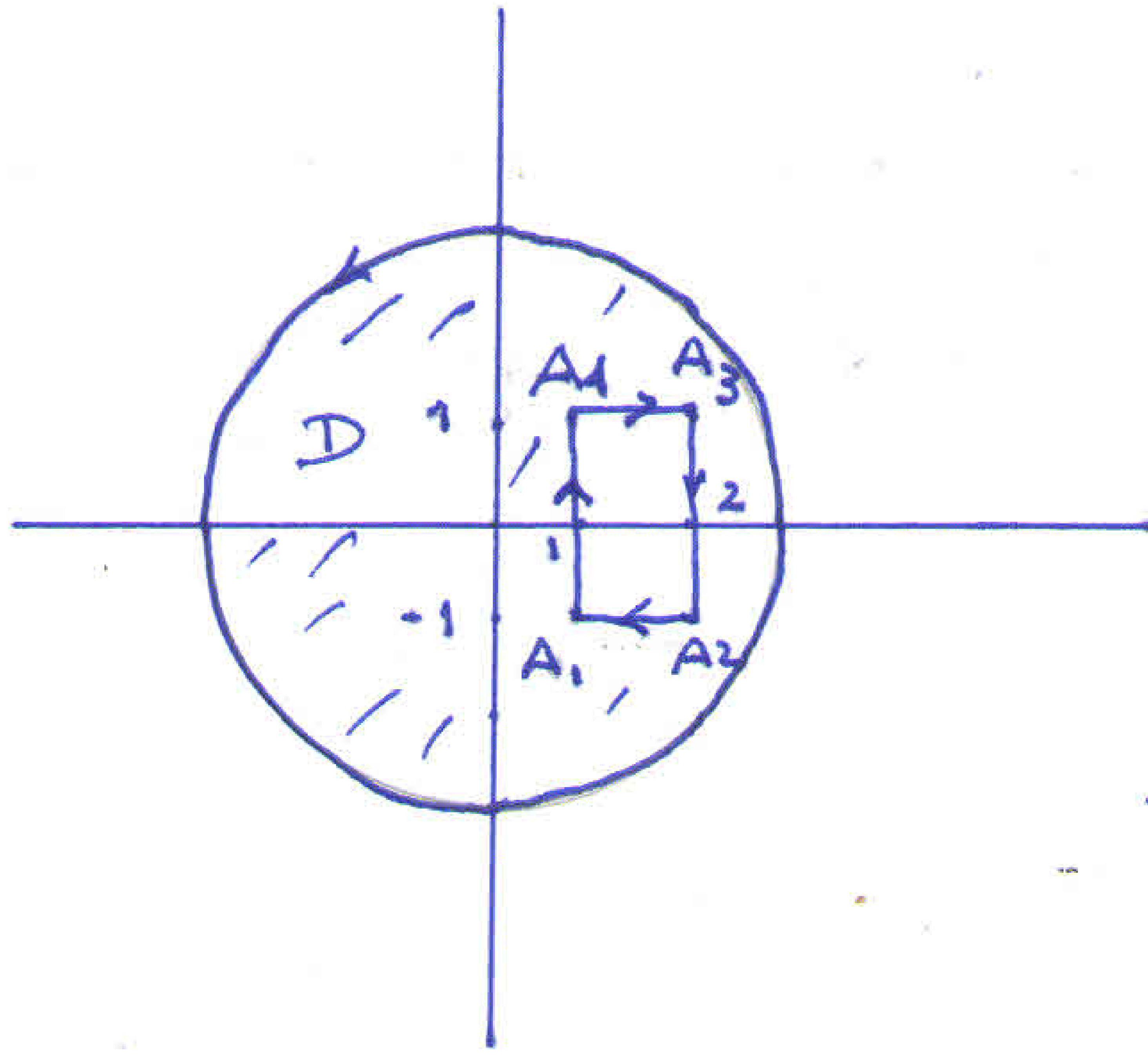
# Cálculo III 2ª Prova 2014.2 (Tarde)

(1)



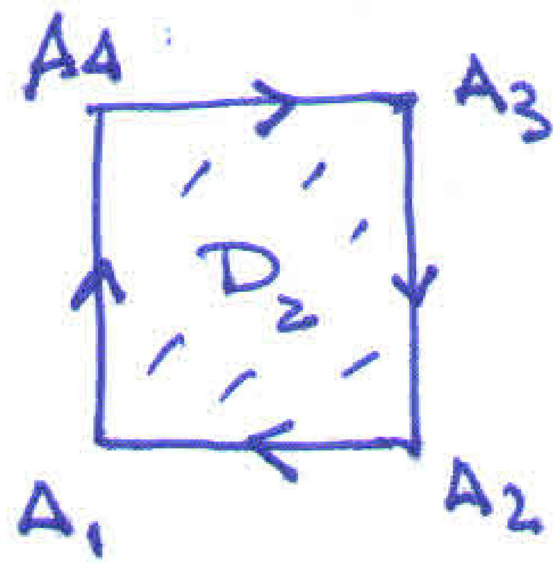

Calcular

$$\int_{\partial D} (2x - y^3)\,dx - xy\,dy$$

Sejam: $C_1^+ : x^2 + y^2 = 9$

$D_1 : x^2 + y^2 \leq 9$ $\qquad \partial D_1 = C_1^+$

$C_2^-$ :




, $D_2$ t.q. $\partial D_2 = C_2^+$

Pelo teor. de Green temos:

$$\int_{C_1^+} (2x - y^3)\,dx - xy\,dy = \iint_{D_1} (-y + 3y^2)\,dx\,dy$$

Em coord. polares, $D_1 : \begin{cases} 0 \leq r \leq 3 \\ 0 \leq \theta \leq 2\pi \end{cases}$ $\qquad x = r\cos\theta$, $y = r\,\mathrm{sen}\,\theta$, $dx\,dy = r\,dr\,d\theta$

$$\iint_{D_1} (3y^2 - y)\,dx\,dy = \int_0^{2\pi}\int_0^3 (3r^2\mathrm{sen}^2\theta - r\,\mathrm{sen}\,\theta)\,r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^3 (3r^3\mathrm{sen}^2\theta - r^2\mathrm{sen}\,\theta)\,dr\,d\theta$$

$$= \int_0^{2\pi} \left( \frac{3r^4}{4} \operatorname{sen}^2\theta - \frac{r^3}{3} \operatorname{sen}\theta \right) \Big|_{r=0}^{r=3} d\theta$$

$$= \int_0^{2\pi} \left( \frac{3 \cdot 3^4}{4} \operatorname{sen}^2\theta - 9 \operatorname{sen}\theta \right) d\theta = \frac{3^5}{4} \int_0^{2\pi} \operatorname{sen}^2\theta \, d\theta - 9 \underbrace{\int_0^{2\pi} \operatorname{sen}\theta \, d\theta}_{0}$$

$$= \frac{3^5}{4} \int_0^{2\pi} \left( \frac{1}{2} - \frac{\cos 2\theta}{2} \right) d\theta$$

$$\boxed{\begin{aligned} \cos 2\theta &= \cos^2\theta - \operatorname{sen}^2\theta \\ \cos 2\theta &= 1 - 2\operatorname{sen}^2\theta \\ \operatorname{sen}^2\theta &= \frac{1}{2} - \frac{1}{2}\cos 2\theta \end{aligned}}$$

$$= \frac{3^5}{4} \cdot \frac{1}{\cancel{2}} \cdot \cancel{2}\pi = \frac{3^5}{4}\pi$$

$$\therefore \boxed{\int_{C_1^+} (2x - y^3)\,dx - xy\,dy = \frac{3^5}{4}\pi}$$

De novo, pelo Teor de Green,

$$\int_{C_2^+} (2x - y^3)\,dx - xy\,dy = \int_{D_2} (3y^2 - y)\,dA$$

$$= \int_{-1}^{1} \int_1^2 (3y^2 - y)\,dx\,dy = \int_{-1}^{1} (3y^2 - y)\, x \Big|_{x=1}^{x=2} dy$$

$$= \int_{-1}^{1} (3y^2 - y)\,dy = \left( y^3 - \frac{1}{2}y^2 \right)\Big|_{-1}^{1} = \left(1 - \tfrac{1}{2}\right) - \left(-1 - \tfrac{1}{2}\right)$$

$$= \tfrac{1}{2} + \tfrac{3}{2} = 2.$$

$$\therefore \boxed{\int_{C_2^+} (2x - y^3)\,dx - xy\,dy = 2}$$

Agora

$$\int_{\partial D} (2x-y^3)\,dx - xy\,dy = \int_{C_1^+} (2x-y^3)\,dx - xy\,dy + \int_{C_2^-} (2x-y^3)\,dx - xy\,dy$$

$$= \frac{3^5}{4}\pi - 2$$

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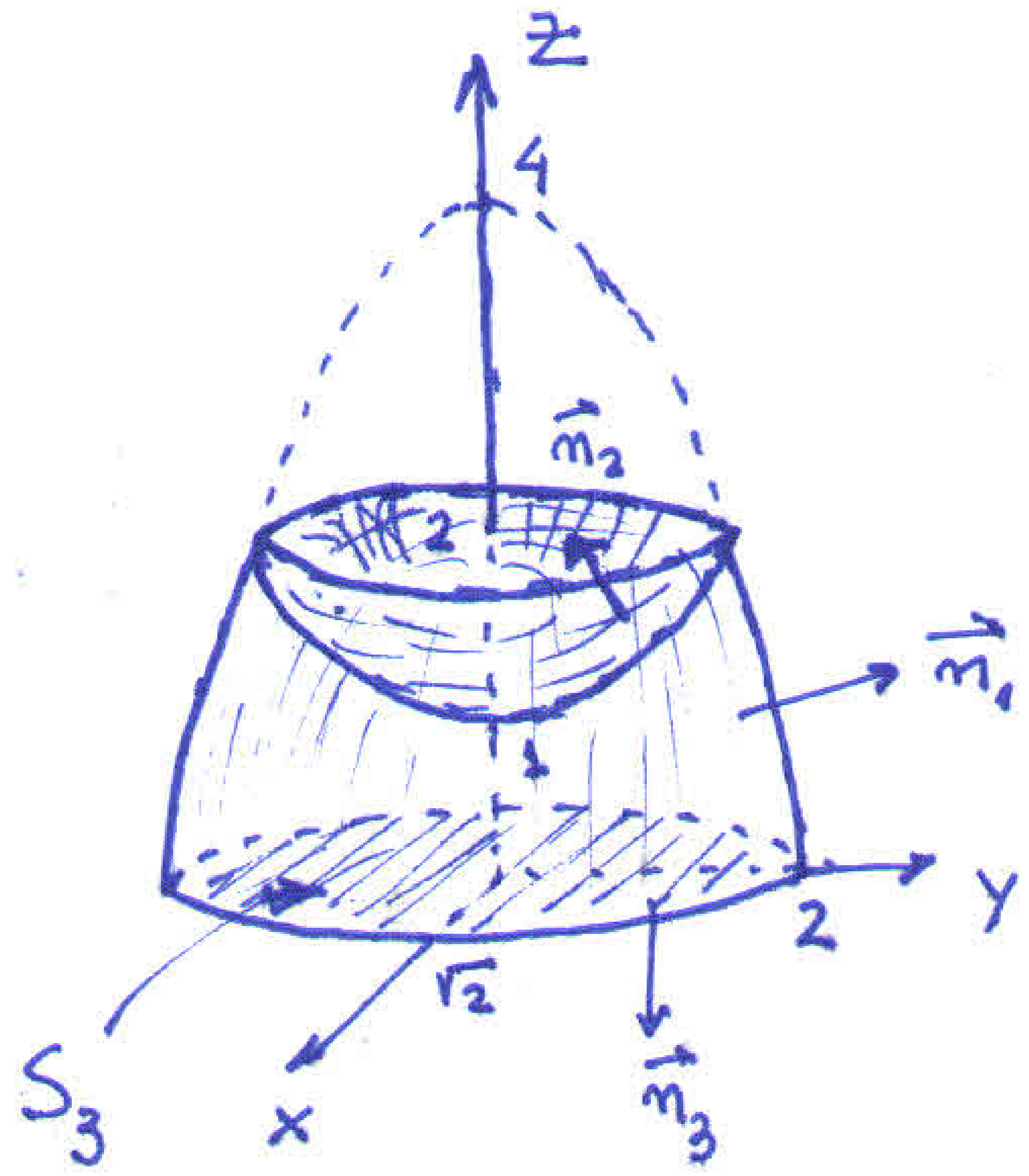
(2) Calcular $\displaystyle\int_S \operatorname{rot}(\vec{F})\cdot\vec{n}\,dS$, onde

$$\vec{F}(x,y,z) = (-y + z e^x)\,\vec{i} + (x + \cos yz)\,\vec{j} + xy\,\vec{k}.$$

$S = S_1 \cup S_2$, $\quad S_1 : \begin{cases} z = 4 - 2x^2 - y^2 \\ 0 \leq z \leq 2 \end{cases}$

$S_2 : \begin{cases} z = 1 + x^2 + \frac{1}{2}y^2 \\ 1 \leq z \leq 2 \end{cases}$

Solução




seja $S_3 : \begin{cases} 2x^2 + y^2 \leq 4 \\ z = 0 \end{cases}$ e seja W o solido limitado

por $S \cup S_3$ $\quad (\partial W = S \cup S_3 = S_1 \cup S_2 \cup S_3)$

Em $S_3$ consideramos a orientação dada por $\vec{n}_3 = -\vec{k}$.

Pelo Teor. de Gauss,

$$\int_{\partial W = S \cup S_3} \operatorname{rot}(\vec{F}) \cdot \vec{n}\, ds = \int_W \operatorname{div}(\operatorname{rot} \vec{F})\, dV$$

e como $\operatorname{div}(\operatorname{rot} \vec{F}) = 0$, temos

$$\int_{S \cup S_3} \operatorname{rot}(\vec{F}) \cdot \vec{n}\, ds = 0$$

$$\therefore \int_S \operatorname{rot}(\vec{F}) \cdot \vec{n}\, ds = -\int_{S_3} \operatorname{rot}(\vec{F}) \cdot \vec{n}_3\, ds .$$

$$\operatorname{rot}(\vec{F}) = \det \begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -y + ze^x & x + \cos yz & xy \end{pmatrix}$$

$$= (x + y \operatorname{sen} yz)\vec{i} + (e^x - y)\vec{j} + 2\vec{k}$$

$$\operatorname{rot}(\vec{F}) \cdot \vec{n}_3 = -2$$

$$\therefore -\int_{S_3} \operatorname{rot}(\vec{F}) \cdot n_3\, ds = -\int_{S_3} -2\, ds = 2\int_{S_3} ds = 2\, \text{área}(S_3)$$

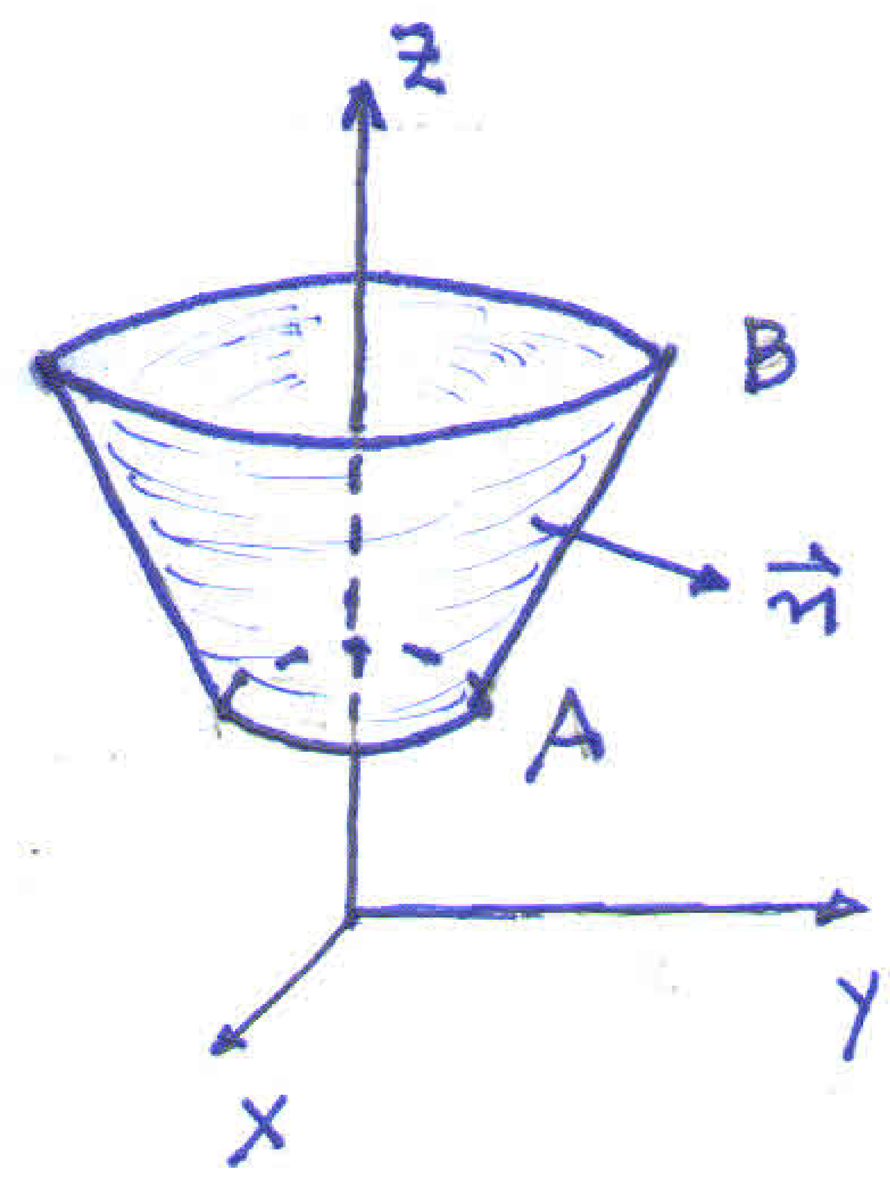
$$= 2 \cdot \sqrt{2} \cdot 2 \cdot \pi = \underline{4\sqrt{2}\,\pi}$$

Logo
$$\int_S \operatorname{rot}(\vec{F}) \cdot \vec{n}\, ds = 4\sqrt{2}\,\pi$$

③ Calcule $\displaystyle\int_S \vec{F} \cdot \vec{n}\, dS$ , onde:

$\vec{F}(x,y,z) = x\vec{i} + y\vec{j} - \dfrac{z^2}{2}\vec{k}$ e $S$ é a superf. de revolução obtida ao girar o segmento $\overline{AB}$ , $A=(0,1,2)$, $B=(0,2,4)$ em torno do eixo $z$ com normal $\vec{n}$ exterior a $S$.

Solução




$A=(0,1,2)$

$B=(0,2,4)$

Uma parametrização para o segmento $AB$ é

$$\vec{r}(t) = A + t(B-A), \quad 0 \leq t \leq 1$$
$$= (0, 1+t, 2+2t).$$

Assim, uma parametrização para $S$ é:

$$\begin{cases} x = (1+t)\cos\theta \\ y = (1+t)\operatorname{sen}\theta \\ z = 2+2t \end{cases} \quad \text{com } (t,\theta) \in D: \begin{cases} 0 \leq t \leq 1 \\ 0 \leq \theta \leq 2\pi. \end{cases}$$

Um vetor normal a $S$ é

$$\vec{N} = (\cos\theta, \operatorname{sen}\theta, 2) \times (-(1+t)\operatorname{sen}\theta, (1+t)\cos\theta, 0)$$
$$= (1+t)(-2\cos\theta, -2\operatorname{sen}\theta, 1)$$

Como $\vec{n}$ aponta para fora, $\vec{n} = \dfrac{-\vec{N}}{\|\vec{N}\|}$

Então,

$$\int_S \vec{F}\cdot\vec{n}\,dS = \iint_D \left((1+t)\cos\theta, (1+t)\,\mathrm{sen}\,\theta, -\frac{(2+2t)^2}{2}\right)\cdot(1+t)\left(2\cos\theta, 2\,\mathrm{sen}\,\theta, -1\right)dt\,d\theta.$$

$$= \iint_D \left[2(1+t)^2\cos^2\theta + 2(1+t)^2\,\mathrm{sen}^2\theta + \frac{4}{2}(1+t)^3\right]dt\,d\theta$$

$$= \int_D \left(2(1+t)^2 + 2(1+t)^3\right)dt\,d\theta$$

$$= 2\int_0^1\int_0^{2\pi}\left[(1+t)^2 + (1+t)^3\right]d\theta\,dt$$

$$= 4\pi\int_0^1\left[(1+t)^2+(1+t)^3\right]dt = 4\pi\left[\frac{(1+t)^3}{3}+\frac{(1+t)^4}{4}\right]_0^1$$

$$= 4\pi\left(\frac{2^3}{3}+\frac{2^4}{4}-\left(\frac{1}{3}+\frac{1}{4}\right)\right)$$

$$= \frac{89}{3}\pi$$

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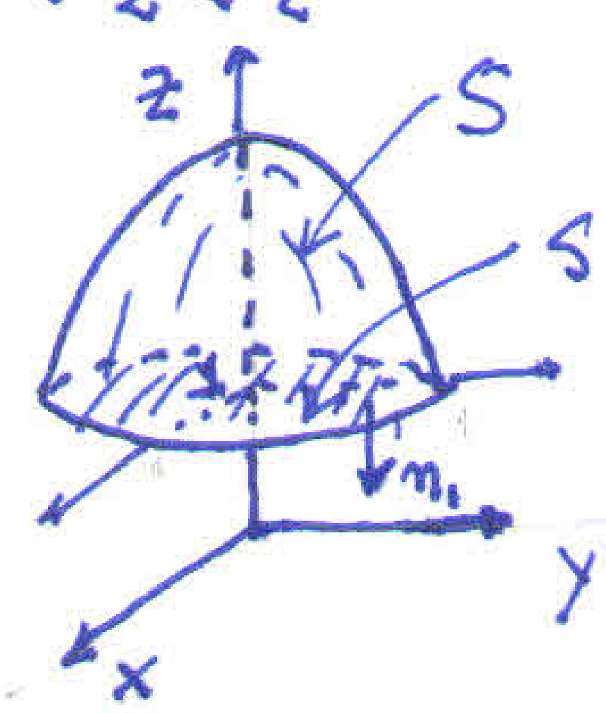
(4) Use o Teor. de Gauss para calcular o fluxo do campo

$$\vec{F} = z\,\mathrm{arctg}(y^2)\,\vec{i} + z^3\ln(x^2+1)\,\vec{j} + z\,\vec{k}$$

através de $S: \begin{cases} z = 2 - x^2 - y^2 \\ 1 < z < 2 \end{cases}$ com $\vec{n}$ exterior.

Solução




Seja $\tilde{S} = S \cup S_1$

$S_1: \begin{cases} x^2+y^2 \le 1 \\ z = 1 \end{cases}$

$S_1$ orientada por $\vec{n}_1 = -\vec{k} = (0,0,-1)$

$$dS_1 = \sqrt{1 + z_x^2 + z_y^2}\, dxdy = dxdy.$$

Seja $W$ o sólido limitado por $\tilde{S}$.

( $\tilde{S} = \partial W$ está orientada positivamente e $\vec{F}$ é de classe $C^1$, então vale o Teor. de Gauss. )

$$\underline{\int_{\tilde{S}} \vec{F} \cdot \vec{n}\, d\tilde{S} = \int_W \operatorname{div} \vec{F}\, dV}$$

$$\operatorname{div}(\vec{F}) = 0 + 0 + 1 = 1$$

Logo $\displaystyle\int_{\tilde{S}} \vec{F} \cdot \vec{n}\, d\tilde{S} = \int_W dV$

$W$ em coord. cilíndricas $\begin{cases} x = r\cos\theta \\ y = r\operatorname{sen}\theta \\ z = z \end{cases}$ $\quad x^2+y^2 = r^2$, $\quad dV = r\,dr\,d\theta\,dz.$

$$W_{r,\theta,z} : \begin{cases} 0 \leq r \leq 1 \\ 0 \leq \theta \leq 2\pi \\ 1 \leq z \leq 2 - r^2 \end{cases}$$

Então

$$\int_W dV = \int_0^1 \int_0^{2\pi} \int_1^{2-r^2} r\, dz\, d\theta\, dr = \int_0^1 \int_0^{2\pi} r(2 - r^2 - 1)\, d\theta\, dr$$

$$= \int_0^1 \int_0^{2\pi} r(1 - r^2)\, d\theta\, dr = 2\pi \int_0^1 (r - r^3)\, dr$$

$$= 2\pi \left( \frac{1}{2} - \frac{1}{4} \right) = \underline{\pi/2}$$