Universidade Federal da Paraíba
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Programa Associado de Pós-Graduação em Matemática
Doutorado em Matemática

# Uniqueness and Stability of Hypersurfaces in semi-Riemannian Spaces

**por**

**Arlandson Matheus Silva Oliveira**

Campina Grande – PB
Dezembro de 2018

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**sob orientação do**

**Prof. Dr. Henrique Fernandes de Lima**

Tese apresentada ao Corpo Docente do Programa Associado de Pós Graduação em Matemática UFPB/UFCG, como requisito parcial para obtenção do título de Doutor em Matemática.

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Campina Grande – PB
Dezembro de 2018

# Dedicatória

À memória dos professores Manfredo Perdigão do Carmo e Claudio Carlos Dias, precursores da Geometria para o Brasil e para mim, respectivamente.

À minha famlia, em sentido amplo, *sine qua non*.

Comme si nous ne restions pas esclaves tant que nous ne disposons pas des problèmes eux-mêmes, d'une participation aux problèmes, d'un droit aux problèmes, d'une gestion des problèmes.

G. Deleuze, Différence et répétition

# Agradecimentos

To be added, but if you, dear reader, enter into here interested in what we have to say in the forthcoming pages of this thesis (and if you preserve the wonderment at existence), then we sincerely welcome and thank you.

# Resumo

Esta tese está dividido em duas partes independentes. Na primeira parte, estudamos a geometria de imersões de variedades de dimensão $n$ em ambientes semi-riemannianos. Os espaços-ambiente consistem em produtos *warped* de um intervalo aberto da reta e de uma variedade riemanniana de dimensão $n$ (chamada *fibra*), em que a função *warping* está definida no intervalo, e são munidos de uma função peso que não depende do parâmetro do intervalo. Tal ambiente é naturalmente folheado por folhas totalmente umbílicas, chamadas *slices*, que são isométricas à fibra do ambiente. Munidas da métrica riemanniana induzida pelo tensor métrico do ambiente, as variedades imersas são também chamadas hipersuperfícies (tipo-espaço no caso de o ambiente ser lorentziano). O objetivo da primeira parte é estudar certas condições suficientes, obtidas da interação das geometrias de uma dada hipersuperfície e do ambiente com a função peso, para garantir que a hipersuperfície é um *slice* do ambiente. Para isso, aplicamos uma série de resultados analíticos às funções peso e altura de uma hipersuperfície, tais como princípios do máximo, condições envolvendo os espaços $L^p$ e critérios de parabolicidade. Na segunda parte, consideramos o problema variacional de minimizar o funcional $s$-área mantendo constante um funcional definido por uma combinação do funcionais $r$-área e balanço de volume. Os pontos críticos desse problema são as hipersuperfícies tais que uma dada razão entre suas funções simétricas de ordem $r$ e $s$ (ou entre as curvaturas médias de ordem superior correspondentes) é constante, o que nos leva à noção de $(r, s, a, b)$-estabilidade (forte ou não). Sob certas condições geométricas e considerando que uma constante que aparece quando encontramos a segunda variação do funcional de Jacobi desse problema variacional  não-positiva, mostramos que as esferas geodésicas são as únicas hipersuperfícies fechadas $(r, s, a, b)$-estáveis das formas espaciais e fortemente $(r, s, a, b)$-estáveis do espaço hiperbólico e que as esferas redondas totalmente umbílicas são as únicas hipersuperfícies fortemente $(r, s, a, b)$-estáveis do espaço de De Sitter.

**Palavras-chave:** produtos *warped*, variedades com peso, imersões isométricas, hipersuperfícies, princípios do máximo, espaços $L^p$, parabolicidade, formas espaciais, espaço hiperbólico, espaço de De Sitter, $(r, s, a, b)$-estabilidade, esferas.

# Abstract

This thesis is divided into two independent parts. In the first one, we study the geometry of immersions of an $n$-dimensional manifold into semi-Riemannian ambient spaces. These ambient spaces consist in warped products of an open interval of the real line and of an $n$-dimensional Riemannian manifold (called the *fiber*), where the warping function is defined on the interval, furnished with a weight funtion that does not deppend on the parameter of the interval. Such an ambient is naturally foliated by means of totally umbilical leaves, called *slices*, which are isometric to the fiber of the ambient. Endowed with the Riemannian metric induced from the metric tensor of the ambient, the immersed manifolds are also called hypersurfaces (spacelike hypersurfaces when the ambient is a Lorentzian one). The aim of the first part is to study certain sufficient conditions, related to the interaction between the geometries of the ambient and of a given hypersurface and the weight function, to guarantee that the hypersurface is a slice of the ambient. To do so, we apply a variety of analytic tools to the height function and to the angle function of a hypersurface, such as maximum principles, conditions involving the $L^p$ spaces, and criteria of parabolicity. In the second part, we consider the variational problem of minimizing the $s$-area funtional while keeping constant a functional defined as a linear combination of the $r$-area functional and the balance of volume. The critical points of this problem are hypersurfaces such that a certain ratio between their symmetric funtions of order $r$ and $s$ (or, equivalently, between their corresponding mean curvatures) is constant, which leads us to the notion of (strong or not) $(r, s, a, b)$-stability. Under certain reasonable geometric conditions, and assuming that a constant, which appears when we compute the second variation of the Jacobi functional associated with this variational problem, is nonpositive, we show that the geodesic sphere are the only $(r, s, a, b)$-stable closed hypersurfaces of the space forms and the only strongly $(r, s, a, b)$-stable closed hypersurfaces of the hyperbolic space, and that the totally umbilical round are the only strongly $(r, s, a, b)$-stable compact hypersurfaces of the De Sitter space.

**Keywords:** warped products, weighted manifolds, isometric immersions, hypersurfaces, maximum principles, $L^p$ spaces, parabolicity, space forms, hyperbolic space, De Sitter space, $(r, s, a, b)$-stability, spheres.

# Contents

# Introduction

This thesis is divided into the following two independent parts.

## Part I: Uniqueness of hypersurfaces in a class of weighted semi-Riemannian warped products

Let $(\Sigma, g)$ be an $n$-dimensional complete Riemannian manifold. The Laplace operator on $\Sigma$, $-\Delta$, can be defined as the differential operator associated to the standard Dirichlet form

$$Q(u) = \int_\Sigma |\nabla u|^2 dV, \quad u \in C_c^\infty(M) \subset L^2(dM),$$

where $|\cdot|$ is the norm induced by the Riemannian inner product $g = \langle \cdot, \cdot \rangle$ and $dV$ is the volume element on $\Sigma$. Now let $\phi \in C^\infty(\Sigma)$, that will be referred as a *weight function*. If we replace the measure $dV$ with the weighted measure $d\mu = \exp(-\phi)dV$ in the definition of $Q$, we obtain a new quadratic form $Q_\phi$, and we shall denote by $\Delta_\phi$ the elliptic operator on $C_c^\infty(\Sigma) \subset L^2(d\mu)$ induced by $Q_\phi$. In this sense, $\Delta_\phi$ arises as a natural generalization of the Laplacian. It is clearly symmetric and positive and extends to a positive operator on $L^2(d\mu)$. By Stokes theorem,

$$\Delta_\phi u = \Delta u - g(\nabla u, \nabla \phi).$$

Thus introducing a weight factor is the first step towards decoupling the leading term and the lower order terms of the operator, which in the case of the Laplace operator are completely determined by the metric of $\Sigma$.

The triple $(\Sigma, g, d\mu)$ and the operator $\Delta_\phi$ defined above and acting over $C^\infty(\Sigma)$ will be called, respectively, the *weighted manifold*, $\Sigma_\phi$, associated with $(\Sigma, g)$ and $\phi$, and the *$\phi$-Laplacian.* A natural question that arise in this setting is what is the right concept of curvature on these spaces. Although there is no canonical choice, good choices are those that reveal interplays between the geometry of $\Sigma$ and the behavior of the weight function $\phi$. We are going to focus on the notion of curvature for weighted manifolds that goes back to Lichnerowicz [66], and it was later developed by Bakry and Émery in their seminal work [23], where they introduced the following modified Ricci curvature

$$\mathrm{Ric}_\phi = \mathrm{Ric} + \mathrm{Hess}\,\phi.$$

As it is common in the current literature, we shall refer to this tensor as being the *Bakry-Émery-Ricci tensor* of $\Sigma$.

Weighted manifolds are closely related to some classical mathematical concepts, as they can be used as a powerful mathematical tool to obtain new results related to them. Specifically, in the case where $\mathrm{Ric}_\phi$ is constant, we can induce on $\Sigma$ a gradient Ricci soliton structure. Its mathematical relevance is due to the Perelman's solution of the Poincaré conjecture since gradient Ricci solitons correspond to self-similar solutions to the Hamilton's Ricci flow and often arise as limits of dilations of singularities developed along the Ricci flow. For an overview of results in this scope one can consult [88]. Furthermore, weighted manifolds have also been considered when studying harmonic heat flows and heat kernels. For instance, Grigor'yan and Saloff-Coste established in [53] a result which relates the heat kernel on a complete, noncompact Riemannian manifold $\Sigma$ with the Dirichlet heat kernel on the exterior of a compact set of $\Sigma$. For further results of geometric investigations concerning these manifolds, we also refer the reader to the articles of Morgan [78] and Wei-Wylie [104].

The Bakry-Émery-Ricci curvature tensor arises in scalar-tensor gravitation theories in the conformal gauge known as the Jordan frame. In Lorentzian geometry, Case [37] has shown that a sign condition on timelike components of the Bakry-Émery-Ricci tensor, a so-called energy condition, will, in an analogous fashion to the Riemannian case, imply that singularity theorems and the timelike splitting theorem hold. Woolgar [105] used these theorems to obtain Jordan-frame singularity and timelike splitting theorems for Brans-Dicke family of scalar-tensor theories and (1-loop) dilaton gravity, including dilaton gravity with totally skew torsion dilaton gravity. Besides that, a connection between the theory of black holes and a Lorentzian Bakry-Émery formulation was established by Rupert and Woolgar [94]. Under an energy condition on the Bakry-Émery-Ricci curvature, Galloway and Woolgar [48] obtained singularity theorems of a cosmological type, both for zero and for positive cosmological constant, that is, they found conditions under which every timelike geodesic is incomplete. The singularity theorems of general relativity (see, for instance, [56]) are some of the deepest statements in modern science, because they imply that the universe has a finite history, beginning in a so-called big bang singularity, provided that we can reliably extrapolate certain features of the known laws of physics back to early times and high energy scales.

On the other hand, a thematic widely approached in the recent years into the theory of isometric immersions is the study of the geometry of hypersurfaces immersed in a warped product of the type $\epsilon I \times_\rho \Sigma^n$, where $\Sigma^n$ is an $n$-dimensional Riemannian manifold, $I \subseteq \mathbb{R}$ is an interval, $\rho$ is a positive smooth function defined on $I$, and $\epsilon \in \{-1, 1\}$. These ambient spaces are foliated by means of the family of totally umbilical (spacelike, in the Lorentzian case) hypersurfaces $S_t := \{t\} \times \Sigma^n$, $t \in I$, that will be called *slices*. In this context, an interesting question to investigate is the uniqueness of such slices among (spacelike) hypersurfaces in $\epsilon I \times_\rho \Sigma^n$, under reasonable assumptions on their geometric data. In Chapter 1, we shall present some further notation and terminology concerning the kind of ambient spaces we shall deal with.

Chapter 2 will be devoted to the study of the uniqueness of slices among hypersurfaces immersed in a Riemannian weighted warped product. We shall consider product manifolds $\overline{M}^{n+1} = I \times M^n$ furnished with the Riemannian metric $\overline{g} = \pi_I^*(dt^2) + \varrho^2(\pi_I)\pi_M^*(g_M)$, where $\pi_I$ and $\pi_M$ are the projections onto the factors $I$ and $M$, respectively. In such a case, we simply write $\overline{M}^{n+1} = I \times_\varrho M^n$. Motivated by the Cheeger-Gromoll type splitting theorems due to Lichnerowicz [66, 67] (see also [104]), and Fang, Li and Zhang [47], the class of Riemannian warped products that will be of our concern in Chapter 2 is the one of *weighted Riemannian warped products* $\overline{M}_\phi = I \times_\varrho M_\phi$ whose weight function $\phi$ does not depend on the parameter $t \in I$, that is, $\overline{g}(\overline{\nabla}\phi, \partial_t) = 0$. Let $x : S^n \to \overline{M}^{n+1}$ be an isometric immersion of an $n$-dimensional connected Riemannian manifold without boundary $S^n$ in $\overline{M}^{n+1}$. Given such an immersion, we shall say that $S^n$ is a hypersurface of $\overline{M}^{n+1}$. Throughout Chapter 2, we shall always assume that $S^n$ is *two-sided* and transversal to the conformal closed vector field $\xi := \varrho(t)\partial_t$ at every point.

We shall consider two particular functions naturally attached to a hypersurface $S^n$ of $\overline{M}^{n+1}$; namely, the height function $h := \pi_I\big|_S$ and the angle function $\Theta := \overline{g}(N, \partial_t)$, where $N$ is the Gauss map of $S^n$. Inspired by Gromov [54], we define the *$\phi$-mean curvature* $H_\phi$ of $S^n$ as being the function $nH_\phi := nH + \overline{g}(\overline{\nabla}\phi, N)$, where $H$ denotes the standard mean curvature of $S^n$ with respect to $N$. In Section 2.1, we compute the $\phi$-Laplacian of these functions and give some sufficient conditions under which we can guarantee that the Bakry-Émery-Ricci curvature of $S^n$ has a lower bound. In Section 2.2, we shall present our uniqueness results concerning slices among two-sided hypersurfaces of $\overline{M}_\phi$. More precisely, we shall first apply a weak Omori-Yau maximum principle to the functions $h$ and $\Theta$. Afterwards, we shall give uniqueness results under $L^p$-conditions. Finally, we shall see some criteria to a hypersurface to inherit the $\phi$-parabolity of the Riemannian covering of the fiber $M^n$ of $\overline{M}^{n+1}$, from which we shall also derive uniqueness results. Proceeding, inspired by Moser [79], who gave a kind of extension of Berteins's theorem [30] by showing that the hyperplanes are the only entire minimal graphs of $\mathbb{R}^n$ whose gradient of the corresponding function has bounded norm, in Section 2.3 from the theorems in 2.2 we shall obtain Moser-Bernstein type results concerning entire graphs $S(u) \subset \overline{M}_\phi$ determined by a function $u \in C^\infty(M)$. The case of constant $\phi$-mean curvature graphs will be approached separately in Section 2.4.

Chapter 3 corresponds to the Lorentzian setting. Whithin it, we shall consider *generalized Robertson-Walker (GRW) spacetimes* of the type $M^{n+1} = -I \times_f F^n$, which are the product manifold of an open interval $(I, dt^2) \subseteq (\mathbb{R}, dt^2)$ and of an $n$-dimensional Riemannian manifold $(F^n, g_F)$, furnished with the Lorentzian metric tensor $\overline{g} = -\pi_I^*(dt^2) + f^2(\pi_I)\pi_F^*(g_F)$, where $\pi_I$ and $\pi_F$ are the projections onto the factors $I$ and $F$. This terminology came from [10]. Motivated by a splitting theorem due to Case (see [37, Theorem 1.2]), throughout Chapter 3, we shall endow $M^{n+1}$ with a weight function $\phi \in C^\infty(M)$ that does not depend on the parameter $t \in I$ and denote by $M_\phi^{n+1}$ the corresponding weighted manifold, which will be called a *spatially weighted GRW spacetime.*

Let $S^n$ be an $n$-dimensional connected manifold. An isometric immersion $\psi : S^n \to M_p^{n+1}hi$ is said to be a *spacelike hypersurface* if $S^n$, furnished with the metric induced from $\overline{g}$ via $\psi$, is a Riemannian manifold. In this setting, it follows from the connectedness of $S^n$ that we can uniquely choose a globally defined timelike unitary normal vector field $N \in \mathfrak{X}(S)^{\perp}$, having the same time-orientation of $\partial_t$, that is, such that $\overline{g}(N, \partial_t) \leq -1$. Following Gromov [54], the *$\phi$-mean curvature* $H_\phi$ of $S^n$ is defined by $nH_\phi = nH - \overline{g}(\overline{\nabla}\phi, N)$, where $H$ denotes the standard mean curvature of $S^n$ with respect to $N$ The family of spacelike hypersurfaces $S_t := \{t\} \times F^n$, $t \in I$, constitutes a foliation of $M$ by means totally umbilical leaves, which we will be called *slices*.

Given a spacelike hypersurface $S^n$ immersed in a spatially weighted Lorentzian GRW spacetime $M^{n+1}_\phi$, we shall consider two functions naturally attached to it; namely, the *height function* $h := \pi_I\big|_S$ and the *angle function* $\Theta = \overline{g}(N, \partial_t)$, where $N$ is the *future-pointing Gauss map* of $N$. Section 3.1 is the dual version of Section 2.1. There we shall prove the same results adapted to the Lorentzian setting. Namely, we shall see useful formulae for the $\phi$-Laplacians of the functions $h$ and $\Theta$ of the spacelike hypersurface $S^n$. We shall also give some conditions under which we can guarantee that the Bakry-Émery-Ricci tensor of $S^n$has a lower bound.

In Section 3.2, we shall present our uniqueness results concerning spacelike slices among spacelike hypersurfaces of a spatially weighted Lorentzian spacetime, first by considering comparison inequalities relating mean curvature of the slices and the $\phi$- mean curvature function of a given spacelike hypersurface and applying a weak Omori-Yau maximum principle to the functions $h$ and $\Theta$, and later by considering $L^p$-conditions and $\phi$-parabolicity criteria. In Section 3.3, we shall apply a mean value inequality for positive subsolutions of the $\phi$-heat operator, obtained from a Sobolev imbedding, to prove a nonexistence result concerning complete noncompact $\phi$-maximal spacelike hypersurfaces in a spatially weighted static GRW spacetime, which is just a weighted Lorentzian product space where the weight function does not deppend on the parameter of the Lorentzian factor. In 1970, Calabi [33] stated the so-called Calabi-Bernstein theorem, which states that the only complete maximal surfaces in the 3-dimensional Lorentz-Minkowski spacetime, that is, spacelike surfaces with zero mean curvature, are the spacelike planes. The nonparametric version of this theorem asserts that the only entire maximal graphs in $\mathbb{L}^3$ are the affine functions. Cheng and Yau [43] extended this result to complete maximal hypersurfaces in $\mathbb{L}^{n+1}$. In Section 3.4, we shall give some generalizations of these results by considering spacelike entire grpahs $S(u) \subset M^{n+1}_\phi$ determined by a function $C^\infty(F)$.

# Part II: Stable closed hypersurfaces as extrema of a linear combination of area and volume

It is well-known that immersions with constant mean curvature in Riemannian space forms are critical points for the variational problem of minimizing the area functional while keeping null balance of volume. A local solution for this variational problem is said to be stable. This concept was introduced and studied by Barbosa and do Carmo [25], and Barbosa, do Carmo and Eschenburg [26], where they proved that spheres are the only stable critical points of the area functional for volume-preserving variations. In [7], Alencar, do Carmo and Colares extended to hypersurfaces with constant scalar curvature the above stability result on constant mean curvature.

The natural generalization of mean and scalar curvatures for an $n$-dimensional hypersurface of a Riemannian space form $\overline{M}^{n+1}(\overline{c})$, $c \in \{-1, 0, 1\}$, are the higher order mean curvatures $H_r$, $r \in \{1, \ldots, n\}$. In fact, $H_1$ is just the mean curvature $H$, and $H_2$ defines a geometric quantity which is directly related to the scalar curvature. In a space form, first Alencar, do Carmo and Rosenberg in [8] and shortly after Barbosa and Colares [27] studied closed hypersurfaces with constant hight order mean curvature $H_{r+1}$ and established the concept of $r$-stability. In this context, they showed that such hypersurfaces are $r$-stable if and only if they are geodesic spheres.

In [57], He and Li treated the stability of closed hypersurfaces having constant quotient $H_{r+1}/H_1$ in a Riemannian space form. They showed that such hypersurfaces are $r$-stable if and only if they are totally umbilical. In [42], Chen and Wang studied the stability of closed hypersurfaces satisfying $(n-1)H_2 + aH_1 = b$, where $a$ and $b$ are real constants, in the Euclidean sphere $\mathbb{S}^{n+1}$. They proved that a such hypersurface $\Sigma^n$ can be characterized as a critical point of an appropriated functional for volume-preserving variations, and that $\Sigma^n$ is stable if and only if it is totally umbilical and non-totally geodesic.

More recently, Velásquez, Sousa and Lima [100] defined the notion of $(r, s)$-stability concerning closed hypersurfaces immersed in $\overline{M}^{n+1}(\overline{c})$. In this setting, they established a characterization result for $(r, s)$-stable closed hypersurfaces contained either in an open hemisphere of the Euclidean sphere $\mathbb{S}^{n+1}$ or in the hyperbolic space $\mathbb{H}^{n+1}$. With a different approach, Silva, Lima and Velásquez [97] completed this result, showing that a closed hypersurface contained in the Euclidean space $\mathbb{R}^{n+1}$ and having higher order mean curvatures linearly related is $(r, s)$-stable if and only if it is a geodesic sphere of $\mathbb{R}^{n+1}$.

In the Lorentzian setting, Barbosa and Oliker [24] obtained an analogous result, proving that constant mean curvature spacelike hypersurfaces in Lorentzian manifolds are also critical points of the area functional for variations that keep the volume constant. They also computed the second variation formula and showed, for the de Sitter space $\mathbb{S}_1^{n+1}$, that spheres maximize the area functional for volume-preserving variations.

Barros, Brasil and Caminha [28] studied the problem of strong stability (that is, stability with respect to not necessarily volume-preserving variations) for spacelike hypersurfaces with constant mean curvature in a generalized RobertsonWalker (GRW) spacetime of constant sectional curvature. They proved that, given a closed spacelike hypersurface $M^n$ of a GRW spacetime $-I \times_f F^n$, if the warping function satisfies $f'' \geq \max\{Hf', 0\}$ and $M^n$ is strongly stable, then $M^n$ is either maximal or a spacelike slice $\{t_0\} \times F^n$ for some $t_0 \in I$. In [34], Camargo et al. obtained an extension of the main result of [28] concerning spacelike hypersurfaces with constant $r$th mean curvature into a GRW spacetime, giving a characterization of $r$-maximal spacelike hypersurfaces and spacelike slices. In particular, they treated the case in which the ambient GRW spacetime is the De Sitter space $\mathbb{S}_1^{n+1}$. Zhang [110] considered the stability of closed linear Weingarten spacelike hypersurfaces in a De Sitter space $\mathbb{S}_1^{n+1}$. This concept of stability arises from considering the variational problem of minimizing a suitable linear combination of the second area for volume-preserving variations. More precisely, he proved that a compact orientable hypersurface

$M^n$ of $\mathbb{S}_1^{n+1}$ satisfying $(n-1)H_2 + aH = b$, for some constants $a > 0$ and $b$, with positive mean curvature, is stable (in the sense he defined) if and only if it is totally umbilical.

Motivated by all these works, in Part II of this thesis we shall introduce the notion of $(r, s, a, b)$-stability, where $r$ and $s$ are integers satisfying the inequalities $0 \leq r < s \leq n-2$, and $a$ and $b$ real numbers (at least one nonzero). More precisely, let $M^n$ be a closed (spacelike, in the Lorentzian case) hypersurface of a semi-Riemannian space form $\overline{M}^{n+1}$, which will be a Riemannian space form or the De Sitter space (details will be given in Chapter 4). We shall first consider the variational problem of *minimizing the $s$-area functional $\mathfrak{A}_s$ for all variations $X : (-\epsilon, \epsilon) \times M^n \to \overline{M}^{n+1}$ of $M^n$ that preserve the functional $\mathfrak{C}_{s,a,b}$*, where $\mathfrak{C}_{s,a,b}$ is the functional given by the following linear combination of the $s$-area and the balance of volume, $\mathfrak{C}_{r,a,b}(t) := a\mathfrak{A}_r(t) + b\mathfrak{V}(t)$, $t \in (-\epsilon, \epsilon)$. The Jacobi functional associated with this problem is given by $\mathfrak{J}_{r,s,a,b}(t) = \mathfrak{A}_s(t) + \alpha\mathfrak{C}_{r,a,b}(t)$, $t \in (-\epsilon, \epsilon)$, where $\alpha$ is a constant to be determined later.

In Chapter 5, where $\overline{M}^{n+1} = \overline{M}^{n+1}(\overline{c})$ is a Riemannian space form of constant sectional curvature $\overline{c}$, we shall see that a closed hypersurface that solves this problem must satisfy

$$\frac{(s+1)S_{s+1}}{a(r+1)S_{r+1} - b} = constant, \tag{1}$$

where $S_{s+1}$ and $S_{r+1}$ are the $(s+1)$th and the $(r+1)$th symmetric functions on the principal curvatures of $M^n$, provided that $a((r+1)S_{r+1} - b)$ never vanishes on $M^n$. When we compute the second variation of $\mathfrak{J}_{r,s,a,b}$, we achieve at

$$\begin{aligned}\mathfrak{J}''_{r,s,a,b}(0)(u) = -(s+1)\int_M \big\{&\mathfrak{L}_{r,s,a,b}(u) + \big(\overline{c}\,\mathrm{trace}(P_s) + \mathrm{trace}(A^2 \circ P_s) \\ &-\Lambda_{r,s,a,b}\left(\overline{c}\,\mathrm{trace}(P_r) + \mathrm{trace}(A^2 \circ P_r)\right)\big)\, u\big\}\, u dM\end{aligned}$$

for $u \in C^\infty(M)$, where $\mathfrak{L}_{r,s,a,b}$ is the second order differential operator

$$\begin{aligned}\mathfrak{L}_{r,s,a,b} : C^\infty(M) &\to C^\infty(M) \\ u &\mapsto \mathfrak{L}_{r,s,a,b}(u) = L_s(u) - \Lambda_{r,s,a,b}\, L_r(u),\end{aligned}$$

and

$$\Lambda_{r,s,a,b} = \frac{a(r+1)S_{s+1}}{a(r+1)S_{r+1} - b}.$$

(Here, $P_j$ denotes the $j$th Newton transformation, and $L_j$ is the second order differential operator associated with $P_j$; see Chapter 4.) We shall say that $x : M^n \to \overline{M}^{n+1}(\overline{c})$ is *$(r, s, a, b)$-stable* if $\mathfrak{A}''_s(0) \geq 0$ for all variations $X : M^n \times (-\epsilon, \epsilon) \to \overline{M}^{n+1}(\overline{c})$ of $x$ that preserve the functional $\mathfrak{C}_{r,a,b}$. We shall prove that the geodesic spheres of $\overline{M}^{n+1}(\overline{c})$ are the only critical points of the functional $\mathfrak{J}_{r,s,a,0}$ (see Theorem 5.3.2) and that a closed hypersurface $M^n$ of $\overline{M}^{n+1}(\overline{c})$ (contained in an open hemisphere of the sphere $\mathbb{S}^{n+1}$ when $\overline{c} = 1$) satisfying (1) and $\Lambda_{r,s,a,b} \leq 0$ is $(r, s, a, b)$-stable for $b \neq 0$ if and only if it is a geodesic sphere (see Theorem 5.3.5).

Proceeding, in Chapter 6, where the ambient space is the hyperbolic space $\mathbb{H}^{n+1}$, motivated by the ideas established in [72], we shall consider the variational problem of *minimizing the*

*Jacobi functional* $\mathfrak{J}_{r,s,a,b}$ *for all variations* $X : (-\epsilon, \epsilon) \times M^n \to \mathbb{H}^{n+1}$ *of a closed hypersurface* $x : M^n \to \mathbb{H}^{n+1}$. We shall say that such a hypersurface $M^n$ whose higher order mean curvatures $H_{r+1}$ and $H_{s+1}$ satisfy

$$\frac{b_s H_{s+1}}{ab_r H_{r+1} - b} = constant,$$

provided that $ab_r H_{r+1} - b \neq 0$ on $M^n$, where $b_j = (j+1)\binom{n}{j+1}$ for $j = r, s$, is *strongly* $(r, s, a, b)$-*stable* if $\mathfrak{J}''_{r,s,a,b}(0)(u) \geq 0$ for all $u \in C^\infty(M)$. In this setting, by using a warped product model for $\mathbb{H}^{n+1}$ as in [77, Example 4.3] and a formula due to Barros and Sousa [29] for computing $\mathfrak{L}_{r,s,a,b}$ at a certain angle function, we shall prove that if a strongly $(r, s, a, b)$-stable closed hypersurface $M^n$ of $\mathbb{H}^{n+1}$ satisfies $H_{j+1} \geq H_j$, for $j = r, s$, and its Gauss image is contained in the chronological future (or past) of an equator of the De Sitter space $\mathbb{S}_1^{n+1}$, then $M^n$ is a geodesic sphere of $\mathbb{H}^{n+1}$ (see Theorem 6.2.4).

Finally, in Chapter 7, $\overline{M}^{n+1}$ is the De Sitter space $\mathbb{S}_1^{n+1}$, we shall consider the variational problem of *maximizing the Jacobi functional* $\mathfrak{J}_{r,s,a,b}$ *for all variations* $X : M^n \times (-\epsilon, \epsilon) \to \mathbb{S}_1^{n+1}$ *of a compact spacelike hypersurface* $x : M^n \to \mathbb{S}_1^{n+1}$. We shall say that compact spacelike hypersurface $x : M^n \to \mathbb{S}_1^{n+1}$ whose higher order mean curvatures $H_{r+1}$ and $H_{s+1}$ satisfy

$$\frac{b_s H_{s+1} + c_s}{a(b_r H_{r+1} + c_r) + b} = constant,$$

with $a(b_r H_{r+1} + c_r) + b \neq 0$ on $M^n$ is *strongly* $(r, s, a, b)$-*stable* provided that $\mathfrak{J}''_{r,s,a,b}(0)(f) \leq 0$ for all $f \in C^\infty(M)$. (Here, the constants $c_k$'s are defined recursively by $c_0 = 0$, $c_1 = 1$, and $c_k = -\frac{(n-k+1)}{k-1} c_{k-2}$ if $k \geq 2$.) Taking into account that the De Sitter space can be described as the spacetime $\mathbb{S}_1^{n+1} = -\mathbb{R} \times_{\cosh t} \mathbb{S}^n$, $t \in \mathbb{R}$, where $\mathbb{S}^n$ is the Euclidean unitary sphere (see, for instance, [77]), and using a computational lemma that provides us with a formula for $\mathfrak{L}_{r,s,a,b}$ at a certain angle function, we shall prove that a compact strongly $(r, s, a, b)$-stable spacelike hypersurface contained in a chronological future (or past) of $\mathbb{S}_1^{n+1}$, and such that $\Lambda_{r,s,a,b} \leq 0$, $H_{s+1} > 0$ and $H_1 \leq 1$, must be a totally umbilical round sphere (see Theorem 7.3.4).

# Part I

# Uniqueness of hypersurfaces in a class of weighted semi-Riemannian warped products

# Chapter 1

# Alice in the well-known things land – Part I

“Then you keep moving round, I suppose?” said Alice.
“Exactly so,” said the Hatter: “as the things get used up.”
“But what happens when you come to the beginning again?” Alice ventured to ask.
Lewis Carroll, Alice in Wonderland

In this chapter, for the sake of clarity we shall introduce several useful definitions and notations that will appear throughout the forthcoming chapters of Part I of this thesis.

## 1.1 Weighted manifolds

Let $(\Sigma, g)$ be an $n$-dimensional complete Riemannian manifold. The Laplace operator on $\Sigma$, $-\Delta$, can be defined as the differential operator associated to the standard Dirichlet form

$$Q(u) = \int_\Sigma |\nabla u|^2 dV, \quad u \in C_c^\infty(M) \subset L^2(dV),$$

where $|\cdot|$ is the norm induced by the Riemannian inner product $g = \langle \cdot, \cdot \rangle$, and $dV$ is the volume element on $\Sigma$. Now let $\phi \in C^\infty(\Sigma)$, which will be referred to as a *weight function*. If we replace the measure $dV$ with the weighted measure $d\mu = \exp(-\phi)dV$ in the definition of $Q$, we obtain a new quadratic form $Q_\phi$, and we shall denote by $\Delta_\phi$ the elliptic operator on $C_c^\infty(\Sigma) \subset L^2(d\mu)$ induced by $Q_\phi$. In this sense, $\Delta_\phi$ arises as a natural generalization of the Laplacian. It is clearly symmetric and positive and extends to a positive operator on $L^2(d\mu)$. By Stokes theorem,

$$\Delta_\phi u = \Delta u - g(\nabla u, \nabla \phi).$$

Thus, introducing a weight factor is the first step towards decoupling the leading term and the lower order terms of the operator, which in the case of the Laplace operator are completely determined by the metric of $\Sigma$.

The triple $(\Sigma, g, d\mu)$ and the operator $\Delta_\phi$ defined above and acting over $C^\infty(\Sigma)$ will be called, respectively, the *weighted manifold*, $\Sigma_\phi$, associated with $(\Sigma, g)$ and $\phi$, and the *$\phi$-Laplacian*.

We recall that a notion of curvature for weighted manifolds goes back to Lichnerowicz [66], and it was later developed by Bakry and Émery in their seminal work [23], where they introduced the following modified Ricci curvature

$$\mathrm{Ric}_\phi = \mathrm{Ric} + \mathrm{Hess}\,\phi. \tag{1.1}$$

As it is common in the current literature, we shall refer to this tensor as being the *Bakry-Émery-Ricci tensor* of $\Sigma$. We note that the interplay between the geometry of $\Sigma$ and the behavior of the weight function $\phi$ is mostly taken into account by means of its Bakry-Émery-Ricci tensor $\mathrm{Ric}_\phi$. We also note that this curvature relates to $\phi$-Laplacian via the following Bochner's formula,

$$\frac{1}{2}\Delta_\phi|\nabla u|^2 = |\mathrm{Hess}\,u|^2 + 2g(\nabla u, \nabla\Delta_\phi u) + \mathrm{Ric}_\phi(\nabla u, \nabla u),$$

which holds for all $u \in C^\infty(\Sigma)$.

## 1.2 Riemannian setting

Let $(M, g_M)$ be an $n$-dimensional Riemannian manifold without boundary, $I \subseteq \mathbb{R}$ an open interval endowed with the metric $dt^2$, and $\varrho : I \to \mathbb{R}$ a positive smooth function. Let us consider the product manifold $\overline{M} = I \times M$ furnished with the Riemannian metric

$$\overline{g} = \pi_I^*(dt^2) + \varrho^2(\pi_I)\pi_M^*(g_M),$$

where $\pi_I$ and $\pi_M$ are the projections onto the factors $I$ and $M$, respectively. In such a case, we simply write

$$\overline{M} = I \times_\varrho M.$$

Then, $(\overline{M}, \overline{g})$ is a Riemannian *warped product* with *fiber* $(M, g_M)$, *base* $(I, dt^2)$, and *warping function* $\varrho$ (see [84, Chapter 7] for details).

Furthermore, we shall consider on $\overline{M}$ the conformal closed vector field $\xi := \varrho(t)\partial_t$, where $t := \pi_I$. In fact, it follows from the relationship between the Levi-Civita connection $\overline{\nabla}$ of $\overline{M}$ and those of $(M, g_M)$ and $(I, dt^2)$ (see [84, Corollary 7.35]) that

$$\overline{\nabla}_X\xi = \varrho'(t)X \quad \forall X \in \mathfrak{X}\left(\overline{M}\right).$$

The family of hypersurfaces $S_t := \{t\} \times M$, $t \in I$, constitutes a foliation of $\overline{M}$ by totally umbilical leaves of constant mean curvature $(\log\varrho)'(t) = \frac{\varrho'(t)}{\varrho(t)}$ (with respect to the orientation given by $N = -\partial_t$) that we shall call *slices*.

Let $x : S \to \overline{M}$ be an isometric immersion of an $n$-dimensional connected complete Riemannian manifold without boundary $S$ in $\overline{M}$, and let us denote by $g_S$ the induced metric on $S$.

Given such an immersion, we shall say that $S$ is a hypersurface of $\overline{M}$. Throughout Chapter 2, we shall always assume that $S$ is *two-sided*, which means that the normal bundle of $S$ is trivial, and thus we can choose a unitary normal vector field $N \in TS^{\perp}$, called the *Gauss map* of $x$, globally defined on $S$. We shall also assume that $S$ is transversal to $\xi$ at every point.

We shall consider two particular functions naturally attached to a hypersurface $S$ immersed in $\overline{M}$; namely, the *height function* $h := (\pi_I)\big|_S$ and the *angle function* $\Theta := \overline{g}(N, \partial_t)$. The transversality condition aforementioned, together with the connectedness of $S$, give that $\Theta$ does not change sign on $S$. From now on, in the Riemannian setting, we shall orient two-sided hypersurfaces in such way that $\Theta \leq 0$, unless stated otherwise. Given any vector field $V \in \mathfrak{X}(\overline{M})$, we denote by $V^M = \pi_M^*(V) = V - \overline{g}(V, \partial_t)\partial_t$ the projection of $V$ onto $M$. In particular, $N^M = N - \Theta\partial_t$, and therefore

$$\left|N^M\right|^2 = \overline{g}(N^M, N^M) = 1 - \Theta^2. \tag{1.2}$$

Let us denote by $\overline{\nabla}$ and $\nabla$ the gradients with respect to the metrics of $\overline{M}$ and $S$, respectively. Then, a simple computation shows that the gradient of $\pi_I$ on $\overline{M}$ is given by

$$\overline{\nabla}\pi_I = \overline{g}(\overline{\nabla}\pi_I, \partial_t)\partial_t = \partial_t,$$

so that the gradient of $h$ on $S$ is

$$\nabla h = (\overline{\nabla}\pi_I)^{\top} = \partial_t^{\top} = \partial_t - \Theta N, \tag{1.3}$$

where $(\cdot)^{\top}$ denotes the tangential component of a vector field in $\mathfrak{X}\left(\overline{M}\right)$ along $S$. Thus, from (1.9) we get

$$|\nabla h|^2 = 1 - \Theta^2, \tag{1.4}$$

where $|\nabla h|^2 = g_S(\nabla h, \nabla h)$.

Motivated by the Cheeger-Gromoll type splitting theorems due to Lichnerowicz [66, 67] (see also [104]), and Fang, Li and Zhang [47], the class of Riemannian manifolds that will be of our concern in Chapter 2 is the one of *weighted warped products* $\overline{M}_\phi = I \times_\varrho M_\phi$ whose weight function $\phi$ does not depend on the parameter $t \in I$, that is, $\overline{g}(\overline{\nabla}\phi, \partial_t) = 0$.

In this setting, for a hypersurface $S$ immersed in $\overline{M}_\phi$, the *$\phi$-divergence operator* of a tangent vector field $X$ on $S$ is defined by

$$\mathrm{div}_\phi(X) = \exp(\phi)\mathrm{div}(\exp(-\phi)X),$$

and, given a smooth function $u : S \to \mathbb{R}$, its *$\phi$-Laplacian* is defined by

$$\Delta_\phi u := \mathrm{div}_\phi(\nabla u) = \Delta u - g_S(\nabla u, \nabla\phi). \tag{1.5}$$

Finally, inspired by Gromov [54], we define the *$\phi$-mean curvature* $H_\phi$ of $S$ by

$$nH_\phi := nH + \overline{g}(\overline{\nabla}\phi, N), \tag{1.6}$$

where $H$ denotes the standard mean curvature of $S$ with respect to its Gauss map $N$. In analogy to the case of the standard mean curvature, the $\phi$-mean curvature $H_\phi$ on $S$ is related to the variational problem for the weighted area functional,

$$vol_\phi(S) := \int_S \exp(-\phi) dS.$$

We observe that, under the assumption on the weight function of the ambient space, the $\phi$-mean curvature of a slice $S_t \subset \overline{M}_\phi$, with respect to the orientation $N = -\partial_t$, is given by $H_\phi(t) = H(t) = \frac{\varrho'}{\varrho}(t)$.

## 1.3 Lorentzian setting

Let $(F, g_F)$ be an $n$-dimensional connected oriented Riemannian manifold, $I \subseteq \mathbb{R}$ an open interval endowed with the metric $-dt^2$, and $f : I \to \mathbb{R}$ a positive smooth function. The product manifold $M = I \times F$ furnished with the Lorentzian metric

$$\overline{g} = -\pi_I^*(dt^2) + f^2(\pi_I)\pi_F^*(g_F),$$

where $\pi_I$ and $\pi_F$ are the projections onto the factors $I$ and $F$, respectively, is a Lorentzian *warped product* with *warping function* $f$ and *fiber* $F$. In this case, we simply write

$$M = -I \times_f F. \tag{1.7}$$

Namely, for all $p \in M = -I \times_f F$ and all $v, w \in T_p(I \times F)$, we have

$$\overline{g}(v, w)\big|_p = -(\pi_I)_*(v)(\pi_I)_*(w) + (f \circ \pi_I)(p)^2 g_F((\pi_F)_*(v), (\pi_F)_*(w))\big|_{\pi_F(p)}.$$

A standard computation shows that $\xi := f(t)\partial_t$, where $t := \pi_I$, is a conformal closed vector field globally defined on $M$ (see [84] for details).

When $F$ has constant sectional curvature, the warped product $M = -I \times_f F$ has been known in the mathematical literature as a *Robertson-Walker (RW) spacetime*. In the case $n = 3$, such spacetimes are known to be exact solutions of the Einstein's field equations for an appropriate choice of the warping function (see [84, Chapter 12]). After [10], warped products of the type (1.7) have usually been referred to as *generalized Robertson-Walker (GRW) spacetimes*, and we shall stick to this terminology along Chapter 3.

Let $S$ be an $n$-dimensional connected manifold. An isometric immersion $\psi : S \to M = -I \times_f F$ is said to be a *spacelike hypersurface* if $S$, furnished with the metric induced from $\overline{g}$ via $\psi$, is a Riemannian manifold. If this is so, we shall always assume that the metric on $S$ is the induced one, which will be denoted by $g_S$. In this setting, it follows from the connectedness of $S$ that we can uniquely choose a globally defined timelike unitary normal vector field $N \in \mathfrak{X}(S)^{\perp}$, having the same time-orientation of $\partial_t$, that is, such that $\overline{g}(N, \partial_t) \leq -1$. We shall say that $N$ is the *future-pointing Gauss map* of $S$. The family of spacelike hypersurfaces $S_t := \{t\} \times F$, $t \in I$, constitutes a foliation of $M$ by means totally umbilical leaves of constant mean curvature $H(t) = (\log f)'(t)$ with respect to the orientation given by $N = \partial_t$, which we shall call *slices*.

Given a spacelike hypersurface $S$ immersed in a Lorentzian warped product $M = -I \times_f F$, we shall consider two functions naturally attached to it; namely, the *height function* $h := \pi_I\big|_S$ and the *angle function* $\Theta = \overline{g}(N, \partial_t)$. It is worth pointing out that $\Theta = -\cosh\theta$, where $\theta$ is the hyperbolic angle between $N$ and $\partial_t$. Given any vector field $V \in \mathfrak{X}(M)$, we denote by $V^F = V + \overline{g}(V, \partial_t)\partial_t$ the projection of $V$ onto $F$. In particular, $N^F = N + \Theta\partial_t$, and therefore

$$g_F(N^F, N^F) = \Theta^2 - 1. \tag{1.8}$$

Let us denote by $\overline{\nabla}$ and $\nabla$ the gradients with respect to the metrics of $M$ and $S$, respectively. Then, a simple computation shows that the gradient of $\pi_I$ on $M$ is given by

$$\overline{\nabla}\pi_I = -\overline{g}(\overline{\nabla}\pi_I, \partial_t)\partial_t = -\partial_t,$$

so that the gradient of $h$ on $S$ is

$$\nabla h = (\overline{\nabla}\pi_I)^{\top} = -\partial_t^{\top} = -\partial_t - \overline{g}(N, \partial_t)N. \tag{1.9}$$

Thus, from (1.9) we get

$$|\nabla h|^2 = \Theta^2 - 1, \tag{1.10}$$

where $|\cdot|$ denotes the norm of a vector field on $S$.

Obviously, by taking the standard product of $(I, -dt^2)$ with a Riemannian weighted manifold $F_\phi = (F, g_F, \exp(-\phi)dF)$, we can construct a Lorentzian weighted manifold $M_{\widetilde{\phi}} = -I \times F_\phi$, where the weight function, defined as being $\widetilde{\phi}(t, x) = \phi(x)$, satisfies $\overline{g}(\overline{\nabla}\phi, \partial_t) = 0$. Motivated by a splitting theorem due to Case (see [37, Theorem 1.2]), throughout Chapter 3, we shall always assume that this later condition holds, that is, the class of Lorentzian manifolds that will be of our concern in Chapter 3 is the one of *spatially weighted GRW spacetimes* $\overline{M}_\phi = -I \times_f F_\phi$ whose weight function $\phi$ does not depend on the parameter $t \in I$.

In this setting, for a spacelike hypersurface $S$ immersed in $\overline{M}_\phi - I \times_f F_\phi$, the *$\phi$-divergence operator* on $S$ of a tangent vector fields $X \in \mathfrak{X}(S)$, $\operatorname{div}_\phi(X)$, and the $\phi$-Laplacian of a function $u \in C^\infty(S)$, $\Delta_\phi u$, are defined, respectively, as in (1.2) and (1.5). Following Gromov [54], the *$\phi$-mean curvature* $H_\phi$ of $S$ is defined by

$$nH_\phi = nH - \overline{g}(\overline{\nabla}\phi, N), \tag{1.11}$$

where $H$ denotes the standard mean curvature of $S$ with respect to its future-pointing Gauss map $N$. As in the Riemannian case, the $\phi$-mean curvature $H_\phi$ on a spacelike hypersurface $S$ is related to the variational problem for the weighted area functional, $vol_\phi(S) := \int_S \exp(-\phi) dS$. We also observe that, under the assumption on the weight function of the ambient spacetime, the $\phi$-mean curvature of a slice $S_t \subset \overline{M}_\phi$, with respect to the orientation $N = \partial_t$, is given by $H_\phi(t) = H(t) = \frac{\varrho'}{\varrho}(t)$.

# Chapter 2

# The Riemannian case

In this chapter, we aim at studying the uniqueness of slices $S_t := \{t\} \times M^n$, $t \in I$, among two-sided hypersurfaces $S^n$ immersed in a Riemannian weighted warped product of the type $\overline{M}_\phi^{n+1} = I \times_\varrho M_\phi^n$, furnished with the metric tensor $\overline{g} = \pi_I^*(dt^2) + \varrho^2(\pi_I)\pi_M^*(g_M)$, and where the weight function $\phi$ does not depend on the parameter $t \in I$, that is, $\overline{g}(\overline{\nabla}\phi, \partial_t) = 0$. Towards this aim, we shall consider a variety of assumptions on the height function and on the angle function of such a $S$, as well as on the $\phi$-mean curvature and on geometric quantities related to $S$, and employ analytic tools such as a weak version of the Omori-Yau maximum principle, $L^{1\leq p<\infty}$-conditions, and $\phi$-parabolicity criteria, in order to guarantee that $S = S_t$ for some $t \in I$. Moser-Bernstein type results concerning entire graphs $S(u) = \{(u(p), p) : p \in M\} \subset \overline{M}_\phi$ of functions $u \in C^\infty(M)$ such that $u(M) \subseteq I$ will also be given.

All manifolds here are supposed to be connected and smooth.

## 2.1 Computational lemmas

In this section, we shall prove two computational lemmas which will be employed with the aim of studying the uniqueness of two-sided hypersurfaces $x : S \to \overline{M}_\phi = I \times_\varrho M_\phi$. The first one contains expressions for the $\phi$-Laplacian of three functions naturally attached to $S$; namely, the height function and its primitive and the angle function. The second auxiliary result gives a lower bound for the Bakry-Émery-Ricci curvature of $S$.

**Lemma 2.1.1.** *Let $S$ be a two-sided hypersurface immersed in a weighted product space $\overline{M}_\phi$, with orientation $N$, height function $h$, and angle function $\Theta$. Then,*

*(i)* $\Delta_\phi \sigma(h) = n\left(\varrho'(h) + \varrho(h)\Theta H_\phi\right)$;

*(ii)* $\Delta_\phi h = (\log \varrho)'(h)(n - |\nabla h|^2) + n\Theta H_\phi$; *and*

*(iii)*

$$\begin{aligned}\Delta_\phi(\varrho(h)\Theta) = &-n\varrho(h)g_S(\nabla H_\phi, \partial_t) - n\varrho'(h)H_\phi - \varrho(h)\Theta|A|^2 - \varrho(h)\Theta\overline{\mathrm{Hess}}\,\phi(N,N)\\ &-\varrho(h)\Theta\left(\mathrm{Ric}^M(N^M,N^M) + (n-1)(\log\varrho)''(h)|\nabla h|^2\right),\end{aligned}$$

*where $A$ is the shape operator of $S$ related to $N$, $|A|$ denotes its Hilbert-Schmidt norm, $\sigma(t) = \int_{t_0}^{t} \varrho(s)ds$, $\mathrm{Ric}^M$ stands for the Ricci curvature tensor of the fiber $M$, $(\cdot)^M$ is the projection of a vector field onto $M$, and $\overline{\mathrm{Hess}}$ is the Hessian operator of $\overline{M}$.*

*Proof.* Items (i) and (ii) were already established in [39, Lemma 1]. Now, from [16, Proposition 6], with the aid of [84, Corollary 7.43], we get the following expression for the Laplacian of the function $\varrho(h)\Theta$,

$$\begin{aligned}\Delta(\varrho(h)\Theta) = &- n\varrho(h)g_S(\nabla H, \partial_t) - n\varrho'(h)H - \varrho(h)\Theta|A|^2 \\ &- \varrho(h)\Theta\left(\mathrm{Ric}^M(N^M, N^M) + (n-1)(\log\varrho)''(h)|\nabla h|^2\right).\end{aligned} \tag{2.1}$$

By definition of the $\phi$-mean curvature (see (1.6)),

$$n\varrho(h)g_S(\nabla H, \partial_t) = n\varrho(h)g_S(\nabla H_\phi, \partial_t) - \varrho(h)\partial_t^\top \overline{g}(\overline{\nabla}\phi, N).$$

Moreover, after a straightforward computation, we get

$$\partial_t^\top \overline{g}(\overline{\nabla}\phi, N) = -\frac{\varrho'}{\varrho}(h)\overline{g}(\overline{\nabla}\phi, N) - \Theta\overline{\mathrm{Hess}}\,\phi(N,N) - g_S(\overline{\nabla}\phi, A\partial_t^\top),$$

and

$$\nabla(\varrho(h)\Theta) = -\varrho(h)A\partial_t^\top.$$

So (2.1) can be written as

$$\begin{aligned}\Delta(\varrho(h)\Theta) = &- n\varrho(h)g_S(\nabla H_\phi, \partial_t) - n\varrho'(h)H_\phi - \varrho(h)\Theta|A|^2 \\ &- \varrho(h)\Theta\overline{\mathrm{Hess}}\,\phi(N,N) + g_S(\overline{\nabla}\phi, \nabla(\varrho(h)\Theta)) \\ &- \varrho(h)\eta\left(\mathrm{Ric}^M(N^M, N^M) + (n-1)(\log\varrho)''(h)|\nabla h|^2\right).\end{aligned} \tag{2.2}$$

Finally, item (iii) follows from (2.2) and the definition of the $\phi$-Laplacian (see (1.5)). □

We shall also need some lower bound for the Bakry-Émery-Ricci tensor of $S$. From now on, we shall ask $\overline{M}_\phi$ to obey the following convergence condition,

$$K^M \geq \sup_I \left(\varrho'^2 - \varrho\varrho''\right), \tag{2.3}$$

where $K^M$ stands for the sectional curvature of the fiber $M$.

**Lemma 2.1.2.** *Let $\overline{M}_\phi = I \times_\varrho M_\phi$ be a weighted warped product obeying (2.3), and such that the Hessian of the weighted function $\phi$ is bounded from below. Let $(S, g_S)$ be a two-sided hypersurface immersed in $(\overline{M}_\phi, \overline{g})$. Suppose that both the second fundamental form and the $\phi$-mean curvature of $S$ are bounded. If the ratio $\frac{\varrho''}{\varrho}(h)$ is also bounded on $S$, then the Ricci-Bakry-Émery tensor of $S$, $\mathrm{Ric}_\phi$, is bounded from below.*

*Proof.* We recall that the curvature tensor $R$ of a hypersurface $x : S \to \overline{M}$ can be described in terms of the shape operator $A$ and the curvature tensor $\overline{R}$ of $\overline{M}$ by the Gauss equation given by

$$R(X,Y)Z = (\overline{R}(X,Y)Z)^\top + g_S(AX,Z)AY - g_S(AY,Z)AX, \tag{2.4}$$

for all tangent vector fields $X,Y,Z \in \mathfrak{X}(S)$. Here, as in [84], the curvature tensor $R$ is given by

$$R(X,Y)Z = \nabla_{[X,Y]}Z - [\nabla_X, \nabla_Y]Z,$$

where $[\cdot\,,\cdot]$ denotes the Lie bracket, and $X,Y,Z \in \mathfrak{X}(S)$.

Let us consider $X \in \mathfrak{X}(S)$ and a local orthonormal frame $\{E_1,\ldots,E_n\}$ of $\mathfrak{X}(S)$. Then, it follows from the Gauss equation (2.4) that

$$\operatorname{Ric}(X,X) = \sum_{i=1}^n \overline{g}(\overline{R}(X,E_i)X,E_i) + nHg_S(AX,X) - g_S(AX,AX). \tag{2.5}$$

Moreover, we have that (see [84] for details)

$$\begin{aligned}\overline{R}(X,Y)Z =& R^M(X^M,Y^M)Z^M - ((\log\varrho)'(h))^2(\overline{g}(X,Z)Y - \overline{g}(Y,Z)X)\\ &+ (\log\varrho)''(h)\overline{g}(Z,\partial_t)(\overline{g}(Y,\partial_t)X - \overline{g}(X,\partial_t)Y)\\ &- (\log\varrho)''(h)(\overline{g}(Y,\partial_t)\overline{g}(X,Z) - \overline{g}(X,\partial_t)\overline{g}(Y,Z))\partial_t,\end{aligned}$$

$R^M$ being the curvature tensor of the fiber $M$, and hence

$$\begin{aligned}\overline{g}(\overline{R}(X,E_i)X,E_i) =& \varrho(h)^2K^M(X^M,E_i^M)(g_M(X^M,X^M)g_M(E_i^M,E_i^M) - g_M(X^M,E_i^M)^2)\\ &- ((\log\varrho)'(h))^2(g_S(X,X) - g_S(X,E_i)^2)\\ &+ (\log\varrho)''(h)g_S(X,\nabla h)(g_S(\nabla h,E_i)g_S(X,E_i) - g_S(X,\nabla h))\\ &- (\log\varrho)''(h)(g_S(\nabla h,E_i)g_S(X,X) - g_S(X,\nabla h)g_S(X,E_i))g_S(\nabla h,E_i).\end{aligned} \tag{2.6}$$

On the other hand, one can easily see that

$$\begin{aligned}&g_M(X^M,X^M)g_M(E_i^M,E_i^M) - g_M(X^M,E_i^M)^2 =\\ &\qquad\frac{1}{\varrho(h)^4}(g_S(X,X) - g_S(X,\nabla h)^2\\ &\qquad - g_S(X,X)g_S(\nabla h,E_i)^2 - g_S(X,E_i)^2\\ &\qquad + 2g_S(X,\nabla h)g_S(X,E_i)g_S(\nabla h,E_i)),\end{aligned}$$

which, jointly with (2.3) and (2.6), implies the following lower bound

$$\sum_{i=1}^n \overline{g}(\overline{R}(X,E_i)X,E_i) \geq \frac{\varrho''}{\varrho}(h)(n-1)|X|^2, \tag{2.7}$$

for all $X \in \mathfrak{X}(\Sigma)$, where, for simplicity of notation, we denote by $|\cdot|$ the norm induced by the

Riemannian metric $g_S$ of $S$. Thus, from (2.5) and (2.7), we get

$$\mathrm{Ric}(X,X) \geq \frac{\varrho''}{\varrho}(h)(n-1)|X|^2 + nHg_S(AX,X) - |AX|^2. \tag{2.8}$$

Since the Hessian of $\phi$ is bounded from below, we have

$$\begin{aligned} \mathrm{Hess}\,\phi(X,X) &= \overline{\mathrm{Hess}}\,\phi(X,X) + \overline{g}(\overline{\nabla}\phi, N)g_S(AX,X) \\ &\geq \beta|X|^2 + \overline{g}(\overline{\nabla}\phi, N)g_S(AX,X), \end{aligned} \tag{2.9}$$

for a certain positive constant $\beta$. Therefore, from the definition of $\mathrm{Ric}_\phi$, (1.6), (2.8), and (2.9), we obtain

$$\mathrm{Ric}_\phi(X,X) \geq \left(\frac{\varrho''}{\varrho}(h)(n-1) + \beta\right)|X|^2 + nH_\phi g_S(AX,X) - |AX|^2, \tag{2.10}$$

for all $X \in \mathfrak{X}(S)$. The proof is complete, taking into account the assumptions that the ratio $\frac{\varrho''}{\varrho}(h)$, the second fundamental form and the $\phi$-mean curvature of $S$ are all bounded. $\square$

## 2.2 Statement and proof of the main results

This section is devoted to present our uniqueness results concerning slices among complete two-sided hypersurfaces immersed in the class of weighted warped product described in Section 1.2. For this, we shall apply some appropriate analytical tools. The results presented here are contained in our paper [74].

### 2.2.1 The weak Omori-Yau maximum principle

It is well-known the validity of the following weak version of the Omori-Yau maximum principle for the $\phi$-Laplacian on a complete weighted manifold $(\Sigma, g, \exp(-\phi)dV)$ (see, for instance, [88, Chapter 2, Remark 2.18]):[1]

**Lemma 2.2.1.** *Let $(\Sigma, g, \exp(-\phi)dV)$ be an $n$-dimensional complete weighted manifold whose Bakry-Émery-Ricci curvature tensor is bounded from below, and let $u \in C^\infty(\Sigma)$ be a function bounded from above (resp., bounded from below) on $\Sigma$. Then, there exists a sequence of points $\{p_k\}_{k\in\mathbb{N}} \subset \Sigma$ such that*

$$\lim_k u(p_k) = \sup_\Sigma u \text{ (resp., } = \inf_\Sigma u) \quad \text{and} \quad \limsup_k \Delta_\phi u(p_k) \leq 0 \text{ (resp., } \liminf_k \Delta_\phi u(p_k) \geq 0).$$

To study the geometry of a hypersurface $S$ immersed in a weighted warped product space of the type $\overline{M}_\phi = I \times_\varrho M_\phi$, initially we shall apply this weak Omori-Yau maximum principle to the functions described in the beginning of the previous section.

By a *slab* of $\overline{M}_\phi$, we mean a region of the form $[t_1, t_2] \times M$, where $t_1 < t_2$ belong to $I$.

[1] For a detailed discussion on maximum principles and geometric applications, we refer the reader to [19, 85]

**Theorem 2.2.2.** *Let $\overline{M}_\phi$ be a weighted warped product space obeying* (2.3)*, and such that the Hessian of the weight function $\phi$ is bounded from below. Let $x : S \to \overline{M}_\phi$ be a complete two-sided hypersurface which lies in a slab of $\overline{M}_\phi$, with bounded second fundamental form, and angle function $\Theta$ bounded away from zero. Suppose that either the $\phi$-mean curvature, $H_\phi$, of $S$ satisfies*

$$0 \leq H_\phi \leq \inf_S (\log \varrho)'(h), \tag{2.11}$$

*or*

$$\alpha \leq H_\phi \leq \inf_S (\log \varrho)'(h), \quad \inf_S (\log \varrho)'(h) \geq 0 \tag{2.12}$$

*for some constant $\alpha \in \mathbb{R}$. If*

$$|\nabla h| \leq \inf_S ((\log \varrho)'(h) - H_\phi), \tag{2.13}$$

*then $S$ is a slice $S_t \subset \overline{M}$.*

*Proof.* From Lema 2.1.1, if (2.11) holds, we have

$$\begin{aligned} \Delta_\phi \sigma(h) &= n\varrho(h)\left((\log \varrho)'(h) + \Theta H_\phi\right) \\ &\geq n\varrho(h)\left((\log \varrho)'(h) - H_\phi\right) \\ &\geq 0. \end{aligned}$$

Since $\sigma(h)$ is bounded from above (because $S$ is contained in a slab of $\overline{M}_\phi$), and, by Lema 2.1.2, $\mathrm{Ric}_\phi$ is bounded from below, the hypotheses of the weak Omori-Yau maximum principle are satisfied for $S$ and $\sigma(h)$, and we can take a sequence of points $\{p_k\} \subset S$ such that $\limsup_k \Delta_\phi \sigma(h(p_k)) \leq 0$. So, up to a subsequence, we have

$$0 \geq \limsup_k \Delta_\phi \sigma(h(p_k)) \geq \lim_k \left[n\varrho(h)\left((\log \varrho)'(h) - H_\phi\right)\right](p_k) \geq 0.$$

Also from the fact that $S$ is contained in a slab of $\overline{M}_\phi$, it follows that there exists a positive constant $C$ such that $\varrho(h(p)) \geq C$ for all $p \in S$. Thus, we have that $\lim_k \left[(\log \varrho)'(h) - H_\phi\right](p_k) = 0$, and, taking into account (2.13), we infer that $S$ is a slice.

If (2.12) holds, then again from Lema 2.1.1, we have that

$$\begin{aligned} \Delta_\phi \sigma(h) &= n\varrho(h)\left((\log \varrho)'(h) + \Theta H_\phi\right) \\ &\geq n\varrho(h)\Theta\left(H_\phi - (\log \varrho)'(h)\right) \\ &\geq 0. \end{aligned}$$

By arguing just as before, we conclude that $\lim_k \left[n\varrho(h)\Theta\left(H_\phi - (\log \varrho)'(h)\right)\right](p_k) = 0$. Since $\varrho(h(p)) \geq C$ for all $p \in S$ and for some positive constant $C$, and $\Theta$ is supposed to be bounded away from zero, (2.13) guarantees that $S$ is a slice. $\square$

We recall that the traceless symmetric tensor $\Gamma = A - HI$ is called *traceless second fundamental form*, where $I$ stands for the identity operator on $\mathfrak{X}(S)$. Observe that $|\Gamma|^2 = \mathrm{trace}(\Gamma^2) =$

$|A|^2 - nH^2 \geq 0$, with equality if and only if $S$ is totally umbilical. For this reason, $\Gamma$ is called the *total umbilicity tensor* of $S$.

**Theorem 2.2.3.** *Let $\overline{M}_\phi$ be a weighted warped product space obeying* (2.3) *with convex weight function $\phi$, that is, $\overline{\operatorname{Hess}}\,\phi \geq 0$.*[2] *Let $x : S \to \overline{M}_\phi$ be a complete two-sided hypersurface which lies in a slab of $\overline{M}_\phi$, with bounded second fundamental form $A$, and angle function $\Theta$ bounded away from zero. Suppose that $\phi$-mean curvature is constant and satisfies*

$$H_\phi \sup_S (\log \varrho)'(h) \leq H^2(p) \tag{2.14}$$

*and*

$$H_\phi (\log \varrho)'(h)(p) \leq -\Theta(p) H_\phi \sup_S (\log \varrho)'(h) \quad \forall p \in S. \tag{2.15}$$

*If either*

$$|\nabla h|^2(p) \leq \inf_S H^2 - H_\phi \sup_S\, (\log \varrho)'(h) \quad \forall p \in S, \tag{2.16}$$

*or*

$$|\nabla h| \leq \inf_S |\Gamma|^2, \tag{2.17}$$

*then $S$ is a slice $S_t \subset \overline{M}$.*

*Proof.* From inequality (2.3) we get, in particular, that

$$\operatorname{Ric}^M(N^M, N^M) + (n-1)(\log \varrho)''(h)|\nabla h|^2 \geq 0.$$

Therefore, from Lemma 2.1.1 and the assumptions of the theorem it holds

$$\Delta_\phi(\varrho(h)\Theta) \geq -n\varrho'(h)H_\phi - \varrho(h)\Theta|A|^2. \tag{2.18}$$

From (2.18) and (2.15) we get

$$\Delta_\phi\left(\varrho(h)\Theta\right) \geq \varrho(h)\Theta\left(nH_\phi \sup_S\,(\log \varrho)'(h) - |A|^2\right).$$

On the other hand, $|A|^2 = n^2H^2 - n(n-1)H_2$, where $H_2$ is the second order mean curvature defined by $\binom{n}{2}H_2 = \sum_{i<j} k_i k_j$, being $k_i$, $i = 1, ..., n$, the main curvatures of $S$. Therefore,

$$\begin{aligned} \Delta_\phi\left(\varrho(h)\Theta\right) &\geq \varrho(h)\Theta\left(nH_\phi \sup_S\,(\log \varrho)'(h) - n^2H^2 + n(n-1)H_2\right) \\ &= n\varrho(h)\Theta\left(H_\phi \sup_S\,(\log \varrho)'(h) - H^2\right) + n(n-1)\varrho(h)\Theta(H_2 - H^2) \geq 0, \end{aligned} \tag{2.19}$$

where we used again (2.14) and the fact that $H^2 - H_2 \geq 0$.

Since $\Theta$ is bounded and $S$ lies in a slab of $\overline{M}$, we have that the function $\varrho(h)\Theta$ is bounded

[2]For a detailed discussion concerning convex functions on Riemannian manifolds, we refer the reader to [99, Chapter 3].

from above. On the other hand, by Lemma 2.1.2, we know that $\mathrm{Ric}_\phi$ is bounded from below, so we can apply the weak Omori-Yau maximum principle for the $\phi$-Laplacian to conclude that there exists a sequence of points $\{p_k\} \subset S$ such that

$$\lim_k \varrho(h(p_k))\Theta(p_k) = \sup_S \varrho(h)\Theta \qquad \text{and} \qquad \limsup_k \Delta_\phi \varrho(h(p_k))\Theta(p_k) \le 0.$$

Thus, up to a subsequence (2.19) implies that

$$\begin{aligned} 0 \ge\ & \limsup_k \Delta_\phi \varrho(h(p_k))\Theta(p_k) \\ \ge\ & n \lim_k \left[ \varrho(h(p_k))\Theta(p_k) \left( H_\phi \sup_S (\log \varrho)'(h) - H^2(p_k) \right) \right] \\ & \quad + n(n-1) \lim_k \left[ \varrho(h(p_k))\Theta(p_k)(H_2 - H^2)(p_k) \right] \\ \ge & 0, \end{aligned}$$

so in particular

$$\lim_k H^2(p_k) - H_\phi \sup_S (\log \varrho)'(h) = 0,$$

since $\Theta$ is bounded away from zero. Consequently, from (2.14) we get

$$\inf_S H^2 - H_\phi \sup_S (\log \varrho)'(h) = 0,$$

and the result follows from (2.16).

Note also that, by combining (2.18), (2.14), (2.15), and $\Theta \le 0$, we achieve at

$$\begin{aligned} \Delta_\phi(\varrho(h)\Theta) &\ge -n\varrho'(h)H_\phi - \varrho(h)\Theta|A|^2 \\ &\ge n\varrho(h)\Theta H_\phi \sup_S (\log \varrho)'(h) - \varrho(h)\Theta|A|^2 \\ &\ge n\varrho(h)\Theta H^2 - \varrho(h)\Theta|A|^2 \\ &= \varrho(h)\Theta(nH^2 - |A|^2) \\ &= -\varrho(h)\Theta|\Gamma|^2 \\ &\ge 0. \end{aligned}$$

If (2.17) holds, then the conclusion follows once more by applying the weak Omori-Yau maximum principle for the $\phi$-Laplacian to the function $\varrho(h)\Theta$. □

### 2.2.2 Uniqueness under $L^{1\le p<\infty}$-conditions

Our next theorems give characterizations of slices under some $L^{1\le p<\infty}$ conditions.

**Theorem 2.2.4.** *Let $\overline{M}_\phi = I \times_\varrho M_\phi$ be a weighted warped product. Let $S$ be a complete two-sided hypersurface immersed in $\overline{M}_\phi$. Suppose that $H_\phi$ and $\varrho'(h)$ satisfy $H_\phi \varrho'(h) \le 0$, and that $\varrho$ is* log-*convex along $S$. Suppose also that $\{p \in S : (\log \varrho)''(h(p)) = 0\}$ is a set of isolated points. Let*

*$d\mu = \exp(-\phi)dS$, where $dS$ stands for the volume element on $S$. If $|\nabla h| \in L^1(d\mu)$ and $\varrho'(h)$ is bounded on $S$, then $S$ is a totally geodesic slice of $\overline{M}$. If $\varrho(h) \in L^p(d\mu)$ for some $p \in (1, +\infty)$, and the weight function is bounded and convex, then $S$ is a compact totally geodesic slice of $\overline{M}$; in particular, the fiber $M$ is compact.*

*Proof.* By Lemma 2.1.1, and under the assumptions of the theorem, we have

$$\begin{aligned} \Delta_\phi \varrho(h) &= \varrho'(h)\Delta_\phi h + \varrho''(h)|\nabla h|^2 \\ &= n\varrho'(h)\left((\log\varrho)'(h) + \Theta H_\phi\right) + \varrho(h)(\log\varrho)''(h)|\nabla h|^2 \\ &\geq 0. \end{aligned} \tag{2.20}$$

If $|\nabla h| \in L^1(d\mu)$ and $\varrho'(h)$ is bounded on $S$, then $|\nabla \varrho(h)| = |\varrho'(h)| \cdot |\nabla h| \in L^1(d\mu)$. In this case, we claim that $\varrho(h)$ is $\phi$-harmonic, that is, $\Delta_\phi \varrho(h) = 0$. This follows from the following extension of a result due to Yau [109] (for a proof, work with [36, Proposition 2.1]):

> Let $(\Sigma, g, d\mu = \exp(-\psi)dV)$ be a complete $n$-dimensional weighted Riemannian manifold, and let $u \in C^\infty(\Sigma)$ be such that $\Delta_\psi u$ does not change sign on $\Sigma$. If $|\nabla u| \in L^1(d\mu)$, then $u$ is $\psi$-harmonic.

Now, returning to (2.20), we obtain in particular that

$$\varrho'(h) \equiv 0 \quad \text{and} \quad (\log\varrho)''(h)|\nabla h|^2 \equiv 0 \quad \text{on } S,$$

which imply that $S$ is a totally geodesic slice, since by hypothesis $\{p \in S : (\log\varrho)''(h(p)) = 0\}$ is a set of isolated points.

If $\varrho(h) \in L^p(d\mu)$ for some $p \in (1, +\infty)$, then $\varrho(h)$ is constant on $S$. In fact, since $\Delta_\phi \varrho(h)$ does not change sign on $S$, this is a consequence of another extension of a Yau's result [109] to the weighted setting. Namely, we have:

> Let $u$ be a nonnegative smooth $\phi$-subharmonic function on a weighted complete Riemannian manifold $(\Sigma, g, d\mu = \exp(-\psi)dV)$. If $u \in L^p(d\mu)$, for some $p \in (1, +\infty)$, then $u$ is constant.

(For a proof, work with Theorem 1.1 of [86].) Reasoning as we just did in the first part, we conclude that $S$ is a totally geodesic slice $\{t_0\} \times M$. Furthermore, because $\varrho(h) \equiv C \in L^p(d\mu)$, we have $vol_\phi(S) = \int_S d\mu < +\infty$. From $(\log\varrho)''(t_0) \geq 0$, it follows that $\varrho''(t_0) \geq 0$. By the lower bound for the Bakry-Émery-Ricci tensor given in Lemma 2.1.2 (see inequality (2.10)), $\mathrm{Ric}_\phi \geq 0$. However, according to Wei and Wylie [104, Theorem 1.2], a complete noncompact Riemannian manifold with nonnegative Bakry-Émery-Ricci tensor for some bounded weight function $\psi$ have at least linear $\psi$-volume growth. Thus, we infer that $S$ is compact (and, hence, so is $M$). □

**Theorem 2.2.5.** *Let $\overline{M}_\phi = I \times M_\phi$ be a weighted product manifold with convex weight function, and whose fiber $M$ has nonnegative sectional curvatures. Let $S$ be a complete two-sided hypersurface immersed in $\overline{M}_\phi$. Suppose that $\Theta$ has strict sign, $H_\phi$ does not change sign, and $|A|$ is bounded on $S$. If $|\nabla h| \in L^1(d\mu)$, then $S$ is a slice of $S_t \subset \overline{M}$.*

*Proof.* Our hypotheses imply that $H_\phi \equiv 0$ on $S$. In fact, this is a consequence of the aforementioned weighted version of Yau's result (see the proof of Theorem 2.2.4), since $\Delta_\phi h = nH_\phi\Theta$ does not change sign on $S$ (see Lemma 2.1.1) and we are assuming that $|\nabla h| \in L^1(d\mu)$.

Consequently, again from Lemma 2.1.1 and the hypothesis of the theorem, we infer that

$$\Delta_\phi\Theta = -\Theta\left(|A|^2 + \mathrm{Ric}_\phi^M(N^M, N^M)\right)$$

does not change sign. Since $|\nabla\Theta| \le |A|\,|\nabla h| \in L^1(d\mu)$, it follows that $\Theta$ is a $\phi$-harmonic function. Thus, we infer that $|A|^2 \equiv 0$ on $S$. From Lemma 2.1.2, it follows, in particular, that the Bakry-Émery-Ricci tensor de $S$ is nonnegative.

We claim that $h$ must be bounded on $S$. We shall argue as in the proof of Theorem 0.16 of [61]. Suppose otherwise. Then, $S \cap S_t \neq 0$, $\forall t \in I$. For a fixed $t \in I$, let $\Sigma_t := \{p \in S : h(p) \ge t\}$. By Sard's theorem, we can suppose that $t$ is a regular value of $h\big|_{\mathrm{int}\, S}$, so that $\Sigma_t$ is a smooth complete manifold with boundary $\partial\Sigma_t = \{p \in S : h(p) = t\}$ and exterior unit normal $\nu_t = -\nabla h/|\nabla h|$. For any $\epsilon > 0$, define on $\Sigma_t$

$$h_\epsilon = \max\{h, t + \epsilon\}.$$

Then $h_\epsilon$ is a $\phi$-harmonic on $\Sigma_t$. In fact, set

$$\begin{aligned}
\Sigma_1 &= \{p \in \Sigma_t : h(p) > t + \epsilon\},\\
\Sigma_2 &= \{p \in \Sigma_t : h(p) = t + \epsilon\},\\
\Sigma_3 &= \{p \in \Sigma_t : t < h(p) < t + \epsilon\}.
\end{aligned}$$

Then, $h_\epsilon = h$ on $\Sigma_1$ and $h_\epsilon$ is constant (equals to $t + \epsilon$) on $\Sigma_3$, so $\Delta_\phi h_\epsilon = 0$ on both $\Sigma_1$ and $\Sigma_3$. Since the transversality of $S$ to $\xi = \varrho(t)\partial_t$ guarantees that $h_\epsilon$ is smooth on $\Sigma_2$ and $\partial\Sigma_t$, we also have that $\Delta_\phi h_\epsilon = 0$ on both $\Sigma_2$ and $\partial\Sigma_t$ by continuity. Whence, on noting that $h_\epsilon \equiv t + \epsilon$ on $\partial\Sigma_t$, by the maximum principle for the $\phi$-Laplacian (see [88, Theorem 2.13]), we obtain that $t \le h \le t+\epsilon$ on $\Sigma_t$. Since this holds for every $\epsilon > 0$, we conclude that $h \equiv t$ on $\Sigma_t$, contradicting the assumption of $h$ being unbounded.

The conclusion follows from a Liouville-type theorem due to Brighton [32], which asserts that a bounded $\psi$-harmonic funcion on a complete manifold $(\Sigma, g)$ endowed with a weight function $\phi$ in such a way that the corresponding Bakry-Émery-Ricci tensor is nonnegative must be constant. □

**Remark 2.2.6.** *We shall sketch another proof of Theorem 2.2.5. By the weighted Bochner's formula (see* [104]*), we have*

$$\frac{1}{2}\Delta_\phi|\nabla h|^2 = |\mathrm{Hess}\, h|^2 + g_S(\nabla h, \nabla\Delta_\phi h) + \mathrm{Ric}_\phi(\nabla h, \nabla h). \tag{2.21}$$

*Since $h$ is a $\phi$-harmonic function, $\Theta \equiv \Theta_0 \in \mathbb{R}$, $S$ is totally geodesic, and* $\mathrm{Ric}_\phi \ge 0$, *from* (1.10) *and* (2.21) *we get*

$$0 = \frac{1}{2}\Delta_\phi|\nabla h|^2 \ge |\mathrm{Hess}\, h|^2 \ge 0.$$

*By the Cauchy-Schwarz inequality, $|\mathrm{Hess}\, h|^2 \geq (\Delta h)^2/n$. Thus, $h$ is harmonic. Reasoning as in the previous proof, we can see that the maximum principle implies that $h$ is bounded. Being totally geodesic, and since $K^M \geq 0$, $S$ has nonnegative Ricci curvature. The conclusion follows from the strong Liouville property (see* [108] *and also* [65]*).*

### 2.2.3 When does a hypersurface inherit the $\phi$-parabolicity from the fiber's covering?

Following the classical terminology in linear potential theory, a weighted manifold $(P, g, d\mu = \exp(-\psi)dP)$ is called *$\psi$-parabolic* if there is no nonconstant bounded function whose $\psi$-Laplacian is globally either nonpositive or nonnegative signed. For any compact subset $K \subset P$, we define the *$\psi$-capacity* of $K$ as

$$\mathrm{cap}_\psi(K) = \inf \left\{ \int_P |\nabla u|^2 d\mu : u \in \mathrm{Lip}_0(P) \,\text{and}\, u\big|_K \equiv 1 \right\},$$

where $\mathrm{Lip}_0(P)$ is the set of all compactly supported Lipschitz functions on $P$. The notion of $\psi$-capacity is related to the $\psi$-parabolicity by the fact that a weighted manifold $(P, g, d\mu = \exp(-\psi)dP)$ is $\psi$-parabolic if and only if $\mathrm{cap}_\psi(K) = 0$ for any compact $K \subset P$ (see [50, Proposition 3] and also [52, Proposition 2.1]).

Let us recall that given two Riemannian manifolds $(P, g)$ and $(P', g')$, a diffeomorphism $f$ from $P$ onto $P'$ is called a *quasi-isometry* if there exists a constant $c \geq 1$ such that

$$c^{-1}|v|_g \leq |df(v)|_{g'} \leq c|v|_g$$

for all $v \in T_pP, p \in P$ (see [62] for details). Suppose that we can endow both $P$ and $P'$ with the same weight function $\phi$. We can reason as in Section 5 of [51] to verify that the $\psi$-capacity changes under a quasi-isometry at most by a constant factor. So, we have that if $(P, g)$ and $(P', g')$ are two quasi-isometric Riemannian manifolds endowed with the same weight function $\psi$, then $P$ and $P'$ are $\psi$-parabolic or not simultaneously.

Inspired by the ideas of [92], we can now state the following criterion of parabolicity.

**Theorem 2.2.7.** *Let $x : S \to \overline{M}_\phi = I \times_\varrho M_\phi$ be a complete two-sided hypersurface. Suppose that the fiber $M$ of $\overline{M}$ has $\phi$-parabolic universal Riemannian covering. If the angle function $\Theta$ of $S$ is bounded away from zero, and the restriction $\varrho(h)$ on $S$ of the warping function $\varrho$ of $\overline{M}$ satisfies*

*(i)* $\sup_S \varrho(h) < \infty$ *and*

*(ii)* $\inf_S \varrho(h) > 0$,

*then $S$ is $\phi$-parabolic.*

*Proof.* First of all, we point out that we can introduce on the universal Riemannian covering $(\tilde{M}, g_{\tilde{M}})$ of the fiber $(M, g_M)$ the notion of $\phi$-parabolicity in a natural way.

Let $\pi := \pi_M \circ x$. By the Cauchy-Schawrz inequality, for any tangent vector field $V \in TS$ we have

$$\begin{aligned} g_S(V,V) &= g_I(h_*V, h_*V) + \varrho(h)^2 g_M(\pi_*V, \pi_*V) \\ &\leq g_S(\nabla h, \nabla h) g_S(V,V) + \varrho(h)^2 g_M(\pi_*V, \pi_*V), \end{aligned}$$

so that,

$$\Theta^2 g_S(V,V) \leq \varrho(h)^2 g_M(\pi_*V, \pi_*V).$$

Thus, we have

$$g_S(V,V) \leq \frac{\varrho(h)^2}{\Theta^2} g_M(\pi_*V, \pi_*V) \leq C_1 g_M(\pi_*V, \pi_*V),$$

where $C_1 := \max\left\{1, \sup_S \frac{\varrho(h)^2}{\Theta^2}\right\}$. Being a map which increases distances between $(S, g_S)$ and $(M, \sqrt{C_1} g_M)$, $\pi$ is a covering map, hence $M$ is complete (see [63, Lemma 8.8.1]). On the other hand, it is clear that

$$g_S(V,V) \geq \varrho(h)^2 g_M(\pi_*V, \pi_*V) \geq \left(\inf_S \varrho(h)^2\right) g_M(\pi_*V, \pi_*V).$$

If we choose $C := \max\{C_1, 1/\inf_S \varrho(h)^2\}$, then it follows that

$$C^{-1} g_M(\pi_*V, \pi_*V) \leq g_S(V,V) \leq C g_M(\pi_*V, \pi_*V). \tag{2.22}$$

Let $(\tilde{S}, g_{\tilde{S}})$ be the universal Riemannian covering of $(S, g_S)$, and denote by $\tilde{\pi}_S : \tilde{S} \to S$ the corresponding Riemannian covering map. A standard result on covering spaces can be then claimed to obtain a lift $\tilde{h} : \tilde{S} \to \tilde{M}$ of the map $h = \pi \circ \tilde{\pi}_S : \tilde{S} \to M$ (see [59] for instance). It is easy to check that $\tilde{h}$ is, in fact, a diffeomorphism from $\tilde{S}$ to $\tilde{M}$. Note that (2.22) gives

$$C^{-1} g_{\tilde{M}}(d\tilde{h}(\tilde{v}), d\tilde{h}(\tilde{v})) \leq g_{\tilde{S}}(\tilde{v}, \tilde{v}) \leq C g_{\tilde{M}}(d\tilde{h}(\tilde{v}), d\tilde{h}(\tilde{v})),$$

for any $\tilde{v} \in T_{\tilde{p}}\tilde{S}, \tilde{p} \in \tilde{S}$, which means that $\tilde{h}$ is a quasi-isometry from $(\tilde{S}, g_{\tilde{S}})$ onto $(\tilde{M}, g_{\tilde{M}})$.

Finally, let $u$ be a nonnegative $\phi$-superharmonic function on $S$, and put $\tilde{u} = u \circ \tilde{\pi}_S$. The function $\tilde{u}$ is a nonnegative $\phi$-superharmonic function on the $\phi$-parabolic Riemannian manifold $\tilde{S}$. Therefore, $\tilde{u}$ must be constant, and, consequently, $u$ is also constant. □

As a direct consequence of Theorem 2.2.7, we get the following corollaries.

**Corollary 2.2.8.** *Let $S$ be a complete two-sided hypersurface in a weighted warped product $\overline{M}_\phi$ with weight function $\phi$ and whose fiber $M$ is simply connected and $\phi$-parabolic. If the angle function $\Theta$ of $S$ is bounded away from zero and the warping function on $S$, $\varrho(h)$, is bounded and it satisfies $\inf_S \varrho(h) > 0$, then $S$ is $\phi$-parabolic.*

**Corollary 2.2.9.** *Let $S$ be a complete two-sided hypersurface in a weighted product $\overline{M}_\phi = I \times M_\phi$ with weight function $\phi$ and whose fiber $M$ has $\phi$-parabolic universal Riemannian covering. If the angle function $\Theta$ of $S$ is bounded away from zero, then $S$ is $\phi$-parabolic.*

A hypersurface $S$ immersed in a weighted warped product $\overline{M}_\phi$ is called *$\phi$-minimal* if $H_\phi = 0$. Proceeding, we obtain the following

**Theorem 2.2.10.** *Let $\overline{M}_\phi$ be a weighted warped product whose fiber $M$ has $\phi$-parabolic universal Riemannian covering, and such that the warping function $\varrho$ is monotone. The only $\phi$-minimal complete two-sided hypersurfaces contained in a slab of $\overline{M}$ and with angle function $\Theta$ bounded away from zero are the slices $S_{t_0} \subset \overline{M}$, where $t_0 \in I$ is such that $\varrho'(t_0) = 0$.*

*Proof.* Let $S$ be such a hypersurface. From Lemma 2.1.1, we obtain

$$\Delta_\phi \sigma(h) = n\varrho'(h).$$

Consequently, the monotonicity of $\varrho$ implies that $\Delta_\phi \sigma(h)$ is globally either nonpositive or nonnegative signed. Since $S$ is contained in a slab of $\overline{M}$, the function $\sigma(h)$ is clearly bounded on $S$. From Theorem 2.2.7 we know that $S$ is $\phi$-parabolic, so $\sigma(h)$ is constant on $S$, and, hence, $h$ must also be constant on $S$. □

**Theorem 2.2.11.** *Let $\overline{M}_\phi$ be a weighted warped product satisfying* (2.3), *with convex weight function $\phi$ and whose fiber $M$ has $\phi$-parabolic universal Riemannian covering. Let $S$ be a $\phi$-minimal complete two-sided hypersurface immersed in $\overline{M}$, with angle function $\Theta$ bounded away from zero and such that the restriction $\varrho(h)$ on $S$ of the warping function $\varrho$ of $\overline{M}$ satisfies*

*(i)* $\sup \varrho(h) < \infty$ *and*

*(ii)* $\inf \varrho(h) > 0$.

*Then $S$ is totally geodesic. In addition, if either the inequality* (2.3) *is strict for all non-zero vector fields on $M$ or $\phi$ is strictly convex on $M$, then $S$ is a slice $S_{t_0} \subset \overline{M}$, where $t_0 \in I$ is such that $\varrho'(t_0) = 0$.*

*Proof.* From Lemma 2.1.1, we have that the $\phi$-Laplacian of the bounded function $\varrho(h)\Theta$ is given by

$$\begin{aligned}\Delta_\phi(\varrho(h)\Theta) = &- \varrho(h)\Theta|A|^2 - \varrho(h)\Theta\overline{\mathrm{Hess}}\,\phi(N,N) \\ &- \varrho(h)\Theta\left(\mathrm{Ric}^M(N^M,N^M) + (n-1)(\log\varrho)''(h)|\nabla h|^2\right).\end{aligned} \tag{2.23}$$

Since $\phi$ is supposed to be convex and we are assuming that the convergence condition (2.3) holds, it follows that $\Delta_\phi(\varrho(h)\Theta) \geq 0$. Theorem 2.2.7 assures that $S$ is $\phi$-parabolic, so $\varrho(h)\Theta$ must be constant. Therefore, returning to (2.23), we infer that $|A|^2 \equiv 0$, that is, $S$ is totally geodesic,

$$\overline{\mathrm{Hess}}\,\phi(N,N) = \mathrm{Hess}^M\phi(N^M,N^M) = 0, \tag{2.24}$$

and

$$\mathrm{Ric}^M(N^M, N^M) + (n-1)(\log \varrho)''(h)|\nabla h|^2 = 0.$$

Consequently, if the inequality (2.3) is strict, or if $\phi$ is strictly convex on $M$, then (2.23) also gives that $|N^M| = |\nabla h| = 0$ on $S$, that is, $S$ is a slice. □

When the ambient space is simply a product manifold, we obtain the following

**Theorem 2.2.12.** *Let $\overline{M}_\phi$ be a weighted product manifold, whose fiber $M$ has $\phi$-parabolic universal Riemannian covering and such that its Bakry-Émery-Ricci tensor $\mathrm{Ric}_\phi^M$ is nonnegative. Let $S$ be a complete two-sided hypersurface immersed in $\overline{M}$ with constant $\phi$-mean curvature $H_\phi$. If the angle function $\Theta$ of $S$ is bounded away from zero, then $S$ is totally geodesic. In addition, if $\mathrm{Ric}_\phi^M$ is definite positive at some point of $S$, then $S$ is a slice $S_t \subset \overline{M}$.*

*Proof.* From Lemma 2.1.1, the definition of $\mathrm{Ric}_\phi$ (see (1.1)), and (2.24) we get that

$$\Delta_\phi \Theta = -(\mathrm{Ric}_\phi^M(N^M, N^M) + |A|^2)\Theta. \tag{2.25}$$

Consequently, since we are supposing that $\mathrm{Ric}_\phi^M$ is nonnegative and that $\Theta$ is negative and bounded away from zero on $S$, we can apply Corollary 2.2.9 to conclude that $\Theta$ is constant on $S$. Thus, returning to (2.25) we get that $|A| \equiv 0$, that is, $S$ is totally geodesic. Moreover, if $\mathrm{Ric}_\phi^M$ is definite positive at some $p \in M$, considering once more equation (2.25), and taking into account relation (1.8), we conclude that $\Theta \equiv -1$ on $S$, which means that $S$ is a slice of $\overline{M}$. □

When $\overline{M}$ belongs to a class of manifolds called in [98] *pseudo-hyperbolic spaces*, that is, warped products of the type $I \times_{\exp(t)} M$, we get the following result.

**Theorem 2.2.13.** *Let $\overline{M}_\phi$ be a weighted pseudo-hyperbolic space, whose fiber $M$ has nonnegative sectional curvatures, and let $S$ be a complete two-sided hypersurface which lies in a slab of $\overline{M}$. Suppose that $|\overline{\nabla}\phi|$ is bounded on $S$. If the $\phi$-mean curvature $H_\phi$ of $S$ is bounded and does not change sign, and $|A|$ is bounded on $S$, then $H_\phi = 1$. If, in addition, the angle function $\Theta$ is bounded away from zero and the universal Riemannian covering of $M$ is $\phi$-parabolic, then $S$ is a slice $S_t \subset \overline{M}$.*

*Proof.* Since $H_\phi$ and $|\overline{\nabla}\phi|$ are supposed to be bounded, it follows immediately from (1.6) that the mean curvature $H$ of $S$ is bounded. Under the hypotheses of Theorem 2.2.13, from estimate (2.8) we conclude that the Ricci curvature of $S$ is bounded from below. By the Omori-Yau maximum principle [83, 108] applied to the bounded function $h$, there exists a sequence of points $\{p_k\}_{k\in\mathbb{N}} \subset S$ such that

$$\lim_k h(p_k) = \sup_S h < +\infty, \quad |\nabla h|^2(p_k) = 1 - \Theta^2(p_k) < 1/k^2,$$

and

$$\Delta_\phi h(p_k) = \left[n - |\nabla h|^2 + n\Theta H_\phi\right](p_k) < 1/k.$$

Therefore $\lim_k \Theta(p_k) = -1$ and taking limits in the last inequality we get $H_\phi \geq 1$.

In a similar way, by applying the Omori-Yau maximum principle to the bounded function $-h$, we obtain another sequence $\{q_k\}_{k\in\mathbb{N}}$ in $S$ such that

$$\lim_k h(q_k) = \inf_S h > -\infty, \quad |\nabla h|^2(q_k) = 1 - \Theta^2(q_k) < 1/k^2$$

and

$$\Delta_\phi h(p_k) = \left[n - |\nabla h|^2 + n\Theta H_\phi\right](q_k) > -1/k.$$

Thus $\lim_k \Theta(p_k) = -1$ and again taking limits in the last inequality we get $H_\phi \leq 1$. As a consequence, $H_\phi = 1$.

Furthermore, from Lemma 2.1.1, we have

$$\Delta_\phi \exp(h) = n\exp(h)(1+\Theta) \geq 0.$$

Assuming that the universal Riemannian covering of $M$ is $\phi$-parabolic and that the angle function is bounded away from zero, Theorem 2.2.7 assures that $S$ is $\phi$-parabolic. Then, we infer that $\exp(h)$ is constant on $S$, which implies that $h$ is constant, and $S$ is a slice. $\square$

## 2.3 Moser-Bernstein type results

In this section, we shall use the theorems of Section 2.2 to establish Moser-Bernstein type results concerning entire graphs in a weighted warped product. Before do so, we need to recall some basic facts related to these graphs.

Let $\Omega \subseteq M$ be a domain. Then, each function $u \in C^\infty(\Omega)$ such that $u(\Omega) \subseteq I$ defines a vertical graph in the Riemannian warped product $I \times_\varrho M$. In such a case, $S(u)$ will denote the graph over $\Omega$ determined by $u$, that is,

$$S(u) = \{(u(p), p) : p \in \Omega\} \subset \overline{M} = I \times_\varrho M.$$

The graph is said to be *entire* if $\Omega = M$. Observe that $h(u(p), p) = u(p), p \in \Omega$. Hence, $h$ and $u$ can be identified in a natural way. The metric induced on $\Omega$ from the Riemannian metric of the ambient space via $S(u)$ is

$$g_{S(u)} = du^2 + \varrho(u)^2 g_M.$$

If $M$ is complete and $\inf_{S(u)} \varrho(u) > 0$, then $S(u)$ furnished with the metric $g_{S(u)}$ is also complete. The unit vector field

$$N(p) = -\frac{\varrho(u(p))}{\sqrt{\varrho(u(p))^2 + |Du(p)|_M^2}} \left(\partial_t\big|_{(u(p),p)} - \frac{Du(p)}{\varrho(u(p))^2}\right), \quad p \in \Omega,$$

where $Du$ stands for the gradient of $u$ in $M$ and $|Du|_M = g_M(Du, Du)^{1/2}$, gives an orientation of $S(u)$ with respect to which we have $\overline{g}(N, \partial_t) < 0$, so that the assumption of transversality to the vector field $\xi = \varrho(t)\partial_t$ is not necessary here. The corresponding shape operator is given by

$$
\begin{aligned}
AX = &-\frac{1}{\varrho(u)\sqrt{\varrho(u)^2+|Du|_M^2}}D_X Du + \frac{\varrho'(u)}{\sqrt{\varrho(u)^2+|Du|_M^2}}X \\
&-\left(\frac{-g_M(D_X Du, Du)}{\varrho(u)\left(\varrho(u)^2+|Du|_M^2\right)^{3/2}} - \frac{\varrho'(u) g_M(Du, X)}{\left(\varrho(u)^2+|Du|_M^2\right)^{3/2}}\right) Du,
\end{aligned} \tag{2.26}
$$

for any vector field $X$ tangent to $\Omega$, where $D$ denotes the Levi-Civita connection in $M$. Consequently, if $S(u)$ is a vertical graph over a domain $\Omega \subseteq M$ of a warped product $M$ endowed with a weight function $\phi$, it is not difficult to verify from (1.6) and (2.26) that the $\phi$-mean curvature function $H_\phi(u)$ of $S(u)$ is given by

$$
nH_\phi(u) = -\mathrm{div}_\phi^M\left(\frac{Du}{\varrho(u)\sqrt{\varrho(u)^2+|Du|_M^2}}\right) + \frac{\varrho'(u)}{\sqrt{\varrho(u)^2+|Du|_M^2}}\left(n - \frac{|Du|_M^2}{\varrho(u)^2}\right).
$$

The non-linear elliptic PDE of $\phi$-divergence form $H_\phi(u) = 0$ is called the *$\phi$-minimal hypersurface equation* in $M$, and its solutions provide $\phi$-minimal graphs in $M$. Here, $\mathrm{div}_\phi^M$ is the $\phi$-divergence operator computed in the metric $g_M$.

Note that if $u \in C^\infty(\Omega)$ is a function such that $u(\Omega) \subseteq I$, then its shape operator $A$ is bounded provided that $u$ has finite $C^2$ norm; namely,

$$
\|u\|_{C^2(M)} := \sup_{|\gamma|\leq 2} |D^\gamma u|_{L^\infty(M)} < +\infty.
$$

Note also that the finiteness of the $C^2$ norm of $u$ implies, in particular, that $u$ is bounded, which, in turn, guarantees that $\inf_{S(u)} \varrho(u) > 0$. We shall use this fact without further comments. The condition $\overline{g}(N, \partial_t)$ bounded away from zero is equivalent to $|Du|_M \leq C\varrho(u)$ for some positive constant $C$.

In this context, we obtain a nonparametric version of Theorem 2.2.2.

**Theorem 2.3.1.** *Let $\overline{M}_\phi$ be a weighted warped product space with complete fiber $M$, obeying* (2.3)*, and such that the Hessian of the weight function $\phi$ is bounded from below. Let $S(u)$ be an entire graph in $\overline{M}$ determined by a function $u \in C^\infty(M)$ with finite $C^2$ norm. Suppose that the $\phi$-mean curvature, $H_\phi$, of $S(u)$ satisfies either*

$$
0 \leq H_\phi \leq \inf_{S(u)} (\log \varrho)'(u),
$$

*or*

$$
\alpha \leq H_\phi \leq \inf_{S(u)} (\log \varrho)'(u), \quad \inf_{S(u)} (\log \varrho)'(u) \geq 0
$$

*for some constant $\alpha$, and that $|Du|_M \leq C\varrho(u)$ for some positive constant $C$. If*

$$
|Du|_M^2 \leq \frac{\inf_{S(u)} \varrho(u)^2 \inf_{S(u)}((\log \varrho)'(u) - H_\phi)}{1 - \inf_{S(u)}((\log \varrho)'(u) - H_\phi)}, \tag{2.27}
$$

*then $u \equiv t_0$ for some $t_0 \in I$.*

*Proof.* This result follows from Theorem 2.2.2 by notting that (2.27) implies (2.13), since

$$|\nabla h|^2 = \frac{|Du|_M^2}{\varrho(u)^2 + |Du|_M^2}. \tag{2.28}$$

$\square$

The nonparametric version of Theorem 2.2.3 can be stated and proved in a similar way as the previous one.

**Theorem 2.3.2.** *Let $\overline{M}_\phi$ be a weighted warped product space obeying (2.3) with complete fiber $M$ and convex weight function $\phi$. Let $S(u)$ be an entire graph in $\overline{M}$ determined by a function $u \in C^\infty(M)$ with finite $C^2$ norm and with constant $\phi$-mean curvature satisfying*

$$H_\phi \sup_{S(u)} (\log \varrho)'(u) \leq H^2(p)$$

*and*

$$H_\phi (\log \varrho)'(u)(p) \leq -\Theta(p) H_\phi \sup_{S(u)} (\log \varrho)'(u) \quad \forall p \in S.$$

*If*

$$|Du(p)|_M^2 \leq \frac{\inf_{S(u)} \varrho(u)^2 \left(\inf_{S(u)} H^2 - H_\phi \sup_{S(u)} (\log \varrho)'(h)\right)}{1 - \left(\inf_{S(u)} H^2 - H_\phi \sup_{S(u)} (\log \varrho)'(h)\right)} \quad \forall p \in S,$$

*then $u \equiv t_0$ for some $t_0 \in I$.*

The nonparametric version of Theorem 2.2.4 reads as follows.

**Theorem 2.3.3.** *Let $\overline{M}_\phi = I \times_\varrho M_\phi$ be a weighted warped product with complete fiber $M$. Let $S(u)$ be an entire graph in $\overline{M}$ determined by a bounded function $u \in C^\infty(M)$. Suppose that $H_\phi$ and $\varrho'(u)$ satisfy $H_\phi \varrho'(u) \leq 0$, and that $\varrho$ is* log*-convex along $S(u)$. Suppose also that $\{p \in M : (\log \varrho)''(u(p)) = 0\}$ is a set of isolated points. Let $d\mu = \exp(-\phi) dM$. If $|Du| \in L^1(d\mu)$ and $\varrho'(u)$ is bounded on $S(u)$, then $u \equiv t_0$, where $t_0 \in I$ is such that $\varrho'(t_0) = 0$. If $\varrho(u) \in L^p(d\mu)$ for some $p \in (1, +\infty)$ and the weight function is bounded and convex, then $u \equiv t_0$, where $t_0 \in I$ is such that $\varrho'(t_0) = 0$, and $M$ is compact.*

*Proof.* Taking into account (2.28), we can see that

$$|Du|_M \in L^1(d\mu) \Longrightarrow |\nabla h| \in L^1(d\mu_u),$$

where $d\mu_u = \exp(-\phi) dS(u)$. Then the first part of Theorem 2.3.3 follows immediately from Theorem 2.2.4. For the second part, we note that the volume element of $S(u)$ is given by

$$dS(u) = \varrho(u)^{n-1} \sqrt{\varrho(u)^2 + |Du|_M^2}\, dM.$$

Consequently, since $\varrho(u) \in L^p(d\mu)$ for some $p \in (1, +\infty)$, and $\varrho(h(q)) = \varrho(u(p))$ for all $q = (u(p), p) \in S(u)$, we infer that $\varrho(h) \in L^p(d\mu_u)$. The conclusion follows by applying the corresponding counterpart in Theorem 2.2.4. □

In a similar way, from Theorem 2.2.5, we obtain the following

**Theorem 2.3.4.** *Let $\overline{M}_\phi = I \times M_\phi$ be a weighted product with convex weight function and whose fiber is complete has nonnegative sectional curvatures. Let $S(u)$ be an entire graph in $\overline{M}$ determined by a function $u \in C^\infty(M)$ with finite $C^2$ norm. Suppose that $H_\phi$ does not change sign on $S(u)$. If $|Du| \in L^1(d\mu)$, then $u \equiv t_0$ for some $t_0 \in I$.*

Now, we shall apply the geometric theorems in Subsection 2.2.3 to its PDE counterparts in order to obtain some Bernstein-Moser type results.

**Theorem 2.3.5.** *Let $\overline{M}_\phi$ be a weighted warped product whose fiber $M$ is complete with $\phi$-parabolic universal Riemannian covering, and such that the warping function $\varrho$ is monotone. The only entire bounded solutions $u \in C^\infty(M)$ to*

$$\operatorname{div}_\phi^M \left( \frac{Du}{\varrho(u)\sqrt{\varrho(u)^2 + |Du|_M^2}} \right) = \frac{\varrho'(u)}{\sqrt{\varrho(u)^2 + |Du|_M^2}} \left( n - \frac{|Du|_M^2}{\varrho(u)^2} \right)$$

*with $|Du| \leq C\varrho(u)$ for some positive constant $C$, are the constant functions.*

**Theorem 2.3.6.** *Let $\overline{M}_\phi$ be a weighted warped product satisfying* (2.3)*, with convex weight function $\phi$ and whose fiber $M$ is complete with $\phi$-parabolic universal Riemannian covering. Let $u \in C^\infty(M)$ be an entire bounded solution to*

$$\operatorname{div}_\phi^M \left( \frac{Du}{\varrho(u)\sqrt{\varrho(u)^2 + |Du|_M^2}} \right) = \frac{\varrho'(u)}{\sqrt{\varrho(u)^2 + |Du|_M^2}} \left( n - \frac{|Du|_M^2}{\varrho(u)^2} \right) \quad \text{in} \quad \overline{M},$$

*with $|Du| \leq C\varrho(u)$ for some positive constant $C$. Then $S(u)$ is totally geodesic. In addition, if either the inequality* (2.3) *is strict for all non-zero vector fields on $M$ or $\phi$ is strictly convex on $M$, then $u \equiv t_0$ for some $t_0 \in I$.*

*Proof.* It suffices to notice that since $u$ is bounded, the restriction $\varrho(u)$ on $S(u)$ of the warping function $\varrho$ of $\overline{M}$ satisfies $\sup \varrho(u) < \infty$ and $\inf \varrho(u) > 0$. □

**Remark 2.3.7.** *If the Riemannian universal covering of the fiber is not $\phi$-parabolic, some counterexamples can be shown. We shall follow* [92, Counterexample 10]*. Consider the manifold $(M, g_M) = (\mathbb{R} \times_k \mathbb{R}, dx^2 + k(x)^2 dy^2)$, where $k(x) = \sqrt{1 + \cosh^4 x}$. We will endow $M$ with a weight function $\phi(x, y) = f(y) \in C^\infty(\mathbb{R})$. This manifold is complete (completeness follows from* [84, Lemma 7.40]*), but it is not $\phi$-parabolic (the function $v(x) = -1/\cosh^2 x, x \in \mathbb{R}$, satisfies $\Delta_\phi v = 2[(\cosh^2 x - 1)^2 + 2]/[\cosh^4 x(1 + \cosh^4 x)] > 0$). The function $u(x) = \tanh x$ defines a minimal graph on $(M, g_M)$ in the ambient space $\overline{M} = \mathbb{R} \times M$. In $\overline{M}$, we consider the weight fucntion $\Phi(t, x, y) = f(y)$. From the identity $\operatorname{div}_\phi(jX) = \operatorname{div}(X) - g_M(X, Dj)$, for all*

*$j \in C^\infty(M)$ and all $X \in \mathfrak{X}(M)$, we have that $S(u)$ is a $\Phi$-minimal graph. Note that it is trivially bounded, and $\overline{g}(N, \partial_t)$ is bounded away from zero. To see that some assumptions are needed on the warping function, recall that minimal graphs are $\phi$-minimal hypersurfaces for $\phi = \text{constant}$. Let $(M, g_M) = (\mathbb{R} \times_k \mathbb{R}, dx^2 + k(x)^2 dy^2)$, where $k(x) = (\sqrt{2x^4 + 6x^2 + 5})/(x^2 + 2)$. Since $\sqrt{5}/2 \leq k(x) \leq \sqrt{2}$, $M$ is quasi-isometric to the Euclidean plane, and therefore, a parabolic Riemannian manifold. The function $w(x) = x + \arctan x$ determines a minimal graph on $(M, g_M)$, with bounded length of its gradient. Note that $w$ is unbounded neither from below nor from above and obeys that $\overline{g}(N, \partial_t)$ is bounded away from zero.*

Finally, when the ambient space is simply a product manifold, we obtain the following

**Theorem 2.3.8.** *Let $\overline{M}_\phi$ be a weighted product manifold, whose fiber $M$ is complete with $\phi$-parabolic universal Riemannian covering and such that its Bakry-Émery-Ricci tensor $\mathrm{Ric}_\phi^M$ is nonnegative. Let $u \in C^\infty(M)$ be an entire bounded solution to*

$$nH_\phi(u) = -\mathrm{div}_\phi^M \left( \frac{Du}{\varrho(u)\sqrt{\varrho(u)^2 + |Du|_M^2}} \right) + \frac{\varrho'(u)}{\sqrt{\varrho(u)^2 + |Du|_M^2}} \left( n - \frac{|Du|_M^2}{\varrho(u)^2} \right),$$

*with $H_\phi(u)$ being constant, and $|Du| \leq C\varrho(u)$ for some positive constant $C$. Then $S(u)$ is totally geodesic. In addition, if $\mathrm{Ric}_\phi^M$ is definite positive at some point of $S(u)$, then $u \equiv t_0$ for some $t_0 \in I$.*

## 2.4 Entire graphs with constant $\phi$-mean curvature

Our aim in this section is to study the rigidity of entire graphs defined over the fiber of a weighted product space $\overline{M}_\phi = I \times M_\phi$ whose Bakry-Émery-Ricci tensor is nonnegative. Assuming that the weighted mean curvature is constant and appropriated constraints on the norm of the gradient of the smooth function $u$ that determines such a graph $S(u)$, we prove that $u$ must be constant. Our approach is based on the formula for the $\phi$-Laplacian of the angle function attached to a hypersurface immersed in $\overline{M}_\phi$ (see Lemma 2.1.1 in Section 2.1) jointly with a weak version of the Omori-Yau maximum principle (see Subsection 2.2.1).

The results presented here are part of our paper [73].

**Theorem 2.4.1.** *Let $\overline{M}_\phi = I \times M_\phi$ be a weighted product space such that its fiber $M$ is complete with sectional curvatures bounded from below and nonnegative Bakry-Émery-Ricci tensor $\mathrm{Ric}_\phi^M$, and the Hessian of the weight function $\phi$ is bounded from below. Let $S(u) \subset \overline{M}_\phi$ be an entire graph over $M$ with constant $\phi$-mean curvature $H_\phi$ and bounded second fundamental form $A$. If*

$$|Du|_M \leq \alpha |A|^\beta, \tag{2.29}$$

*for some positive constants $\alpha$ and $\beta$, then $u \equiv t_0$ for some $t_0 \in I$.*

*Proof.* First, we observe that $S(u)$ is, in fact, complete. Indeed, an entire vertical graph is properly immersed into the Riemannian product space $I \times M$, which is obviously complete when the fiber $M$ is complete.

Consider on $S(u)$ the orientation given by

$$N = \frac{1}{\sqrt{1 + |Du|_M^2}}(\partial_t - Du). \tag{2.30}$$

Note that, with respect to this orientation, we have $\overline{g}(N, \partial_t) > 0$. Since $H_\phi$ is supposed to be constant, from Lemma 2.1.1 we obtain

$$\Delta_\phi \Theta = -\left(\mathrm{Ric}_\phi^M(N^M, N^M) + |A|^2\right)\Theta. \tag{2.31}$$

On the other hand, from equation (1.3) it is not difficult to see that $\left(N^M\right)^\top = \Theta \nabla u$ and $|\nabla u|^2 = g_M(N^M, N^M)$. Here, we are taking into account that the height function $h$ of $S(u)$ is nothing but the function $u$ regarded as a function on $S(u)$. Thus, from (2.30) we obtain that

$$|\nabla u|^2 = \frac{|Du|_M^2}{1 + |Du|_M^2}. \tag{2.32}$$

Leting $C = \alpha \sup_{p \in S(u)} |A(p)|^\beta$, from (1.4), (2.29) and (2.32) we have that

$$\Theta \geq \frac{1}{\sqrt{1 + C^2}} > 0. \tag{2.33}$$

Since by hypothesis $H_\phi$ is constant, $\sup_{p \in S(u)} |A(p)| < +\infty$, and the fiber $M^n$ has sectional curvatures bounded from below, from Lemma 2.1.2 we have that the Bakry-Émery-Ricci curvature of $S(u)$ is also bounded from below. Hence, we can apply Lemma 2.2.1 to the function $\Theta$, obtaining a sequence of points $\{p_k\}_{k \in \mathbb{N}} \subset S(u)$ such that $\lim_k \Theta(p_k) = \inf_{S(u)} \Theta$ and $\liminf_k \Delta_\phi \Theta(p_k) \geq 0$.

Consequently, since we are also assuming that the Bakry-Émery-Ricci curvature of $M$ is nonnegative, from (2.31) and (2.33), up to a subsequence we have that

$$\begin{aligned} 0 \leq \liminf_k \Delta_\phi \Theta(p_k) &= \liminf_k \left[-\left(\mathrm{Ric}_\phi^M\left(N^M, N^M\right) + |A|^2\right)\Theta\right](p_k) \\ &= \lim_k \left(\mathrm{Ric}_\phi^M\left(N^M, N^M\right) + |A|^2\right)(p_k)\left(-\inf_{p \in S(u)} \Theta(p)\right) \leq 0. \end{aligned} \tag{2.34}$$

Thus, since $\inf_{p \in S(u)} \Theta(p) > 0$, from (2.34) we get that $\lim_k |A(p_k)| = 0$. So, taking into account our hypothesis (2.29), from (2.32) we get that $\lim_k |\nabla u(p_k)|^2 = 0$. Therefore, from relation (1.4) we infer that $\inf_{p \in S(u)} \Theta(p) = 1$, and, hence, we must have that $u \equiv t_0$ for some $t_0 \in I$. $\square$

Proceeding, we also establish a Moser type result when the fiber of the ambient space has positive Bakry-Émery-Ricci tensor.

**Theorem 2.4.2.** *Let $\overline{M}_\phi = I \times M_\phi$ be a weighted product space, whose fiber $M$ is complete, has* *sectional curvatures bounded from below, and its Bakry-Émery-Ricci tensor satisfies* $\mathrm{Ric}_{\phi}^{M} \geq c$ *for some positive constant* $c$*, and such that the Hessian of the weight function* $\phi$ *is bounded from below. Let* $S(u) \subset \overline{M}_{\phi}$ *be an entire graph over* $M$ *with constant* $\phi$*-mean curvature* $H_{\phi}$ *and bounded second fundamental form* $A$*. If* $|Du|_{M} \leq C$ *for some positive constant* $C$*, then* $u \equiv t_0$ *for some* $t_0 \in I$*.*

*Proof.* We orient $S(u)$ by choosing $N$ as in (2.30) and note that our constraint on the Bakry-Émery-Ricci tensor of $M$ amounts to

$$\mathrm{Ric}_{\phi}^{M}\left(N^{M}, N^{M}\right) \geq c\left|N^{M}\right|_{M}^{2} = c|\nabla u|^{2}. \tag{2.35}$$

Consequently, reasoning as in the proof of Theorem 2.4.1, from (2.34) and (2.35) we get that $\lim_k |\nabla u(p_k)|^2 = 0$. Therefore, taking into account once more the relation (1.4), we conclude again that $\inf_{p \in S(u)} \Theta(p) = 1$, and, hence, $u \equiv t_0$ for some $t_0 \in I$. □

**Remark 2.4.3.** *Related to Theorem 2.4.2, it is worth pointing out that a complete weighted manifold* $(\Sigma, g, \exp(-\phi)dV)$ *whose Bakry-Émery-Ricci tensor satisfies* $\mathrm{Ric}_{\phi} \geq c$ *for some positive constant* $c$ *is not necessarily compact. Indeed, it is easy to verify that the Gaussian space* $\mathbb{G}^n$ *works as a counterexample (see Corollary 2.4.5 below). On the other hand, under the additional hypothesis that the weight function* $\phi$ *is bounded, the extension of Myers' theorem due to Wei and Willie (see* [104, Theorem 1.4]*) guarantees the compactness of* $\Sigma$*.*

We recall that, according to the classical terminology in linear potential theory, a weighted manifold $(\Sigma, g, \exp(-\phi)dV)$ is said to be *$\phi$-parabolic* if any bounded solution $f$ of $\Delta_{\phi} f \geq 0$ must be constant. In this setting, we obtain the following

**Theorem 2.4.4.** *Let* $\overline{M}_{\phi} = I \times M_{\phi}$ *be a weighted product space, whose fiber* $M$ *is complete and such that its Bakry-Émery-Ricci tensor* $\mathrm{Ric}_{\phi}^{M}$ *is nonnegative. Let* $S(u) \subset \overline{M}_{\phi}$ *be an entire graph over* $M$ *with constant* $\phi$*-mean curvature* $H_{\phi}$*. If* $S(u)$ *is* $\phi$*-parabolic, then* $S(u)$ *is totally geodesic. Moreover, if* $\mathrm{Ric}_{\phi}^{M}$ *is strictly positive, then* $u \equiv t_0$ *for some* $t_0 \in I$*.*

*Proof.* As in the proof of the previous theorems, we consider on $S(u)$ the orientation given by (2.30). We also consider on $S(u)$ the bounded function $\theta_{-} = -\Theta$. From Lemma 2.1.1 we get

$$\Delta_f \theta_{-} = \left(\mathrm{Ric}_{\phi}^{M}\left(N^{M}, N^{M}\right) + |A|^2\right)\Theta \geq 0. \tag{2.36}$$

Consequently, since we are assuming that $S(u)$ is $\phi$-parabolic, from (2.36) we obtain that $\theta_{-}$ is constant on $S(u)$. Thus, since $\Theta > 0$ on $S(u)$, returning to (2.36) we get that $|A| \equiv 0$, that is, $S(u)$ is totally geodesic. Moreover, when $\mathrm{Ric}_{\phi}^{M}$ is strictly positive, we conclude that $N^{M}$ vanishes identically on $S(u)$, which means that $N = \partial_t$ on $S(u)$. So, in this case, $u \equiv t_0$ for some $t_0 \in I$. □

Recently, concerning the weighted product space $\mathbb{R} \times \mathbb{G}^n$, which is just the Euclidean space

$\mathbb{R}^{n+1} = \mathbb{R} \times \mathbb{R}^n$ endowed with the Euclidean-Gaussian density

$$\exp(-\phi) = (2\pi)^{-\frac{n}{2}} \exp\left(-\frac{|x|}{2}\right), \tag{2.37}$$

Hieu and Nam extended the classical Bernstein's theorem [30] showing that the only weighted minimal graphs $S(u)$ (that is, with identically zero weighted mean curvature) of smooth functions $u(x_2, \ldots, x_{n+1}) = x_1$ over $\mathbb{G}^n$ are the affine hyperplanes $x_1 = \textit{constant}$ (see [58, Theorem 4]). The *Gaussian space* $\mathbb{G}^n$ is the Euclidean space $\mathbb{R}^n$ with the Gaussian probability density (2.37). Furthermore, we also note that Corollary 3 of [58] assures that entire weighted minimal graphs in $\mathbb{R} \times \mathbb{G}^n$ have finite $\phi$-volume. As it was observed by Impera and Rimoldi in [60, Remark 3], the $\phi$-parabolicity holds if $S(u)$ has finite $\phi$-volume. Consequently, taking this into account, these graphs are $\phi$-parabolic. Hence, from Theorem 2.4.4 we obtain the following extension of Theorem 4 of [58].

**Corollary 2.4.5.** *The affine hyperplanes $x_1 = constant$ of $\mathbb{R} \times \mathbb{G}^n$ are the only entire graphs of smooth functions $u(x_2, \ldots, x_{n+1}) = x_1$ over the Gaussian space $\mathbb{G}^n$ that have constant $\phi$-mean curvature and are $\phi$-parabolic.*

# Chapter 3

# The Lorentzian case

In this chapter, we aim at studying the uniqueness of spacelike slices $S_t := \{t\} \times M^n$, $t \in I$, among spacelike hypersurfaces $S^n$ immersed in a spatially weighted GRW spacetime of the type $M_\phi^{n+1} = -I \times_f F_\phi^n$, furnished with the metric tensor $\overline{g} = -\pi_I^*(dt^2) + f^2(\pi_I)\pi_F^*(g_F)$, and where the weight function $\phi$ does not depend on the parameter $t \in I$, that is, $\overline{g}(\overline{\nabla}\phi, \partial_t) = 0$. Towards this aim, we shall consider a variety of assumptions on the height function and on the angle function of such a $S$, as well as on the $\phi$-mean curvature and on geometric quantities related to $S$, and employ analytic tools such as a weak version of the Omori-Yau maximum principle, $L^{1\leq p<\infty}$-conditions, and $\phi$-parabolicity criteria, in order to guarantee that $S = S_t$ for some $t \in I$. Calabi-Bernstein type results concerning entire graphs $S(u) = \{(u(p), p) : p \in M\} \subset \overline{M}_\phi$ of functions $u \in C^\infty(M)$ such that $u(M) \subseteq I$ will also be given.

## 3.1 Auxiliary lemmas

This section is the dual version of Section 2.1. Here we shall prove the same results adapted to the Lorentzian setting. Namely, we shall see useful formulae for the $\phi$-Laplacians of the height function, $h = \pi_I\big|_S$, and its primitive and of the angle function, $\overline{g}(N, \partial_t)$, of a spacelike hypersurface $S$ with normal field $N$ immersed in a spatially weighted GRW spacetime $M_\phi = -I \times_f F_\phi$. We shall also give some conditions under which we can guarantee that the Bakry-Émery-Ricci tensor has a lower bound.

**Lemma 3.1.1.** *Let $S$ be a oriented spacelike hypersurface with normal field $N$ immersed in a spatially weighted GRW spacetime $M_\phi$ with weight function $\phi$. Then we have*

*(i)*

$$\Delta_\phi h = -(\log f)'(h)(n + |\nabla h|^2) - n\Theta H_\phi; \tag{3.1}$$

*(ii)*

$$\Delta_\phi \mathcal{F}(h) = -n(f'(h) + f(h)\Theta H_\phi), \tag{3.2}$$

*where $\mathcal{F}(t) = \int_{t_0}^t f(s)ds$; and*

*(iii)*

$$
\begin{aligned}
\Delta_\phi(f(h)\Theta) =& nf(h)\overline{g}(\nabla H_\phi, \partial_t) + nf'(h)H_\phi + f(h)\Theta|A|^2 \\
&+ f(h)\Theta\overline{\operatorname{Hess}}\,\phi(N,N) \\
&+ f(h)\Theta\left(\operatorname{Ric}^F(N^F,N^F) - (n-1)(\log f)''(h)|\nabla h|^2\right),
\end{aligned}
\tag{3.3}
$$

*where* $\operatorname{Ric}^F$ *stands for the Ricci curvature tensor of the fiber* $F$*,* $A$ *is the shape operator of* $S$ *related to* $N$ *and* $|A|$ *denotes its norm.*

*Proof.* Equations (3.1) and (3.2) correspond to [38, Lemma 1], and (3.3) corresponds to Lema 1 of our paper [4]. For the sake of completeness we present here a proof of (3.3).

In [12, Corollary 8.2] it is proved that

$$
\begin{aligned}
\Delta(f(h)\Theta) =& nf(h)\overline{g}(\nabla H, \partial_t) + nf'(h)H + f(h)\Theta|A|^2 \\
&+ f(h)\Theta\left(\operatorname{Ric}^F(N^F,N^F) - (n-1)(\log f)''(h)|\nabla h|^2\right).
\end{aligned}
\tag{3.4}
$$

Taking into account the definition of the $\phi$-mean curvature (see (1.11)), we get

$$
nf(h)\overline{g}(\nabla H, \partial_t) = nf(h)\overline{g}(\nabla H_\phi, \partial_t) + f(h)\partial_t^\top\overline{g}(\overline{\nabla}\phi, N).
$$

Moreover, from a straightforward computation we get

$$
\partial_t^\top\left(\overline{g}\left(\overline{\nabla}\phi, N\right)\right) = -\frac{f'}{f}(h)\overline{g}(\overline{\nabla}\phi, N) + \Theta\,\overline{\operatorname{Hess}}\,\phi(N,N) - \overline{g}(\overline{\nabla}\phi, A\partial_t^\top),
$$

and $\nabla(f(h)\Theta) = -f(h)A\partial_t^\top$. So (3.4) can be written as

$$
\begin{aligned}
\Delta(f(h)\Theta) =& nf(h)\overline{g}(\nabla H_\phi, \partial_t) - f'(h)\overline{g}(\overline{\nabla}\phi, N) + f(h)\Theta\overline{\operatorname{Hess}}\,\phi(N,N) \\
&+ \overline{g}(\overline{\nabla}\phi, \nabla(f(h)\Theta)) + nf'(h)H + f(h)\Theta|A|^2 \\
&+ f(h)\Theta\left(\operatorname{Ric}^F(N^F,N^F) - (n-1)(\log f)''(h)|\nabla h|^2\right).
\end{aligned}
\tag{3.5}
$$

Finally, equation (3.3) follows from (3.5) and (1.5). □

In what follows, a *slab* $[t_1,t_2]\times F = \{(t,q) \in -I\times_f F : t_1 \leq t \leq t_2\}$ is called a *timelike bounded region* of the spatially weighted GRW spacetime $M_\phi$. Following the terminology established by Alías and Colares [12], we say that $M_\phi$ obeys the *strong null convergence condition* (SNCC) when the sectional curvatures $K^F$ of its Riemannian fiber $F$ satisfy the following inequality,

$$
K^F \geq \sup_I(f^2(\log f)''). \tag{3.6}
$$

The following lemma establishes some sufficient conditions that guarantee that the Bakry-Émery-Ricci curvature of a spacelike hypersurface $S$ of $M_\phi$ is bounded from below.

**Lemma 3.1.2.** *Let $(M_\phi^{n+1} = -I \times_f F_\phi^n, \overline{g})$ be a spatially weighted GRW spacetime obeying the SNCC* (3.6)*, and such that the Hessian of the weight function $\phi$ is bounded from below, that is, $\overline{\operatorname{Hess}}\,\phi(U,U) \geq \beta\,\overline{g}(U,U)$ for some real constant $\beta$ and for all $U \in \mathfrak{X}(-I \times_f F)$. Let $(S^n, g)$ be a spacelike hypersurface that lies in a timelike bounded region of $M_\phi^{n+1}$. Suppose that the $\phi$-mean curvature $H_\phi$ is bounded on $S^n$. Then the Bakry-Émery-Ricci curvature* $\operatorname{Ric}_\phi$ *of $S^n$ is bounded from below.*

*Proof.* We recall that the curvature tensor $R$ of a spacelike hypersurface $S^n$ can be described in terms of its shape operator $A$ and the curvature tensor $\overline{R}$ of $-I \times_f F^n$ by the so-called Gauss equation given by

$$R(X,Y)Z = (\overline{R}(X,Y)Z)^\top - g(AX,Z)AY + g(AY,Z)AX, \tag{3.7}$$

for all tangent vector fields $X, Y, Z \in \mathfrak{X}(S)$. Here, as in [84], the curvature tensor $R$ is given by

$$R(X,Y)Z = \nabla_{[X,Y]}Z - [\nabla_X, \nabla_Y]Z,$$

where $[\,,\,]$ denotes the Lie bracket, and $X, Y, Z \in \mathfrak{X}(S)$.

Let us consider $X \in \mathfrak{X}(S)$ and a local orthonormal frame $\{E_1, \ldots, E_n\}$ of $\mathfrak{X}(S)$. Then, it follows from the Gauss equation (3.7) that

$$\operatorname{Ric}(X,X) = \sum_{i=1}^{n} \overline{g}(\overline{R}(X,E_i)X,E_i) + nHg(AX,X) + |AX|^2, \tag{3.8}$$

where $|\,|$ is the norm induced by $g$. Moreover, we have that (see [84, Proposition 7.42] for details)

$$\begin{aligned}\overline{R}(X,Y)Z =& R^F(X^F,Y^F)Z^F + ((\log f)'(h))^2(\overline{g}(X,Z)Y - \overline{g}(Y,Z)X) \\ &+ (\log f)''(h)\overline{g}(Z,\partial_t)(\overline{g}(Y,\partial_t)X - \overline{g}(X,\partial_t)Y) \\ &- (\log f)''(h)(\overline{g}(Y,\partial_t)\overline{g}(X,Z\rangle - \overline{g}(X,\partial_t)\overline{g}(Y,Z)\partial_t,\end{aligned}$$

$R^F$ being the curvature tensor of $F$ and $(\,)^F$ being the projection of a vector field onto $F$; hence

$$\begin{aligned}\overline{g}(\overline{R}(X,E_i)X,E_i) =& f(h)^2K^F(X^F,E_i^F)(|X^F|_F^2|E_i^F|_F^2 - g_F(X^F,E_i^F)^2) \\ &+ ((\log f)'(h))^2(|X|^2 - g(X,E_i)^2) \\ &+ (\log f)''(h)g(X,\nabla h)(g(\nabla h,E_i)g(X,E_i) - g(X,\nabla h)) \\ &- (\log f)''(h)(g(\nabla h,E_i)|X|^2 - g(X,\nabla h)g(X,E_i))g(\nabla h,E_i),\end{aligned} \tag{3.9}$$

where $g_F$ is the metric tensor of $F$ and $|\,|_F$ is the norm induced by $g_F$. On the other hand, one can easily see that

$$
\begin{aligned}
&|X^F|_F^2|E_i^F|_F^2 - g_F(X^F, E_i^F)^2 = \\
&\qquad \frac{1}{f(h)^4}(|X|^2 + g(X, \nabla h)^2 + |X|^2 g(\nabla h, E_i)^2 - g(X, E_i)^2 \\
&\qquad - 2g(X, \nabla h)g(X, E_i)g(\nabla h, E_i)),
\end{aligned}
$$

which jointly with (3.6) and (3.9) implies the following lower bound

$$
\sum_{i=1}^{n} \overline{g}(\overline{R}(X, E_i)X, E_i) \geq \frac{f''}{f}(h)(n-1)|X|^2, \tag{3.10}
$$

for all $X \in \mathfrak{X}(S)$. Thus, from (3.15) and (3.10), we get

$$
\mathrm{Ric}(X, X) \geq \frac{f''}{f}(h)(n-1)|X|^2 + nHg(AX, X) + |AX|^2. \tag{3.11}
$$

Since the Hessian of $\phi$ is bounded from below, we have

$$
\begin{aligned}
\operatorname{Hess}\phi(X, X) &= \overline{\operatorname{Hess}}\,\phi(X, X) - \overline{g}(\overline{\nabla}\phi, N)g(AX, X) \\
&\geq \beta\,\overline{g}(X, X) - \overline{g}(\overline{\nabla} f, N)g(AX, X).
\end{aligned} \tag{3.12}
$$

Therefore, from (1.1), (1.11), (3.39) and (3.20), we obtain

$$
\mathrm{Ric}_\phi(X, X) \geq \left(\frac{f''}{f}(h)(n-1) + \beta\right)|X|^2 + nH_\phi g(AX, X) + |AX|^2. \tag{3.13}
$$

Now, we can write

$$
nH_\phi g(AX, X) + |AX|^2 = \left|AX + \frac{nH_\phi}{2}X\right|^2 - \frac{n^2H_\phi^2}{4}|X|^2,
$$

so inequality (3.13) becomes

$$
\mathrm{Ric}_\phi(X, X) \geq \left(\frac{f''}{f}(h)(n-1) + \beta\right)|X|^2 + \left|AX + \frac{nH_\phi}{2}X\right|^2 - \frac{n^2H_\phi^2}{4}|X|^2 \tag{3.14}
$$

for all $X \in \mathfrak{X}(S)$. Finally, the assumptions that $H_\phi$ is bounded and that $S$ is contained in a timelike bounded region of $-I \times F$ guarantee that the Bakry-Émery-Ricci curvature $\mathrm{Ric}_\phi$ of $S$ is bounded from below. □

When the ambient spacetime is a spatially weighted Lorentzian product space $-I \times F_\phi$, under a different set of assumptions, we can also guarantee a lower bound for the Bakry-Émery-Ricci tensor of an oriented spacelike hypersurface $\psi : S^n \to -I \times F^n$.

**Lemma 3.1.3.** *Let $S^n$ be a spacelike hypersurface immersed in a spatially weighted Lorentzian product space $-I \times F_\phi^n$ such that both the sectional curvatures $K^F$ of the Riemannian fiber $F^n$ and the Hessian of the weighted function $\phi$ are bounded from below. If the $\phi$-mean curvature*

$H_\phi$ *and the angle function* $\Theta$ *are bounded on* $S$*, then the Bakry-Émery-Ricci tensor* $\mathrm{Ric}_\phi$ *of* $S$ *is bounded from below.*

*Proof.* One can follow the same steps of the proof of the previous lemma. Let us consider $X \in \mathfrak{X}(S)$ and a local orthonormal frame $\{E_1, \ldots, E_n\}$ of $\mathfrak{X}(S)$. It follows from the Gauss equation (3.7) that

$$\mathrm{Ric}(X, X) = \sum_{i=1}^{n} \overline{g}(\overline{R}(X, E_i)X, E_i) + nHg(AX, X) + |AX|^2. \tag{3.15}$$

Moreover, we have that

$$\begin{aligned}\overline{g}(\overline{R}(X, E_i)X, E_i) =& g_F(R(X^F, E_i^F)X^F, E_i^F) \\ =& K^F(X^F, E_i^F)(g_F(X^F, X^F)g_F(E_i^F, E_i^F) - g_F(X^F, E_i^F)^2).\end{aligned} \tag{3.16}$$

On the other hand, since $X^F = X + \overline{g}(X, \partial_t)\partial_t$, $E_i^F = E_i + \overline{g}(E_i, \partial_t)\partial_t$ and $\nabla h = -\partial_t^\top$, after a straightforward computation we see that

$$g_F(X^F, X^F)g_F(E_i^F, E_i^F) = (1 + g(E_i, \nabla h)^2)(|X|^2 + g(X, \nabla h)^2)$$

and

$$\begin{aligned} g_F(X^F, E_i^F)^2 &= g(X, E_i)^2 + 2g(X, \nabla h)g(E_i, \nabla h)g(X, E_i) \\ &\quad + g(X, \nabla h)g(E_i, \nabla h)^2. \end{aligned}$$

Hence, since we are assuming that $K^F \geq -\kappa$ for some positive constant $\kappa$, from (3.16) we obtain

$$\sum_{i=1}^{n} \overline{g}(\overline{R}(X, E_i)X, E_i) \geq -\kappa \left((n-1)|X|^2 + (n-2)g(X, \nabla h)^2 + |X|^2|\nabla h|^2\right). \tag{3.17}$$

Consequently, from (1.10) and (3.17) we get

$$\sum_{i=1}^{n} \langle \overline{R}(X, E_i)X, E_i \rangle \geq -(n-1)\kappa\Theta^2|X|^2. \tag{3.18}$$

Thus, from (3.15) and (3.18) we obtain

$$\mathrm{Ric(X, X)} \geq -(n-1)\kappa\Theta^2|X|^2 + nHg(AX, X) + |AX|^2. \tag{3.19}$$

On the other hand, taking into account that the Hessian of $\phi$ is bounded from below, we have

$$\mathrm{Hess}\,\phi(X, X) \geq -\beta|X|^2 - \overline{g}(\overline{\nabla} f, N)g(AX, X), \tag{3.20}$$

for all $X \in \mathfrak{X}(S)$ and some positive constant $\beta$.

It follows from (1.1), (3.19) and (3.20) that

$$\mathrm{Ric}_\phi(X,X) \geq -\left((n-1)\kappa\Theta^2+\beta\right)|X|^2 + nH_\phi g(AX,X) + |AX|^2. \tag{3.21}$$

We also observe that we can write

$$nH_\phi g(AX,X) + |AX|^2 = \left|AX + \frac{nH_\phi}{2}X\right|^2 - \frac{n^2H_\phi^2}{4}|X|^2. \tag{3.22}$$

Thus, from (3.21) and (3.22) we get the following lower bound for $\mathrm{Ric}_\phi$,

$$\mathrm{Ric}_\phi(X,X) \geq -\left((n-1)\kappa\Theta^2+\beta+\frac{n^2H_\phi^2}{4}\right)|X|^2 \tag{3.23}$$

for all $X \in \mathfrak{X}(S)$.

Therefore, since both $H_\phi$ and $\Theta$ are supposed to be bounded on $S^n$, from equation (3.23) we infer that $\mathrm{Ric}_\phi$ is bounded from below. □

## 3.2 Statement and proof of the main results

This section is devoted to present our uniqueness results concerning spacelike slices among spacelike hypersurfaces immersed in a spatially weighted Lorentzian spacetime ambient.

### 3.2.1 Comparison inequalities between curvatures in the presence of a Omori-Yau maximum principle

The results presented here are part of our paper [4].

We recall that the following version of the Omori-Yau maximum principle for the $\phi$-Laplacian holds (see [88, Remark 2.18]).

**Lemma 3.2.1.** *Let $(\Sigma^n, g, \exp(-\phi)dV)$ be a complete weighted manifold whose Bakry-Émery-Ricci curvature tensor is bounded from below, and let $u:\Sigma^n\to\mathbb{R}$ be a smooth function bounded from above (resp., bounded from below) on $\Sigma^n$. Then, there exists a sequence of points $\{p_k\}_{k\in\mathbb{N}}\subset\Sigma^n$ such that*

$$\lim_k u(p_k) = \sup_\Sigma u \ (\text{resp., } = \inf_\Sigma u) \quad \text{and} \quad \limsup_k \Delta_\phi u(p_k) \leq 0 \ (\text{resp., } \liminf_k \Delta_\phi u(p_k) \geq 0).$$

We know that the $\phi$-mean curvature of a slice $S_t := \{t\}\times F^n$ in a spatially weighted GRW spacetime, $-I\times_f F^n_\phi$, is given by $H_\phi(t) = (\log f)'(t)$ (with respect to the orientation $N=\partial_t$). The results presented in this subsection use the previous lemma to study the uniqueness of spacelike slices under some comparison inequalities relating their mean curvature and the $\phi$-mean curvature function of a given spacelike hypersurface $S^n$. We also require some bounds on the norm of the gradient of the height function, which can be interpreted as a measure of how much a spacelike hypersurface is no longer a slice. Controls on $H_\phi$ involving geometric inequalities have already been employed in the literature (see, for instance, [48]).

Our first result here generalizes Theorem 4.3 of [35] and Theorem 2 of [38].

**Theorem 3.2.2.** *Let $M^{n+1}_\phi = -I \times_f F^n_\phi$ be a spatially weighted GRW spacetime obeying the SNCC* (3.6) *and such that the Hessian of the weight function $\phi$ is bounded from below. Let $\psi : S^n \to M^{n+1}_\phi$ be a complete spacelike hypersurface that lies in a timelike bounded region of $M^{n+1}_\phi$. Suppose that the $\phi$-mean curvature $H_\phi$ of $S^n$ satisfies*

$$(\log f)'(h) \leq H_\phi \leq \alpha \quad \text{and} \quad H_\phi \geq 0 \tag{3.24}$$

*for some positive constant $\alpha$. If*

$$|\nabla h| \leq \beta \inf_S |H_\phi - (\log f)'(h)|^\gamma \tag{3.25}$$

*for some constants $\beta > 0$ and $\gamma \neq 0$, then $S^n$ is a slice $S_t$ for some $t \in I$.*

*Proof.* From Lemma 3.1.1-(ii), for the $\phi$-Laplacian of the function $\mathcal{F}(t) = \int_{t_0}^t f(s)ds$ we get

$$\begin{aligned}\Delta_\phi \mathcal{F}(h) &= -nf(h)\left((\log f)'(h) + \Theta H_\phi\right)\\ &\geq nf(h)\left(H_\phi - (\log f)'(h)\right).\end{aligned}$$

Thus, by (3.24), we conclude that $\Delta_\phi \mathcal{F}(h) \geq 0$ on $S^n$. Since the smooth function $\mathcal{F}$ is bounded from above, and, by Lemma 3.1.3, $\mathrm{Ric}_\phi$ is bounded from below, the hypotheses of Lemma 3.2.1 are verified, and we can take a sequence of points $\{p_k\}_{k\in\mathbb{N}} \subset S^n$ such that $\limsup_k \Delta_\phi \mathcal{F}(h(p_k)) \leq 0$. Thus, up to a subsequence, we have

$$0 \geq \limsup_k \Delta_\phi \mathcal{F}(h(p_k)) \geq \lim_k \left[nf(h)\left(H_\phi - (\log f)'(h)\right)\right](p_k) \geq 0.$$

Since $S^n$ is contained in a timelike bounded region of $M^{n+1}_\phi$, there exists a positive constant $C$ such that $f(h(p)) \geq C$ for all $p \in S^n$. Therefore, we have that $\lim_k (H_f - (\log \varrho)'(h))(p_k) = 0$, and, taking into account our hypothesis (3.25), the proof is complete. $\square$

By noting that, when the function $\phi$ is constant, the $\phi$-mean curvature is just the usual mean curvature $H$, our next result can be seen as a generalization of [3, Theorem 3.3]. It is also interesting to observe that, although the authors of [3] deal with hypersurfaces in RW spacetimes, the same conclusions can be obtained when considering GRW spacetimes $-I \times_f F$ such that the sectional curvatures of the fiber are bounded from below, as it happens under the SNCC.

**Theorem 3.2.3.** *Let $M^{n+1}_\phi = -I \times_f F^n_\phi$ be a spatially weighted GRW spacetime obeying the SNCC* (3.6) *with convex weight function $\phi$, that is, $\overline{\mathrm{Hess}}\,\phi \geq 0$). Let $\psi : S^n \to M^{n+1}_\phi$ be a complete spacelike hypersurface that lies in a timelike bounded region of $M^{n+1}_\phi$ and with constant*

*$\phi$-mean curvature $H_\phi$ satisfying*

$$0 \le H_\phi \sup_S (\log f)'(h) \le H^2. \tag{3.26}$$

*If*

$$|\nabla h|^2 \le \alpha \left( \inf_S H^2 - H_\phi \sup_S (\log f)'(h) \right)^\beta \tag{3.27}$$

*for some constants $\alpha > 0$ and $\beta \neq 0$, then $S^n = S_t$ for some $t \in I$.*

*Proof.* From the SNCC (3.6) we get that

$$\mathrm{Ric}^F(N^F, N^F) - (n-1)(\log f)''(h)|\nabla h|^2 \ge 0.$$

Therefore, from Lemma 3.1.1-(iii) and the assumptions of the theorem, the following holds,

$$\Delta_\phi(f(h)\Theta) \le nf'(h)H_\phi + f(h)\Theta|A|^2. \tag{3.28}$$

By using (3.28) and (3.26), we obtain

$$\Delta_\phi \left(f(h)\Theta\right) \le f(h)\Theta \left( |A|^2 - nH_\phi \sup_S (\log f)'(h) \right).$$

On the other hand, a simple algebraic computation shows that $|A|^2 = n^2H^2 - n(n-1)H_2$, where $H_2$ is the second order mean curvature defined by $\binom{n}{2}H_2 = \sum_{i<j} k_i k_j$, being $k_i$, $i = 1, \ldots, n$, the principal curvatures of $S^n$. Therefore,

$$\begin{aligned} \Delta_\phi \left(f(h)\Theta\right) &\le f(h)\Theta \left( n^2H^2 - n(n-1)H_2 - nH_\phi \sup_S (\log f)'(h) \right) \\ &= n(n-1)f(h)\Theta(H^2 - H_2) + nf(h)\Theta \left( H^2 - H_\phi \sup_S (\log f)'(h) \right) \le 0, \end{aligned} \tag{3.29}$$

where we used again (3.26) and the fact that $H^2 - H_2 \ge 0$.

From (3.27) and (1.10) we have that $\Theta$ is bounded, which jointly with the fact that $S^n$ lies in a timelike bounded region of $M^{n+1}_\phi$ implies that the function $f(h)\Theta$ is bounded from below. On the other hand, by Lemma 3.1.3, we know that $\mathrm{Ric}_\phi$ is also bounded from below. So, we can apply the Omori-Yau maximum principle for the $\phi$-Laplacian (see Lemma 3.2.1) to get a sequence of points $\{p_k\} \subset S^n$ such that

$$\lim_k \left(f(h(p_k))\Theta(p_k)\right) = \inf_S f(h)\Theta \qquad \text{and} \qquad \liminf_k \Delta_\phi \left(f(h(p_k))\Theta(p_k)\right) \ge 0.$$

Thus, up to a subsequence (3.29) implies that

$$
\begin{aligned}
0 \leq\ & \liminf_k \Delta_\phi f(h(p_k))\Theta(p_k) \\
\leq\ & n(n-1) \lim_k \big(f(h(p_k))\Theta(p_k)(H^2 - H_2)(p_k)\big) \\
& + n \lim_k \left( f(h(p_k))\Theta(p_k) \left( H^2(p_k) - H_\phi \sup_S (\log f)'(h) \right) \right) \leq 0,
\end{aligned}
$$

so, in particular,

$$\lim_k \left( H^2(p_k) - H_\phi \sup_S (\log f)'(h) \right) = 0.$$

Consequently, from (3.26) we get

$$\inf_S \left( H^2 - H_\phi \sup_S (\log f)'(h) \right) = 0,$$

and the result follows from (3.27). □

### 3.2.2 Uniqueness under $L^{1\leq p<\infty}$-conditions

The result presented in this subsection is also part of our paper [4].

Let $(\Sigma^n, g, d\mu = \exp(-\phi)dV)$ be a weighted manifold. For $1 \leq p < \infty$, let us consider $L^p(d\mu) := \{u : \Sigma^n \to \mathbb{R} : \int_\Sigma |u|^p d\mu < +\infty\}$. We recall the following result, which is a consequence of [85, Theorem 1.1].

**Lemma 3.2.4.** *Let $u$ be a nonnegative smooth $\phi$-subharmonic function on a complete Riemannian manifold $\Sigma^n$. If $u \in L^p(d\mu)$ for some $1 \leq p < \infty$, then $u$ is constant.*

The next result is due to Wei and Wylie [104].

**Lemma 3.2.5.** *Let $\Sigma^n$ be a noncompact complete Riemannian manifold with nonnegative Bakry-Émery-Ricci curvature for some bounded weight function $\phi \in C^\infty(\Sigma)$. Then $\Sigma^n$ has at least linear $\phi$-volume growth, i.e., for any $p \in \Sigma^n$, $vol_\phi(B(p, R))$ has at least linear growth on $R$, where $B(p, R)$ is the geodesic ball in $\Sigma^n$ centered at $p$ with radius $R$.*

Combining Lemmas 3.2.4 and 3.2.5, we can prove the main result of this subsection, which reads as follows.

**Theorem 3.2.6.** *Let $M^{n+1}_\phi = -I \times_f F^n_\phi$ be a spatially weighted GRW spacetime such that the Hessian of the weight function $\phi$ is bounded from below, and let $\psi : S^n \to M^{n+1}_\phi$ be a complete spacelike hypersurface. Suppose that $H_\phi > 0$, that $f'(h) > 0$, and that the following inequalities are satisfied,*

$$\frac{n^2}{4}((\log f)'(h))^2 \leq \frac{n^2 H_\phi^2}{4} \leq (n-1)\frac{f''}{f}(h). \tag{3.30}$$

*If $f(h) \in L^p(d\mu)$ for some $1 \le p < \infty$, where $d\mu = \exp(-\phi)dS$, then $S^n$ is a slice of $M_\phi^{n+1}$ with $vol_\phi(S) < +\infty$. In addition, if $M_\phi^{n+1}$ obeys the* SNCC (3.6) *and $\phi$ is bounded and convex, then $S^n$ is compact.*

*Proof.* From Lemma 3.1.1-(i) we get

$$\Delta_\phi h = -(\log f)'(h)(n + |\nabla h|^2) - nH_\phi \Theta. \tag{3.31}$$

Moreover, it is not difficult to see that hypothesis (3.30) implies that $(\log f)''(h) \ge 0$ on $S^n$. Consequently, using the assumptions of the theorem, from (3.31) we get

$$\begin{aligned} \Delta_\phi f(h) &= f'(h)\Delta_\phi h + f''(h)|\nabla h|^2 \\ &\ge nf'(h)\left(H_\phi - (\log f)'(h)\right) \ge 0. \end{aligned} \tag{3.32}$$

Thus, since we are assuming that $f(h) \in L^p(d\mu)$, in view of (3.32) we can apply Lemma 3.2.4 to conclude that $f(h)$ is constant on $S^n$, and so $vol_\phi(S) < +\infty$. Hence, since $f'(h) > 0$ on $S^n$, we get that $h$ is also constant, and, consequently, $S^n$ must be a slice of $M_\phi^{n+1}$.

Furthermore, if $M_\phi^{n+1}$ obeys the SNCC and the weight function $\phi$ is convex, from (3.23) we obtain

$$\mathrm{Ric}_\phi(X, X) \ge \left((n-1)\frac{f''}{f}(h) - \frac{n^2 H_\phi^2}{4}\right)|X|^2 \tag{3.33}$$

for all $X \in \mathfrak{X}(S)$. Thus, putting (3.30) into (3.33), we see that $\mathrm{Ric}_\phi$ is nonnegative. Therefore, Lemma 3.2.5 guarantees that $S^n$ must be compact. □

**Remark 3.2.7.** *By* [10, Proposition 3.2]*, if a GRW spacetime admits a compact spacelike hypersurface, then it is spatially closed. Therefore, the conclusion of the second part of Theorem 3.52 implies in particular that the fiber $F^n$ must also be compact. Moreover, the compactness of $S^n$ could be obtained in a more direct way if we add the assumption that $F^n$ is compact. In fact, again by* [10, Proposition 3.2]*, if $F^n$ is compact and $S^n$ is a complete spacelike hypersurface on which the function $f(h)$ is bounded, then $S^n$ is compact. That last assumption holds since, as a consequence of the hypotheses of Theorem 3.30, we proved that $f(h)$ is, in fact, constant.*

**Remark 3.2.8.** *Following* [95]*, given a spacelike hypersurface $S^n$ immersed in $M_\phi^{n+1} = -I \times_f F_\phi^n$, with future-pointing Gauss map $N$, for each $q \in S^n$ we let $\xi_q = E(q)N_q + \xi_q^\top$, where $E(q) := -\overline{g}(\xi_q, N_q) = -f(h(q))\Theta(q) > 0$ and $\xi_q^\top$ are, respectively, the energy and the $n$-momentum that the instantaneous observer $N_q$ measures for $\xi_q$.*

*Extending the concept of total energy established in* [70]*, for $1 \le p < \infty$, we say that $S^n$ has finite total $(\phi, p)$-energy when $\int_S E^p d\mu < +\infty$. So, since $E(q) \ge f(h(q))$ for all $q \in S^n$, if we assume in Theorem 3.2.6 that $S^n$ has finite total $(\phi, p)$-energy instead of $f(h) \in L^p(d\mu)$, the conclusion of this theorem still holds.*

### 3.2.3 Uniqueness under $\phi$-parabolicity criteria

In this subsection, we extend a technique due to Romero, Rubio and Salamanca [89–91] establishing sufficient conditions to guarantee the $\phi$-parabolicity of complete spacelike hypersurfaces immersed in a weighted generalized Robertson-Walker spacetime whose fiber has $\phi$-parabolic universal Riemannian covering. As application of these criteria, we obtain uniqueness results concerning spacelike slices among spacelike hypersurfaces immersed in a spatially weighted generalized Robertson-Walker spacetimes. The results presented here are contained in our paper [5].

We shall say that a smooth function $u$ on a weighted manifold $P_\phi = (P, g, d\mu = \exp(-\phi)dP)$ is *$\phi$-superharmonic* if $\Delta_\phi u \leq 0$. Taking this into account, the weighted manifold $P_\phi$ is called *$\phi$-parabolic* if there is no nonconstant, nonnegative, $\phi$-superharmonic function on it. On the other hand, for any compact subset $K \subset P$, we define the *$\phi$-capacity* of $K$ as

$$\mathrm{cap}_\phi(K) = \inf\left\{\int_P |\nabla u|^2 d\mu : u \in \mathrm{Lip}_0(P) \,\mathrm{and}\, u\big|_K \equiv 1\right\},$$

where $\mathrm{Lip}_0(P)$ is the set of all compactly supported Lipschitz functions on $P$. The following statement relates the notion of $\phi$-capacity to the concept of $\phi$-parabolicity (see [52, Proposition 2.1]).

**Lemma 3.2.9.** *The weighted manifold $(P, g, d\mu = \exp(-\phi)dP)$ is $\phi$-parabolic if and only if $\mathrm{cap}_\phi(K) = 0$ for any compact set $K \subset P$.*

Let us recall that given two Riemannian manifolds $(P, g)$ and $(P', g')$, a diffeomorphism $\psi$ from $P$ onto $P'$ is called a *quasi-isometry* if there exists a constant $c \geq 1$ such that

$$c^{-1}|v|_g \leq |d\psi(v)|_{g'} \leq c|v|_g$$

for all $v \in T_pP, p \in P$ (see [62] for more details). Suppose that we can endow both $P$ and $P'$ with the same weight function $\phi$. We can reason as in Section 5 of [51] to verify that the $\phi$-capacity changes under a quasi-isometry at most by a constant factor. So, from Lemma 3.2.9, we can state the following result.

**Lemma 3.2.10.** *Let $(P, g)$ and $(P', g')$ be two Riemannian manifolds endowed with the same weight function $\phi$. If $P$ and $P'$ are quasi-isometric, then $P$ and $P'$ are $\phi$-parabolic or not simultaneously.*

Given a spacelike hypersurface $x : S \to M$ in a GRW spacetime $M = -I \times_f F$, the following lemma provides sufficient conditions to guarantee that the hypersurface $S$ and the fiber $F$ are quasi-isometric (see [89, Lemma 4.1]).

**Lemma 3.2.11.** *Let $x : S \to M$ be a spacelike hypersurface in a GRW spacetime $M = -I \times_f F$, whose hyperbolic angle function $\Theta$ is bounded. If the warping function $f$ on $S$ satisfies*

*(i)* $\sup_S f(h) < \infty$ *and*

*(ii)* $\inf_S f(h) > 0$,

*then* $\pi = \pi_F \circ x$ *is a quasi-isometry from* $S$ *onto* $F$.

We can now present our main criterion of $\phi$-parabolicity.

**Theorem 3.2.12.** *Let* $S$ *be a complete spacelike hypersurface in a spatially weighted GRW spacetime* $M_\phi$ *with weight function* $\phi$*, whose fiber* $F$ *is complete with* $\phi$*-parabolic universal Riemannian covering. If the angle function* $\Theta$ *of* $S$ *is bounded and the restriction* $f(h)$ *on* $S$ *of the warping function* $f$ *of* $M$ *satisfies*

*(i)* $\sup_S f(h) < \infty$ *and*

*(ii)* $\inf_S f(h) > 0$,

*then* $S$ *is* $\phi$*-parabolic.*

The proof of Theorem 3.2.12 follows the same steps of the proof of [89, Theorem 4.4] (see also [90, Theorem 1]). We shall need the following standard result on covering spaces (see [59] for instance).

**Lemma 3.2.13.** *Let* $\rho : (\tilde{E}, \tilde{x}_0) \to (E, x_0)$ *be a covering space and let* $h : (W, y_0) \to (E, x_0)$ *be a continuous map, where* $W$ *is a path connected and locally path connected topological space. Then, there exists a lift* $\tilde{h} : (W, y_0) \to (\tilde{E}, \tilde{x}_0)$ *of* $h$ *if and only if* $h_*(\pi_1(W, y_0)) \subset \rho_*(\pi_1(\tilde{E}, \tilde{x}_0))$*.*

*Proof of Theorem 3.2.12.* Using [10, Lemma 3.1], we know that the projection on the fiber, $\pi : S \to F$, is a covering map. Moreover, by Lemma 3.2.11, we can find a constant $c \geq 1$ such that

$$c^{-1} g_F(d\pi(v), d\pi(v)) \leq g_S(v, v) \leq c\, g_F(d\pi(v), d\pi(v)) \tag{3.34}$$

for all $v \in T_pS$ and all $p \in S$.

Let $(\tilde{S}, g_{\tilde{S}})$ be the universal Riemannian covering of $(S, g_S)$, and denote by $\tilde{\pi}_S : \tilde{S} \to S$ the corresponding Riemannian covering map. From Lemma 3.2.13 we conclude that there exists a lift $\tilde{h} : \tilde{S} \to \tilde{F}$ of the map $h = \pi \circ \tilde{\pi}_S : \tilde{S} \to F$. It is easy to check that $\tilde{h}$ is, in fact, a diffeomorphism from $\tilde{S}$ to $\tilde{F}$. Note that (3.34) implies

$$c^{-1} g_{\tilde{F}}(d\tilde{h}(\tilde{v}), d\tilde{h}(\tilde{v})) \leq g_{\tilde{S}}(\tilde{v}, \tilde{v}) \leq c\, g_{\tilde{F}}(d\tilde{h}(\tilde{v}), d\tilde{h}(\tilde{v}))$$

for any $\tilde{v} \in T_{\tilde{p}}\tilde{S}, \tilde{p} \in \tilde{S}$, which means that $\tilde{h}$ is a quasi-isometry from $(\tilde{S}, g_{\tilde{S}})$ onto $(\tilde{F}, g_{\tilde{F}})$.

Finally, let $u$ be a nonnegative $\phi$-superharmonic function on $S$, and put $\tilde{u} = u \circ \tilde{\pi}_S$. The function $\tilde{u}$ is a nonnegative $\phi$-superharmonic function on the $\phi$-parabolic Riemannian manifold $\tilde{S}$. Therefore, $\tilde{u}$ must be constant, and, consequently, $u$ is also constant. □

As a direct consequence of Theorem 3.2.12 we get the following corollaries.

**Corollary 3.2.14.** *Let* $S$ *be a complete spacelike hypersurface in a spatially weighted GRW spacetime* $M_\phi$ *with weight function* $\phi$ *and whose fiber* $F$ *is complete, simply connected and* $\phi$*-parabolic. If the angle function* $\Theta$ *of* $S$ *is bounded, and the warping function on* $S$*,* $f(h)$*, is bounded and satisfies* $\inf_S f(h) > 0$*, then* $S$ *is* $\phi$*-parabolic.*

We recall that a GRW is said to be *static* when its warping function is constant, which, without loss of generality, can be supposed equal to 1.

**Corollary 3.2.15.** *Let $S$ be a complete spacelike hypersurface in a static spatially weighted GRW spacetime $M_\phi$ with weight function $\phi$ and whose fiber $F$ is complete with $\phi$-parabolic universal Riemannian covering. If the angle function $\Theta$ of $S$ is bounded, then $S$ is $\phi$-parabolic.*

As an application of Theorem 3.2.12, we shall prove some uniqueness results concerning spacelike slices among spacelike hypersurfaces immersed in a spatially weighted GRW spacetime.

**Theorem 3.2.16.** *Let $M_\phi$ be a spatially weighted GRW spacetime whose fiber $F$ is complete with $\phi$-parabolic universal Riemannian covering, and such that the warping function $f$ is monotone. The only $\phi$-maximal complete spacelike hypersurfaces contained in a timelike bounded region of $M$ and with bounded angle function $\Theta$ are the slices $\{t_0\}\times F$, where $t_0 \in I$ is such that $f'(t_0) = 0$.*

*Proof.* Let $S$ be such a spacelike hypersurface. From Lemma 3.1.1, we obtain

$$\Delta_\phi \mathcal{F}(h) = -nf'(h).$$

Consequently, the monotonicity of $f$ implies that $\Delta_\phi \mathcal{F}(h)$ is globally either nonpositive or nonnegative signed. Since $S$ is contained in a timelike bounded region of $M$ and the warping function $f$ is monotone, the function $\mathcal{F}(h)$ is clearly bounded on $S$. From Theorem 3.2.12 we know that $S$ is $\phi$-parabolic, so $\mathcal{F}(h)$ is constant in $S$, and, hence, $h$ must also be constant in $S$. $\square$

In what follows, we shall assume that the ambient space obeys the so-called *null convergence condition* (NCC) meaning that

$$\mathrm{Ric}^F \geq (n-1)f^2(\log f)'' g_F, \tag{3.35}$$

which is equivalent to the Ricci curvature of $M$ being nonnegative on null or lightlike directions (see [77]). In our next result we shall also assume that the weight function $\phi$ is convex, that is, $\overline{\mathrm{Hess}}\,\phi \geq 0$.

**Theorem 3.2.17.** *Let $M_\phi$ be a spatially weighted GRW spacetime satisfying* (3.35)*, with convex weight function $\phi$, and whose fiber $F$ is complete with $\phi$-parabolic universal Riemannian covering. Let $S$ be a complete $\phi$-maximal spacelike hypersurface immersed in $M$, with bounded angle function $\Theta$ and such that the restriction $f(h)$ on $S$ of the warping function $f$ of $M$ satisfies*

*(i)* $\sup f(h) < \infty$ *and*

*(ii)* $\inf f(h) > 0$.

*Then $S$ is totally geodesic. In addition, if either the inequality (3.35) is strict for all non-zero vector fields on $F$ or $\phi$ is strictly convex on $F$, then $S$ is a slice $S_{t_0}$, where $t_0 \in I$ is such that $f'(t_0) = 0$.*

*Proof.* From Lemma 3.1.1, we have that the $\phi$-Laplacian of the bounded function $f(h)\Theta$ is given by

$$\begin{aligned}\Delta_\phi(f(h)\Theta) =& f(h)\Theta|A|^2 + f(h)\Theta\overline{\operatorname{Hess}}\,\phi(N,N)\\ &+ f(h)\Theta\left(\operatorname{Ric}^F(N^F,N^F) - (n-1)(\log f)''(h)|\nabla h|^2\right).\end{aligned} \tag{3.36}$$

Since $\phi$ is convex and we are under the null convergence condition (3.35), it follows that $\Delta_\phi(f(h)\Theta) \leq 0$. Theorem 3.2.12 assures that $S$ is $\phi$-parabolic, so $f(h)\Theta$ must be constant. Therefore, returning to (3.36), we infer that $|A| \equiv 0$, that is, $S$ is totally geodesic,

$$\overline{\operatorname{Hess}}\,\phi(N,N) = \operatorname{Hess}^F\phi(N^F,N^F) = 0, \tag{3.37}$$

and

$$\operatorname{Ric}^F(N^F,N^F) - (n-1)(\log f)''(h)|\nabla h|^2 = 0.$$

Consequently, if the inequality (3.35) is strict, or if $\phi$ is strictly convex on $F$, then (3.36) also gives that $\left|N^F\right| = |\nabla h| = 0$ on $S$, that is, $S$ is a slice. $\square$

We recall that the Gaussian space $\mathbb{G}^n$ corresponds to the Euclidean space $\mathbb{R}^n$ endowed with the Gaussian probability density $\exp(-\phi(x)) = (2\pi)^{-\frac{n}{2}}\exp\left(-\frac{|x|^2}{2}\right)$. From Corollary 3 of [58] we have that $\mathbb{G}^n$ has finite $\phi$-volume. Consequently, taking into account Remark 3.8 of [60], we conclude that $\mathbb{G}^n$ is $\phi$-parabolic. On the other hand, it is not difficult to verify that the weight function $\phi$ of $\mathbb{G}^n$ is is strictly convex. Consequently, from Theorem 3.2.17 we get

**Corollary 3.2.18.** *The only complete $\phi$-maximal spacelike hypersurfaces of $-\mathbb{R}\times\mathbb{G}^n$ having bounded angle function are the spacelike hyperplanes $\{t\}\times\mathbb{G}^n$.*

Now, considering as ambient space a static GRW spacetime, we obtain the following result.

**Theorem 3.2.19.** *Let $M_\phi$ be a static spatially weighted GRW spacetime, whose fiber $F$ is complete with $\phi$-parabolic universal Riemannian covering and such that its Bakry-Émery-Ricci tensor $\operatorname{Ric}_\phi^F$ is nonnegative. Let $S$ be a complete spacelike hypersurface immersed in $M_\phi$ with constant $\phi$-mean curvature, $H_\phi$. If the angle function $\Theta$ of $S$ is bounded, then $S$ is totally geodesic. In addition, if $\operatorname{Ric}_\phi^F$ is definite positive at some point of $S$, then $S$ is a slice $S_t \subset M_\phi$.*

*Proof.* From Lemma 3.1.1, (1.1), and (3.37) we get that

$$\Delta_\phi\Theta = (\operatorname{Ric}_\phi^F(N^F,N^F) + |A|^2)\Theta. \tag{3.38}$$

Consequently, since we are supposing that $\operatorname{Ric}_\phi^F$ is nonnegative and that $\Theta$ is negative and bounded on $S$, we can apply Corollary 3.2.15 to conclude that $\Theta$ is constant on $S$. Thus, returning to (3.38), we get that $|A| \equiv 0$, that is, $S$ is totally geodesic. Moreover, if $\operatorname{Ric}_\phi^F$ is definite positive at some $p \in F$, considering once more equation (3.38), and taking into account relation (1.8), we conclude that $\Theta \equiv -1$ on $S$, which means that $S$ is a slice of $M$. $\square$

We recall that Xin [106] and Aiyama [1] proved simultaneously and independently that the only spacelike hypersurfaces of the Lorentz-Minkowski space $\mathbb{L}^{n+1} = -\mathbb{R} \times \mathbb{R}^n$, with constant mean curvature and having bounded angle function, are the spacelike hyperplanes. Hence, from Theorem 3.2.19 we get the following extension of this Xin-Aiyama result.

**Corollary 3.2.20.** *The only complete spacelike hypersurfaces of $-\mathbb{R}\times\mathbb{G}^n$, with constant $\phi$-mean curvature and having bounded angle function, are the spacelike hyperplanes $\{t\} \times \mathbb{G}^n$.*

Proceeding, we shall use a Bochner's formula due to Wei and Wylie [104] to obtain the following theorem.

**Theorem 3.2.21.** *Let $M_\phi$ be a static spatially weighted GRW spacetime endowed with a convex weight function $\phi$ and whose fiber $F$ is complete, with nonnegative sectional curvature, and such that its universal Riemannian covering is $\phi$-parabolic. Let $S$ be a complete spacelike hypersurface lying in a semi-space of $M$ and with constant $\phi$-mean curvature $H_\phi$. If the angle function $\Theta$ is bounded, then $S$ is a slice $S_t$.*

*Proof.* Since we are supposing that $F$ has nonnegative sectional curvature, it follows from inequalities (3.3) and (3.4) of [69] that

$$\mathrm{Ric}(X,X) \geq nHg_S(AX,X) + |AX|^2. \tag{3.39}$$

On the other hand, taking into account that

$$\operatorname{Hess}\phi(X,X) = \overline{\operatorname{Hess}}\,\phi(X,X) - \overline{g}(\overline{\nabla}\phi, N)g_S(AX,X),$$

from the convexity of the weight function $\phi$ we get

$$\operatorname{Hess}\phi(X,X) \geq -\overline{g}(\overline{\nabla}\phi, N)g_S(AX,X). \tag{3.40}$$

From (3.39) and (3.40) we get the following lower bound for $\mathrm{Ric}_\phi$,

$$\mathrm{Ric}_\phi(X,X) \geq nH_\phi g_S(AX,X) + |AX|^2. \tag{3.41}$$

Inequality (3.41) provides us with

$$\mathrm{Ric}_\phi(\nabla h,\nabla h) \geq nH_\phi g_S(A(\nabla h),\nabla h) + |A(\nabla h)|^2. \tag{3.42}$$

Since $H_\phi$ is constant, we have

$$\nabla\Delta_\phi h = -nH_\phi A(\nabla h). \tag{3.43}$$

On the other hand, from Bochner's formula (see [104]),

$$\frac{1}{2}\Delta_\phi|\nabla h|^2 = |\operatorname{Hess} h|^2 + g_S(\nabla h, \nabla\Delta_\phi h) + \mathrm{Ric}_\phi(\nabla h,\nabla h). \tag{3.44}$$

Consequently, from (1.10), (3.42), (3.43) and (3.44) we get

$$\frac{1}{2}\Delta_\phi\Theta^2 = \frac{1}{2}\Delta_\phi|\nabla h|^2 \geq |\mathrm{Hess}\, h|^2 \geq 0. \tag{3.45}$$

Thus, from Corollary 3.2.15, we have that $\Theta$ is constant, and, returning to (3.45), we get $|\mathrm{Hess}\, h|^2 = 0$ in $S$. Then, since $n|\mathrm{Hess}\, h|^2 \geq (\Delta h)^2$, we have that $h$ is harmonic. So, since $\Delta h = -nH\Theta$, we also get that $H = 0$ in $S$, and from (3.39) we have that $S$ has nonnegative Ricci curvature. Therefore, since we are supposing that $S$ lies in a semi-space of $M$, we can apply the strong Liouville property due to Yau [108] (see also [65, Theorem 4.8]) to conclude that $h$ must be constant and, hence, $S$ is a slice of $M$. $\square$

Extending the ideas of [2], we now consider *spatially weighted steady state type spacetimes*, that is, GRW spacetimes of the type $-\mathbb{R}\times_{\exp(t)} F$ whose fiber $F$ is endowed with a weight function $\phi$. In this setting, our next result is an extension of Theorem 8 of [2].

**Theorem 3.2.22.** *Let $M_\phi$ be a spatially weighted steady state type spacetime, whose fiber $F$ is complete, with nonnegative sectional curvature, and let $S$ be a complete spacelike hypersurface that lies in a timelike bounded region of $M$. Suppose that $|\overline{\nabla}\phi|$ is bounded on $S$. If the $\phi$-mean curvature $H_\phi$ of $S$ is constant and the angle function $\Theta$ is bounded, then $H_\phi = 1$. In addition, if the universal Riemannian covering of $F$ is $\phi$-parabolic, then $S$ is a slice $\{t\}\times F$.*

*Proof.* We claim that the mean curvature $H$ of $S$ is bounded. Indeed, since $\overline{g}(\overline{\nabla}\phi, \partial_t) = 0$, from (1.11) we have that

$$\begin{aligned} n|H| &\leq n|H_\phi| + |\overline{g}(\overline{\nabla}\phi, N)| \\ &= n|H_\phi| + |\overline{g}(\overline{\nabla}\phi, N^F)|. \end{aligned} \tag{3.46}$$

Thus, from (1.8) and (3.46) we get

$$n|H| \leq n|H_\phi| + |\overline{\nabla}\phi|(\Theta^2 - 1). \tag{3.47}$$

Consequently, since $H_\phi$ is constant, and both $\overline{\nabla}\phi$ and $\Theta$ are supposed to be bounded on $S$, it follows from (3.47) that $H$ is also bounded on $S$.

On the other hand, from inequality (16) of [2] we have that the Ricci curvature of $S$ satisfies

$$\mathrm{Ric}(X,X) \geq n - 1 - \frac{n^2H^2}{4}.$$

So, we conclude that the Ricci curvature of $S$ is bounded from below.

Thus, we can apply the maximum principle of Omori [83] and Yau [108] to guarantee that there exists a sequence $\{p_k\}$ in $S$ such that

$$\lim_k h(p_k) = \sup_S h, \quad \lim_k |\nabla h(p_k)| = 0 \quad \text{and} \quad \limsup_k \Delta h(p_k) \leq 0.$$

But, from (1.10) and (1.5), we also get that

$$\lim_k \Theta(p_k) = -1 \quad \text{and} \quad \limsup_k \Delta_\phi h(p_k) \leq 0.$$

Hence, taking into account formula (3.1), we can reason in a similar way to the proof of Theorem 8 of [2] to obtain that $H_\phi = 1$. Furthermore, from formula (3.2) we also have that

$$\Delta_\phi \exp(h) = -n \exp(h)(1+\Theta) \geq 0.$$

Therefore, assuming now that the universal Riemannian covering of $F$ is $\phi$-parabolic, we can apply Theorem 3.2.12 to conclude that $h$ is constant on $S$. $\square$

Our next result extends Theorem 5.3 of [44].

**Theorem 3.2.23.** *Let $M_\phi$ be a spatially weighted steady state type spacetime whose fiber $F$ is complete, with $\phi$-parabolic universal Riemannian covering, and let $S$ be a complete spacelike hypersurface that lies in a timelike bounded region of $M$, with $1 \leq H_\phi \leq \alpha$ for a certain constant $\alpha \geq 1$. If the angle function $\Theta$ of $S$ satisfies $-\Theta \leq H_\phi$, then $S$ is a slice $S_t$.*

*Proof.* From formula (3.1) we get

$$\begin{aligned} \Delta_\phi \exp(-h) &= \exp(-h)\left(|\nabla h|^2 - \Delta_\phi h\right) \\ &\leq n \exp(-h)\left(|\nabla h|^2 + 1 + H_\phi \Theta\right) \\ &= n \exp(-h)\Theta\left(\Theta + H_\phi\right). \end{aligned} \tag{3.48}$$

Hence, our hypothesis on $\Theta$ guarantees that the function $\exp(-h)$ is a $\phi$-superharmonic positive function on $S$. Therefore, we can apply once more Theorem 3.2.12 to get that $h$ is constant on $S$. $\square$

### 3.2.4 Constant $\phi$-mean curvature spacelike hypersurfaces

Our aim in this subsection is to study the uniqueness of spacelike slices among complete spacelike hypersurfaces immersed in a weighted Lorentzian product space of type $-\mathbb{R}\times F^n_\phi$, whose Riemannian fiber $F^n$ has nonnegative Bakry-Émery-Ricci tensor and such that the Hessian of the weight function $\phi$ is bounded from below. By assuming that the weighted mean curvature $H_\phi$ of a given spacelike hypersurface $S$ is constant, as well as some appropriated constraints on the norm of the gradient of the height function of $S$, we prove that such a hypersurface must be a slice of the ambient spacetime. The results presented here are part of our paper [71].

**Theorem 3.2.24.** *Let $-\mathbb{R}\times F^n_\phi$ be a weighted Lorentzian product space, whose Riemannian fiber $F^n$ is complete, has sectional curvatures bounded from below, and its Bakry-Émery-Ricci tensor satisfies $\mathrm{Ric}^F_\phi \geq c$ for some positive constant $c$. Suppose that the Hessian of the weighted function $\phi$ is bounded from below. Let $S^n$ be a complete spacelike hypersurface immersed in $-\mathbb{R}\times F^n_\phi$ with constant $\phi$-mean curvature $H_f$. If $\Theta$ is bounded on $S^n$, then $S^n$ is a slice $S_t$ for some $t \in \mathbb{R}$.*

*Proof.* Since we are assuming that $H_\phi$ is constant, Lemma 3.1.1 gives

$$\Delta_\phi \Theta = (\mathrm{Ric}_\phi^F(N^F, N^F) + |A|^2)\Theta. \tag{3.49}$$

Moreover, from our assumptions that $\Theta$ is bounded and that the fiber $F^n$ has sectional curvatures bounded from below, by Lemma 3.1.3 we have that the Bakry-Émery-Ricci curvature of $S^n$ is also bounded from below. Thus, we can apply Lemma 3.2.1 to the function $\Theta$ to obtain a sequence of points $\{p_k\}_{k\in\mathbb{N}} \subset S^n$ such that $\lim_k \Theta(p_k) = \inf_S \Theta$ and $\liminf_k \Delta_\phi\Theta(p_k) \geq 0$.

Consequently, since we are also assuming that the Bakry-Émery-Ricci curvature $\mathrm{Ric}_\phi^F$ of $F^n$ is positive, and taking into account that $\Theta \leq -1$, from (3.49) up to a subsequence we get

$$0 \leq \liminf_k \Delta_\phi \Theta(p_k) = \lim_k \left[\mathrm{Ric}_\phi^F(N^F, N^F) + |A|^2\right](p_k) \inf_{p\in S} \Theta(p) \leq 0. \tag{3.50}$$

Furthermore, we note that our constraint on the Bakry-Émery-Ricci tensor of $F^n$ amounts to

$$\mathrm{Ric}_\phi^F(N^F, N^F) \geq c|N^F|_F^2 = c|\nabla h|^2. \tag{3.51}$$

Thus, from (3.50) and (3.51) we get that $\lim_k |\nabla h(p_k)| = 0$. Therefore, taking into account relation (1.10), we conclude that $\inf_{p\in S} \Theta(p) = -1$, and hence $S^n$ must be a slice $S_t = \{t\} \times F^n$ for some $t \in \mathbb{R}$. □

**Remark 3.2.25.** *Regarding Theorem 3.2.24, we point out that a complete weighted manifold* $(\Sigma^n, g, \exp(-\phi)dV)$ *whose Bakry-Émery-Ricci tensor satisfies* $\mathrm{Ric}_\phi \geq c$ *for some positive constant c is not necessarily compact. Indeed, it is easy to verify that the Gaussian space* $\mathbb{G}^n$ *works as a counterexample. On the other hand, under the additional hypothesis that the weighted function* $\phi$ *is bounded, the extension of Myers' theorem due to Wei and Willie guarantees the compactness of* $\Sigma^n$ *(see* [104, Theorem 1.4]*).*

Proceeding, we also get the following

**Theorem 3.2.26.** *Let* $-\mathbb{R} \times F_\phi^n$ *be a weighted Lorentzian product space. Suppose that the fiber* $F^n$ *is complete and has sectional curvatures bounded from below and nonnegative Bakry-Émery-Ricci tensor,* $\mathrm{Ric}_\phi^F$*, and that the Hessian of the weight function* $\phi$ *is bounded from below. Let* $S^n$ *be a complete spacelike hypersurface immersed in* $-\mathbb{R} \times F_\phi^n$ *with constant* $\phi$*-mean curvature* $H_\phi$ *and bounded second fundamental form A. If*

$$|\nabla h| \leq \alpha |A|^\beta, \tag{3.52}$$

*for some positive constants* $\alpha$ *and* $\beta$*, then* $S^n$ *is a slice* $S_t = \{t\} \times F^n$ *for some* $t \in \mathbb{R}$*.*

*Proof.* By noting that hypothesis (3.52) assures, in particular, that $\Theta$ is bounded on $S^n$, we can reason as in the proof of Theorem 3.2.24 to infer that (3.50) still holds. Consequently, we get that $\lim_k |A(p_k)| = 0$. Thus, taking into account once more hypothesis (3.52), we obtain that $\lim_k |\nabla h(p_k)| = 0$. Therefore, from relation (1.10) we conclude that $\inf_{p\in S} \Theta(p) = -1$, and hence $S^n$ must be a slice $S_t = \{t\} \times F^n$ for some $t \in \mathbb{R}$, as desired. □

Under the assumption that $S^n$ is $\phi$-parabolic, we get the following

**Theorem 3.2.27.** *Let $-\mathbb{R} \times F^n_\phi$ be a weighted Lorentzian product space whose fiber $F^n$ is complete and has nonnegative Bakry-Émery-Ricci tensor, $\mathrm{Ric}^F_\phi$. Let $S^n$ be a complete spacelike hypersurface immersed in $-\mathbb{R} \times F^n_\phi$ with constant $\phi$-mean curvature $H_\phi$. If $S^n$ is $\phi$-parabolic and $\Theta$ is bounded on $S^n$, then $S^n$ is totally geodesic. In addition, if $\mathrm{Ric}^F_\phi$ is strictly positive, then $S^n$ is a slice $S_t$ for some $t \in \mathbb{R}$.*

*Proof.* Considering on $S^n$ the bounded function $u = -\Theta$, from Lemma 3.1.1 we have that

$$\Delta_\phi u = -(\mathrm{Ric}^F_\phi(N^F, N^F) + |A|^2)\Theta \geq 0. \tag{3.53}$$

By hypothesis, $S^n$ is $\phi$-parabolic; so, from (3.53), we get that $u$ is constant. Thus, since $\Theta < 0$ on $S^n$, and taking into account that $\mathrm{Ric}^F_\phi \geq 0$, returning to (3.53) we get that $|A| \equiv 0$, that is, $S^n$ is totally geodesic. Moreover, in the case that $\mathrm{Ric}^F_\phi$ is strictly positive, we conclude that $N^F$ vanishes identically on $S^n$, which means that $N = \partial_t$ on $S^n$. So, in this case, $S^n$ must be a slice $S_t$ for some $t \in \mathbb{R}$. □

## 3.3 On the existence of $\phi$-maximal spacelike hypersurfaces

In this section, we prove the following nonexistence result, which is the main theorem of our paper [82].

**Theorem 3.3.1.** *Let $M_\phi = -I \times F_\phi$ be a weighted Lorentzian product space whose fiber $F$ is noncomplete with nonnegative sectional curvature and such that the weight function $\phi$ is bounded and convex. Then there is no complete noncompact $\phi$-maximal spacelike hypersurfaces immersed in $M_\phi$.*

As a consequence of the proof of Theorem 3.3.1, which is presented in Subsection 3.3.2, we get the following Calabi-Bernstein type result.

**Theorem 3.3.2.** *Let $\overline{M} = -I \times M$ be a Lorentzian product space whose fiber $M$ is complete with nonnegative sectional curvature. The only maximal noncompact spacelike hypersurfaces of $\overline{M}$ are the slices.*

### 3.3.1 Key lemmas

In this subsection, we present two key lemmas which will be used to prove Theorem 3.3.1. Before quoting the first one, we shall recall some important facts.

Let $(\Sigma, g, d\mu = \exp(-\phi)dV)$ be an $n$-dimensional weighted complete Riemannian manifold. Take any point $x \in \Sigma$ and denote the volume form in geodesic polar coordinates centered at $x$ by

$$dV\big|_{\mathrm{Exp}_x(r\zeta)} = J(x, r, \zeta)drd\zeta,$$

where $r > 0$ and $\zeta \in S_x\Sigma$ is a unitary tangent vector at $x$. It is well-known that if $y \in \Sigma$ is any point such that $y = \mathrm{Exp}_x(r\zeta)$, then

$$\Delta_\phi d(x,y) = \frac{J'_\phi(x,r,\zeta)}{J_\phi(x,r,\zeta)},$$

where $J_\phi(x,r,\zeta) := \exp(-\phi)J(x,r,\zeta)$ is the $\phi$-volume form in geodesic polar coordinates. For a fixed point $p \in \Sigma$ and $R > 0$, define

$$A(R) := \sup_{B_p(3R)} |\phi(x)|. \tag{3.54}$$

For a subset $\Omega \subseteq \Sigma$, we will denote by $V(\Omega)$ the volume of $\Omega$ with respect to the usual volume form $dV$, and by $V_\phi(\Omega)$ the $\phi$-volume of $\Omega$, $V_\phi(\Omega) := \int_\Omega d\mu$. If $\Sigma$ has nonnegative Bakry-Émery-Ricci curvature, then along any minimizing geodesic starting from $x \in B_p(R)$ we have

$$\frac{J_\phi(x,r_2,\zeta)}{J_\phi(x,r_1,\zeta)} \leq e^{4A}\left(\frac{r_2}{r_1}\right)^{n-1} \tag{3.55}$$

for any $0 < r_1 < r_2 < R$; in particular, for any $0 < r_1 < r_2 < R$,

$$\frac{V_\phi(B_x(r_2))}{V_\phi(B_x(r_1))} \leq e^{4A}\left(\frac{r_2}{r_1}\right)^{n}, \tag{3.56}$$

where $A = A(R)$ is defined in (3.54) (see [81, Lemma 2.1]). If $\Sigma$ is noncompact, the comparison inequality (3.55) guarantees that there exist constants $\nu > 2$, $C_1$ and $C_2$, depending only on $n$, such that for all $u \in C_0^\infty(B_x(r))$ the following local Sobolev inequality holds,

$$\left(\int_{B_x(r)} |u|^{\frac{2\nu}{\nu-2}} d\mu\right)^{\frac{\nu-2}{\nu}} \leq C_1 \frac{e^{C_2 A} r^2}{V_\phi(B_x(r))^{\frac{2}{\nu}}} \int_{B_x(r)} \left(|\nabla u|^2 + r^{-2}|u|^2\right) d\mu, \tag{3.57}$$

where $x \in B_p(R)$ and $0 < r < R$ (see [40, Lemma 2.3]). Such a family of Sobolev inequalities can be used to obtain a mean value inequality for subsolutions to the $\phi$-heat equation as in [96, Theorem 5.2.9] (see [40, Lemma 2.5]).

**Lemma 3.3.3.** *Let $(\Sigma, g, d\mu = \exp(-\phi)dV)$ be an $n$-dimensional weighted complete Riemannian manifold which satisfies the local Sobolev inequality (3.57) for all $u \in C_0^\infty(B_o(\rho))$ and $0 < \rho \leq R$, where $o \in \Sigma$ is a fixed origin. Fix $q \in (0, +\infty)$, and let $v$ be a positive subsolution of the $\phi$-heat equation, that is,*

$$Lv \geq 0,$$

*in the cylinder $Q = B_o(r) \times (s - r^2, s)$ for some $s \in \mathbb{R}$ and $0 < r < R$, where $L := \Delta_\phi + \frac{\partial}{\partial t}$. Then for any $0 < \delta < \delta' \leq 1$ there exist constants $C_3 = C_3(n, \sup|\phi|, \nu, q)$ and $C_4 = C_4(n, \nu, q)$ such that*

$$\sup_{Q_\delta} v^q \leq C_3 \frac{e^{C_4 A(R)}}{(\delta' - \delta)^{\nu+2} r^2 V_\phi(B_o(r))} \int_{Q_{\delta'}} v^q d\mu dt,$$

*where* $Q_z = B_o(zr) \times (s - zr^2, s)$.

Our second key lemma gives sufficient conditions to guarantee that the Bakry-Émery-Ricci curvature of a $\phi$-maximal spacelike hypersurface immersed in a weighted Lorentzian product be nonnegative.

**Lemma 3.3.4.** *Let* $M^{n+1}_\phi = -I \times F^n_\phi$ *be a Lorentzian weighted product whose fiber* $F^n$ *has nonnegative sectional curvarures and with convex weight function. Let* $\psi : (S^n, g_S) \to (M^{n+1}_\phi, \overline{g})$ *be an* $\phi$*-maximal spacelike hypersurface with orientation* $N$ *and shape operator* $A$*. Then the Bakry-Émery-Ricci curvature* $\mathrm{Ric}_\phi$ *of* $S$ *is nonnegative.*

*Proof.* It follows from Gauss equation (3.7) that

$$\mathrm{Ric}(X,X) = \sum_{i=1}^{n} \overline{g}(\overline{R}(X,E_i)X,E_i) + nHg_S(AX,X) + g_S(AX,AX) \tag{3.58}$$

for all $X, Y, Z \in \mathfrak{X}(S)$, where Ric and $\overline{R}$ stand for the Ricci curvature and the curvature tensor of $M^{n+1}$, respectively, and $H$ is the mean curvature function of $S^n$. Moreover, we have

$$\overline{R}(X,Y)Z = R^F(X^F, Y^F)Z^F,$$

$R^F$ and $(\cdot)^F$ being, respectively, the curvature tensor and the projection of a vector field onto the fiber $F^n$ (see [84, Proposition 7.42] for details); hence,

$$\overline{g}(\overline{R}(X,E_i)X,E_i) = K^F(X^F, E_i^F)(g_F(X^F,X^F)g_F(E_i^F,E_i^F) - g_F(X^F,E_i^F)),$$

where $K^F$ stands for the sectional curvature of $F^n$. Since $F^n$ is nonnegatively curved, considering this previous equation into (3.58) we get

$$\mathrm{Ric}(X,X) \geq nHg_S(AX,X) + g_S(AX,AX). \tag{3.59}$$

Since $\phi$ is convex, we have

$$\begin{aligned} \mathrm{Hess}\,\phi(X,X) &= \overline{\mathrm{Hess}}\,\phi(X,X) - \overline{g}(\overline{\nabla}\phi, N)g_S(AX,X) \\ &\geq -\overline{g}(\overline{\nabla}\phi, N)g_S(AX,X). \end{aligned} \tag{3.60}$$

Therefore, from (1.1), (1.11), (3.59) and (3.60), we obtain

$$\mathrm{Ric}_\phi(X,X) \geq nH_\phi g_S(AX,X) + g_S(AX,AX). \tag{3.61}$$

The result follows from (3.61) taking into account the hypothesis that $S$ is $\phi$-maximal. ☐

### 3.3.2 Proof of Theorem 3.3.1

Let $S^n$ be a complete noncompact spacelike $\phi$-maximal hypersurface immersed in $M^{n+1}_\phi = -I \times F^n_\phi$. We observe that the height function of $S$, $h = \pi_I\big|_S$, is an $\phi$-harmonic function.

By Lemma 3.1.1-(i), the $\phi$-Laplacian of $h$ on $S^n$ is given by $\Delta_\phi h = -nH_\phi \overline{g}(N, \partial_t)$, and the $\phi$-harmonicity of $h$ follows from the fact that $S^n$ is supposed to be $\phi$-maximal. We claim that $h$ must be bounded from above. We shall argue as in the proof of [61, Theorem 0.13]. Suppose otherwise. Then, $S \cap S_t \neq 0 \, \forall t \in I$. For a fixed $t \in I$, let $\Sigma_t := \{p \in S^n : h(p) \geq t\}$. By Sard theorem, we can suppose that $t$ is a regular value of $h\big|_{\operatorname{int} S}$, so that $\Sigma_t$ is a smooth complete manifold with boundary $\partial\Sigma_t = \{p \in S : h(p) = t\}$ and exterior unit normal $\nu_t = -\nabla h/|\nabla h|$. For any $\epsilon > 0$, define on $\Sigma_t$

$$h_\epsilon = \max\{h, t + \epsilon\}.$$

Then $h_\epsilon$ is $\phi$-harmonic on $\Sigma_t$. Indeed, set

$$\begin{aligned}
\Sigma_1 &= \{p \in \Sigma_t : h(p) > t + \epsilon\}, \\
\Sigma_2 &= \{p \in \Sigma_t : h(p) = t + \epsilon\}, \\
\Sigma_3 &= \{p \in \Sigma_t : t < h(p) < t + \epsilon\}.
\end{aligned}$$

Then, $h_\epsilon = h$ on $\Sigma_1$ and $h_\epsilon$ is constant (equals to $t + \epsilon$) on $\Sigma_3$, so $\Delta_\phi h_\epsilon = 0$ on both $\Sigma_1$ and $\Sigma_3$. Now, we note that the tranversality of $S$ and $\partial_t$, by which we mean the fact that the function $\overline{g}(N, \partial_t)$ is negatively signed globally on $S$, implies a certain kind of monotonic behavior of the height function $h$, which in turn guarantees that $h_\epsilon$ is smooth on $\Sigma_2$ and on $\partial\Sigma_t$. So, we also have that $\Delta_\phi h_\epsilon = 0$ on both $\Sigma_2$ and $\partial\Sigma_t$ by continuity. Whence, on noting that $h_\epsilon \equiv t + \epsilon$ on $\partial\Sigma_t$, by the maximum principle for the $\phi$-Laplacian, we obtain that $t \leq h \leq t + \epsilon$ on $\Sigma_t$. Since this holds for every $\epsilon > 0$, we conclude that $h \equiv t$ on $\Sigma_t$, contradicting the assumption of $h$ being unbounded from above. Obviously, the previous reasoning also works for proving that $h$ is bounded from below. We now observe that $h$ has sublinear growth, that is,

$$\lim_{p \to \infty} \frac{|h|(p)}{r(p)} = 0, \tag{3.62}$$

where $r(p) := d(p, p_0)$ is the distance from a fixed point $p_0 \in S^n$. This follows from the noncompactness of $S^n$ and from the fact just established that $\alpha := \sup_S |h| < +\infty$, so that

$$0 \leq \frac{|h|(p)}{r(p)} \leq \frac{\alpha}{r(p)} \xrightarrow{p \to \infty} 0.$$

From $\Delta_\phi h = 0$, Lemma 3.3.4 and the weighted Bochner formula (3.44), we conclude that $|\nabla h|^2$ is $\phi$-suharmonic. By setting $q = 1$ in Lemma 3.3.3 and observing that an $\phi$-subharmonic function is also a subsolution to the $\phi$-heat equation, we get the following mean value inequality

$$\sup_{B_p(R/2)} |\nabla h|^2 \leq \frac{\beta}{R^2 V_f(B_p(R))} \int_{B_p(R)} |\nabla h|^2 d\mu, \tag{3.63}$$

where $\beta$ is a constant depending only on $n$ and $\sup_S |\phi|$.

We now will closely follow the proof of [81, Theorem 3.2] and apply a standard cut-off argument. We choose a cut-off function $\chi$ such that $\chi = 1$ on $B_p(R)$, $\chi = 0$ on $S^n \setminus B_p(2R)$ and

$|\nabla\chi| \leq \frac{\beta}{R}$. Integrating by parts and using $\Delta_\phi h = 0$, we obtain

$$\begin{aligned}
\int_S |\nabla h|^2\chi^2 d\mu &= -2\int_S h\chi\langle\nabla h, \nabla\chi\rangle d\mu \\
&\leq 2\int_S |h|\chi|\langle\nabla h, \nabla\chi\rangle| d\mu \\
&\leq 2\left(\int_S |\nabla h|^2\chi^2 d\mu\right)^{1/2}\left(\int_S h^2|\nabla\chi|^2 d\mu\right)^{1/2}.
\end{aligned}$$

It follows that

$$\begin{aligned}
\int_S |\nabla h|^2\chi^2 d\mu &\leq 4\int_S h^2|\nabla\chi|^2 d\mu \\
&\leq \frac{\beta^2}{R^2}\int_{B_p(2R)\setminus B_p(R)} h^2 d\mu \\
&\leq \frac{\beta^2}{R^2}\left(\sup_{B_p(2R)} h^2\right) V_\phi(B_p(2R)) \\
&\leq \frac{\gamma}{R^2}\left(\sup_{B_p(2R)} h^2\right) V_\phi(B_p(R)),
\end{aligned}$$

for a positive constant $\gamma$, where in the last line we used (3.56).

Taking into account (3.62), we obtain

$$\lim_{R\to\infty} \frac{\beta}{R^2 V_\phi(B_p(R))}\int_{B_p(R)} |\nabla h|^2 d\mu = 0,$$

so that, by (3.63), $|\nabla h| = 0$ on $S^n$, that is, $S^n$ is a slice of $M^{n+1}$. But, since we are assuming that the fiber $M$ is noncomplete, this cannot occur.

**Remark 3.3.5.** *As a matter of fact, Munteanu and Wang* [81] *already stablished that a sublinear growth $\phi$-harmonic function in a complete noncompact weighted manifold with bounded weight function $\phi$ must be constant. Here, for the sake of completeness, and since we employ a mean value inequality slightly different from inequality (3.14) in* [81]*, we have decided to present the full argument.*

## 3.4 Calabi-Bernstein type results

Let $\Omega \subseteq F$ be a connected domain, and let $u \in C^\infty(\Omega)$ be a smooth function. Then, $S(u)$ will denote the vertical graph over $\Omega$ determined by $u$, that is,

$$S(u) = \{(u(x), x) : x \in \Omega\} \subset M = -I \times_f F.$$

The graph is said to be *entire* if $\Omega = F$. The metric induced on $\Omega$ from the Lorentzian metric of the ambient space via $S(u)$ is

$$g_{S(u)} = -du^2 + f^2(u)g_F. \tag{3.64}$$

It can be easily seen that a graph $S(u)$ is a spacelike hypersurface if and only if $|Du|_F^2 < f^2(u)$, $Du$ being the gradient of $u$ in $F$ and $|Du|_F$ its norm, both with respect to the metric $g_F$. It is well known (see [10, Lemma 3.1]) that in the case where $F$ is a simply connected manifold, every complete spacelike hypersurface $S$ immersed in $M$ such that the warping function $f$ is bounded on $S$ is an entire spacelike graph over $F$. In particular, this happens for complete spacelike hypersurfaces contained in a timelike bounded region of $M$. It is interesting to observe that, in contrast to the case of graphs into a Riemannian space, an entire spacelike graph $S(u)$ in a Lorentzian spacetime is not necessarily complete, in the sense that the induced Riemannian metric is not necessarily complete on $F$. However, it can be proved that if $F$ is complete and $|Du|_F^2 \leq f^2(u) - c$ for a certain positive constant $c$, then $S(u)$ is complete. Although a particular case of this claim is proven in [3, Theorem 4.1], and the general proof is given in ([4, Proposition 9]), we shall expose it here for the sake of completeness.

**Proposition 3.4.1.** *Let $F$ be a complete Riemannian manifold, and let $S(u)$ an entire spacelike vertical graph in $M = -I \times_f F$. If*

$$|Du|_F^2 \leq f^2(u) - c$$

*for a certain positive constant $c$, then $S(u)$ is complete.*

*Proof.* From (3.64), the Cauchy-Schwarz inequality and the assumptions of the proposition we get

$$g_{S(u)}(X,X) = -g_F(Du,X)^2 + f^2(u)g_F(X,X) \geq \left(f^2(u) - |Du|_F^2\right) g_F(X,X) \geq c g_F(X,X)$$

for every $X \in \mathfrak{X}(S(u))$. This implies that $L \geq \sqrt{c}L_F$, where $L$ and $L_F$ denote the length of a curve on $S(u)$ with respect to the Riemannian metrics $g_{S(u)}$ and $g_F$, respectively. As a consequence, since $F$ is complete by assumption, the induced metric on $S(u)$ from the metric of $M$ is also complete. □

**Remark 3.4.2.** *Let us observe that such a constant $c$ clearly exists if $F$ is assumed to be compact and the spacelike graph is entire. Moreover, when $F$ is complete and noncompact, on an entire graph the existence of such a constant prevents that the tangent vector field to a divergent curve in $S$ asymptotically approaches to a lightlike direction in the ambient spacetime.*

The future-pointing Gauss map of a spacelike vertical graph $S(u)$ over $\Omega$ is given by the vector field

$$N(x) = \frac{f(u(x))}{\sqrt{f^2(u(x)) - |Du(x)|_F^2}} \left( \partial_t|_{(u(x),x)} + \frac{1}{f^2(u(x))} Du(x) \right), \quad x \in \Omega. \tag{3.65}$$

Moreover, the shape operator $A$ of $S(u)$ with respect to the orientation (3.65) given by

$$
\begin{aligned}
AX = & -\frac{1}{f(u)\sqrt{f^2(u)-|Du|_F^2}} D_X Du - \frac{f'(u)}{\sqrt{f^2(u)-|Du|_F^2}} X \\
& + \left( \frac{-g_F(D_X Du, Du)}{f(u)\left(f^2(u)-|Du|_F^2\right)^{3/2}} + \frac{f'(u) g_F(Du, X)}{\left(f^2(u)-|Du|_F^2\right)^{3/2}} \right) Du
\end{aligned} \tag{3.66}
$$

for any tangent vector field $X$ tangent to $\Omega$. Consequently, if $S(u)$ is a spacelike vertical graph over a domain $\Omega$ of the fiber $F$ of a spatially weighted GRW spacetime $M$ endowed with a weight function $\phi$, it is not difficult to verify from (1.11) and (3.66) that the $\phi$-mean curvature function $H_\phi(u)$ of $S(u)$ is given by

$$
nH_\phi(u) = -\mathrm{div}_\phi \left( \frac{Du}{f(u)\sqrt{f(u)^2-|Du|_F^2}} \right) - \frac{f'(u)}{\sqrt{f(u)^2-|Du|_F^2}} \left( n + \frac{|Du|_F^2}{f(u)^2} \right).
$$

The differential equation $H_\phi(u) = 0$ with the constraint $|Du|_F < f(u)$ is called the *$\phi$-maximal spacelike hypersurface equation* on $M$, and its solutions provide $\phi$-maximal spacelike graphs in $M$.

In this context, we can establish a nonparametric version of Theorem 3.2.2.

**Theorem 3.4.3.** *Let $F$ be a complete $n$-dimensional Riemannian manifold, and consider a spatially weighted GRW spacetime $M_\phi = -I \times_f F_\phi$ obeying the SNCC* (3.6) *and such that the Hessian of the weight function $\phi$ is bounded from below. Let $S(u)$ be an entire vertical graph in $M_\phi$ determined by a bounded smooth function $u \in C^\infty(F)$. Suppose that the $\phi$-mean curvature $H_\phi$ of $S(u)$ satisfies*

$$
(\log f)'(u) \le H_\phi \le \alpha \qquad \text{and} \qquad H_\phi \ge 0
$$

*for some positive constant $\alpha$. If*

$$
|Du|_F^2 \le \frac{\beta \inf_{S(u)} f^2(u) \inf_{S(u)} |H_\phi - (\log f)'(u)|^\gamma}{1 + \beta \inf_{S(u)} |H_\phi - (\log f)'(u)|^\gamma} \tag{3.67}
$$

*for some constants $\beta > 0$ and $\gamma \neq 0$, then $u \equiv t_0$ for some $t_0 \in I$.*

*Proof.* Let us observe first that, under the assumptions of the theorem, $S(u)$ is complete. In fact, from (3.67) we easily obtain

$$
f^2(u) - |Du|_F^2 \ge c := \frac{\inf_{S(u)} f^2(u)}{1 + \beta \inf_{S(u)} |H_\phi - (\log f)'(h)|^\gamma} > 0,
$$

and the completeness of $S^n(u)$ follows from Proposition 3.4.1.

On the other hand, from (1.8), (1.10) and (3.65) we get

$$
|\nabla h|^2 = \frac{|Du|_F^2}{f^2(u) - |Du|_F^2}. \tag{3.68}
$$

Therefore, (3.67) implies (3.25), and the result follows from Theorem 3.2.2. $\square$

The nonparametric version of Theorem 3.2.3 can be stated and proved in a similar way as for the previous theorem.

**Theorem 3.4.4.** *Let $F$ be a complete $n$-dimensional Riemannian manifold, and consider a spatially weighted GRW spacetime $M_\phi = -I \times_f F_\phi$ obeying the SNCC (3.6) with convex weight function $\phi$. Let $S(u)$ be an entire vertical graph in $M_\phi$ determined by a bounded smooth function $u \in C^\infty(F)$ with constant $\phi$-mean curvature $H_\phi$ satisfying*

$$0 \leq H_\phi \sup_{S(u)} (\log f)'(u) \leq H^2.$$

*If*

$$|Du|_F^2 \leq \frac{\alpha \inf_{S(u)} f^2(u) \left(\inf H^2 - H_\phi \sup_{S(u)} (\log f)'(u)\right)^\beta}{1 + \alpha \left(\inf H^2 - H_\phi \sup_{S(u)} (\log f)'(u)\right)^\beta}$$

*for some constants $\alpha > 0$ and $\beta \neq 0$, then $u \equiv t_0$ for some $t_0 \in I$.*

In the case of Theorem 3.2.6, we need some extra assumptions in order to obtain a nonparametric version of it, since we cannot assure the completeness of an entire graph only under its assumptions.

**Theorem 3.4.5.** *Let $F$ be a complete $n$-dimensional Riemannian manifold, and consider a spatially weighted GRW spacetime $M_\phi = -I \times_f F_\phi$. Let $S(u)$ be an entire vertical graph in $M_\phi$ of a bounded smooth function $u \in C^\infty(F)$, and suppose that $H_\phi > 0$, $f'(u) > 0$, and that the following inequalities are satisfied,*

$$\frac{n^2}{4} (\log f)'^2(u) \leq \frac{n^2 H_\phi^2}{4} \leq (n-1) \frac{f''}{f}(u).$$

*If $|Du|_F^2 \leq \alpha f^2(u)$ for some constant $0 < \alpha < 1$, and $f(u) \in L^p(d\mu)$, where $d\mu = \exp(-\phi) dF$, for some $1 < p < +\infty$, then $u \equiv t_0$ for some $t_0 \in I$, with $vol_\phi(S(u)) < +\infty$. In addition, if $M_\phi$ obeys (3.6) and the weight function $\phi$ is bounded and convex, then $S(u)$ is compact.*

*Proof.* Since we are assuming that $u$ is bounded and $|Du|_F^2 \leq \alpha f^2(u)$ for some constant $0 < \alpha < 1$, it is easy to see that the assumptions of Proposition 3.4.1 are satisfied, so $S(u)$ is complete. Moreover, from equation (5.9) of [13] we get that

$$dS(u) = f^{n-1}(u) \sqrt{f^2(u) - |Du|_F^2}\, dF.$$

Consequently, since $f(u) \in L^p(d\mu)$ for $1 < p < +\infty$, and $f(h(q)) = f(u(x))$ for all $q = (u(x), x) \in S(u)$, we infer that $f(h) \in L^p(d\mu_u)$ $1 < p < +\infty$, where $d\mu_u = \exp(-\phi) dS(u)$. Therefore, we can apply Theorem 3.2.6 to conclude the desired result. □

In [20] An et al. obtained a Calabi-Bernstein type result for $-\mathbb{R} \times \mathbb{G}^n$ showing that the graph $S(u)$ of a function $u(x) = t$ over $\mathbb{G}^n$, with $|Du|_{\mathbb{G}^n}$ bounded away from 1, is $\phi$-maximal if and only if $u$ is constant. Now, we present an extension of this result.

**Theorem 3.4.6.** *Let $M_\phi$ be a spatially weighted GRW spacetime, whose fiber $F$ is complete with $\phi$-parabolic universal Riemannian covering, and such that the warping function $f$ is monotone. The only entire bounded solutions on $M_\phi$ to the following modified $\phi$-maximal spacelike hypersurface equation*

$$(\mathbf{E})\ \begin{cases} \operatorname{div}_\phi\left(\dfrac{Du}{f(u)\sqrt{f(u)^2-|Du|_F^2}}\right) = -\dfrac{f'(u)}{\sqrt{f(u)^2-|Du|_F^2}}\left(n+\dfrac{|Du|_F^2}{f(u)^2}\right) \\ \\ |Du|_F \leq \alpha f(u) \qquad (0<\alpha<1) \end{cases}$$

*are the constant functions $u = t_0$ with $f'(t_0) = 0$.*

*Proof.* We observe that the constraint on $|Du|_F$ assures the boundedness of the angle function $\Theta(u)$ of $S(u)$. Indeed, this follows from (3.68). Hence, by using (1.10) and (3.68), we can see that $|Du|_F \leq \alpha f(u)$ implies $\Theta(u) \geq \dfrac{-1}{\sqrt{1-\alpha^2}}$.

Moreover, since we are looking for bounded solutions to $(\mathbf{E})$, taking $c := (1-\alpha^2)\inf_{S(u)} f^2(u)$, we can apply Proposition 3.4.1 to see that such a solution must be complete. Therefore, the result follows from Theorem 3.2.16. □

It is not difficult to see that we can reason as in the proof of the previous result to obtain nonparametric versions of all others theorems of Subsection 3.2.3. For instance, we have the following nonparametric version of Theorem 3.2.23.

**Theorem 3.4.7.** *Let $M_\phi$ be a spatially weighted steady state type spacetime, whose fiber $F$ is complete with $\phi$-parabolic universal Riemannian covering. The only entire bounded solutions on $M_\phi$ to the equation*

$$(\mathbf{E}')\ \begin{cases} \operatorname{div}_\phi\left(\dfrac{Du}{e^u\sqrt{e^{2u}-|Du|_F^2}}\right) = -nH_\phi(u) - \dfrac{e^u}{\sqrt{e^{2u}-|Du|_F^2}}\left(n+\dfrac{|Du|_F^2}{e^{2u}}\right) \\ 1 \leq H_\phi(u) \leq \alpha \qquad (\alpha \geq 1) \\ |Du|_F \leq \sqrt{1-\dfrac{1}{H_\phi^2(u)}}\,e^u \end{cases}$$

*are the constant functions $u = t_0$.*

*Proof.* Let us observe first that, under the assumptions of the theorem, the solutions to $(\mathbf{E}')$ determine complete entire graphs $S(u)$. In fact, from the last inequality in $(\mathbf{E}')$ we easily obtain that

$$e^{2u} - |Du|_F^2 \geq c = e^{2\inf_{S(u)} u} \sup_{S(u)} \frac{1}{H_\phi^2(u)} > 0,$$

and the completeness of $S(u)$ follows again from Proposition 3.4.1.

Therefore, using (3.68) and the last inequality of $(\mathbf{E}')$, we can easily verify that the angle function $\Theta(u)$ of $S(u)$ satisfies $-\Theta(u) \leq H_\phi(u)$, and, hence, the result follows from Theorem 3.2.23. □

**Remark 3.4.8.** *As it was observed by the referee of our paper [5], an important property of Robertson-Walker spacetimes is that they are locally conformally flat, a condition that does not hold in the generalized Robertson-Walker spacetimes. Hence, a nice generalization of Robertson-Walker spacetimes could be, instead of weakening the assumption on the geometry of the fiber, a more general assumption on the warped product structure, and to consider locally conformally flat multiply warped products. Therefore, it remains as an interesting open problem to obtain analogous versions of our previous results for this other context.*

Moving forward, we apply Theorem 3.2.24 to get the following

**Theorem 3.4.9.** *Let $M_\phi = -\mathbb{R} \times F_\phi$ be a weighted Lorentzian product space, whose fiber $F$ is complete with sectional curvature bounded from below and its Bakry-Émery-Ricci tensor satisfies $\mathrm{Ric}_\phi^F \geq c$ for some positive constant $c$, and such that the Hessian of the weight function $\phi$ is bounded from below. Let $S(u) \subset M_\phi$ be an entire graph over $F$ with constant $\phi$-mean curvature, $H_\phi$. If $|Du|_F \leq C$ for some constant $0 < C < 1$, then $u \equiv t_0$ for some $t_0 \in \mathbb{R}$.*

*Proof.* Proposition 3.4.1 guarantees that such $S(u)$ is a complete spacelike hypersurface of $M_\phi$. We observe that the choice of the future-pointing Gauss map of $S(u)$ given in (3.65) amounts to $\Theta \leq -1$. Moreover, from here and (3.68) we have that

$$-\frac{1}{\sqrt{1-C^2}} \leq \Theta \leq -1.$$

Therefore, we are in position to apply Theorem 3.2.24 to conclude that $S(u)$ is a slice $S_{t_0}$ for some $t_0 \in \mathbb{R}$, which means that $u \equiv t_0$. □

To establish our next result, we recall that

$$|u|_{C^2(F)} = \max_{|\gamma| \leq 2} |D^\gamma u|_{L^\infty(F)}.$$

So, from Theorem 3.2.26 we get

**Theorem 3.4.10.** *Let $M_\phi = -\mathbb{R} \times F_\phi$ be a weighted Lorentzian product space such that its fiber $F$ is complete with sectional curvature bounded from below and nonnegative Bakry-Émery-Ricci tensor $\mathrm{Ric}_\phi^F$, and such that the Hessian of the weight function $\phi$ is bounded from below. Let $S(u) M_\phi$ be an entire graph over $F$ with constant $\phi$-mean curvature $H_\phi$. If $|u|_{C^2(F)} < +\infty$ and $|Du|_F \leq \min\{\alpha |A|^\beta, C\}$, for some positive constants $\alpha$, $\beta$ and $C < 1$, then $u \equiv t_0$ for some $t_0 \in \mathbb{R}$.*

*Proof.* From (3.66) the shape operator $A$ of $S(u)$ with respect to $N$ defined in (3.65) is given by

$$AX = -\frac{1}{\sqrt{1-|Du|_F^2}} D_X Du - \frac{g_F(D_X Du, Du)}{(1-|Du|_F^2)^{3/2}} Du \tag{3.69}$$

for every tangent vector field $X \in \mathfrak{X}(S(u))$. Thus, since we are supposing that $|u|_{C^2(F)} < +\infty$ and $|Du|_F \leq C$ for some positive constant $C < 1$, from (3.69) we get that $|A|$ is bounded on $S(u)$. Therefore, reasoning as in the proof of Theorem 3.4.9, the result follows from Theorem 3.2.26. □

From Theorem 3.2.27 we obtain the following

**Corollary 3.4.11.** *Let $M_\phi = -\mathbb{R} \times F_\phi$ be a weighted Lorentzian product space, whose fiber $F$ is complete and such that its Bakry-Émery-Ricci tensor $\mathrm{Ric}_\phi^F$ is nonnegative. Let $S(u) \subset M_\phi$ be an entire graph over $F$ with constant $\phi$-mean curvature $H_\phi$. If $S(u)$ is $\phi$-parabolic and $|Du|_F \leq C$ for some positive constant $C < 1$, then $S(u)$ is totally geodesic. In addition, if $\mathrm{Ric}_\phi^F$ is strictly positive, then $u \equiv t_0$ for some $t_0 \in \mathbb{R}$.*

Last but not least, we get the following nonparametric version of Theorem 3.3.2.

**Corollary 3.4.12.** *Let $F$ be an $n$-dimensional complete noncompact manifold with nonnegative sectional curvatures. The only maximal entire functions into $-I \times F$ such that $|Du|_F \leq c$ for some constant $c \in (0,1)$, are the constant ones.*

# Part II

# Stable closed hypersurfaces as extrema of a linear combination of area and volume

# Chapter 4

# Alice in the well-known things land – Part II

In this chapter we shall briefly introduce some basic facts and notations that will appear along Part II of this thesis, namely: the higher order mean curvatures of a hypersurface immersed in a semi-Riemannian space form; the Newton transformations, some useful formulae for their traces and the second order linear differential operator associated to each of them; among others.

Let $(M^n, g)$ be a connected $n$-dimensional Riemannian manifold, and $(\overline{M}^{n+1}, \overline{g})$ be an $(n+1)$-dimensional semi-Riemannian manifold. An isometric immersion $x : M^n \to \overline{M}^{n+1}$ is an immersion such that $x^*\overline{g} = g$. In such a case, it is customary to denote both metric tensors of $M^n$ and $\overline{M}^{n+1}$ by the same symbol $\langle\, ,\, \rangle$, and so we shall do. Given an isometric immersion $x : M^n \to \overline{M}^{n+1}$, we shall say that $M^n$ is a hypersurface of $\overline{M}^{n+1}$. When $\overline{M}^{n+1}$ is a Lorentzian space, $M^n$ is said to be a spacelike hypersurface of $\overline{M}^{n+1}$.

When $\overline{M}^{n+1}$ is a Riemannian manifold, we shall assume that $M^n$ is a two-sided hypersurface, which means that its normal bundle is trivial, that is, there exists a unitary normal vector field $N$ globally defined on $M^n$. Otherwise, we shall assume that $\overline{M}^{n+1}$ is a temporally oriented Lorentzian manifold. Under this assumption, one can show that there exists a unitary future-pointing normal $C^\infty$-vector field globally defined on $M^n$; in particular, $M^n$ is orientable.

Given an isometric immersion $x : M^n \to \overline{M}^{n+1}$, let $A$ be the shape operator of $x$ associated to a chosen unitary normal vector field $N \in \mathfrak{X}^\perp(M)$ which gives the orientation of $M^n$. Such an $N$ is named the Gauss map of $x$. Namely, we have that $A : \mathfrak{X}(M) \to \mathfrak{X}(M)$ is defined by $AX = -\overline{\nabla}_X N$, $X \in \mathfrak{X}(M)$, where $\mathfrak{X}(M)$ stands for the ring of the smooth tangent vector fields on $M^n$, and $\overline{\nabla}$ stands for the Levi-Civita connection of $\overline{M}^{n+1}$. It is well-known that the curvature tensor $R$ of the hypersurface $M^n$ can be described in terms of the shape operator $A$ and of the curvature tensor $\overline{R}$ of the ambient space $\overline{M}^{n+1}$ by the so-called Gauss equation as follows

$$R(X,Y)Z = (\overline{R}(X,Y)Z)^\top + \varepsilon_N \left(\langle AX, Z\rangle AY - \langle AY, Z\rangle AX\right) \tag{4.1}$$

for every tangent vector fields $X, Y, Z \in \mathfrak{X}(M)$, where $(\ )^\top$ denotes the tangential component of a vector field in $\mathfrak{X}(\overline{M})$ along $M$ and $\varepsilon_N = \langle N, N\rangle$ (see [84]).

It is also well-known that $A$ defines a self-adjoint linear operator on each tangent space

$T_pM$, $p \in M^n$, and its eigenvalues $\kappa_1(p), \dots, \kappa_n(p)$ are the principal curvatures of the hypersurface. Associated to the shape operator there are $n$ algebraic invariants given by

$$S_k(p) = \sigma_k(\kappa_1(p), \dots, \kappa_n(p)), \quad 1 \leq k \leq n,$$

where $\sigma_k : \mathbb{R}^n \to \mathbb{R}$ is the elementary symmetric function in $\mathbb{R}^n$ given by

$$\sigma_k(x_1, \dots, x_n) = \sum_{1 \leq i_1 < \cdots < i_k \leq n} x_{i_1} \cdots x_{i_k}.$$

Observe that the characteristic polynomial of $A$ can be written in terms of the $S_k$'s as

$$\det(tI - A) = \sum_{k=0}^{n} (-1)^k S_k t^{n-k}, \tag{4.2}$$

where $S_0 = 1$ by definition. The $k$th-mean curvature $H_k$ of the hypersurface is then defined by

$$\binom{n}{k} H_k = \varepsilon_N^k S_k = \sigma_k(\varepsilon_N \kappa_1, \dots, \varepsilon_N \kappa_n) \tag{4.3}$$

for every $0 \leq k \leq n$, where $\varepsilon_N = \langle N, N \rangle$. In particular, when $r = 1$,

$$H = H_1 = \varepsilon_N \frac{1}{n} \sum_{i=1}^{n} \kappa_i = \varepsilon_N \frac{1}{n} \mathrm{trace}(A)$$

is the (extrinsec) mean curvature of $M^n$. When $r = 2$, $H_2$ defines a geometric quantity that relates to the (intrinsic) normalized scalar curvature $S$ of the hypersurface. For instance, if the ambient space $\overline{M}^{n+1}$ has constant sectional curvature $\bar{c}$, then it follows from Gauss equation (4.1) that

$$S = \bar{c} + \epsilon_N H_2. \tag{4.4}$$

These functions satisfy some very useful algebraic inequalities, often called *Newton inequalities*. Because of their value, we shall collect them here. A proof of these inequalities for positive real numbers can be found in [55]. For a different proof of a more general version of them, jointly with a sufficient condition for the equality between the $\kappa_k$'s to hold, we refer the reader to [80].

**Lemma 4.0.1.** *Let $n > 1$ be an integer, and let $\kappa_1, \kappa_2, \dots, \kappa_n$ be real numbers. For $0 \leq k \leq n$, define $S_k = \sigma_k(\kappa_1, \kappa_2, \dots, \kappa_n)$ and $H_k = \binom{n}{k}^{-1} S_k$ as before.*

*(i) For $1 \leq k < n$, we have $H_k^2 \geq H_{k-1}H_{k+1}$. Moreover, if the equality holds for $k = 1$ or for some $1 < k < n$ with $H_{k+1} \neq 0$, then $\kappa_1 = \kappa_2 = \cdots = \kappa_n$.*

*(ii) If $H_1, H_2, \dots, H_k > 0$ for some $1 < k \leq n$, then $H_1 \leq H_2^{1/2} \leq \cdots \leq H_k^{1/k}$. Moreover, if equality holds for some $1 \leq j < n$, then $\kappa_1 = \kappa_2 = \cdots = \kappa_n$.*

Returning to the isometric immersions, from the ideas of Montiel and Ros in [76, Lemma 1] and their use of Gårding's inequalities [49], and taking into account our sign convention in the definition of the higher order mean curvatures, we derive the following result.

**Lemma 4.0.2.** *Let $x : M^n \to \overline{M}^{n+1}$ be a (spacelike) hypersurface. Suppose that there exists an elliptic point in $M^n$. If $H_k$, $k > 1$, is positive on $M^n$, we have that the same holds for every $H_j$, $j = 1, \ldots, k-1$.*

Here, by an *elliptic point*, we mean a point $p_0 \in M^n$ where all the principal curvatures $\kappa_j(p_0)$ have the same sign. We point out that, if $H_k > 0$ on $M^n$ for some $1 \leq k < n$ and $M^n$ has an elliptic point, then, since by Lemma 4.0.2 $H_j >$ for every $j = 1, \ldots, k-1$, the inequality in Lemma 4.0.1-(i) is equivalent to

$$\frac{H_{k+1}}{H_k} \leq \frac{H_k}{H_{k-1}} \leq \cdots \leq \frac{H_{j+1}}{H_j} \leq \cdots \leq \frac{H_2}{H_1} \leq H_1. \tag{4.5}$$

For each $0 \leq r \leq n$, the Newton transformation $P_k : \mathfrak{X}(M) \to \mathfrak{X}(M)$ is defined inductively from $A$ by

$$P_0 = I \quad \text{and} \quad P_k = \varepsilon_N^k S_k I - \varepsilon_N A \circ P_{k-1},$$

where $I$ denotes the identity in $\mathfrak{X}(M)$. Note that by the Cayley-Hamilton theorem, from (7.1) we get $P_n = 0$. A trivial induction shows that

$$P_k = \varepsilon_N^k (S_k I - S_{k-1} A + S_{k-2} A^2 + \cdots + (-1)^k A^k).$$

Being a polynomial in $A$, $P_k$ is also self-adjoint and commutes with $A$, for every $k$. Moreover, $A(p)$ and $P_k(p)$ can be simultaneously diagonalized: if $e_1, \ldots, e_n$ are the eigenvectors of $A(p)$ corresponding to the eigenvalues $\kappa_1(p), \ldots, \kappa_n(p)$, respectively, then they are also the eigenvectors of $P_k(p)$ with corresponding eigenvalues given by

$$\mu_{i,k}(p) = \varepsilon_N^k \frac{\partial \sigma_{k+1}}{\partial x_i}(\kappa_1(p), \ldots, \kappa_n(p)) = \varepsilon_N^k \sum_{\substack{i_1 < \cdots < i_k \\ i_j \neq i}} \kappa_{i_1}(p) \cdots \kappa_{i_k}(p), \tag{4.6}$$

for every $1 \leq i \leq n$. From here, it can be easily seen that (see [27, Lemma 2.1])

$$\left\{ \begin{array}{rcl} \operatorname{trace}(P_k) & = & \varepsilon_N^k (n-k) S_k = b_k H_k; \\ \operatorname{trace}(A \circ P_k) & = & \varepsilon_N^k (k+1) S_{k+1} = b_k H_{k+1}; \\ \operatorname{trace}(A^2 \circ P_k) & = & \varepsilon_N^k (S_1 S_{k+1} - (k+2) S_{k+2}) \\ & & = n \dfrac{b_k}{k+1} H_1 H_{k+1} - b_{k+1} H_{k+2}, \end{array} \right. \tag{4.7}$$

where $b_k = (k+1)\binom{n}{k+1} = (n-k)\binom{n}{k}$ and $H_k = 0$ if $k > n$. Moreover, denoting by $A_i$ the restriction of $A$ to $\langle e_i \rangle^\top \subset T_p M$, it is easy to see that

$$\det(tI - A_i) = \sum_{k=0}^{n-1} (-1)^k S_k(A_i) t^{n-1-k},$$

where

$$S_k(A_i) = \sum_{\substack{1\leq j_1<\ldots<j_k\leq n\\ j_1,\ldots,j_k\neq i}} \lambda_{j_1}\cdots\lambda_{j_k}.$$

With the previous notations, it is also immediate to check that

$$P_k e_i = (-1)^k S_k(A_i) e_i. \tag{4.8}$$

Let $\nabla$ stand for the Levi-Civita connection of $M^n$. Associated to each Newton transformation $P_k$, we consider the second order linear differential operator $L_k : C^\infty(M) \to C^\infty(M)$ given by

$$L_k(f) = \mathrm{trace}(P_k \circ \nabla^2 f).$$

Here $\nabla^2 f : \mathfrak{X}(M) \to \mathfrak{X}(M)$ denotes the self-adjoint linear operator metrically equivalent to the Hessian of $f$, and it is given by

$$\langle \nabla^2 f(X), Y\rangle = \langle \nabla_X(\nabla f), Y\rangle, \quad X, Y \in \mathfrak{X}(M).$$

In particular, $L_0(f) = \mathrm{trace}(\nabla^2 f) = \Delta f$. When $\overline{M}^{n+1}$ is a Riemannian manifold of constant sectional curvature, H. Rosenberg [93] showed that

$$L_k(f) = \mathrm{div}(P_k(\mathrm{grad}\, f)),$$

where div denotes the divergence of a vector field on $M^n$. When $\overline{M}^{n+1}$ is the De Sitter space $\mathbb{S}^{n+1}$, we shall also show that this happens (see Chapter 7). Thus, assuming that the boundary of $M^n$ is empty, it follows from the divergence theorem that if $f \in C_c^\infty(M)$ we get

$$\int_M L_k(f) dM = 0.$$

Moreover, by noting the $\mathrm{div}(f P_k(\nabla f)) = f L_k(f) + \langle P_k(\nabla f), \nabla f\rangle$, we can once again apply the divergence theorem to get

$$\int_M f L_k(f) dM = -\int_M \langle P_k(\nabla f), \nabla f\rangle dM. \tag{4.9}$$

# Chapter 5

# Hypersurfaces in space forms

Concerning $n$-dimensional closed hypersurfaces isometrically immersed in a $(n{+}1)$-dimensional Riemannian space form, we define the notion of $(r,s,a,b)$-stability, where $r$ and $s$ are entire numbers satisfying the inequalities $0 \leq k < r \leq n-2$, and $a$ and $b$ are real numbers (at least one nonzero). In this setting, when $b=0$, we provide a characterization of the geodesic spheres as critical points of the Jacobi functional associated with the $(r,s,a,0)$-stability variational problem. Moreover, in the case $b \neq 0$, by assuming that a hypersurface $M^n$ is contained either in an open hemisphere of the Euclidean sphere or in the Euclidean space or in the hyperbolic space, under some appropriate restrictions on the constants $a$ and $b$, we prove that $M^n$ is $(r,s,a,b)$-stable if and only if $M^n$ is a geodesic sphere. The results presented here are part of our paper [101].

## 5.1 The operator $\mathfrak{L}_{r,s,a,b}$

Let $\overline{M}^{n+1}(\overline{c})$ be an orientable simply connected Riemannian manifold with constant sectional curvatures $\overline{c}$. Consider a isometric immersion $x : M^n \to \overline{M}^{n+1}(\overline{c})$ of a closed hypersurface $M^n$. Let $r$ and $s$ be entire numbers satisfying the inequalities $0 \leq r < s \leq n-2$, and let $a$ and $b$ be real numbers (at least one nonzero). Suppose that $a(r+1)S_{r+1} - b \neq 0$ on $M^n$, where $S_{r+1}$ is the elementary symmetric function of $x$ defined in Chapter 4.

We define the following second order differential operator

$$\begin{array}{rcl} \mathfrak{L}_{r,s,a,b} : C^\infty(M) & \to & C^\infty(M) \\ u & \mapsto & \mathfrak{L}_{r,s,a,b}(u) = L_s(u) - \Lambda_{r,s,a,b}\, L_r(u), \end{array} \tag{5.1}$$

where

$$\Lambda_{r,s,a,b} = \frac{a(s+1)S_{s+1}}{a(r+1)S_{r+1} - b}. \tag{5.2}$$

Now, the interesting second order differential operators are the elliptic ones. We remark that $L_0 = \Delta$ is always elliptic. On the other hand, by Lemma 3.10 of [46], $L_1$ is elliptic when $H_2 > 0$. Then, from (5.1) and (5.2) we get the following criterion of ellipticity.

**Proposition 5.1.1.** *Let $\overline{M}^{n+1}(\overline{c})$ be the Euclidean sphere $\mathbb{S}^{n+1}$ (when $\overline{c}=1$), the Euclidean space $\mathbb{R}^{n+1}$ (when $\overline{c}=0$), or the hyperbolic space $\mathbb{H}^{n+1}$ (when $\overline{c}=-1$), and let $x: M^n \to \overline{M}^{n+1}(\overline{c})$ be a closed hypersurface with $H_2>0$. If either $a=0$ or $a\neq 0$ and $H<\dfrac{b}{na}$, then the differential operator $\mathfrak{L}_{1,0,a,b}$ defined in* (5.1) *is elliptic.*

On the other hand, from Proposition 3.2 of [27] we obtain the following result, which assures us that $\mathfrak{L}_{r,s,a,b}$ is elliptic under certain hypotheses that will be used later in this chapter.

**Proposition 5.1.2.** *Let $\overline{M}^{n+1}(\overline{c})$ be an open hemisphere of the Euclidean sphere $\mathbb{S}^{n+1}$ (when $\overline{c}=1$), the Euclidean space $\mathbb{R}^{n+1}$ (when $\overline{c}=0$), or the hyperbolic space $\mathbb{H}^{n+1}$ (when $\overline{c}=-1$), and let $x: M^n \to \overline{M}^{n+1}(\overline{c})$ be a closed hypersurface. If the function $\Lambda_{r,s,a,b}$ defined in* (5.2) *is nonpositive, then the differential operator $\mathfrak{L}_{r,s,a,b}$ defined in* (5.1) *is elliptic.*

## 5.2 Description of the variational problem

Let $\overline{M}^{n+1}(\overline{c})$ be an orientable simply connected Riemannian manifold with constant sectional curvatures $\overline{c}$, and let $x: M^n \to \overline{M}^{n+1}(\overline{c})$ be an isometric immersion of a closed hypersurface $M^n$ with boundary $\partial M$ (possibly empty). A *variation* of $x$ is a smooth map $X: M^n\times(-\epsilon,\epsilon)\to \overline{M}^{n+1}(\overline{c})$ satisfying the following conditions:

(i) For each $t\in(-\epsilon,\epsilon)$, the map $X_t: M^n\to\overline{M}^{n+1}(\overline{c})$ given by $X_t(p)=X(p,t)$ is a immersion such that $X_0=x$.

(ii) $X_t\big|_{\partial M}=x\big|_{\partial M}$ for all $t\in(-\epsilon,\epsilon)$.

In what follows, we let $dM_t$ denote the volume element of the metric induced on $M^n$ by $X_t$ and $N_t$ denote the Gauss map of $X_t$. The *variational field* associated to the variation $X$ is the vector field $\frac{\partial X}{\partial t}\big|_{t=0}$.

Letting $f=-\left\langle \frac{\partial X}{\partial t}, N_t\right\rangle$, we get

$$\frac{\partial X}{\partial t}=fN_t+\left(\frac{\partial X}{\partial t}\right)^{\top},$$

where $(\cdot)^{\top}$ stands for tangential components.

The *balance of volume* of $X$ is the functional

$$\begin{array}{rcl}\mathfrak{V}:(-\epsilon,\epsilon) & \to & \mathbb{R}\\ t & \mapsto & \mathfrak{V}(t)=\displaystyle\int_{M\times[0,t]}X^*(dV),\end{array}$$

where $dV$ is the volume element of $\overline{M}^{n+1}(\overline{c})$, and we say that $X$ is *volume-preserving* if $\mathfrak{V}(t)=\mathfrak{V}(0)=0$, for all $t\in(-\epsilon,\epsilon)$.

The following result is well-known and can be found, for instance, in [27] (see also [107]).

**Lemma 5.2.1.** *Let $x : M^n \to \overline{M}^{n+1}(\overline{c})$ a closed hypersurface. If $X : M^n \times (-\epsilon,\epsilon) \to \overline{M}^{n+1}(\overline{c})$ is a variation of $x$, then*

$$\frac{\partial \mathfrak{V}}{\partial t} = \int_M f\, dM_t.$$

*In particular, $X$ is volume-preserving if and only if $\int_M f\, dM_t = 0$ for all $t \in (-\epsilon,\epsilon)$.*

We define the *$k$-area functional* $\mathfrak{A}_k : (-\epsilon,\epsilon) \to \mathbb{R}$ associated to the variation $X$ by

$$\mathfrak{A}_k(t) = \int_M F_k(S_1,\dots,S_k) dM_t,$$

where $S_k = S_k(t)$ and $F_k$ is recursively defined by setting $F_0 = 1, F_1 = S_1$ and for $2 \le k \le n$,

$$F_k = S_k + \overline{c}\,\frac{n-k+1}{k-1} F_{k-2}.$$

**Lemma 5.2.2** (Proposition 4.1 of [27]; see also Lemma 2.2 of [68])**.** *Let $x : M^n \to \overline{M}^{n+1}(\overline{c})$ be a closed hypersurface. If $X : M^n \times (-\epsilon,\epsilon) \to \overline{M}^{n+1}(\overline{c})$ is a variation of $x$, then*

$$\frac{\partial S_{k+1}}{\partial t} = L_k(f) + \big(\overline{c}\,\mathrm{trace}(P_k) + \mathrm{trace}(A^2 \circ P_k)\big)\, f + \left\langle \left(\frac{\partial X}{\partial t}\right)^{\top}, \mathrm{grad}\, S_{k+1} \right\rangle. \tag{5.3}$$

The previous lemma leads us to the first variation of the $k$-area functional.

**Lemma 5.2.3** (Proposition 4.3 of [68])**.** *Let $x : M^n \to \overline{M}^{n+1}(\overline{c})$ be a closed hypersurface. If $X : M^n \times (-\epsilon,\epsilon) \to \overline{M}^{n+1}(\overline{c})$ is a variation of $x$, then*

$$\mathfrak{A}_k'(t) = -(r+1)\int_M S_{k+1} f dM_t. \tag{5.4}$$

**Remark 5.2.4.** *We point out that Lemmas 5.2.2 and 5.2.3 were first proved by R. Reilly* [87].

Let $r$ and $s$ be entire numbers satisfying the inequalities $0 \le r < s \le n-2$, and consider real numbers $a$ and $b$ (at least one nonzero). We define the following function

$$\begin{array}{rccl} \mathfrak{C}_{r,a,b}: & (-\epsilon,\epsilon) & \to & \mathbb{R} \\ & t & \mapsto & \mathfrak{C}_{r,a,b}(t) = a\,\mathfrak{A}_r(t) + b\,\mathfrak{V}(t), \end{array} \tag{5.5}$$

and we say that the variation $X : M^n \times (-\epsilon,\epsilon) \to \overline{M}^{n+1}(\overline{c})$ of $x : M^n \to \overline{M}^{n+1}(\overline{c})$ *preserve the linear combination* $\mathfrak{C}_{r,a,b}$ if $\mathfrak{C}_{r,a,b}(t) = \mathfrak{C}_{r,a,b}(0)$ for all $t \in (-\epsilon,\epsilon)$. From Lemmas 5.2.1 and 5.2.3, we have that $X$ preserves $\mathfrak{C}_{r,a,b}$ if and only if

$$\int_M \{a(r+1)S_{r+1} - b\}\, f dM_t = 0 \quad \text{for all } t \in (-\epsilon,\epsilon),$$

provided that $a(r+1)S_{r+1} - b \neq 0$ on $M^n$. From now on, we shall always assume that the function $a(r+1)S_{r+1} - b$ never vanishes on $M^n$, unless we specify otherwise.

**Remark 5.2.5.** *From Lemma 7 of* [45], *we have that if* $\widetilde{f} \in C^\infty(M)$ *is such that* $\int_M (a(r+1)S_{r+1} - b)\widetilde{f}\, dM = 0$, *then there exists a variation* $X : M^n \times (-\epsilon, \epsilon) \to \overline{M}^{n+1}(\overline{c})$ *of* $x : M^n \to \overline{M}^{n+1}(\overline{c})$ *preserving* $\mathfrak{C}_{r,a,b}$ *whose variational field is* $\widetilde{f}N$.

Now, we consider the variational problem of minimizing the $s$-area functional $\mathfrak{A}_s$ for all variations that preserve the functional $\mathfrak{C}_{s,a,b}$. The *Jacobi functional* associated wtih this problem is given by

$$\begin{array}{rcl} \mathfrak{J}_{r,s,a,b} : (-\epsilon, \epsilon) & \to & \mathbb{R} \\ t & \mapsto & \mathfrak{J}_{r,s,a,b}(t) = \mathfrak{A}_s(t) + \varrho\, \mathfrak{C}_{r,a,b}(t), \end{array} \tag{5.6}$$

where $\varrho$ is a constant to be determined. As an immediate consequence of Lemmas 5.2.1 and 5.2.3, we get

$$\mathfrak{J}'_{r,s,a,b}(t) = -\int_M \{(s+1)S_{s+1} + \varrho\,(a(r+1)S_{r+1} - b)\}\, f dM_t.$$

To choose $\varrho$, let

$$\mathfrak{H} = \frac{1}{\mathfrak{A}_0(0)} \int_M \left\{ \frac{(s+1)S_{s+1}(0)}{a(r+1)S_{r+1}(0) - b} \right\} dM$$

be a mean of the function $(s+1)S_{s+1}(0)/\{a(r+1)S_{r+1}(0) - b\}$ along $M^n$. We call attention to the fact that, in case $(s+1)S_{s+1}(0)/\{a(r+1)S_{r+1}(0) - b\}$ is constant, one has

$$\mathfrak{H} = \frac{(s+1)S_{s+1}(0)}{a(r+1)S_{r+1}(0) - b} = \frac{(s+1)S_{s+1}}{a(r+1)S_{r+1} - b}, \tag{5.7}$$

and this notation will be used in what follows without further comments. Therefore, if we choose $\varrho = -\mathfrak{H}$, we arrive at

$$\mathfrak{J}'_{r,s,a,b}(t) = -\int_M \{(s+1)S_{s+1} - \mathfrak{H}\,(a(r+1)S_{r+1} - b)\}\, f dM_t. \tag{5.8}$$

Now, proceeding as in the proof of Proposition 2.7 of [25], we can establish, from (5.8), the following result (see also Proposition 3.6 of [100]).

**Proposition 5.2.6.** *Consider a closed hypersurface* $x : M^n \to \overline{M}^{n+1}(\overline{c})$. *Let* $r$ *and* $s$ *be entire numbers satisfying the inequalities* $0 \le r < s \le n-2$, *and let* $a$ *and* $b$ *be real numbers (at least one nonzero). Suppose that* $a(r+1)S_{r+1} - b \neq 0$ *on* $M^n$. *The following statements are equivalent.*

*(i) The elementary symmetric functions* $S_{r+1}$ *and* $S_{s+1}$ *of* $x$ *satisfy*

$$\frac{(s+1)S_{s+1}}{a(r+1)S_{r+1} - b} = \text{constant}.$$

*(ii) For all variations* $X : M^n \times (-\epsilon, \epsilon) \to \overline{M}^{n+1}(\overline{c})$ *of* $x$ *that preserve the functional* $\mathfrak{C}_{r,a,b}$, *we have* $\mathfrak{A}'_s(0) = 0$.

*(iii) For all variations* $X : \Sigma^n \times (-\epsilon, \epsilon) \to \overline{M}^{n+1}(\overline{c})$ *of* $x$, *we have* $\mathfrak{J}'_{r,s,a,b}(0) = 0$.

From Proposition 5.2.6, we infer that a hypersurface $x : M^n \to \overline{M}^{n+1}(\overline{c})$ is a critical point of the variational problem described above if and only if its elementary symmetric functions $S_{r+1}$

and $S_{s+1}$ satisfy

$$\frac{(s+1)S_{s+1}}{a(r+1)S_{r+1}-b} = \text{constant},$$

and, hence, it is not difficult to see that the function $\Lambda_{r,s,a,b}$ defined in (5.2) is a constant. This fact will be used throughout this chapter without further comments. This motivates the following

**Definition 5.2.7.** *Consider a closed hypersurface $x : M^n \to \overline{M}^{n+1}(\overline{c})$. Let $r$ and $s$ be entire numbers satisfying the inequalities $0 \le r < s \le n-2$, and let $a$ and $b$ be real numbers (at least one nonzero). Suppose that $a(r+1)S_{r+1} - b \neq 0$ on $M^n$ and that the elementary symmetric functions $S_{r+1}$ and $S_{s+1}$ of $x$ satisfy*

$$\frac{(s+1)S_{s+1}}{a(r+1)S_{r+1}-b} = \textit{constant}.$$

*We say that $x : M^n \to \overline{M}^{n+1}(\overline{c})$ is $(r,s,a,b)$-stable if $\mathfrak{A}_s''(0) \ge 0$ for all variations $X : M^n \times (-\epsilon, \epsilon) \to \overline{M}^{n+1}(\overline{c})$ of $x$ that preserve the functional $\mathfrak{C}_{r,a,b}$.*

**Remark 5.2.8.** *Let $x : M^n \to \overline{M}^{n+1}(\overline{c})$ be a hypersurface as described in Definition 5.2.7. Just as in [27], [68] and [100], we can establish the following criterion of $(r,s,a,b)$-stability: $x$ is $(r,s,a,b)$-stable if and only if $\mathfrak{J}_{r,s,a,b}''(0) \ge 0$ for all $f \in C^\infty(M)$ such that*

$$\int_M \{a(r+1)S_{r+1} - b\} f dM = 0. \tag{5.9}$$

For such hypersurfaces, it is natural to compute the second variation of $\mathfrak{J}_{r,s,a,b}$.

**Proposition 5.2.9.** *Consider a closed hypersurface $x : M^n \to \overline{M}^{n+1}(\overline{c})$. Let $r$ and $s$ be entire numbers satisfying the inequalities $0 \le r < s \le n-2$, and let $a$ and $b$ be real numbers (at least one nonzero). Suppose that $a(r+1)S_{r+1} - b \neq 0$ on $M^n$ and that the elementary symmetric functions $S_{r+1}$ and $S_{s+1}$ satisfy of $x$ satisfy*

$$\frac{(s+1)S_{s+1}}{a(r+1)S_{r+1}-b} = \textit{constant}.$$

*If $X : M^n \times (-\epsilon, \epsilon) \to \overline{M}^{n+1}(\overline{c})$ is a variation of $x$, then $\mathfrak{J}_{r,s,a,b}''(0)$ is given by*

$$\begin{aligned} \mathfrak{J}_{r,s,a,b}''(0)(u) = -(s+1)\int_M \big\{ & \mathfrak{L}_{r,s,a,b}(f) + \big(\overline{c}\,\mathrm{trace}(P_s) + \mathrm{trace}(A^2 \circ P_s) \\ & -\Lambda_{r,s,a,b}\left(\overline{c}\,\mathrm{trace}(P_r) + \mathrm{trace}(A^2 \circ P_r)\right)\big) f \big\} f dM \end{aligned} \tag{5.10}$$

*for all $u \in C^\infty(M)$.*

*Proof.* From (5.8), (5.7), and (5.3), we get

$$\begin{aligned}
\mathfrak{J}''_{r,s,a,b}(0)(u) &= \frac{\partial}{\partial t}\left(-\int_M \{(s+1)S_{s+1} - \mathfrak{H}\,(a(r+1)S_{r+1} - b)\}\, u dM_t\right)\Big|_{t=0} \\
&= -\int_M \left((s+1)\frac{\partial S_{s+1}}{\partial t}\Big|_{t=0} - \mathfrak{H} a(r+1)\frac{\partial S_{r+1}}{\partial t}\Big|_{t=0}\right) u dM \\
&\quad - \int_M \Bigg(\underbrace{(s+1)S_{s+1} - \mathfrak{H}\,(a(r+1)S_{r+1} - b)}_{=0}\Bigg)\frac{\partial}{\partial t}(u dM_t)\Big|_{t=0} \\
&= -(r+1)\int_M \big\{(L_s - \Lambda_{r,s,a,b}L_r)(u) + \big(\bar{c}\,\mathrm{trace}(P_s) + \mathrm{trace}(A^2\circ P_s) \\
&\qquad -\Lambda_{r,s,a,b}\left(\bar{c}\,\mathrm{trace}(P_r) + \mathrm{trace}(A^2\circ P_r)\right)\big)\, u\big\}\, u dM \\
&\quad - \int_M \Big\langle \left(\frac{\partial X}{\partial t}\right)^{\top}, \underbrace{\mathrm{grad}\left((s+1)S_{s+1} - \mathfrak{H}\,(a(r+1)S_{r+1} - b)\right)}_{=0}\Big\rangle u dM.
\end{aligned}$$

To finish the proof, we observe that the above expression depends only on the hypersurface $x : M^n \to \overline{M}^{n+1}(\bar{c})$ and on the function $u \in C^\infty(M)$. □

## 5.3 The geodesic spheres solve the problem!

Initially, we state a result similar to the Proposition 5.1 of [27], Proposition 4.1 of [100], Proposition 5.3 of [57], and Theorem 1 of [45].

**Proposition 5.3.1.** *Let $\overline{M}^{n+1}(\bar{c})$ be an orientable simply connected Riemannian manifold with constant sectional curvatures $\bar{c}$. If $\Lambda_{r,s,a,b}$ is nonpositive, then the geodesic spheres of $\overline{M}^{n+1}(\bar{c})$ are $(r,s,a,b)$-stable.*

*Proof.* Let $x : M^n \to \overline{M}^{n+1}(\bar{c})$ be a geodesic sphere of $\overline{M}^{n+1}(\bar{c})$. Since $M^n$ is totally umbilical, its principal curvatures are all equal to a certain constant $\lambda$. By choosing the appropriate normal vector, we may assume that $\lambda > 0$. Thus, we have

$$S_j = \binom{n}{j}\lambda^j, \quad S_j(A_i) = \binom{n-1}{j}\lambda^j,$$

and if $e_1, \dots, e_n$ are the principal directions of $M^n$,

$$L_j(u) = \sum_{i=1}^{n}\left\langle \nabla^2(u)(e_i), P_j(e_i)\right\rangle = \binom{n-1}{j}\lambda^j \Delta u,$$

for any $j \in \{0, \dots, n\}$ and all $u \in C^\infty(M)$, where we use (4.8) in the last step. Next, for entire numbers $r$ and $s$ satisfying the inequalities $0 \le r < s \le n-2$, and real numbers $a$ and $b$ (at least one nonzero) such that $a(r+1)\binom{n}{r+1}\lambda^{r+1} \neq b$, we have

$$\frac{(s+1)S_{s+1}}{a(r+1)S_{r+1} - b} = \frac{(s+1)\binom{n}{s+1}\lambda^{s+1}}{a(r+1)\binom{n}{r+1}\lambda^{r+1} - b} = \text{constant}. \tag{5.11}$$

Now, if $u \in C^\infty(M)$ satisfies the condition (5.9), then from (5.10) and (7.3), we get

$$\begin{aligned}\mathfrak{J}''_{r,s,a,b}(0)(u) &= -(s+1)\int_M \{\Gamma_{r,s,a,b}\,\Delta u + (\bar{c}(n-s)S_s + S_1S_{s+1}\\ &\quad -(s+2)S_{s+2})\,u - \Lambda_{r,s,a,b}\,(\bar{c}(n-r)S_r + S_1S_{r+1}\\ &\quad -(r+2)S_{r+2})\,u\}\,udM,\end{aligned}$$

where

$$\Gamma_{r,s,a,b} = \binom{n-1}{s}\lambda^s - \Lambda_{r,s,a,b}\binom{n-1}{r}\lambda^r.$$

It follows that

$$\begin{aligned}\mathfrak{J}''_{r,s,a,b}(0)(u) &= -(s+1)\int_M \left\{\Gamma_{r,s,a,b}\,\Delta u + \left(\bar{c}(n-s)\binom{n}{s}\lambda^s\right.\right.\\ &\quad \left. +n\binom{n}{s+1}\lambda^{s+2} - (s+2)\binom{n}{s+2}\lambda^{s+2}\right)u\\ &\quad -\Lambda_{r,s,a,b}\left(\bar{c}(n-r)\binom{n}{r}\lambda^r + n\binom{n}{r+1}\lambda^{r+2}\right.\\ &\quad \left.\left. -(r+2)\binom{n}{r+2}\lambda^{r+2}\right)u\right\}udM\end{aligned}$$

$$\begin{aligned}&= -(s+1)\int_M \left\{\Gamma_{r,s,a,b}\,\Delta u\right.\\ &\quad +\bar{c}\left((n-s)\binom{n}{s}\lambda^s - \Lambda_{r,s,a,b}(n-r)\binom{n}{r}\lambda^r\right)u\\ &\quad +\lambda^{s+2}\left(n\binom{n}{s+1} - (s+2)\binom{n}{s+2}\right)u\\ &\quad \left.-\lambda^{r+2}\Lambda_{r,s,a,b}\left(n\binom{n}{r+1} - (r+2)\binom{n}{r+2}\right)u\right\}udM\end{aligned}$$

$$\begin{aligned}&= -(s+1)\int_M \left\{\Gamma_{r,s,a,b}\,\Delta u + n\bar{c}\,\Gamma^{n-1}_{r,s,a,b}\,u + n\Gamma_{r,s,a,b}\,\lambda^2 u\right\}udM\\ &= (s+1)\Gamma_{r,s,a,b}\int_M \left\{-u\Delta u - n(\bar{c}+\lambda^2)u^2\right\}dM.\end{aligned}$$

Thus, if $\eta_1$ denotes the first eigenvalue of the Laplacian on $M^n$, and considering the assumption on the constant $\Lambda_{r,s,a,b}$, we get

$$\mathfrak{J}''_{r,s,a,b}(0)(u) \;\geq\; (s+1)\Gamma_{r,s,a,b}\int_M \left\{\eta_1 - n(\bar{c}+\lambda^2)\right\}u^2dM \;=\; 0,$$

where the last equality was obtained by observing that $M^n$ is isometric to a $n$-dimensional Euclidean sphere with constant sectional curvature equal to $\lambda^2+\bar{c}$; hence $\eta_1 = n(\lambda^2+\bar{c})$.[1] Therefore, taking into account Remark 5.2.8, we conclude that $x : M^n \to \overline{M}^{n+1}(\bar{c})$ is $(r,s,a,b)$-

[1] For details on the computation of $\eta_1$, see [41].

stable. $\square$

The next result gives a characterization of the geodesic spheres of $\overline{M}^{n+1}(\overline{c})$ as critical points of the variational problem of minimizing the $r$-area functional $\mathfrak{A}_r$ for all variations of a hypersurface $x : M^n \to \overline{M}^{n+1}(\overline{c})$ that preserve the $k$-area functional $\mathfrak{A}_k$, that is, the variational problem described in the previous section when $a \in \mathbb{R} \setminus \{0\}$ and $b = 0$. From Proposition 5.2.6, we can infer that the critical points for this variational problem are exactly the closed hipersurfaces $x : M^n \to \overline{M}^{n+1}(\overline{c})$ whose elementary symmetric functions $S_{r+1}$ and $S_{s+1}$ satisfy $S_{s+1}/S_{r+1} =$ constant, provided that $S_{k+1} \neq 0$ on $M^n$. Our next result also yields a generalization of the main result of Koh [64].

**Theorem 5.3.2.** *Let $M^{n+1}(\overline{c})$ be either an open hemisphere of the sphere $\mathbb{S}^{n+1}$ (when $\overline{c} = 1$), or the Euclidean space $\mathbb{R}^{n+1}$ (when $\overline{c} = 0$), or the hyperbolic space $\mathbb{H}^{n+1}$ (when $\overline{c} = -1$). Then, a closed hypersurface $x : M^n \to \overline{M}^{n+1}(\overline{c})$ is a critical point of the functional $\mathfrak{J}_{r,s,a,0}$ if and only if $M^n$ is a sphere, and $x$ is its inclusion as a geodesic sphere.*

*Proof.* By choosing $b = 0$ in (5.11), we have that every geodesic sphere of $M^{n+1}(\overline{c})$ is a critical point of $\mathfrak{J}_{r,s,a,0}$.

Conversely, let $x : M^n \to \overline{M}^{n+1}(\overline{c})$ be a critical point of $\mathfrak{J}_{r,s,a,0}$. This means that $H_{k+1} \neq 0$ on $M^n$, and $H_{s+1}/H_{r+1} =$ constant. Since $M^n$ is closed in $\overline{M}^{n+1}(\overline{c})$, its principal curvatures are positive at some point $p_0 \in M^n$. Let

$$0 < \omega := \frac{H_{s+1}}{H_{r+1}}(p_0) \equiv \frac{H_{s+1}}{H_{r+1}}. \tag{5.12}$$

Since $H_{r+1} \neq 0$ on $M^n$ and $H_{r+1}(p_0) > 0$, by continuity we have that $H_{r+1} > 0$ on $M^n$, which guarantees that $H_{s+1} = \omega H_{r+1} > 0$ on $M^n$. It follows from Lemma 4.0.2 that $H_j > 0$ everywhere on $M^n$ for all $j \in \{0, \dots, s+1\}$.

Moreover, if $s = s(\cdot) : M^n \to \mathbb{R}$ denotes the distance function $d(\cdot, q_0)$ from a fixed point $q_0 \in \overline{M}^{n+1}(\overline{c})$, and $\theta_{\overline{c}} = (d/ds)\, S_{\overline{c}}(s)$, where

$$S_{\overline{c}}(s) = \begin{cases} \sin s & \text{if } \overline{c} = 1, \\ s & \text{if } \overline{c} = 0, \\ \sinh s & \text{if } \overline{c} = -1, \end{cases}$$

then for $x : M^n \to \overline{M}^{n+1}(\overline{c})$ we have the following Minkowski type formulae (see [9]),

$$\int_M (H_r\, \theta_{\overline{c}} + H_{r+1}\langle Z, N\rangle)\, dM = 0, \tag{5.13}$$

and

$$\int_M (H_s\, \theta_{\overline{c}} + H_{s+1}\langle Z, N\rangle)\, dM = 0, \tag{5.14}$$

where $Z = S_{\overline{c}}(s)\, \overline{\operatorname{grad}}\, s$.

So, since $H_{s+1}/H_{r+1}$ = constant, by multiplying the left hand side of (5.13) by $H_{s+1}/H_{r+1}$ and subtracting it from the left hand side of (5.14) we arrive at

$$\int_M \theta_{\overline{c}} \left( H_{s+1} \frac{H_r}{H_{r+1}} - H_s \right) dM = 0. \tag{5.15}$$

On the order hand, by applying the classical Newton inequalities to the higher order mean curvatures of $x : M^n \to \overline{M}^{n+1}(\overline{c})$ and after some algebraic manipulations (see (5.16)), we achieve at

$$H_s \;\geq\; H_{s+1} \frac{H_r}{H_{r+1}}, \tag{5.16}$$

with equality only at umbilical points of $x(M^n)$. Now, the assumptions on the ambient space $\overline{M}^{n+1}(\overline{c})$ assure us that $\theta_{\overline{c}} > 0$ on $M^n$. Therefore, from (5.15) and (5.16) we get $H_s = H_{s+1}(H_r/H_{r+1})$, and hence $x(M^n)$ is a geodesic sphere of $\overline{M}^{n+1}(\overline{c})$. □

For what follows, we recall that the $(n+2)$-dimensional *Lorentz-Minkowski space* is the real vector space $\mathbb{R}^{n+2}$ endowed with the metric tensor

$$\langle \, , \rangle = -dx_0^2 + dx_1^2 + \cdots + dx_{n+1}^2,$$

where $(x_0, \ldots, x_{n+1})$ are the canonical coordinates of $\mathbb{R}^{n+2}$. Moreover, the *hyperbolic space*,

$$\mathbb{H}^{n+1} = \left\{ p \in \mathbb{L}^{n+2} ; \, \langle p, p \rangle = -1 \right\},$$

is a hypersurface of $\mathbb{L}^{n+2}$ such that the induced metric via the inclusion $\iota : \mathbb{H}^{n+1} \to \mathbb{L}^{n+2}$ is a Riemannian metric on $\mathbb{H}^{n+1}$. Furnished with this metric, $\mathbb{H}^{n+1}$ is the simply connected Riemannian space form of constant sectional curvatures equal to $-1$. In addition, we also recall that the *Euclidean space* $\mathbb{R}^{n+1}$ and the *Euclidean sphere* $\mathbb{S}^{n+1} \subset \mathbb{R}^{n+2}$ are the simply connected Riemannian space forms of constant sectional curvatures equal to 0 and 1, respectively.

Now, for a hypersurface $x : M^n \to \mathbb{R}^{n+1}$, let u be a fixed vector of $\mathbb{R}^{n+1}$; for a hypersurface $x : M^n \to \mathbb{S}^{n+1}$, let u be a fixed vector of $\mathbb{R}^{n+2}$; and for a hypersurface $x : M^n \to \mathbb{H}^{n+1}$, let u be a fixed vector of $\mathbb{L}^{n+2}$. We define the following maps associated to $x : M^n \to \overline{M}^{n+1}(\overline{c})$ and u,

$$\begin{array}{rccl} f_{\mathrm{u}} : & M^n & \to & \mathbb{R} \\ & p & \mapsto & f_{\mathrm{u}}(p) = \langle N(p), \mathrm{u} \rangle_p, \end{array}$$

and

$$\begin{array}{rccl} g_{\mathrm{u}} : & M^n & \to & \mathbb{R} \\ & p & \mapsto & g_{\mathrm{u}}(p) = \langle x(p), \mathrm{u} \rangle_p, \end{array}$$

which will be refer to as the *height functions* in the direction u. We will need the following result, whose proof can be found in [27].

**Lemma 5.3.3.** *If $f_{\mathrm{u}}$ and $g_u$ are the height functions of a hypersurface $x : M^n \to \overline{M}^{n+1}(\overline{c})$ in* *the direction* u *defined as above, then*

$$L_j(g_{\mathrm{u}}) \;=\; (j+1)S_{j+1}f_{\mathrm{u}} - c\,(n-j)S_j\,g_{\mathrm{u}}, \tag{5.17}$$

$$L_j(f_{\mathrm{u}}) \;=\; -\left(S_1S_{j+1} - (j+2)S_{j+2}\right)f_{\mathrm{u}} + \,c\,(j+1)S_{j+1}\,g_{\mathrm{u}} - \left\langle \operatorname{grad} S_{j+1}, \mathrm{u}\right\rangle \tag{5.18}$$

*for any* $j \in \{0, \dots, n-2\}$.

To state the main result of this chapter when the ambient space is the hyperbolic space, we shall also need the following Minkowski-type inequality for integrals. The proof is due to M.A.L. Velásquez.

**Lemma 5.3.4.** *Let* $\Sigma^n$ *be a Riemannian manifold. If* $\psi, \varphi_0, \varphi_1, \dots, \varphi_m : \Sigma^n \to \mathbb{R}$ *are continuous and compactly supported functions, then*

$$\left(\int_\Sigma |\psi| \sqrt{\sum_{\alpha=0}^{m} \varphi_\alpha^2}\, d\Sigma\right)^2 \;\geq\; \sum_{\alpha=0}^{m}\left(\int_\Sigma |\psi||\varphi_\alpha|\, d\Sigma\right)^2.$$

*Proof.* We shall show this inequality by induction on $m$. For the case $m = 1$, note that

$$\begin{aligned}
&\left(\int_\Sigma |\psi|\sqrt{\varphi_0^2+\varphi_1^2}\, d\Sigma\right)^2 - \left(\int_\Sigma |\psi||\varphi_0|\, d\Sigma\right)^2 = \\
&= \left(\int_\Sigma |\psi|\sqrt{\varphi_0^2+\varphi_1^2}\, d\Sigma - \int_\Sigma |\psi||\varphi_0|\, d\Sigma\right)\left(\int_\Sigma |\psi|\sqrt{\varphi_0^2+\varphi_1^2}\, d\Sigma + \int_\Sigma |\psi||\varphi_0|\, d\Sigma\right) \\
&= \left(\int_\Sigma \left(|\psi|\sqrt{\varphi_0^2+\varphi_1^2} - |\psi||\varphi_0|\right) d\Sigma\right)\left(\int_\Sigma \left(|\psi|\sqrt{\varphi_0^2+\varphi_1^2} + |\psi||\varphi_0|\right) d\Sigma\right).
\end{aligned}$$

Then, putting

$$f = \sqrt{|\psi|\sqrt{\varphi_0^2+\varphi_1^2} - |\psi||\varphi_0|} \quad \text{and} \quad g = \sqrt{|\psi|\sqrt{\varphi_0^2+\varphi_1^2} + |\psi||\varphi_0|}$$

into the Schwarz integral inequality

$$\left(\int_\Sigma |fg|\, d\Sigma\right)^2 \leq \left(\int_\Sigma f^2\, d\Sigma\right)\left(\int_\Sigma g^2 d\Sigma\right),$$

we get

$$
\begin{aligned}
\left(\int_\Sigma |\psi|\sqrt{\varphi_0^2+\varphi_1^2}\,d\Sigma\right)^2 - \left(\int_\Sigma |\psi||\varphi_0|\,d\Sigma\right)^2 &= \left(\int_\Sigma f^2 d\Sigma\right)\left(\int_\Sigma g^2 d\Sigma\right)\\
&\geq \left(\int_\Sigma |fg|\,d\Sigma\right)^2\\
&= \left(\int_\Sigma \sqrt{\left(|\psi|\sqrt{\varphi_0^2+\varphi_1^2}-|\psi||\varphi_0|\right)\left(|\psi|\sqrt{\varphi_0^2+\varphi_1^2}+|\psi||\varphi_0|\right)d\Sigma}\right)^2\\
&= \left(\int_\Sigma \sqrt{|\psi|^2(\varphi_0^2+\varphi_1^2)-|\psi|^2|\varphi_0|^2\,d\Sigma}\right)^2\\
&= \left(\int_\Sigma \sqrt{|\psi|^2\varphi_0^2+|\psi|^2\varphi_1^2-|\psi|^2|\varphi_0|^2\,d\Sigma}\right)^2 = \left(\int_\Sigma |\psi||\varphi_1|\,d\Sigma\right)^2, \quad .
\end{aligned}
$$

Thus,

$$
\left(\int_\Sigma |\psi|\sqrt{\varphi_0^2+\varphi_1^2}\,d\Sigma\right)^2 \geq \left(\int_\Sigma |\psi||\varphi_0|\,d\Sigma\right)^2 + \left(\int_\Sigma |\psi||\psi_1|\,d\Sigma\right)^2
$$

Now, consider $k \in \mathbb{N}$, $k \geq 2$, and assume that the result is valid for all $m \in \{1, \ldots, k-1\}$. We claim that the result is also valid for $m = k$. In fact, if $\zeta^2 = \varphi_0^2 + \cdots + \varphi_{k-1}^2$, then, from the case $m = 1$, we have

$$
\begin{aligned}
\left(\int_\Sigma |\psi|\sqrt{\varphi_0^2+\cdots+\varphi_{k-1}^2+\varphi_k^2}\,d\Sigma\right)^2 &= \left(\int_\Sigma |\psi|\sqrt{\zeta^2+\varphi_k^2}\,d\Sigma\right)^2 \geq\\
&\geq \left(\int_\Sigma |\psi||\zeta|\,d\Sigma\right)^2 + \left(\int_\Sigma |\psi||\varphi_k|\,d\Sigma\right)^2\\
&= \left(\int_\Sigma |\psi|\sqrt{\varphi_0^2+\cdots+\varphi_{k-1}^2}\,d\Sigma\right)^2 + \left(\int_\Sigma |\psi||\varphi_k|\,d\Sigma\right)^2\\
&= \sum_{\alpha=0}^{k-1}\left(\int_\Sigma |\psi||\varphi_\alpha|\,d\Sigma\right)^2 + \left(\int_\Sigma |\psi||\varphi_k|\,d\Sigma\right)^2 = \sum_{\alpha=0}^{k}\left(\int_\Sigma |\psi||\varphi_\alpha|\,d\Sigma\right)^2.
\end{aligned}
$$

□

Now, we can state the main result of this chapter, which is a sort of extension of the main results of [27], [45] and [57].

**Theorem 5.3.5.** *Let $\overline{M}^{n+1}(\overline{c})$ be either an open hemisphere of the sphere $\mathbb{S}^{n+1}$ (when $\overline{c}=1$), or the Euclidean space $\mathbb{R}^{n+1}$ (when $\overline{c}=0$), or the hyperbolic space $\mathbb{H}^{n+1}$ (when $\overline{c}=-1$). Consider a closed hypersurface $x : M^n \to \overline{M}^{n+1}(\overline{c})$. Let $r$ and $s$ be entire numbers satisfying the inequalities $0 \leq r < s \leq n-2$, and let $a$ and $b$ be real numbers (at least one nonzero). Suppose that $a(r+1)S_{r+1} - b \neq 0$ on $M^n$ and that the elementary symmetric functions $S_{r+1}$ and $S_{s+1}$ of $x$ satisfy*

$$
\frac{(s+1)S_{s+1}}{a(r+1)S_{r+1}-b} = \textit{constant.}
$$

*Suppose in addition that $\Lambda_{r,s,a,b}$ is nonpositive. Then, $x : M^n \to \overline{M}^{n+1}(\bar{c})$ is $(r,s,a,b)$-stable if and only if $M^n$ is a sphere, and $x$ is its inclusion as a geodesic sphere.*

*Proof.* According to Proposition 5.3.1, the condition is sufficient.

Now, we prove that it is also necessary. Let $x : M^n \to \overline{M}^{n+1}(\bar{c})$ be such a hypersurface. Since $M^n$ is closed in $\overline{M}^{n+1}(\bar{c})$, we may assume that the orientation is given by the choice of the inward pointed unitary normal vector field $N$ to $M^n$. So, the principal curvatures of $x$ are positive at some point $p_0 \in M^n$. Let

$$\beta := \frac{b_s H_{s+1}}{a(r+1)S_{r+1} - b}(p_0) \equiv \frac{(s+1)S_{s+1}}{a(r+1)S_{r+1} - b}, \tag{5.19}$$

where $b_r = (s+1)\binom{n}{s+1}$. If $a(r+1)S_{r+1} - b < 0$, we have $\beta < 0$, so, by (5.19), $S_{s+1} = \beta\{a(r+1)S_{r+1} - b\}(s+1)^{-1} > 0$, which implies $H_{s+1} > 0$ on $M^n$. On the other hand, if $a(r+1)S_{r+1} - b > 0$ then $\beta > 0$, so, again by (5.19), $S_{s+1} = \beta\{a(r+1)S_{r+1} - b\}(s+1)^{-1} > 0$, which guarantees that $H_{s+1} > 0$ on $M^n$. In any circumstance, we have $H_{s+1} > 0$ everywhere on $M^n$. Thus, since $\Lambda_{r,s,a,b} \leq 0$, from Proposition 5.1.2 we obtain that the operator $\mathfrak{L}_{r,s,a,b}$ defined in (5.1) is elliptic. Moreover, it follows from Lemma 4.0.2 that $H_j > 0$ everywhere on $M^n$ for all $j \in \{0, \ldots, r\}$.

We shall consider separately the three cases.

*Case 1.* Suppose $\overline{M}^{n+1}(\bar{c})$ is an open hemisphere of $\mathbb{S}^{n+1} \subset \mathbb{R}^{n+2}$. Set

$$\overline{N} = \int_M N\{a(r+1)S_{r+1} - b\}dM,$$

where the integration is done coordinatewise. We claim that $\overline{N} \neq 0$. In fact, if we assume that $\overline{N} = 0$, then

$$\int_M \Big\langle N\{a(r+1)S_{r+1} - b\}, \mathrm{u}\Big\rangle dM = 0$$

for any constant vector $\mathrm{u} \in \mathbb{R}^{n+2}$. Fixing an orthonormal basis $\{\mathrm{u}_0, \ldots, \mathrm{u}_{n+1}\}$ of $\mathbb{R}^{n+2}$, we consider, for any $\alpha \in \{0, \ldots, n+1\}$, the height functions in the direction $\mathrm{u}_\alpha$ associate to the hypersurface, given by

$$\begin{array}{rcl} f_\alpha : M^n & \to & \mathbb{R} \\ p & \mapsto & f_\alpha(p) = \langle N(p), \mathrm{u}_\alpha \rangle_p, \end{array} \tag{5.20}$$

and

$$\begin{array}{rcl} g_\alpha : M^n & \to & \mathbb{R} \\ p & \mapsto & g_\alpha(p) = \langle x(p), \mathrm{u}_\alpha \rangle_p. \end{array} \tag{5.21}$$

The hypothesis of $(r,s,a,b)$-stability implies that

$$\mathfrak{J}''_{r,s,a,b}(0)(f_\alpha) \geq 0$$

for each $\alpha \in \{0, \ldots, n+1\}$. Hence, using (4.7), (5.10) and (5.18), we obtain

$$
\begin{aligned}
0 \leq & -(s+1) \int_M \{L_s(f_\alpha) - \Lambda_{r,s,a,b} L_r(f_\alpha) \\
& \qquad + ((n-s)S_s + S_1 S_{s+1} - (s+2)S_{s+2})\, f_\alpha \\
& \qquad - \Lambda_{r,s,a,b} ((n-r)S_r + S_1 S_{r+1} - (r+2)S_{r+2})\, f_\alpha\} f_\alpha dM \\
= & -(s+1) \int_M \{-(S_1 S_{s+1} - (s+2)S_{s+2})\, f_\alpha + (s+1)S_{s+1}\, g_\alpha \\
& \qquad - \Big\langle \operatorname{grad} S_{s+1}, \mathrm{u}_\alpha \Big\rangle - \Lambda_{r,s,a,b} (-(S_1 S_{r+1} - (r+2)S_{r+2})\, f_\alpha \\
& \qquad + (r+1)S_{r+1}\, g_\alpha - \Big\langle \operatorname{grad} S_{r+1}, \mathrm{u}_\alpha \Big\rangle \Big) \\
& \qquad + ((n-s)S_s + S_1 S_{s+1} - (s+2)S_{s+2})\, f_\alpha \\
& \qquad - \Lambda_{r,s,a,b} ((n-r)S_r + S_1 S_{r+1} - (r+2)S_{r+2})\, f_\alpha\} f_\alpha dM \\
= & -(s+1) \int_M \big\{(b_s H_{s+1} + b_r H_{r+1})\, g_\alpha f_\alpha + (b_s H_s - \Lambda_{r,s,a,b}\, b_r H_r)\, f_\alpha^2\big\}\, dM \\
& \quad + \int_M \Big\langle \underbrace{\operatorname{grad}(-S_{s+1} + \Lambda_{r,s,a,b}\, S_{r+1})}_{=0}, \mathrm{u}_\alpha \Big\rangle f_\alpha dM,
\end{aligned}
$$

where $b_j = (j+1)\binom{n}{j+1} = (n-j)\binom{n}{j}$ and $\Lambda_{r,s,a,b}$ is the constant defined in (5.2). Thus,

$$
\begin{aligned}
0 \leq -(s+1) \int_M \{ & (b_s H_{s+1} - \Lambda_{r,s,a,b}\, b_r H_{r+1})\, g_\alpha f_\alpha \\
& + (b_s H_s - \Lambda_{r,s,a,b}\, b_r H_r)\, f_\alpha^2\}\, dM
\end{aligned} \tag{5.22}
$$

for any $\alpha \in \{0, \ldots, n+1\}$. By adding,

$$
\begin{aligned}
0 \leq & -(s+1) \int_M \left\{ (b_s H_{s+1} - \Lambda_{r,s,a,b}\, b_r H_{r+1}) \sum_{\alpha=0}^{n+1} g_\alpha f_\alpha \right. \\
& \qquad \left. + (b_s H_s - \Lambda_{r,s,a,b}\, b_r H_r) \sum_{\alpha=0}^{n+1} f_\alpha^2 \right\} dM \\
= & -(s+1) \int_M \{(b_s H_{s+1} - \Lambda_{r,s,a,b}\, b_r H_{r+1})\, \langle x, N \rangle \\
& \qquad + (b_s H_s - \Lambda_{r,s,a,b}\, b_r H_r)\, |N|^2\}\, dM \\
= & -(s+1) \int_M (b_s H_s - \Lambda_{r,s,a,b}\, b_r H_r)\, dM \;<\; 0\,,
\end{aligned}
$$

since $H_s > 0$, $H_r > 0$ and $\Lambda_{r,s,a,b} \leq 0$, which is absurd, and our assertion is shown.

Thus, we can take an orthonormal basis $\{\mathrm{u}_0, \ldots, \mathrm{u}_{n+1}\}$ of $\mathbb{R}^{n+2}$ such that $\mathrm{u}_0 = \overline{N}/|\overline{N}|$. Define $f_\alpha$ and $g_\alpha$ as in (5.20) and (5.21). Now, we have that

$$
\int_M \{a(r+1)S_{r+1} - b\} f_\alpha dM = 0,
$$

for each $\alpha \in \{1, \ldots, n+1\}$. For these functions, the hypothesis of $(r, s, a, b)$-stability ensures the validity of (5.22). We may add them from 1 to $n+1$ to get

$$\begin{aligned} 0 \leq -(r+1)\int_M \Bigg\{ & (b_s H_{s+1} - \Lambda_{r,s,a,b}\, b_r H_{r+1}) \sum_{\alpha=1}^{n+1} g_\alpha f_\alpha \\ & + (b_s H_s - \Lambda_{r,s,a,b}\, b_r H_r) \sum_{\alpha=1}^{n+1} f_\alpha^2 \Bigg\}\, dM \\ = (s+1)\int_M & \{(b_s H_{s+1} - \Lambda_{r,s,a,b}\, b_r H_{r+1})\, g_0 f_0 \\ & - (b_s H_s - \Lambda_{r,s,a,b}\, b_r H_r)\left(1 - f_0^2\right)\}\, dM \end{aligned} \tag{5.23}$$

Note that

$$1 - f_0^2 \geq g_0^2, \tag{5.24}$$

since $1 = |\mathrm{u}_0|^2 = \sum_{i=0}^{n+1}\langle \mathrm{u}_0, e_i\rangle^2 \geq \langle \mathrm{u}_0, x\rangle^2 + \langle \mathrm{u}_0, N\rangle^2 = g_0^2 + f_0^2$ for any local frame field $\{x = e_0, e_1, \ldots, e_n, e_{n+1} = N\}$ adapted to the hypersurface. Substituting (5.24) into (5.23) and considering (5.17),

$$\begin{aligned} 0 &\leq (s+1)\int_M \left\{(b_s H_{s+1} - \Lambda_{r,s,a,b}\, b_r H_{r+1})\, g_0 f_0 - (b_s H_s - \Lambda_{r,s,a,b}\, b_r H_r)\, g_0^2\right\} dM \\ &= (s+1)\int_M g_0 \left\{b_s H_{s+1} f_0 - b_s H_s g_0 - \Lambda_{r,s,a,b}\,(b_r H_{r+1} f_0 - b_r H_r g_0)\right\} dM \\ &= (s+1)\int_M g_0 \left\{L_s(g_0) - \Lambda_{r,s,a,b} L_r(g_0)\right\} dM \;=\; (s+1)\int_M g_0\, \mathfrak{L}_{r,s,a,b}(g_0)\, dM \\ &= -(s+1)\int_M \Big\langle (P_s - \Lambda_{r,s,a,b} P_r)\,(\operatorname{grad} g_0), \operatorname{grad} g_0 \Big\rangle dM \;\leq\; 0, \end{aligned}$$

where we used (4.9) and the ellipticity of $\mathfrak{L}_{r,s,a,b}$, respectively, in the last two steps. In particular, we infer that $\operatorname{grad} g_0 = 0$, so $g_0$ is constant. Therefore, $x(M^n)$ is contained in the intersection of a hyperplane of $\mathbb{R}^{n+2}$ and $\mathbb{S}^{n+1}$, and hence $x(M^n)$ is a geodesic sphere of $\mathbb{S}^{n+1}$.

*Case 2.* When $\overline{M}^{n+1}(\overline{c}) = \mathbb{R}^{n+1}$, the result follows as a particular case of Theorem 3 of [45].

*Case 3.* Suppose $\overline{M}^{n+1}(\overline{c}) = \mathbb{H}^{n+1} \subset \mathbb{L}^{n+2}$. We define

$$\overline{x} = \int_M x|a(r+1)S_{r+1} - b|dM,$$

where the integration is done coordinatewise. From $-1 = \langle x, x\rangle = -x_0^2 + \sum_{\alpha=1}^{n+1} x_\alpha^2$ and $x_0 > 0$, we have $x_0 = \left(1 + \sum_{\alpha=1}^{n+1} x_\alpha^2\right)^{1/2}$. Then, making $\psi = a(r+1)S_{r+1} - b$, $\varphi_0 = 1$ and $\varphi_\alpha = x_\alpha$ in

Lemma 5.3.4, we get

$$\left(\int_M x_0|a(r+1)S_{r+1}-b|dM\right)^2=\left(\int_M\left(1+\sum_{\alpha=1}^{n+1}x_\alpha^2\right)^{1/2}|a(r+1)S_{r+1}-b|dM\right)^2$$
$$\geq\left(\int_M|a(r+1)S_{r+1}-b|dM\right)^2+\sum_{\alpha=1}^{n+1}\left(\int_M|x_\alpha||a(r+1)S_{r+1}-b|dM\right)^2.$$

Thus,

$$\begin{aligned}\langle\overline{x},\overline{x}\rangle&=-\left(\int_M x_0|a(r+1)S_{r+1}-b|dM\right)^2+\sum_{\alpha=1}^{n+1}\left(\int_M x_\alpha|a(r+1)S_{r+1}-b|dM\right)^2\\&\leq-\left(\int_M x_0|a(r+1)S_{r+1}-b|dM\right)^2+\sum_{\alpha=1}^{n+1}\left(\int_M |x_\alpha||a(r+1)S_{r+1}-b|dM\right)^2\\&\leq-\left(\int_M|a(r+1)S_{r+1}-b|dM\right)^2<0.\end{aligned}$$

We choose $\mathrm{u}_0=\overline{x}/|\overline{x}|$ and complete it to an orthonormal basis $\{\mathrm{u}_0,\dots,\mathrm{u}_{n+1}\}$ of $\mathbb{L}^{n+2}$. For such a basis, we define $f_\alpha$ and $g_\alpha$ as in (5.20) and (5.21). It is now clear that

$$\int_M\{a(r+1)S_{r+1}-b\}g_\alpha dM=0$$

for each $\alpha\in\{1,\dots,n+1\}$. Thus, for such $g_\alpha$, the hypothesis of $(r,s,a,b)$-stability and formula (5.17) assures us that

$$\begin{aligned}0\leq&-(s+1)\int_M\{L_s(g_\alpha)-\Lambda_{r,s,a,b}L_r(g_\alpha)\\&\qquad+(-(n-s)S_s+S_1S_{s+1}-(s+2)S_{s+2})\,g_\alpha\\&\qquad-\Lambda_{r,s,a,b}\left(-(n-r)S_r+S_1S_{r+1}-(r+2)S_{r+2}\right)g_\alpha\}\,g_\alpha dM\\=&-(s+1)\int_M\{(s+1)S_{s+1}f_\alpha+(n-s)S_s\,g_\alpha\\&\qquad-\Lambda_{r,s,a,b}\left((r+1)S_{r+1}f_\alpha+(n-r)S_r\,g_\alpha\right)\\&\qquad+(-(n-s)S_s+S_1S_{s+1}-(s+2)S_{s+2})\,g_\alpha\\&\qquad-\Lambda_{r,s,a,b}\left(-(n-r)S_r+S_1S_{r+1}-(r+2)S_{r+2}\right)g_\alpha\}\,g_\alpha dM\\=&-(s+1)\int_M\{((s+1)S_{s+1}-\Lambda_{r,s,a,b}(r+1)S_{r+1})\,f_\alpha g_\alpha\\&\qquad+((S_1S_{s+1}-(s+2)S_{s+2})-\Lambda_{r,s,a,b}(S_1S_{r+1}-(r+2)S_{r+2}))\,g_\alpha^2\}\,dM,\end{aligned}$$

where $\Lambda_{r,s,a,b}$ is the constant defined in (5.2). Adding for $\alpha \in \{1, \ldots, n+1\}$, we get

$$0 \leq -(s+1) \int_M \left\{ ((s+1)S_{s+1} - \Lambda_{r,s,a,b}(r+1)S_{r+1}) \sum_{\alpha=1}^{n+1} f_\alpha g_\alpha \right.$$
$$\left. + ((S_1 S_{s+1} - (s+2)S_{s+2}) - \Lambda_{r,s,a,b}(S_1 S_{r+1} - (r+2)S_{r+2})) \sum_{\alpha=1}^{n+1} g_\alpha^2 \right\} dM,$$

and using that $0 = \langle N, x\rangle = -f_0 g_0 + \sum_{\alpha=1}^{n+1} f_\alpha g_\alpha$ and $-1 = \langle x, x\rangle = -g_0^2 + \sum_{\alpha=1}^{n+1} g_\alpha^2$, we obtain that

$$\begin{aligned} 0 \leq (s+1) \int_M & \{ -((s+1)S_{s+1} - \Lambda_{r,s,a,b}(r+1)S_{r+1}) f_0 g_0 \\ & + ((S_1 S_{s+1} - (s+2)S_{s+2}) \\ & -\Lambda_{r,s,a,b}(S_1 S_{r+1} - (r+2)S_{r+2})) \left(1 - g_0^2\right) \} \, dM. \end{aligned} \tag{5.25}$$

We note that

$$1 - g_0^2 \leq -f_0^2, \tag{5.26}$$

since $-1 = \langle \mathrm{u}_0, \mathrm{u}_0\rangle = -\langle \mathrm{u}_0, x\rangle^2 + \sum_{i=1}^n \langle \mathrm{u}_0, e_i\rangle^2 + \langle \mathrm{u}_0, N\rangle^2 \geq -g_0^2 + f_0^2$ for any local frame field $\{x = e_0, e_1, \ldots, e_n, e_{n+1} = N\}$ adapted to the hypersurface. Moreover, since, under our hypotheses, both $P_s$ and $P_r$ are positive operators, from (4.7) we can guarantee that the factor multiplying $1 - g_0^2$ in (5.25) is nonnegative. So, substituting (5.26) into (5.25), we get

$$\begin{aligned} 0 \leq \; & (s+1) \int_M \{ -((s+1)S_{s+1} - \Lambda_{r,s,a,b}(r+1)S_{r+1}) f_0 g_0 \\ & - ((S_1 S_{s+1} - (s+2)S_{s+2}) \\ & -\Lambda_{r,s,a,b}(S_1 S_{r+1} - (r+2)S_{r+2})) f_0^2 - \left\langle \mathrm{grad}\,(0)\,, \mathrm{u}_0 \right\rangle f_0 \Big\} \, dM \\ = \; & (s+1) \int_M f_0 \{ -((s+1)S_{s+1} - \Lambda_{r,s,a,b}(r+1)S_{r+1}) g_0 \\ & - ((S_1 S_{s+1} - (s+2)S_{s+2}) \\ & -\Lambda_{r,s,a,b}(S_1 S_{r+1} - (r+2)S_{r+2})) f_0 \\ & - \left\langle \mathrm{grad} \left( S_{s+1} - \frac{S_{s+1}}{a(r+1)S_{r+1} - b}(a(r+1)S_{r+1} - b) \right), \mathrm{u}_0 \right\rangle \Big\} \, dM \\ = \; & (s+1) \int_M f_0 \{ -(S_1 S_{s+1} - (s+2)S_{s+2}) f_0 - (s+1)S_{s+1}\, g_0 \\ & -\Lambda_{r,s,a,b} \left( -(S_1 S_{r+1} - (r+2)S_{r+2}) f_0 - (r+1)S_{r+1}\, g_0 \right) \\ & - \left\langle \mathrm{grad} \left( S_{s+1} - \frac{a(r+1)S_{s+1}}{a(r+1)S_{r+1} - b}\, S_{r+1} \right), \mathrm{u}_0 \right\rangle \Big\} \, dM \end{aligned}$$

$$
\begin{aligned}
&= (s+1)\int_M f_0\Big\{-\left(S_1S_{s+1}-(s+2)S_{s+2}\right)f_0-(s+1)S_{s+1}\,g_0-\Big\langle \mathrm{grad}S_{s+1},\mathrm{u}_0\Big\rangle\\
&\quad -\Lambda_{r,s,a,b}\Big(-\left(S_1S_{r+1}-(r+2)S_{r+2}\right)f_0-(r+1)S_{r+1}\,g_0-\Big\langle \mathrm{grad}S_{r+1},\mathrm{u}_0\Big\rangle\Big)\Big\}\,dM\\
&= (s+1)\int_M f_0\left\{L_s(f_0)-\Lambda_{r,s,a,b}L_r(f_0)\right\}dM \;=\; (s+1)\int_M f_0\,\mathfrak{L}_{r,s,a,b}(f_0)\,dM\\
&= -(s+1)\int_M \Big\langle \left(P_s-\Lambda_{r,s,a,b}P_r\right)(\mathrm{grad}\,f_0),\mathrm{grad}\,f_0\Big\rangle dM \;\leq\; 0,
\end{aligned}
$$

where we used (5.18), (4.9) and the ellipticity of $\mathfrak{L}_{r,s,a,b}$, respectively, in the last three steps. In particular, $\mathrm{grad}\,f_0 = 0$ and $1-g_0^2=-f_0^2$. This implies that $g_0$ is constant. Thus, $x(M^n)$ is contained in the intersection of a Euclidean hyperplane of $\mathbb{L}^{n+2}$ and $\mathbb{H}^{n+1}$, and hence $x(M^n)$ is a geodesic sphere of $\mathbb{H}^{n+1}$. □

We recall that a hypersurface $x: M^n \to \overline{M}^{n+1}(\overline{c})$ is called *linear Weingarten* when the mean curvature $H$ and normalized scalar curvature $S$ satisfy

$$\delta_0 H + \delta_1 S = \delta_2,$$

for some constants $\delta_0,\delta_1,\delta_2\in\mathbb{R}$. Then, making $r=1$ and $k=0$ in Theorem 5.3.5, from Proposition 5.1.1 and formulae (4.3) and (4.4), we get the following

**Corollary 5.3.6.** *Let $\overline{M}^{n+1}(\overline{c})$ be either an open hemisphere of the sphere $\mathbb{S}^{n+1}$ (when $\overline{c}=1$), or the Euclidean space $\mathbb{R}^{n+1}$ (when $\overline{c}=0$), or the hyperbolic space $\mathbb{H}^{n+1}$ (when $\overline{c}=-1$). Consider a closed linear Weingarten hypersurface $x: M^n\to\overline{M}^{n+1}(\overline{c})$ such that*

$$n(n-1)(S-\overline{c}) - naH\delta = -b\delta,$$

*where $a\in\mathbb{R}$ and $b,\delta\in\mathbb{R}\setminus\{0\}$. Suppose that either $a=0$, or $a\neq 0$ and $H<\dfrac{b}{na}$. Then $x: M^n\to\overline{M}^{n+1}(\overline{c})$ is $(1,0,a,b)$-stable if and only if $M^n$ is a sphere, and $x$ is its inclusion as a geodesic sphere.*

# Chapter 6

# Hypersurfaces in hyperbolic space

In this chapter, we establish the notion of strong $(r, s, a, b)$-stability related to closed hypersurfaces isometrically immersed in the hyperbolic space $\mathbb{H}^{n+1}$, where $r$ and $s$ are nonnegative integers satisfying the inequalities $0 \leq r < s \leq n-2$, and $a$ and $b$ are real numbers (at least one nonzero). Under some appropriate restrictions on the constants $a$ and $b$, we show that geodesic spheres are strongly $(r, s, a, b)$-stable. Afterwards, under a suitable restriction on the higher order mean curvatures $H_{s+1}$ and $H_{r+1}$, we prove that if a closed hypersurface into the hyperbolic space $\mathbb{H}^{n+1}$ is strongly $(r, k, a, b)$-stable, then it must be a geodesic sphere, provided that the image of its Gauss map is contained in the chronological future (or past) of an equator of the De Sitter space. The results presented here are part of our paper [102].

## 6.1 The notion of strong $(r, s, a, b)$-stability

First of all, we shall summarize here the variational concepts introduced in Section 5.2 of Chapter 5.

A *variation* of a closed hypersurface $x : M^n \to \mathbb{H}^{n+1}$ is a smooth map $X : (-\epsilon, \epsilon) \times M^n \to \mathbb{H}^{n+1}$ that satisfies the following condition: for $t \in (-\epsilon, \epsilon)$, the map $X_t : M^n \to \mathbb{H}^{n+1}$ given by $X_t(p) = X(t, p)$ is an immersion such that $X_0 = x$.

The *balance of volume* of the variation $X : (-\epsilon, \epsilon) \times M^n \to \mathbb{H}^{n+1}$ is the following functional

$$\begin{array}{rcl} \mathfrak{V} : (-\epsilon, \epsilon) & \to & \mathbb{R} \\ t & \mapsto & \mathfrak{V}(t) = \displaystyle\int_{M^n \times [0,t]} X^*(d\mathbb{H}^{n+1}), \end{array}$$

where $d\mathbb{H}^{n+1}$ denotes the volume element of $\mathbb{H}^{n+1}$.

We define the *k-area functional* associated to the variation $X : (-\epsilon, \epsilon) \times M^n \to \mathbb{H}^{n+1}$ by

$$\begin{array}{rcl} \mathfrak{A}_k : (-\epsilon, \epsilon) & \to & \mathbb{R} \\ t & \mapsto & A_k(t) = \displaystyle\int_{M} F_k(S_1, S_2, \ldots, S_k) dM_t, \end{array}$$

where $dM_t$ denotes the volume element of the metric induced on $M^n$ by $X_t$, $N_t$ is the unitary normal vector field along $X_t$, $S_k = S_k(t)$, and $F_k$ is recursively defined by setting $F_0 = 1$, $F_1 = S_1$ and, for $2 \leq k \leq n-1$,

$$F_k = S_k - \frac{(n-k+1)}{k-1} F_{k-2}.$$

Let $r$ and $s$ be integers satisfying the inequalities $0 \leq r < s \leq n-2$, and consider real numbers $a$ and $b$ (with at least one non zero). We define the following functional,

$$\begin{array}{rccl} \mathfrak{C}_{r,a,b}: & (-\epsilon,\epsilon) & \to & \mathbb{R} \\ & t & \mapsto & \mathfrak{C}_{r,a,b}(t) = a\,\mathfrak{A}_r(t) + b\,\mathfrak{V}(t), \end{array}$$

and we say that the variation $X : (-\epsilon,\epsilon) \times M^n \to \mathbb{H}^{n+1}$ of $x : M^n \to \mathbb{H}^{n+1}$ *preserves the functional* $\mathfrak{C}_{r,a,b}$ if $\mathfrak{C}_{r,a,b}(t) = \mathfrak{C}_{r,a,b}(0)$ for all $t \in (-\epsilon,\epsilon)$.

Now, consider the variational problem of *minimizing the s-area functional* $\mathfrak{A}_s$ *preserving the functional* $\mathfrak{C}_{r,a,b}$. To solve this problem we assume that one of its critical points is a given immersion $x$. Hence, with respect to variations that preserve the functional $\mathfrak{C}_{r,a,b}$, it must be a critical point of the function $\mathfrak{A}_k(t) = \int_M F_k(S_1, S_2, \ldots, S_k)dM_t$. To find the corresponding Euler equation, we then use Lagrange's multipliers rule. This means to consider the *Jaboci operator*

$$\begin{array}{rccl} \mathfrak{J}_{r,s,a,b}: & (-\epsilon,\epsilon) & \to & \mathbb{R} \\ & t & \mapsto & \mathfrak{J}_{r,s,a,b}(t) = \mathfrak{A}_s(t) + \varrho\, \mathfrak{C}_{r,a,b}(t), \end{array}$$

where $\varrho$ is a constant to be determined. As an immediate consequence of Lemmas 5.2.3 and 5.2.1, we get

$$\mathfrak{J}'_{r,s,a,b}(t) = -\int_M \{b_s H_{s+1} + \varrho\,(a\, b_r H_{r+1} - b)\}\, f dM_t,$$

where $f$ is the function given by $f = \left\langle \frac{\partial X}{\partial t}, N_t \right\rangle$. (Recall that the *variational field* associated to a variation $X : (-\epsilon,\epsilon) \times M^n \to \mathbb{H}^{n+1}$ is $\frac{\partial X}{\partial t}|_{t=0}$. So, we have $\frac{\partial X}{\partial t} = fN_t + \left(\frac{\partial X}{\partial t}\right)^{\top}$, where $(\,\cdot\,)^{\top}$ stands for tangential components.) Reasoning as in Section 5.2 of Chapter 5, we choose

$$\varrho = -\frac{b_s H_{s+1}(0)}{a\, b_r H_{r+1}(0) - b} = -\frac{b_s H_{s+1}}{a\, b_r H_{r+1} - b}, \tag{6.1}$$

where $b_k = (k+1)\binom{n}{k+1}$, $k = r, s$.

In the setting of this chapter, Proposition 5.2.6 reads as follows.

**Proposition 6.1.1.** *Consider a closed hypersurface $x : M^n \to \mathbb{H}^{n+1}$. Let $r$ and $s$ be entire numbers satisfying the inequalities $0 \leq r < s \leq n-2$, and let $a$ and $b$ be real numbers (at least one nonzero). Suppose that $a(r+1)S_{r+1} - b \neq 0$ on $M^n$. The following statements are equivalent.*

*(i) The higher order mean curvatures $H_{r+1}$ and $H_{s+1}$ of $x$ satisfy*

$$\frac{b_s H_{s+1}}{ab_r H_{r+1} - b} = \text{constant}.$$

*(ii) For all variations $X : M^n \times (-\epsilon,\epsilon) \to \mathbb{H}^{n+1}$ of $x$ that preserve the functional $\mathfrak{C}_{r,a,b}$, we*

*have* $\mathfrak{A}'_s(0) = 0$.

*(iii) For all variations* $X : M^n \times (-\epsilon, \epsilon) \to \mathbb{H}^{n+1}$ *of* $x$, *we have* $\mathfrak{J}'_{r,s,a,b}(0) = 0$.

Note that the variational problems described in items (ii) and (iii) of Proposition 6.1.1 are equivalent, since the critical points of both problems are closed hypersurfaces $M^n$ of $\mathbb{H}^{n+1}$ such that

$$\frac{b_s H_{s+1}}{ab_r H_{r+1} - b} = \text{constant}, \tag{6.2}$$

provided that $ab_r H_{r+1} - b \neq 0$ on $M^n$. From now on, we shall assume that this later condition holds.

At this point, motivated by the ideas established in [72], we exchange our problem, and now we want *to detect closed hypersurfaces* $x : M^n \to \mathbb{H}^{n+1}$ *that minimize the Jacobi functional* $\mathfrak{J}_{r,s,a,b}$ *for all variations* $X : (-\epsilon, \epsilon) \times M^n \to \mathbb{H}^{n+1}$ *of* $x$. Proposition 6.1.1 shows that the critical points of $\mathfrak{J}_{r,s,a,b}$ are hypersurfaces $x : M^n \to \mathbb{H}^{n+1}$ whose higher order mean curvatures $H_{r+1}$ and $H_{s+1}$ satisfy the condition (6.2). This motivates us to establish the following notion of stability.

**Definition 6.1.2.** *Consider a closed hypersurface* $x : M^n \to \mathbb{H}^{n+1}$. *Let* $r$ *and* $s$ *be entire numbers satisfying the inequalities* $0 \leq r < s \leq n-2$, *and let* $a$ *and* $b$ *be real numbers (at least one nonzero). Suppose that* $ab_r H_{r+1} - b \neq 0$ *on* $M^n$, *and that the higher order mean curvatures* $H_{r+1}$ *and* $H_{s+1}$ *of* $x$ *satisfy*

$$\frac{b_s H_{s+1}}{ab_r H_{r+1} - b} = \textit{constant},$$

*where* $b_k = (k+1)\binom{n}{k+1}$, $k = r, s$. *We say that* $x : M^n \to \mathbb{H}^{n+1}$ *is* strongly $(r, s, a, b)$-stable *if* $\mathfrak{J}''_{r,s,a,b}(0)(u) \geq 0$ *for all* $u \in C^\infty(M)$.

For such hypersurfaces, it is natural to compute the second variation of $\mathfrak{J}_{r,s,a,b}$. The proof of the next proposition follows the same steps as the proof of Proposition 5.2.9.

**Proposition 6.1.3.** *Consider a closed hypersurface* $x : M^n \to \mathbb{H}^{n+1}$. *Let* $r$ *and* $s$ *be entire numbers satisfying the inequalities* $0 \leq r < s \leq n-2$, *and let* $a$ *and* $b$ *be real numbers (at least one nonzero). Suppose that* $a\, b_r H_{r+1} - b \neq 0$ *on* $M^n$, *and that the higher order mean curvatures* $H_{r+1}$ *and* $H_{s+1}$ *of* $x$ *satisfy the condition*

$$\frac{b_s H_{s+1}}{a\, b_r H_{r+1} - b} = \text{constant},$$

*where* $b_k = (k+1)\binom{n}{k+1}$ *for* $k = r, s$. *If* $X : (-\epsilon, \epsilon) \times M^n \to \mathbb{H}^{n+1}$ *is a variation of* $x$, *then* $\mathfrak{J}''_{r,s,a,b}(0)$ *is given by*

$$\begin{aligned} \mathfrak{J}''_{r,s,a,b}(0)(u) = -(s+1)\int_M \Big\{ \mathfrak{L}_{r,s,a,b}(u) + \Big( &- \text{trace}(P_s) + \text{trace}(A^2 \circ P_s) \\ &-\Lambda_{r,s,a,b}\big( - \text{trace}(P_r) + \text{trace}(A^2 \circ P_r)\big)\Big) u \Big\} u dM \end{aligned} \tag{6.3}$$

*for any $u \in C^\infty(\Sigma^n)$, where $\mathfrak{L}_{r,s,a,b}$ is the second order differential operator*

$$\begin{array}{rcl} \mathfrak{L}_{r,s,a,b} : C^\infty(M) & \to & C^\infty(M) \\ u & \mapsto & \mathfrak{L}_{r,s,a,b}(u) = L_s(u) - \Lambda_{r,s,a,b}\, L_r(u), \end{array} \tag{6.4}$$

*and*

$$\Lambda_{r,s,a,b} := \frac{a(r+1)b_s H_{s+1}}{a(s+1)b_r H_{r+1} - (s+1)b}. \tag{6.5}$$

## 6.2 Strongly $(r,s,a,b)$-stable hypersurfaces in $\mathbb{H}^{n+1}$

We first establish a result similar to the statements found in Proposition 5.1 of [27] and Proposition 4.1 of [100].

**Proposition 6.2.1.** *If $\Lambda_{r,s,a,b}$ is nonpositive, then the geodesic spheres of $\mathbb{H}^{n+1}$ are strongly $(r,s,a,b)$-stable.*

*Proof.* Let $x : M^n \to \mathbb{H}^{n+1}$ be a geodesic sphere of $\mathbb{H}^{n+1}$. Since $M^n$ is totally umbilical, its principal curvatures are all equal to a certain constant $\lambda$. By choosing a suitable normal vector, we may assume that $\lambda > 0$. Thus, we have

$$S_j = \binom{n}{j}\lambda^j, \quad H_j = \lambda^j, \quad S_j(A_i) = \binom{n-1}{j}\lambda^j,$$

and, if $e_1, \dots, e_n$ are the principal directions of $M^n$,

$$L_j(u) = \sum_{i=1}^n \langle \nabla^2 f(e_i), P_j(e_i)\rangle = \binom{n-1}{j}\lambda^j \Delta u$$

for any $j \in \{0, \dots, n\}$ and all $u \in C^\infty(M)$, where we used (4.8) in the last step. Next, for integers $r$ and $s$ satisfying the inequalities $0 \le r < s \le n-2$, and real numbers $a$ and $b$ (with at least one non zero) such that $a(r+1)\binom{n}{r+1}\lambda^{r+1} \neq b$, we have $b_s H_{s+1}/\{a\, b_r H_{r+1} - b\} = b_s \lambda^{s+1}/\{a\, b_r \lambda^{r+1} - b\} =$ constant, where $b_j = (j+1)\binom{n}{j+1}$ for $j = r, s$. Then, from (4.7) and (6.3) we get

$$\begin{aligned} \mathfrak{J}''_{r,s,a,b}(0)(u) = -(s+1)\int_M \Big\{ & \Gamma_{r,s,a,b}\,\Delta u + \Big( -(n-s)S_s + S_1 S_{s+1} \\ & -(s+2)S_{s+2}\Big)u - \Lambda_{r,s,a,b}\Big( -(n-r)S_r + S_1 S_{r+1} \\ & -(r+2)S_{r+2}\Big)u\Big\} u\, dM, \end{aligned}$$

where

$$\Gamma_{r,s,a,b} = \binom{n-1}{s}\lambda^s - \Lambda_{r,s,a,b}\binom{n-1}{r}\lambda^r. \tag{6.6}$$

Thus, for any $u \in C^\infty(M)$ we have

$$\begin{aligned}
\mathfrak{J}''_{r,s,a,b}(0)(u) &= -(s+1)\int_M \Big\{\Gamma_{r,s,a,b}\,\Delta u + \Big(-(n-s)\binom{n}{s}\lambda^s \\
&\quad + n\binom{n}{s+1}\lambda^{s+2} - (s+2)\binom{n}{s+2}\lambda^{s+2}\Big)\,u \\
&\quad - \Lambda_{r,s,a,b}\Big(-(n-r)\binom{n}{r}\lambda^r + n\binom{n}{r+1}\lambda^{r+2} \\
&\quad - (r+2)\binom{n}{r+2}\lambda^{r+2}\Big)\,u\Big\} u dM \\
&= -(s+1)\int_M \Big\{\Gamma_{r,s,a,b}\,\Delta u \\
&\quad - \Big((n-s)\binom{n}{s}\lambda^s - \Lambda_{r,s,a,b}(n-r)\binom{n}{r}\lambda^r\Big)\,u \\
&\quad + \lambda^{s+2}\Big(n\binom{n}{s+1} - (s+2)\binom{n}{s+2}\Big)\,u \\
&\quad - \lambda^{r+2}\Lambda_{r,s,a,b}\Big(n\binom{n}{r+1} - (r+2)\binom{n}{r+2}\Big)\,u\Big\} u dM \\
&= -(s+1)\int_M\Big\{\Gamma_{r,s,a,b}\,\Delta u - n\,\Gamma_{r,s,a,b}\,u + n\Gamma_{r,s,a,b}\,\lambda^2 u\Big\} u dM \\
&= (s+1)\Gamma_{r,s,a,b}\int_M \left\{-u\Delta u - n(-1+\lambda^2)u^2\right\} dM.
\end{aligned} \tag{6.7}$$

Therefore, if $\eta_1$ denotes the first eigenvalue of the Laplacian of $M^n$, taking into account the assumption on the function $\Lambda_{r,s,a,b}$, from (6.6) and (6.7) we get

$$\mathfrak{J}''_{r,s,a,b}(0)(u) \;\geq\; (s+1)\Gamma_{r,s,a,b}\int_M \left\{\eta_1 - n(-1+\lambda^2)\right\} u^2 dM \;=\; 0,$$

for any $u \in C^\infty(M)$, where the last equality was obtained by observing that $M^n$ is isometric to a $n$-dimensional Euclidean sphere with constant sectional curvatures equal to $\lambda^2 - 1$; hence $\eta_1 = n(\lambda^2 - 1)$. It follows that $x : M^n \to \mathbb{H}^{n+1}$ is strongly $(r, s, a, b)$-stable. $\square$

To prove the main result of this chapter, we note that, according to Example 4.3 of [77], the hyperbolic space $\mathbb{H}^{n+1}$ (minus a certain point $q$) can be regarded of as the following warped product

$$\mathbb{H}^{n+1} \setminus \{q\} \;\simeq\; (0, +\infty) \times_{\sinh t} \mathbb{S}^n, \quad t \in (0, +\infty), \tag{6.8}$$

($\simeq$ means *isometric to*) where $\mathbb{S}^n$ stands for the Euclidean unitary sphere. More precisely, if $dt^2$ and $d\sigma^2$ denote the metrics of $(0, +\infty)$ and $\mathbb{S}^n$, respectively, then we endowed $(0, +\infty) \times \mathbb{S}^n$ with the metric tensor

$$\langle\, , \,\rangle = (\pi_1)^*\left(dt^2\right) + (\sinh t)^2 (\pi_{\mathbb{S}^n})^*\left(d\sigma^2\right),$$

where $\pi_1$ and $\pi_{\mathbb{S}^n}$ denote the projections onto the $(0, +\infty)$ and $\mathbb{S}^n$ factors, respectively. The gradient field of $\pi_1$ in $(0, +\infty) \times_{\sinh t} \mathbb{S}^n$ will be denoted by $\partial_t$.

In this warped product model, the *slices*

$$M^n_{t_0} := \{t_0\} \times \mathbb{S}^n, \quad t_0 \in (0, +\infty),$$

are exactly the geodesic spheres of $\mathbb{H}^{n+1}$. Moreover, if we orient such slices by the unitary normal vector field $-\partial_t$, then the $j$-th mean curvature of $M^n_{t_0}$ is constant and equals to

$$H_j = (\coth t_0)^j, \quad j = 1, \ldots, n. \tag{6.9}$$

Finally, we observe that

$$W = \sinh t \, \partial_t \tag{6.10}$$

is a conformal closed vector field (in the sense that its dual 1-form is closed), namely, $\overline{\nabla}_Y W = \cosh t \, Y$ for any tangent vector field $Y$ defined in $\mathbb{H}^{n+1} \setminus \{q\}$.

Given a hypersurface $x : M^n \to (0, +\infty) \times_{\sinh t} \mathbb{S}^n$ with Gauss map $N$, we define a function $\eta$ by $\eta(p) = \langle W(p), N(p), p \in M^n$. The following result gives a formula for $\mathfrak{L}_{r,s,a,b}(\eta)$.

**Lemma 6.2.2.** *Let $r$ and $s$ be integer numbers satisfying the inequalities $0 \leq r < s \leq n-2$. Consider a hypersurface $x : M^n \to (0, +\infty) \times_{\sinh t} \mathbb{S}^n$ whose higher order mean curvatures $H_{r+1}$ and $H_{s+1}$ satisfy the condition*

$$\frac{b_s H_{s+1}}{a \, b_r H_{r+1} - b} = \text{constant},$$

*with $a \, b_r H_{r+1} - b \neq 0$ on $M^n$, where $a$ and $b$ are real numbers (with at least one non zero) and $b_j = (j+1)\binom{n}{j+1}$ for $j = r, s$. If $N$ is the Gauss map of $M^n$ and $\eta = \langle W, N \rangle$, then*

$$\begin{aligned} \mathfrak{L}_{r,s,a,b}(\eta) = & -\big(-\mathrm{trace}(P_s) + \mathrm{trace}(A^2 \circ P_s)\big)\,\eta \\ & -\Lambda_{r,s,a,b}\big(-\mathrm{trace}(P_r) + \mathrm{trace}(A^2 \circ P_r)\big)\,\eta \\ & -\{b_s H_s - \Lambda_{r,s,a,b}\, b_r H_r\}\,\langle W, N\rangle \\ & -\{b_s H_{s+1} - \Lambda_{r,s,a,b}\, b_r H_{r+1}\}\cosh t, \end{aligned} \tag{6.11}$$

*where $\mathfrak{L}_{r,s,a,b}$ is the differential operator defined in* (6.4)*, and $\Lambda_{r,s,a,b}$ is defined in* (6.5)*.*

*Proof.* From Theorem 2 of [29], for $j = r, s$ we have

$$\begin{aligned} L_j(\eta) = & -\big\{\mathrm{trace}(A^2 \circ P_j) - \mathrm{trace}(P_j)\big\}\,\eta - b_j H_j N(\cosh t) - b_j H_{j+1}\cosh t \\ & -\frac{b_j}{j+1}\langle W, \mathrm{grad}\, H_{j+1}\rangle. \end{aligned}$$

It follows that

$$
\begin{aligned}
\mathfrak{L}_{r,s,a,b}(\eta) &= L_s(\eta) - \Lambda_{r,s,a,b}L_r(\eta) \qquad (6.12)\\
&= -\left\{-\mathrm{trace}(P_s) + \mathrm{trace}(A^2 \circ P_s)\right\}\eta \\
&\quad -b_sH_sN(\cosh t) - b_sH_{s+1}\cosh t - \frac{b_s}{s+1}\langle W, \mathrm{grad}\, H_{s+1}\rangle \\
&\quad -\Lambda_{r,s,a,b}\big(-\left\{-\mathrm{trace}(P_r) + \mathrm{trace}(A^2 \circ P_r)\right\}\eta - b_rH_rN(\cosh t) \\
&\quad -b_rH_{r+1}\cosh t - \frac{b_r}{r+1}\langle W, \mathrm{grad}\, H_{r+1}\rangle\Big) \\
&= -\big(-\mathrm{trace}(P_s) + \mathrm{trace}(A^2 \circ P_s)\big) \\
&\quad -\Lambda_{r,s,a,b}\big(-\mathrm{trace}(P_r) + \mathrm{trace}(A^2 \circ P_r)\big)\eta \\
&\quad -\{b_sH_s - \Lambda_{r,s,a,b}\, b_rH_r\}\, N(\cosh t) \\
&\quad -\{b_sH_{s+1} - \Lambda_{r,s,a,b}\, b_rH_{r+1}\}\cosh t \\
&\quad -\Big\langle W, \underbrace{\mathrm{grad}\left(-\frac{b_s}{s+1}H_{s+1} + \Lambda_{r,s,a,b}\,\frac{b_r}{r+1}H_{r+1}\right)}_{=0}\Big\rangle.
\end{aligned}
$$

Now, observe that

$$\overline{\mathrm{grad}}\,\cosh t = \left\langle \overline{\mathrm{grad}}\,\cosh t, \partial_t \right\rangle \partial_t = (\cosh t)'\partial_t = W;$$

so we have that

$$N(\cosh t) = \left\langle \overline{\mathrm{grad}}\,\cosh t, N \right\rangle = \langle W, N\rangle. \qquad (6.13)$$

Finally, substituting (6.13) into (6.12) we obtain (6.11). □

Returning to the hyperquadric model of $\mathbb{H}^{n+1} \subset \mathbb{L}^{n+2}$, let $\mathbf{a} \in \mathbb{L}^{n+2}$ be an unitary timelike vector (that is, $\langle \mathbf{a}, \mathbf{a}\rangle = -1$). Then, we easily verify that

$$V(p) = \mathbf{a} + \langle p, \mathbf{a}\rangle p, \quad p \in \mathbb{H}^{n+1}, \qquad (6.14)$$

is a closed conformal vector field globally defined in $\mathbb{H}^{n+1}$. Consequently, from Proposition 1 in [77], we have that such a vector field $V$ foliates $\mathbb{H}^{n+1}$ by means of totally umbilical spheres, which can be characterized as the following level sets

$$L_\delta = \{\ p \in \mathbb{H}^{n+1}\ :\ \langle p, \mathbf{a}\rangle = \delta\ \}, \quad \delta^2 > 1.$$

We recall that the $(n+1)$-dimensional De Sitter space $\mathbb{S}_1^{n+1}$ is defined as being the following hyperquadric of $\mathbb{L}^{n+2}$,

$$\mathbb{S}_1^{n+1} = \left\{p \in \mathbb{L}^{n+2}\ :\ \langle p, p\rangle = 1\right\}.$$

The induced metric from $\langle\ ,\ \rangle$ makes $\mathbb{S}_1^{n+1}$ into a Lorentz manifold with constant sectional curvature one.

In a dual manner of that of the hyperbolic space, taking again a unitary timelike vector

$\mathbf{a} \in \mathbb{L}^{n+2}$, we have the vector field

$$K(p) = \mathbf{a} - \langle p, \mathbf{a} \rangle p \,, \quad p \in \mathbb{S}^{n+1}_1,$$

is a conformal closed timelike vector field globally defined in $\mathbb{S}^{n+1}_1$. From Proposition 1 in [77], we see that such a vector field $K$ foliates $\mathbb{S}^{n+1}_1$ by means of totally umbilical round spheres, which are described as the following level sets

$$\mathcal{L}_\varepsilon = \{ \; p \in \mathbb{S}^{n+1}_1 : \langle p, \mathbf{a} \rangle = \varepsilon \; \} \,, \quad \varepsilon \in \mathbb{R}.$$

In particular, the level set $\{ \; p \in \mathbb{S}^{n+1}_1 : \langle p, \mathbf{a} \rangle = 0 \; \}$ defines a round sphere of radius one which is a totally geodesic hypersurface in $\mathbb{S}^{n+1}_1$. Following [6], we shall refer to that sphere as the *equator* of $\mathbb{S}^{n+1}_1$ determined by $\mathbf{a}$. This equator divides $\mathbb{S}^{n+1}_1$ into two connected components, the *chronological future*, which is given by

$$\{ \; p \in \mathbb{S}^{n+1}_1 \; : \; \langle \mathbf{a}, p \rangle < 0 \; \}, \tag{6.15}$$

and the *chronological past*, given by

$$\{ \; p \in \mathbb{S}^{n+1}_1 \; : \; \langle \mathbf{a}, p \rangle > 0 \; \}.$$

The reason for this terminology is due to the fact that, in the time-orientation of $\mathbb{S}^{n+1}$ determined by $\mathbf{a}$, the subset $\{p \in \mathbb{S}^{n+1}_1 : \langle \mathbf{a}, p \rangle < 0\}$ represents the events which are in the chronological future of the equator determined by $\mathbf{a}$.

We observe that the unitary normal vector field $N$ of $x : M^n \to \mathbb{H}^{n+1}$ can be regarded as a map $N : M^n \to \mathbb{S}^{n+1}_1$, called *Gauss map* of $x$. The image $N(M^n)$ will be called the *Gauss image* of $x$.

**Remark 6.2.3.** *By fixing a unit timelike vector* $\mathbf{a} \in \mathbb{L}^{n+2}$ *and considering in* $\mathbb{H}^{n+1}$ *as well as in* $\mathbb{S}^{n+1}_1$ *the foliations previously described, it is easy to verify that the Gauss map of a geodesic sphere* $L_\delta$ *of* $\mathbb{H}^{n+1}$ *is given by*

$$N(p) = -\frac{1}{\sqrt{\delta^2 - 1}} (\mathbf{a} + \delta p) \,, \qquad p \in L_\delta.$$

*Consequently, we have that* $N(L_\delta) \subset \mathcal{L}_\varepsilon$ *for* $\varepsilon = -\sqrt{\delta^2 - 1} < 0$. *Therefore, we conclude that the Gauss image of a geodesic sphere of* $\mathbb{H}^{n+1}$ *is contained in the chronological future (or past) of the equator of* $\mathbb{S}^{n+1}_1$ *determined by* $\mathbf{a}$.

Now, we are in position to state and prove the main result of this chapter.

**Theorem 6.2.4.** *Let* $r$ *and* $s$ *be integer numbers satisfying the inequalities* $0 \le r < s \le n-2$, *and let* $a$ *and* $b$ *be real numbers with* $b \neq 0$. *Consider a strongly* $(r, s, a, b)$*-stable closed hypersurface* $x : M^n \to \mathbb{H}^{n+1}$. *Suppose that* $\Lambda_{r,s,a,b}$ *is nonpositive and that the higher order mean curvatures*

*$H_{r+1}$ and $H_{s+1}$ of $x$ satisfy*

$$H_{j+1} \;\geq\; H_j, \quad j = r, s. \tag{6.16}$$

*If the Gauss image of $x$ is contained in the chronological future (or past) of an equator of $\mathbb{S}_1^{n+1}$, then $x(M^n)$ is a geodesic sphere of $\mathbb{H}^{n+1}$.*

*Proof.* First we claim that $H_j > 0$ everywhere on $M^n$, for all $j = 0, \ldots, s+1$. In fact, since $M^n$ is closed in $\mathbb{H}^{n+1}$, we may assume that the orientation $N$ of $M^n$ is taken in such a way that the principal curvatures of $x$ are positive at some point $p_0 \in M^n$. Moreover, it follows from the strong $(r, s, a, b)$-stability of $x$ that the ratio $b_s H_{s+1}/\{a\, b_r H_{r+1} - b\}$ is constant. Let

$$\beta := \frac{b_s H_{s+1}}{a\, b_r H_{r+1} - b}(p_0) \equiv \frac{b_s H_{s+1}}{a\, b_r H_{r+1} - b}. \tag{6.17}$$

If $a\, b_r H_{r+1} - b < 0$, then $\beta < 0$; so, by (6.17), $H_{s+1} = b_s^{-1}\beta\{a\, b_r H_{r+1} - b\} > 0$ on $M^n$. On the other hand, if $a\, b_r H_{r+1} - b > 0$, we have $\beta > 0$; so, again by (6.17), $H_{s+1} = b_s^{-1}\beta\{a\, b_r H_{r+1} - b\} > 0$ on $M^n$. In any scenario, we have that $H_{s+1} > 0$ on $M^n$. Finally, our assertion follows directly from Lemma 4.0.2.

Now, without loss of generality, we may assume that the Gauss image $N(M^n)$ of the hypersurface $M^n$ is contained in the chronological future of the equator of $\mathbb{S}_1^{n+1}$ determined by a unit timelike vector $\mathbf{a} \in \mathbb{L}^{n+2}$. For such a vector $\mathbf{a}$, let us also consider the warped product given in (6.8), which models the hyperbolic space $\mathbb{H}^{n+1}$ (minus a point) as being $(0, +\infty) \times_{\sinh t} \mathbb{S}^n$.

In this setting, we define the *normal angle* $\theta$ of $x$ as being the smooth function $\theta : M^n \to [0, \pi]$ given by

$$\cos\theta(p) = -\langle \Phi_* N, \partial_t \rangle_{(\Phi\circ x)(p)}, \quad p \in M^n, \tag{6.18}$$

where $\Phi$ stands for an isometry between the hyperquadric and the warped product models of $\mathbb{H}^{n+1}$. From (6.18) and (6.10), for any $p \in M^n$ we have that

$$\begin{aligned} \cos\theta(p) &= -\left\langle \Phi_* N((\Phi\circ x)(p)), \frac{W((\Phi\circ x)(p))}{|W((\Phi\circ x)(p))|} \right\rangle \\ &= -\frac{1}{|\Phi_*^{-1} W(x(p))|} \left\langle N(x(p)), \Phi_*^{-1} W(x(p)) \right\rangle. \end{aligned} \tag{6.19}$$

Since $\Phi_*^{-1} W = V$, where $V$ is the closed conformal vector field given by (6.14), from (6.19) we get

$$\begin{aligned} \cos\theta(p) &= -\frac{1}{|V(x(p))|} \Big\langle N(x(p)), \mathbf{a} + \langle \mathbf{a}, x(p)\rangle x(p) \Big\rangle \\ &= -\frac{1}{|V(x(p))|} \Big\langle N(x(p)), \mathbf{a} \Big\rangle. \end{aligned} \tag{6.20}$$

Hence, since we are supposing that the Gauss image $N(M^n)$ is contained in the chronological future of $\mathbb{S}_1^{n+1}$ determined by $\mathbf{a}$, from (6.15) and (6.20) we conclude that

$$0 < \cos\theta \leq 1. \tag{6.21}$$

On the other hand, since $x$ is strongly $(r,k,a,b)$-stable, from Definition 6.1.2 and formula (6.3) we obtain

$$\begin{aligned} 0 \ \leq \ \mathfrak{J}''_{r,s,a,b}(0)(u) = -(s+1)\int_M \big\{ \mathfrak{L}_{r,s,a,b}(u) + \big(-\mathrm{trace}(P_s) + \mathrm{trace}(A^2\circ P_s) \\ -\Lambda_{r,s,a,b}\left(-\mathrm{trace}(P_r) + \mathrm{trace}(A^2\circ P_r)\right)\big) u\big\}\, u dM \end{aligned}$$

for all $u \in C^\infty(M)$, where $\mathfrak{L}_{r,s,a,b}$ is the second order differential operator defined in (6.4), and $\Lambda_{r,s,a,b}$ is defined in (6.5). In particular, taking

$$u = \eta\circ\Phi^{-1} = \langle W, \Phi_* N\rangle = \langle \sinh t\,\partial_t, \Phi_* N\rangle = -\sinh t\,\cos\theta$$

and for simplicity of notation writing $N = \Phi_* N$ and $H_j = H_j\circ\Phi^{-1}$ for $j = r, r+1, s, r+1$, from Lemma 6.2.2 we obtain

$$\begin{aligned} 0 &\leq (s+1)\int_{\Phi(M^n)} \{(b_s H_s - \Lambda_{r,s,a,b}\, b_r H_r)\sinh t\,\cos\theta \\ &\qquad - (b_s H_{s+1} - \Lambda_{r,s,a,b}\, b_r H_{r+1})\cosh t\}\sinh t\,\cos\theta\, d\Phi(M) \\ &\leq (s+1)\int_{\Phi(M^n)} \{(b_s H_s - \Lambda_{r,s,a,b}\, b_r H_r)\cos\theta \\ &\qquad - (b_s H_{s+1} - \Lambda_{r,s,a,b}\, b_r H_{r+1})\}\cosh t\,\sinh t\,\cos\theta\, d\Phi(M), \end{aligned} \tag{6.22}$$

where $b_j = (j+1)\binom{n}{j+1} = (n-j)\binom{n}{j}$ for $j = r, s$. But, since $\Lambda_{r,s,a,b} \leq 0$ by hypothesis, from (6.16) we also have that

$$b_r\left(H_{r+1} - H_r\right)\Lambda_{r,s,a,b} - b_s\left(H_{s+1} - H_s\right) \leq 0.$$

Equivalently,

$$b_s H_{s+1} - \Lambda_{r,s,a,b}\, b_r H_{r+1} \geq b_s H_s - \Lambda_{r,s,a,b}\, b_r H_r. \tag{6.23}$$

Substituting (6.23) into (6.22), we get

$$0 \ \leq \ (r+1)\int_{\Phi(\Sigma^n)} (b_r H_r - \Lambda_{r,k,a,b}\, b_k H_k)(\cos\theta - 1)\cosh\tau\sinh\tau\cos\theta\, d\Phi(\Sigma) \ \leq \ 0,$$

where we used (6.21) and the facts that $\Lambda_{r,s,a,b} \leq 0$, $H_s > 0$, $H_r > 0$ and $t > 0$. Hence, $\cos\theta \equiv 1$, and, consequently, there exists $t_0 \in (0, +\infty)$ such that

$$(\Phi\circ x)(M^n) = \{t_0\}\times\mathbb{S}^n.$$

□

**Remark 6.2.5.** *We would like to point out that, taking into account that the higher order mean curvatures of each slice $M^n_{t_0} = \{t_0\}\times\mathbb{S}^n$ verify $H_{s+1} > H_s \geq H_{r+1} > H_r > 1$ (as can be observed from (6.9)) for any entire numbers $r$ and $s$ such that $0 \leq r < s \leq n-2$, our restriction on the values of the higher order mean curvatures $H_{r+1}$ and $H_{s+1}$ in Theorem 6.2.4 constitutes a* *reasonable hypothesis in the sense that, in view of Proposition 6.2.1 and Remark 6.2.3, it is natural if we want to detect geodesic spheres of* $\mathbb{H}^{n+1}$.

We recall that a hypersurface $x : M^n \to \mathbb{H}^{n+1}$ is said to a *linear Weingarten hypersurface* when the mean curvature $H$ and normalized scalar curvature $S$ satisfy the relation

$$\delta_0 H + \delta_1 S = \delta_2,$$

for some constants $\delta_0, \delta_1, \delta_2 \in \mathbb{R}$. Then, making $r = 1$ and $k = 0$ in Theorem 6.2.4, from (4.4) we get the following

**Corollary 6.2.6.** *Let $a$ and $b$ be real numbers with $b \neq 0$, and let $x : M^n \to \mathbb{H}^{n+1}$ be a strongly $(1, 0, a, b)$-stable closed linear Weingarten hypersurface such that*

$$n(n-1)(S+1) - naH\delta = -b\delta,$$

*where $\delta \in \mathbb{R} \setminus \{0\}$. Suppose that $\Lambda_{1,0,a,b}$ is nonpositive and that $1 \leq H \leq S + 1$. If the Gauss image of $x$ is contained in the chronological future (or past) of an equator of $\mathbb{S}_1^{n+1}$, then $M^n$ is a sphere, and $x$ is its inclusion as a geodesic sphere of* $\mathbb{H}^{n+1}$.

# Chapter 7

# Hypersurfaces in De Sitter space

In this chapter, we define the notion of strong $(r, s, a, b)$-stability concerning compact spacelike hypersurfaces immersed in the De Sitter space $\mathbb{S}_1^{n+1}$. We study the variational problem of maximizing a certain Jacobi functional given by a linear combination of area and volume. Under a suitable constraint on a constant that appears in the computation of the second variation of this functional, we prove that a compact spacelike hypersurface $M^n$ contained in a chronological future (or past) of $\mathbb{S}_1^{n+1}$, with positive $(s+1)$th curvature and such that $H \leq 1$, must be a totally umbilical round sphere. The results presented here are part of our paper [103].

## 7.1 Preliminaries

Let $\mathbb{L}^{n+2}$ denote the $(n+2)$-dimensional Lorentz-Minkowski space ($n \geq 2$), that is, the real vector space $\mathbb{R}^{n+2}$ endowed with the Lorentzian metric

$$\langle v, w \rangle = \sum_{i=1}^{n+1} v_i w_i - v_{n+2} w_{n+2},$$

for all $v, w \in \mathbb{R}^{n+2}$. We define the $(n+1)$-dimensional De Sitter space $\mathbb{S}_1^{n+1}$ as the following hyperquadric of $\mathbb{L}^{n+2}$,

$$\mathbb{S}_1^{n+1} = \{p \in \mathbb{L}^{n+2} : \langle p, p \rangle = 1\}.$$

From the previous definition, it is easy to show that the metric induced from $\langle\, , \,\rangle$ turns $\mathbb{S}_1^{n+1}$ into a Lorentzian manifold with constant sectional curvatures equal to 1.

Choosing a unitary timelike vector $v \in \mathbb{L}^{n+2}$, we have that

$$V(p) = v - \langle p, v \rangle p, \quad p \in \mathbb{S}_1^{n+1},$$

is a conformal Killing closed timelike vector field, namely, $\overline{\nabla}_Y V = \psi Y$ for any smooth vector field $Y$ and for some $\psi \in C^\infty(\mathbb{S}_1^{n+1})$, where $\overline{\nabla}$ stands for the Levi-Civita connection of $\mathbb{S}_1^{n+1}$. The smooth function $\psi$ is called the *conformal factor* associated with $V$. Such a vector field $V$

foliates the De Sitter space by means of totally umbilical round spheres,

$$\mathbb{M}^n_\delta = \{p \in \mathbb{S}^{n+1}_1 : \langle p, v\rangle = \delta\}, \quad \delta \in \mathbb{R}.$$

The level set given by

$$\mathbb{M}^n_0 = \{p \in \mathbb{S}^{n+1}_1 : \langle p, v\rangle = 0\}$$

defines a round sphere of radius 1 which is a totally geodesic hypersurface in $\mathbb{S}^{n+1}_1$. We will refer to that sphere as the *equator* of $\mathbb{S}^{n+1}_1$ determined by $v$. This equator decomposes $\mathbb{S}^{n+1}_1$ into two connected components, the *chronological future*, given by

$$\{p \in \mathbb{S}^{n+1}_1 : \langle p, v\rangle < 0\},$$

and the *chronological past*, given by

$$\{p \in \mathbb{S}^{n+1}_1 : \langle p, v\rangle > 0\}.$$

Consider a smooth immersion $x : M^n \to \mathbb{S}^{n+1}_1$ of an $n$-dimensional connected manifold $M^n$ into $\mathbb{S}^{n+1}_1$, and assume that the induced metric via $x$ is a Riemannian metric on $M^n$, that is, $M^n$ is a spacelike hypersurface. In that case, since $\mathbb{S}^{n+1}_1$ is time-orientable, $M^n$ is orientable, and one can choose a globally defined unitary normal timelike vector field $N$ on $M^n$ having the same time orientation as $\mathbb{S}^{n+1}_1$. Such an $N$ is named the *future-pointing Gauss map* of $x$.

Let $A : \mathfrak{X}(M) \to \mathfrak{X}(M)$ be the shape operator (or Weingarten endomorphism) of $x : M^n \to \mathbb{S}^{n+1}_1$ with respect to its future-pointing Gauss map $N$, which is defined by $A(X) := -\overline{\nabla}_X N$ for $X \in \mathfrak{X}(M)$, where $\mathfrak{X}(M)$ stands for the ring of the smooth tangent vector fields on $M^n$. It is well known that $A$ defines a self-adjoint linear operator on each tangent space $T_pM, p \in M^n$, and its eigenvalues $\kappa_1(p), \dots, \kappa_n(p)$ are the principal curvatures of the hypersurface. Associated to the shape operator there are $n$ algebraic invariants given by

$$S_k(p) = \sigma_k(\kappa_1(p), \dots, \kappa_n(p)), \quad 1 \leq k \leq n,$$

where $\sigma_k : \mathbb{R}^n \to \mathbb{R}$ is the elementary symmetric function in $\mathbb{R}^n$ given by

$$\sigma_k(x_1, \dots, x_n) = \sum_{1\leq i_1 < \cdots < i_k \leq n} x_{i_1} \cdots x_{i_k}.$$

Observe that the characteristic polynomial of $A$ can be written in terms of the $S_k$'s as

$$\det(tI - A) = \sum_{k=0}^{n} (-1)^k S_k t^{n-k}, \tag{7.1}$$

where $S_0 = 1$ by definition. The $k$th-mean curvature $H_k$ of the hypersurface is then defined by

$$\binom{n}{k} H_k = (-1)^k S_k = \sigma_k(-\kappa_1, \dots, -\kappa_n),$$

for every $0 \leq k \leq n$.

The classical Newton transformations $P_k : \mathfrak{X}(M) \to \mathfrak{X}(M)$ are defined inductively from $A$ by

$$P_0 = I \quad \text{and} \quad P_k = \binom{n}{k} H_k I + A \circ P_{k-1},$$

for every $k = 1, \ldots, n$, where $I$ denotes the identity in $\mathfrak{X}(M)$. Note that by the Cayley-Hamilton theorem, from (7.1) we get $P_n = 0$. A trivial induction shows that

$$P_r = (-1)^r (S_r I - S_{r-1} A + S_{r-2} A^2 + \cdots + (-1)^r A^r).$$

Being a polynomial in $A$, $P_r$ is also self-adjoint and commutes with $A$, for every $r$. Moreover, $A(p)$ and $P_k(p)$ can be simultaneously diagonalized: if $\{e_1, \ldots, e_n\}$ are the eigenvectors of $A(p)$ corresponding to the eigenvalues $\kappa_1(p), \ldots, \kappa_n(p)$, respectively, then they are also the eigenvectors of $P_k(p)$ with corresponding eigenvalues given by

$$\mu_{i,k}(p) = (-1)^k \frac{\partial \sigma_{k+1}}{\partial x_i}(\kappa_1(p), \ldots, \kappa_n(p)) = (-1)^k \sum_{\substack{i_1 < \cdots < i_k \\ i_j \neq i}} \kappa_{i_1}(p) \cdots \kappa_{i_k}(p), \tag{7.2}$$

for every $1 \leq i \leq n$. From here, it can be easily seen that (see [27, Lemma 2.1])

$$\begin{cases} \operatorname{trace}(P_k) &= (-1)^k (n-k) S_k = b_k H_k; \\ \operatorname{trace}(A \circ P_k) &= (-1)^k (k+1) S_{k+1} = b_k H_{k+1}; \\ \operatorname{trace}(A^2 \circ P_k) &= (-1)^k (S_1 S_{k+1} - (k+2) S_{k+2}) \\ & = n \dfrac{b_k}{k+1} H_1 H_{k+1} - b_{k+1} H_{k+2}, \end{cases} \tag{7.3}$$

where $b_k = (k+1)\binom{n}{k+1} = (n-k)\binom{n}{k}$ and $H_k = 0$ if $k > n$. Moreover, denoting by $A_i$ the restriction of $A$ to $\langle e_i \rangle^\top \subset T_p M$, it is easy to see that

$$\det(tI - A_i) = \sum_{k=0}^{n-1} (-1)^k S_k(A_i) t^{n-1-k},$$

where

$$S_k(A_i) = \sum_{\substack{1 \leq j_1 < \ldots < j_k \leq n \\ j_1, \ldots, j_k \neq i}} \lambda_{j_1} \cdots \lambda_{j_k}.$$

With the previous notations, it is also immediate to check that

$$P_k e_i = (-1)^k S_k(A_i) e_i. \tag{7.4}$$

Let $\nabla$ stand for the Levi-Civita connection of $M^n$. Associated to each Newton transformation $P_k$, we consider the second order linear differential operator $L_k : C^\infty(M) \to C^\infty(M)$ given by

$$L_k(f) = \operatorname{trace}(P_k \circ \nabla^2 f).$$

Here $\nabla^2 f : \mathfrak{X}(M) \to \mathfrak{X}(M)$ denotes the self-adjoint linear operator metrically equivalent to the Hessian of $f$, and it is given by

$$\langle \nabla^2 f(X), Y\rangle = \langle \nabla_X(\nabla f), Y\rangle, \quad X, Y \in \mathfrak{X}(M).$$

Observe that

$$\begin{aligned} L_k(f) &= \operatorname{trace}(P_k \circ \nabla^2 f) = \sum_{i=1}^{n} \langle P_k(\nabla_{E_i}\nabla f), E_i\rangle \\ &= \sum_{i=1}^{n} \langle \nabla_{E_i}\nabla f, P_k(E_i)\rangle = \sum_{i=1}^{n} \langle \nabla_{P_k(E_i)}\nabla f, E_i\rangle \\ &= \operatorname{trace}(\nabla^2 f \circ P_k), \end{aligned} \tag{7.5}$$

where $\{E_1, \ldots, E_n\}$ is a local orthonormal frame on $M^n$.

**Remark 7.1.1.** *The equality between the terms in the second line of* (7.5) *can be verified as follows. Fix $p \in M$. Let $\{E_1, \ldots, E_n\} \subset \mathfrak{X}(M)$ be a local orthonormal frame in a neighborhood of $p$, such that at $p$ this frame is geodesic and diagonalizes the shape operator $A$ (and, thus, every $P_k$). We have to prove that*

$$\sum_{i=1}^{n} \langle \nabla_{E_i}\nabla f, P_k(E_i)\rangle = \sum_{i=1}^{n} \langle \nabla_{P_k(E_i)}\nabla f, E_i\rangle. \tag{7.6}$$

*We claim that $\langle \nabla_{E_i}\nabla f, P_k(E_i)\rangle = \langle \nabla_{P_k(E_i)}\nabla f, E_i\rangle$ for every $i = 1, \ldots, n$.*

*Note that*

$$\begin{cases} E_i\langle \nabla f, P_k(E_i)\rangle &= \langle \nabla_{E_i}\nabla f, P_k(E_i)\rangle + \langle \nabla f, \nabla_{E_i}(P_k(E_i))\rangle, \\ P_k(E_i)\langle \nabla f, E_i\rangle &= \langle \nabla_{P_k(E_i)}\nabla f, E_i\rangle + \langle \nabla f, \nabla_{P_k(E_i)}E_i\rangle. \end{cases}$$

*Writing $P_k(E_i) = \mu_{i,k}E_i$ at $p$, where $\mu_{i,k}$ is the eigenvalue given in* (7.2)*, we get*

$$\nabla_{P_k(E_i)}E_i = \nabla_{\mu_{i,k}E_i}E_i = \mu_{i,k}\nabla_{E_i}E_i = 0,$$

*and*

$$\nabla_{E_i}(P_k(E_i)) = \nabla_{E_i}(\mu_{i,k}E_i) = E_i(\mu_{i,k})E_i + \mu_{i,k}\underbrace{\nabla_{E_i}E_i}_{=0} = E_i(\mu_{i,k})E_i,$$

*so*

$$\begin{cases} E_i\langle \nabla f, P_k(E_i)\rangle &= \langle \nabla_{E_i}\nabla f, P_k(E_i)\rangle + E_i(\mu_{i,k})\langle \nabla f, E_i\rangle, \\ P_k(E_i)\langle \nabla f, E_i\rangle &= \langle \nabla_{P_k(E_i)}\nabla f, E_i\rangle. \end{cases} \tag{7.7}$$

*On the other hand, also at $p$ we have*

$$\begin{aligned}
P_k(E_i)\langle\nabla f, E_i\rangle &= \mu_{i,k}E_i\langle\nabla f, E_i\rangle\\
&= \mu_{i,k}(\langle\nabla_{E_i}\nabla f, E_i\rangle + \langle\nabla f, \underbrace{\nabla_{E_i}E_i}_{=0}\rangle)\\
&= \mu_{i,k}\langle\nabla_{E_i}\nabla f, E_i\rangle;
\end{aligned}$$

*and*

$$\begin{aligned}
E_i\langle\nabla f, P_k(E_i)\rangle &= E_i(\mu_{i,k}\langle\nabla f, E_i\rangle)\\
&= E_i(\mu_{i,k})\langle\nabla f, E_i\rangle + \mu_{i,k}E_i\langle\nabla f, E_i\rangle\\
&= E_i(\mu_{i,k})\langle\nabla f, E_i\rangle + \mu_{i,k}(\langle\nabla_{E_i}\nabla f, E_i\rangle + \langle\nabla f, \underbrace{\nabla_{E_i}E_i}_{=0}\rangle)\\
&= E_i(\mu_{i,k})\langle\nabla f, E_i\rangle + \mu_{i,k}\langle\nabla_{E_i}\nabla f, E_i\rangle\\
&= E_i(\mu_{i,k})\langle\nabla f, E_i\rangle + P_k(E_i)\langle\nabla f, E_i\rangle.
\end{aligned}$$

*It follows that*

$$E_i\langle\nabla f, P_k(E_i)\rangle - E_i(\mu_{i,k})\langle\nabla f, E_i\rangle - P_k(E_i)\langle\nabla f, E_i\rangle = 0. \tag{7.8}$$

*Subtracting the last equality in* (7.7) *from the first one and using* (7.8)*, we get*

$$\langle\nabla_{E_i}\nabla f, P_k(E_i)\rangle - \langle\nabla_{P_k(E_i)}\nabla f, E_i\rangle = 0,$$

*as claimed.*

For $X, Y \in \mathfrak{X}(M)$, define

$$(\nabla_X P_k)(Y) := \nabla_X(P_k(Y)) - P_k(\nabla_X Y).$$

Fix $X \in \mathfrak{X}(M)$. Then it can be easily verified that the operator $\nabla_X P_k$ is self-adjoint.

Now,

$$\begin{aligned}
\mathrm{div}(P_k(\nabla f)) &= \sum_{i=1}^n \langle\nabla_{E_i}(P_k(\nabla f)), E_i\rangle \\
&= \sum_{i=1}^n \langle(\nabla_{E_i}P_k)(\nabla f) + P_k(\nabla_{E_i}\nabla f), E_i\rangle\\
&= \sum_{i=1}^n \langle(\nabla_{E_i}P_k)(\nabla f), E_i\rangle + \sum_{i=1}^n \langle P_k(\nabla_{E_i}\nabla f), E_i\rangle,
\end{aligned} \tag{7.9}$$

where $\{E_1, \ldots, E_n\} \subset \mathfrak{X}(M)$ is a local orthonormal frame, and, in the second equality, we used that $\nabla_{E_i}(P_k(\nabla f)) = (\nabla_{E_i}P_k)(\nabla f) - P_k(\nabla_{E_i}\nabla f)$.

For the first summand in the last line of (7.9), note that for each $i$ we get

$$\langle(\nabla_{E_i}P_k)(\nabla f), E_i\rangle = \langle(\nabla_{E_i}P_k)(E_i), \nabla f\rangle,$$

since $\nabla_{E_i} P_k$ is self-adjoint. So, that summand becomes

$$\begin{aligned}\sum_{i=1}^{n} \langle (\nabla_{E_i} P_k)(\nabla f), E_i \rangle &= \sum_{i=1}^{n} \langle (\nabla_{E_i} P_k)(E_i), \nabla f \rangle \\ &= \langle \sum_{i=1}^{n} (\nabla_{E_i} P_k)(E_i), \nabla f \rangle.\end{aligned}$$

Summing up, we also have that

$$\operatorname{div}(P_k(\nabla f)) = \langle \operatorname{div} P_k, \nabla f \rangle + L_k(f),$$

where

$$\operatorname{div} P_k := \operatorname{trace}(\nabla\, P_k) = \sum_{i=1}^{n} (\nabla_{E_i} P_k)(E_i).$$

In particular, if $P_k$ is divergence-free, or, in another words, $\operatorname{div} P_k = 0$, then $L_k(f) = \operatorname{div}(P_k(\nabla f))$ for every $f \in C^{\infty}(M)$. This happens trivially when $k = 0$ (indeed, $L_0 = \Delta$ is the Laplacian operator). On the other hand, by virtue of [11, Corollary 3.2], this also happens for every $k = 1, \ldots, n-1$ since $\mathbb{S}_1^{n+1}$ has constant sectional curvature. Then, assuming that the boundary of $M^n$ is empty, it follows from the divergence theorem that if $f \in C_c^{\infty}(M)$ we get

$$\int_M L_k(f) dM = 0.$$

Moreover, by noting the $\operatorname{div}(f P_k(\nabla f)) = f L_k(f) + \langle P_k(\nabla f), \nabla f \rangle$, we can once again apply the divergence theorem to get

$$\int_M f L_k(f) dM = -\int_M \langle P_k(\nabla f), \nabla f \rangle dM.$$

## 7.2 Description of the variational problem

Let $x : M^n \to \mathbb{S}_1^{n+1}$ be an isometric immersion of a compact spacelike hypersurface $M^n$ with boundary $\partial M$ (possibly empty). A *variation* of $x$ is a smooth map $X : M^n \times (-\epsilon, \epsilon) \to \mathbb{S}_1^{n+1}$ satisfying the following conditions:

(1) For each $t \in (-\epsilon, \epsilon)$, the map $X_t : M^n \to \mathbb{S}_1^{n+1}$ given by $X_t(p) = X(p,t)$ is a spacelike immersion such that $X_0 = x$.

(2) $X_t\big|_{\partial M} = x\big|_{\partial M}\, \forall t \in (-\epsilon, \epsilon)$.

In what follows, we let $dM_t$ denote the volume element of the metric induced on $M^n$ by $X_t$, and $N_t$ denote the future-pointing Gauss map of $X_t$. The *variational field* associated to the variation $X$ is the vector field $\frac{\partial X}{\partial t}\big|_{t=0}$.

Letting $f = -\left\langle \frac{\partial X}{\partial t}, N_t \right\rangle$, we get

$$\frac{\partial X}{\partial t} = f N_t + \left( \frac{\partial X}{\partial t} \right)^\top,$$

where $(\cdot)^\top$ stands for tangential components.

The *balance of volume* of $X$ is the functional

$$\begin{aligned} \mathfrak{V} : (-\epsilon, \epsilon) &\to \mathbb{R} \\ t &\mapsto \mathfrak{V}(t) = \int_{M \times [0,t]} X^*(dV), \end{aligned}$$

where $dV$ is the volume element of $\mathbb{S}^{n+1}_1$, and we say that $X$ is *volume-preserving* if $\mathfrak{V}(t) = \mathfrak{V}(0) = 0$, for all $t \in (-\epsilon, \epsilon)$.

The following result is well-known and can be found, for instance, in [107].

**Lemma 7.2.1.** *Let $x : M^n \to \mathbb{S}^{n+1}_1$ a compact spacelike hypersurface. If $X : M^n \times (-\epsilon, \epsilon) \to \mathbb{S}^{n+1}_1$ is a variation of $x$, then*

$$\frac{\partial \mathfrak{V}}{\partial t} = \int_M f \, dM_t.$$

*In particular, $X$ is volume-preserving if and only if $\int_M f \, dM_t = 0$, $\forall t \in (-\epsilon, \epsilon)$.*

We define the *$k$-area functional* $\mathfrak{A}_k : (-\epsilon, \epsilon) \to \mathbb{R}$ associated to the variation $X$ by

$$\mathfrak{A}_k(t) = \int_M F_k(S_1, \dots, S_k) dM_t,$$

where $S_k = S_k(t)$ and $F_k$ is recursively defined by setting $F_0 = 1, F_1 = -S_1$ and for $2 \leq k \leq n$,

$$F_k = (-1)^k S_k + \frac{n-k+1}{k-1} F_{k-2}.$$

The following result follows from [34, Lemma 2.2].

**Lemma 7.2.2.** *Let $x : M^n \to \mathbb{S}^{n+1}_1$ a compact spacelike hypersurface. If $X : M^n \times (-\epsilon, \epsilon) \to \mathbb{S}^{n+1}_1$ is a variation of $x$, then*

$$\frac{\partial H_{k+1}}{\partial t} = \frac{k+1}{b_k} \{ L_k(f) + \mathrm{trace}(P_k) f - \mathrm{trace}(A^2 \circ P_k) f \} + \left\langle \left( \frac{\partial X}{\partial t} \right)^\top, \mathrm{grad}\, H_{k+1} \right\rangle, \quad (7.10)$$

*where $b_k = (k+1)\binom{n}{k+1}$.*

The previous lemma allows us to compute the first variation of the $r$-area functional (see [34, Proposition 2.3]).

**Lemma 7.2.3.** *Let $x : M^n \to \mathbb{S}^{n+1}_1$ a compact spacelike hypersurface. If $X : M^n \times (-\epsilon, \epsilon) \to \mathbb{S}^{n+1}_1$ is a variation of $x$, then*

$$\mathfrak{A}'_k(t) = \int_M (b_k H_{k+1} + c_k) f \, dM_t, \quad (7.11)$$

*where* $c_0 = 0$, $c_1 = 1$ *and* $c_k = -\frac{(n-k+1)}{k-1}c_{k-2}$, *if* $k \geq 2$.

Let $r$ and $s$ be entire numbers satisfying the inequalities $0 \leq r < s \leq n-1$, and consider real numbers $a$ and $b$ (at least one nonzero). We define the following functional

$$\begin{aligned} \mathfrak{C}_{r,a,b} : (-\epsilon, \epsilon) &\rightarrow \mathbb{R} \\ t &\mapsto \mathfrak{C}_{r,a,b}(t) = a\mathfrak{A}_r(t) + b\mathfrak{V}(t), \end{aligned} \tag{7.12}$$

and we say that the variation $X : M^n \times (-\epsilon, \epsilon) \rightarrow \mathbb{S}_1^{n+1}$ *preserves* $\mathfrak{C}_{r,a,b}$ if $\mathfrak{C}_{r,a,b}(t) = \mathfrak{C}_{r,a,b}(0)$ for all $t \in (-\epsilon, \epsilon)$.

Now, we consider the variational problem of *maximizing the $s$-area functional* $\mathfrak{A}_s$ *for all variations which preserve the functional* $\mathfrak{C}_{r,a,b}$. The *Jacobi functional* associated with this problem is given by

$$\begin{aligned} \mathfrak{J}_{r,s,a,b} : \ (-\epsilon, \epsilon) &\rightarrow \qquad \mathbb{R} \\ t \quad &\mapsto \mathfrak{J}_{r,s,a,b}(t) \ = \ \mathfrak{A}_s(t) + \gamma\, \mathfrak{C}_{r,a,b}(t), \end{aligned}$$

where $\gamma$ is a constant to be chosen later. As an immediate consequence of Lemmas 7.2.1 and 7.2.3 we get

$$\mathfrak{J}'_{r,s,a,b}(t) = \int_M \{b_s H_{s+1} + c_s + \gamma\,(a(\,b_r H_{r+1} + c_r) + b)\}\, f dM_t.$$

In order to make a convenient choice of $\gamma$, supposing that the function $a(b_r H_{r+1} + c_r) + b$ never vanishes on $M^n$, let

$$\mathfrak{H} = \frac{1}{\mathfrak{A}_0(0)} \int_M \frac{b_s H_{s+1}(0) + c_s}{a(b_r H_{r+1}(0) + c_r) + b} dM$$

be an integral mean of the function $(b_s H_{s+1}(0) + c_s)/\,[a(b_r H_{r+1}(0) + c_r) + b]$ over $M^n$. When the function $(b_s H_{s+1}(0) + c_s)/\,[a(b_r H_{r+1}(0) + c_r) + b]$ is constant, one has

$$\mathfrak{H} = \frac{b_s H_{s+1}(0) + c_s}{a(b_r H_{r+1}(0) + c_r) + b} = \frac{b_s H_{s+1} + c_s}{a(b_r H_{r+1} + c_r) + b}, \tag{7.13}$$

and this notation will be used in what follows without further comments. Thus, if we choose $\gamma = -\mathfrak{H}$, we arrive at

$$\mathfrak{J}'_{r,s,a,b}(t) = \int_M \{b_s H_{s+1} + c_s - \mathfrak{H}\,(a(\,b_r H_{r+1} + c_r) + b)\}\, f dM_t. \tag{7.14}$$

Let $r$ and $s$ be two entire numbers satisfying the inequalities $0 \leq r < s \leq n-1$, and $a$ and $b$ two real numbers, at least one nonzero. Let also $x : M^n \rightarrow \mathbb{S}_1^{n+1}$ be a compact hypersurface such that the function $a(b_r H_{r+1} + c_r) + b$ never vanishes on $M^n$. Following the same ideas of [25, Proposition 2.7], we can establish from (7.14) the following

**Proposition 7.2.4.** *The following statements are equivalent.*

(*a*) $x : M^n \rightarrow \mathbb{S}_1^{n+1}$ *have higher order mean curvatures* $H_{r+1}$ *and* $H_{s+1}$ *verifying*

$$\frac{b_s H_{s+1} + c_s}{a(\,b_r H_{r+1} + c_r) + b} = \text{constant}, \tag{7.15}$$

*where* $b_j = (j+1)\binom{n}{j+1}$ *for* $j \in \{r, s\}$.

(*b*) *For all variations* $X : M^n \times (-\epsilon, \epsilon) \to \mathbb{S}_1^{n+1}$ *of* $x$ *that preserve the functional* $\mathfrak{C}_{r,a,b}$*, we have* $\mathfrak{A}'_s(0) = 0$.

(*c*) *For all variations* $X : M^n \times (-\epsilon, \epsilon) \to \mathbb{S}_1^{n+1}$ *of* $x$*, we have* $\mathfrak{J}'_{r,s,a,b}(0) = 0$.

**Remark 7.2.5.** *In the case of a surface, we have* $0 \le r < s \le n-1 = 2-1 = 1$*, that is,* $r = 0, s = 1$*. Then the Jacobi functional becomes*

$$J(t) = J_{0,1,a,b}(t) = A_1(t) + \text{Constant},$$

*since we are only considering variations that preserve the functional* $C_{0,a,b}$*. The stability condition* (7.15) *becomes*

$$\frac{b_1 H_2 + c_1}{a(b_0 H_1 + c_0) + b} = \frac{2H_2 + 1}{2aH_1 + b} = \text{constant},$$

*which is equivalent to*

$$H_2 - CH_1 = D,$$

*where* $C = a \times \text{constant}$ *and* $D = (b \times \text{constant} - 1)/2$*. In the general case of a hypersurface* $M^n$ *and* $r = 0, s = 1$*, the Jacobi functional is the same as before. The stability condition becomes*

$$(n-1)H_2 - CH_1 = D,$$

*where* $C = a \times \text{constant}$ *and* $D = (b \times \text{constant} - 1)/n$*. In this sense, the results presented here can be regarded as extensions of those in* [110].

Motivated by the ideas established in [97], we exchanged our study problem, and now we want *to detect compact hypersurfaces* $x : M^n \to \mathbb{S}_1^{n+1}$ *which maximize the Jacobi functional* $\mathfrak{J}_{r,s,a,b}$ *for all variations* $X : M^n \times (-\epsilon, \epsilon) \to \mathbb{S}_1^{n+1}$ *of* $x$. Proposition 7.2.4 shows that the critical points of $\mathfrak{J}_{r,s,a,b}$ are hypersurfaces $x : M^n \to \mathbb{S}_1^{n+1}$ whose higher order mean curvatures $H_{r+1}$ and $H_{s+1}$ verify

$$\frac{b_s H_{s+1} + c_s}{a(\, b_r H_{r+1} + c_r) + b} = \text{constant},$$

with $a(\, b_r H_{r+1} + c_r) + b \neq 0$ on $M^n$. This motivates us to define the following notion of stability.

**Definition 7.2.6.** *Let* $x : M^n \to \mathbb{S}_1^{n+1}$ *be a compact spacelike hypersurface whose higher order mean curvatures* $H_{r+1}$ *and* $H_{s+1}$ *satisfy*

$$\frac{b_s H_{s+1} + c_s}{a(b_r H_{r+1} + c_r) + b} = \text{constant},$$

*with* $a(b_r H_{r+1} + c_r) + b \neq 0$ *on* $M^n$*, where* $0 \le r < s \le n-1$ *are natural numbers and* $a$ *and* $b$ *are real numbers (at least one nonzero). We say that* $x$ *is* strongly $(r, s, a, b)$-stable *provided that* $\mathfrak{J}''_{r,s,a,b}(0)(f) \le 0$ *for all* $f \in C^\infty(M)$.

For such a hypersurface, we aim at computing the second variation of $\mathfrak{J}_{r,s,a,b}$. As a direct consequence of Lemmas 7.2.2 and 7.2.3, we get the following

**Proposition 7.2.7.** *Let $x : M^n \to \mathbb{S}_1^{n+1}$ a spacelike hypersurface whose higher order mean curvatures $H_{s+1}$ and $H_{r+1}$ satisfy*

$$\frac{b_s H_{s+1} + c_s}{a(b_r H_{r+1} + c_r) + b} = \text{constant},$$

*with $ab_r H_{r+1} + c_r - b \neq 0$ on $M^n$, where $0 \leq r < s \leq n-1$ are natural numbers, and $a$ and $b$ are real numbers (at least one nonzero). If $X$ is a variation of $x$, then $\mathfrak{J}''_{r,s,a,b}$ is given by*

$$\begin{aligned}\mathfrak{J}''_{r,s,a,b}(0)(f) = (s+1)\int_M \big\{\mathfrak{L}_{r,s,a,b}(f) + \big(&\text{trace}(P_s) - \text{trace}(A^2 \circ P_s)\\ &-\Lambda_{r,s,a,b}\left(\text{trace}(P_r) - \text{trace}(A^2 \circ P_r)\right)\big) f\big\} f dM, \end{aligned} \tag{7.16}$$

*for all $f \in C^\infty(M)$, where $\mathfrak{L}_{r,s,a,b}$ is the differential operator*

$$\begin{aligned} \mathfrak{L}_{r,s,a,b} : C^\infty(M) &\to C^\infty(M) \\ f &\mapsto \mathfrak{L}_{r,s,a,b}(f) = L_s(f) - \Lambda_{r,s,a,b}\, L_r(f), \end{aligned} \tag{7.17}$$

*and*

$$\Lambda_{r,s,a,b} = \frac{a(r+1)(b_s H_{s+1} + c_s)}{(s+1)(a(b_r H_{r+1} + c_r) + b)}. \tag{7.18}$$

*Proof.* From (7.14), (7.13), and (7.10), we obtain

$$\begin{aligned}
\mathfrak{J}''_{r,s,a,b}(0) &= \frac{\partial}{\partial t}\left(\int_M \{b_s H_{s+1} + c_s - \mathfrak{H}\,(a(b_r H_{r+1} + c_r) + b)\} f dM_t\right)\Big|_{t=0} \\
&= \int_M \left(b_s \frac{\partial H_{s+1}}{\partial t}\Big|_{t=0} - \mathfrak{H} a b_r \frac{\partial H_{r+1}}{\partial t}\Big|_{t=0}\right) f dM \\
&\quad + \int_M \underbrace{(b_s H_{s+1} + c_s - \mathfrak{H}\,(a(b_r H_{r+1} + c_r) + b))}_{=0} \frac{\partial}{\partial t}(f dM_t)\Big|_{t=0} \\
&= (s+1)\int_M \big\{(L_s - \Lambda_{r,s,a,b} L_r)(f) + \big(\text{trace}(P_s) - \text{trace}(A^2 P_s) \\
&\qquad -\Lambda_{r,s,a,b}\left(\text{trace}(P_r) - \text{trace}(A^2 P_r)\right)\big) f\big\} f dM \\
&\quad + \int_M \left\langle \left(\frac{\partial X}{\partial t}\right)^\top, \underbrace{\text{grad}\,(b_s H_{s+1} - \mathfrak{H}\,(a(b_r H_{r+1} + c_r) + b))}_{=0} \right\rangle f d\Sigma.
\end{aligned}$$

To finish the proof, we observe that the above expression depends only on the hypersurface $x : M^n \to \mathbb{S}_1^{n+1}$ and on the function $f \in C^\infty(M)$. □

## 7.3 Statement and proof of main result

We start this section by showing that the totally umbilical round spheres of $\mathbb{S}_1^{n+1}$ are strongly $(r, s, a, b)$-stable.

**Proposition 7.3.1.** *If $\Lambda_{r,s,a,b}$ is nonpositive, then the totally umbilical spheres of $\mathbb{S}_1^{n+1}$ are strongly $(r,s,a,b)$-stable.*

*Proof.* Let $x : M^n \to \mathbb{S}_1^{n+1}$ be a totally umbilical sphere. Since $M^n$ is totally umbilical, then its principal curvatures are all equal to a certain constant $\omega$. By choosing the orientation, we may assume that $\omega < 0$. Thus we have

$$S_k = \binom{n}{k}\omega^k, \qquad H_k = (-\omega)^k > 0, \qquad S_k(A_j) = \binom{n-1}{k}\omega^k,$$

and if $e_1, \ldots, e_n$ are the principal directions of $M^n$,

$$L_k(f) = \sum_{j=1}^{n} \left\langle \nabla^2 f(e_j), P_k(e_j) \right\rangle = \binom{n-1}{k}(-\omega)^k \Delta f,$$

for any $k \in \{0, \ldots, n\}$ and all $f \in C^\infty(M)$, where we use (7.4) in the last step. Next, for entire numbers $r$ and $s$ satisfying the inequality $0 \le r < s \le n-2$, and real numbers $a$ and $b$ (with at least one nonzero) such that $a\left[(r+1)\binom{n}{r+1}(-\omega)^{r+1} + c_r\right] + b \neq 0$, we have

$$\frac{b_s H_{s+1} + c_s}{a(\, b_r H_{r+1} + c_r) + b} = \frac{b_s(-\omega)^{s+1} + c_s}{a\left[b_r(-\omega)^{r+1} + c_r\right] + b} = \text{constant},$$

where $b_k = (k+1)\binom{n}{k+1}$ for $k \in \{r,s\}$, and $c_0 = 0$, $c_1 = 1$ and $c_k = -\frac{(n-k+1)}{k-1} c_{k-2}$, if $k \ge 2$. Then, from (7.16) and (7.3) we obtain

$$\begin{aligned}\mathfrak{J}''_{r,s,a,b}(0)(f) = (s+1)\int_M \Bigg\{ \Gamma_{r,s,a,b}\,\Delta f + \Bigg( b_s H_s - n\frac{b_s}{s+1} H_1 H_{s+1} \\ + b_{s+1} H_{s+2} \Bigg) f - \Lambda_{r,s,a,b} \Bigg( b_r H_r - n\frac{b_r}{r+1} H_1 H_{r+1} \\ + b_{r+1} H_{r+2} \Bigg) f \Bigg\} f dM,\end{aligned}$$

where

$$\Gamma_{r,s,a,b} = \binom{n-1}{s}(-\omega)^s - \Lambda_{r,s,a,b}\binom{n-1}{r}(-\omega)^r. \tag{7.19}$$

Thus,

$$
\begin{aligned}
\mathfrak{J}''_{r,s,a,b}(0)(f) &= (s+1)\int_M \Bigg\{\Gamma_{r,s,a,b}\,\Delta f + \Bigg(b_s(-\omega)^s \\
&\qquad -\frac{nb_s}{s+1}(-\omega)^{s+2} + b_{s+1}(-\omega)^{s+2}\Bigg)f \\
&\qquad -\Lambda_{r,s,a,b}\Bigg(b_r(-\omega)^r - \frac{nb_r}{r+1}(-\omega)^{r+2} \\
&\qquad +b_{r+1}(-\omega)^{r+2}\Bigg)f\Bigg\}f\,dM \\
&= (s+1)\int_M \Bigg\{\Gamma_{r,s,a,b}\,\Delta f \qquad\qquad (7.20)\\
&\qquad +\left((s+1)\binom{n}{s+1}(-\omega)^s - \Lambda_{r,s,a,b}(r+1)\binom{n}{r+1}(-\omega)^r\right)f \\
&\qquad +(-\omega)^{s+2}\left(-n\binom{n}{s+1} + (s+2)\binom{n}{s+2}\right)f \\
&\qquad -(-\omega)^{r+2}\Lambda_{r,s,a,b}\left(-n\binom{n}{r+1} + (r+2)\binom{n}{r+2}\right)f\Bigg\}f\,dM \\
&= (s+1)\int_M \{\Gamma_{r,s,a,b}\,\Delta f + n\,\Gamma_{r,s,a,b}\,f - n\Gamma_{r,s,a,b}\omega^2 f\}f\,dM \\
&= (s+1)\Gamma_{r,s,a,b}\int_M \left\{f\Delta f + n(1-\omega^2)f^2\right\}dM,
\end{aligned}
$$

for any $f \in C^\infty(M)$. Hence, if $\eta_1$ denotes the first nonzero eigenvalue of the Laplacian of $M^n$, and considering the assumption on the function $\Lambda_{r,k,a,b}$, from (7.19) and (7.20) we get

$$\mathfrak{J}''_{r,s,a,b}(0)(f) \;\le\; (s+1)\Gamma_{r,s,a,b}\int_M \left\{-\eta_1 + n(1-\omega^2)\right\}f^2 dM \;\le\; 0,$$

for any $f \in C^\infty(M)$, where the last inequality was obtained by observing that $M^n$ is isometric to an $n$-dimensional Euclidean sphere with constant sectional curvature equals to $\omega^2+1$; consequently, $\eta_1 = n(\omega^2+1)$. Therefore, we conclude that $M^n$ is strongly $(r,s,a,b)$-stable. □

**Remark 7.3.2.** *Note that the constant $\Lambda_{r,s,a,b}$ is supposed to be nonpositive. If we also require that $a>0$, then, when $r=0$ and $s=1$, our results in fact generalize Zhang's results* [110], *since the stability condition* (7.15) *in that case takes the form $(n-1)H_2 - CH_1 = D$, where $C>0$ and $D$ are constants (see Remark 7.2.5 above).*

The de Sitter space can be described as the spacetime

$$\mathbb{S}_1^{n+1} = -\mathbb{R}\times_{\cosh t}\mathbb{S}^n, \quad t\in\mathbb{R},$$

where $\mathbb{S}^n$ is the Euclidean unitary sphere. We observe that there are many possible choices for the unitary timelike vector $a \in \mathbb{L}^{n+2}$ and, hence, a lot of ways to describe $\mathbb{S}_1^{n+1}$ as such a warped product (see, for instance, [77]). We note that in this model the totally umbilical round spheres of $\mathbb{S}_1^{n+1}$ are the slices

$$M_{t_0}^n = \{t_0\} \times \mathbb{S}^n, \quad t_0 \in \mathbb{R}.$$

When oriented by the unitary normal vector field $\partial_t$, $M_{t_0}^n$ has constant $k$th mean curvature

$$H_k = (\tanh t_0)^k.$$

Moreover, the equator of $\mathbb{S}_1^{n+1}$ is the slice $M_0^n$; consequently, $(\cosh t)' = \sinh t$ vanishes only on this slice. Finally, the vector field

$$K = \cosh t \, \partial_t$$

is conformal Killing, timelike and closed in $\mathbb{S}_1^{n+1}$, with associated conformal factor $\psi = \sinh t$.

Toward the aim of proving the main result of this chapter, we shall need the following computational lemma.

**Lemma 7.3.3.** *Let $x : M^n \to \mathbb{S}_1^{n+1}$ be a compact spacelike hypersurface whose higher order mean curvatures $H_{r+1}$ and $H_{s+1}$ satisfy*

$$\frac{b_s H_{s+1} + c_s}{a(b_r H_{r+1} + c_r) + b} = \text{constant}, \tag{7.21}$$

*with $a(b_r H_{r+1} + c_r) + b \neq 0$ on $M^n$, where $0 \leq r < s \leq n-1$ are natural numbers, and $a$ and $b$ are real numbers (at least one nonzero). If $N$ is the future-pointing Gauss map of $x$, then*

$$\begin{aligned}
\mathfrak{L}_{r,s,a,b}(\langle N, K\rangle) \;=\;& \big(\text{trace}(A^2 \circ P_s) - \text{trace}(P_s)\big) \langle N, K\rangle \\
& -\Lambda_{r,s,a,b} \big(\text{trace}(A^2 \circ P_r) - \text{trace}(P_r)\big) \langle N, K\rangle \\
& + (b_s H_s - \Lambda_{r,s,a,b} b_r H_r) \langle N, K\rangle \\
& + (b_s H_{s+1} - \Lambda_{r,s,a,b} b_r H_{r+1}) \sinh t.
\end{aligned} \tag{7.22}$$

*Proof.* From [12, Lemma 8.1], we get

$$\begin{aligned}
L_k(\langle N, K\rangle) \;=\;& (\text{trace}(A^2 \circ P_k) - \text{trace}(P_k))\langle N, K\rangle - b_k H_k N(\sinh t) \\
& + b_k H_{k+1} \sinh t + \frac{b_k}{k+1} \langle \cosh t \, \partial_t, \text{grad}\, H_{k+1}\rangle,
\end{aligned}$$

for $k \in \{r, s\}$. From here and (7.17), we arrive at

$$\begin{aligned}
\mathfrak{L}_{r,s,a,b}(\langle N, K\rangle) \;=\;& \big(\text{trace}(A^2 \circ P_s) - \text{trace}(P_s)\big) \langle N, K\rangle \quad (7.23)\\
& -\Lambda_{r,s,a,b} \big(\text{trace}(A^2 \circ P_r) - \text{trace}(P_r)\big) \langle N, K\rangle \quad (7.24)\\
& + (b_s H_{s+1} - \Lambda_{r,s,a,b} b_r H_{r+1}) \sinh t \\
& (-b_s H_s + \Lambda_{r,s,a,b} b_r H_r)\, N(\sinh t) \\
& + \left\langle \cosh t \, \partial_t, \text{grad}\left(\frac{b_s}{s+1} H_{s+1} - \Lambda_{r,s,a,b} \frac{b_r}{r+1} H_{r+1}\right)\right\rangle
\end{aligned}$$

Now, in view of (7.21) and (7.18), a straightforward algebraic computation shows that

$$\operatorname{grad}\left(\frac{b_s}{s+1}H_{s+1} - \Lambda_{r,s,a,b}\frac{b_r}{r+1}H_{r+1}\right) = 0. \tag{7.25}$$

Finally, we observe that

$$\overline{\operatorname{grad}}\,\sinh t = -\langle\overline{\operatorname{grad}}\,\sinh t, \partial_t\rangle = -(\sinh t)'\partial_t = -\cosh t\partial_t,$$

so that

$$N(\sinh t) = \langle N, \overline{\operatorname{grad}}\,\sinh t\rangle = -\langle N, K\rangle. \tag{7.26}$$

Substituting (7.25) and (7.26) into (7.23), we obtain (7.22). □

Now, we are read to state and prove the main result of this chapter.

**Theorem 7.3.4.** *Let $x : M^n \to \mathbb{S}_1^{n+1}$ be a compact strongly $(r,s,a,b)$-stable spacelike hypersurface that is contained in a chronological future (or past) of $\mathbb{S}_1^{n+1}$. Suppose that $\Lambda_{r,s,a,b}$ is nonpositive. If $H_{s+1} > 0$ and $H_1 \leq 1$, then $x(M^n)$ is a totally umbilical round sphere.*

*Proof.* Without loss of generality, we can assume that the closed spacelike hypersurface $x : M^n \to \mathbb{S}_1^{n+1} \hookrightarrow \mathbb{L}^{n+2}$ is contained in the chronological future of the equator determined by an unitary timelike vector $v \in \mathbb{L}^{n+2}$. By hypothesis, we have

$$\begin{aligned}0 \geq \mathfrak{J}''_{r,s,a,b}(0)(f) = (s+1)\int_M \big\{\mathfrak{L}_{r,s,a,b}(f) + \big(\operatorname{trace}(P_s) - \operatorname{trace}(A^2 \circ P_s)\\ -\Lambda_{r,s,a,b}\left(\operatorname{trace}(P_r) - \operatorname{trace}(A^2 \circ P_r)\right)\big)\, f\big\}\, f d\Sigma,\end{aligned}$$

for all $f \in C^\infty(M)$. In particular, by letting $f = \langle\cosh t\,\partial_t, N\rangle$, from Lemma 7.3.3 we get

$$\begin{aligned}0 \quad \geq \quad \int_M \{&(b_s H_s - \Lambda_{r,s,a,b} b_r H_r)\cosh\theta\,\cosh t\\ &- (b_s H_{s+1} - \Lambda_{r,s,a,b} b_r H_{r+1})\sinh t\}\cosh\theta\,\cosh t\, dM,\end{aligned} \tag{7.27}$$

where $\theta$ is the hyperbolic angle between the future-pointing Gauss map $N$ of $x$ and $\partial_t$, that is, $\theta$ is the only nonnegative real number such that $\cosh\theta = -\langle N, \partial_t\rangle$.

Now, since $M^n$ is compact and contained in the chronological future of $\mathbb{S}_1^{n+1}$, we can reason as in the proof of [6, Theorem 7] in order to guarantee the existence of a elliptic point in $M^n$. Thus, since $H_{s+1} > 0$, from Lemmas 4.0.1 and 4.0.2 we conclude that the following holds.

(i) Each $H_k, k \in \{1, \ldots, s+1\}$, is positive.

(ii) $H_{k-1} \geq H_k^{(k-1)/k}$ and $H_1 \geq H_k^{1/k}, k \in \{1, \ldots, s+1\}$. If $k \geq 2$, then in the above inequalities, the equality holds only at umbilical points.

Therefore, taking into account that $H_1 \leq 1$ by assumption, we have

$$1 \geq H_1 \geq H_{k+1}^{1/(k+1)}, \quad k \in \{0, \ldots, s\};$$

hence $H_{k+1} \leq 1$ for all $k \in \{1, \ldots, s+1\}$. Consequently, we get

$$H_k \geq H_{k+1}^{k/(k+1)} \geq H_{k+1}, \quad k \in \{0, \ldots, s\}.$$

Moreover, we know $\sinh y \leq \cosh y$ and $\cosh y \geq 1$, for all $y \in \mathbb{R}$. Putting all these facts into (7.27), we arrive at

$$0 \geq \int_M (\cosh\theta - 1)(b_s H_s - \Lambda_{r,s,a,b} b_r H_r) \cosh^2 t \, \cosh\theta \, dM \geq 0.$$

We infer that $\cosh\theta \equiv 1$ on $M^n$. Consequently, there exists $t_0 \in \mathbb{R} \setminus \{0\}$ such that $x(M^n) = \{t_0\} \times \mathbb{S}^n$. $\square$

**Remark 7.3.5.** *Barbosa and Oliker* [24] *proved that a complete, connected and orientable hypersurface $x : M^n \to \mathbb{S}_1^{n+1}$, with constat mean curvature $H$, is stable (in the sense they defined) provided that either (i) $M^n$ is compact; (ii) $H^2 \geq 1$; or (iii) $H^2 < 4(n-1)/n^2$. It is worth noting that, when $a = 0$, our stability condition* (7.15) *becomes $H_k = \text{constant}$ for some $k \in \{2, \ldots, n\}$. Thus, our Theorem 7.3.4 can be seen as a kind of generalization of Theorem 4.7 of* [24].

**Remark 7.3.6.** *According to Theorem 4 of* [75], *a compact spacelike hypersurface of $\mathbb{S}_1^{n+1}$, with constant mean curvature, is umbilical and coincides, up to a rigid motion of $\mathbb{S}_1^{n+1}$, with an $n$-dimensional sphere described in Example 1 (c) of the same paper. The square of the mean curvature, $H^2$, of such spheres assumes all possible values in the interval $[0, 1)$. Taking these facts into account, we would like to point out that our assumption $H_1 \leq 1$ in Theorem 7.3.4 is a mild hypothesis, since we are afterall interested in detecting totally umbilical round spheres of $\mathbb{S}_1^{n+1}$.*

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