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Multiplicity of solutions to ressonat elliptic problems.

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In this notes we discuss the existence and multiple solutions of the Dirichlet boundary value problem

$$\begin{cases} -\Delta_p u = \lambda_1 |u|^{p-2}u + f(x,u) \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{0.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded open domain with smooth boundary $\partial\Omega$, $1 < p < N$ and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carath*é*odory function such that

$$\lim_{|t|\to\infty} \frac{f(x,t)}{|t|^{p-1}} = 0. \tag{0.2}$$

Here $\triangle_p$ denotes the p-Laplacian operator, that is, $\triangle_p u = div(|\nabla u|^{p-2}\nabla u)$. When $p = 2$, it is the usual Laplacian operator.

From a variational stand point of view, finding solutions of (0.1) in $W_0^{1,p}(\Omega)$ is equivalent to finding critical points of the C^1 functional J given by

$$J(u) = \frac{1}{p}\int_\Omega |\nabla u|^p dx - \frac{\lambda_1}{p}\int_\Omega |u|^p dx - \int_\Omega F(x,u)dx, \ \forall\, u \in W_0^{1,p}(\Omega), \tag{0.3}$$

where $F(x,t) = \int_0^t f(x,s)ds$ and the Sobolev space $W_0^{1,p}(\Omega)$ is a Banach space endowed with the norm $\|u\| = (\int_\Omega |\nabla u|^p dx)^{\frac{1}{p}}$.

First, we can prove the following result

Teorema 0.1. *(Existence) Suppose $(SR), (H0)$. Then the problem (0.1) has at least one solution $u_0 \in W_0^{1,p}(\Omega)$.*

Now, we take $F(x,0) \equiv 0, f(x,0) \equiv 0$ which implies that $u = 0$ is a trivial solution of problem (0.1). In this case the key point is assure the existence of nontrivial solutions. We need some additional hypothesis

$(H1)$ There are $\delta > 0$ and $\alpha \in (0, \lambda_1)$ such that

$$F(x,t) \leq \frac{\alpha - \lambda_1}{p}|t|^p, \ \forall\ |t| \leq \delta, \ \forall\, x \in \Omega.$$

$(H2)$ There is $t_\star \in \mathbb{R}\backslash\{0\}$ such that

$$\int_\Omega F(x, t_\star \Phi_1(x))dx > 0.$$

Thus, combining Ekeland's Variational Principle and Mountain Pass Theorem, we can prove the following multiplicity result

Teorema 0.2. *Suppose $(SR), (H0), (H1), (H2)$. Then the problem (0.1) has at least two nontrivial solutions $u_0, u_1 \in W_0^{1,p}(\Omega)$.*

Next, we consider the following hypothesis

$(H3)$ There are $r > 0$ and $\epsilon \in (0, \lambda_2 - \lambda_1)$ such that

$$0 \leq F(x,t) \leq \frac{\lambda_2 - \lambda_1 - \epsilon}{p}|t|^p, \ \forall\ |t| \leq r, \ \forall\, x \in \Omega.$$

Then, using the Three-Critical Point Theorem, we can show the following result

Teorema 0.3. *Suppose $(SR), (H0), (H3)$. Then the problem (0.1) has at least two nontrivial solutions.*

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