Prova 1 2013.1 Calculo III

(1) $$W: \begin{cases} 0 \leq z \leq h & h = \text{cte.} > 0 \\ a \leq x^2+y^2 \leq b, & b > a > 0 \end{cases}$$

$$I_z = \iiint_W (x^2+y^2) \cdot K \, dx\,dy\,dz$$

Em coord. cilíndricas W é descrito como:

$$W: \begin{cases} \sqrt{a} \leq r \leq \sqrt{b} \\ 0 \leq \theta \leq 2\pi \\ 0 \leq z \leq h. \end{cases}$$

$dx\,dy\,dz = r\,dr\,d\theta\,dz$

$$I_z = \int_0^h \int_0^{2\pi} \int_{\sqrt{a}}^{\sqrt{b}} K \cdot r^2 \cdot r\,dr\,d\theta\,dz$$

$$= \int_0^h \int_0^{2\pi} K \left. \frac{r^4}{4} \right|_{\sqrt{a}}^{\sqrt{b}} d\theta\,dz = \frac{K}{4}(b^2-a^2) \int_0^h \int_0^{2\pi} d\theta\,dz$$

$$= \frac{K\,h}{4} \cdot 2\pi\,(b^2-a^2)$$

$$= \frac{K\,h}{2} \pi\,(b^2-a^2)$$

(2)

(a) $\iiint_W \frac{1}{z^2}\, dx\,dy\,dz$.

W: $z=\sqrt{x^2+y^2}$, $z=\sqrt{1-x^2-y^2}$, $z=\sqrt{4-x^2-y^2}$

Em coord esféricas W é dado por

$$W: \begin{cases} 0 \leq \theta \leq 2\pi \\ 1 \leq \rho \leq 2 \\ 0 \leq \phi \leq \pi/4 \end{cases}$$

$$\begin{cases} x = \rho \cos\theta \operatorname{sen}\phi \\ y = \rho \operatorname{sen}\theta \operatorname{sen}\phi \\ z = \rho \cos\phi \end{cases}$$

$$\iiint_W \frac{1}{z^2}\, dx\,dy\,dz = \int_0^{2\pi}\int_1^2\int_0^{\pi/4} \frac{1}{\cancel{\rho^2}\cos^2\phi}\, \cancel{\rho^2} \operatorname{sen}\phi \, d\phi\, d\rho\, d\theta$$

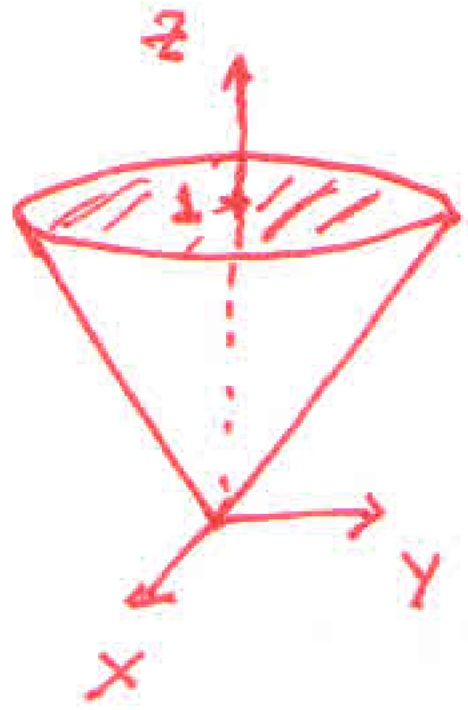
$$= \int_0^{2\pi}\int_1^2\int_0^{\pi/4} \underbrace{\frac{\operatorname{sen}\phi}{\cos\phi}}_{\operatorname{tg}\phi}\, \underbrace{\frac{1}{\cos\phi}}_{\sec\phi}\, d\phi\, d\rho\, d\theta$$

$$= \int_0^{2\pi}\int_1^2\int_0^{\pi/4} \sec\phi \cdot \operatorname{tg}\phi \, d\phi\, d\rho\, d\theta = \int_0^{2\pi}\int_1^2 \sec\phi \Big|_0^{\pi/4} d\rho\, d\theta$$

$$= \int_0^{2\pi}\int_1^2 (\sqrt{2}-1)\, d\rho\, d\theta = 2\pi(\sqrt{2}-1)$$

② (b) $\iiint_W \left(1+\sqrt{x^2+y^2}\right)dx\,dy\,dz$

$$W: \left\{ z=\sqrt{x^2+y^2} \ , \quad z=1 \right.$$



W em coord. cilíndricas:

$$W: \begin{cases} 0 \leq r \leq 1 \\ 0 \leq \theta \leq 2\pi \\ r \leq z \leq 1 \end{cases} \qquad \begin{array}{l} x = r\cos\theta \\ y = r\,\mathrm{sen}\,\theta \\ z = z \\ x^2+y^2 = r^2 \end{array}$$

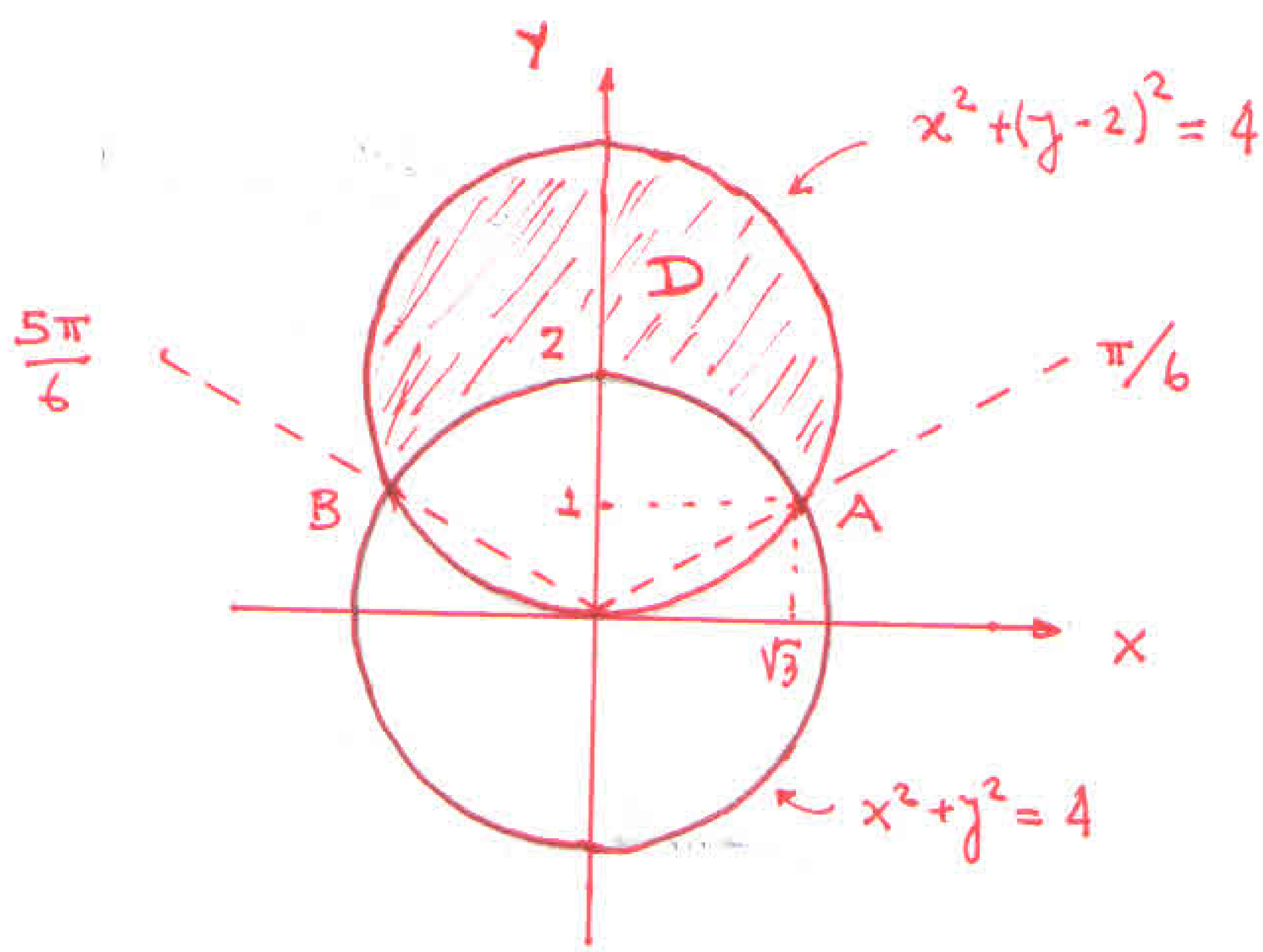
$$\iiint_W \left(1+\sqrt{x^2+y^2}\right)dx\,dy\,dz = \int_0^1\int_0^{2\pi}\int_r^1 (1+r)\cdot r\,dz\,d\theta\,dr$$

$$= \int_0^1\int_0^{2\pi} (1+r)\cdot r(1-r)\,d\theta\,dr = 2\pi\int_0^1 (r-r^3)\,dr$$

$$= 2\pi\left(\frac{r^2}{2}-\frac{r^4}{4}\right)\Big|_0^1 = 2\pi\left(\frac{1}{2}-\frac{1}{4}\right)$$

$$= \pi/2$$

3

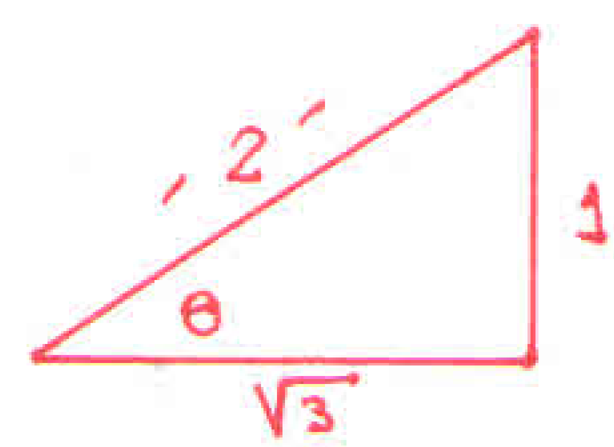


$A = (\sqrt{3}, 1)$

$B = (-\sqrt{3}, 1)$

$$\left.\begin{array}{l} x^2+y^2=4 \\ x^2+(y-2)^2=4 \end{array}\right\} \Rightarrow y=1$$

$y=1 \Rightarrow x = \pm\sqrt{3}$



$tg\,\theta = \frac{1}{\sqrt{3}}$

$sen\,\theta = 1/2$

$\cos\theta = \frac{\sqrt{3}}{2}$

$\theta = 30° = \pi/6$

$\delta(x,y) = \sqrt{x^2+y^2}$

$$M = \text{Massa} = \iint_D \delta(x,y)\,dx\,dy = \iint_D \sqrt{x^2+y^2}\,dx\,dy$$

Em coord. polares

$$x = r\cos\theta \qquad y = r\,sen\,\theta \qquad dx\,dy = r\,dr\,d\theta$$

$$\left\{\begin{array}{l} x^2+y^2=4 \Rightarrow r=2 \\ x^2+(y-2)^2=4 \Rightarrow r = 4\,sen\,\theta \end{array}\right.$$

D em coord. polares:

$$D: \left\{\begin{array}{l} \frac{\pi}{6} \leq \theta \leq 5\pi/6 \\ 2 \leq r \leq 4\,sen\,\theta \end{array}\right.$$

$$M = \int_{\pi/6}^{5\pi/6} \int_{2}^{4\,sen\,\theta} r \cdot r\,dr\,d\theta = \int_{\pi/6}^{5\pi/6} \left.\frac{r^3}{3}\right|_{2}^{4\,sen\,\theta} d\theta .$$

$$M = \int_{\pi/6}^{5\pi/6} \left(\frac{4^3 \, sen^3\theta}{3} - \frac{2^3}{3} \right) d\theta$$

$$= -\frac{8}{3} \cdot \frac{\cancel{4}^{2}\pi}{\cancel{6}_{3}} + \frac{4^3}{3} \int_{\pi/6}^{5\pi/6} sen^3\theta \, d\theta$$

$$= -\frac{16}{9}\pi + \frac{4^3}{3} \int_{\pi/6}^{5\pi/6} sen^3\theta \, d\theta$$

$$\boxed{\begin{aligned} \int sen^3\theta \, d\theta &= \int sen\theta(1-\cos^2\theta)\,d\theta = \int sen\theta \, d\theta + \int \cos^2\theta(-sen\theta \, d\theta) \\ &= -\cos\theta + \frac{1}{3}\cos^3\theta + C \end{aligned}}$$

$$M = \frac{-16}{9}\pi + \frac{4^3}{3}\left(-\cos\theta \Big|_{\pi/6}^{5\pi/6} + \frac{1}{3}\cos^3\theta \Big|_{\pi/6}^{5\pi/6} \right)$$

$$\boxed{\cos\frac{5\pi}{6} = -\cos\pi/6}$$

$$M = -\frac{16}{9}\pi + \frac{4^3}{3} \cdot 2\cos\pi/6 - \frac{4^3}{3} \cdot \frac{2}{3}\cos^3\pi/6$$

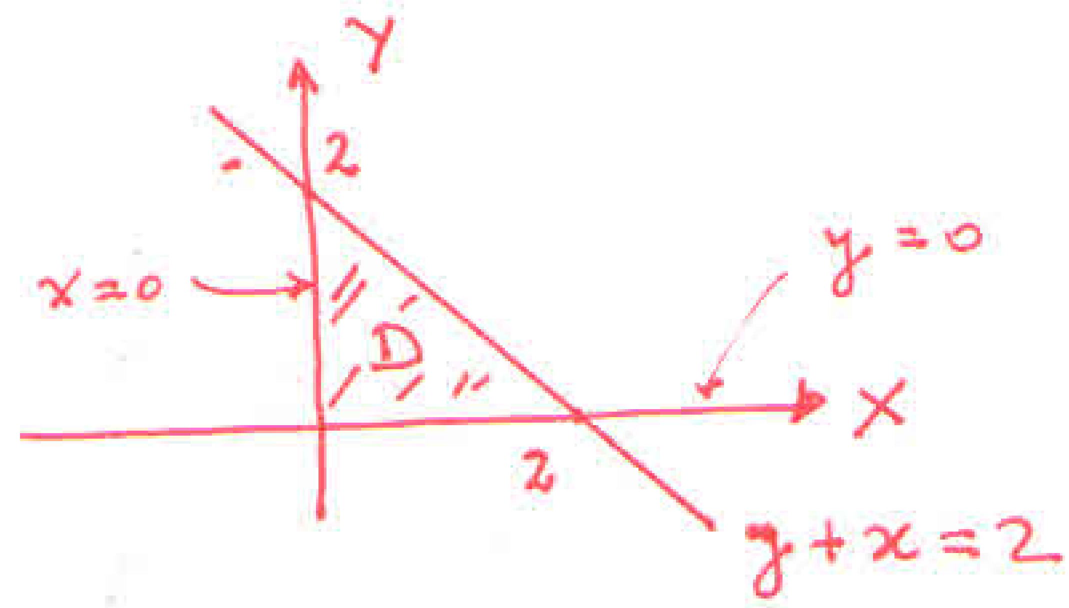
$$\left(\cos\pi/6 = \frac{\sqrt{3}}{2} \right)$$

$$M = \frac{-16}{9}\pi + \frac{4^3}{3} \cdot \cancel{2} \cdot \frac{\sqrt{3}}{\cancel{2}} - \frac{4^3}{3} \cdot \frac{2}{3}\left(\frac{\sqrt{3}}{2}\right)^3$$

$$= \frac{-16}{9}\pi + \frac{56}{3}\sqrt{3}$$

④ $\displaystyle\iint_D e^{\frac{y-x}{x+y}}\,dx\,dy$

D :

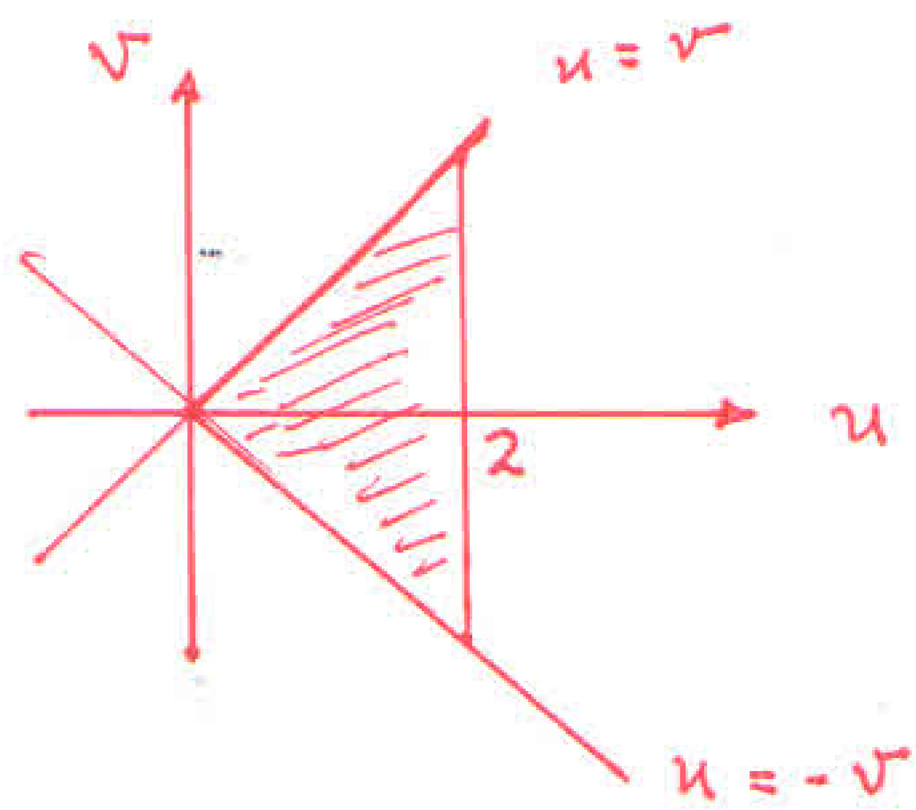


Faça $\begin{cases} u = x+y \\ v = y-x \end{cases}$

$x=0 \Rightarrow u=v$

$y=0 \Rightarrow u=-v$

$y+x=2 \Rightarrow u=2$



$$\iint_D e^{\frac{y-x}{x+y}}\,dx\,dy = \int_0^2\int_{-u}^{u} e^{v/u}\cdot|J|\,dv\,du$$

$$J^{-1} = \det\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = 2 \Rightarrow \underline{J = 1/2}$$

$$\int_0^2\int_{-u}^{u} 1/2\, e^{v/u}\,dv\,du = 1/2\int_0^2 u\, e^{v/u}\Big|_{v=-u}^{v=u} du$$

$$= 1/2\int_0^2 u\,(e - e^{-1})\,du = 1/2\left(e - \tfrac{1}{e}\right)\int_0^2 u\,du$$

$$= 1/2\,(e - 1/e)\,\frac{u^2}{2}\Big|_0^2$$

$$= \underline{\underline{e - 1/e}}$$