Cálculo III Prova 1 (tarde) 2014.2

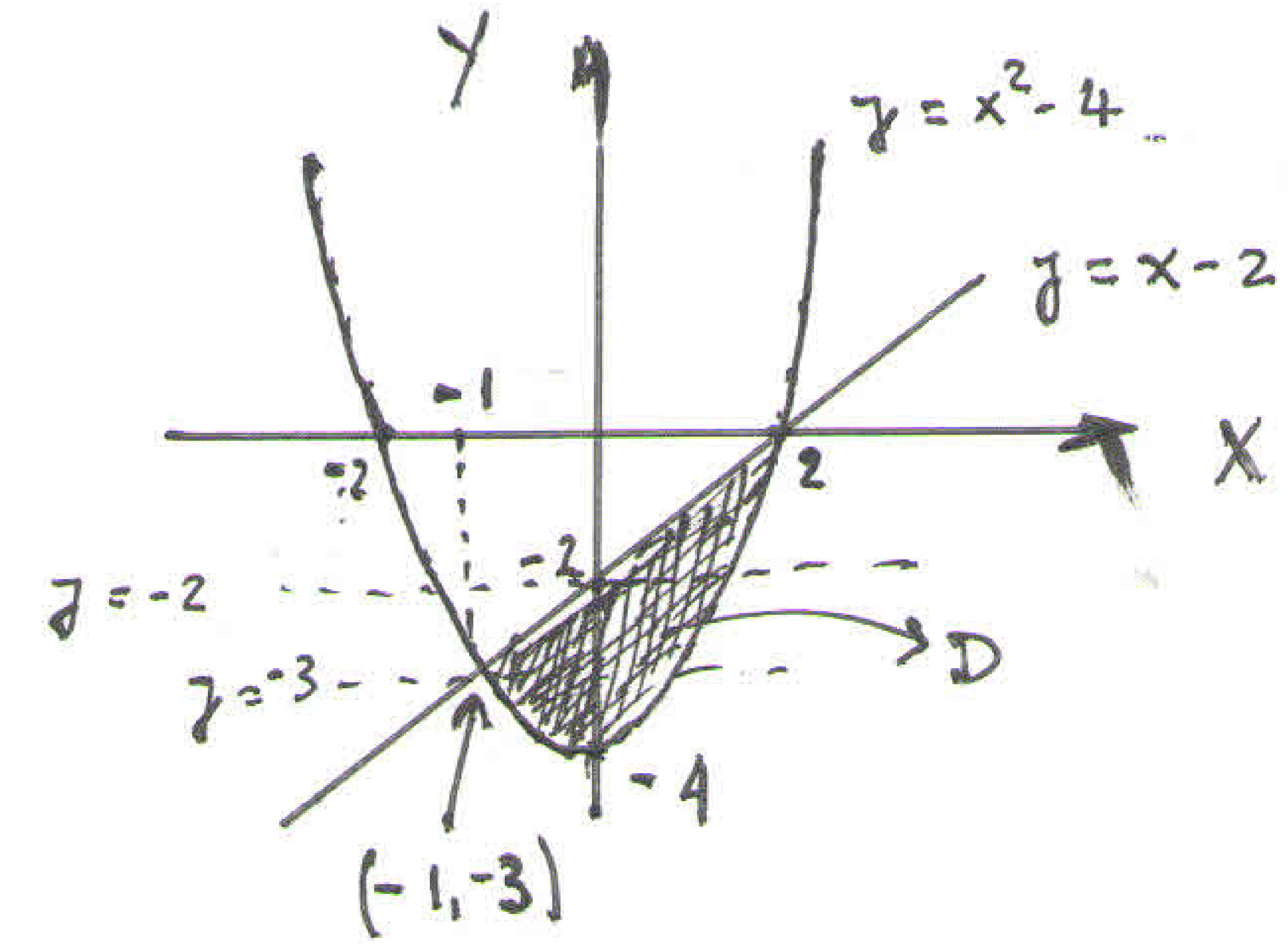
① $I = \int_D f\,dA = \int_{-1}^{2}\int_{x^2-4}^{x-2} f(x,y)\,dy\,dx$

(a) Esboce a região D

(b) Inverta a ordem de integração

Solução:

(a)



$D: \begin{cases} -1 \le x \le 2 \\ x^2 - 4 \le y \le x - 2 \end{cases}$

(b)

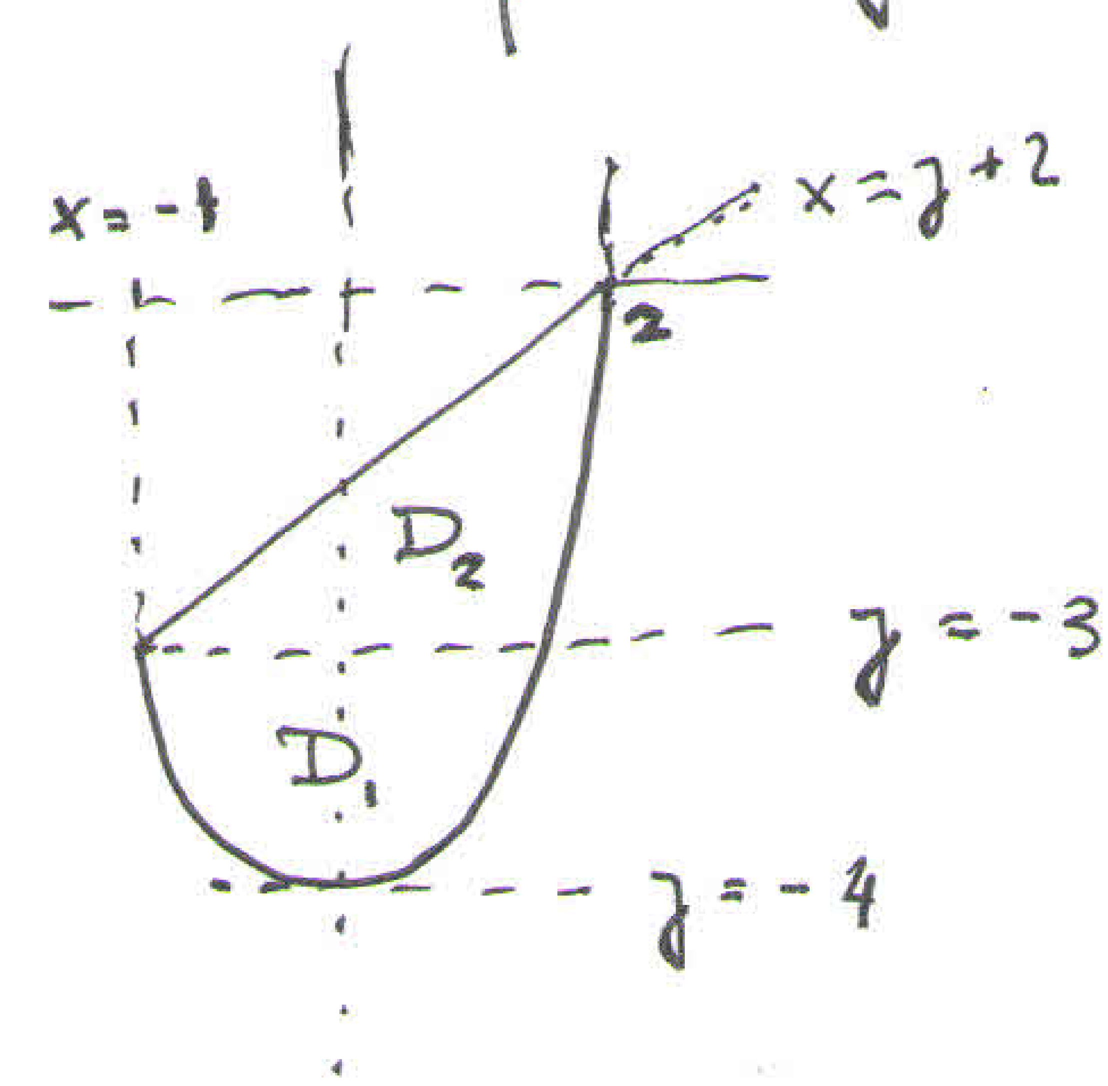
$x - 2 = x^2 - 4$

$x^2 - x - 2 = 0$

$(x+1)(x-2) = 0$

$x = -1, \quad x = 2$

D



$D = D_1 \cup D_2$

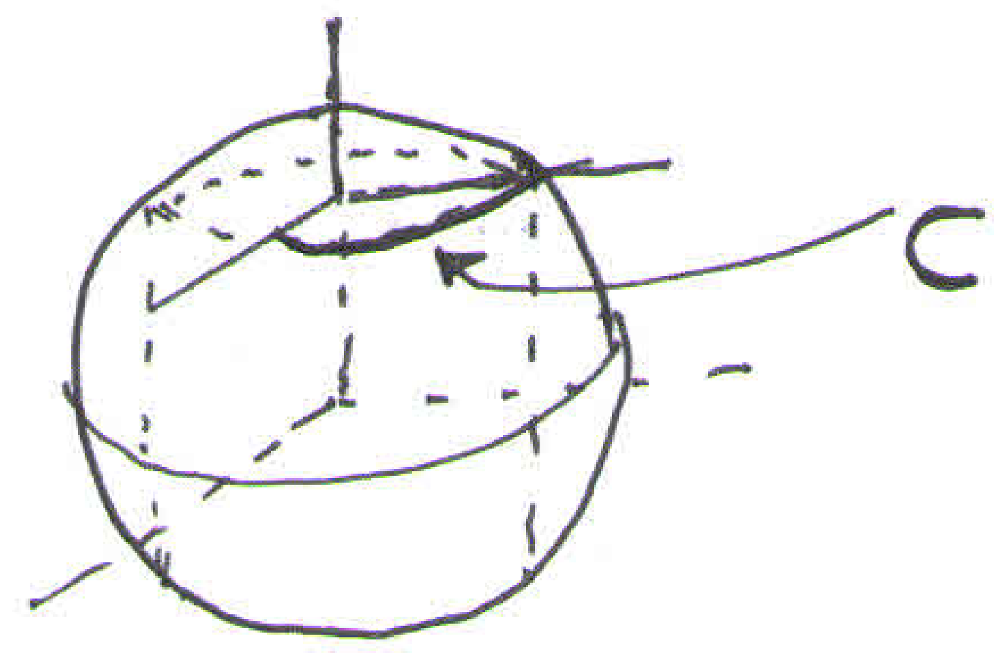
$D_1: \begin{cases} -4 \le y \le -3 \\ -\sqrt{y+4} \le x \le \sqrt{y+4} \end{cases}$ $\qquad D_2: \begin{cases} -3 \le y \le 0 \\ y+2 \le x \le \sqrt{y+4} \end{cases}$

$$I = \int_{-4}^{-3}\int_{-\sqrt{y+4}}^{\sqrt{y+4}} f(x,y)\,dx\,dy + \int_{-3}^{0}\int_{y+2}^{\sqrt{y+4}} f(x,y)\,dx\,dy$$

(2) C : curva interseção da esfera $x^2+y^2+z^2=16$ com o cilindro $x^2+y^2=4$ no 1º octante.

Calcular o momento de inércia de C em relação ao eixo Z. A densidade em cada pto $(x,y,z) \in C$ é dada por $\delta(x,y,z) = xy$.

Solução :



$$C : \begin{cases} z = 2\sqrt{3} \\ x^2+y^2 = 4 \\ x \geq 0, \ y \geq 0 \end{cases}$$

Uma parametrização para C é $\gamma : [0, \pi/2] \longrightarrow \mathbb{R}^3$

$$\gamma(t) = (2\cos t, 2\,\text{sen}\, t, 2\sqrt{3}).$$

$$\gamma'(t) = (-2\,\text{sen}\, t, 2\cos t, 0), \quad \|\gamma'(t)\| = 2$$

$$I_z = \int_C (x^2+y^2)\,xy\,ds = \int_0^{2\pi} \underbrace{4}_{x^2+y^2} \cdot \underbrace{4\,\text{sen}\, t \cos t}_{2xy} \cdot \underbrace{2}_{\|\gamma'\|}\, dt$$

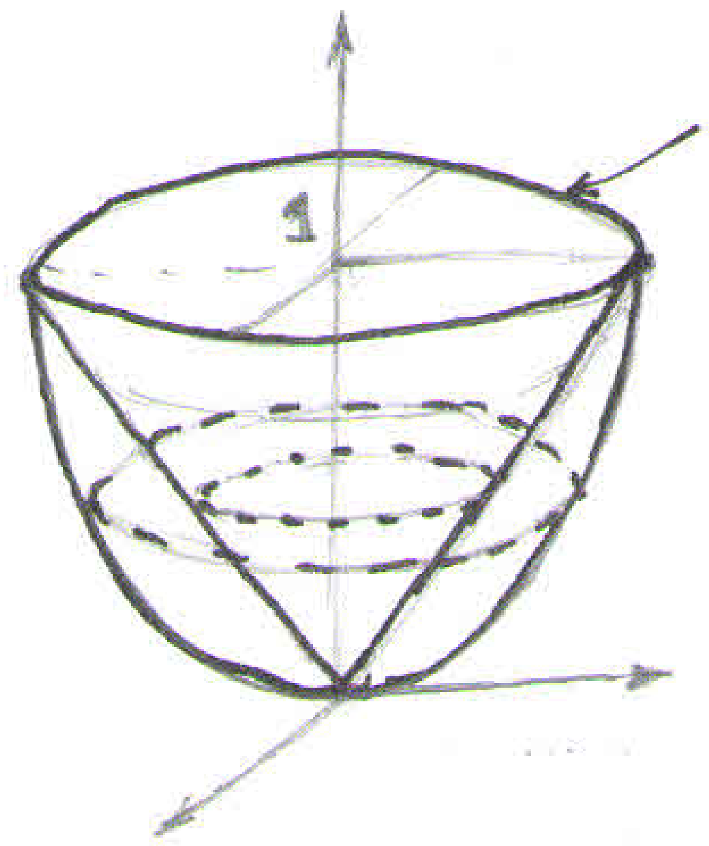
$$= 32 \int_0^{\pi/2} \text{sen}\, t \cos t \, dt = 32 \cdot \frac{\text{sen}^2 t}{2} \Big|_0^{\pi/2}$$

$$= \frac{32}{2}(1-0) = 16$$

(3) Calcular o volume de W.

W limitado por cima por $z = \sqrt{x^2+y^2}$ e por baixo por $z = x^2+y^2$

Solução:



$$W: \begin{cases} (x,y) \in D \\ x^2+y^2 \leq z \leq \sqrt{x^2+y^2} \end{cases}$$

$$D: \{ x^2+y^2 \leq 1 \qquad D_{r\theta}: \begin{cases} 0 \leq r \leq 1 \\ 0 \leq \theta \leq 2\pi \end{cases}$$

$$\text{vol}(W) = \int\int_D \int_{x^2+y^2}^{\sqrt{x^2+y^2}} dz\, dA \qquad \begin{aligned} z &= z \\ x &= r\cos\theta \\ y &= r\,\text{sen}\theta \\ x^2+y^2 &= r^2, \quad J = r \end{aligned}$$

$$= \int_0^1 \int_0^{2\pi} \int_{r^2}^{r} r\, dz\, d\theta\, dr$$

$$= \int_0^1 \int_0^{2\pi} r(r-r^2)\, d\theta\, dr = 2\pi \int_0^1 (r^2-r^3)\, dr$$

$$= 2\pi \left(\frac{1}{3}r^3 - \frac{1}{4}r^4 \right)\Big|_0^1 = 2\pi\left(\frac{1}{3}-\frac{1}{4}\right)$$

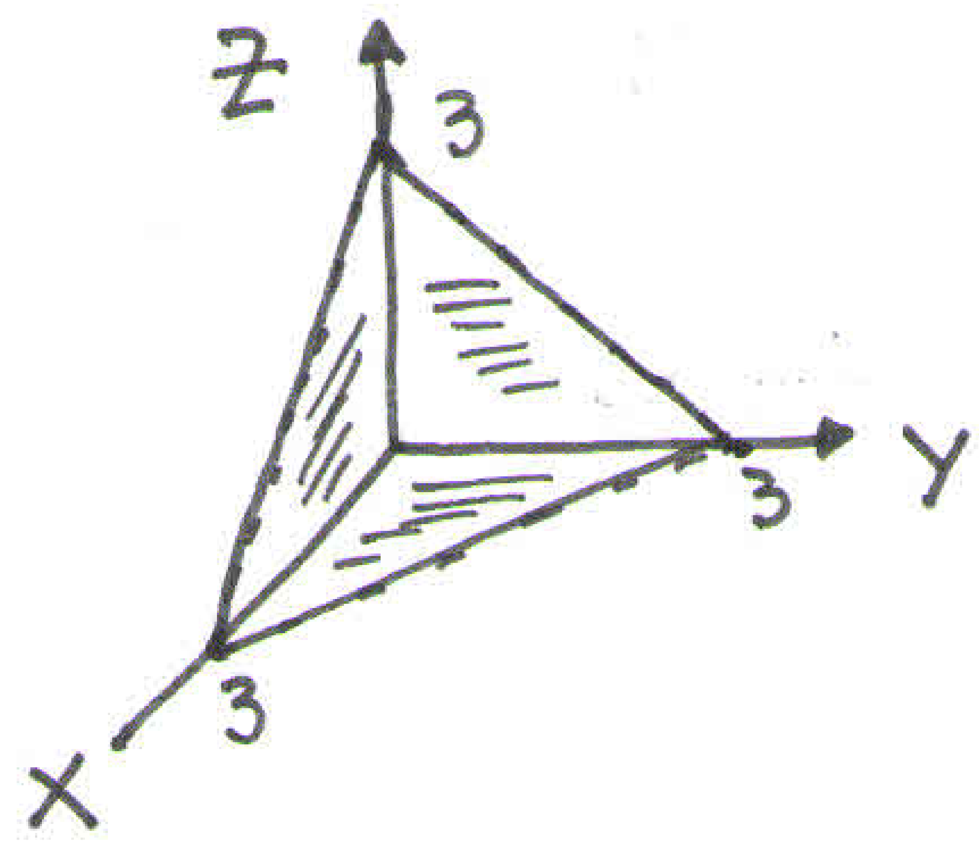
$$= \pi/6$$

④ Calcule $\int_W f\,dV$.

(a) $f(x,y,z) = x - y$.

W limitado pelos planos coord. e pelo plano $x+y+z=3$

Solução



$$W: \begin{cases} 0 \leq x \leq 3 \\ 0 \leq y \leq 3-x \\ 0 \leq z \leq 3-x-y \end{cases}$$

$$\int_W f\,dV = \int_0^3 \int_0^{3-x} \int_0^{3-x-y} (x-y)\,dz\,dy\,dx$$

$$= \int_0^3 \int_0^{3-x} (x-y)(3-x-y)\,dy\,dx = \int_0^3 \int_0^{3-x} (3x - 3y - x^2 + y^2)\,dy\,dx$$

$$= \int_0^3 \left(3xy - \frac{3}{2}y^2 - x^2 y + \frac{1}{3}y^3\right)\Big|_{y=0}^{y=3-x} dx$$

$$= \int_0^3 \left[3x(3-x) - \frac{3}{2}(3-x)^2 - x^2(3-x) + \frac{1}{3}(3-x)^3\right] dx$$

$$= \int_0^3 (3-x)\left[3x - \frac{3}{2}(3-x) - x^2 + \frac{1}{3}(3-x)^2\right] dx$$

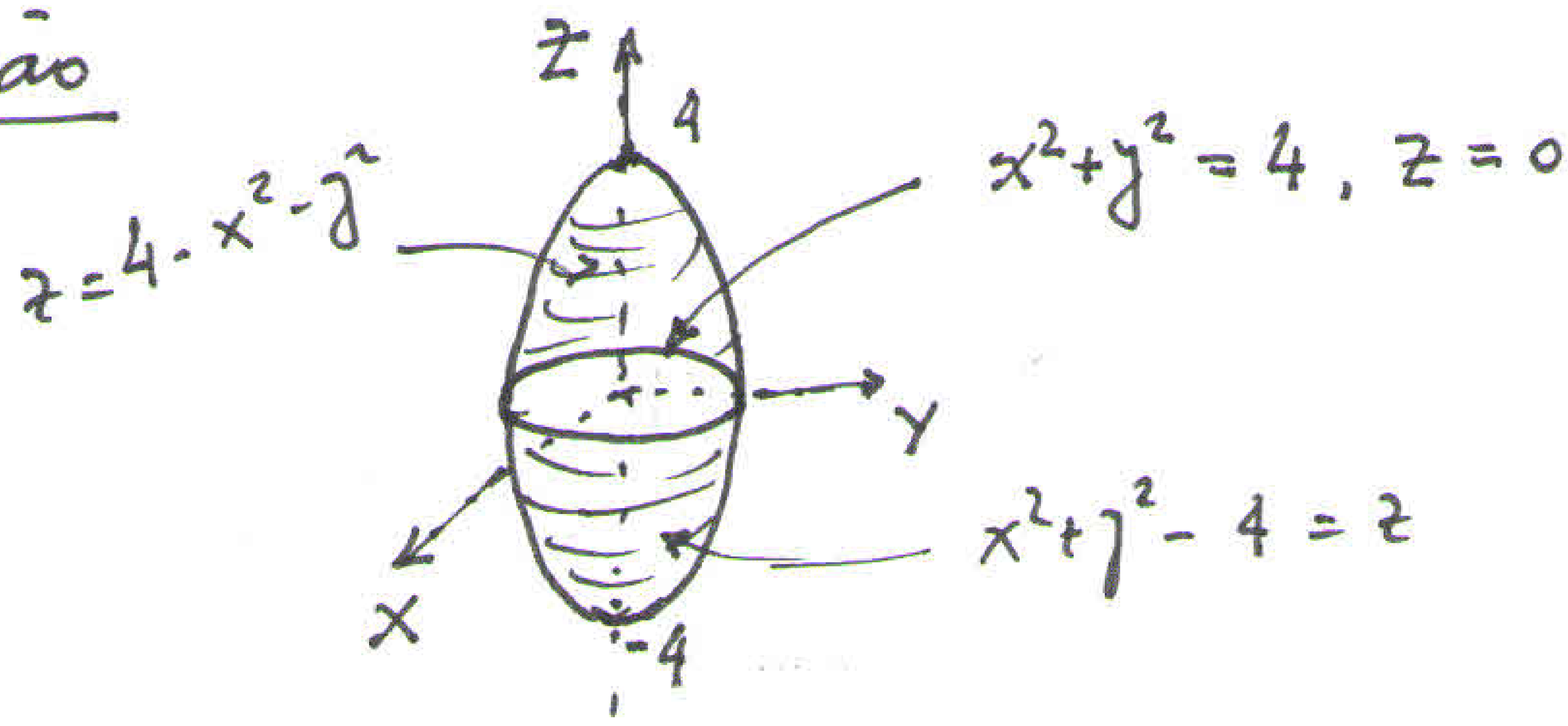
$$= \int_0^3 \left(-\frac{9}{2}x^2 + \frac{2}{3}x^3 + 9x - \frac{9}{2}\right) dx$$

$$= \left(-\frac{9}{6}x^3 + \frac{2}{12}x^4 + \frac{9}{2}x^2 - \frac{9}{2}x\right)\Big|_0^3 = \dots = \underline{\underline{0}}$$

(b) $f(x,y,z) = \sqrt{x^2+y^2}$

W limitado por $z = x^2+y^2-4$ e $z = 4-x^2-y^2$

<u>Solução</u>



W em coord. cilíndricas : $0 \le r \le 2$

$0 \le \theta \le 2\pi$

$x^2+y^2-4 \le z \le 4-x^2-y^2$

$$\begin{cases} x = r\cos\theta \\ y = r\,\text{sen}\,\theta \\ z = z \end{cases}$$

$x^2+y^2 = r^2$

$dx\,dy\,dz = r\,dz\,dr\,d\theta$

$$\int_W f\,dV = \int_0^2 \int_0^{2\pi} \int_{r^2-4}^{4-r^2} r \cdot r\,dz\,d\theta\,dr$$

$$\int_W f\,dV = \int_0^2 \int_0^{2\pi} r^2\left(4-r^2-(r^2-4)\right)d\theta\,dr$$

$$= \int_0^2 \int_0^{2\pi} r^2(8-2r^2)\,d\theta\,dr = 4\pi\int_0^2 r^2(4-r^2)\,dr$$

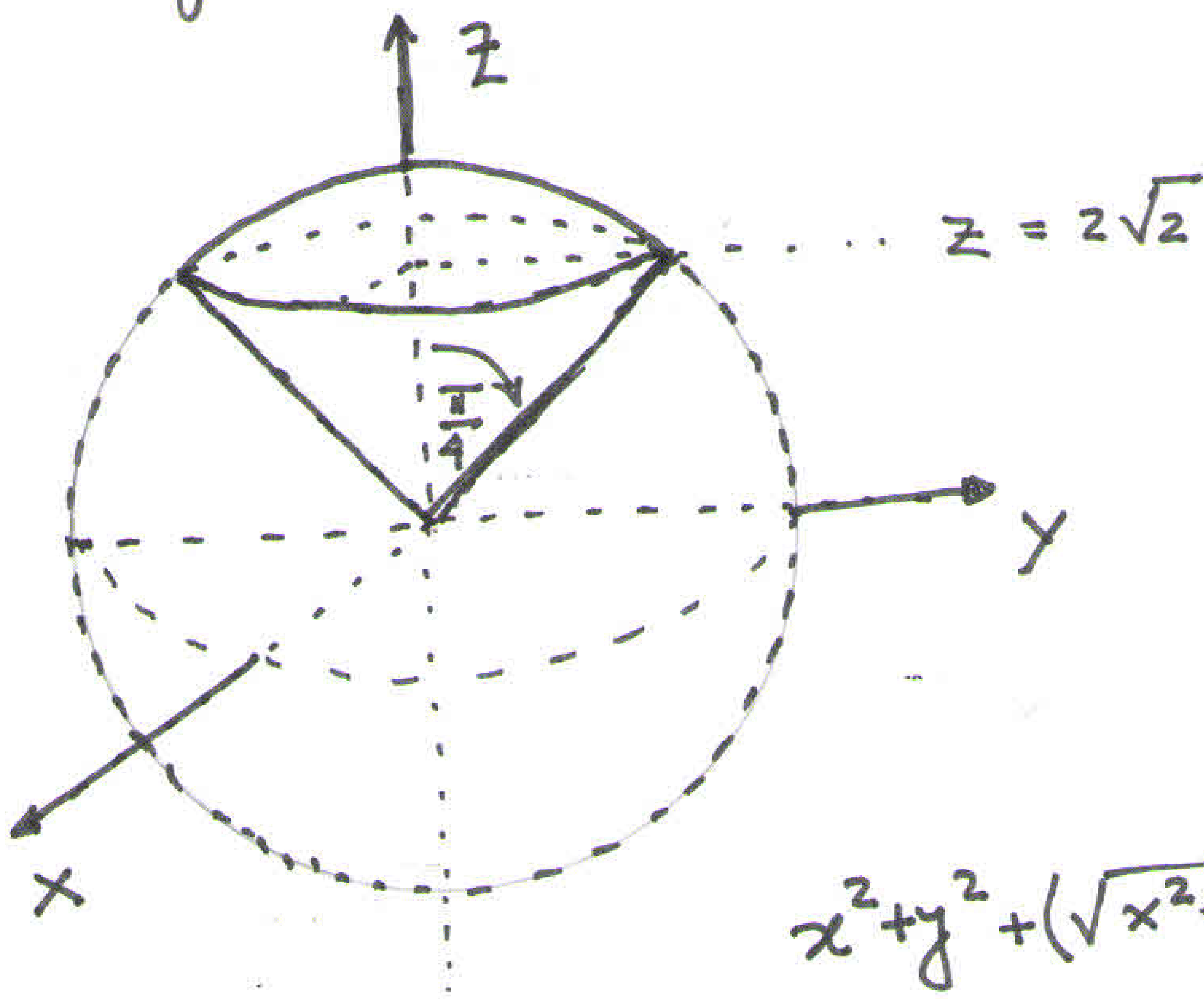
$$= 4\pi\left(\frac{4}{3}r^3 - \frac{1}{5}r^5\right)\Big|_0^2 = 4\pi\left(\frac{4\cdot 8}{3} - \frac{1\cdot 32}{5}\right)$$

$$= 4\pi\cdot\frac{64}{15} = \frac{256}{15}\pi$$

(c) $f(x,y,z) = x^2+y^2+z^2$

W limitado superiormente por $x^2+y^2+z^2=16$ e inferiormente por $z=\sqrt{x^2+y^2}$

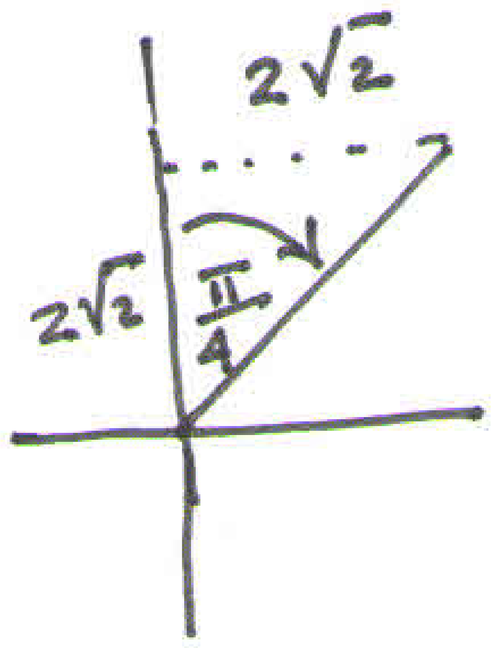
Solução :



$$x^2+y^2+\left(\sqrt{x^2+y^2}\right)^2=16$$

$$\Rightarrow \quad x^2+y^2=8$$

$$8+z^2=16 \Rightarrow z=\pm\sqrt{8}=\pm 2\sqrt{2}$$



W em coord. esféricas

$$W: \begin{cases} 0 \leq \rho \leq 4 \\ 0 \leq \theta \leq 2\pi \\ 0 \leq \phi \leq \pi/4 \end{cases}$$

$$x = \rho \operatorname{sen}\phi \cos\theta$$
$$y = \rho \operatorname{sen}\phi \operatorname{sen}\theta$$
$$z = \rho\cos\phi$$
$$x^2+y^2+z^2=\rho^2$$
$$J = \rho^2 \operatorname{sen}\phi .$$

$$\int_W f\,dV = \int_0^4\int_0^{2\pi}\int_0^{\pi/4} \rho^2 . \rho^2 \operatorname{sen}\phi \, d\phi\, d\theta\, d\rho$$

$$= \int_0^4\int_0^{2\pi} -\rho^4\cos\phi \Big|_0^{\pi/4} d\theta\, d\rho = \left(1-\frac{\sqrt{2}}{2}\right)\int_0^4\int_0^{2\pi}\rho^4\, d\theta\, d\rho$$

$$= 2\pi\left(1-\frac{\sqrt{2}}{2}\right)\int_0^4 \rho^4\, d\rho = 2\pi\left(1-\frac{\sqrt{2}}{2}\right)\frac{1}{5}\rho^5\Big|_0^4$$

$$= \frac{2}{5}\pi\left(1-\frac{\sqrt{2}}{2}\right) . 4^5 = \frac{2048}{5}\pi\left(1-\frac{\sqrt{2}}{2}\right)$$