{\partial x_i}\right)^2 \mathrm{d}x = 2\pi \frac{1}{2\pi \log n} \int_{r^2/n}^{r^2} \frac{1}{\tilde{r}^2} \tilde{r} \mathrm{d}\tilde{r} = 1, \tag{2.33}$$

where $\tilde{r} = |x - x_0|$.

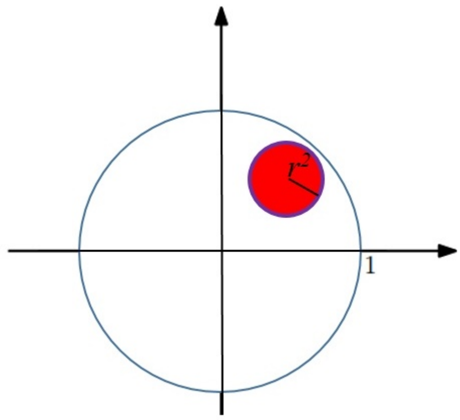

Figure 2.1: Support of z_n^r.

This sequence will be used to guarantee the existence of a minimax level lying under an appropriate constant which will allow us to recover the compactness properties for J_λ that are lost when dealing with critical growth ranges. Before that, we see in the next proposition that for large n we still have the same geometric condition proved in Proposition 2.5.2 with z_n^r taking the place of ϕ_1.

Proposition 2.6.2. *Suppose that* $(g_0) - (g_2)$ *and the condition* (2.2)*. Then, there exists* $R_n = R(n) > \rho$ *such that* $J_\lambda(R_n z_n^r) \leq 0$.

Proof: We fix $R_0 > \rho$ and $N \in \mathbb{N}$, such that for all $n > N$, we have

$$z_n^r(x) \geq \frac{2\|\psi\|_\infty}{R_0} \text{ almost everywhere in } B_{r^2/n}(x_0).$$

Using (2.27), let us take $R > R_0$. We get

$$J_\lambda(Rz_n^r) \leq \frac{R^2}{2} - \int_{B_1} |x|^\alpha G((Rz_n^r + \psi)_+)\,\mathrm{d}x \leq \frac{R^2}{2} - R^\vartheta C_\sigma \int_{B_1} |x|^\alpha \left(z_n^r + \frac{\psi}{R}\right)_+^\sigma - D_\sigma\,\mathrm{d}x.$$

Let $R \geq R_0$ be, we estimate the last integral

$$\int_{B_1} |x|^\alpha \left(z_n^r + \frac{\psi}{R}\right)_+^\sigma \mathrm{d}x \geq \int_{B_{r^2/n}(x_0)} |x|^\alpha \left(z_n^r - \frac{\|\psi\|_\infty}{R}\right)^\sigma \mathrm{d}x$$

$$\geq \left(\frac{\|\psi\|_\infty}{R}\right)^\sigma \int_{B_{r^2/n}(x_0)} |x|^\alpha\,\mathrm{d}x = \left(\frac{\|\psi\|_\infty}{R}\right)^\sigma \cdot \frac{2\pi r^{2(2+\alpha)}}{(2+\alpha)n^{2+\alpha}} = \tau = \tau(n, \alpha).$$

Thus we obtain

$$J_\lambda(Rz_n^r) \leq \frac{R^2}{2} - C_\sigma R^\sigma \tau + \frac{r^4}{n^2}\pi D_\sigma.$$

Since $\sigma > 2$, we can choose $R_n \geq R_0$ such that

$$J_\lambda(R_n z_n^r) \leq 0$$

and the proof is complete. ■

Now we consider $\lambda_k < \lambda < \lambda_{k+1}$ and $H = H_0^1(B_1)$. Before proving the geometry of linking we want split the supports of the Moser sequence and the k^{th} eigenfunctions. Consequently, we can make the proof easier to be carried on.

Let us take x_0 and $B_r(x_0)$ as in (2.31) and set $\zeta_r : B_1 \to \mathbb{R}$ by

$$\zeta_r(x) = \begin{cases} 0 & \text{in } B_{r^2}(x_0); \\ \dfrac{|x-x_0|^{\sqrt{r}} - r^{2\sqrt{r}}}{r^{\sqrt{r}} - r^{2\sqrt{r}}} & \text{in } B_r(x_0) \setminus B_{r^2}(x_0); \\ 1 & \text{in } B_1 \setminus B_r(x_0). \end{cases}$$

Define

$$\phi_j^r = \zeta_r \phi_j$$

and consider the following finite-dimensional subspace

$$H_k^r = [\phi_1^r, \phi_2^r, ..., \phi_k^r].$$

We observe that if $v \in H_k^r$, then supp$v \subset B_1 \setminus B_{r^2}$, as we can see in the next figure.

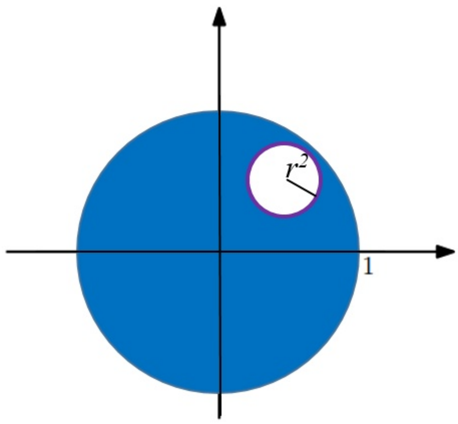

Figure 2.2: Support of $v \in H_k^r$ is a subset of $B_1 \setminus B_{r^2}$.

Now we will prove the next result, which is a two-dimensional version of [32, Lemma 2].

Lemma 2.6.1. If $r \to 0$, then $\phi_j^r \to \phi_j$ in H for all $j = 1, ..., k$. Moreover, for each r small enough, we have that there exists c_k such that

$$\|v\|^2 \leq (\lambda_k + \varrho_r c_k)\|v\|_2^2 \text{ for all } v \in H_k^r,$$

where $\lim\limits_{r \to 0} \varrho_r = 0$.

Proof: For $j \in \{1, ..., k\}$ fixed, we have

$$\begin{aligned} \|\phi_j - \phi_j^r\|^2 &= \int_{B_1} |\nabla(\phi_j - \phi_j^r)|^2 \,\mathrm{d}x \\ &= \int_{B_1} |\nabla\phi_j(1-\zeta_r) - \phi_j\nabla\zeta_r|^2 \,\mathrm{d}x \\ &\leq \int_{B_r} |\nabla\phi_j|^2|1-\zeta_r|^2 \,\mathrm{d}x + 2\int_{B_r\setminus B_{r^2}} |\nabla\phi_j|(1-\eta_r)|\phi_j||\nabla\zeta_r| \,\mathrm{d}x \\ &\quad + \int_{B_r\setminus B_{r^2}} |\phi_j|^2|\nabla\zeta_r|^2 \,\mathrm{d}x \\ &\leq C_1\|\nabla\phi_j\|_\infty^2 \frac{r^{2+2\sqrt{r}}}{(r^{\sqrt{r}} - r^{2\sqrt{r}})^2} + C_2\|\phi_j\|_\infty\|\nabla\phi_j\|_\infty \frac{r^{2+2\sqrt{r}+1/2}}{(r^{\sqrt{r}} - r^{2\sqrt{r}})^2} \\ &\quad + C_3\|\phi_j\|_\infty^2 \frac{r^{2+2\sqrt{r}-1}}{(r^{\sqrt{r}} - r^{2\sqrt{r}})^2} \\ &\leq C\frac{r}{(1-r^{\sqrt{r}})^2}. \end{aligned} \tag{2.34}$$

It is straightforward to prove that

$$\lim_{r\to 0} \frac{r}{(1-r^{\sqrt{r}})^2} = 0.$$

Consequently, $\|\phi_j - \phi_j^r\| \to 0$.

Now, let us take $v_r \in H_k^r$ such that $\|v_r\|_2 = 1$. Notice that $v_r = \sum_{j=1}^k c_j\zeta_r\phi_j = \zeta_r\bar{v}$ where $\bar{v} = \sum_{j=1}^k c_j\phi_j \in H_k$ and $\|v_r - \bar{v}\| = o(1)$ when $r \to 0$. Thus,

$$\begin{aligned}\|v_r\|^2 = &\ (\|v_r\|^2 - \|\bar{v}\|^2) + \|\bar{v}\|^2 \le \tilde{C}_k\tilde{\varrho}_r + \lambda_k\|\bar{v}\|_2^2 \le \tilde{C}_k\tilde{\varrho}_r + \lambda_k[(\|\bar{v}\|_2^2 - \|v_r\|_2^2) + \|v_r\|_2^2] \\ = &\ c_k\varrho_r + \lambda_k,\end{aligned}$$

as desired. ■

As usual, we must continue by choosing an appropriate decomposition for H. Notice that for r small enough, we can split the space $H = H_k^r \oplus H_k^\perp$.

Fix r small enough such that we obtain

$$\frac{\lambda}{\lambda_k + c_k\varrho_r} - 1 > 0 \tag{2.35}$$

and $\delta > 0$ such that

$$\delta^2 < \frac{\lambda}{\lambda_k + c_k\varrho_r} - 1. \tag{2.36}$$

The next two propositions regard the geometric conditions of the Linking Theorem using this non-orthogonal direct sum.

Proposition 2.6.3. *Suppose that $\lambda_k < \lambda < \lambda_{k+1}$, $(g_0) - (g_2)$ and (2.2) hold. Then there exist $\rho, a > 0$ such that $J_\lambda(u) \ge a$ if $u \in H_k^\perp$ with $\|u\| = \rho$.*

Proof: We can now proceed analogously to the proof of Proposition 2.5.3. ■

For the following proposition, let us remark that, since

$$|\text{supp} z_n^r \cap \text{supp} v| = |\partial B_{r^2}(x_0)| = 0 \quad \text{for all} \quad v \in H_k^r. \tag{2.37}$$

This fact is easily seen if we observe Figures 3.1 and 3.2. Thus, one has $z_n^r \in (H_k^r)^\perp$.

Proposition 2.6.4. *Suppose that $\lambda_k < \lambda < \lambda_{k+1}$, $(g_0) - (g_2)$ and (2.2) hold. For each n large enough, there exists $R_n = R(n) > 0$ such that*

$$J_\lambda(u) \le 0 \text{ for all } u \in \partial Q_r^\delta,$$

where

$$Q_r^\delta := \{v + s\delta z_n^r : v \in H_k^r,\ \|v\| \le R_n \text{ and } 0 \le s \le R_n\}.$$

Proof: Fix $R_0 > \rho$ and $N \in \mathbb{N}$ such that for all $n > N$, we have

$$z_n^r(x) \ge \frac{2\|\psi\|_{\infty,r}}{\delta R} \text{ in } B_{r^2/n}(x_0) \text{ for all } R > R_0. \tag{2.38}$$

Let us take $R > R_0$ and as in Proposition 2.5.4, we split ∂Q_r^δ as follows

$$Q_1 = \{v \in H_k^r : \|v\| \leq R\};$$
$$Q_2 = \{v + s\delta z_n^r : v \in H_k^r, \|v\| = R \text{ and } 0 \leq s \leq R\};$$
$$Q_3 = \{v + R\delta z_n^r : v \in H_k^r, \|v\| \leq R\}.$$

If $v \in Q_1$, from Lemma 2.6.1, we obtain

$$J_\lambda(v) \leq \frac{1}{2}\left(1 - \frac{\lambda}{\lambda_k + c_k \varrho_r}\right)\|v\|^2 \text{ for all } v \in H_k^r.$$

Since $\lambda > \lambda_k$ and due to the choice of r in (2.35), we can take $C_1 > 0$ such that

$$J_\lambda(v) \leq -C_1\|v\|^2 \leq 0 \tag{2.39}$$

for all $v \in H_k^r$ independently of $R > 0$.

Using (2.37), we observe that

$$J_\lambda(v + s\delta z_n^r) = J_\lambda(v) + J_\lambda(s\delta z_n^r) \text{ for all } v \in H_k^r. \tag{2.40}$$

For Q_2, using (2.33), and the choice of δ in (2.36), we get

$$J_\lambda(v + s\delta z_n^r) = J_\lambda(v) + J_\lambda(R\delta z_n^r) \leq \frac{1}{2}R^2\left(1 - \frac{\lambda}{\lambda_k + c_k\varrho_r} + \delta^2\right) < 0,$$

independently of $R > 0$.

For Q_3, using (2.27), (2.39) and (2.40), we obtain

$$\begin{aligned}
J_\lambda(v + R\delta \bar{z}^2) \leq\ & J_\lambda(v) + J_\lambda(s\delta \bar{z}_n) \\
\leq\ & -C_1\|v\|^2 + \frac{R^2}{2}\delta^2 - \int_{B_1} |x|^\alpha G(R\bar{z^r}_n + \psi)_+ \,\mathrm{d}x \\
\leq\ & \frac{\delta^2 R^2}{2} - R^\sigma C_\sigma \int_{B_{r^2/n}(x_0)} |x|^\alpha \left(\delta z_n^r - \frac{\|\psi\|_{\infty,n}}{R}\right)_+^\sigma \mathrm{d}x - D_\sigma.
\end{aligned}$$

Let us choose n sufficiently large that

$$\delta\bar{z}_n = \frac{\delta}{\sqrt{2\pi}}\log^{1/2} n \geq \frac{2\|\psi\|_{\infty,n}}{R_0} \text{ for all } x \in B_{r^2/n}(x_0).$$

Taking $R \geq R_0$, we can estimate the last integral as it follows

$$\int_{B_1} |x|^\alpha \left(\delta z_n^r + \frac{\psi}{R}\right)_+^\sigma \,\mathrm{d}x \geq \int_{B_{r^2/n}(x_0)} |x|^\alpha \left(\delta z_n^r - \frac{\|\psi\|_{\infty,n}}{R}\right)^\sigma \,\mathrm{d}x$$

$$\geq \left(\frac{\|\psi\|_{\infty,n}}{R}\right)^\sigma \int_{B_{r^2/n}(x_0)} |x|^\alpha \,\mathrm{d}x$$

$$\geq \left(\frac{\|\psi\|_{\infty,n}}{R}\right)^\sigma \frac{2\pi r^{2(2+\alpha)}}{(2+\alpha)n^{2+\alpha}} = \tau(n) > 0.$$

Thus, we obtain

$$J_\lambda(v + \delta\bar{z}) \leq \frac{R^2\delta^2}{2} - C_\sigma R^\sigma \tau - D_\sigma.$$

Since $\sigma > 2$, this complete the proof. ■

2.6.2 Estimate of minimax levels

For the Mountain–Pass case, we define the minimax level of J_λ by

$$\tilde{c} = \tilde{c}(n) = \inf_{v\in\Gamma} \max_{w\in v([0,1])} J_\lambda(w) \tag{2.41}$$

where

$$\Gamma = \{v \in C([0,1], H) : v(0) = 0 \text{ and } v(1) = R_n z_n^r\},$$

R_n being such that $J_\lambda(R_n z_n^r) \leq 0$ as in Proposition 2.6.2.

Proposition 2.6.5. *Let $\tilde{c}(n)$ be given as in* (2.41)*. Then there exists n large enough such that*

$$\tilde{c}(n) < \frac{2\pi}{\beta_0}.$$

Proof: We claim that there exists n such that

$$\max_{t\geq 0} J_\lambda(tz_n^r) < \frac{2\pi}{\beta_0}. \tag{2.42}$$

Let us fix some constants that we shall use in this proof. We can assume, without loss of generality, that there exist $C_0 > 0$

$$\varepsilon_0 \leq \frac{\log(h(s))}{s} \leq C_0 \tag{2.43}$$

for all s large enough, where ε_0 is given in (2.30).

Indeed, note that if $h(s)$ satisfies (2.5), there exists $\tilde{h}(s)$ such that $h(s) \geq \tilde{h}(s)$ for all s large enough and

$$0 < \liminf_{s\to+\infty} \frac{\log(\tilde{h}(s))}{s} \leq \limsup_{s\to+\infty} \frac{\log(\tilde{h}(s))}{s} < +\infty.$$

By (2.31), we have

$$\|\psi\|_{\infty,r} \leq \frac{\varepsilon_0}{2\beta_0}. \tag{2.44}$$

Finally, we consider γ given by (2.4) such that

$$\gamma > \frac{2^{2+\alpha}}{r^4\beta_0} \exp\left(\frac{C_0^2}{4\beta_0}\right). \tag{2.45}$$

Now suppose by contradiction that (2.42) is not true. So, for all n, this maximum is larger than or equal to $2\pi/\beta_0$ (it is indeed a maximum, in view of Proposition 2.6.2). Let $t_n > 0$ be such that

$$J_\lambda(t_n z_n^r) = \max_{t\geq 0} J_\lambda(t z_n^r). \tag{2.46}$$

Then

$$J_\lambda(t_n z_n^r) \geq \frac{2\pi}{\beta_0} \text{ for all } n \in \mathbb{N}, \tag{2.47}$$

and, consequently, from (2.33),

$$t_n^2 \geq \frac{4\pi}{\beta_0} \text{ for all } n \in \mathbb{N}. \tag{2.48}$$

Let us prove that $t_n^2 \to 4\pi/\beta_0$. From (2.46) we get

$$\frac{d}{dt}(J_\lambda(t z_n^r))\Big|_{t=t_n} = 0.$$

So we have

$$t_n^2 \geq \int_{B_{r^2/n}(x_0)} |x|^\alpha g(t_n z_n^r + \psi)_+ t_n z_n^r \,\mathrm{d}x \geq \int_{B_{r^2/n}(x_0)} |x|^\alpha g(t_n z_n^r + \psi)_+ (t_n z_n^r + \psi)_+ \,\mathrm{d}x.$$

Then, (2.4) and (2.31) imply that there exists s_0 large enough such that

$$t_n^2 \geq \frac{1}{2^\alpha}\gamma \int_{B_{r^2/n}(x_0)} h\left(\frac{t_n}{\sqrt{2\pi}}\log^{1/2} n + \psi\right) \exp\left(\beta_0\left(\frac{t_n}{\sqrt{2\pi}}\log^{1/2} n + \psi\right)^2\right) \,\mathrm{d}x,$$

where we have taken n sufficiently large such that

$$\left(\frac{t_n}{\sqrt{2\pi}}\log^{1/2} n + \psi\right) \geq \left(\frac{t_n}{\sqrt{2\pi}}\log^{1/2} n - \|\psi\|_{\infty,r}\right) \geq s_0 \quad \text{in} \quad B_{r^2/n}(x_0).$$

We still have

$$\begin{aligned} t_n^2 &\geq \frac{1}{2^\alpha}\gamma \int_{B_{r^2/n}(x_0)} \exp\left[-\left(\frac{\log[h((t_n/\sqrt{2\pi})\log^{1/2} n + \psi)]}{2\sqrt{\beta_0}((t_n/\sqrt{2\pi})\log^{1/2} n + \psi)} \right)^2 + \right. \\ &\left. +\beta_0 \left((t_n/\sqrt{2\pi})\log^{1/2} n + \psi + \frac{\log[h((t_n/\sqrt{2\pi})\log^{1/2} n + \psi)]}{2\sqrt{\beta_0}((t_n/\sqrt{2\pi})\log^{1/2} n + \psi)} \right)^2 \right] \mathrm{d}x \end{aligned}$$

and taking an even larger n, (2.43) shows that

$$t_n^2 \geq \frac{1}{2^\alpha}\gamma \frac{\pi r^4}{n^2} \exp\left(-\frac{C_0^2}{4\beta_0}\right) \exp\left(\beta_0 \left(\frac{t_n}{\sqrt{2\pi}} \log^{1/2} n - \|\psi\|_{\infty,r} + \frac{\varepsilon_0}{2\beta_0} \right)^2 \right).$$

By (2.44), we see that

$$\begin{aligned} t_n^2 &\geq \frac{1}{2^\alpha} \exp\left(-\frac{C_0^2}{4\beta_0}\right) \gamma \frac{\pi r^4}{n^2} \exp\left(\beta_0 \frac{t_n^2}{2\pi} \log n\right) \\ &= \frac{1}{2^\alpha} \exp\left(-\frac{C_0^2}{4\beta_0}\right) \gamma \pi r^4 \exp\left(\left(\beta_0 \frac{t_n^2}{2\pi} - 2 \right) \log n \right), \end{aligned} \tag{2.49}$$

which implies that t_n is bounded. Moreover, (2.49) together with (2.48) give $t_n^2 \to 4\pi/\beta_0$.

Letting $n \to \infty$ in (2.49), one gets

$$\gamma \leq \frac{2^{2+\alpha}}{r^4 \beta_0} \exp\left(\frac{C_0^2}{4\beta_0}\right),$$

which is contrary to the choice of γ in (2.45). This contradiction happens because we are supposing $\tilde{c}(n) \geq 2\pi/\beta_0$, so we conclude this proof. ■

Now we can define the minimax level for the Linking case

$$\hat{c} = \hat{c}(n) = \inf_{\nu \in \Gamma} \max_{w \in Q_r^\delta} J_\lambda(\nu(w)) \tag{2.50}$$

where

$$\Gamma = \{\nu \in C(Q_r^\delta; H) : \nu(w) = w \text{ if } w \in \partial Q\}.$$

Proposition 2.6.6. *Let $\hat{c}(n)$ be given as in* (2.50)*. Then there exists n large enough such that*

$$\hat{c}(n) < \frac{2\pi}{\beta_0}.$$

Proof: Since we split the support of the functions in H_k^r of the support of z_n^r, we have

$$\begin{aligned} \hat{c}(n) &\leq \max\{J_\lambda(v + t z_n^r) : v \in H_k^r, \|v\| \leq R_n \text{ and } t \geq 0\} \\ &= \max\{J_\lambda(v) + J_\lambda(t z_n^r) : v \in H_k^r, \|v\| \leq R_n \text{ and } t \geq 0\} \\ &\leq \max\{J_\lambda(v) : v \in H_k^r \text{ and } \|v\| \leq R_n\} + \max\{J_\lambda(t z_n^r); t \geq 0\} \end{aligned}$$

By (2.39), we see that $J_\lambda(v) \leq 0$ for all $v \in H_k^r$. It follows that

$$\hat{c}(n) \leq \max\{J_\lambda(tz_n^r) : t \geq 0\}.$$

From now on, we can proceed analogously to the proof of Proposition 2.6.5. ■

2.6.3 Proof of Theorem 2.2.3 completed

Let us take n such that $c(n) < 2\pi/\beta_0$, where $c(n) = \tilde{c}(n)$, if $\lambda < \lambda_1$ or $c(n) = \hat{c}(n)$, if $\lambda > \lambda_1$. Let us consider (u_m) a $(PS)-$sequence at level $c(n)$. Since it is bounded by Lemma 2.4.1, Then, up to a subsequence, we may assume that $u_m \rightharpoonup u$ weakly in H, $u_m \to u$ strongly in $L^p(B_1)$ for all $p \geq 1$ and almost everywhere in B_1. Therefore, we notice that u is a solution for (2.7). Indeed, for each $v \in C_c^\infty(B_1)$ we have

$$0 \leftarrow \langle J'_\lambda(u_m), v\rangle = \int_{B_1} \nabla u_m \nabla v \, \mathrm{d}x - \lambda \int_{B_1} u_m v \, \mathrm{d}x - \int_{B_1} |x|^\alpha g(u_m + \psi)_+ v \, \mathrm{d}x.$$

To see this fact we use that

$$\int_{B_1} \nabla u_m \nabla v \, \mathrm{d}x \to \int_{B_1} \nabla u \nabla v \, \mathrm{d}x,$$

$$\int_{B_1} u_m v \, \mathrm{d}x \to \int_{B_1} uv \, \mathrm{d}x \text{ and}$$

$$\int_{B_1} |x|^\alpha g(u_m + \psi)_+ v \, \mathrm{d}x \to \int_{B_1} |x|^\alpha g(u + \psi)_+ v \, \mathrm{d}x,$$

(this last one because of (2.15), and using Lemma 2.1 in [28] and (g_1)). Thus, we can easily conclude that $\langle J'_\lambda(u), v\rangle = 0$ for all $v \in C_c^\infty(B_1)$. Consequently u is a weak solution for (2.7). We still need to ensure that $u \not\equiv 0$.

Suppose, instead, that $u \equiv 0$. So, we must have $\|u_m\|_2 \to 0$ and, again using (2.15), (g_1) and Lemma 2.1 in [28], we have

$$\int_{B_1} |x|^\alpha G(u_m + \psi)_+ \, \mathrm{d}x \to \int_{B_1} |x|^\alpha G(u + \psi)_+ \, \mathrm{d}x = 0.$$

Moreover, since

$$c(n) = \lim_{m\to\infty} J_\lambda(u_m) = \frac{1}{2} \lim_{m\to\infty} \|u_m\|^2,$$

and $c(n) < 2\pi/\beta_0$, one can find $\delta > 0$ and m_0 such that

$$\|u_m\|^2 \leq \frac{4\pi}{\beta_0} - \delta \quad \text{for all} \quad m \geq m_0.$$

Consider a small $\varepsilon > 0$ and $p > 1$ (sufficiently close to 1) in order to have

$$p(\beta_0 + \varepsilon)(4\pi/\beta_0 - \delta) \leq 4\pi.$$

By (2.2), one can take $C > 0$ sufficiently large such that

$$g(s)^p \leq e^{p(\beta_0+\varepsilon)s^2} + C \text{ for all } s \geq 0. \tag{2.51}$$

From the fact that $\langle J'_\lambda(u_m), u_m\rangle = \varepsilon_m \to 0$, we can see that

$$\|u_m\|^2 \leq \lambda\|u_m\|_2^2 + \int_{B_1} |x|^\alpha g(u_m + \psi)_+ u_m \,\mathrm{d}x + \varepsilon_m.$$

We need to estimate the integral on the right of this last inequality. Hölder's inequality and (2.51) give

$$\begin{aligned}\int_{B_1} |x|^\alpha g(u_m + \psi)_+ u_m \,\mathrm{d}x \leq & \left(\int_{B_1} (g(u_m + \psi)_+)^p \,\mathrm{d}x\right)^{\frac{1}{p}} \left(\int_{B_1} |u_m|^{p'} \,\mathrm{d}x\right)^{\frac{1}{p'}} \\ \leq & \left[\left(\int_{B_1} \exp\left(p(\beta_0 + \varepsilon)u_m^2\right) \,\mathrm{d}x\right)^{\frac{1}{p}} + C\pi^{\frac{1}{p}}\right] \|u_m\|_{p'}\end{aligned}$$

and then we have

$$\begin{aligned}\|u_m\|^2 \leq & \lambda\|u_m\|_2^2 + \varepsilon_m + \left[\left(\int_{B_1} \exp\left(p(\beta_0 + \varepsilon)\left(4\frac{\pi}{\beta_0} - \delta\right)\left(\frac{u_m}{\|u_m\|}\right)^2\right) \mathrm{d}x\right)^{\frac{1}{p}} + C\pi^{\frac{1}{p}}\right] \|u_m\|_{p'} \\ \leq & \lambda\|u_m\|_2^2 + \varepsilon_m + \left[\left(\int_{B_1} \exp\left(4\pi\left(\frac{u_m}{\|u_m\|}\right)^2\right) \mathrm{d}x\right)^{\frac{1}{p}} + C\pi^{\frac{1}{p}}\right] \|u_m\|_{p'}.\end{aligned}$$

Since the last integral in the estimates above is bounded (because of Trudinger-Moser inequality), we get $\|u_m\| \to 0$. Hence $u_m \to 0$ in H and then $J_\lambda(u_m) \to 0 = c(n)$. This is impossible since $c(n) \geq a > 0$ for all n. Thus $u \not\equiv 0$ is the desired solution. ■

2.7 Proof of Theorem 2.2.4: Critical case in $H^1_{0,\mathrm{rad}}(B_1)$

In this section, we treat the radial case with g having critical growth. In this case, a solution u for Problem (2.7) is in $H = H^1_{0,\mathrm{rad}}(B_1)$, which will force us to change some calculations.

2.7.1 The geometric conditions

If $\lambda < \lambda_1$, we observe that the geometric conditions follow from Proposition 2.6.1 and Proposition 2.6.2, replacing z_n^r, given in (2.32), with z_n, given in (2.29). If $\lambda_k < \lambda < \lambda_{k+1}$, we use the same arguments developed in Proposition 2.6.3 in order to prove that 0 is a local minimum of J_λ in $\left((H_l^*)^\perp \cap H^1_{0,\text{rad}}(B_1)\right)$. However, we cannot take the Moser sequence with disjoint support from the eigenfunctions, as we did in Proposition 2.6.4. Indeed, since now we work in an environment of radial functions, we need to use the sequence set in (2.29), instead of the one given in (2.32). This replacement brings some difficulties because we lose the advantage of being close to the boundary, where the interference of ψ could be considered negligible.

Initially, let $l = \max\{j : H_j^* \subset H_k\}$ and we set $T_l : H^1_{0,\text{rad}}(B_1) \to \left((H_l^*)^\perp \cap H^1_{0,\text{rad}}(B_1)\right)$ as the orthogonal projection. We consider (2.29) and define

$$w_n(x) = T_l z_n(x).$$

Since $\left((H_l^*)^\perp \cap H^1_{0,\text{rad}}(B_1)\right) \subset H_k^\perp$, we have that estimates of [48, Lemma 2] hold (see Lemma 0.7.2).

Proposition 2.7.1. *Suppose that $\lambda_k < \lambda < \lambda_{k+1}$, $(g_0) - (g_2)$ and (2.2) hold. For each n large enough, there exists $R_n = R(n) > 0$ such that if*

$$Q := \{v \in H_l^* : \|v\| \le R_n\} \oplus \{s\delta w_n : 0 \le s \le R_n\},$$

with

$$\delta^2 \le \frac{\lambda}{\lambda_k} - 1, \tag{2.52}$$

then $J_\lambda(u) \le 0$ for all $u \in \partial Q$; moreover, $R_n \to \infty$ when $n \to \infty$.

Proof: Fix $R_0 > \rho$ and let us take $R > R_0$. As it is usual, we split ∂Q as follows

$$\begin{aligned}
Q_1 &= \{v \in H_l^* : \|v\| \le R\};\\
Q_2 &= \{v + s\delta w_n : v \in H_l^*, \|v\| = R \text{ and } 0 \le s \le R\};\\
Q_3 &= \{v + R\delta w_n : v \in H_l^*, \|v\| \le R\}.
\end{aligned}$$

Recalling that $G(s) \ge 0$ for all $s \in \mathbb{R}$ and considering $v \in Q_1$ we get

$$J_\lambda(v) \le \left(1 - \frac{\lambda}{\lambda_k}\right) \|v\|^2 \le 0, \tag{2.53}$$

independently of R.

For Q_2, let us take δ satisfying (2.52). By (14), we obtain

$$J_\lambda(v + s\delta w_n) \leq \left(1 - \frac{\lambda}{\lambda_k} + \delta^2\right)\frac{R^2}{2} \leq 0,$$

independently of $R > 0$.

For Q_3, we take $\delta > 0$ given in (2.52) and $v + R\delta w_n$ with $\|v\| \leq R$ and $v \in H_l^*$. Using (2.27), we obtain

$$J_\lambda(v + R\delta w_n) \leq \frac{1+\delta^2}{2}R^2 - R^\sigma \int_{B_1} |x|^\alpha \left(\frac{-(\|v\|_\infty + \|\psi\|_\infty)}{R} + \delta w_n\right)_+^\sigma \mathrm{d}x + D_\sigma.$$

Since $H_l^* \subset H_k$ has finite dimension, it follows that $\|v\|_\infty \leq C_k R$ for some $C_k > 0$. We can suppose that R_0, previously fixed, satisfies

$$\frac{\|\psi\|_\infty}{R_0} \leq C_k.$$

Then, by (15) and considering $R > R_0$, we have

$$J_\lambda(v + R\delta w_n) \leq \frac{1+\delta^2}{2}R^2 - \bar{C}_n R^\sigma + \pi D_\sigma,$$

where

$$\bar{C}_n = \left(-2C_k + \delta\left(\frac{(\log n)^{1/2}}{\sqrt{2\pi}} - \frac{B_k}{(\log n)^{1/2}}\right)\right)_+^\sigma \frac{2\pi}{(2+\alpha)n^{2+\alpha}} > 0.$$

Since $\sigma > 2$, the result is then proved by taking R_n large enough so that $\delta\|w_n\|R_n > \rho$ and, thus, we obtain

$$J_\lambda(v + R_n\delta w_n) \leq \frac{1+\delta^2}{2}R_n - \bar{C}_n R_n^\sigma + \pi D_\sigma \leq 0$$

and since $\bar{C}_n \to 0$ when $n \to \infty$, we can see that $R_n \to \infty$. This concludes the proof. ■

2.7.2 Estimate of minimax levels

Here we can notice another important difference between the cases studied in $H_0^1(B_1)$ and in $H_{0,\mathrm{rad}}^1(B_1)$. Since, in the radial case, we need to work with radially symmetric functions whose supports have the center at $0 \in \mathbb{R}^2$, we cannot neglect the weight $|x|^\alpha$ as we did in the critical case. Actually, although the estimates are harder to obtain, it turns out to be an advantage because this environment of symmetric functions changes the boundedness of the minimax levels: in the critical case, we had that they were bounded by the constant

$2\pi/\beta_0$. On the other hand, working in $H^1_{0,\mathrm{rad}}(B_1)$, this boundedness can be obtained by a greater value that depends on β_0 and α.

For the Mountain–Pass problem, we define the minimax level of J_λ by

$$\bar{c} = \bar{c}(n) = \inf_{\upsilon\in\Gamma} \max_{w\in\upsilon([0,1])} J_\lambda(w), \tag{2.54}$$

where

$$\Gamma = \{\upsilon \in C([0,1], H) : \upsilon(0) = 0 \text{ and } \upsilon(1) = R_n z_n\},$$

R_n being such that $J_\lambda(R_n z_n) \leq 0$.

Proposition 2.7.2. *Let $\bar{c}(n)$ be given in* (2.54). *Then there exists n large enough such that*

$$\bar{c}(n) < \frac{(2+\alpha)\pi}{\beta_0}. \tag{2.55}$$

Proof: We claim that exists n such that

$$\max_{t\geq 0} J_\lambda(tz_n) < \frac{(2+\alpha)\pi}{\beta_0}. \tag{2.56}$$

First of all, let us fix some constants that we shall use in this proof.

Analogously as we have done in Proposition 2.6.5, we can assume, without loss of generality, that

$$K_0 \leq \frac{\log(h(s))}{s} \leq C_0 \tag{2.57}$$

for all s large enough.

By (2.6), K_0 can be taken large enough in order to have

$$\|\psi\|_\infty \leq \frac{K_0}{2\beta_0}. \tag{2.58}$$

Finally, consider γ such that

$$\gamma > \frac{(2+\alpha)^2}{\beta_0} \exp\left(\frac{C_0^2}{4\beta_0}\right). \tag{2.59}$$

We suppose by contradiction that (2.56) is not true. So, for all n, this maximum is large or equal to $(2+\alpha)\pi/\beta_0$. Let $t_n > 0$ be such that

$$J_\lambda(t_n z_n) = \max_{t\geq 0} J_\lambda(tz_n). \tag{2.60}$$

Then

$$J_\lambda(t_n z_n) \geq \frac{(2+\alpha)\pi}{\beta_0} \text{ for all } n \in \mathbb{N},$$

and, consequently,

$$t_n^2 \geq \frac{2(\alpha+2)\pi}{\beta_0} \text{ for all } n \in \mathbb{N}. \tag{2.61}$$

Let us prove that $t_n^2 \to 2(2+\alpha)\pi/\beta_0$. From (2.60) we get

$$\frac{d}{dt}(J_\lambda(tz_n))\bigg|_{t=t_n} = 0.$$

Thus,

$$t_n\|z_n\|^2 - \lambda t\|z_n\|_2^2 - \int_{B_1} |x|^\alpha g((t_n z_n + \psi)_+) z_n \, \mathrm{d}x = 0.$$

Multiplying this last equation by t_n and since $\|z_n\| = 1, \lambda \geq 0, \psi < 0$ and $g \geq 0$, for n large enough, we have

$$t_n^2 \geq \int_{B_{1/n}} |x|^\alpha g((t_n z_n + \psi)_+) t_n z_n \, \mathrm{d}x \geq \int_{B_{1/n}} |x|^\alpha g((t_n z_n + \psi)_+)(t_n z_n + \psi)_+ \, \mathrm{d}x.$$

Since $t_n \geq \sqrt{2(2+\alpha)\pi/\beta_0} > 0$, we can take n large enough that $(t_n(\sqrt{2\pi})^{-1}\log^{1/2} n - \|\psi\|_\infty) \geq c_\lambda$, where c_λ is given in (2.4). Since

$$(t_n z_n + \psi)_+ = (t_n z_n + \psi) \geq t_n(\sqrt{2\pi})^{-1}\log^{1/2} n - \|\psi\|_\infty$$

in $B_{1/n}$ and so (2.4) implies that

$$t_n \geq \gamma \int_{B_{1/n}} |x|^\alpha h\left(\frac{t_n}{\sqrt{2\pi}}\log^{1/2} n - \|\psi\|_\infty\right) \exp\left(\beta_0\left(\frac{t_n}{\sqrt{2\pi}}\log^{1/2} n - \|\psi\|_\infty\right)^2\right) \, \mathrm{d}x.$$

We have

$$\begin{aligned} t_n^2 \;&\geq \gamma \int_{B_{1/n}} |x|^\alpha \exp\left[-\left(\frac{\log[h((t_n/\sqrt{2\pi})\log^{1/2} n - \|\psi\|_\infty)]}{2\sqrt{\beta_0}((t_n/\sqrt{2\pi})\log^{1/2} n - \|\psi\|_\infty)}\right)^2\right. \\ &\left. +\beta_0\left((t_n/\sqrt{2\pi})\log^{1/2} n - \|\psi\|_\infty + \frac{\log[h((t_n/\sqrt{2\pi})\log^{1/2} n - \|\psi\|_\infty)]}{2\beta_0((t_m/\sqrt{2\pi})\log^{1/2} n - \|\psi\|_\infty)}\right)^2\right] \, \mathrm{d}x. \end{aligned}$$

But if n is large, (2.57) shows that

$$-\left(\frac{\log[h((t_n/\sqrt{2\pi})\log^{1/2} n - \|\psi\|_\infty)]}{2\sqrt{\beta_0}((t_n/\sqrt{2\pi})\log^{1/2} n - \|\psi\|_\infty)}\right)^2 \geq -\frac{C_0^2}{4\beta_0}$$

and

$$\frac{\log[h((t_n/\sqrt{2\pi})\log^{1/2} n - \|\psi\|_\infty)]}{2\beta_0((t_m/\sqrt{2\pi})\log^{1/2} n - \|\psi\|_\infty)} \geq \frac{K_0}{2\beta_0},$$

from which we obtain

$$t_n^2 \geq \gamma \, \frac{2\pi}{(2+\alpha)n^{2+\alpha}} \exp\left(-\frac{C_0^2}{4\beta_0}\right) \exp\left(\beta_0\left(\frac{t_n}{\sqrt{2\pi}}\log^{1/2} n - \|\psi\|_\infty + \frac{K_0}{2\beta_0}\right)^2\right).$$

By (2.58), we see that

$$
\begin{aligned}
t_n^2 &\geq \exp\left(-\frac{C_0^2}{4\beta_0}\right)\gamma\frac{2\pi}{(2+\alpha)n^{2+\alpha}}\exp\left(\beta_0\frac{t_n}{2\pi}\log n\right) \\
&= \exp\left(-\frac{C_0^2}{4\beta_0}\right)\gamma\frac{2\pi}{(2+\alpha)}\exp\left(\beta_0\frac{t_n}{2\pi}-(2+\alpha)\right)\log n,
\end{aligned}
\tag{2.62}
$$

which implies that t_n is bounded. Moreover, (2.62) together with (2.61) gives that $t_n^2 \to 2(2+\alpha)\pi/\beta_0$.

Now, letting $n \to \infty$ in (2.62), one gets

$$\gamma \leq \frac{(2+\alpha)^2}{\beta_0}\exp\left(\frac{C_0^2}{4\beta_0}\right).$$

It contradicts (2.59), then we have that (2.55) holds. ■

Remark 2.7.1. *This last proposition explains why we had to assume* (2.6) *instead of* (2.5). *In the* $H_0^1(B_1)$ *environment, we could control the* L^∞ *norm of* ψ *by moving the supports of our Moser sequence far away from* 0 *and close enough to* ∂B_1 *so that* $\|\psi\|_\infty$ *would be sufficiently small and then it would not interfere in the estimates. Since this could not be done in the radial case, the interference was avoided by using* (2.58).

Remark 2.7.2. *We also point out the different choices we have made for the constant* γ *in* (2.45) *and* (2.59), *which explains, in part, the role of* $|x|^\alpha$ *in the radial case. Notice that* γ *in* (2.45) *must be greater than the one given in* (2.59).

Now, for the linking problem, we can define the minimax level

$$\check{c} = \check{c}(n) = \inf_{\nu\in\Gamma}\max_{w\in\nu(Q)} J_\lambda(\nu(w)) \tag{2.63}$$

where $\Gamma = \{\nu \in C(Q, H) : \nu(w) = w \text{ if } w \in \partial Q\}$.

We remark that since we did not separate the supports of the eigenfunctions from the Moser sequence, the estimates done in Proposition 2.6.6 will not work in this radial case. Therefore, we need do handle more delicate arguments in order to achieve analogous results. This is done in the following proposition.

Proposition 2.7.3. *Let* $\check{c}(n)$ *be given as in* (2.63). *Then there exists* n *large enough such that*

$$\check{c}(n) < \frac{(2+\alpha)\pi}{\beta_0}.$$

Proof: Suppose by contradiction that for all n we have $\check{c}(n) \geq (2+\alpha)\pi/\beta_0$. We notice that

$$\check{c}(n) \leq \max\{J_\lambda(v + tw_n) : v \in H_l^* \text{ with } \|v\| \leq R_n, t \geq 0\}$$

and it follows that for each n there exist $v_n \in H_l^*$ and $t_n > 0$ such that

$$J_\lambda(v_n + t_n w_n) = \max\{J_\lambda(v + tw_n) : v \in H_l^* \text{ with } \|v\| \leq R_n, t \geq 0\}. \tag{2.64}$$

Therefore, we have

$$J_\lambda(v_n + t_n w_n) \geq \frac{(2+\alpha)\pi}{\beta_0} \text{ for all } n \in \mathbb{N}. \tag{2.65}$$

So, since $w_n \in \left((H_l^*)^\perp \cap H^1_{0,\mathrm{rad}}(B_1)\right)$, from (2.53) we obtain

$$t_n^2 \geq \frac{2(2+\alpha)\pi}{\beta_0} \text{ for all } n \in \mathbb{N}. \tag{2.66}$$

Let us assume the following claims whose proofs we give later:

Claim 5. *(v_n) and (t_n) are bounded sequences.*

Claim 6. *$t_n^2 \to \dfrac{2(2+\alpha)\pi}{\beta_0}$ in $\mathbb{R}$ and $v_n \to 0$ in H.*

Since $v_n \in H_l^*$, in view of Claim 6, we also get $\|v_n\|_\infty \to 0$. However, we have $v_n + t_n w_n \to \infty$ uniformly in $B_{1/n}$.

As we also have done in Proposition 2.7.2, let us observe that we can assume, without loss of generality, that

$$K_0 \leq \frac{\log(h(s))}{s} \leq C_0 \tag{2.67}$$

for all s large enough. From (2.6), K_0 can be taken large enough in order to have

$$\|\psi\|_\infty \leq \frac{K_0}{4\beta_0}$$

and

$$\|v_n\|_\infty \leq \frac{K_0}{4\beta_0}$$

for n large enough.

Finally, consider γ such that

$$\gamma > \frac{(2+\alpha)^2}{\beta_0} \exp\left(\frac{4(2+\alpha)\pi B_k}{\sqrt{2\pi}} + \frac{C_0^2}{4\beta_0}\right). \tag{2.68}$$

From (2.64), using the fact that derivative of J_λ, restricted to $H_l^*\oplus\mathbb{R}w_n$ is zero at $v_n+t_nw_n$, we obtain

$$\|v_n+t_nw_n\|^2-\lambda\|v_n+t_nw_n\|_2^2-\int_{B_1}|x|^\alpha g(v_n+t_nw_n+\psi)_+(v_n+t_nw_n)\,\mathrm{d}x=0, \quad (2.69)$$

and we can see, for n large enough, we have

$$t_n^2\geq\int_{B_{1/n}}|x|^\alpha g((v_n+t_nw_n+\psi)_+)(v_n+t_nw_n+\psi)_+\,\mathrm{d}x$$

Since $t_n\geq\sqrt{2(2+\alpha)\pi/\beta_0}>0$ and $v_n\to 0$, by (15) we can take n large enough that

$$(t_nw_n-(\|v\|_\infty+\|\psi\|_\infty)\geq c_\lambda,$$

where c_λ is given in (2.4). Thus, we obtain

$$t_n^2\geq\gamma\int_{B_{1/n}}|x|^\alpha h\left(t_nw_n-(\|v\|_\infty+\|\psi\|_\infty\right)\exp\left(\beta_0\left(t_nw_n-(\|v\|_\infty+\|\psi\|_\infty\right)^2\right)\,\mathrm{d}x,$$

consequently,

$$\begin{aligned}t_n^2\ &\geq\gamma\int_{B_{1/n}}|x|^\alpha\exp\left[-\left(\frac{\log[h(t_nw_n-(\|v\|_\infty+\|\psi\|_\infty)]}{2\sqrt{\beta_0}(t_nw_n-(\|v\|_\infty+\|\psi\|_\infty)}\right)^2\right.\\&\left.+\beta_0\left(t_nw_n-(\|v\|_\infty+\|\psi\|_\infty+\frac{\log[h(t_nw_n-(\|v\|_\infty+\|\psi\|_\infty)]}{2\beta_0(t_nw_n-(\|v\|_\infty+\|\psi\|_\infty)}\right)^2\right]\,\mathrm{d}x.\end{aligned}$$

If n is large, (2.67) shows that

$$-\left(\frac{\log[h(t_nw_n-(\|v\|_\infty+\|\psi\|_\infty)]}{2\sqrt{\beta_0}(t_nw_n-(\|v\|_\infty+\|\psi\|_\infty)}\right)^2\geq-\frac{C_0^2}{4\beta_0}$$

and

$$\frac{\log[h(t_nw_n-(\|v\|_\infty+\|\psi\|_\infty)]}{2\beta_0(t_nw_n-(\|v\|_\infty+\|\psi\|_\infty)}\geq\frac{K_0}{2\beta_0},$$

from which we obtain

$$\begin{aligned}t_n^2\ &\geq\gamma\exp\left(-\frac{C_0^2}{4\beta_0}\right)\int_{B_{1/n}}|x|^\alpha\exp\left(\beta_0\left(t_nw_n-(\|v\|_n+\|\psi\|_\infty)+\frac{K_0}{2\beta_0}\right)^2\right)\,\mathrm{d}x\\&\geq\gamma\exp\left(-\frac{C_0^2}{4}\right)\int_{B_{1/n}}|x|^\alpha\exp(\beta_0t_n^2w_n^2)\,\mathrm{d}x.\end{aligned}$$

Consequently, from (15) and (2.66) we can see that

$$
\begin{aligned}
t_n^2 &\geq \gamma \exp\left(-\frac{C_0^2}{4}\right) \int_{B_{1/n}} |x|^\alpha \exp[2(2+\alpha)\pi w_n^2]\, \mathrm{d}x \\
&= \gamma \exp\left(-\frac{C_0^2}{4}\right) \frac{2\pi}{(2+\alpha)n^{2+\alpha}}) \exp\left[2(2+\alpha)\pi\left(\frac{(\log n)^{1/2}}{\sqrt{2\pi}} - \frac{B_k}{(\log n)^{1/2}}\right)^2\right] \\
&= \gamma e^{\left(-\frac{C_0^2}{4}\right)\frac{2\pi}{(2+\alpha)n^{2+\alpha}})+[(2+\alpha)\log n]+(-4(2+\alpha)\pi B_k/\sqrt{2\pi})+[2(2+\alpha)\pi B_k^2/\log n]}.
\end{aligned}
$$

Now, letting $n \to \infty$ in the inequality above, we can conclude that

$$
\gamma \leq \frac{(2+\alpha)^2}{\beta_0} \exp\left(\frac{4(2+\alpha)\pi B_k}{\sqrt{2\pi}} + \frac{C_0^2}{4\beta_0}\right),
$$

which is contrary to the choice of γ in (2.68). This contradiction follows from the assumption $\check{c}(n) \geq (2+\alpha)\pi/\beta_0$ for all $n \in \mathbb{N}$, which concludes the proof. ■

Proof of Claim 5: It is sufficient to prove that all subsequences of (t_n) and (v_n) have bounded subsequences. Let us suppose that this is not true. So, we can find subsequences, which by convenience we still denote by (t_n) and (v_n), respectively, such that all of their subsequences are unbounded. That means, we can assume that

$$
t_{n_k} + \|v_{n_k}\| \to \infty \text{ for all subsequences } (n_k). \tag{2.70}
$$

Therefore, one of the following two possibilities has to hold:

(i) either there exists a constant $C_0 > 0$ such that $t_n/\|v_n\| \geq C_0$, or

(ii) there are subsequences such that $t_n/\|v_n\| \to 0$.

Assume that **(i)** holds and using (2.70), we have that $t_n \to \infty$. Now we can see from (2.4) that

$$
\begin{aligned}
t_n^2 \geq & \int_{B_{1/n}} |x|^\alpha g(v_n + t_n w_n + \psi)_+ (v_n + t_n w_n + \psi)_+\, \mathrm{d}x \\
\geq & \ \gamma \int_{B_{1/n}} |x|^\alpha h(v_n + t_n w_n + \psi) \exp\left(\beta_0 (v_n + t_n w_n + \psi)^2\right)\, \mathrm{d}x
\end{aligned}
$$

for n large enough and we can see from (2.6) that

$$
h(s) \geq \tilde{C} \tag{2.71}
$$

for all s large enough. So we have

$$t_n^2 \geq \gamma\tilde{C}\int_{B_{1/n}} |x|^\alpha \exp\left(\beta_0(v_n + t_n w_n + \psi)^2\right)\,\mathrm{d}x.$$

We notice that since H_l^* has finite dimension, we have that $\|v_n\|_\infty/t_n$ is bounded for all $x \in B_{1/n}$. We also know that $\|\psi\|_\infty/t_n$ is bounded as well. These facts together with (15), give us

$$\begin{aligned}
&v_n(x) + t_n w_n(x) + \psi(x)\\
&= t_n w_n(x)\left(1 + \frac{v_n(x)+\psi(x)}{t_n}\frac{1}{w_n}\right)\\
&\geq \frac{t_n}{\sqrt{2\pi}}\left((\log n)^{1/2} - \frac{\sqrt{2\pi}B_k}{(\log n)^{1/2}}\right)\left(1 - C\left(\frac{(\log n)^{1/2}}{\sqrt{2\pi}} - \frac{B_k}{(\log n)^{1/2}}\right)^{-1}\right)\\
&\geq \frac{t_n}{2}\frac{1}{\sqrt{2\pi}}\left((\log n)^{1/2} - \frac{\sqrt{2\pi}B_k}{(\log n)^{1/2}}\right)
\end{aligned}$$

and taking n such that $(\log n)^{1/2} - \sqrt{2\pi}B_k/(\log n)^{1/2} \geq (1/2)(\log n)^{1/2}$, we obtain

$$v_n(x) + t_n w_n(x) + \psi(x) \geq \frac{t_n}{4\sqrt{2\pi}}(\log n)^{1/2}$$

and by (2.71) it follows that

$$\begin{aligned}
t_n^2 \geq\ & \gamma\tilde{C}\int_{B_{1/n}} |x|^\alpha \exp\left(\beta_0\left(\frac{t_n}{4\sqrt{2\pi}}(\log n)^{1/2}\right)^2\right)\,\mathrm{d}x\\
&= \gamma\tilde{C}\frac{2\pi}{2+\alpha}\exp\left(\left(\beta_0\frac{t_n^2}{32\pi} - (2+\alpha)\right)\log n\right).
\end{aligned}$$

Consequently, t_n must be bounded in case **(i)**, which contradicts $t_n \to \infty$ in case **(i)**.

So **(ii)** occurs. Since $\lim_{n\to\infty} t_n/\|v_n\| = 0$, by (2.70) we conclude that $\|v_n\| \to \infty$. By (2.69), we get

$$\|t_n w_n + v_n\|^2 \geq \int |x|^\alpha g(t_n w_n + v_n + \psi)_+(t_n w_n + v_n)_+\,\mathrm{d}x.$$

Using (2.4) and (2.71), for n large enough, we have

$$\|t_n w_n + v_n\|^2 \geq \gamma\tilde{C}\int_{\{t_n w_n + v_n + \psi \geq c_\gamma\}} |x|^\alpha e^{\beta_0(t_n w_n + v_n + \psi)^2}\,\mathrm{d}x.$$

Since we are supposing **(ii)**, it follows that

$$\begin{aligned}1 &\geq \tilde{C}\gamma \int_{\{t_n w_n + v_n + \psi \geq c_\gamma\}} |x|^\alpha \frac{e^{\beta_0 (t_n w_n + v_n + \psi)^2}}{\|t_n w_n + v_n\|^2}\, \mathrm{d}x \\ &\geq \tilde{C}\frac{\gamma}{2} \int_{\{t_n w_n + v_n + \psi \geq c_\gamma\}} |x|^\alpha \frac{e^{\beta_0 (t_n w_n + v_n + \psi)^2}}{\|v_n\|^2}\, \mathrm{d}x.\end{aligned} \tag{2.72}$$

On the other hand, we notice that

$$\frac{t_n w_n + v_n + \psi}{\|v_n\|} \chi_{\{t_n w_n + v_n + \psi \geq c_\gamma\}}$$

$$= \frac{v_n}{\|v_n\|} + \frac{t_n}{\|v_n\|} w_n - \frac{t_n w_n + v_n}{\|v_n\|} \chi_{\{t_n w_n + v_n + \psi \leq c_\gamma\}} + \frac{\psi}{\|v_n\|} \chi_{\{t_n w_n + v_n + \psi \geq c_\gamma\}}.$$

Hence, we can see that

$$\frac{t_n w_n + v_n(x) + \psi}{\|v_n(x)\|} \chi_{\{t_n w_n + v_n + \psi \geq c_\gamma\}}(x) \to \widehat{v} \text{ a. e. in } H_0^1(B_1),$$

where $\widehat{v} \in H_k$, with $v_n/\|v_n\| \to \widehat{v}$ and $\|\widehat{v}\| = 1$. So using Fatou's Lemma in (2.72) and since we have assumed that $\|v_n\| \to \infty$, we reach a contradiction. So $\|v_n\|$ is bounded and, consequently, t_n is also bounded. ■

Proof of Claim 6: First, we notice for some appropriated subsequences we have $v_n \to v_0$ in H and $t_n \to t_0$ and since $z_n \rightharpoonup 0$ we get $w_n \rightharpoonup 0$ and $w_n \to 0$ for all $x \in B_1$. Then it follows

$$v_n + t_n w_n \to v_0 \text{ almost everywhere in } B_1. \tag{2.73}$$

Moreover, in view of (2.69) we see that

$$\int_{B_1} |x|^\alpha g(v_n + t_n w_n \psi)_+ (v_n + t_n w_n \psi)\, \mathrm{d}x \leq \|v_n + t_n w_n\|^2 \leq C. \tag{2.74}$$

However, using [28, Lemma 2.1] and recalling (g_1), (2.73) and (2.74), we have

$$\int_{B_1} |x|^\alpha G(v_n + t_n w_n + \psi)_+\, \mathrm{d}x \to \int_{B_1} |x|^\alpha G(v_0 + \psi)_+\, \mathrm{d}x. \tag{2.75}$$

From (2.65) and (2.75) we can see that

$$J_\lambda(v_0) + \frac{t_0^2}{2} \geq \frac{(2+\alpha)\pi}{\beta_0}. \tag{2.76}$$

and since $v_0 \in H_l^*$, in view of $J_\lambda(v_0) \leq 0$ we have

$$t_0^2 \geq \frac{2(2+\alpha)\pi}{\beta_0}.$$

Now we prove that $t_0^2 = (2(2+\alpha)\pi)/\beta_0$. Let us suppose that this is not true. We have $t_0^2 > (2(2+\alpha)\pi)/\beta_0$. Thus we can take small enough $\varepsilon > 0$ so that

$$t_n^2 > (1+\varepsilon)\frac{2(2+\alpha)\pi}{\beta_0}$$

for all large n. We consider

$$\varepsilon_n = \sup_{B_{1/n}} \frac{|v_n(x) + \psi(x)|}{t_n w_n(x)},$$

clearly we see that $\varepsilon_n \to 0$, which, together with (2.71), yields

$$\begin{aligned} C &\geq \gamma \tilde{C} \int_{B_{1/n}} |x|^\alpha \exp[\beta_0(v_n + \psi + t_n w_n)^2] \,\mathrm{d}x \\ &\geq \gamma \tilde{C} \int_{B_{1/n}} |x|^\alpha \exp[\beta_0(-\varepsilon_n t_n w_n + t_n w_n)^2] \,\mathrm{d}x. \end{aligned}$$

Using (2.4) and (2.74) and n large enough, we see that

$$\begin{aligned} C &\geq \gamma \tilde{C} \frac{2\pi}{(2+\alpha)n^{2+\alpha}} \exp\left(\beta_0(1-\varepsilon_n)^2 t_n^2 \left[\frac{(\log n)^{1/2}}{\sqrt{2\pi}} - \frac{B_k}{(\log n)^{1/2}}\right]^2\right) \\ &= \gamma \tilde{C} \frac{2\pi}{(2+\alpha)e^{(2+\alpha)\log n}} \exp\left(\beta_0(1-\varepsilon_n)^2 t_n^2 \left[\frac{B_k^2}{\log n} - 2\frac{B_k}{\sqrt{2\pi}}\right]\right) \exp\left(\beta_0 \frac{(1-\varepsilon_n)^2 t_n^2 \log n}{2\pi}\right). \end{aligned}$$

We notice that

$$\exp\left(\beta_0(1-\varepsilon_n)^2 t_n^2 \left[\frac{B_k^2}{\log n} - 2\frac{B_k}{\sqrt{2\pi}}\right]\right) > C_1$$

for n large enough and some $C_1 > 0$, due to the facts $t_n^2 > 2\pi(2+\alpha)/\beta_0$ and $\varepsilon_n \to 0$. Thus, using $t_n^2 > (1+\varepsilon)2\pi(2+\alpha)/\beta_0$, we have

$$\begin{aligned} C &\geq C_1 \gamma \tilde{C} \frac{2\pi}{(2+\alpha)} \exp\left(\beta_0(1-\varepsilon_n)^2 t_n^2 \frac{\log n}{2\pi} - (2+\alpha)\log n\right) \\ &\geq C_1 \gamma \tilde{C} \frac{2\pi}{(2+\alpha)} \exp\left((2+\alpha)\log n[(1-\varepsilon_n)^2(1+\varepsilon) - 1]\right) \to \infty, \end{aligned}$$

which is a contradiction. Consequently, we must have $t_0^2 = 2\pi(2+\alpha)/\beta_0$ as desired. So by (2.76) we get $J_\lambda(v_0) \geq 0$. But we know that $v_0 \in H_l^*$, so by (2.53), we have $J_\lambda(v_0) = 0$.

Now we must show that if $v_0 \in H_l^*$ and $J_\lambda(v_0) = 0$, then $v_0 = 0$ and we finish the proof of Claim 6.

Consider $v_0 \in H_l^*$, then

$$\begin{aligned} 0 &= J_\lambda(v_0) = \frac{1}{2}\|v_0\|^2 - \frac{\lambda}{2}\|v_0\|_2^2 - \int_{B_1} |x|^\alpha G(v_0+\psi)_+ \, \mathrm{d}x \\ &\le \frac{1}{2}\left(1 - \frac{\lambda}{\lambda_k}\right)\|v_0\|^2 - \int_{B_1} |x|^\alpha G(v_0+\psi)_+ \, \mathrm{d}x \\ &\le -\int_{B_1} |x|^\alpha G(v_0+\psi)_+ \, \mathrm{d}x. \end{aligned}$$

Since $G \ge 0$, we can see that $\int_{B_1} |x|^\alpha G(v_0+\psi)_+ \, \mathrm{d}x = 0$. Thus

$$\begin{aligned} 0 = J_\lambda(v_0) = &\ \frac{1}{2}\|v_0\|^2 - \frac{\lambda}{2}\|v_0\|_2^2 \\ \le &\ \frac{1}{2}\left(1 - \frac{\lambda}{\lambda_k}\right)\|v_0\|^2. \end{aligned}$$

So we can see

$$\frac{1}{2}\left(1 - \frac{\lambda}{\lambda_k}\right)\|v_0\|^2 \ge 0$$

which gives us $v_0 = 0$. ■

2.7.3 Proof of Theorem 2.2.4 completed

Let us take n such that the minimax level is below $(2+\alpha)\pi/\beta_0$. Consider that (u_m) is a $(PS)_c$ sequence that it is bounded by Lemma 2.4.1, so there exists a subsequence of (u_m) and $u \in H$ such that $u_m \rightharpoonup u$ weakly in H, $u_m \to u$ in $L^p_{\mathrm{rad}}(B_1)$ for all $p \ge 1$ and almost everywhere in B_1. In the same way of the Theorem 2.2.3, we conclude that u is a solution for (2.7) and $u \neq 0$. ■

Chapter 3

Hénon type equation with jumping nonlinearity in critical growth range

In this Chapter, we search for two nontrivial radially symmetric solutions of the Dirichlet problem involving a Hénon-type equation of the form

$$\begin{cases} -\Delta u &= \lambda u + |x|^\alpha k(u_+) + f(x) \quad \text{in} \quad B_1, \\ u &= 0 \qquad\qquad\qquad\qquad\quad \text{on} \quad \partial B_1, \end{cases} \tag{P3}$$

where $\lambda > 0$, $\alpha \geq 0$, B_1 is a unity ball centered at the origin of $\mathbb{R}^N$ $(N \geq 3)$, $k(s) = s^{2^*_\alpha - 1} + g(s)$ with $2^*_\alpha = 2(N+\alpha)/(N-2)$ and $g(s)$ is a C^1 function in $[0, +\infty)$ which is assumed to be in the subcritical growth range.

The proofs are based on variational methods and to ensure that the considered minimax levels lie in a suitable range, special classes of approximating functions which have disjoint support with Talenti functions (Hénon version) are constructed.

We notice that Problem (P3) is naturally a Sobolev version of Problem (P2). However, here we only seek radially symmetric solutions because for obtaining solutions in $H^1_0(B_1)$, we would apply arguments very close to those used by CALANCH and RUF in [15].

As in Chapter 3, we begin studying a linear problem related to our equation. This is an important step to guarantee a first solution for the problem, which will be denoted by ψ and is nonpositive. This case is almost identical to the previous results in [15, 29], but we also give a little remark concerning existence of a radially symmetric solution when the forcing term is also radial. After that, we notice that to find a second solution of

(P3), we could set $u = v + \psi$, then v is a solution for

$$\begin{cases} -\Delta v = & \lambda v + |x|^\alpha (v+\psi)_+^{2^*_\alpha - 1} + |x|^\alpha g(v+\psi)_+ \quad \text{in} \quad B_1; \\ \quad v = & 0 \qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{on} \quad \partial B_1. \end{cases} \tag{3.1}$$

Obviously, $v = 0$ is a solution of (3.1) and this gives back the negative solution ψ for (P3). In [29], the authors searched for critical points of the associated functional of a similar problem to (3.1). This translation has the advantage of working with a homogeneous problem, but the interference of ψ in the estimates of minimax level was a great setback.

Here we use a different technique. In the same spirit of [32], we separate the supports of the negative solution and the function given in (3.7) in order to make some estimates easier to handle. This is done by cutting a small hole into the function ψ . Initially, we consider $B_r \subset B_1$ as a ball of radius r and center in origin. Let us take $\eta_r \in C^1(B_1)$, given by

$$\eta_r(x) = \begin{cases} 0 & \text{in } B_{r^2}; \\ \dfrac{|x|^{\sqrt{r}} - r^{2\sqrt{r}}}{r^{\sqrt{r}} - r^{2\sqrt{r}}} & \text{in } B_r \setminus B_{r^2}; \\ 1 & \text{in } B_1 \setminus B_r. \end{cases}$$

Define the functions

$$\psi_r(x) = \eta_r(x)\psi(x) \quad \text{in} \quad B_1 \tag{3.2}$$

and set

$$f_r(x) = -\Delta \psi_r(x) - \lambda \psi_r(x). \tag{3.3}$$

We can notice that the supports of ψ_r and f_r are $B_1 \setminus B_r^2$. See the Figure 4.1.

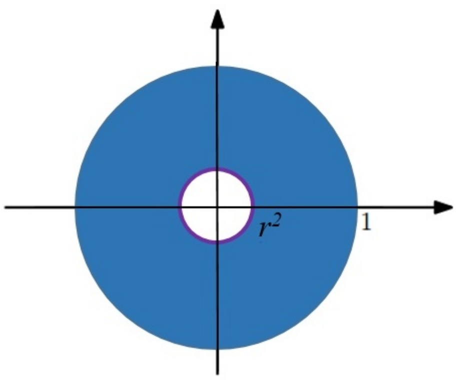

Figure 3.1: Supports of ψ_r and f_r.

Using as before $u = v + \psi_r$ in (P3), we see that $v = \psi - \psi_r$ solves the following equation

$$\begin{cases} -\Delta v = & \lambda v + |x|^\alpha (v+\psi_r)_+^{2^*_\alpha - 1} + |x|^\alpha g(v+\psi_r)_+ + (f - f_r) \quad \text{in } B_1; \\ \quad v = & 0 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{on } \partial B_1. \end{cases} \tag{3.4}$$

Clearly $v = \psi - \psi_r$ corresponds to the trivial solution of this problem. We also observe that if $u \not\equiv \psi - \psi_r$ is another solution, then it is straightforward to show that $v = u - (\psi - \psi_r)$ solves (3.4) and, consequently, (P3). Our aim then is to prove that there exists such u.

Problem (3.4) seems more complex than Problem (3.1), because the translation used in (3.1) makes us work with a homogeneous problem. Nevertheless, we can see (3.4) as an approximate equation of (3.1). This approximation creates an error, given by terms $f - f_r$, which, due to behavior of $|\nabla \eta|$, is easily estimated. In this direction, we have as reference the work of TARANTELLO [55], which uses heavily the fact that the forcing term has small enough norm in $L^1(\Omega)$.

In [15], Calanchi and Ruf used similar technique while studying the same problem treated in [29] and showed the existence of at least two solutions provided $N \geq 6$ without assuming (g_2). But they created the hole near $\partial\Omega$, where the first solution is small, in order to estimate the error. Here, we do not work with the negative solution ψ near ∂B_1, because, in our case, we are searching for a radially symmetric solution for (3.4).

Another different point which used here was the kind of functions used by estimate the minimax levels. More precisely, CALANCHI and RUF in [15] and FIGUEIREDO and YANG in [29], in order to prove the existence of a second solution of a similar problem to (P3), follow the Brezis–Niremberg approach to estimate the minimax levels with the help of the Talenti functions,

$$U_\varepsilon(x) = \left[\frac{N(N-2)\varepsilon}{\varepsilon + |x|^2}\right]^{(N-2)/4}, \tag{3.5}$$

which are solutions of the problem

$$\begin{cases} -\Delta u &= |u|^{2^*-2}u \quad \text{in} \quad \mathbb{R}^N; \\ u(x) &\to 0 \qquad\quad \text{as} \quad |x| \to \infty. \end{cases}$$

It is well-know that they realize the best constant Sobolev embedding constant $H^1(\mathbb{R}^N) \subset L^{2^*}(\mathbb{R}^N)$ given by

$$S = \inf_{\substack{u \in H_0^1(B_1) \\ u \neq 0}} \frac{\|u\|^2}{\|u\|_{2^*}^2}.$$

Using U_ε one can prove that the minimax level of the associated functional of their problem belongs to the interval where the Palais-Smale compactness condition holds. However, they reached a technical difficult when trying to prove that the (PS) sequence obtained in fact converged to a nontrivial solution and could overcome it by considering only $N \geq 7$.

Here, since we search radially symmetric solutions for Hénon-type equations and the weight $|x|^\alpha$ modifies the critical exponent in $H^1_{0,\mathrm{rad}}(B_1)$, that becomes $2^*_\alpha \geq 2^*$ for all $\alpha \geq 0$. Consequently, we need to invoke a different family of function adapted for the radial context. More precisely, since we are searching for radial solution for (P3) with critical growth, we let S_α be the best constant for the Sobolev-Hardy embedding

$$H^1_{0,\mathrm{rad}}(\mathbb{R}^N) \to L^{2^*_\alpha}(\mathbb{R}^N, |x|^\alpha).$$

The constant

$$S_\alpha = \inf_{\substack{u \in H^1_{0,\mathrm{rad}}(B_1) \\ u \neq 0}} \frac{\int_{\mathbb{R}^N} |\nabla u|^2 \, \mathrm{d}x}{\left(\int_{\mathbb{R}^N} |x|^\alpha |u|^{2^*_\alpha} \, \mathrm{d}x\right)^{2/2^*_\alpha}} \tag{3.6}$$

is achieved by the family of functions

$$u_\varepsilon(x) = \frac{[(N+\alpha)(N-2)\varepsilon]^{(N-2)/2(2+\alpha)}}{(\varepsilon + |x|^{2+\alpha})^{(N-2)/(2+\alpha)}} \tag{3.7}$$

defined for each $\varepsilon > 0$. Indeed, these functions are minimizers of S_α in the set of radial functions in case $\alpha > -2$. Furthermore, u_ε are the only positive radial solutions of

$$\begin{cases} -\Delta u &=& |x|^\alpha |u|^{2^*_\alpha - 2} u & \text{in} \quad \mathbb{R}^N; \\ u(x) &\to& 0 & \text{as} \quad |x| \to \infty. \end{cases} \tag{3.8}$$

For details and more general results, see [6, 14, 20, 41].

This new approach allows us to guarantee the existence of at least two radially symmetric solutions for a problem with jumping nonlinearity to a Hénon type equation with Sobolev critical growth using variational methods. We get a first solution ψ using the Fredholm Alternative in a related linear equation. A second solution is obtained via the Mountain–Pass Theorem, if $\lambda < \lambda_1$, or the Linking Theorem, if $\lambda > \lambda_1$. In order to use these well-known critical point results, we need to prove some geometric conditions satisfied by the functional associated to the problem and, we need to estimate the minimax levels, which have to lie below some appropriate constants. In our case, we can obtain this estimated for $N \geq 3$, when we add a lower order growth term to the nonlinearity.

3.1 Hypotheses

Before stating our main results, we shall introduce the following assumptions on the nonlinearity g:

(g_0) $g \in C(\mathbb{R}, \mathbb{R}^+)$, $g(s) = 0$ for all $s \leq 0$.

(g_1) There exists c_1 and $2 < p+1 < 2^*_\alpha$ such that $g(s) \leq c_1 s^p$ for all $s \in \mathbb{R}^+$.

(g_2) There exists c_2 and q such that $g(s) \geq c_2 s^q$ for all $s \in \mathbb{R}^+$, where

$$\begin{cases} 2^*_\alpha - \dfrac{2N-8}{3N-8} < q+1 < 2^*_\alpha & \text{for } N \geq 5; \\ (4+\alpha) - \dfrac{2}{5} < q+1 < 4+\alpha = 2^*_\alpha & \text{for } N = 4; \\ (6+2\alpha) - \dfrac{2}{5} < q+1 < 6+2\alpha = 2^*_\alpha & \text{for } N = 3. \end{cases}$$

Let us consider $\lambda_1 < \lambda_2 \leq \lambda_3 \leq ... \leq \lambda_j \leq ...$ the sequence of eigenvalues of $(-\Delta, H^1_0(B_1))$ and ϕ_j is a j^{th} eigenfunction of $(-\Delta, H^1_0(B_1))$.

Assuming (g_0) and that $\lambda \neq \lambda_j$ for all j, one can prove that ψ is a nonpositive solution of (P3) if and only if it is a nonpositive solution for the linear problem

$$\begin{cases} -\Delta\psi = \lambda\psi + f(x) & \text{in } B_1, \\ \psi = 0 & \text{on } \partial B_1. \end{cases} \tag{3.9}$$

In order to obtain such solutions for (3.9), we assume that

(f_1) $f(x) = h(x) + t\phi_1(x)$,

where $h \in L^\mu(B_1)$, $\mu > N$ and

$$\int_{B_1} h\phi_1 \, dx = 0.$$

The parameter t will be used in the proof of the first Theorem of this Chapter.

3.2 Statement of main results

We divide our results in two theorems. The first one deals with the first solution of the problem, which is nonpositive and is obtained by a simple remark about a linear problem related to our equation. The other theorem concerns the second solution and we need to consider the dimension which we are working. On condition (f_1), for $N \geq 5$, we only need to assume $\mu > N$ in order to recover the compactness of the functional associated to Problem (P3). In dimensions $N = 4$ and 3, we should consider $\mu \geq 8$ and $\mu \geq 12$ [3], respectively, for this purpose.

Theorem 3.2.1 (The Linear Problem)**.** Assume (f_1) and $\lambda \neq \lambda_j$ for all $j \in \mathbb{N}$. Then there exists a constant $T = T(h) > 0$ such that:

(i) If $\lambda < \lambda_1$, there exists $\psi_t < 0$, a solution for (3.9) and, consequently, for (P3), for all $t < -T$.

(ii) If $\lambda > \lambda_1$, there exists $\psi_t < 0$, a solution for (3.9) and, consequently, for (P3), for all $t > T$.

Furthermore, if f is radially symmetric, then ψ_t is radially symmetric as well.

Theorem 3.2.2. Assume the existence of nonpositive radial solution ψ of (P3), conditions $(g_0)-(g_2)$ and $\lambda \neq \lambda_j$ for all $j \in \mathbb{N}$. Then, (P3) possesses a second radial solution provided that $f \in L^\mu(B_1)$ with $\mu \geq 12$ if $N = 3$, $\mu \geq 8$ if $N = 4$ and $\mu > N$ if $N \geq 5$.

Remark 3.2.1. *Some comments on our assumptions and theorems are in order.*

(a) *The condition* (f_1) *of Theorem 3.2.1 ensure the existence of nonpositive solution of* (P3).

(b) *The hypothesis* (g_2) *will play a crucial role in the estimate on the minimax levels. However, we can consider* $g \equiv 0$ *for high dimensions. For details see Remarks 3.5.2 and 3.5.3.*

(c) *In this work, we added a subcritical perturbation* $g(s)$ *to obtain existence of solution for* (P3). *An interesting question that arises here is to classify the kind of perturbation and determine the optimal growth for which we can reach similar results.*

(d) *We notice that if* $\lambda < \lambda_1$*, we shall use the Mountain–Pass Theorem in the proof of Theorem 3.2.2. On the other hand, if* $\lambda > \lambda_1$*, we need to use the Linking Theorem.*

(e) *In Theorem 3.2.2, a second solution will be given by* $u + \psi$*, where* ψ *is a nonpositive solution of* (P3) *and* u *is a nontrivial solution of a translated problem, which will be explained later (see* (3.1) *and* (3.4)*).*

[3] These values are $2N = 8$ and $3N = 12$, respectively.

3.2.1 Outline

This Chapter is organized as follows. In Sect. 3.3, we study a linear problem related to our equation. This is an important step to find a first solution for the problem, which will be denoted by ψ and is nonpositive. To this end we use some standard ideas contained in the literature, for instance in [29]. Here, using that the forcing term is radial we can prove that the solution ψ is also radially symmetric. In Sect. 3.4, we introduce the variational framework and prove the boundedness of Palais-Smale sequences of the functional associated to the approximated Problem (3.4). First, we prove some estimate that will be used in order to show the conditions of the Mountain–Pass Theorem, for $\lambda < \lambda_1$, and the Linking Theorem, for $\lambda > \lambda_1$. In Sect. 3.5, we obtain the geometric conditions for the functional in order to prove the existence of a second solution to Problem (3.4) and since we are searching for a radial solution for a problem with critical Sobolev growth nonlinearity, we need to prove the boundedness of the minimax levels by appropriate constants depending on α and S_α.

3.3 The Linear Problem

In this section, we show a proof of Theorem 3.2.1. We consider the linear problem

$$\begin{cases} -\Delta\psi &= \lambda\psi + f(x) \quad \text{in} \quad B_1, \\ \psi &= 0 \qquad\qquad\quad \text{on} \quad \partial B_1. \end{cases} \tag{3.10}$$

It is easy to see that if f is such that this linear problem admits a nonpositive solution, then it will also be a solution for Problem (P3). Considering f decomposed as in (f_1), we will see that the sign of the (unique) solution of (3.10) can be established depending on t. Moreover, we give an idea of how to obtain radial solutions as well.

Proof: Since $\lambda \neq \lambda_j$ for all $j \in \mathbb{N}$, by the Fredholm Alternative, we obtain a unique solution ψ of (3.10) in $H_0^1(B_1)$. Using (f_1), we can write $\psi = \psi_t = \omega + s_t\phi_1$, with $\int_{B_1} \omega\phi_1 \, \mathrm{d}x = 0$ and

$$s_t = \frac{t}{\lambda_1 - \lambda}.$$

Now, we need to find a parameter t such that ψ is negative. We recall that $f \in L^\mu(B_1)$, with μ always larger than N for $N \geq 3$. Thus in both cases, by elliptic regularity, we have $\omega \in C^{1,1-N/\mu}$. Then

$$\left\| \frac{\lambda_1 - \lambda}{t}\psi - \phi_1 \right\|_{C^1} = \left| \frac{\lambda_1 - \lambda}{t} \right| \|\omega\|_{C^1}.$$

It is known that the (exterior) normal derivative $\frac{\partial \phi_1}{\partial n}|_{\partial B_1}$ is negative. So, let $\varepsilon > 0$ be such that if $\phi \in C^1(B_1)$ and $\|\phi - \phi_1\|_{C^1} < \varepsilon$, then $\phi > 0$ (we are assuming $\phi_1 > 0$). Since we want $\psi < 0$, we must have

$$\frac{\lambda - \lambda_1}{t} > 0 \quad \text{and} \quad \frac{\lambda - \lambda_1}{t} \|\omega\|_{C^1} < \varepsilon.$$

Therefore, for $\lambda < \lambda_1$, $t < -\varepsilon^{-1}(\lambda_1 - \lambda)\|\omega\|_{C^1}$ and, for $\lambda > \lambda_1$, $t > \varepsilon^{-1}(\lambda - \lambda_1)\|\omega\|_{C^1}$.

We also notice that ψ is a solution for (3.10) if only if it is a critical point of the following functional $I : H_0^1(B_1) \to \mathbb{R}$, given by

$$I(\psi) := \frac{1}{2}\int_{B_1} |\nabla\psi|^2 \, \mathrm{d}x - \lambda\frac{1}{2}\int_{B_1} \psi^2 \, \mathrm{d}x - \int_{B_1} f(x)\psi \, \mathrm{d}x.$$

Considering I restricts to $H^1_{0,\mathrm{rad}}(B_1)$, we also obtain a critical point of this functional on this subspace. If f is radially symmetric, by the Principle of Symmetric Criticality of Palais (see [42]), we can see that all critical points on $H^1_{0,\mathrm{rad}}(B_1)$ are also critical points on $H_0^1(B_1)$. So, due to the fact that I admits only one critical point in the whole space, we get that ψ is also radially symmetric. This completes the proof of Theorem 3.2.1. ■

Remark 3.3.1. *This proof follows exactly the same steps of the proof of Theorem 2.2.1. We observe that in Theorem 2.2.1 we only need $\mu > 2$ and here we should assume $\mu > N \geq 3$ in order to obtain sufficient regularity of the solution.*

3.4 The variational formulation

Here, we will denote $H = H^1_{0,\mathrm{rad}}(B_1)$, with the norm

$$\|u\| = \left(\int_{B_1} |\nabla u|^2 \, \mathrm{d}x \right)^{1/2}$$

for all $u \in H$. We also denote the L^z space with weigh $|x|^\gamma$ on B_1 by $L^z(B_1, |x|^\gamma)$ with the norm

$$\|u\|_{z,|x|^\gamma} = \left(\int_{B_1} |x|^\gamma |u|^z \, \mathrm{d}x \right)^{1/z},$$

where $0 \leq \gamma \leq \alpha$ and $1 \leq z \leq 2^\alpha_*$.

In short, for find out other (weak) solutions of (3.4) we should look for critical points of functional $J_{\lambda,r} : H \to \mathbb{R}$,

$$J_{\lambda,r}(v) = \frac{1}{2}\int_{B_1} (|\nabla v|^2 - \lambda v^2)\,\mathrm{d}x - \frac{1}{2^*_\alpha}\int_{B_1} |x|^\alpha (v+\psi_r)_+^{2^*_\alpha}\,\mathrm{d}x$$

$$- \int_{B_1} |x|^\alpha G(v+\psi_r)_+\,\mathrm{d}x - \int_{B_1} (f - f_r)v\,\mathrm{d}x,$$

which is $C^1(H, R)$, once g is continuous.

This will be done by proving some geometric properties of this functional which will satisfy the geometric hypotheses of the Mountain–Pass Theorem without the (PS) condition, for $\lambda < \lambda_1$ and the Linking Theorem without (PS) condition, if $\lambda_k < \lambda < \lambda_{k+1}$ for some $k \in \mathbb{N}$. However, we do not apply the standard variational arguments once the imbedding H in $L^{2^*_\alpha}(B_1, |x|^\alpha)$ is not compact, which implies that the functional $J_{\lambda,r}$ does not satisfy the Palais-Smale condition.

Thus, we need to adapt an idea introduced by H. Brezis and L. Nirenberg [11], which used the Talenti functions, given in (3.5), in order to prove that a functional associated to a problem with critical Sobolev growth nonlinearity satisfies the (PS) condition in the interval $(0, S^{N/2}/N)$.

Here, in radial context for a Hénon type equation, we should construct minimax levels for the functional $J_{\lambda,r}$ which lie in the interval

$$\left(0, \frac{2+\alpha}{2(N+\alpha)} S_\alpha^{(N+\alpha)/(2+\alpha)}\right).$$

For this purpose, we use the crucial fact that positive solutions of (3.7) realize the constant S_α in the imbedding $H^1_{\mathrm{rad}}(\mathbb{R}^N)$ in $L^{2^*_\alpha}(\mathbb{R}^N, |x|^\alpha)$. Before that, we make some estimates which will be used to show the geometric conditions, the boundedness of minimax level and that $\|f - f_r\|_1$ is small.

3.4.1 Preliminary properties

Initially, we recall (3.2) and (3.3). Here, we must estimate the errors created in the approximation problem (3.4), namely $\psi - \psi_r$ and $f - f_r$. The following lemma has the results needed to continue.

Lemma 3.4.1. For r small enough, there exists δ_r such that

$$\|\psi - \psi_r\| \le \tilde{C}\delta_r \tag{3.11}$$

and

$$\left|\int_{B_1}(f-f_r)\phi\,\mathrm{d}x\right|\le c\|\phi\|\delta_r \text{ for all } \phi\in H. \tag{3.12}$$

Moreover, δ_r can be chosen satisfying $\delta_r=o(1)$ when $r\to 0$.

Proof: We notice that

$$\begin{aligned}
\|\psi-\psi_r\|^2 = & \int_{B_1}|\nabla(\psi-\psi_r)|^2\,\mathrm{d}x\\
= & \int_{B_1}|\nabla\psi(1-\eta_r)-\psi\nabla\eta_r|^2\,\mathrm{d}x\\
\le & \int_{B_r}|\nabla\psi|^2|1-\eta_r|^2\,\mathrm{d}x+2\int_{B_r\setminus B_{r^2}}|\nabla\psi|(1-\eta_r)|\psi||\nabla\eta_r|\,\mathrm{d}x\\
& +\int_{B_r\setminus B_{r^2}}|\psi|^2|\nabla\eta_r|^2\,\mathrm{d}x\\
\le & C_1\|\nabla\psi\|_\infty^2 r^{2N}+C_1\|\nabla\psi\|_\infty^2\frac{r^{N+2\sqrt{r}}}{(r^{\sqrt{r}}-r^{2\sqrt{r}})^2}\\
& +C_2\|\nabla\psi\|_\infty\|\psi\|_\infty\frac{r^{N+2\sqrt{r}-1/2}}{(r^{\sqrt{r}}-r^{2\sqrt{r}})^2}+C_3\|\psi\|_\infty^2\frac{r^{N+2\sqrt{r}-1}}{(r^{\sqrt{r}}-r^{2\sqrt{r}})^2}\\
\le & \tilde{C}\frac{r^{N-1}}{(1-r^{\sqrt{r}})^2},
\end{aligned}$$

where

$$\tilde{C}\ge\max\{C_1\|\nabla\psi\|_\infty^2, C_2\|\nabla\psi\|_\infty\|\psi\|_\infty, C_3\|\psi\|_\infty^2\}. \tag{3.13}$$

It is straightforward to prove that for all $\vartheta\ge 1$, we have

$$\lim_{r\to 0}\frac{r^\vartheta}{(1-r^{\sqrt{r}})^2}=0. \tag{3.14}$$

Thus, we choose δ_r such that

$$\delta_r^2=\tilde{C}\frac{r^{N-1}}{(1-r^{\sqrt{r}})^2}.$$

Since $N\ge 3$, by (3.14), we obtain (3.11).

From (3.2), using the Hölder inequality and the variational characterization of λ_1, we have

$$\begin{aligned}
\left|\int_{B_1}(f-f_r)\phi\,\mathrm{d}x\right| = & \left|\int_{B_1}\nabla(\psi-\psi_r)\nabla\phi-\lambda(\psi-\psi_r)\phi\,\mathrm{d}x\right|\\
\le & c\|\psi-\psi_r\|\|\phi\|\le c\|\phi\|\delta_r,
\end{aligned}$$

for all $\phi\in H$. Consequently, (3.12) follows. ■

Now, we take $\xi_r\in C_0^\infty(B_r^2,[0,1])$, a radial cut-off function such that $\xi_r=1$ in $B_{r^2/2}$ and $|\nabla\xi_r|\le 4/r^2$, set $u_\varepsilon^r(x)=\xi(x)u_\varepsilon$. The support of u_ε^r can be seen in Figure 4.2.

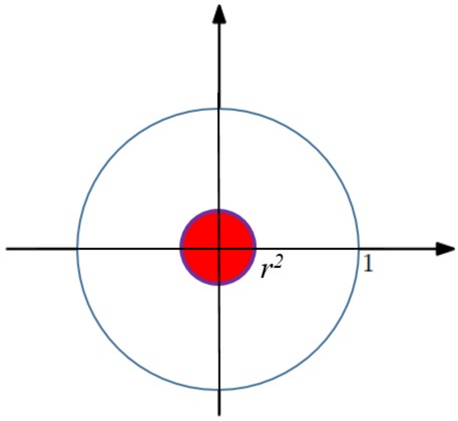

Figure 3.2: Support of u_ε^r.

Now, we will make some estimates similar to Brezis-Niremberg Lemma [11, Lemma 1.2].

Lemma 3.4.2. For fixed $r > 0$ and $0 \le \gamma \le \alpha$, we have

(a) $\|u_\varepsilon^r\|^2 = S_\alpha^{(N+\alpha)/(2+\alpha)} + O\left(\varepsilon^{(N-2)/(2+\alpha)}\right)$;

(b) $\|u_\varepsilon^r\|_{2_\alpha^*,|x|^\alpha}^{2_\alpha^*} = S_\alpha^{(N+\alpha)/(2+\alpha)} + O\left(\varepsilon^{(N+\alpha)/(2+\alpha)}\right)$;

(c) $\|u_\varepsilon^r\|_{2,|x|^\gamma}^2 = K_1\varepsilon^{(\gamma+2)/(2+\alpha)} + O\left(\varepsilon^{(N-2)/(2+\alpha)}\right) \quad \text{if} \quad N > 4+\gamma$;

(d) $\|u_\varepsilon^r\|_{s,|x|^\gamma}^s \ge K_2\varepsilon^{[N+\gamma-(N-2)s/2]/(2+\alpha)}$.

Furthermore, for $r \to 0$ and $\varepsilon = o(r^{2(2+\alpha)})$

(e) $\|u_\varepsilon^r\|^2 = S_\alpha^{(N+\alpha)/(2+\alpha)} + O\left((\varepsilon/r^{2(2+\alpha)})^{(N-2)/(2+\alpha)}\right)$;

(f) $\|u_\varepsilon^r\|_{2_\alpha^*,|x|^\alpha}^{2_\alpha^*} = S_\alpha^{(N+\alpha)/(2+\alpha)} + O\left((\varepsilon/r^{2(2+\alpha)})^{(N+\alpha)/(2+\alpha)}\right)$.

Proof of (a): First, we can notice that, since u_ε is a solution of Problem (3.8) for each $\varepsilon > 0$, we have

$$\|\nabla u_\varepsilon\|_2^2 = \|u_\varepsilon\|_{2_\alpha^*,|x|^\alpha}^{2_\alpha^*} = S_\alpha^{(N+\alpha)/(2+\alpha)}. \tag{3.15}$$

Straightforward calculations show that

$$\int_{B_1} |\nabla u_\varepsilon^r|^2 \,\mathrm{d}x = \int_{B_1} |\nabla u_\varepsilon \xi_r|^2 \,\mathrm{d}x = \int_{\mathbb{R}^N} |\nabla u_\varepsilon|^2 \,\mathrm{d}x + O\left(\varepsilon^{(N+\alpha)/(2+\alpha)}\right)$$

and using (3.15), we conclude that

$$\int_{B_1} |\nabla u_\varepsilon^r|^2 \, \mathrm{d}x = S_\alpha^{(N+\alpha)/(2+\alpha)} + O\left(\varepsilon^{(N-2)(2+\alpha)}\right).$$

Proof of (b): Similarly to **(a)**, we have

$$\int_{B_1} |x|^\alpha (u_\varepsilon^r)^{2^*_\alpha} \, \mathrm{d}x = \int_{\mathbb{R}^N} |x|^\alpha u_\varepsilon^{2^*_\alpha} \, \mathrm{d}x + O\left(\varepsilon^{(N+\alpha)(2+\alpha)}\right) = S_\alpha^{(N+\alpha)/(2+\alpha)} + O\left(\varepsilon^{(N+\alpha)(2+\alpha)}\right).$$

Proof of (c): Initially, observe that

$$\begin{aligned}\int_{B_1} |x|^\gamma |u_\varepsilon^r|^2 \, \mathrm{d}x = & \int_{B_1} |x|^\gamma \xi_r^2 |u_\varepsilon|^2 \, \mathrm{d}x \\ = & \int_{B_{r^2}} |x|^\gamma |u_\varepsilon|^2 \, \mathrm{d}x + \int_{B_{r^2}} |x|^\gamma (\xi_r^2 - 1)|u_\varepsilon|^2 \, \mathrm{d}x \\ = & \int_{B_{r^2}} |x|^\gamma |u_\varepsilon|^2 \, \mathrm{d}x + O\left(\varepsilon^{(N-2)/(2+\alpha)}\right).\end{aligned}$$

Now, we calculate

$$\begin{aligned}\int_{B_{r^2}} |x|^\gamma |u_\varepsilon|^2 \, \mathrm{d}x = & \int_{B_{r^2}} |x|^\gamma \frac{[(N+\alpha)(N-2)\varepsilon]^{(N-2)/(2+\alpha)}}{(\varepsilon + |x|^{2+\alpha})^{2(N-2)/(2+\alpha)}} \, \mathrm{d}x \\ = & \quad K_1 \varepsilon^{(\gamma+2)/(2+\alpha)} \quad \text{if} \quad N > 4+\gamma.\end{aligned}$$

Proof of (d): We have for ε small enough

$$\begin{aligned}\int_{B_1} |x|^\gamma |u_\varepsilon^r|^s \, \mathrm{d}x \geq & \int_{B_{r^2/2}} |x|^\gamma \left(\frac{[(N+\alpha)(N-2)\varepsilon]^{(N-2)/2(2+\alpha)}}{(\varepsilon + |x|^{2+\alpha})^{(N-2)/(2+\alpha)}} \right)^s \mathrm{d}x \\ \geq & \; C \int_0^{\varepsilon^{1/(2+\alpha)}} \frac{\varepsilon^{(N-2)s/2(2+\alpha)}}{(\varepsilon + t^{2+\alpha})^{(N-2)s/(2+\alpha)}} t^{N-1+\gamma} \, \mathrm{d}t \\ \geq & \; C \varepsilon^{[N+\gamma-(N-2)s/2]/(2+\alpha)}.\end{aligned}$$

Proof of (e) and (f): Initially, we observe that

$$\begin{aligned}\int_{B_1} |\nabla u_\varepsilon^r|^2 \, \mathrm{d}x = & \int_{B_1} |\nabla u_\varepsilon \xi_r|^2 \, \mathrm{d}x \\ = & \int_{\mathbb{R}^N} |\nabla u_\varepsilon|^2 \, \mathrm{d}x - \int_{\mathbb{R}^N \setminus B_{r^2/2}} |\nabla u_\varepsilon|^2 \, \mathrm{d}x + \int_{B_{r^2} \setminus B_{r^2/2}} |\nabla(\xi_r u_\varepsilon)|^2 \, \mathrm{d}x\end{aligned}$$

and using (3.15), we must have

$$\int_{B_1} |\nabla u_\varepsilon^r|^2 \, \mathrm{d}x = S_\alpha^{(N+\alpha)/(2+\alpha)} - \int_{\mathbb{R}^N \setminus B_{r^2/2}} |\nabla u_\varepsilon|^2 \, \mathrm{d}x + \int_{B_{r^2} \setminus B_{r^2/2}} |\nabla(\xi_r u_\varepsilon)|^2 \, \mathrm{d}x.$$

Now, let us show that

$$-\int_{\mathbb{R}^N\setminus B_{r^2/2}} |\nabla u_\varepsilon|^2 \, \mathrm{d}x + \int_{B_{r^2}\setminus B_{r^2/2}} |\nabla(\xi_r u_\varepsilon)|^2 \, \mathrm{d}x = O\left(\frac{\varepsilon}{r^{2(2+\alpha)}}\right)^{(N-2)/(2+\alpha)}.$$

Indeed,

$$\begin{aligned}&\left|-\int_{\mathbb{R}^N\setminus B_{r^2/2}} |\nabla u_\varepsilon|^2 \, \mathrm{d}x + \int_{B_{r^2}\setminus B_{r^2/2}} |\nabla(\xi_r u_\varepsilon)|^2 \, \mathrm{d}x\right| \\ &\leq \left|-\int_{\mathbb{R}^N\setminus B_{r^2/2}} |\nabla u_\varepsilon|^2 \, \mathrm{d}x\right| + \left|\int_{B_{r^2}\setminus B_{r^2/2}} |\nabla(\xi_r u_\varepsilon)|^2 \, \mathrm{d}x\right|\end{aligned}$$

First, we observe that

$$\begin{aligned}&\left|\int_{B_{r^2}\setminus B_{r^2/2}} |\nabla(\xi_r u_\varepsilon)|^2 \, \mathrm{d}x\right| \\ &\leq C_{\alpha,N}\varepsilon^{(N-2)/(2+\alpha)} \int_{B_{r^2}\setminus B_{r^2/2}} \left|\frac{\nabla \xi_r}{(\varepsilon+|x|^{2+\alpha})^{(N-2)/(2+\alpha)}} + \frac{(N-2)\xi_r|x|^\alpha x}{(\varepsilon+|x|^{2+\alpha})^{(N+\alpha)/(2+\alpha)}}\right|^2 \mathrm{d}x \\ &\leq C_{\alpha,N}\varepsilon^{(N-2)/(2+\alpha)} \int_{B_{r^2}\setminus B_{r^2/2}} \frac{1}{(\varepsilon+|x|^{2+\alpha})^{2(N-2)/(2+\alpha)}}\left(|\nabla\xi_r|^2 + \frac{2|\nabla\xi_r|C_N\xi_r|x|^{\alpha+1}}{\varepsilon+|x|^{2+\alpha}}\right) \mathrm{d}x \\ &+C_{\alpha,N}\varepsilon^{(N-2)/(2+\alpha)} \int_{B_{r^2}\setminus B_{r^2/2}} \frac{1}{(\varepsilon+|x|^{2+\alpha})^{2(N-2)/(2+\alpha)}} + \frac{C_N^2\xi_r^2|x|^{2(\alpha+1)}}{(\varepsilon+|x|^{2+\alpha})^2} \, \mathrm{d}x.\end{aligned}$$

Observe that, since $|\nabla \xi_r| \leq C/r^2$, $0 \leq \xi_r \leq 1$ and $|x| \geq r^2/2$, one has

$$\begin{aligned}&|\nabla\xi_r|^2 + \frac{2|\nabla\xi_r|C_N\xi_r|x|^{\alpha+1}}{(\varepsilon+|x|^{2+\alpha})} + \frac{C_N^2\xi_r^2|x|^{2(\alpha+1)}}{(\varepsilon+|x|^{2+\alpha})^2} \\ &\leq \frac{C_1}{r^4} + \frac{C_2}{r^4} + \frac{C_3}{r^4},\end{aligned}$$

So we have that

$$\begin{aligned}&\varepsilon^{(N-2)/(2+\alpha)} \int_{B_{r^2}\setminus B_{r^2/2}} \left(\frac{1}{(\varepsilon+|x|^{2+\alpha})^{2(N-2)/(2+\alpha)}}\right)\left(|\nabla\xi_r|^2 + \frac{2|\nabla\xi_r|C_N\xi_r|x|^{\alpha+1}}{(\varepsilon+|x|^{2+\alpha})}\right) \mathrm{d}x \\ &+\varepsilon^{(N-2)/(2+\alpha)} \int_{B_{r^2}\setminus B_{r^2/2}} \left(\frac{1}{(\varepsilon+|x|^{2+\alpha})^{2(N-2)/(2+\alpha)}}\right)\left(\frac{C_N^2\xi_r^2|x|^{2(\alpha+1)}}{(\varepsilon+|x|^{2+\alpha})^2}\right) \mathrm{d}x \\ &\leq C\frac{\varepsilon^{2(N-2)/(2+\alpha)}}{r^4} \int_{B_{r^2}\setminus B_{r^2/2}} \left(\frac{1}{(\varepsilon+|x|^{2+\alpha})^{2(N-2)/(2+\alpha)}}\right) \mathrm{d}x \\ &\leq C\frac{\varepsilon^{(N-2)/(2+\alpha)}}{r^4} \int_{B_{r^2}\setminus B_{r^2}/2} \frac{1}{|x|^{2N-4}} \, \mathrm{d}x = C\frac{\varepsilon^{(N-2)/(2+\alpha)}}{r^4} \int_{r^2/2}^{r^2} \frac{s^{N-1}}{s^{2N-4}} \, \mathrm{d}s\end{aligned}$$

$$= C\frac{\varepsilon^{(N-2)/(2+\alpha)}}{r^4}\int_{r^2/2}^{r^2}\frac{1}{s^{N-3}}\,\mathrm{d}s$$

$$\leq C\tfrac{\varepsilon^{(N-2)/(2+\alpha)}}{r^4}\begin{cases} r^2 & \text{if} \quad N=3 \\ 2\log r & \text{if} \quad N=4 \\ \dfrac{1}{r^{2N-8}} & \text{if} \quad N\geq 5 \end{cases}$$

$$\leq C\left(\frac{\varepsilon}{r^{2(2+\alpha)}}\right)^{(N-2)/(2+\alpha)}.$$

Notice that, since u_ε^r is a radially symmetric function, we have that

$$\left|-\int_{\mathbb{R}^N\setminus B_{r^2/2}}|\nabla u_\varepsilon|^2\,\mathrm{d}x\right|\leq\left|C\int_{\infty}^{r^2/2}|(\zeta_r(s)u_\varepsilon(s))'|^2 s^{N-1}\,\mathrm{d}x\right|.$$

So, analogously as before, we obtain

$$\left|-\int_{R^N\setminus B_{r^2/2}}|\nabla u_\varepsilon^r|^2\mathrm{d}x\right|\leq C\frac{\varepsilon^{(N-2)/(2+\alpha)}}{r^4}\int_{\infty}^{r^2/2}\frac{1}{s^{N-3}}\,\mathrm{d}s$$

$$\leq C\frac{\varepsilon^{(N-2)/(2+\alpha)}}{r^4}\begin{cases} r^2 & \text{if} \quad N=3 \\ 2\log r & \text{if} \quad N=4 \\ \dfrac{1}{r^{2N-8}} & \text{if} \quad N\geq 5 \end{cases}\leq C\left(\frac{\varepsilon}{r^{2(2+\alpha)}}\right)^{(N-2)/(2+\alpha)}.$$

And item (e) follows as desired.

Using a similar argument, we conclude that item (f) holds. ∎

3.4.2 Palais-Smale sequences

Let us notice that the proof of the Palais-Smale condition for the functional associated to Problem (3.4) follows traditional methods. So we will present a brief proof for this condition.

Lemma 3.4.3. Suppose $(g_0)-(g_1)$. Let $(u_m)\subset H$ be a $(PS)_c$ sequence of $J_{\lambda,r}$. Then (u_m) is bounded in H.

Proof: Let $(u_m)\subset H$ be a $(PS)_c$ sequence, i.e.

$$\begin{aligned} J_{\lambda,r}(u_m) = {} & \frac{1}{2}\|u_m\|^2-\frac{\lambda}{2}\|u_m\|_2^2-\frac{1}{2_\alpha^*}\int_{B_1}|x|^\alpha(u_m+\psi_r)_+^{2_\alpha^*}\,\mathrm{d}x \\ & -\int_{B_1}|x|^\alpha G(u_m+\psi_r)_+\,\mathrm{d}x-\int_{B_1}(f-f_r)u_m\,\mathrm{d}x \\ = {} & c+o(1) \end{aligned} \tag{3.16}$$

and

$$\begin{aligned}\langle J'_{\lambda,r}(u_m), v\rangle = & \int_{B_1} \nabla u_m \nabla v \,\mathrm{d}x - \lambda \int_{B_1} u_m v \,\mathrm{d}x - \int_{B_1} |x|^\alpha (u_m + \psi_r)_+^{2^*_\alpha - 1} v \,\mathrm{d}x \\ & + \int_{B_1} |x|^\alpha g(u_m + \psi_r)_+ v \,\mathrm{d}x - \int_{B_1} (f - f_r) v \,\mathrm{d}x \\ = & \; o(1)\|v\|\end{aligned} \tag{3.17}$$

for all $v \in H$.

From (3.16) and (3.17), it follows that

$$\begin{aligned}J_{\lambda,r}(u_m) - \frac{1}{2}\langle J'_{\lambda,r}(u_m), u_m\rangle = & \frac{2^*_\alpha - 2}{2 \cdot 2^*_\alpha} \int_{B_1} |x|^\alpha (u_m + \psi_r)_+^{2^*_\alpha} \,\mathrm{d}x - \frac{1}{2}\int_{B_1} |x|^\alpha (u_m + \psi_r)_+^{2^*_\alpha - 1} \psi_r \,\mathrm{d}x \\ & - \int_{B_1} |x|^\alpha G(u_m + \psi_r)_+ \,\mathrm{d}x + \frac{1}{2}\int_{B_1} |x|^\alpha g(u_m + \psi_r)_+ u_m \,\mathrm{d}x \\ & - \frac{1}{2}\int_{B_1} (f - f_r) u_m \,\mathrm{d}x \\ = & \; c + o(1) + o(1)\|u_m\|.\end{aligned}$$

Thus, due to $\psi_r \leq 0$ and (3.12), we have

$$\begin{aligned}\frac{2^*_\alpha - 2}{2 \cdot 2^*_\alpha} \int_{B_1} |x|^\alpha (u_m + \psi_r)_+^{2^*_\alpha} \,\mathrm{d}x \leq & \int_{B_1} |x|^\alpha G(u_m + \psi_r)_+ \,\mathrm{d}x \\ & - \frac{1}{2}\int_{B_1} |x|^\alpha g(u_m + \psi_r)_+ (u_m + \psi_r) \,\mathrm{d}x \\ & + c + (o(1) + c\delta_r)\|u_m\|.\end{aligned} \tag{3.18}$$

From (g_1) and making use of Höder inequality we obtain

$$\begin{aligned}\int_{B_1} |x|^\alpha (u_m + \psi_r)_+^{2^*_\alpha} \,\mathrm{d}x \leq & \; C\int_{B_1} |x|^\alpha (u_m + \psi_r)_+^{p+1} \,\mathrm{d}x + C + (o(1) + C\delta_r^2)\|u_m\| \\ \leq & \left(C\int_{B_1} |x|^\alpha (u_m + \psi_r)_+^{2^*_\alpha} \,\mathrm{d}x\right)^{\frac{p+1}{2^*_\alpha}} + C + C(o(1) + c\delta_r^2)\|u_m\|.\end{aligned}$$

Since $p + 1 < 2^*_\alpha$, we get

$$\int_{B_1} (u_m + \psi_r)_+^{2^*_\alpha} \,\mathrm{d}x \leq c + (o(1) + c\delta_r^2)\|u_m\| \leq c_1 + c_2\|u_m\|. \tag{3.19}$$

Considering $0 < \lambda < \lambda_1$, by the variational characterization of λ_1, we have

$$\begin{aligned}\langle J'_{\lambda,r}(u_m), u_m\rangle \geq & \left(1 - \frac{\lambda}{\lambda_1}\right)\|u_m\|^2 - \int_{B_1} |x|^\alpha (u + \psi_r)^{2^*_\alpha} \,\mathrm{d}x - \int_{B_1} |x|^\alpha g(u_m + \psi_r)_+ u_m \,\mathrm{d}x \\ & - \int_{B_1} (f - f_r) u_m \,\mathrm{d}x.\end{aligned}$$

Hence, we obtain

$$\|u_m\|^2 \leq C_1 + C_2\|u_m\|.$$

Thus, (u_m) is a bounded sequence.

Now we consider $\lambda_k < \lambda < \lambda_{k+1}$. It is convenient to decompose H into appropriate subspaces. For $H_0^1(B_1)$ is usual the following decomposition

$$H_0^1(B_1) = H_k \oplus H_k^{\perp},$$

where H_k is finite dimensional and defined by

$$H_k = [\phi_1, ..., \phi_k]. \tag{3.20}$$

Since here we are working in $H_{0,\text{rad}}(B_1)$, we also need to split this Hilbert Space into two orthogonal subspaces. Initially we recall the notation introduced in (3.20) and for each $k \in \mathbb{N}$ we define

$$H_k^* = H_k \cap H_{0,\text{rad}}^1(B_1) \text{ for all } k \in \mathbb{N}.$$

Notice that

$$H_1^* = H_1 \cap H_{0,\text{rad}}^1(B_1) = H_1,$$

because ϕ_1 is radially symmetric.

Thus, analogously to $H_0^1(B_1)$, we can write

$$H_{0,\text{rad}}^1(B_1) = \bigcup_{k=1}^{\infty} H_k^*.$$

Moreover, it is straightforward to prove that the spectrum of $(-\Delta, H_{0,\text{rad}}^1(B_1))$ is a subsequence of (λ_k) that we will denote by $\lambda_1^* = \lambda_1 \leq \lambda_2^* \leq \lambda_3^* \leq ... \leq \lambda_k^* \leq ...$ where $\lambda_j^* \geq \lambda_j$ for all $j = 1, 2, 3....$

For all u in H, let us take $u = u^k + u^{\perp}$, where $u^k \in H_k^*$ and $u^{\perp} \in \big((H_k^*)^{\perp} \cap H_{0,\text{rad}}^1(B_1)\big)$. We notice that

$$\int_{B_1} \nabla u \nabla u^k \,\mathrm{d}x - \lambda \int_{B_1} u u^k \,\mathrm{d}x = \|u^k\|^2 - \lambda \|u^k\|_2^2$$

and

$$\int_{B_1} \nabla u \nabla u^{\perp} \,\mathrm{d}x - \lambda \int_{B_1} u u^{\perp} \,\mathrm{d}x = \|u^{\perp}\|^2 - \lambda \|u^{\perp}\|_2^2.$$

By (3.17), we can see that

$$\begin{aligned} \langle J_{\lambda,r}(u_m), u_m^{\perp} \rangle = \; & \|u_m^{\perp}\|^2 - \lambda \|u_m^{\perp}\|_2^2 \\ & - \int_{B_1} |x|^{\alpha} (u_m + \psi_r)_+^{2_\alpha^* - 1} u_m^{\perp} \,\mathrm{d}x - \int_{B_1} |x|^{\alpha} g(u_m + \psi_r)_+ u_m^{\perp} \,\mathrm{d}x \\ & - \int_{B_1} (f - f_r) u_m^{\perp} \,\mathrm{d}x \\ = \; & o(1) \|u_m^{\perp}\|. \end{aligned}$$

Then, from (g_1), the variational characterization of λ_{k+1}, the Hölder and Young inequalities and (3.19), we get

$$\begin{aligned}\left(1-\frac{\lambda}{\lambda_{k+1}}\right)\|u_m^{\perp}\|^2 \leq\ & \int_{B_1}|x|^{\alpha}(u_m+\psi_r)_+^{2_\alpha^*-1}u_m^{\perp}\,\mathrm{d}x + c\int_{B_1}(u_m+\psi_r)^p|u_m^{\perp}|\,\mathrm{d}x \\ & +o(1)\|u_m^{\perp}\| + c\delta_r\|u_m^{\perp}\| \\ \leq\ & c\left(\int_{B_1}|u_m^{\perp}|^{2_\alpha^*}\,\mathrm{d}x\right)^{2/2_\alpha^*} + c\left(\int_{B_1}(u_m+\psi_r)^{2_\alpha^*}\,\mathrm{d}x\right)^{\frac{2(2_\alpha^*-1)}{2_\alpha^*}} \\ & +c\left(\int_{B_1}(u_m+\psi_r)^{2_\alpha^*}\,\mathrm{d}x\right)^{p/2_\alpha^*}\left(\int_{B_1}|u_m^{\perp}|^{\frac{2_\alpha^*}{2_\alpha^*-p}}\,\mathrm{d}x\right)^{\frac{2_\alpha^*-p}{2_\alpha^*}} + c\|u_m^{\perp}\| \\ \leq\ & c\|u_m^{\perp}\|^2 + c\left(\int_{B_1}(u_m+\psi_r)^{2_\alpha^*}\,\mathrm{d}x\right)^{\frac{2(2_\alpha^*-1)}{2_\alpha^*}} + c\left(\|u_m^{\perp}\|^{2\frac{2_\alpha^*-p}{p}}\right) \\ & +c\left(\int_{B_1}(u_m+\psi_r)_+^{2_\alpha^*}\,\mathrm{d}x\right)^{2p/2_\alpha^*} + c\|u_m^{\perp}\|.\end{aligned}$$

By (3.19) and [41, Compactness Lemma], which guarantees the compact embedding of H in $L^z(B_1,|x|^\alpha)$ for $2 \leq z < 2_\alpha^*$, we have

$$\|u_m^{\perp}\|^2 \leq c + c(\|u_m\|^{\frac{2_\alpha^*-1}{2_\alpha^*}} + \|u_m\|^{\frac{2p}{2_\alpha^*}}) + c\|u_m^{\perp}\|. \tag{3.21}$$

For $u_m^k \in H_k$, using the variational characterization of λ_k, similarly to (3.21), we obtain

$$\|u_m^{k}\|^2 \leq \bar{c} + c(\|u_m\|^{\frac{2_\alpha^*-1}{2_\alpha^*}} + \|u_m\|^{\frac{2p}{2_\alpha^*}}) + c\|u_m^{k}\|. \tag{3.22}$$

By summing the inequalities in (3.21) and (3.22), we reach

$$\|u_m\|^2 \leq C + C(\|u_m\|^{\frac{2_\alpha^*-1}{2_\alpha^*}} + \|u_m\|^{\frac{2p}{2_\alpha^*}}) + C\|u_m\|.$$

proving the boundedness of the sequence (u_m) as desired. ■

3.5 Proof of Theorem 3.2.2

In this section, we search for a solution u of Problem (3.4) in $H^1_{0,\mathrm{rad}}(B_1)$. First we need to prove the geometric condition for $J_{\lambda,r}$. After that, we make some estimates for minimax levels.

3.5.1 The geometric conditions

Initially we consider $\lambda < \lambda_1$. We will prove the geometric condition of the Mountain–Pass Theorem.

Proposition 3.5.1. *Let $\lambda < \lambda_1$, suppose $(g_0), (g_1)$. Then there exist $a, \rho > 0$ such that*

$$J_{\lambda,r}(u) \geq a \text{ if } \|u\| = \rho.$$

Proof: Using (g_1), (3.12), the characterization variational of λ_1 and the fact that H is continuously embedded in $L^z(B_1, |x|^\alpha)$ if $1 \leq z \leq 2^*_\alpha$, we have

$$\begin{aligned} J_{\lambda,r}(u) \geq\ & \frac{1}{2}\left(1-\frac{\lambda}{\lambda_1}\right)\|u\|^2 - c\int_{B_1}|x|^\alpha(u+\psi_r)_+^{2^*_\alpha}\,\mathrm{d}x \\ & -c\int_{B_1}|x|^\alpha(u+\psi_r)_+^{p+1}\,\mathrm{d}x - \int_{B_1}(f-f_r)u\,\mathrm{d}x \\ \geq\ & C_1\|u\|^2 - c\|u\|^{2^*_\alpha}_{2^*_\alpha,|x|^\alpha} - c\|u\|^{p+1}_{p+1,|x|^\alpha} - \int_{B_1}(f-f_r)u\,\mathrm{d}x \\ \geq\ & C_1\|u\|^2 - C_2\|u\|^{2^*_\alpha} - C_3\|u\|^{p+1} - C_4\delta_r\|u\|. \end{aligned}$$

Now we choose $\rho > 0$ as the point where the function $h_r(s) = C_1 s^2 - C_2 s^{2^*_\alpha} - C_3 s^{p+1} - C_4\delta_r s$ assumes its maximum for $s \geq 0$. We can see that there exists r_0 such that $\max_{\mathbb{R}} h_r(s) = M_r > M_{r_0}$ for all $r \leq r_0$. Thus, there exist $a, \rho > 0$ such that $J_{\lambda,r}(u) \geq a > 0$ for all $\|u\| = \rho$. ■

Proposition 3.5.2. *Suppose (g_0) and $\lambda < \lambda_1$. Then, there exists $R_\varepsilon > \rho$ such that $J_{\lambda,r}(Ru^r_\varepsilon) \leq 0$.*

Proof: By (3.2), we can see that $|\mathrm{supp} u^r_\varepsilon \cap \mathrm{supp}\psi_r| = 0$ (See Figures 4.1 and 4.2). Consequently,

$$\int_{B_1}|x|^\alpha(Ru^r_\varepsilon+\psi_r)_+^{2^*_\alpha}\,\mathrm{d}x = R^{2^*_\alpha}\int_{B_1}|x|^\alpha(u^r_\varepsilon)^{2^*_\alpha}\,\mathrm{d}x$$

and from (3.3) we conclude that

$$\int_{B_1} f_r(su^r_\varepsilon)\,\mathrm{d}x = 0 \text{ for all } s \in \mathbb{R}.$$

Now, we observe that

$$\begin{aligned} J_{\lambda,r}(Ru^r_\varepsilon) \leq\ & R^2\|u^r_\varepsilon\|^2 - CR^{2^*_\alpha}\int_{B_{r^2/2}}|x|^\alpha(u^r_\varepsilon)_+^{2^*_\alpha}\,\mathrm{d}x + R\|f\|_2\|u^r_\varepsilon\|_2 \\ \leq\ & CR^2 - CR^{2^*_\alpha} + CR. \end{aligned} \tag{3.23}$$

Then, we can choose $R_0(\varepsilon)$ sufficiently large to have $J_{\lambda,r}(R_\varepsilon u^r_\varepsilon) \leq 0$ for each $R_\varepsilon > R_0(\varepsilon)$. ■

Before proving the geometric conditions in the saddle point case, we need to introduce notations and to make estimates. Take r as before and $\zeta_r : B_1 \to \mathbb{R}$ smooth radially symmetric functions such that $0 \leq \zeta_r \leq 1$, $|\nabla \zeta_r| \leq 4/r$ and

$$\zeta_r(x) = \begin{cases} 0 & \text{if} \quad x \in B_r; \\ 1 & \text{if} \quad x \in B_1 \setminus B_{2r}. \end{cases}$$

Define

$$\phi_j^r = \zeta_r \phi_j$$

and consider the following finite-dimensional subspaces

$$H_k^r = [\phi_1^r, \phi_2^r, ..., \phi_k^r] \cap H_0^1(B_1)$$

and

$$H_k^{r,*} = H_k^r \cap H_{0,\text{rad}}^1(B_1).$$

Lemma 3.5.1. If $r \to 0$, then $\phi_j^r \to \phi_j$ in $H_0^1(B_1)$ for all $j = 1, ..., k$. Moreover, for each r small enough, we have that there exists c_k such that

$$\|v\|^2 < (\lambda_k + c_k r^{N-2})\|v\|_2^2 \text{ for all } v \in [\phi_1^r, ..., \phi_r^k].$$

Proof: For $i \in \{1, ..., k\}$ fixed we have

$$\|\phi_i^r - \phi_i\|^2 \quad \leq c_1\|\phi_i\|_\infty^2 r^{N-2} + c_1\|\nabla\phi_i\|_\infty\|\phi_i\|_\infty r^{N-1} + c_1\|\nabla\phi_i\|_\infty^2 r^N = c\|\phi_i\|_\infty r^{N-2}.$$

where c_0, c_1 and c are constants independent of ϕ_i.

Consequently, $\|\phi_i - \phi_i^r\| \to 0$.

Now, let us take $v_r \in H_k^r$ such that $\|v_r\|_2 = 1$. Notice that $v_r = \sum_{j=1}^k c_j \zeta_r \phi_j = \zeta_r \bar{v}$ where $\bar{v} = \sum_{j=1}^k c_i \phi_i \in H_k$ and $\|v_r - \bar{v}\| = o(1)$ when $r \to 0$. Thus,

$$\begin{aligned} \|v_r\|^2 = & \quad (\|v_r\|^2 - \|\bar{v}\|^2) + \|\bar{v}\|^2 \leq \tilde{C}_k r^{N-2} + \lambda_k\|\bar{v}\|_2^2 \leq \tilde{C}_k r^{N-2} + \lambda_k[(\|\bar{v}\|_2^2 - \|v_r\|_2^2) + \|v_r\|_2^2] \\ = & \quad c_k r^{N-2} + \lambda_k, \end{aligned}$$

as desired. ■

Again, we should choose a suitable decomposition for H.

Notice that for r small enough, we can split the space

$$H = H_k^{r,*} \oplus \left((H_k^*)^\perp \cap H_{0,\text{rad}}^1(B_1)\right).$$

Now we can prove the geometric conditions of the Linking Theorem using this non-orthogonal direct sum.

Proposition 3.5.3. *Suppose that* $\lambda_k < \lambda < \lambda_{k+1}$, $(g_0) - (g_1)$. *Then there exist* $\rho, a > 0$ *such that* $J_\lambda(u) \geq a$ *if* $u \in \left((H_k^*)^\perp \cap H^1_{0,\mathrm{rad}}(B_1)\right)$ *with* $\|u\| = \rho$.

Proof: Using (g_1), (3.12), the characterization variational of λ_k and continuous embedding of H in $L^z(B_1, |x|^\alpha)$ for $2 \leq z \leq 2^*_\alpha$, we have

$$\begin{aligned} J_{\lambda,r}(u) \geq\ & \frac{1}{2}\left(1 - \frac{\lambda}{\lambda_{k+1}}\right)\|u\|^2 - \frac{1}{2^*_\alpha}\int_{B_1} |x|^\alpha (u+\psi_r)_+^{2^*_\alpha}\,\mathrm{d}x \\ & - \int_{B_1} |x|^\alpha (u+\psi_r)_+^{p+1}\,\mathrm{d}x - \int_{B_1} (f - f_r)u\,\mathrm{d}x \\ \geq\ & C_1\|u\|^2 - C_2\|u\|^{2^*_\alpha} - C_3\|u\|^{p+1} - C_4\delta_r\|u\|. \end{aligned}$$

Thus, the result follows the same way of Proposition 3.5.2. ■

Proposition 3.5.4. *Suppose that* $\lambda_k < \lambda < \lambda_{k+1}$ *and* $(g_0) - (g_1)$. *For each* ε *small enough, there exists* $R_\varepsilon > \rho$ *such that if*

$$Q := \{v + su_\varepsilon^r : v \in H_k^{r,*} \text{ with } \|v\| \leq R_\varepsilon \text{ and } 0 \leq s \leq R_\varepsilon\},$$

then $J_\lambda(u) \leq 0$ *for all* $u \in \partial Q$.

Proof: Let us take $R > R_0$ and we split ∂Q as follows

$$\begin{aligned} Q_1 &= \{v \in H_k^{r,*} : \|v\| \leq R\}; \\ Q_2 &= \{v + su_\varepsilon^r : v \in H_k^{r,*}, \|v\| = R \text{ and } 0 \leq s \leq R\}; \\ Q_3 &= \{v + Ru_\varepsilon^r : v \in H_k^{r,*}, \|v\| \leq R\}. \end{aligned}$$

First, we notice that, for all $v \in H_k^*$, one has

$$\int_{B_1} (f - f_r)v\,\mathrm{d}x = \int_{B_1} \nabla(\psi - \psi_r)\nabla v\,\mathrm{d}x - \lambda \int_{B_1} (\psi - \psi_r)v\,\mathrm{d}x = 0.$$

The last equality is true because supp$v \subset \overline{B_1 \setminus B_r}$ for all $v \in H_k^{r,*}$ and supp$(\psi - \psi_r) \subset B_{r^2}$. Consequently, if $v \in Q_1$, from Lemma 3.5.1, we have

$$J_{\lambda,r}(v) \leq \left(1 - \frac{\lambda}{\lambda_k + c_k r^{N-2}}\right)\|v\|^2.$$

Since $\lambda > \lambda_k$ and due to the choice of r small enough, we can take $C_1 > 0$ such that

$$J_\lambda(v) \leq -C_1\|v\|^2 \leq 0. \tag{3.24}$$

for all $v \in H_k^{r,*} \subset H_k^r$ independently of $R > 0$.

Since, $|\text{supp}v \cap \text{supp}u_\varepsilon| = 0$, we have that

$$J_{\lambda,r}(v + su_\varepsilon) = J_{\lambda,r}(v) + J_{\lambda,r}(su_\varepsilon). \tag{3.25}$$

For Q_2, using (3.25), (3.24) and Lemma 3.4.2, we get

$$\begin{aligned} J_{\lambda,r}(v + su_\varepsilon^r) \leq & -C_1R^2 + C_2s^2\|u_\varepsilon^r\|^2 - Cs^{2_\alpha^*}\int_{B_1}|x|^\alpha(u_\varepsilon^r)^{2_\alpha^*}\,\mathrm{d}x + Cs \\ \leq & -C_1R^2 + S_\alpha^{(N+\alpha)/(2+\alpha)}(C_2s^2 - C_3s^{2_\alpha^*}) + C_2s^2O(\varepsilon^{(N-2)/(2+\alpha)}) \\ & +C_3s^{2_\alpha^*}O(\varepsilon^{(N+\alpha)/(2+\alpha)}) + Cs \\ \leq & -CR^2 + CR + C + C\varepsilon^{(N-2)/(2+\alpha)}R^{2_\alpha^*}. \end{aligned}$$

Take $R_\varepsilon = \max\{R, -CR^2+CR+C+C_4\varepsilon^{(N-2)/(2+\alpha)}R^{2_\alpha^*} = 0\}$ and we can see that $R_\varepsilon \to \infty$ if $\varepsilon \to 0$.

For Q_3, using (3.24) and (3.25), we obtain

$$\begin{aligned} J_{\lambda,r}(v + Ru_\varepsilon^r) \leq & \; J_{\lambda,r}(v) + J_{\lambda,r}(v + Ru_\varepsilon^r) \\ \leq & \; CR^2\|u_\varepsilon^r\|^2 - \frac{\lambda}{2}R^{2_\alpha^*}\|u_\varepsilon^r\|_{2_\alpha^*}^{2_\alpha^*} + CR \\ \leq & -CR^{2_\alpha^*} + CR^2 + CR. \end{aligned}$$

Hence, $J(v + Ru_\varepsilon^r) \leq 0$ for R sufficiently large and ε small. This completes the proof. ■

3.5.2 Estimates of minimax levels

For the Mountain–Pass case, we define the minimax level of $J_{\lambda,r}$ by

$$\bar{c} = \inf_{v\in\Gamma}\max_{w\in v([0,1])} J_\lambda(w) \tag{3.26}$$

where

$$\Gamma = \{v \in C([0,1], H) : v(0) = 0 \text{ and } v(1) = R_\varepsilon u_\varepsilon^r\},$$

R_ε being such that $J_\lambda(R_\varepsilon u_\varepsilon^r) \leq 0$ as in Proposition 3.5.2.

We need to show that the minimax levels are below a suitable constant. For this purpose, it will be necessary to make a estimate which will allows us to simplify some calculations needed ahead. Initially, we consider a Palais-Smale sequence (u_n), thus, by Lemma 3.4.3, we may assume that (eventually passing to a subsequence)

$$u_n \rightharpoonup u \in H \quad \text{and} \quad \|u_n - u\| \quad \text{is convergent.}$$

We should check that u is a solution for (3.4). However, we have to ensure that u_n is not a trivial solution. More specifically, we should show that $u_n \neq \psi - \psi_r$. In order to prove this we need to show the following lemma.

Lemma 3.5.2. Let

$$K_\alpha := \lim \|u - u_n\|^2.$$

Then

$$J_{\lambda,r}(u) + \frac{(2+\alpha)}{2(N+\alpha)} K_\alpha = \bar{c}.$$

Furthermore, if $K_\alpha > 0$ then $K_\alpha \geq S_\alpha^{(N+\alpha)/(2+\alpha)}$.

Proof: Returning to relation (3.17), since $\|u_n\|$ is bounded, we assume that $u_n \to u$ in $L^z(B_1, |x|^\alpha)$ for $2 \leq z \leq 2^*_\alpha$ and $u_n \to u$ almost everywhere in B_1. In particular u is a weak solution of (3.4), so we have

$$\begin{aligned}\|u\|^2 - \lambda\|u\|_2^2 - \int_{B_1} |x|^\alpha (u+\psi_r)_+^{2^*_\alpha}\,\mathrm{d}x + \int_{B_1} |x|^\alpha (u+\psi_r)^{2^*_\alpha - 1}\,\mathrm{d}x \\ - \int_{B_1} |x|^\alpha g(u+\psi_r)_+\,\mathrm{d}x - \int_{B_1} (f - f_r)u\,\mathrm{d}x &= 0.\end{aligned}$$

By the Brezis-Lieb Lemma (see [10]), it follows that

$$\int_{B_1} |x|^\alpha (u_n+\psi_r)_+^{2^*_\alpha}\,\mathrm{d}x = \int_{B_1} |x|^\alpha (u_n-u)_+^{2^*_\alpha}\,\mathrm{d}x + \int_{B_1} |x|^\alpha (u+\psi_r)_+^{2^*_\alpha}\,\mathrm{d}x + o(1). \quad (3.27)$$

Since H is a Hilbert Space, we obtain

$$\|u_n\|^2 = \|u_n - u\|^2 + \|u\|^2 + o(1). \quad (3.28)$$

Due to (g_1) and $u_n \to u$ in $L^z(B_1, |x|^\alpha)$ for $2 \leq z < 2^*_\alpha$, we have

$$\int_{B_1} |x|^\alpha G(u_n+\psi_r)\,\mathrm{d}x - \int_{B_1} |x|^\alpha G(u+\psi_r)\,\mathrm{d}x = o(1).$$

Using (3.16), (3.27) and (3.28), we get

$$\begin{aligned}\bar{c} + o(1) = &\ J_{\lambda,r}(u_n) \\ = &\ J_{\lambda,r}(u) + \frac{1}{2}\|u_n - u\|^2 - \frac{1}{2^*_\alpha}\int_{B_1} |x|^\alpha (u_n-u)_+^{2^*_\alpha}\,\mathrm{d}x + o(1).\end{aligned} \quad (3.29)$$

Since $J'_{\lambda,r}(u) = 0$, analogously, we conclude that

$$\begin{aligned}\langle J'_{\lambda,r}(u_n), u_n\rangle = &\ \|u_n - u\|^2 - \int_{B_1} |x|^\alpha (u_n-u)_+^{2^*_\alpha}\,\mathrm{d}x - \int_{B_1} |x|^\alpha (u_n-u)^{2^*_\alpha-1}\psi_r\,\mathrm{d}x \\ &- \int_{B_1} |x|^\alpha g((u_n+\psi_r)_+)u_n\,\mathrm{d}x + \int_{B_1} |x|^\alpha g((u+\psi_r)_+)u\,\mathrm{d}x + o(1).\end{aligned}$$

Moreover, we notice that

$$\int_{B_1} |x|^\alpha (u_n-u)^{2^*_\alpha-1}\psi_r\,\mathrm{d}x = o(1).$$

Using (g_1) and [28, Lemma 2.1], we have

$$\int_{B_1} |x|^\alpha g((u_n+\psi_r)_+)u_n \,\mathrm{d}x - \int_{B_1} |x|^\alpha g((u+\psi_r)_+)u \,\mathrm{d}x = o(1).$$

Then, we can see that

$$\|u_n - u\|^2 = \int_{B_1} |x|^\alpha (u_n-u)_+^{2^*_\alpha} \,\mathrm{d}x + o(1). \tag{3.30}$$

Since

$$K_\alpha = \lim \|u_n - u\|^2,$$

if $K_\alpha > 0$, from (3.6) and (3.30), we obtain that

$$\begin{aligned}\|u_n - u\|^2 \geq\ & S_\alpha \left(\int_{B_1} |x|^\alpha |u_n - u|^{2^*_\alpha} \,\mathrm{d}x \right)^{2/2^*_\alpha} \\ \geq\ & S_\alpha \left(\|u_n-u\|^2 + o(1) \right)^{2/2^*_\alpha}.\end{aligned}$$

This implies that $K_\alpha \geq S_\alpha K_\alpha^{(N-2)/(N+\alpha)}$, consequently, $K_\alpha \geq S_\alpha^{(N+\alpha)/(2+\alpha)}$. ■

Now we take $0 < d < 1/2(2+\alpha)$ (to be chosen precisely later) and by item (e) and (f) of Lemma 3.4.2 we can make $\varepsilon^d = r$. We will prove the following result.

Lemma 3.5.3. Fix $\varepsilon^d = r$. Then

$$|J_{\lambda,r}(\psi - \psi_r)| \leq C\varepsilon^{d(N-2)} \quad \text{for all} \quad N \geq 3. \tag{3.31}$$

Proof: First, we notice that

$$J_{\lambda,r}(\psi - \psi_r) = -\frac{1}{2}\left(\|\psi - \psi_r\|^2 - \lambda \|\psi - \psi_r\|_2^2 \right).$$

From variational characterization of λ_1 and (3.11), we obtain

$$|J_{\lambda,r}(\psi - \psi_r)| \leq C\delta_r^2.$$

By (3.14), we notice that, for $\vartheta \geq 1$ and r small enough, one has

$$r^\vartheta \leq (1 - r^{\sqrt{r}})^2.$$

Hence, we have that

$$|J_{\lambda,r}(\psi - \psi_r)| \leq C\frac{r^{N-1}}{(1-r^{\sqrt{r}})^2} \leq C\frac{r^{N-1}}{r^\vartheta} = Cr^{N-1-\vartheta}.$$

Choosing $\vartheta = 1$, the result is proved. ■

Remark 3.5.1. *Recall* (3.13) *and notice that*

$$\delta_r^2 \leq C \max\{\|\nabla\psi\|_\infty^2, \|\psi\|_\infty^2\} \frac{r^{N-1}}{(1-r^{\sqrt{r}})^2}.$$

If we search a solution for (3.4) *in* $H_0^1(B_1)$, *we can use a similar way, changing the ball with center in origin into a small ball near of boundary of* B_1 *where* $\max\{\|\nabla\psi\|_\infty^2, \|\psi\|_\infty^2\} < r^2$, *which is possible because* ψ *vanish on* ∂B_1. *This allows us work with a new estimate for Lemma 3.5.3, more specifically*

$$|J_{\lambda,r}(\psi - \psi_r)| \leq C r^N,$$

which helps us to improve hypotheses of our problem in high dimensions. (See Remark 3.5.3).

Now, we need to fix some constants which shall be used in the proof of boundedness of minimax levels. First of all, let us define $\beta > 1$ as

$$\frac{1}{\beta} + \frac{1}{\mu} + \left(\frac{N+2}{2N}\right) = 1, \tag{3.32}$$

where μ is given in (f_1). Consequently,

$$\frac{1}{\beta} = 1 - \frac{1}{\mu} - \frac{N+2}{2N}. \tag{3.33}$$

For $N \geq 5$, we recall that $\mu > N$. Thus, we obtain that

$$\frac{2N}{N-2} < \beta < \frac{2N}{N-4}. \tag{3.34}$$

Initially, take β as above in order to prove the following proposition.

Proposition 3.5.5. *Assume* (g_2) *and* $\mu > N \geq 5$. *If*

$$[2_\alpha^* - (q+1)]\left(\frac{N-2}{2N(2+\alpha)}\right)\beta < d < \frac{1}{2(2+\alpha)} - \frac{2_\alpha^* - (q+1)}{4(2+\alpha)},$$

then

$$\bar{c} < \frac{(2+\alpha)}{2(N+\alpha)} S_\alpha^{(N+\alpha)/(2+\alpha)} - C\varepsilon^{[2_\alpha^* - (q+1)](N-2)/2(2+\alpha)}.$$

Proof: Due to (3.26) and (3.23), for $\varepsilon > 0$ fixed, we can choose $t_\varepsilon > 0$ such that

$$\sup_{t\geq 0} J_{\lambda,r}(t u_\varepsilon^r) = J_{\lambda,r}(t_\varepsilon u_\varepsilon^r).$$

Furthermore, one can prove that $t_\varepsilon > k$ for some $k > 0$ and for all $\varepsilon > 0$. Then we only need to show that

$$J_{\lambda,r}(t_\varepsilon u_\varepsilon^r) < \frac{(2+\alpha)}{2(N+\alpha)} S_\alpha^{(N+\alpha)/(2+\alpha)} - C\varepsilon^{[2_\alpha^* - (q+1)](N-2)/2(2+\alpha)}.$$

For this matter we use (g_2) and Lemma 3.4.2. First, we notice that

$$\begin{aligned} J_{\lambda,r}(t_\varepsilon u_\varepsilon^r) \leq\ & \frac{t_\varepsilon^2}{2}\|u_\varepsilon^r\|^2 - \frac{t_\varepsilon^{2_\alpha^*}}{2_\alpha^*}\|u_\varepsilon^r\|_{2_\alpha^*,|x|^\alpha}^{2_\alpha^*} - \frac{\lambda t_\varepsilon^2}{2}\|u_\varepsilon^r\|_2^2 \\ & - t^{q+1}\|u_\varepsilon^r\|_{q+1}^{q+1} + t_\varepsilon \int_{B_{r^2}} |f - f_r||u_\varepsilon^r|\, \mathrm{d}x. \end{aligned} \tag{3.35}$$

From items (e) and (f) of Lemma 3.4.2, we have

$$\frac{t_\varepsilon^2}{2}\|u_\varepsilon^r\|^2 - \frac{t_\varepsilon^{2_\alpha^*}}{2_\alpha^*}\|u_\varepsilon^r\|_{2_\alpha^*,|x|^\alpha}^{2_\alpha^*} \leq \left(\frac{t_\varepsilon^2}{2} - \frac{t_\varepsilon^{2_\alpha^*}}{2_\alpha^*}\right)(S_\alpha^{(N+\alpha)/(2+\alpha)} + O((\varepsilon/r^{2(2+\alpha)})^{(N-2)/(2+\alpha)})).$$

Notice that the polynomial function $t^2/2 - t^{2_\alpha^*}/2_\alpha^*$ attains its maximum at $t = 1$, so we obtain that

$$\begin{aligned} \frac{t_\varepsilon^2}{2}\|u_\varepsilon^r\|^2 - \frac{t_\varepsilon^{2_\alpha^*}}{2_\alpha^*}\|u_\varepsilon^r\|_{2_\alpha^*,|x|^\alpha}^{2_\alpha^*} &\leq \left(\frac{1}{2} - \frac{1}{2_\alpha^*}\right)(S^{(N+\alpha)/(2+\alpha)} + O((\varepsilon/r^{2(2+\alpha)})^{(N-2)/(2+\alpha)})) \\ &\leq \frac{(2+\alpha)}{2(N+\alpha)} S_\alpha^{(N+\alpha)/(2+\alpha)} + c(\varepsilon/r^{2(2+\alpha)})^{(N-2)/(2+\alpha)}. \end{aligned}$$

By the Hölder inequality, we can estimate the last term in (3.35). Indeed, let β be given in (3.32) and since $\mathrm{supp} f_r \subset \overline{B_1 \setminus B_{r^2}}$, we have that

$$\int_{B_{r^2}} |f - f_r||u_\varepsilon^r|\, \mathrm{d}x \leq \left(\int_{B_{r^2}} |f|^\mu\, \mathrm{d}x\right)^{1/\mu} \left(\int_{B_{r^2}} |u_\varepsilon^r|^{2N/(N+2)}\, \mathrm{d}x\right)^{(N+2)/2N} \left(\int_{B_{r^2}} 1\, \mathrm{d}x\right)^{1/\beta}.$$

We note that $2N/(N+2) < 2$, thus $L^2(B_1)$ is continuously embedded in $L^{2N/(N+2)}(B_1)$, for this there is a constant $C > 0$ such that $\|u\|_{2N/(N+2)} \leq C\|u\|_2$ and consequently $\|u\|_{2N/(N+2)}$ is bounded. Since f belongs to $L^\mu(B_1)$, we can see that

$$\int_{B_{r^2}} |f - f_r||u_\varepsilon^r|\, \mathrm{d}x \leq Cr^{2N/\beta}.$$

Therefore, we conclude that

$$J_{\lambda,r}(u_\varepsilon^r) \leq \frac{(2+\alpha)}{2(N+\alpha)} S_\alpha^{(N+\alpha)/(2+\alpha)} + c(\varepsilon/r^{2(2+\alpha)})^{(N-2)/(2+\alpha)} - c\|u_\varepsilon^r\|_{q+1}^{q+1} + Cr^{2N/\beta}. \tag{3.36}$$

We remember that we are working with $\varepsilon^d = r$, consequently, from (3.36) and item (d) of Lemma 3.4.2 (with $\gamma = \alpha$ and $s = q+1$), we obtain that

$$\begin{aligned} J_{\lambda,r}(u_\varepsilon^r) &\leq \tfrac{(2+\alpha)}{2(N+\alpha)} S_\alpha^{(N+\alpha)/(2+\alpha)} + c(\varepsilon^{1-2(2+\alpha)d})^{(N-2)/(2+\alpha)} \\ &\quad + c\varepsilon^{2dN/\beta} - C\varepsilon^{[2_\alpha^* - (q+1)](N-2)/2(2+\alpha)}. \end{aligned} \tag{3.37}$$

Choosing d such that $(1-2(2+\alpha)d)(N-2)/(2+\alpha) > [2^*_\alpha-(q+1)](N-2)/2(2+\alpha)$ and $2dN/\beta > [2^*_\alpha-(q+1)](N-2)/2(2+\alpha)$, it follows that

$$[2^*_\alpha-(q+1)]\left(\frac{N-2}{4N(2+\alpha)}\right)\beta < d < \frac{1}{2(2+\alpha)} - \frac{[2^*_\alpha-(q+1)]}{4(2+\alpha)}. \tag{3.38}$$

Since, $\beta < 2N/(N-4)$, we have that

$$[2^*_\alpha-(q+1)]\frac{N-2}{2(N-4)(2+\alpha)} < d < \frac{1}{2(2+\alpha)} - \frac{[2^*_\alpha-(q+1)]}{4(2+\alpha)},$$

which is possible only for $N \geq 5$ (because the above boundedness of β) and q given in (g_2).

Thus, for ε small enough, there is a constant $C>0$ such that

$$J_{\lambda,r}(u^r_\varepsilon) \leq \frac{(2+\alpha)}{2(N+\alpha)}S_\alpha^{(N+\alpha)/(2+\alpha)} - C\varepsilon^{[2^*_\alpha-(q+1)](N-2)/2(2+\alpha)},$$

as desired. ■

Remark 3.5.2. *In order to obtain* (3.36)*, we neglected the term* $-\|u^r_\varepsilon\|^{q+1}_{q+1}$ *in* (3.35)*, which allows to prove Proposition 3.5.5 to* $N \geq 5$ *using item* (d) *of Lemma 3.4.2. For* $N \geq 5$*, if we change* $-\|u^r_\varepsilon\|^{q+1}_{q+1}$ *for* $-\|u^r_\varepsilon\|^2_2$*, considering items* (c) *and* (d) *of Lemma 3.4.2 (with* $s=2$ *and* $\gamma=0$*), we have that*

$$K_2\varepsilon^{2/(2+\alpha)} \leq \|u^r_\varepsilon\|^2_2 \leq K_1\varepsilon^{2/(2+\alpha)}.$$

Thus, we get an analogous inequality to (3.37)*. In this case, we do not need to assume* (g_2) *and following the same steps of proof of Proposition 3.5.5 to obtain a similar result.*

However, in this away we have a greater restriction in dimension, this argument only works for $N \geq 9$*.*

For a while, we cannot use β given in (3.33) for $N=4,3$, since it is not bounded from above for $N=4$ (see (3.34)) and it is impossible to obtain (3.34) for $N=3$. However, if we have $\mu \geq 2N = 8$, then by (3.33) we get

$$\frac{2N}{N-2} < \beta < \frac{2N}{N-3}. \tag{3.39}$$

So we obtain a boundedness of β when $N \geq 4$, but we do not for $N=3$. Similarity, if we consider $\mu \geq 4N = 12$, we have β in interval

$$\frac{2N}{N-2} < \beta < \frac{4N}{2N-5}, \tag{3.40}$$

which also allows an analogous boundedness for β for $N \geq 3$. Thus, we can make similar results to Proposition 3.5.5 for $N=4,3$ and we have the next propositions.

Proposition 3.5.6. *Assume* (g_2) *and* $\mu \geq 2N = 8$. *If*

$$\frac{(4+\alpha)-(q+1)}{8(2+\alpha)}\beta < d < \frac{1}{2(2+\alpha)} - \frac{(4+\alpha)-(q+1)}{4(2+\alpha)},$$

then

$$\bar{c} < \frac{(2+\alpha)}{2(4+\alpha)} S_\alpha^{(4+\alpha)/(2+\alpha)} - C\varepsilon^{[(4+\alpha)-(q+1)]/(2+\alpha)}.$$

Proof: The proof is identical to Proposition 3.5.5, using (3.39) instead of (3.34). More precisely, when we consider the assumpions of this proposition, we also obtain (3.38). Here, $2^*_\alpha = (4+\alpha)$ because $N = 4$. Thus, we have

$$\frac{(4+\alpha)-(q+1)}{8(2+\alpha)}\beta < d < \frac{1}{2(2+\alpha)} - \frac{(4+\alpha)-(q+1)}{4(2+\alpha)}.$$

From (3.39), $\beta < 8$. Consequently, we have that

$$\frac{[(4+\alpha)-(q+1)]}{8(2+\alpha)} \cdot 8 < d < \frac{1}{2(2+\alpha)} - \frac{[(4+\alpha)-(q+1)]}{4(2+\alpha)}.$$

Hence

$$(4+\alpha) - \frac{2}{5} < (q+1),$$

which is possible once (g_2) is satisfied for $N = 4$. ■

Proposition 3.5.7. *Assume* (g_2) *and* $\mu \geq 4N = 12$. *If*

$$\frac{(6+2\alpha)-(q+1)}{12(2+\alpha)}\beta < d < \frac{1}{2(2+\alpha)} - \frac{(6+2\alpha)-(q+1)}{4(2+\alpha)},$$

then

$$\bar{c} < \frac{(2+\alpha)}{2(N+\alpha)} S_\alpha^{(N+\alpha)/(2+\alpha)} - C\varepsilon^{[(6+2\alpha)-(q+1)]/2(2+\alpha)}.$$

Proof: The proof follows exactly the same steps which were used in proof of Proposition 3.5.5, changing (3.34) into (3.40). Here, we also obtain (3.38) and, since $N = 3$, $2^*_\alpha = (6+2\alpha)$. Thus, we get

$$\frac{(6+2\alpha)-(q+1)}{12(2+\alpha)}\beta < d < \frac{1}{2(2+\alpha)} - \frac{(6+2\alpha)-(q+1)}{4(2+\alpha)}.$$

By (3.40), $\beta < 12$. Hence, we have that

$$\frac{(6+2\alpha)-(q+1)}{12(2+\alpha)} \cdot 12 < d < \frac{1}{2(2+\alpha)} - \frac{(6+2\alpha)-(q+1)}{4(2+\alpha)}.$$

So we should have

$$(6+2\alpha) - \frac{2}{5} < (q+1).$$

Since $N = 3$ by (g_2), it follows this result.

■

In the Linking case, we define the minimax level

$$\hat{c} = \inf_{\nu\in\Gamma} \max_{w\in Q} J_\lambda(\nu(w)) \tag{3.41}$$

where

$$\Gamma = \{\nu \in C(Q;H) : \nu(w) = w \text{ if } w \in \partial Q\}.$$

Since we split the support of the functions in H_k^r of the support of u_ε^r, we have

$$\begin{aligned}\hat{c} &\leq \max\{J_\lambda(v + u_\varepsilon^r) : v \in H_k^r, \|v\| \leq R_n \text{ and } t \geq 0\}\\ &= \max\{J_\lambda(v) + J_\lambda(u_\varepsilon^r) : v \in H_k^n, \|v\| \leq R_n \text{ and } t \geq 0\}\\ &\leq \max\{J_\lambda(v) : v \in H_k^r \text{ and } \|v\| \leq R_n\} + \max\{J_\lambda(tz_n^r); t \geq 0\}.\end{aligned}$$

By (3.24), we see that $J_\lambda(v) \leq 0$ for all $v \in H_k^r$. It follows that

$$\hat{c} \leq \max\{J_\lambda(tz_n^r) : t \geq 0\}.$$

Thus, we can proceed analogously to the Mountain–Pass case in order to see that Propositions 3.5.5, 3.5.6 and 3.5.7 are available to the Linking case.

3.5.3 Proof of Theorem 3.2.2 completed

To simplify the notation, here let us denote $\dot{c} = \bar{c}$, given in (3.26), or $\dot{c} = \hat{c}$, given in (3.41). Now we are ready to prove that $u \neq \psi - \psi_r$, where u is the weak limit of the PS-sequence in the minimax level $\dot{c}$.

Let us take K_α given in Lemma 3.5.2. If $K_\alpha = 0$, then $u_n \to u$ strongly in H and in $L^{2^*_\alpha}(B_1)$, then (3.31) and (3.29), we have

$$J_{\lambda,r}(u) = \dot{c} \geq a > c\varepsilon^{d(N-2)} \geq J_{\lambda,r}(\psi - \psi_r).$$

Hence, $u \neq \psi - \psi_r$.

If $K_\alpha > 0$, suppose, by contradiction, that $u = \psi - \psi_r$, then from Lemmas 3.5.2 and 3.5.3, we obtain that

$$\frac{(2+\alpha)}{2(N+\alpha)} S^{(N+\alpha)/(2+\alpha)} - c\varepsilon^{d(N-2)} \leq \frac{(2+\alpha)}{2(N+\alpha)} K_\alpha + J_{\lambda,r}(u) = \dot{c},$$

By Propositions 3.5.5, 3.5.6 and 3.5.7, we have that

$$\dot{c} < \frac{(2+\alpha)}{2(N+\alpha)} S^{(N+\alpha)/(2+\alpha)} - C\varepsilon^{[2^*_\alpha-(q+1)](N-2)/2(2+\alpha)}.$$

Thus, we obtain

$$d < \frac{2^*_\alpha - (q+1)}{2(2+\alpha)}.$$

On other hands, for $N \geq 3$, we choose d and β such that

$$[2^*_\alpha - (q+1)] \left(\frac{N-2}{4N(2+\alpha)} \right) \beta < d \quad \text{and} \quad \frac{2N}{N-2} < \beta.$$

Hence, we should have

$$[2^*_\alpha - (q+1)] \left(\frac{N-2}{4N(2+\alpha)} \right) \left(\frac{2N}{N-2} \right) < [2^*_\alpha - (q+1)] \left(\frac{N-2}{4N(2+\alpha)} \right) \beta < d < \frac{2^*_\alpha - (q+1)}{2(2+\alpha)},$$

which is a contradiction. ■

Remark 3.5.3. *From Remark 3.5.1, if we work in $H^1_0(B_1)$, we can get $J_{\lambda,r}(\psi - \psi_r) \leq c\varepsilon^{dN}$. Thus, in order to obtain a similar result this theorem, we should have β as it follows*

$$\frac{1}{\beta} + \frac{1}{\mu} + \frac{1}{2} = 1$$

and, consequently for $\mu > N$, we get $2 < \beta < 2^*$. Using this fact, together with Remark 3.5.2, we can do without (g_2) for $N \geq 6$. For details see [15, Lemma 5.2].

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