QUALITATIVE PROPERTIES OF SADDLE-SHAPED SOLUTIONS TO THE ALLEN-CAHN EQUATION

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Abstract. In these lectures we will study the existence, uniqueness, and stability properties of saddle-shaped solutions to the equation $-\Delta u = f(u)$ in all of $\mathbb{R}^{2m}$, where f is of bistable type —for instance $f(u) = u - u^3$ in the Allen-Cahn equation. Saddle-shaped solutions are relevant in connection with a conjecture of De Giorgi on 1d symmetry of minimizers.

Saddle-shaped solutions are odd with respect to the Simons cone $\mathcal{C} = \{(x^1, x^2) \in \mathbb{R}^m \times \mathbb{R}^m : |x^1| = |x^2|\}$ and depend only on the two radial variables $|x^1|$ and $|x^2|$. Thus, they solve a PDE in a quarter of the plane $\mathbb{R}^2$.

In all even dimensions $2m$, we will prove their existence, derive their asymptotic behavior at infinity, and also establish some of their monotonicity and convexity properties. All this will allow to prove the uniqueness of saddle-shaped solution in every dimension —a very recent result.

In addition, we will show their instability in dimensions 2, 4, and 6, as well as their stability in dimensions $2m \geq 14$.

Their stability in dimensions 8, 10, and 12 remains an open question. In addition, since the Simons cone minimizes area when $2m \geq 8$, saddle-shaped solutions are expected to be global minimizers when $2m \geq 8$, or at least in higher dimensions. This is a property stronger than stability which is not yet established in any dimension.

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