5. Mudança de variáveis em integrais triplas

Como no caso de integrais duplas temos um resultado análogo para integrais triplas:

$$\int_W f(x,y,z)\,dx\,dy\,dz = \int_{\tilde{W}} f(x(u,v,w),\, y(u,v,w),\, z(u,v,w))\,|J|\,du\,dv\,dw$$

onde

$$J = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \det\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{pmatrix}$$

é o jacobiano da mudança de variáveis

$$T: \begin{cases} x = x(u,v,w) \\ y = y(u,v,w) \\ z = z(u,v,w \end{cases}$$

e

$$W = T(\tilde{W})$$

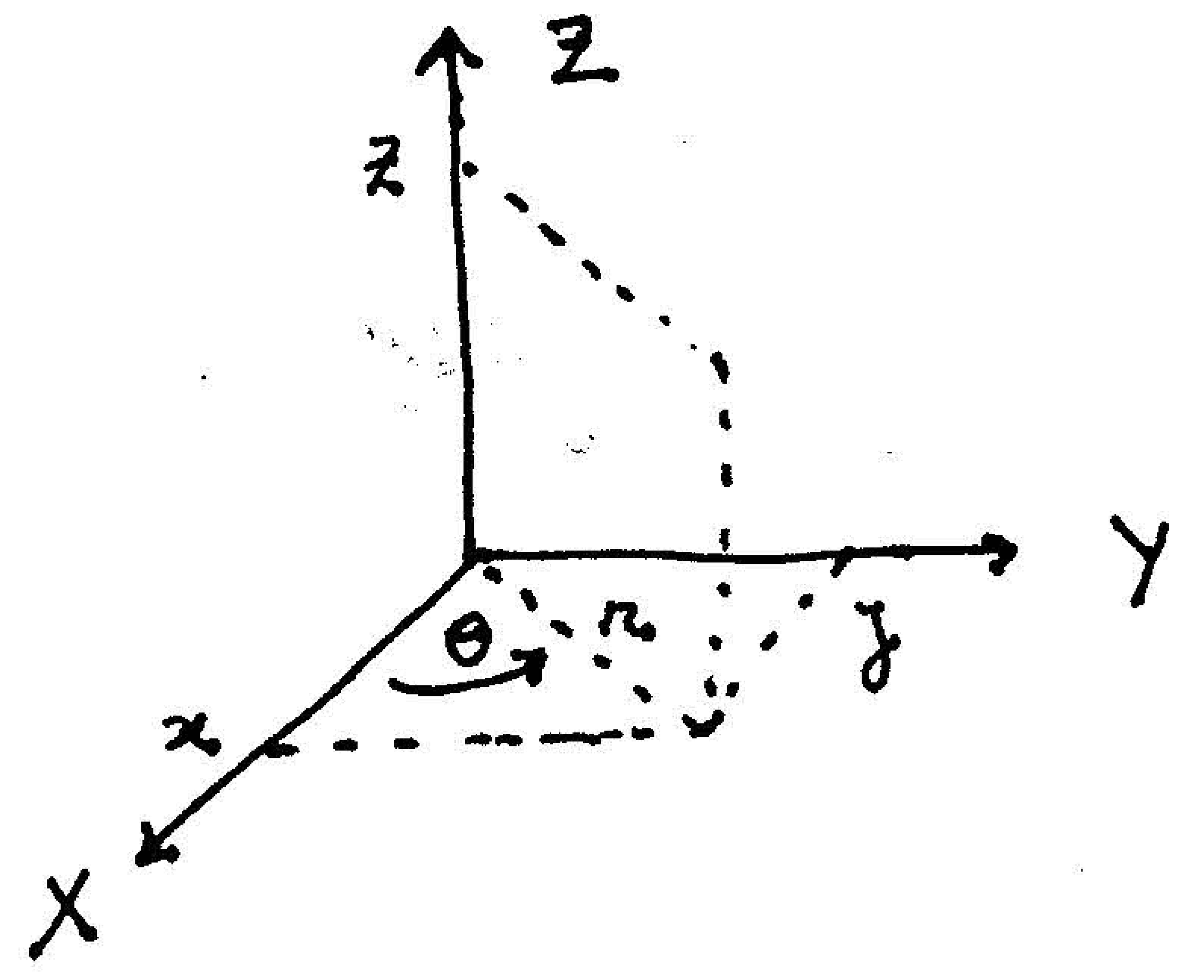
Casos particulares:

5.1 Coordenadas cilíndricas

As coord. cilíndricas de um pto $(x,y,z) \in \mathbb{R}^3$ em coord. cartesianas são (r,θ,z) definidas por

$$\begin{cases} x = r\cos\theta \\ y = r\,\text{sen}\,\theta \\ z = z \end{cases}$$

com $r > 0$ e $\theta_0 < \theta < \theta_0 + 2\pi$ para algum $\theta_0 \in \mathbb{R}$



O jacobiano da transformação é

$$J = \det\begin{pmatrix} \cos\theta & -r\,\text{sen}\,\theta & 0 \\ \text{sen}\,\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} = r$$

Assim, neste caso:

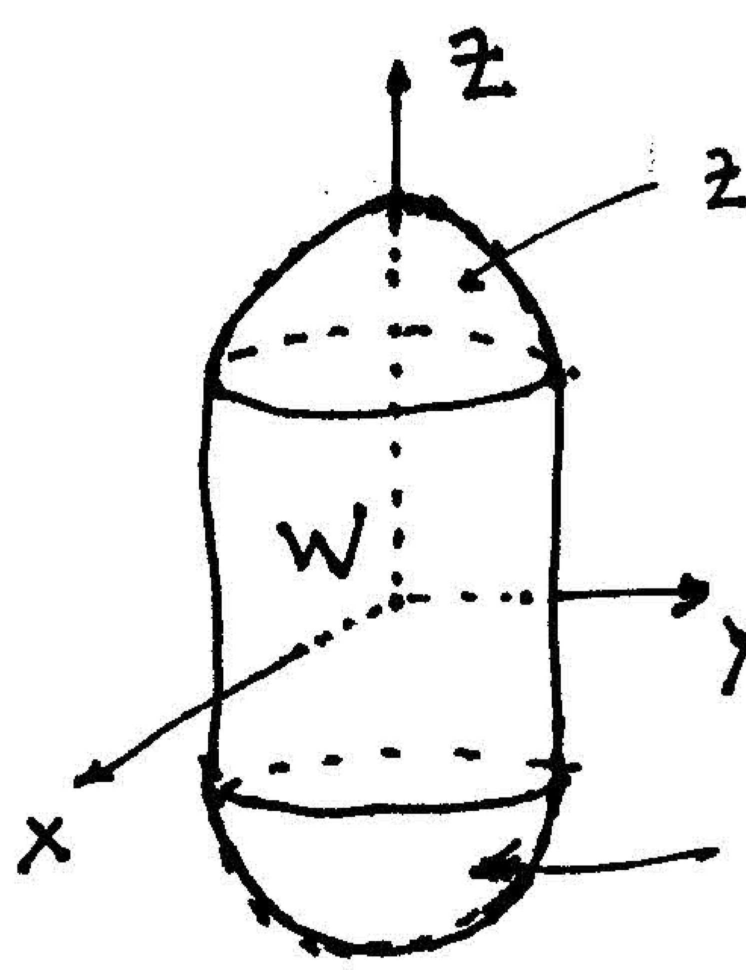
$$\int_W f(x,y,z)\,dx\,dy\,dz = \int_{W_{r\theta z}} f(r\cos\theta, r\,\text{sen}\,\theta, z)\, r\, dr\, d\theta\, dz$$

5.2 <u>Exemplos</u>:

(a) Calcule $\int_W (x^2+y^2)\,dV$ onde W é a região interior ao cilindro $x^2+y^2=1$ e à esfera $x^2+y^2+z^4=4$

<u>Solução</u>:

$$\begin{cases} x = r\cos\theta \\ y = r\,\text{sen}\,\theta \\ z = z \end{cases}$$



$z = \sqrt{4-x^2-y^2} = \sqrt{4-r^2}$

$z = -\sqrt{4-x^2-y^2} = -\sqrt{4-r^2}$

$$W_{r\theta z} : \begin{cases} 0 \leq \theta \leq 2\pi \\ 0 \leq r \leq 1 \\ -\sqrt{4-r^2} \leq z \leq \sqrt{4-r^2} \end{cases}$$

$$\int_W (x^2+y^2)\,dV = \int_{W_{r\theta z}} r^2 \cdot r\,dr\,d\theta\,dz$$

$$= \int_0^1 \int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} \int_0^{2\pi} r^3\,d\theta\,dz\,dr$$

$$= 2\pi \int_0^1 \int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} r^3\,dz\,dr$$

$$= 4\pi \int_0^1 r^3\sqrt{4-r^2}\,dr$$

Fazendo $u = 4-r^2$ Temos $\begin{cases} du = -2r\,dr \\ r^2 = 4-u \end{cases}$

Logo

$$4\pi \int_0^1 r^3\sqrt{4-r^2}\,dr = 4\pi \int_4^3 \sqrt{u}\,.(4-u)\left(\frac{-1}{2}\right)du$$

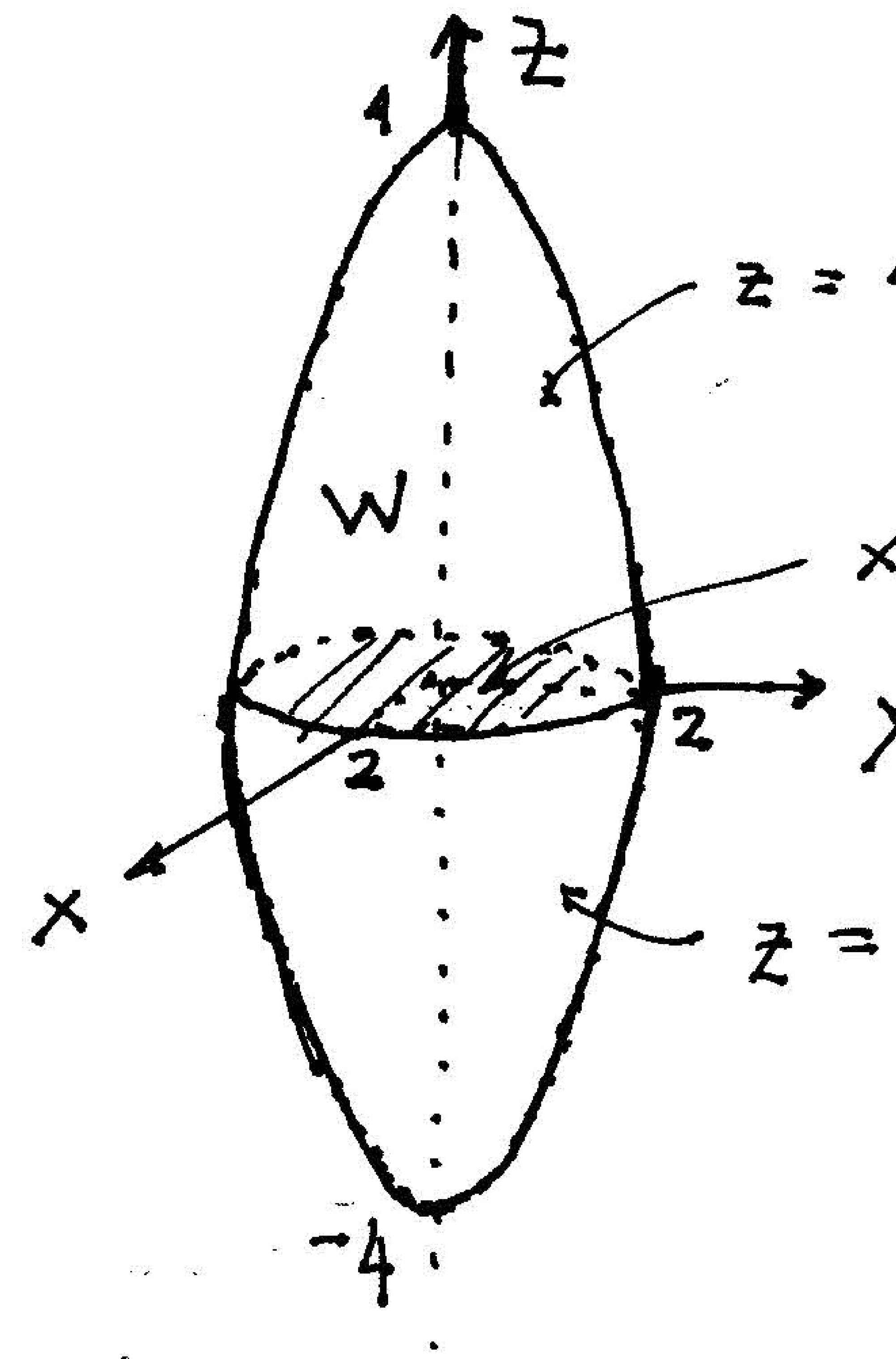
$$= -2\pi \int_4^3 (4-u)\,u^{1/2}\,du = 2\pi \int_3^4 \left(4u^{1/2} - u^{3/2}\right)du$$

$$= 2\pi \cdot \left[4\cdot\frac{2}{3}u^{3/2} - \frac{2}{5}u^{5/2}\right]_3^4$$

$$= \cdots = \pi\left(\frac{256}{15} - \frac{44\sqrt{3}}{5}\right)$$

(b) Calcule $\int_W \sqrt{x^2+y^2}\, dV$, onde W é a região limitada por $z = x^2+y^2-4$ e $z = 4-x^2-y^2$.

<u>solução</u> :



$z = 4-x^2-y^2 = 4-r^2$

$x^2+y^2 \leq 4 \longrightarrow r \leq 2$

$z = x^2+y^2-4 = r^2-4$

coord. cilíndricas

$$\begin{cases} x = r\cos\theta \\ y = r\,\text{sen}\,\theta \\ z = z \end{cases} \qquad x^2+y^2 = r^2 \qquad W_{r\theta z}: \begin{cases} 0 \leq \theta \leq 2\pi \\ 0 \leq r \leq 2 \\ r^2-4 \leq z \leq 4-r^2 \end{cases}$$

$$\int_W \sqrt{x^2+y^2}\, dV = \int_0^2 \int_0^{2\pi} \int_{r^2-4}^{4-r^2} \sqrt{r^2} \cdot r \, dz\, d\theta\, dr$$

$$= \int_0^2 \int_0^{2\pi} r^2 \left(4-r^2 - (r^2-4) \right) d\theta\, dr$$

$$= 2\pi \int_0^2 \left(8r^2 - 2r^4 \right) dr$$

$$= 4\pi \left(\frac{4r^3}{3} - \frac{r^5}{5} \right) \Bigg|_0^2 = \dots\dots$$

$$= \frac{256}{15}\pi$$

5.3. Coordenadas Esféricas

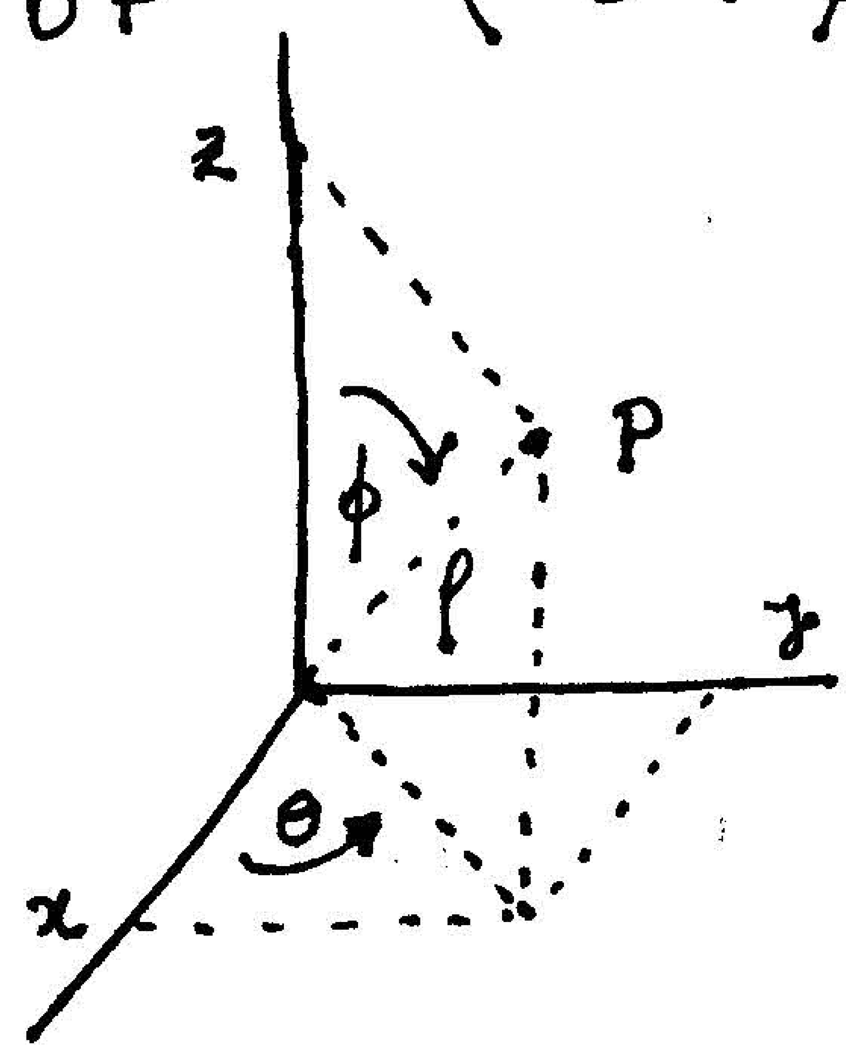
As coord. esféricas de um pto $(x,y,z) \in \mathbb{R}^3$, em coord. cartesianas, são (ρ, ϕ, θ) definidas por:

$$\begin{cases} x = \rho \operatorname{sen}\phi \cos\theta \\ y = \rho \operatorname{sen}\phi \operatorname{sen}\theta \\ z = \rho \cos\phi \end{cases} \quad \text{com} \quad \begin{array}{c} \rho \geq 0 \\ 0 \leq \phi \leq \pi \\ \theta_0 \leq \theta \leq \theta_0 + 2\pi \\ (\theta_0 \in \mathbb{R}) \end{array}$$

ρ mede a distância do ponto P à origem $(\therefore \rho \geq 0)$

θ é como nas coord. cilíndricas, então podemos encontrar sua variação na projeção de W sobre o plano XY.

ϕ é o ângulo entre o eixo positivo Z $(\phi = 0)$ e a semirreta $\overrightarrow{OP}$ $(0 \leq \phi \leq \pi)$



o jacobiano da mudança de coord é

$$J = \frac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)} = \det \begin{pmatrix} \operatorname{sen}\phi\cos\theta & \rho\cos\phi\cos\theta & -\rho\operatorname{sen}\phi\operatorname{sen}\theta \\ \operatorname{sen}\phi\operatorname{sen}\theta & \rho\cos\phi\operatorname{sen}\theta & \rho\operatorname{sen}\phi\cos\theta \\ \cos\phi & -\rho\operatorname{sen}\phi & 0 \end{pmatrix}$$

$$J = \rho^2 \operatorname{sen}^3\phi \operatorname{sen}^2\theta + \rho^2 \operatorname{sen}\phi \cos^2\phi \cos^2\theta + \rho^2 \operatorname{sen}\phi \cos^2\phi \operatorname{sen}^2\theta$$
$$+ \rho^2 \operatorname{sen}^3\phi \cos^2\theta$$

$$= \rho^2 \operatorname{sen}^3\phi \underbrace{\left(\operatorname{sen}^2\theta + \cos^2\theta\right)}_{1} + \rho^2 \operatorname{sen}\phi \cos^2\phi \underbrace{\left(\cos^2\theta + \operatorname{sen}^2\theta\right)}_{1}$$

$$= \rho^2 \operatorname{sen}\phi \left(\operatorname{sen}^2\phi + \cos^2\phi\right)$$

$$= \rho^2 \operatorname{sen}\phi \qquad \therefore \boxed{J = \rho^2 \operatorname{sen}\phi}$$

(Note que $\rho^2 \operatorname{sen}\phi \geq 0$ já que $0 \leq \phi \leq \pi$.)

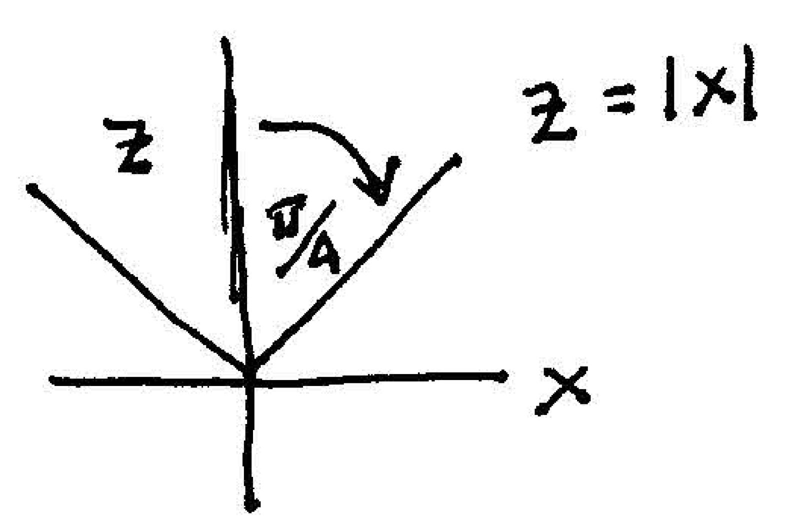
Assim,

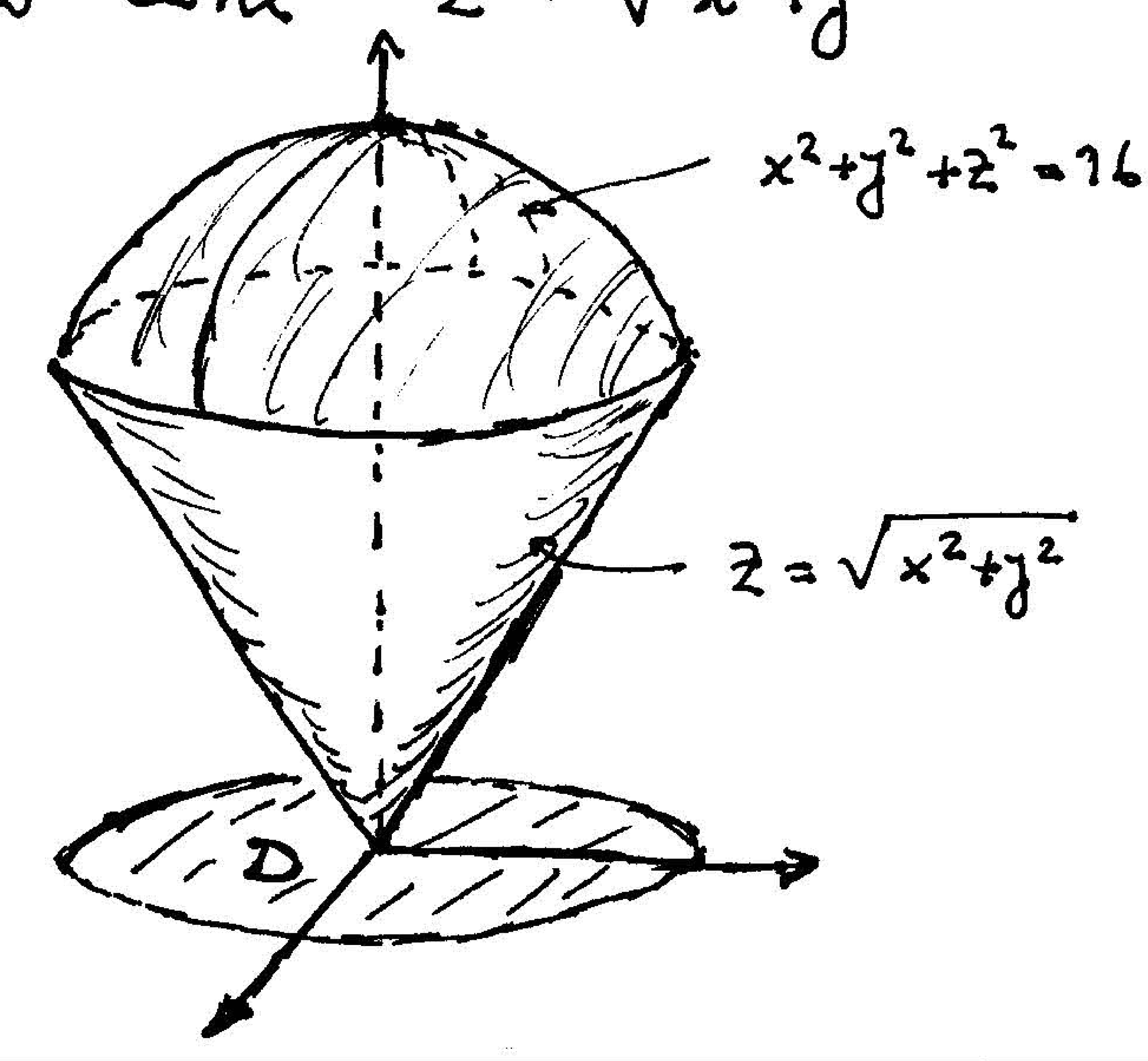
$$\boxed{\int_W f(x,y,z)\,dx\,dy\,dz = \int_{\tilde{W}} f(\rho,\phi,\theta) \cdot \rho^2 \operatorname{sen}\phi \, d\rho \, d\phi \, d\theta}$$

5.4. Exemplos

(a) Calcule $\int_W (x^2+y^2+z^2)\,dV$ sendo W a região limitada superiormente pela esfera $x^2+y^2+z^2=16$ e inferiormente pelo cone $z = \sqrt{x^2+y^2}$

Solução





Em coord esféricas

$$W_{\rho\phi\theta} : \begin{cases} 0 \leq \theta \leq 2\pi \\ 0 \leq \rho \leq 4 \\ 0 \leq \phi \leq \pi/4 \end{cases}$$

Logo,

$$\int_W (x^2+y^2+z^2)\,dV = \int_0^{\pi/4}\int_0^4\int_0^{2\pi} \rho^2 \cdot \rho^2 \operatorname{sen}\phi \, d\theta\, d\rho\, d\phi$$

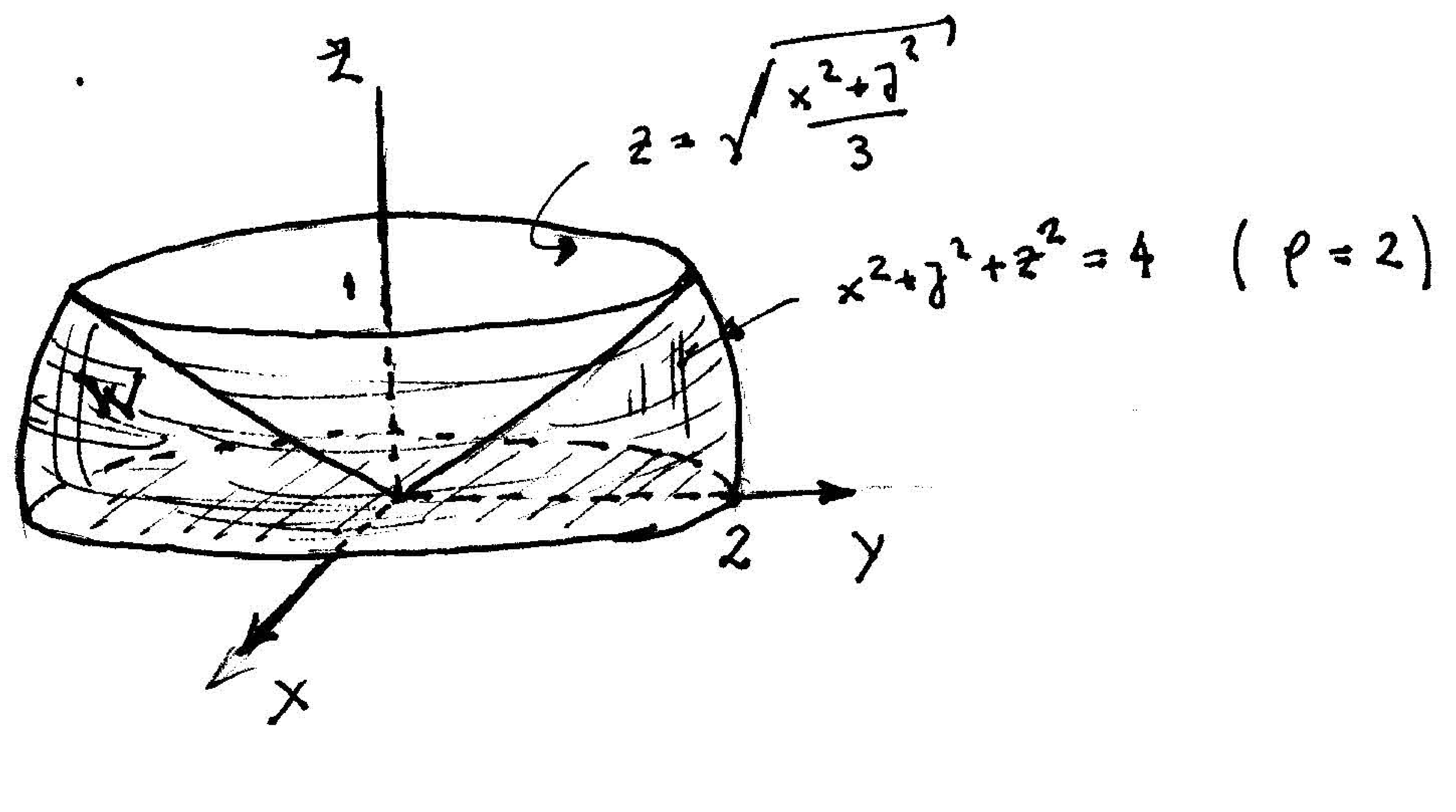
$$= 2\pi \int_0^{\pi/4}\int_0^4 \rho^4 \operatorname{sen}\phi\, d\rho\, d\phi = 2\pi \int_0^{\pi/4} \frac{\rho^5}{5} \operatorname{sen}\phi \Big|_{\rho=0}^{\rho=4} d\phi$$

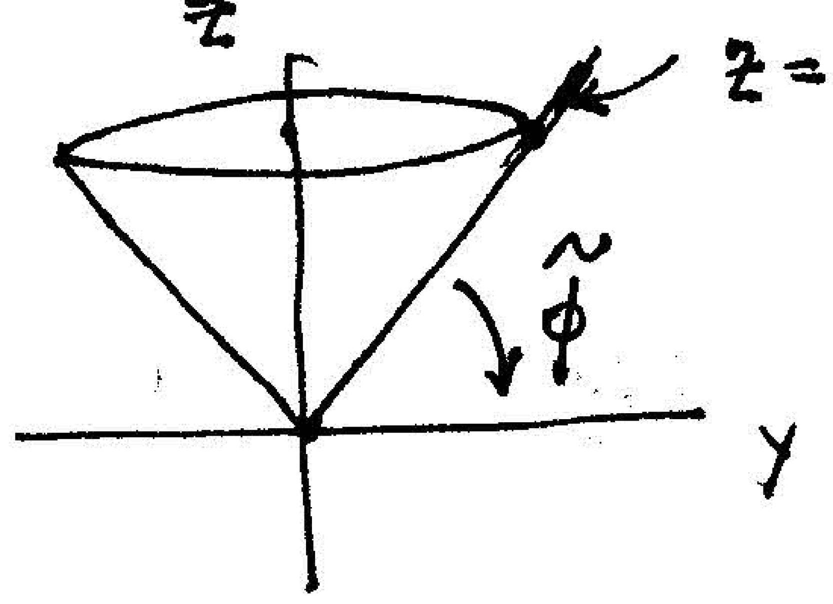
$$= \frac{2\cdot 4^5}{5}\pi \int_0^{\pi/4} \operatorname{sen}\phi\, d\phi = \frac{2\cdot 4^5}{5}\pi \left(-\cos\phi \Big|_0^{\pi/4}\right)$$

$$= \frac{2\cdot 4^5}{5}\pi\left(-\frac{\sqrt{2}}{2}+1\right) = \frac{4^5}{5}\left(2-\sqrt{2}\right)\pi$$

(b) Calcule o volume do sólido W que está dentro da esfera $x^2+y^2+z^2=4$, acima do plano $z=0$ e abaixo do cone $z=\sqrt{\dfrac{x^2+y^2}{3}}$.

Solução





$z = \frac{|y|}{\sqrt{3}}$

$$\operatorname{tg}\tilde{\phi} = \frac{1}{\sqrt{3}} \Rightarrow \tilde{\phi} = \pi/6$$

$$\phi_0 \leq \phi \leq \pi/2$$

$$\phi_0 = \frac{\pi}{2} - \tilde{\phi} = \frac{\pi}{2} - \frac{\pi}{6} = \pi/3$$

$$\underline{\phi_0 = \pi/3}$$

$$W_{\rho\phi\theta} : \begin{cases} 0 \leq \theta \leq 2\pi \\ 0 \leq \rho \leq 2 \\ \frac{\pi}{3} \leq \phi \leq \pi/2 \end{cases}$$

$$\operatorname{vol}(W) = \int_W dV = \int_{W_{\rho\phi\theta}} \rho^2 \operatorname{sen}\phi \, d\rho \, d\theta \, d\phi$$

$$= \int_0^2 \int_{\pi/3}^{\pi/2} \int_0^{2\pi} \rho^2 \operatorname{sen}\phi \, d\theta \, d\phi \, d\rho$$

$$= 2\pi \int_0^2 \int_{\pi/3}^{\pi/2} \rho^2 \operatorname{sen}\phi \, d\phi \, d\rho = 2\pi \int_0^2 \rho^2 \left(-\cos\phi \Big|_{\pi/3}^{\pi/2}\right) d\rho$$

$$= 2\pi \int_0^2 \rho^2 \left(-0 + \frac{1}{2}\right) d\rho = \pi \cdot \frac{1}{3} \rho^3 \Big|_0^2$$

$$= \frac{8}{3}\pi \quad u.v.$$