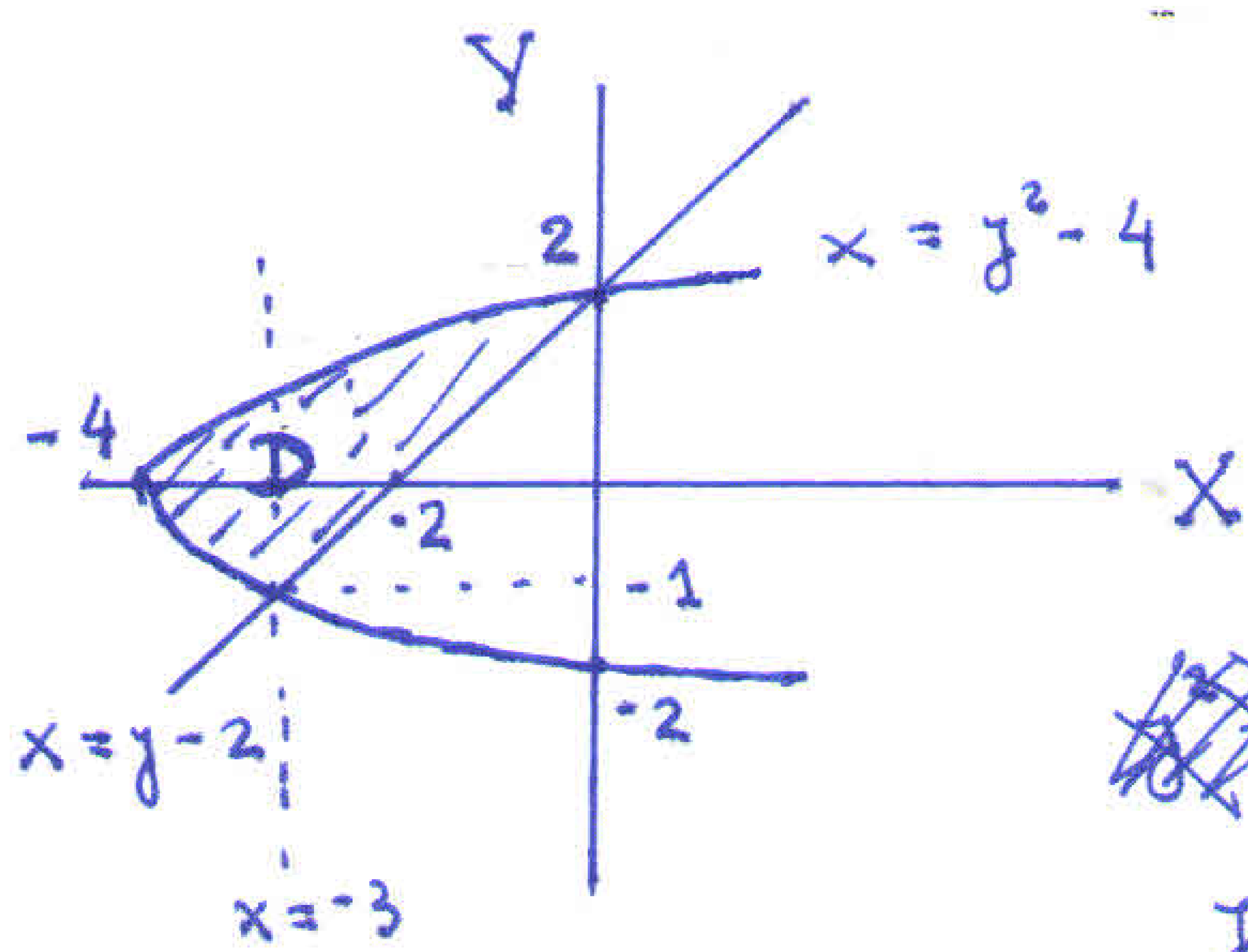
Calculo III 2014.2 1ª Prova (Manhã)

(1) $I = \int_D f\,dA = \int_{-1}^{2}\int_{y^2-4}^{y-2} f(x,y)\,dx\,dy$

(a) Esboce a região D

(b) Inverta a ordem de integração

Solução:



$y - 2 = y^2 - 4$

$y^2 - y - 2 = 0$

$(y-2)(y+1) = 0$

$$D: \begin{cases} -1 \leq y \leq 2 \\ y^2 - 4 \leq x \leq y - 2 \end{cases}$$

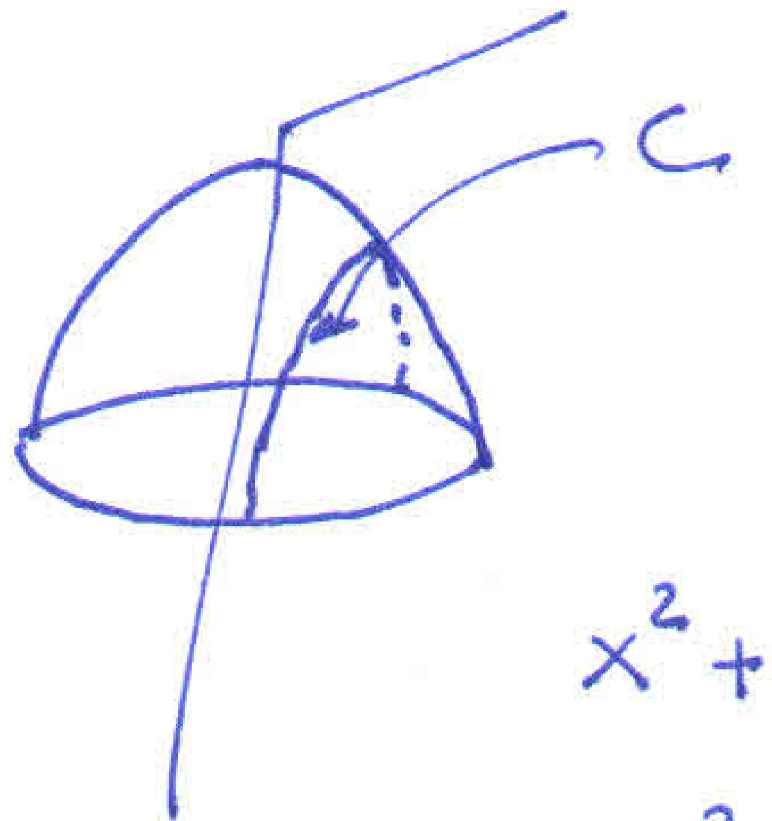
$D = D_1 \cup D_2$

$$D_1: \begin{cases} -4 \leq x \leq -3 \\ -\sqrt{x+4} \leq y \leq \sqrt{x+4} \end{cases} \qquad D_2: \begin{cases} -3 \leq x \leq 0 \\ x+2 \leq y \leq \sqrt{x+4} \end{cases}$$

Assim,

$$\int_{-1}^{2}\int_{y^2-4}^{y-2} f(x,y)\,dx\,dy = \int_{-4}^{-3}\int_{-\sqrt{x+4}}^{\sqrt{x+4}} f(x,y)\,dy\,dx + \int_{-3}^{0}\int_{x+2}^{\sqrt{x+4}} f(x,y)\,dy\,dx$$

(2) $C : \begin{cases} x^2+y^2+z^2=5, & z \geq 0 \\ x+y=1 \end{cases}$



$$x^2+y^2+z^2=5 \;, \; \underline{\underline{z \geq 0}}$$

$$x+y=1 \;\Rightarrow\; y=1-x$$

$$x^2+(1-x)^2+z^2=5$$

$$2x^2-2x+1+z^2=5$$

$$2x^2-2x+z^2=4$$

$$2\left(x^2-x+\frac{1}{4}\right)-\frac{1}{2}+z^2=4$$

$$2(x-1/2)^2+z^2=9/2$$

$$\frac{4}{9}(x-1/2)^2+\frac{2}{9}z^2=1$$

$$\frac{(x-1/2)^2}{9/4}+\frac{z^2}{9/2}=1 \;, \; z \geq 0$$

ou

$$\boxed{\frac{(x-1/2)^2}{(3/2)^2}+\frac{z^2}{\left(\frac{3\sqrt{2}}{2}\right)^2}=1 \;, \; z \geq 0} \quad (*)$$

Ou seja, a projeção da curva C sobre o plano XZ é a semielipse (*) acima.

Assim,

$$\frac{x-1/2}{3/2}=\cos t \quad e \quad \frac{z}{\frac{3\sqrt{2}}{2}}=\operatorname{sen} t$$

Como $z \geq 0$, temos $\operatorname{sen} t \geq 0$ e logo $0 \leq t \leq \pi$

$$\therefore \begin{cases} x = \frac{1}{2} + \frac{3}{2}\cos t \\ y = 1 - x = \frac{1}{2} - \frac{3}{2}\cos t \\ z = \frac{3\sqrt{2}}{2}\,\text{sen}\, t \end{cases} \quad \text{com} \quad 0 \leq t \leq \pi$$

Então uma parametrização para C é dada por

$$\alpha(t) = \left(\frac{1}{2} + \frac{3}{2}\cos t, \frac{1}{2} - \frac{3}{2}\cos t, \frac{3\sqrt{2}}{2}\,\text{sen}\, t\right), \quad 0 \leq t \leq \pi$$

Daí,

$$\alpha'(t) = \left(-\frac{3}{2}\,\text{sen}\, t, \frac{3}{2}\,\text{sen}\, t, \frac{3\sqrt{2}}{2}\cos t\right) \quad \text{e}$$

$$\|\alpha'(t)\|^2 = \frac{9}{4}\,\text{sen}^2 t + \frac{9}{4}\,\text{sen}^2 t + \frac{9}{2}\cos^2 t$$

$$= \frac{9}{2}\left(\text{sen}^2 t + \cos^2 t\right) = 9/2$$

$$\therefore \boxed{\|\alpha'(t)\| = \frac{3\sqrt{2}}{2}}$$

O momento de inércia em relação ao eixo z é dado por

$$I_z = \int_C (x^2 + y^2)\,\delta(x,y,z)\,ds$$

onde $\delta(x,y,z) = k|z| = kz \quad (z \geq 0)$, $k = \text{cte}$

(δ é proporcional à distância ao plano xy)

$$ds = \|\alpha'(t)\|\,dt = \frac{3\sqrt{2}}{2}\,dt.$$

Logo,

$$I_z = \int_C (x^2+y^2)\cdot k\,z\,ds$$

$$= k\int_0^{\pi}\left[\left(\frac{1}{2}+\frac{3}{2}\cos t\right)^2+\left(\frac{1}{2}-\frac{3}{2}\cos t\right)^2\right]\left(\frac{3\sqrt{2}}{2}\overset{\text{sen}\,t}{\cancel{\cos t}}\right)\frac{3\sqrt{2}}{2}\,dt$$

$$= k\int_0^{\pi}\cancel{2}\left(\frac{1}{4}+\frac{9}{4}\cos^2 t\right)\cdot\frac{9}{\cancel{2}}\,\text{sen}\,t\,dt$$

$$= k\int_0^{\pi}\frac{9}{4}\left(1+9\cos^2 t\right)\cdot\text{sen}\,t\,dt$$

$$= \frac{9}{4}k\left\{\int_0^{\pi}\text{sen}\,t\,dt + 9\int_0^{\pi}\cos^2 t\,\text{sen}\,t\,dt\right\}$$

$$= \frac{9}{4}k\left[\left(-\cos t\Big|_0^{\pi}\right)+9\left(-\frac{\cos^3 t}{3}\right)\Big|_0^{\pi}\right]$$

$$= \frac{9}{4}k\,(2+6) = 18k$$

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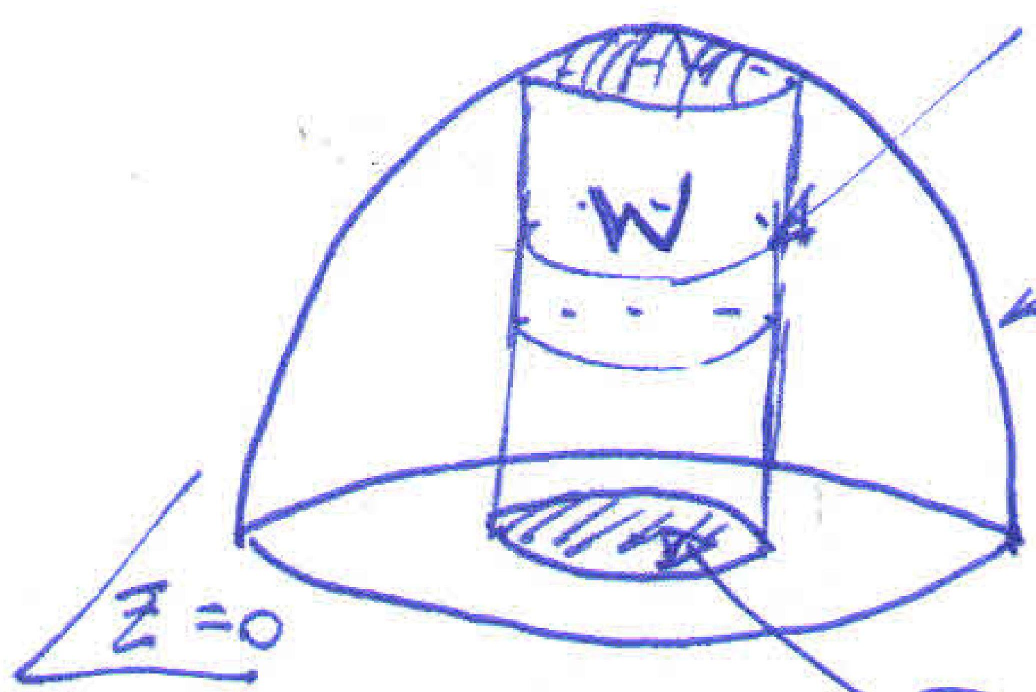


$x^2+y^2=1$

$x^2+y^2+z^2=4, \ z \geq 0$

$D: \ x^2+y^2 \leq 1$

volume de W

$$vol(W) = \iint_D \int_0^{\sqrt{4-x^2-y^2}} dz\, dA.$$

Em coord. cilíndricas, $W: \begin{cases} 0 \leq z \leq \sqrt{4-r^2} \\ 0 \leq r \leq 1 \\ 0 \leq \theta \leq 2\pi \end{cases}$

$$\begin{cases} x = r\cos\theta \\ y = r \operatorname{sen}\theta \\ z = z \end{cases} \qquad x^2+y^2 = r^2$$

$$dx\,dy\,dz = r\,dr\,d\theta\,dz$$

Daí,

$$vol(W) = \int_0^1 \int_0^{2\pi} \int_0^{\sqrt{4-r^2}} r\, dz\, d\theta\, dr$$

$$= \int_0^1 \int_0^{2\pi} r\sqrt{4-r^2}\, d\theta\, dr$$

$$= 2\pi \int_0^1 r\sqrt{4-r^2}\, dr$$

$u = 4-r^2$
$du = -2r\,dr$
$r=0 \Rightarrow u=4$
$r=1 \Rightarrow u=3$

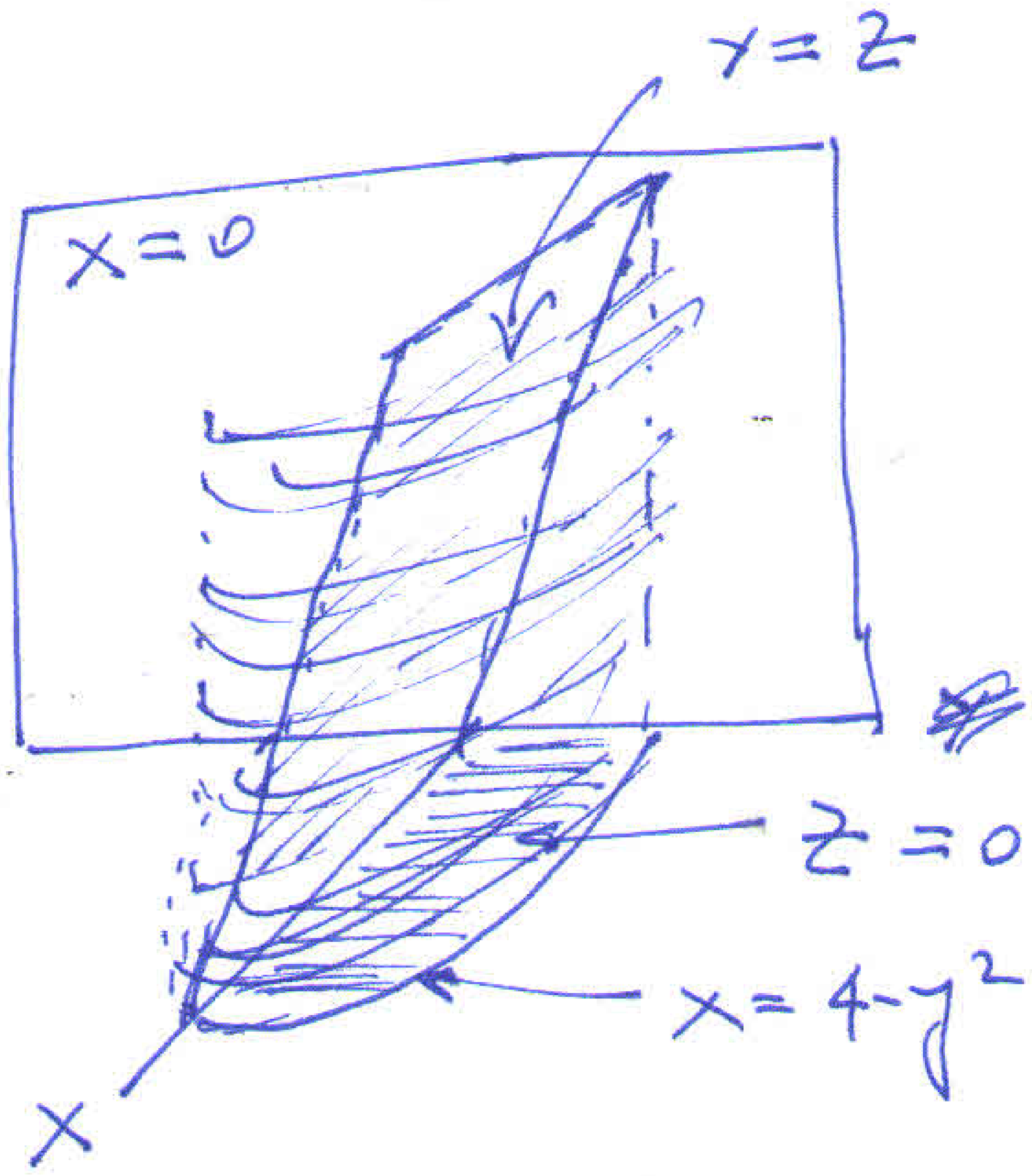
$$= -\pi \int_4^3 \sqrt{u}\, du = \pi \int_3^4 \sqrt{u}\, du$$

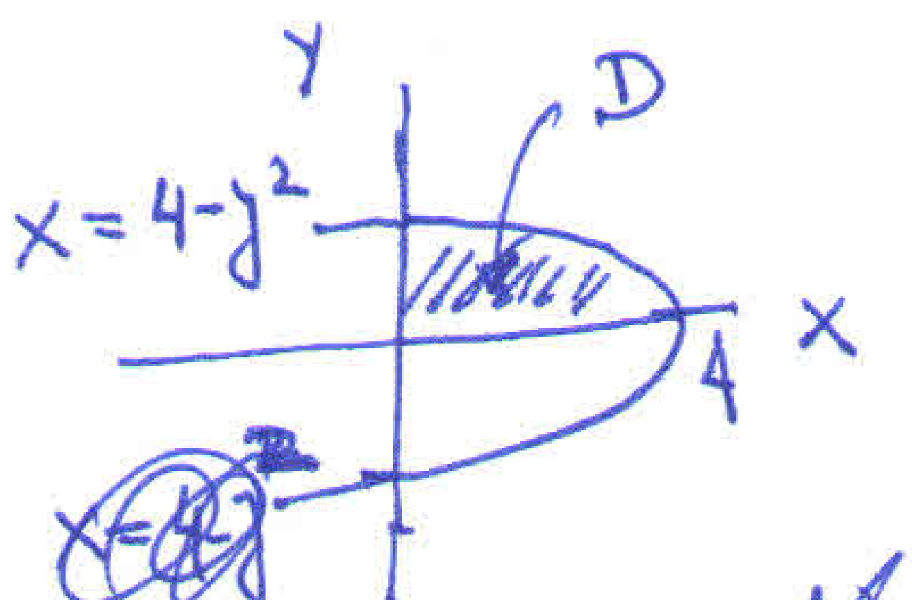
$$= \pi \cdot \frac{2}{3} u^{3/2} \Big|_3^4 = \frac{2}{3}\pi\left(8 - 3\sqrt{3}\right)$$

(4) Calcular $\int_W f\,dV$

(a) $f(x,y,z) = 1$, $W: \begin{cases} x = 4-y^2 \\ y = z \\ x = 0 \\ z \geq 0 \end{cases}$

Solução:





$W: \begin{cases} (x,y) \in D \\ 0 \leq z \leq y \end{cases}$

$D: \begin{cases} 0 \leq x \leq 4 \\ 0 \leq y \leq \sqrt{4-x} \end{cases}$

$$\int_W f\,dV = \int_0^4 \int_0^{\sqrt{4-x}} \int_0^y dz\,dy\,dx$$

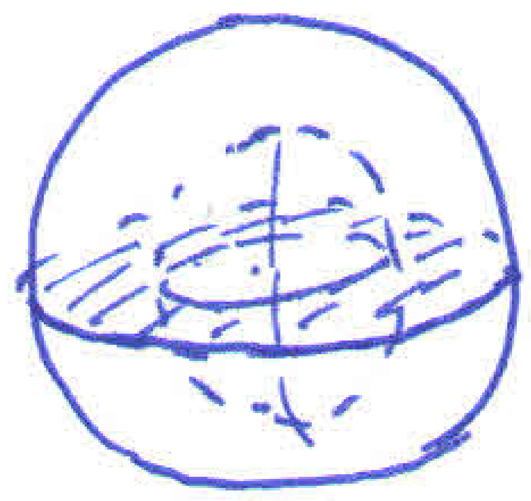
$$= \int_0^4 \int_0^{\sqrt{4-x}} y\,dy\,dx = \int_0^4 \frac{1}{2}(4-x)\,dx$$

$$= \frac{1}{2}\left(4x - \frac{x^2}{2}\Big|_0^4\right) = \frac{1}{2}\cdot(16-8) = 4$$

(b) $f(x,y,z) = \sqrt{x^2+y^2+z^2}$

W é a coroa limitada por $x^2+y^2+z=1$ e $x^2+y^2+z^2=4$

Solução



Em coord. esféricas : $\begin{cases} x = \rho \operatorname{sen}\phi \cos\theta \\ y = \rho \operatorname{sen}\phi \operatorname{sen}\theta \\ z = \rho\cos\phi \end{cases}$

$W_{\rho\phi\theta} : \begin{cases} 1 \leq \rho \leq 2 \\ 0 \leq \theta \leq 2\pi \\ 0 \leq \phi \leq \pi \end{cases}$

$x^2+y^2+z^2 = \rho^2$

$J = \rho^2 \operatorname{sen}\phi$

$$\int_W f\,dV = \int_1^2 \int_0^{\pi} \int_0^{2\pi} \sqrt{\rho^2} \cdot \rho^2 \operatorname{sen}\phi \, d\theta\, d\phi\, d\rho$$

$$= 2\pi \int_1^2 \int_0^{\pi} \rho^3 \operatorname{sen}\phi \, d\phi\, d\rho$$

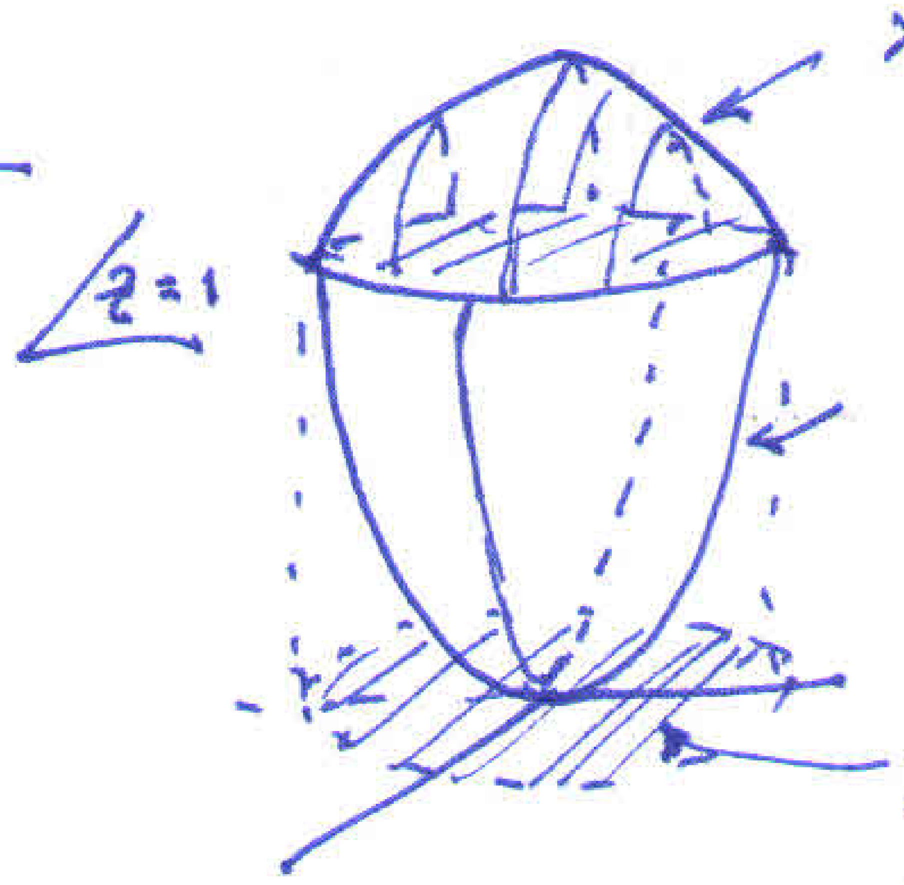
$$= 2\pi \int_1^2 \rho^3 \left(-\cos\phi \Big|_0^{\pi}\right) d\rho = 4\pi \int_1^2 \rho^3\, d\rho$$

$$= 4\pi \left.\frac{\rho^4}{4}\right|_1^2 = \pi\,(16-1) = 15\pi$$

(c) $f(x,y,z) = z$

W limitada por cima por $x^2+y^2+z^2=5$ e por baixo por $z = \frac{1}{4}(x^2+y^2)$

Solução



$D : x^2+y^2 \leq 1$

Coord. cilíndricas:

$$\begin{cases} x = r\cos\theta \\ y = r\,\text{sen}\,\theta \\ z = 0 \end{cases} \qquad W: \begin{cases} \frac{1}{4}r^2 \leq z \leq \sqrt{5-r^2} \\ 0 \leq r \leq 1 \\ 0 \leq \theta \leq 2\pi \end{cases}$$

$$\int_W f\,dV = \int_0^1 \int_0^{2\pi} \int_{\frac{1}{4}r^2}^{\sqrt{5-r^2}} z \cdot r \, dz\, d\theta\, dr$$

$$= \int_0^1 \int_0^{\pi} \frac{1}{2} r \left[(5-r^2) - \frac{r^4}{16} \right] d\theta\, dr$$

$$= \pi \int_0^1 \left(5r - r^3 - \frac{r^5}{16} \right) dr$$

$$= \pi \left(\frac{5}{2} - \frac{1}{4} - \frac{1}{\cancel{5} \cdot 16} \right) = \cancel{\frac{179}{80}\pi}$$

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~~216~~

$$\frac{215}{96}\pi$$