**Universidade Federal da Paraíba**
**Universidade Federal de Campina Grande**
**Programa Associado de Pós-Graduação em Matemática**
**Doutorado em Matemática**

# Existência de solução de energia mínima para uma classe de problemas fortemente indefinidos em $\mathbb{R}^N$

**por**

**Geilson Ferreira Germano**

**Campina Grande - PB**

**Fevereiro/2018**

# Existência de solução de energia mínima para uma classe de problemas fortemente indefinidos em $\mathbb{R}^N$


**por**

**Geilson Ferreira Germano** †

**sob orientação do**

**Prof. Dr. Claudianor Oliveira Alves**


Tese apresentada ao Corpo Docente do Programa Associado de Pós-Graduação em Matemática - UFPB/UFCG, como requisito parcial para obtenção do título de Doutor em Matemática.

**Campina Grande - PB**

**Fevereiro/2018**


†Este trabalho contou com apoio financeiro da CAPES

**Universidade Federal da Paraíba**

**Universidade Federal de Campina Grande**

**Programa Associado de Pós-Graduação em Matemática**

**Doutorado em Matemática**

Área de Concentração: Análise

Aprovada em: 21/02/18

---

**Prof. Dr. Marco Aurélio Soares Souto - UFCG**

---

**Prof. Dr. Marcelo Fernandes Furtado - UNB**

---

**Prof. Dr. Carlos Alberto Pereira dos Santos - UNB**

---

**Prof. Dr. Edcarlos Domingos da Silva - UFG**

---

**Prof. Dr. Claudianor Oliveira Alves - UFCG**

**Orientador**

Tese apresentada ao Corpo Docente do Programa Associado de Pós-Graduação em Matemática - UFPB/UFCG, como requisito parcial para obtenção do título de Doutor em Matemática.

**Fevereiro/2018**

# Resumo


Nesta tese estamos interessados na existência e concentração de soluções de energia mínima para a classe de problema

$$\begin{cases} -\Delta u + V(x)u = A(\epsilon x)f(u), & x \in \mathbb{R}^N \\ u \in H^1(\mathbb{R}^N). \end{cases}$$

Quando $\epsilon \approx 0^+$, supondo que $V$ é uma função contínua $\mathbb{Z}^N$-periódica, supondo que $0 \notin \sigma(-\Delta + V)$ e $f : \mathbb{R} \to \mathbb{R}$ é uma função contínua com crescimento subcrítico e crítico para $N \geq 2$. Aqui $A : \mathbb{R}^N \to \mathbb{R}$ é uma função contínua que verifica

$$0 < A_0 = \inf_{x \in \mathbb{R}^N} A(x) \leq \lim_{|x| \to +\infty} A(x) < \sup_{x \in \mathbb{R}^N} A(x).$$

Quando $A \equiv 1$ também mostramos a existência de soluções de energia mínima.



**Palavras-chave:** Equação de Schrödinger não linear (NLSE), métodos variacionais, equações elípticas, funcional fortemente indefinido, concentração de soluções.

# Abstract

In this thesis we are interested in the existence and concentration of ground state solutions for the following class of problem

$$\begin{cases} -\Delta u + V(x)u = A(\epsilon x)f(u), & x \in \mathbb{R}^N \\ u \in H^1(\mathbb{R}^N). \end{cases}$$

When $\epsilon \approx 0^+$, by supposing that $V$ is $\mathbb{Z}^N$-periodic continuous function, with $0 \notin \sigma(-\Delta + V)$ and $f : \mathbb{R} \to \mathbb{R}$ is a continuous function with subcritical or critical growth for $N \geq 2$. Here $A : \mathbb{R}^N \to \mathbb{R}$ is continuous function that verifies

$$0 < A_0 = \inf_{x \in \mathbb{R}^N} A(x) \leq \lim_{|x| \to +\infty} A(x) < \sup_{x \in \mathbb{R}^N} A(x).$$

When $A \equiv 1$ we have also shown the existence of ground state solution.

**Keywords:** Nonlinear Schrödinger Equation (NLSE), variational methods, elliptic equations, indefinite strongly functional, concentration of solutions.

# Agradecimentos

Gostaria de agradecer primeiramente ao Verbo Divino, que sustenta todo o universo com apenas um fragmento do seu infinito amor, que nos faz perceber a importância da vida nas alegrias e dificuldades, nos sucessos e insucessos, nos ganhos e nas perdas. Que ama com um amor além humano que atravessa circunstâncias, tempos e dimensões. Que Deus abençoe a tudo e a todos sempre.

Muitos amigos contribuíram enormemente no decorrer do doutorado, e por mais que tenham feito atitudes tidas como simples, foram atitudes que ajudaram a construir este caminho e acredito profundamente que merecem serem citados em suas mais diversas atitudes.

Gostaria de agradecer ao meu Orientador Claudianor Oliveira Alves que foi um orientador imensamente focado e dedicado na orientação e que buscou sempre estar fazendo o melhor por meu aprendizado. Agradeço ao professor Marco Aurélio Soares Souto que, apesar de não ser o orientador, sempre esteve de coração aberto para fornecer sugestões, conselhos e disponibilidade. Agradeço aos professores da banca que são os professores Marcelo Fernandes Furtado, Carlos Alberto Pereira dos Santos, Edcarlos Domingos da Silva que de bom grado aceitaram participar da banca de defesa de tese e compartilharam suas diversas sugestões.

Alguns grandes amigos estiveram muitos distantes fisicamente, por que moram longe, mas foram sempre pessoas que estiveram prontas para ajudar com toda sinceridade, com uma sinceridade que venceu toda e qualquer distância. São amigos muitíssimos queridos. São os amigos Laís Oliveira, Odacir Seghetto, Shirlei Tatsukawa, Sandra Monforte (A San).

Agradeço ao apoio proporcionado pela minha família, meus pais João Germano e Cicera Germano, meus irmãos Gerson Germano e Gessiane Germano, a minha avó Luiza Dias e ao meu sobrinho Guilherme Ribeiro.

Algumas pessoas me prestaram apoio e me fizeram ter conversas abençoadas, proveitosas e muito agradáveis, meus amigos Paulo Maia, Victor Rafael, Renato Bezerra e Valquíria Oliveira (Val).

Sou grato por alguns colegas bastante antigos e que me auxiliaram em diversos momentos durante o doutorado, Romildo Nascimento, Itailma da Rocha (Tatá), Ailton Rodrigues, José Fernando, Wenia Félix, Ronaldo César e sua família: Josefa Tomaz (Dona Deta), Sérgio Roberto, Alexssandro Duarte (Sandro), Micarla Oliveira.

O doutorado me proporcionou a oportunidade de encontrar amigos como Navilta Veras, Natan Lima, Rivaldo Dantas, Weiller Barboza. Algumas pessoas as quais conheci por meio do doutorado os considero como irmãos, como Luciano Barros, Sânzia Raline e Luan Max.

Agradeço aos amigos da área de geometria Jogli Gidel e Fábio Reis que me deram grande suporte, principalmente durante os exames de qualificação.

Há dois amigos que os conheço ainda por pouco tempo, porém tenho construído uma amizade muito abençoada. Me incentivaram e me ajudaram de maneira bastante sincera. São os amigos César Torres e Rebe Callejas.

Sou grato a todos os funcionários do departamento por ajudarem sempre como podem de muito bom grado, dentre eles Aninha, Renato, Davi e Andrezza.

Enfim, sou grato a todos que me incentivaram, me auxiliaram, me ajudaram de alguma maneira para o desenvolvimento deste trabalho de maneira direta ou indireta. Que Deus os encham de plenitude.

*"Os que conservam a equanimidade, já neste mundo se unem com Brama, porque Ele é imutável e eternamente o mesmo.*
*Não te deixes arrebatar, quando te acontece algo desagradável, nem percas o ânimo, quando tens má sorte. Levanta o teu pensamento à claridade limpa da esfera divina, imerge-te em Deus e n'Ele vive."*

*(Krishna, Bhagavad Gita)*

*"Mas a dúvida é o preço da pureza e é inútil ter certeza."*

*(Humberto Gessinger)*

# Dedicatória

Ao dom da Vida...

# Sumário

# Introdução

Desenvolvida por Erwin Schrödinger, a equação de Schrödinger descreve a evolução temporal de partículas massivas subatômicas em sua natureza ondulatória e não relativística. Isto significa que é uma interpretação matemática para o comportamento de partículas subatômicas. Por seus trabalhos em direção ao entendimento quântico Schrödinger, em 1933, ganha o prêmio Nobel da física. Desde então se tem explorado bastante suas equações para os cientistas entenderem as nuances do mundo quântico. Destacamos aqui o trabalho [50].

Nos últimos anos, vários artigos têm sido publicados utilizando a equação de Schrödinger. Muitos desses trabalhos têm abordado a equação de Schrödinger não linear independente do tempo com diversos tipos de função potencial e diversos tipos de não linearidade que são equações com o seguinte formato:

$$E\Psi(x) = \left(-\frac{\hbar^2}{2\mu}\nabla^2 + V(x)\right)\Psi(x) + f(\Psi(x))$$

No caso em que a não linearidade $f$ é uma função nula tal equação descreve exatamente a energia total do sistema como uma adição da energia cinética $\left(-\frac{\hbar^2}{2\mu}\nabla^2\right)$ com a energia potencial $V(x)$.

Floer e Weinstein [19] estudaram uma equação de Schrödinger de dimensão 1 da forma

$$-i\epsilon\frac{\partial\Psi}{\partial t} = -\frac{\epsilon^2}{2m}\Psi_{xx} + V(x)\Psi - \gamma|\Psi|^2\Psi, \quad x \in \mathbb{R} \tag{$FW$}$$

onde $\gamma, \epsilon > 0$, e encontraram uma solução no formato

$$\Psi(x,t) = exp(-iEt/\epsilon)v(x) \tag{$SW$}$$

denominada soluções de onda estacionária (em inglês, standing wave), onde $v : \mathbb{R} \to \mathbb{R}$ é uma função a ser encontrada. Note que para $\Psi$ ser uma solução para $(FW)$ uma condição necessária e suficiente é que

$$Ev(x) = -\frac{\epsilon^2}{2m}v''(x) + V(x)v(x) - \gamma|v(x)|^2v(x), \quad x \in \mathbb{R}$$

que é o formato da equação de Schrödinger independente do tempo. Quando a não linearidade for $\gamma|\Psi|^{p-1}\Psi$, com $p \in (1, 2^* - 1)$, a solução do tipo onda estacionária deve satisfazer

$$-\frac{\epsilon^2}{2m}v''(x) + (V(x) - E)v(x) = \gamma|v(x)|^{p-1}v(x), \quad x \in \mathbb{R}. \tag{$SWE$}$$

Motivados pelos estudos realizados em [19], Oh em [36], estudou a equação

$$-i\epsilon\frac{\partial\Psi}{\partial t} = -\frac{\epsilon^2}{2m}\Delta\Psi + V(x)\Psi - \gamma|\Psi|^{p-1}\Psi, \; x \in \mathbb{R}^N \tag{$OH$}$$

e obteve resultados similares a [19]. Após os estudos realizados por Oh [36] diversos trabalhos foram publicados com o intuito de encontrar soluções do tipo onda estacionária da equação $(OH)$ quando $2m = 1$ e uma não linearidade $f(\Psi)$, desta forma $(SWE)$ toma forma

$$\begin{cases} -\epsilon^2\Delta u + V(x)u = f(u), \; x \in \mathbb{R}^N \\ u \in H^1(\mathbb{R}^N). \end{cases} \tag{$(S)_\epsilon$}$$

O problema $(S)_\epsilon$ tem sido abordado para diversos tipos de potenciais e não linearidades. Dentre estes trabalhos alguns abordam o comportamento dos valores de máximo das soluções de $(S)_\epsilon$, geralmente demonstrando que esses valores se concentram em pontos críticos não degenerados de $V$. Nesta direção citamos os trabalhos de Wang [54], del Pino e Felmer [16], Ambrosetti, Badiale e Cingolani [12], Ambrosetti e Malchiodi [11], Alves e Souto [8], Gui [22], Wang e Zeng [55], Alves e Soares [9] e [10], Noussair e Wei [35].

Nesta abordagem dos problemas de concentração, como dito acima, geralmente as soluções estão concentradas no conjunto

$$\mathcal{V} = \left\{ x \in \mathbb{R}^N \; ; \; V(x) = \min_{z \in \mathbb{R}^N} V(z) \right\}.$$

Além disso, em muitos trabalhos a multiplicidade de soluções tem uma associação direta com a riqueza topológica de $\mathcal{V}$ e a geometria do potencial $V$.

Em [42], Rabinowitz prova a existência de soluções positivas para $(S)_\epsilon$ quando

$$\liminf_{|x| \to +\infty} V(x) > \inf_{x \in \mathbb{R}^N} V(x) = V_0 > 0 \tag{$R$}$$

e com algumas condições sobre a não linearidade que engloba o caso em que a não linearidade é $f(u) = \gamma|u|^{p-1}u$. Continuando o estudo, em [54], Wang provou que tais soluções se concentram em $\mathcal{V}$ quando $\epsilon \to 0$.

Em [16], del Pino e Felmer melhoram os resultados encontrados em [42] e [54] generalizando a condição $(R)$ para a condição

$$\min_{x\in\partial\Lambda} V(x) > \inf_{x\in\Lambda} V(x) \quad \text{e} \quad V(x) \geq \alpha > 0$$

onde $\Lambda \subset \mathbb{R}^N$ é um domínio compactamente contido em $\mathbb{R}^N$ e com a não linearidade satisfazendo as condições

(f1) $\frac{f(t)}{t} \to 0$ quando $t \to 0$;

(f2) $\displaystyle\lim_{|t|\to+\infty} \frac{|f(t)|}{|t|^p} = 0$ para algum $p \in (1, 2^* - 1)$;

(f3) existe $\theta > 2$ tal que

$$0 < \theta F(t) \leq f(t)t, \quad \text{para todo } t \in \mathbb{R} \setminus \{0\}$$

onde $F(t) := \int_0^t f(s)ds$;

(f4) a função $t \mapsto \frac{f(t)}{t}$ é crescente em $\mathbb{R}^+$ e decrescente em $\mathbb{R}^-$.

Para estabelecer a existência de solução para $(S)_\epsilon$ quando $\epsilon \approx 0^+$ foi usado um método denominado método de penalização e foi estabelecido que as soluções se concentram no ponto mínimo de $V$ quando $\epsilon \to 0$. Observe que $(f2)$ é equivalente a condição

(f2) existe $p \in (1, 2^* - 1)$ tal que $\displaystyle\limsup_{|t|\to+\infty} \frac{|f(t)|}{|t|^p} < +\infty$.

Vale a pena destacar que outras geometrias sobre o potencial $V$ foram consideradas no estudo da existência de solução para $(S)_\epsilon$, como por exemplo potenciais coercivos, periódicos e assintoticamnte periódicos. Novamente em [42], Rabinowitz estabelece existência de solução não nula como um primeiro resultado de soluções de $(S)_1$ para um potencial $V$ coercivo, isto é,

$$V(x) \to +\infty \quad \text{quando} \quad |t| \to +\infty$$

e algumas hipóteses sobre a não linearidade que englobam o caso $f(u) = \gamma|u|^{p-1}u$ com $p \in (1, 2^* - 1)$. Em [59], Coti Zelati estabelece existência de solução positiva de energia mínima para $(S)_1$ com a não linearidade satisfazendo as condições (f1)-(f4) e $V : \mathbb{R}^N \to \mathbb{R}$ um potencial contínuo e $\mathbb{Z}^N$-periódico, isto é,

$$V(x+z) = V(x), \quad \text{para todo } (x,z) \in \mathbb{R}^N \times \mathbb{Z}^N.$$

Para contornar a falta de compacidade é utilizado o Teorema do Passo da Montanha e um lema devido a Lions.

Em [3], Alves, Carrião e Miyagaki estudaram o problema $(P)$ para dimensões $N \geq 3$, onde o potencial possui o formato $V - W$ onde $V$ é $\mathbb{Z}^N$-periódico, contínuo e positivo e $W$ é não negativa e assintoticamente nula no infinito, além da não linearidade possuir crescimento subcrítico com algumas condições técnicas. Na literatura, problemas com esses tipos de potenciais são chamados problemas com potencial assintoticamente periódico, os quais são uma generalização dos problemas com potencial assintoticamente constante.

Em [2], Alves, do Ó e Miyagaki motivados pela desigualdade de Trundiger-Moser e utilizando uma desigualdade devido a Cao [13] estudaram o problema $(P)$ e estabeleceram existência de solução para o caso em que o potencial $V$ é contínuo, positivo e assintoticamente periódico e uma condição sobre a não linearidade que engloba casos em que $f$ tem crescimento crítico exponencial, de uma forma mais precisa:

(f5*) existe $\Gamma$ tal que $|f(x,t)| \leq \Gamma e^{4\pi t^2}$ para todo $(x,t) \in \mathbb{R}^N \times \mathbb{R}$;

e mais algumas condições técnicas sobre a não linearidade, como por exemplo:

(f1*) $\frac{f(x,t)}{t} \to 0$ uniformemente em $x$ quando $t \to 0$;

(f3*) existe $\theta > 2$ tal que

$$0 < \theta F(x,t) \leq f(x,t)t, \ \forall \ (x,t) \in \mathbb{R}^N \times \mathbb{R}^*$$

onde $F(x,t) := \int_0^t f(x,s)ds$.

Lembramos aqui que a definição de $f$ possuir crescimento crítico exponencial significa que existe $\alpha_0 > 0$ tal que

$$\lim_{|t|\to+\infty} \frac{|f(t)|}{e^{\alpha|t|^2}} = 0, \ \text{ para todo } \alpha > \alpha_0, \lim_{|t|\to+\infty} \frac{|f(t)|}{e^{\alpha|t|^2}} = +\infty, \ \text{ para todo } \alpha < \alpha_0 \text{ (ver [18])}$$

Em grande parte dos artigos mencionados acima o potencial $V$ possui a condição $\inf_{x\in\mathbb{R}^N} V(x) > 0$ o que implica em

$$\inf(\sigma(-\Delta + V)) \geq 0, \qquad (\mathcal{I})$$

caracterizando o problema como sendo fortemente definido.

O estudo dos problemas periódicos e assintoticamente periódicos também tem sido feitos para problemas fortemente indefinidos. Em [27], Kryszewski e Szulkin estudaram o problema

$$\begin{cases} -\Delta u + V(x)u = f(x,u), \ x \in \mathbb{R}^N \\ u \in H^1(\mathbb{R}^N) \end{cases} \qquad (P)$$

onde $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ é uma função contínua, $\mathbb{Z}^N$-periódica na coordenada $x$, possui crescimento subcrítico, isto é,

(f2*) existe $C > 0$ tal que $|f(x,t)| \leq C(1 + |t|^{p-1})$ onde $p \in (2, 2^*)$,

além das condições (f1*) e (f3*). Além disso, o potencial $V : \mathbb{R}^N \to \mathbb{R}$ satisfaz a seguinte hipótese

$V$ é contínua, $\mathbb{Z}^N$-periódica e $0 \notin \sigma(-\Delta + V)$, o espectro de $-\Delta + V$ $\qquad (V)$.

O funcional energia $J : H^1(\mathbb{R}^N) \to \mathbb{R}$ associado ao problema $(P)$ é definido por

$$J(u) = \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2)dx - \int_{\mathbb{R}^N} F(x,u)dx, \quad u \in H^1(\mathbb{R}^N)$$

e sabemos, por argumentos usuais, que $J$ é um funcional de classe $C^1$ com

$$J'(u)v = \int_{\mathbb{R}^N} (\nabla u \nabla v + V(x)uv)dx - \int_{\mathbb{R}^N} f(x,u)vdx, \ u, v \in H^1(\mathbb{R}^N).$$

Note que a forma bilinear, definida por

$$B(u,v) = \int_{\mathbb{R}^N} (\nabla u \nabla v + V(x)uv)dx,$$

não é necessariamente positiva definida. O que caracteriza o problema como sendo fortemente indefinido.

Com a condição $(V)$ conseguimos encontrar subespaços $E^+$ e $E^-$ fechados de $H^1(\mathbb{R}^N)$ tais que $H^1(\mathbb{R}^N) = E^+ \oplus E^-$ e que satisfazem:

- $B$ é positiva definida sobre $E^+$ e negativa definida sobre $E^-$.

- $E^+$ e $E^-$ são ortogonais com o produto interno usual de $H^1(\mathbb{R}^N)$ e também ortogonais em relação a forma bilinear $B$.

- Existe uma norma $||\cdot||$ que provém de um produto interno sobre $H^1(\mathbb{R}^N)$ equivalente a norma usual e tal que

$$B(u,u) = ||u||^2, \text{ para todo } u \in E^+ \text{ e } B(u,u) = -||u||^2, \text{ para todo } u \in E^-.$$

Maiores detalhes das afirmações podem ser vistas no Apêndice A. É importante mencionar aqui que grande parte das ideias que aparecem nesse apêndice surgiram de notas de estudos individuais dos professores Marco Aurélio e Claudianor Alves.

As condições mencionadas acima garantem que o funcional energia do problema $(P)$ possui o seguinte formato

$$J(u) = \frac{1}{2}||u^+||^2 - \frac{1}{2}||u^-||^2 - \int_{\mathbb{R}^N} F(x,u)dx, \quad u \in H^1(\mathbb{R}^N).$$

Kryszewski e Szulkin introduzem um teorema muito semelhante ao Teorema de Link devido a Rabinowitz, distinto principalmente pelas dimensões infinitas dos espaços vetoriais da decomposição. Em [31], Li e Szulkin utilizam o Teorema de Link devido a Kryzewski e Szulkin para estabelecer solução para a equação $(P)$ supondo $(V)$ e a não linearidade $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ assintoticamente linear no infinito, isto é,

$$f(x,u) = V_\infty(x)u + f_\infty(x,u), \text{ onde } V_\infty \text{ é periódica e } \frac{f_\infty(x,u)}{u} \to 0 \text{ quando } |u| \to \infty.$$

Muitos trabalhos na literatura utilizam o Teorema de Link acima mencionado, como exemplo: Chabrowski e Szulkin [14] para não linearidade com crescimento crítico; Furtado e Marchi [20] e os trabalhos de Tang [51] e [52] para não linearidade com crescimento subcrítico e suas referências.

Em [39], Pankov e Pfluger trabalharam no problema $(P)$ com hipóteses similares a Kryszewski e Szulkin em [27], mas utilizando o Teorema de Link devido a Rabinowitz [40]. Continuando tal estudo, em [38], Pankov estudou problemas do tipo

$$\begin{cases} -\Delta u + V(x)u = \pm f(x,u), \ x \in \mathbb{R}^N \\ u \in H^1(\mathbb{R}^N) \end{cases} \qquad (P')$$

com $f$ satisfazendo $(f1)$, $(f2)$ e a condição $(V)$. É importante ressaltar que tanto [39] como [38] estabelecem soluções não nulas de energia mínima, denominada soluções ground state, mediante a condição

$$0 < \frac{f(x,t)}{t} \leq \theta f_t(x,t), \ (x,t) \in \mathbb{R}^N \times \mathbb{R}^*. \qquad (f^*)$$

Para isso é utilizado o método de minimização do funcional energia $J$ sobre o conjunto

$$\mathcal{O} := \{u \in H^1(\mathbb{R}^N) \setminus E^- \ ; \ J'(u)u = J'(u)v = 0, \ \forall v \in E^-\}.$$

É interessante observar que no caso em que $E^- = \{0\}$ então $\mathcal{O}$ é exatamente a variedade de Nehari associado ao funcional energia $J$.

Em [45], Szulkin e Weth complementaram os estudos de Pankov [38] estabelecendo soluções do tipo ground state para $(P)$, porém com condições mais fracas sobre a não linearidade, sem utilizar condições sobre a derivada da $f$ e também enfraquecendo a condição (f3*) de Ambrosetti Rabinowitz para

$$\frac{F(x,t)}{t^2} \to +\infty, \quad \text{quando} \ \ t \to +\infty$$

que é conhecida como condição de super quadraticidade. Para encontrar solução que possui energia mínima é crucial a utilização da seguinte condição:

$$t \mapsto \frac{f(x,t)}{|t|} \ \text{ é crescente sobre o conjunto } \ \mathbb{R} \setminus \{0\}. \qquad \text{(f4*)}$$

Na literatura observamos que existem poucos estudos sobre problemas fortemente indefinidos de equações do tipo $(P)$ cuja não linearidade possui crescimento crítico. Podemos citar para $N \geq 4$ os trabalhos de Chabrowski e Szulkin [14], Zhang, Xu e Zhang [61] e Schechter e Zou [49]. Nestes três trabalhos a não linearidade possui o formato

$$f(x,t) = k(x)|t|^{2^*-2}t + g(x,t), \qquad (F)$$

onde $g$ possui crescimento subcrítico e $k : \mathbb{R}^N \to \mathbb{R}$ é uma função positiva. Para o caso $N = 2$ encontramos apenas o trabalho de do Ó e Ruf [17], que trata do caso em que a não linearidade possui crescimento crítico exponencial.

Motivados por [45] e [3], no Capítulo 1 desta tese encontramos soluções de energia mínima para o problema

$$\begin{cases} -\Delta u + (V - W)u = f(x,u), \ x \in \mathbb{R}^N \\ u \in H^1(\mathbb{R}^N) \end{cases} \qquad (P_W)$$

onde a não linearidade $f$ possui crescimento critico e satisfaz (f4*), $V$ cumpre a condição $(V)$ e $W \geq 0$ verifica:

$(W_1)$ $W:\mathbb{R}^N \to \mathbb{R}$ é contínua e $\lim\limits_{|x|\to+\infty} W(x) = 0$.

$(W_2)$ $0 \leq W(x) \leq \Theta = \sup\limits_{x\in\mathbb{R}^N} W(x) < \overline{\Lambda} := \inf(\sigma(-\Delta + V) \cap [0,+\infty)), \quad \forall x \in \mathbb{R}^N$.

No caso específico $N \geq 3$ a não linearidade possui o formato $(F)$ com $g(x,t) = h(x)|t|^{p-1}t$ onde $p \in (1, 2^* - 1)$. No caso $N = 2$ a não linearidade satisfaz (f1*), (f3*), (f4*) e (f5*) e mais algumas condições técnicas. Ressaltamos que não existem trabalhos similares para o caso $N = 3$. O resultado principal deste capítulo é

**Teorema 1.1.1** *Assuma que o potencial $V$ satisfaz $(V)$, e $W : \mathbb{R}^N \to \mathbb{R}$ satisfaz $(W_1) - (W_2)$, com não linearidade $(x,t) \mapsto f(x,t)$, no caso $N \geq 3$, satisfazendo (F) com $g(x,t) = h(x)|t|^{q-1}t$ com $q \in (1, 2^* - 1)$ e*

$(C_1)$ *$h(x) = h_0(x) + h_*(x)$ e $k(x) = k_0(x) + k_*(x)$, onde $h_0, h_*, k_0, k_* : \mathbb{R}^N \to \mathbb{R}$ são funções contínuas, $h_0, k_0$ são $\mathbb{Z}^N$-periódicas, $\lim\limits_{|x|\to+\infty} h_*(x) = \lim\limits_{|x|\to+\infty} k_*(x) = 0$ e $h_0, h_*, k_0, k_*$ são não negativas;*

$(C_2)$ *Existe $x_0 \in \mathbb{R}^N$ tal que*

$$k(x_0) = \max_{x\in\mathbb{R}^N} k(x) \quad e \quad k(x) - k(x_0) = o(|x - x_0|^2) \quad quando \quad x \to x_0;$$

$(C_3)$ *Se $\inf_{x\in\mathbb{R}^N} h(x) = 0$, assumimos que $V(x_0) < 0$,*

*no caso $N = 2$ a não linearidade $f$ cumpre $f(x,t) = f_0(x,t) + f^*(x,t)$, (f1*), (f3*) e (f4*) onde $f_0$ é uma função contínua não negativa $\mathbb{Z}^2$-periódica em relação a coordenada $x$, satisfazendo (f1*), (f3*), (f4*), (f5*) e com a condição de que existem $q > 2$ e $D : \mathbb{R}^2 \to \mathbb{R}$ tal que*

$$F_0(x,t) \geq D(x)|t|^q, \ \forall\ (x,t) \in \mathbb{R}^2 \times \mathbb{R}, \qquad e \qquad \inf_{x\in\mathbb{R}^2} D(x) > 0$$

*e $f^*$ uma função contínua não negativa satisfazendo:*

$(D1)$ *Existe $\tau \in (1,2)$ tal que $|f^*(x,t)| \leq H(x)e^{4\pi|t|^{\tau-2}t}$ para todo $(x,t) \in \mathbb{R}^2 \times \mathbb{R}$, onde $H \in L^2(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$;*

$(D2)$ *Para todos $\epsilon > 0$ e $\beta > 0$, existe $R > 0$ tal que*

$$|f^*(x,t)| \leq \epsilon(e^{\beta t^2} - 1) \quad para \quad |t| > R \quad e \quad x \in \mathbb{R}^2 \setminus B_R(0).$$

*Então, o problema $(P_W)$ tem uma solução de energia mínima.*

No caso $N = 3$ existem restrições técnicas que vem de restrições de argumentos devidos a Brezis e Nirenberg.

Após uma revisão bibliográfica, percebemos que não existem artigos para problemas fortemente indefinidos onde

$$f(x,t) = A(\epsilon x)f(t), \quad (x,t) \in \mathbb{R}^N \times \mathbb{R},$$

e $A$ satisfazendo

$$0 < A_0 = \inf_{x \in \mathbb{R}^N} A(x) \leq \lim_{|x| \to \infty} A(x) < \sup_{x \in \mathbb{R}^N} A(x). \tag{$A$}$$

Para os Capítulos 2 e 3 fomos motivados pelas idéias de Rabinowitz [42], Wang [54] e Alves e Germano [5] para estudar a existência e concentração de solução para o problema

$$\begin{cases} -\Delta u + V(x)u = A(\epsilon x)f(u), \ x \in \mathbb{R}^N \\ u \in H^1(\mathbb{R}^N) \end{cases} \tag{$(P)_\epsilon$}$$

com as condições $(A)$ e $(V)$ satisfeitas.

O funcional energia $I_\epsilon : H^1(\mathbb{R}^N) \to \mathbb{R}$ do problema $(P)_\epsilon$ é definido por

$$I_\epsilon(u) = \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2)dx - \int_{\mathbb{R}^N} A(\epsilon x)F(u)dx, \quad u \in H^1(\mathbb{R}^N) \tag{$I_\epsilon$}$$

onde $F(t) = \int_0^t f(s)ds$. Por argumentos usuais temos que $I_\epsilon \in C^1(H^1(\mathbb{R}^N), \mathbb{R})$ com

$$I'_\epsilon(u)v = \frac{1}{2}\int_{\mathbb{R}^N} (\nabla u \nabla v + V(x)uv)dx - \int_{\mathbb{R}^N} A(\epsilon x)f(u)vdx.$$

Nestes dois ultimos capítulos da tese faremos um estudo sobre o comportamento do número

$$c_\epsilon = \inf_{u \in \mathcal{M}_\epsilon} I_\epsilon,$$

onde

$$M_\epsilon = \{u \in H^1(\mathbb{R}^N) \ ; \ I'_\epsilon(u)u = I'_\epsilon(u)v = 0, \ \text{para todo } v \in E^-\}$$

e a não linearidade $f : \mathbb{R} \to \mathbb{R}$ satisfaz (f1), (f3), (f4). Especificamente no Capítulo 2 consideramos a não linearidade $f$ com crescimento subcrítico. Enquanto no Capítulo 3 a não linearidade assume a condição de crescimento crítico. Além disso, para $N \geq 3$ especificamente, consideramos

$$f(t) = \xi|t|^{q-1}t + |t|^{2^*-2}t, \ \xi > 0, \ q \in (1, 2^* - 1), \ t \in \mathbb{R}$$

e no caso $N = 2$ as condições sobre a não linearidade são (f5) Existe $\Gamma > 0$ tal que $|f(t)| \leq \Gamma e^{4\pi t^2}$

(f6) Existem $\tau > 0$, $q > 2$ tal que $F(t) \geq \tau|t|^q$, para todo $t \in \mathbb{R}$.

Nos Capítulos 2 e 3 estabelecemos existência de soluções de energia mínima para $(P)_\epsilon$ e mostramos a concentração no conjunto

$$\mathcal{A} = \left\{ x \in \mathbb{R}^N \ ; \ A(x) = \max_{x \in \mathbb{R}^N} A(x) \right\}.$$

Mais especificamente o teorema principal do Capítulo 2 é

**Teorema 2.1.1** *Assuma $V, A : \mathbb{R}^N \to \mathbb{R}$ satisfazendo $(V), (A)$ e não linearidade $f$ satisfazendo as condições como mencionadas acima, isto é, (f1)-(f4). Então, existe $\epsilon_0 > 0$ tal que $(P)_\epsilon$ tem uma solução de energia mínima $u_\epsilon$ para todo $\epsilon \in (0, \epsilon_0)$. Além disso, se $x_\epsilon \in \mathbb{R}^N$ denota o ponto de máximo global de $|u_\epsilon|$, então*

$$\lim_{\epsilon \to 0} A(\epsilon x_\epsilon) = \sup_{x \in \mathbb{R}^N} A(x).$$

Enquanto que no Capítulo 3 o teorema principal é

**Teorema 3.1.1** *Assuma $V, A : \mathbb{R}^N \to \mathbb{R}$ satisfazendo $(V), (A)$ e não linearidade satisfazendo as condições como mencionadas acima. Então, existe $\xi_0, \tau_0, \epsilon_0 > 0$ tal que $(P)_\epsilon$ tem uma solução de energia mínima $u_\epsilon$ para todo $\epsilon \in (0, \epsilon_0)$, com $\xi < \xi_0$ para $N = 3$ e com $\tau < \tau_0$ para $N = 2$. Além disso, se $x_\epsilon \in \mathbb{R}^N$ denota o ponto de máximo global de $|u_\epsilon|$, então*

$$\lim_{\epsilon \to 0} A(\epsilon x_\epsilon) = \sup_{x \in \mathbb{R}^N} A(x).$$

# Capítulo 1

# Soluções de energia mínima para uma classe de problemas variacionais indefinidos com crescimento crítico.

## Ground state solution for a class of indefinite variational problems with critical growth

CLAUDIANOR O. ALVES and GEILSON F. GERMANO

### Abstract

In this paper we study the existence of ground state solution for an indefinite variational problem of the type

$$\begin{cases} -\Delta u + (V(x) - W(x))u = f(x,u) \quad \text{in} \quad \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \end{cases} \tag{$P$}$$

where $N \geq 2$, $V, W : \mathbb{R}^N \to \mathbb{R}$ and $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ are continuous functions verifying some technical conditions and $f$ possesses critical growth. Here, we will consider the case where the problem is asymptotically periodic, that is, $V$ is $\mathbb{Z}^N$-periodic, $W$ goes to 0 at infinity and $f$ is asymptotically periodic.

**Mathematics Subject Classifications (2010):** 35B33, 35A15, 35J15 .

**Keywords:** critical growth, variational methods, elliptic equations, indefinite strongly functional.

## 1.1 Introduction

In this paper we study the existence of ground state solution for an indefinite variational problem of the type

$$\begin{cases} -\Delta u + (V(x) - W(x))u = f(x,u), \ \text{ in } \ \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \end{cases} \tag{$P$}$$

where $N \geq 2$, $V, W : \mathbb{R}^N \to \mathbb{R}$ are continuous functions verifying some technical conditions and $f$ has critical growth. Here, we will consider the case where the problem is asymptotically periodic, that is, $V$ is $\mathbb{Z}^N$-periodic, $W$ goes to 0 at infinity and $f$ is asymptotically periodic.

In [27], Kryszewski and Szulkin have studied the existence of ground state solution for an indefinite variational problem of the type

$$\begin{cases} -\Delta u + V(x)u = f(x,u), \quad \text{in} \quad \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \end{cases} \tag{$P_1$}$$

where $V : \mathbb{R}^N \to \mathbb{R}$ is a $\mathbb{Z}^N$-periodic continuous function such that

$$0 \notin \sigma(-\Delta + V), \text{ the spectrum of } -\Delta + V. \tag{$V_1$}$$

Related to the function $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$, they assumed that $f$ is continuous, $\mathbb{Z}^N$-periodic in $x$ with

$$|f(x,t)| \leq c(|t|^{q-1} + |t|^{p-1}), \quad \forall t \in \mathbb{R} \quad \text{and} \quad x \in \mathbb{R}^N \tag{$h_1$}$$

and

$$0 < \alpha F(x,t) \leq t f(x,t) \quad \forall t \in \mathbb{R}, \quad F(x,t) = \int_0^t f(x,s)\,ds \tag{$h_2$}$$

for some $c > 0$, $\alpha > 2$ and $2 < q < p < 2^*$ where $2^* = \frac{2N}{N-2}$ if $N \geq 3$ and $2^* = +\infty$ if $N = 2$. The above hypotheses guarantee that the energy functional associated with $(P_1)$ given by

$$J(u) = \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)|u|^2\,dx) - \int_{\mathbb{R}^N} F(x,u)\,dx, \ u \in H^1(\mathbb{R}^N),$$

is well defined and belongs to $C^1(H^1(\mathbb{R}^N),\mathbb{R})$. By $(V_1)$, there is an equivalent inner product $\langle\ ,\ \rangle$ in $H^1(\mathbb{R}^N)$ such that

$$J(u) = \frac{1}{2}\|u^+\|^2 - \frac{1}{2}\|u^-\|^2 - \int_{\mathbb{R}^N} F(x,u)\,dx,$$

where $\|u\| = \sqrt{\langle u,u\rangle}$ and $H^1(\mathbb{R}^N) = E^+ \oplus E^-$ corresponds to the spectral decomposition of $-\Delta + V$ with respect to the positive and negative part of the spectrum with $u = u^+ + u^-$, where $u^+ \in E^+$ and $u^- \in E^-$. In order to show the existence of solution for $(P_1)$, Kryszewski and Szulkin introduced a new and interesting generalized link theorem. In [31], Li and Szulkin have improved this generalized link theorem to prove the existence of solution for a class of indefinite problem with $f$ being asymptotically linear at infinity.

The link theorems above mentioned have been used in a lot of papers. We would like to cite Chabrowski and Szulkin [14], do Ó and Ruf [17], Furtado and Marchi [20], Tang [51, 52] and their references.

Pankov and Pflüger [39] also have considered the existence of solution for problem $(P_1)$ with the same conditions considered in [27], however the approach is based on an approximation technique of periodic function together with the linking theorem due to Rabinowitz [40]. After, Pankov [38] has studied the existence of solution for problems of the type

$$\begin{cases} -\Delta u + V(x)u = \pm f(x,u), & \text{in} \quad \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \end{cases} \tag{$P_2$}$$

by supposing $(V_1)$, $(h_1) - (h_2)$ and employing the same approach explored in [39]. In [38] and [39], the existence of ground state solution has been established by supposing that $f$ is $C^1$ and there is $\theta \in (0,1)$ such that

$$0 < t^{-1}f(x,t) \le \theta f_t'(x,t), \quad \forall t \neq 0 \quad \text{and} \quad x \in \mathbb{R}^N. \tag{$h_3$}$$

However, in [38], Pankov has found a ground state solution by minimizing the energy functional $J$ on the set

$$\mathcal{O} = \left\{u \in H^1(\mathbb{R}^N) \setminus E^-\ ;\ J'(u)u = 0 \text{ and } J'(u)v = 0, \forall\ v \in E^-\right\}.$$

The reader is invited to see that if $J$ is definite strongly, that is, when $E^- = \{0\}$, the set $\mathcal{O}$ is exactly the Nehari manifold associated with $J$. Hereafter, we say that

$u_0 \in H^1(\mathbb{R}^N)$ is a *ground state solution* if

$$J'(u_0) = 0, \quad u_0 \in \mathcal{O} \quad \text{and} \quad J(u_0) = \inf_{w \in \mathcal{O}} J(w).$$

In [45], Szulkin and Weth have established the existence of ground state solution for problem $(P_1)$ by completing the study made in [38], in the sense that, they also minimize the energy function on $\mathcal{O}$, however they have used weaker conditions on $f$, for example $f$ is continuous, $\mathbb{Z}^N$-periodic in $x$ and satisfies

$$|f(x,t)| \leq C(1 + |t|^{p-1}), \;\; \forall t \in \mathbb{R} \quad \text{and} \quad x \in \mathbb{R}^N \tag{$f_1$}$$

for some $C > 0$ and $p \in (2, 2^*)$.

$$f(x,t) = o(t) \;\text{ uniformly in }\; x \text{ as } |t| \to 0 \tag{$f_2$}$$

$$F(x,t)/|t|^2 \to +\infty \;\text{ uniformly in }\; x \text{ as } |t| \to +\infty \tag{$f_3$}$$

and

$$t \mapsto f(x,t)/|t| \;\text{ is strictly increasing on }\; \mathbb{R} \setminus \{0\}. \tag{$f_4$}$$

The same approach has been used by Zhang, Xu and Zhang [60, 61] to study a class of indefinite and asymptotically periodic problem.

After a bibliography review, we have observed that there are few papers involving indefinite problem whose the nonlinearity has critical growth. For example, the critical case for $N \geq 4$ was considered in [14], [49] and [61] when $f$ is given by

$$f(x,t) = g(x,t) + k(x)|t|^{2^*-2}t,$$

with $g : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ being a function with subcritical growth and $k : \mathbb{R}^N \to \mathbb{R}$ be a continuous function satisfying some conditions. For the case $N = 2$, we only know the paper [17] which considered the periodic case with $f$ having an exponential critical growth, namely there is $\alpha_0 > 0$ such that

$$\lim_{|t| \to +\infty} \frac{|f(t)|}{e^{\alpha |t|^2}} = 0, \;\; \forall \alpha > \alpha_0, \lim_{|t| \to +\infty} \frac{|f(t)|}{e^{\alpha |t|^2}} = +\infty, \;\; \forall \alpha < \alpha_0.$$

Motivated by ideas found in Szulkin and Weth [45, 46] together with the fact that there are few papers involving critical growth for $N = 2$ and $N \geq 3$ and indefinite problem, we intend in the present paper to study the existence of ground state solution for $(P)$, with the nonlinearity $f$ having critical growth and the problem being asymptotically periodic. Since we will work with the dimensions $N = 2$ and $N \geq 3$, we will state our conditions in two blocks, however the conditions on $V$ and $W$ are the same for any these dimensions.

**The conditions on $V$ and $W$.**

On the functions $V$ and $W$, we assume the following conditions:

$(V_1)$ $V : \mathbb{R}^N \to \mathbb{R}$ is continuous and $\mathbb{Z}^N$-periodic.

$(V_2)$ $\underline{\Lambda} := \sup(\sigma(-\Delta + V) \cap (-\infty, 0]) < 0 < \overline{\Lambda} := \inf(\sigma(-\Delta + V) \cap [0, +\infty))$.

$(W_1)$ $W : \mathbb{R}^N \to \mathbb{R}$ is continuous and $\displaystyle\lim_{|x| \to +\infty} W(x) = 0$.

$(W_2)$ $0 \leq W(x) \leq \Theta = \displaystyle\sup_{x \in \mathbb{R}^N} W(x) < \overline{\Lambda}, \quad \forall x \in \mathbb{R}^N$.

Concerning the function $f$, we assume the following conditions:

**Dimension $N \geq 3$:**

For this case, we suppose that $f$ is the form

$$f(x, t) = h(x)|t|^{q-1}t + k(x)|t|^{2^*-2}t$$

with $1 < q < 2^* - 1$ and

$(C_1)$ $h(x) = h_0(x) + h_*(x)$ and $k(x) = k_0(x) + k_*(x)$, where $h_0, h_*, k_0, k_* : \mathbb{R}^N \to \mathbb{R}$ are continuous function, $h_0, k_0$ are $\mathbb{Z}^N$-periodic, $\displaystyle\lim_{|x| \to +\infty} h_*(x) = \lim_{|x| \to +\infty} k_*(x) = 0$ and $h_0, h_*, k_0, k_*$ are nonnegative;

$(C_2)$ there is $x_0 \in \mathbb{R}^N$ such that

$$k(x_0) = \max_{x \in \mathbb{R}^N} k(x) \quad \text{and} \quad k(x) - k(x_0) = o(|x - x_0|^2) \quad \text{as} \quad x \to x_0;$$

$(C_3)$ if $\inf_{x \in \mathbb{R}^N} h(x) = 0$, we assume that $V(x_0) < 0$.

**Dimension $N = 2$:**

($f_1$) there exist functions $f_0, f^* : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ such that

$$f(x,t) = f_0(x,t) + f^*(x,t),$$

where $f_0$ and $f^*$ are continuous functions, $f_0$ is $\mathbb{Z}^2$-periodic with respect to $x$, $f^*$ is nonnegative and satisfies the following condition: given $\epsilon > 0$ and $\beta > 0$, there exists $R > 0$ such that

$$|f^*(x,t)| \leq \epsilon(e^{\beta t^2} - 1) \quad \text{for} \quad |t| > R \quad \text{and} \quad x \in \mathbb{R}^2 \setminus B_R(0);$$

($f_2$) $\frac{f(x,t)}{t}, \frac{f_0(x,t)}{t} \to 0$ as $t \to 0$ uniformly with respect to $x \in \mathbb{R}^2$;

($f_3$) for each fixed $x \in \mathbb{R}^2$, the functions $t \mapsto \frac{f(x,t)}{t}$ and $t \mapsto \frac{f_0(x,t)}{t}$ are increasing on $(0, +\infty)$ and decreasing on $(-\infty, 0)$;

($f_4$) there exist $\theta, \mu > 2$ such that

$$0 < \theta F_0(x,t) \leq t f_0(x,t) \qquad \text{and} \qquad 0 < \mu F(x,t) \leq t f(x,t)$$

for all $(x,t) \in \mathbb{R}^2 \times \mathbb{R}^*$, where

$$F_0(x,t) := \int_0^t f_0(x,s)ds \quad \text{and} \quad F(x,t) := \int_0^t f(x,s)ds;$$

($f_5$) There exist $\Gamma > 0$ and $\tau \in (1,2)$ such that $|f_0(x,t)| \leq \Gamma e^{4\pi t^2}$ and $|f^*(x,t)| \leq \Gamma H(x) e^{4\pi |t|^{\tau-2} t}$ for all $(x,t) \in \mathbb{R}^2 \times \mathbb{R}$, where $H \in L^2(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$;

($f_6$) $F_0(x,t) \geq D(x)|t|^q,\ \forall\ (x,t) \in \mathbb{R}^2 \times \mathbb{R}$, for some positive continuous function $D$ with $\inf_{x\in\mathbb{R}^2} D(x) > 0$ and $q > 2$.

An example of a function $f$ verifying $(f_1) - (f_6)$ is

$$f(x,t) = \lambda(3 - sin((x_1 + x_2)2\pi))|t|^{q-2} t e^{\alpha_0 t^2} + \frac{1}{x_1^2 + x_2^2 + 1}|t|^{p-2} t e^{4\pi|t|^{\tau-2}t}, \quad \forall t \in \mathbb{R}$$

with $x = (x_1, x_2), \lambda > 0, \alpha_0 \in (0, 4\pi), q, p \in (2, +\infty)$ and $\tau \in (1,2)$.

The above conditions imply that $f$ has a critical growth if $N = 2$ or $N \geq 3$.

Our main theorem is the following:

**Theorem 1.1.1** *Assume that* $(V_1)-(V_2)$, $(W_1)-(W_2)$, $(C_1)-(C_3)$ *and* $(f_1)-(f_6)$ *hold. Then, problem* $(P)$ *has a ground state solution for* $N \geq 4$. *If* $N = 2, 3$, *there is* $\lambda^* > 0$ *such that if* $\inf_{x\in\mathbb{R}^2} D(x), \inf_{x\in\mathbb{R}^N} h(x) \geq \lambda^*$, *then problem* $(P)$ *has a ground state solution.*

The Theorem 1.1.1 completes the study made in some of the papers above mentioned, in the sense that we are considering others conditions on $V$ and $f$. For example, for the case $N \geq 3$, it completes the study made in [45], because the critical case was not considered for $N \geq 3$ or $N = 2$, and the case asymptotically periodic was not also analyzed. The Theorem 1.1.1 also completes [17], because in that paper was proved the existence of a solution only for periodic case, while we are finding ground state solution for the periodic and asymptotically periodic case by using a different method. Finally, the above theorem completes the main result of [49] and [60], because the authors considered only the case $W = 0$, and also the paper [14], because the dimension $N = 3$ was not considered as well as the asymptotically periodic case. Moreover, in [14] and [49] the authors considered only the

$$V(x_0) < 0 \quad \text{and} \quad k(x) - k(x_0) = o(|x - x_0|^2) \quad \text{as} \quad x \to x_0.$$

In Theorem 1.1.1 this condition was not assumed if $\inf_{x\in\mathbb{R}^N} h(x) > 0$.

Before concluding this introduction, we would like point out that the reader can find others interesting results involving indefinite variational problem in Jeanjean [25], Schechter [47, 48], Lin and Tang [32], Willem and Zou [57], Yang [58] and their references.

**Notation:** In this paper, we use the following notations:

- The usual norms in $H^1(\mathbb{R}^N)$ and $L^p(\mathbb{R}^N)$ will be denoted by $\|\ \ \|_{H^1(\mathbb{R}^N)}$ and $|\ \ |_p$ respectively.
- $C$ denotes (possible different) any positive constant.
- $B_R(z)$ denotes the open ball with center $z$ and radius $R$ in $\mathbb{R}^N$.
- We say that $u_n \to u$ in $L^p_{loc}(\mathbb{R}^N)$ when

$$u_n \to u \quad \text{in} \quad L^p(B_R(0)), \quad \forall R > 0.$$

- If $g$ is a mensurable function, the integral $\int_{\mathbb{R}^N} g(x)\,dx$ will be denoted by $\int g(x)\,dx$.

The plan of the paper is as follows: In Section 2 we will show some technical lemmas and prove the Theorem 1.1.1 for $N \geq 3$, while in Section 3 we will focus our attention to the dimension $N = 2$.

## 1.2 The case $N \geq 3$

In this section, our intention is to prove the Theorem 1.1.1 for the case $N \geq 3$. Some technical lemmas this section also are true for dimension $N = 2$ and they will be used in Section 3.

In this section, our focus is the indefinite problem

$$\begin{cases} -\Delta u + (V(x) - W(x))u = h(x)|u|^{q-1}u + k(x)|u|^{2^*-2}u, \;\text{ in } \mathbb{R}^N \\ u \in H^1(\mathbb{R}^N), \end{cases} \tag{2.1}$$

whose the energy functional $\Phi_W : H^1(\mathbb{R}^N) \to \mathbb{R}$ given by

$$\Phi_W(u) = \frac{1}{2}B(u,u) - \frac{1}{2}\int W(x)|u|^2 dx - \frac{1}{q+1}\int h(x)|u|^{q+1} dx - \frac{1}{2^*}\int k(x)|u|^{2^*} dx \tag{2.2}$$

is well defined, $\Phi_W \in C^1(H^1(\mathbb{R}^N), \mathbb{R})$ and its critical points are precisely weak solutions of (2.1). Here, $B$ is the bilinear form

$$B(u,v) = \int (\nabla u \nabla v + V(x)uv)\,dx. \tag{2.3}$$

Note that the bilinear form $B$ is not positive definite, therefore it does not induce a norm. As in [45], there is an inner product $\langle\ ,\ \rangle$ in $H^1(\mathbb{R}^N)$ such that

$$\Phi_W(u) = \frac{1}{2}\|u^+\|^2 - \frac{1}{2}\|u^-\|^2 - \frac{1}{2}\int W(x)|u|^2\,dx - \int F(x,u)\,dx, \tag{2.4}$$

where $\|u\| = \sqrt{\langle u, u\rangle}$ and $H^1(\mathbb{R}^N) = E^+ \oplus E^-$ corresponds to the spectral decomposition of $-\Delta + V$ with respect to the positive and negative part of the spectrum with $u = u^+ + u^-$, where $u^+ \in E^+$ and $u^- \in E^-$. It is well known that $B$ is positive definite on $E^+$, negative definite on $E^-$ and the norm $\|\ \|$ is equivalent to the usual norm in $H^1(\mathbb{R}^N)$, that is, there are $a, b > 0$ such that

$$b||u|| \leq ||u||_{H^1(\mathbb{R}^N)} \leq a||u||, \quad \forall\, u \in H^1(\mathbb{R}^N). \tag{2.5}$$

Hereafter, we denote by $\Phi : H^1(\mathbb{R}^N) \to \mathbb{R}$ the functional defined by

$$\Phi(u) = \frac{1}{2}B(u,u) - \frac{1}{q+1}\int h_0(x)|u|^{q+1}dx - \frac{1}{2^*}\int k_0(x)|u|^{2^*}dx,$$

or equivalently,

$$\Phi(u) = \frac{1}{2}\|u^+\|^2 - \frac{1}{2}\|u^-\|^2 - \frac{1}{q+1}\int h_0(x)|u|^{q+1}dx - \frac{1}{2^*}\int k_0(x)|u|^{2^*}dx. \tag{2.6}$$

Note that the critical points of $\Phi$ are weak solutions of the periodic problem

$$\begin{cases} -\Delta u + V(x)u = h_0(x)|u|^{q-1}u + k_0(x)|u|^{2^*-2}u, \ \text{ in } \ \mathbb{R}^N, \\ \qquad\qquad u \in H^1(\mathbb{R}^N). \end{cases} \tag{2.7}$$

In the sequel, $\mathcal{M}, E(u)$ and $\hat{E}(u)$ denote the following sets

$$\mathcal{M} := \{u \in H^1(\mathbb{R}^N) \setminus E^- \ ; \ \Phi'_W(u)u = 0 \text{ and } \Phi'_W(u)v = 0, \forall\ v \in E^-\}$$

and

$$E(u) := E^- \oplus \mathbb{R}u \ \text{ and } \ \hat{E}(u) := E^- \oplus [0,+\infty)u.$$

Therefore

$$E(u) = E^- \oplus \mathbb{R}u^+ \ \text{ and } \ \hat{E}(u) = E^- \oplus [0,+\infty)u^+.$$

Moreover, we denote by $\gamma_W$ and $\gamma$ the real numbers

$$\gamma_W := \inf_{\mathcal{M}} \Phi_W \quad \text{and} \quad \gamma := \inf_{\mathcal{M}} \Phi. \tag{2.8}$$

### 1.2.1 Technical lemmas

In this section we are going to show some lemmas which will be used in the proof of main Theorem 1.1.1.

**Lemma 1.2.1** *If $u \in \mathcal{M}$ and $w = su + v$ where $s \geq 1$, $v \in E^-$ and $w \neq 0$, then*

$$\Phi_W(u + w) < \Phi_W(u).$$

**Proof.** In the sequel, we fix

$$G(x,t) := \frac{1}{2}W(x)t^2 + \frac{1}{q+1}h(x)|t|^{q+1} + \frac{1}{2^*}k(x)|t|^{2^*}$$

and

$$g(x,t) := W(x)t + h(x)|t|^{q-1}t + k(x)|t|^{2^*-2}t.$$

Then by a simple computation,

$$\Phi_W(u+w) - \Phi_W(u) =$$
$$-\frac{1}{2}||v||^2 + \int \Big( g(x,u) \Big[ \Big( \frac{s^2}{2} + s \Big) u + (s+1)v \Big] + G(x,u) - G(x,u+w) \Big)\, dx.$$

Now, the proof follows by adapting the ideas explored in [45, Proposition 2.3]. ■

**Lemma 1.2.2** *Let $\mathcal{K} \subset E^+ \setminus \{0\}$ be a compact subset, then there exists $R > 0$ such that $\Phi_W(w) \le 0$, $\forall\, w \in E(u) \setminus B_R(0)$ and $u \in \mathcal{K}$.*

**Proof.** Setting the functional

$$\Psi_*(u) = \frac{1}{2}B(u,u) - \frac{1}{2^*}\int |u|^{2^*}\, dx$$

we have

$$\Phi_W(u) \le \Psi_*(u), \quad \forall u \in H^1(\mathbb{R}^N).$$

Now, we apply the same idea from [45, Lemma 2.2] to the functional $\Psi_*$ to get the desired result. ■

**Lemma 1.2.3** *For all $u \in H^1(\mathbb{R}^N)$, the functional $\Phi_W|_{E(u)}$ is weakly upper semicontinuous.*

**Proof.** First of all, note that $E(u)$ is weakly closed, because it is convex strongly closed. Now, we claim that the functional

$$\begin{array}{rcll} \widetilde{\Phi} & : & E(u) & \to \ \mathbb{R} \\ & & w & \mapsto \ \frac{1}{2}\int W(x)|w|^2\, dx + \frac{1}{q+1}\int h(x)|w|^{q+1}\, dx + \frac{1}{2^*}\int k(x)|w|^{2^*}\, dx \end{array}$$

is weakly lower semicontinuous. Indeed, if $w_n \rightharpoonup w$ on $E(u)$, then after passing to a subsequence $w_n(x) \to w(x)$ a.e. in $\mathbb{R}^N$. Then by Fatou's Lemma,

$$\widetilde{\Phi}(w) = \int W(x)w^2\, dx + \frac{1}{q+1}\int h(x)|w|^{q+1}\, dx + \frac{1}{2^*}\int k(x)|w|^{2^*}\, dx \le$$

$$\le \liminf_{n\to+\infty} \left[ \int W(x)w_n^2\, dx + \frac{1}{q+1}\int h(x)|w_n|^{q+1}\, dx + \frac{1}{2^*}\int k(x)|w_n|^{2^*}\, dx \right],$$

leading to

$$\widetilde{\Phi}(w) \le \liminf_{n\to+\infty} \widetilde{\Phi}(w_n).$$

Furthermore, the functional

$$\begin{array}{rcll} \widetilde{\Psi} & : & E(u) & \to \ \mathbb{R} \\ & & w & \mapsto \ \frac{1}{2}B(w,w) \end{array}$$

is weakly upper semicontinuous. In fact, since

$$\widetilde{\Psi}(w) = \frac{1}{2}(||w^+||^2 - ||w^-||^2),$$

if $w_n = s_n u^+ + v_n \rightharpoonup w = su^+ + v$ with $v_n, v \in E^-$, then $s_n \to s$ in $\mathbb{R}$ and $v_n \rightharpoonup v$ in $H^1(\mathbb{R}^N)$. Thus,

$$\widetilde{\Psi}(w) = \frac{1}{2}(s^2||u^+||^2 - ||v||^2) \geq \limsup_{n\to+\infty} \frac{1}{2}(s_n^2||u^+||^2 - ||v_n||^2) = \limsup_{n\to+\infty} \widetilde{\Psi}(w_n).$$

As $\Phi_W|_{E(u)} = \widetilde{\Psi} - \widetilde{\Phi}$, the result is proved. ■

**Lemma 1.2.4** *For each $u \in H^1(\mathbb{R}^N)\backslash E^-$, $\mathcal{M}\cap\hat{E}(u)$ is a singleton set and the element of this set is the unique global maximum of $\Phi_W|_{\hat{E}(u)}$*

**Proof.** The proof follows very closely the proof of [45, Lemma 2.6]. ■

**Lemma 1.2.5** *There exists $\rho > 0$ such that $\inf\limits_{B_\rho(0)\cap E^+} \Phi_W > 0$.*

**Proof.** In what follows, let us fix $\overline{h} := \sup_{x\in\mathbb{R}^N} h(x)$ and $\overline{k} := \sup_{x\in\mathbb{R}^N} k(x)$. For $u \in E^+$,

$$\begin{aligned}
\Phi_W(u) = & \tfrac{1}{2}||u||^2 - \tfrac{1}{2}\int W(x)|u|^2dx - \tfrac{1}{q+1}\int h(x)|u|^{q+1}dx - \tfrac{1}{2^*}\int k(x)|u|^{2^*}dx \\
& \geq \tfrac{1}{2}||u||^2 - \tfrac{\Theta}{2}\int |u|^2dx - \tfrac{\overline{h}}{q+1}\int |u|^{q+1}dx - \tfrac{\overline{k}}{2^*}\int |u|^{2^*}dx \\
& \geq \tfrac{1}{2}||u||^2 - \tfrac{\Theta}{2\overline{\Lambda}}||u||^2 - \tfrac{\overline{h}c_1}{q+1}||u||^{q+1} - \tfrac{\overline{k}c_2}{2^*}||u||^{2^*} \\
& = \tfrac{1}{2}\left(1 - \tfrac{\Theta}{\overline{\Lambda}}\right)||u||^2 - \tfrac{\overline{h}c_1}{q+1}||u||^{q+1} - \tfrac{\overline{k}c_2}{2^*}||u||^{2^*}.
\end{aligned}$$

Thereby, the lemma follows by taking $\rho > 0$ satisfying

$$\frac{1}{2}\left(1 - \frac{\Theta}{\overline{\Lambda}}\right)\rho^2 - \frac{\overline{h}c_1}{q+1}\rho^{q+1} - \frac{\overline{k}c_2}{2^*}\rho^{2^*} > 0.$$

■

**Lemma 1.2.6** *The real number $\gamma_W$ given in (2.8) is positive. In addition, if $u \in \mathcal{M}$ then $||u^+|| \geq \max\{||u^-||, \sqrt{2\gamma_W}\}$.*

**Proof.** By Lemma 1.2.5, there is $\rho > 0$ such that

$$l := \inf_{B_\rho(0)\cap E^+} \Phi_W > 0.$$

For all $u \in \mathcal{M}$, we know that $u^+ \neq 0$, then by Lemma 1.2.4,

$$\Phi_W(u) \geq \Phi_W\left(\frac{\rho}{||u^+||}u^+\right) \geq l,$$

from where it follows that

$$\gamma_W = \inf_{\mathcal{M}} \Phi_W \geq l > 0.$$

In addition, for all $u \in \mathcal{M}$,

$$\gamma_W \leq \Phi_W(u) \leq \frac{1}{2}B(u,u) = \frac{1}{2}(||u^+||^2 - ||u^-||^2),$$

implying that $||u^+|| \geq \max\{||u^-||, \sqrt{2\gamma_W}\}$. ■

Next we will show a boundedness from above for $\gamma_W$ which will be crucial in our approach. However, before doing this we need to prove two technical lemmas. The first one is true for $N \geq 2$ and it has the following statement

**Lemma 1.2.7** *Consider $N \geq 2$ and let $u \in E^+ \setminus \{0\}$, $p \in (2, 2^*)$ and $r, s_0 > 0$. Then there exists $\xi > 0$ such that*

$$\xi |su|_p \leq |su + v|_p, \tag{2.9}$$

*for all $s \geq s_0$ and $v \in E^-$ with $||su + v|| \leq r$.*

**Proof.** If the lemma does not hold, there are $s_n \geq s_0$ and $v_n \in E^-$ satisfying

$$||s_n u + v_n|| \leq r \quad \text{and} \quad |s_n u|_p \geq n |s_n u + v_n|_p, \ \forall n \in \mathbb{N}.$$

Setting $\alpha_n := |s_n u|_p$, we obtain

$$\left| \frac{u}{|u|_p} + \frac{v_n}{\alpha_n} \right|_p \leq \frac{1}{n}.$$

Thus, passing to a subsequence if necessary,

$$w_n := \frac{v_n}{\alpha_n} \to -\frac{u}{|u|_p} \qquad \text{a.e. in} \quad \mathbb{R}^N. \tag{2.10}$$

On the other hand,

$$||w_n||^2 = \frac{||v_n||^2}{s_n^2 |u|_p^2} \leq \frac{||s_n u + v_n||^2}{s_0^2 |u|_p^2} \leq \frac{r^2}{s_0^2 |u|_p^2} \quad \forall n \in \mathbb{N},$$

showing that $(w_n)$ is a bounded sequence in $H^1(\mathbb{R}^N)$. As $w_n \in E^-$, there is $w \in E^-$ such that for some subsequence (not renamed) $w_n \rightharpoonup w$ in $E^-$. Then by (2.10),

$$\frac{u}{|u|_p} = -w \in E^-,$$

which is absurd, since $u \in E^+ \setminus \{0\}$. ■

**Lemma 1.2.8** *Let $u \in E^+ \setminus \{0\}$ be fixed. Then there are $r, s_0 > 0$ satisfying*

$$\sup_{w\in\widehat{E}(u)} \Phi_W(w) = \sup_{\substack{||su+v|| \le r \\ s \ge s_0,\, v\in E^-}} \Phi_W(su+v). \tag{2.11}$$

**Proof.** From Lemma 1.2.2,

$$\sup_{\widehat{E}(u)} \Phi_W = \sup_{\widehat{E}(u)\cap B_r(0)} \Phi_W$$

for some $r > 0$. Hence, there are $(s_n) \subset [0, +\infty)$ and $(v_n) \subset E^-$ with $||s_n u + v_n|| \le r$ and

$$\Phi_W(s_n u + v_n) \to \sup_{\widehat{E}(u)\cap B_r(0)} \Phi_W. \tag{2.12}$$

Next, we will prove that there exists $s_0 > 0$ such that

$$\sup_{\widehat{E}(u)\cap B_r(0)} \Phi_W = \sup_{\substack{||su+v|| \le r \\ s \ge s_0,\, v\in E^-}} \Phi_W(su+v).$$

Arguing by contradiction, suppose that for all $s_0 > 0$

$$\sup_{\widehat{E}(u)\cap B_r(0)} \Phi_W > \sup_{\substack{||su+v|| \le r \\ s \ge s_0,\, v\in E^-}} \Phi_W(su+v). \tag{2.13}$$

Such supposition permit us to conclude that $s_n \to 0$. On the other hand, recalling that

$$\Phi_W(s_n u + v_n) \le \frac{1}{2} s_n^2 ||u||^2,$$

we are leading to

$$0 < \gamma_W = \inf_{\mathcal{M}} \Phi_W \le \sup_{\widehat{E}(u)} \Phi_W = \Phi_W(s_n u + v_n) + o_n(1) \le \frac{1}{2} s_n^2 ||u||^2 + o_n(1),$$

which is a contradiction. This completes the proof. ■

Now, we are ready to show the estimate from above involving the number $\gamma_W$ given in (2.8)

**Proposition 1.2.9** *Assume the conditions of Theorem 1.1.1. If $N \ge 4$, then*

$$\gamma_W < \frac{1}{N|k_0|_\infty^{\frac{N-2}{2}}} S^{N/2}. \tag{2.14}$$

*If $N = 3$, there is $\lambda^* > 0$ such that the estimate (2.14) holds for $\inf_{x\in\mathbb{R}^N} h(x) > \lambda^*$.*

**Proof.** Since $\gamma_W \leq \gamma$, it is enough to prove that

$$\gamma < \frac{1}{N|k_0|_\infty^{\frac{N-2}{2}}} S^{N/2}.$$

If $N \geq 4$ and $\inf_{x\in\mathbb{R}^N} h(x) = 0$, the estimate is made in [14, Proposition 4.2]. Next we will do the proof for $N \geq 4$ and $\inf_{x\in\mathbb{R}^N} h(x) > 0$. To this end, we follow the same notation used in [14]. Let

$$\varphi_\epsilon(x) = \frac{c_N \psi(x) \epsilon^{\frac{N-2}{2}}}{(\epsilon^2 + |x|^2)^{\frac{N-2}{2}}}$$

where $c_N = (N(N-2))^{\frac{N-2}{4}}, \epsilon > 0$ and $\psi \in C_0^\infty(\mathbb{R}^N)$ is such that

$$\psi(x) = 1 \quad \text{for} \quad |x| \leq \frac{1}{2} \quad \text{and} \quad \psi(x) = 0 \quad \text{for} \quad |x| \geq 1.$$

From [56], we know that the estimates below hold

$$\begin{aligned} &|\nabla\varphi_\epsilon|_2^2 = S^{\frac{N}{2}} + O(\epsilon^{N-2}), \quad |\nabla\varphi_\epsilon|_1 = O(\epsilon^{\frac{N-2}{2}}), \quad |\varphi_\epsilon|_{2^*}^{2^*} = S^{\frac{N}{2}} + O(\epsilon^N), \\ &|\varphi_\epsilon|_{2^*-1}^{2^*-1} = O(\epsilon^{\frac{N-2}{2}}), \quad |\varphi_\epsilon|_q^q = O(\epsilon^{\frac{N-2}{2}}), \quad |\varphi_\epsilon|_1 = O(\epsilon^{\frac{N-2}{2}}) \end{aligned} \tag{2.15}$$

and

$$|\varphi_\epsilon|_2^2 = \begin{cases} b\epsilon^2|log\epsilon| + O(\epsilon^2), & \text{if} \quad N = 4 \\ b\epsilon^2 + O(\epsilon^{N-2}), & \text{if} \quad N \geq 5. \end{cases} \tag{2.16}$$

Adapting the same idea explored in [14, Proposition 4.2], for each $u \in E^-$ we obtain

$$\Phi(s\varphi_\epsilon + u) \leq \Phi(s\varphi_\epsilon) + O(\epsilon^{N-2}), \quad \forall s \geq 0,$$

where $O(\epsilon^{N-2})$ does not depend on $u$. Now, arguing as in [1], we get

$$\sup_{s\geq 0} \Phi(s\varphi_\epsilon) \leq \frac{1}{N|k_0|_\infty^{\frac{N-2}{2}}} S^{N/2} + O(\epsilon^{N-2}) + c_1 \int_{B_1(0)} |\varphi_\epsilon|^2\, dx - c_2 \int_{B_1(0)} |\varphi_\epsilon|^{q+1}\, dx,$$

implying that

$$\sup_{s\geq 0,\, u\in E^-} \Phi(s\varphi_\epsilon + u) \leq \frac{1}{N|k_0|_\infty^{\frac{N-2}{2}}} S^{N/2} + c_1 \int_{B_1(0)} |\varphi_\epsilon|^2\, dx - c_2 \int_{B_1(0)} |\varphi_\epsilon|^{q+1}\, dx + O(\epsilon^{N-2}).$$

Moreover, in [1], we also find that

$$\lim_{\epsilon\to 0} \frac{1}{\epsilon^{N-2}} \left( c_1 \int_{B_1(0)} |\varphi_\epsilon|^2\, dx - c_2 \int_{B_1(0)} |\varphi_\epsilon|^{q+1}\, dx \right) = -\infty,$$

from where it follows that there exists $\epsilon > 0$ small enough verifying

$$c_1 \int_{B_1(0)} |\varphi_\epsilon|^2 \, dx - c_2 \int_{B_1(0)} |\varphi_\epsilon|^{q+1} \, dx + O(\epsilon^{N-2}) < 0,$$

and so,

$$\sup_{s \geq 0, \, u \in E^-} \Phi(s\varphi_\epsilon + u) < \frac{1}{N|k_0|_\infty^{\frac{N-2}{2}}} S^{N/2}$$

for some $\epsilon > 0$ small enough.

Now, we will consider the case $N = 3$. For each $u \in E^+ \setminus \{0\}$, the Lemma 1.2.8 guarantees the existence of $r, s_0 > 0$ satisfying

$$\sup_{w \in \widehat{E}(u)} \Phi(w) = \sup_{\substack{||su+v|| \leq r \\ s \geq s_0, v \in E^-}} \Phi(su+v).$$

Therefore, applying Lemma 1.2.7,

$$\begin{aligned} \sup_{\widehat{E}(u)} \Phi = \; & \sup_{\substack{||su+v|| \leq r \\ s \geq s_0, v \in E^-}} \Phi(su+v) \\ & \leq \sup_{\substack{||su+v|| \leq r \\ s \geq s_0, v \in E^-}} \left( \tfrac{s^2||u||^2}{2} - \tfrac{\lambda}{q+1} \textstyle\int |su+v|^{q+1} dx \right) \\ & \leq \sup_{\substack{||su+v|| \leq r \\ s \geq s_0, v \in E^-}} \left( \tfrac{s^2||u||^2}{2} - \tfrac{\lambda\xi}{q+1} \textstyle\int |su|^{q+1} dx \right) \\ & \leq \max_{s \geq 0} (As^2 - \lambda B s^{q+1}), \end{aligned}$$

where

$$\lambda = \inf_{x \in \mathbb{R}^N} h(x), \quad A = \frac{||u||^2}{2} \quad \text{and} \quad B = \frac{\xi}{q+1} \int |u|^{q+1} dx.$$

As

$$\max_{s \geq 0} (As^2 - \lambda B s^{q+1}) \to 0 \quad \text{as} \quad \lambda \to +\infty,$$

there is $\lambda^* > 0$ such that

$$\sup_{w \in \widehat{E}(u)} \Phi(w) < \frac{1}{N|k_0|_\infty^{\frac{N-2}{2}}} S^{N/2} \quad \forall \lambda \geq \lambda^*,$$

showing the desired result. ■

**Lemma 1.2.10** *Let $(u_n) \subset H^1(\mathbb{R}^N)$ be a sequence verifying*

$$\Phi_W(u_n) \leq d, \quad \pm\Phi'_W(u_n)u_n^\pm \leq d||u_n|| \quad \textit{and} \quad -\Phi'_W(u_n)u_n \leq d||u_n||$$

*for some $d > 0$. Then, $(u_n)$ is bounded in $H^1(\mathbb{R}^N)$.*

**Proof.** In the sequel, let $\theta := \chi_{[-1,1]} : \mathbb{R} \to \mathbb{R}$ be the characteristic function on interval $[-1, 1]$,

$$g(x,t) := \theta(t)f(x,t) \quad \text{and} \quad j(x,t) := (1-\theta(t))f(x,t),$$

where $f(x,t) = h(x)|t|^{q-1}t + k(x)|t|^{2^*-2}t$. Fixing

$$r := \frac{q+1}{q} \quad \text{and} \quad s = \frac{2^*}{2^*-1},$$

it follows that

$$(r-1)q = (s-1)(2^*-1) = 1.$$

Note that

$$\begin{aligned} |g(x,t)|^{r-1} &= \theta(t)^{r-1}|f(x,t)|^{r-1} \leq \theta(t)(|h|_\infty |t|^q + |k|_\infty |t|^{2^*-1})^{r-1} \\ &\leq \theta(t)2^{r-1}C(|t|^{(r-1)q} + |t|^{(r-1)(2^*-1)}) \leq K|t| \end{aligned}$$

for some $C > 0$ sufficiently large. So

$$|g(x,t)|^{r-1} \leq C|t|, \forall\ (x,t) \in \mathbb{R}^{N+1}. \tag{2.17}$$

Analogously,

$$|j(x,t)|^{s-1} \leq C|t|, \forall\ (x,t) \in \mathbb{R}^{N+1}. \tag{2.18}$$

Since $tf(x,t) \geq 0,\ (x,t) \in \mathbb{R}^{N+1}$, the inequalities (2.17) and (2.18) give

$$|g(x,t)|^r \leq Ctg(x,t) \ \text{ and } \ |j(x,t)|^s \leq Ctj(x,t), \quad \forall (x,t) \in \mathbb{R}^{N+1}. \tag{2.19}$$

The last two inequalities lead to

$$\begin{aligned} & d + d||u_n|| \geq \Phi_W(u_n) - \tfrac{1}{2}\Phi'_W(u_n)u_n = \\ & \left(\tfrac{1}{2} - \tfrac{1}{q+1}\right) \int h(x)|u|^{q+1}dx + \left(\tfrac{1}{2} - \tfrac{1}{2^*}\right) \int k(x)|u|^{2^*}dx \geq \\ & \left(\tfrac{1}{2} - \tfrac{1}{q+1}\right) \int h(x)|u|^{q+1}dx + \left(\tfrac{1}{2} - \tfrac{1}{q+1}\right) \int k(x)|u|^{2^*}dx = \\ & \left(\tfrac{1}{2} - \tfrac{1}{q+1}\right) \int (g(x,u_n)u_n + j(x,u_n)u_n)dx \geq \\ & \left(\tfrac{1}{2} - \tfrac{1}{q+1}\right) \tfrac{1}{C} \left(\int |g(x,u_n)|^r dx + \int |j(x,u_n)|^s dx\right), \end{aligned}$$

from where it follows

$$|g(x,u_n)|_r^r + |j(x,u_n)|_s^s \leq C(1 + ||u_n||) \tag{2.20}$$

for some $C > 0$. On the other hand,

$$\begin{aligned}
||u_n^-||^2 &= -\Phi'_W(u_n)u_n^- - \int W(x)u_n u_n^- dx - \int f(x,u_n)u_n^- dx \\
&\le d||u_n^-|| - \int W(x)u_n u_n^- dx + |g(x,u_n)|_r |u_n^-|_{q+1} + |j(x,u_n)|_s |u_n^-|_{2^*} \\
&\le -\int W(x)u_n u_n^- dx + C||u_n^-|| \left(1 + |g(x,u_n)|_r + |j(x,u_n)|_s\right) \\
&\le -\int W(x)u_n u_n^- dx + C||u_n^-|| \left(1 + (1+||u_n||)^{1/r} + (1+||u_n||)^{1/s}\right) \\
&\le -\int W(x)u_n u_n^- dx + C||u_n^-|| \left(1 + ||u_n||^{1/r} + ||u_n||^{1/s}\right).
\end{aligned}$$

Thus,

$$||u_n^-||^2 \le -\int W(x)u_n u_n^- dx + C||u_n|| \left(1 + ||u_n||^{1/r} + ||u_n||^{1/s}\right). \tag{2.21}$$

The same argument works to prove that

$$||u_n^+||^2 \le \int W(x)u_n u_n^+ dx + C||u_n|| \left(1 + ||u_n||^{1/r} + ||u_n||^{1/s}\right). \tag{2.22}$$

Recalling that $||u_n||^2 = ||u_n^+||^2 + ||u_n^-||^2$, the estimates (2.21) and (2.22) combined give

$$||u_n||^2 \le \int W(x)u_n(u_n^+ - u_n^-)dx + C||u_n|| \left(1 + ||u_n||^{1/r} + ||u_n||^{1/s}\right). \tag{2.23}$$

On the other hand, we know that

$$\begin{aligned}
\int W(x)u_n(u_n^+ - u_n^-)dx &= \int W(x)(u_n^+ + u_n^-)(u_n^+ - u_n^-)dx \\
&= \int W(x)(u_n^+)^2 dx - \int W(x)(u_n^-)^2 dx \\
&\le \int W(x)(u_n^+)^2 dx \le \Theta \int (u_n^+)^2 dx \le \tfrac{\Theta}{\overline{\Lambda}}||u_n^+||^2
\end{aligned}$$

that is,

$$\int W(x)u_n(u_n^+ - u_n^-)dx \le \frac{\Theta}{\overline{\Lambda}}||u_n||^2, \tag{2.24}$$

where $\overline{\Lambda}$ was fixed in $(W_2)$. Now, (2.23) combines with (2.24) to give

$$\left(1 - \frac{\Theta}{\overline{\Lambda}}\right) ||u_n||^2 \le C||u_n|| \left(1 + ||u_n||^{1/r} + ||u_n||^{1/s}\right).$$

This concludes the verification of Lemma 1.2.10. ■

As a byproduct of the last lemma, we have the corollaries below

**Corollary 1.2.11** *If $(u_n)$ is a (PS) sequence for $\Phi_W$, then $(u_n)$ is bounded. In addition, if $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^N)$, then $u$ is a solution of* (2.1).

**Corollary 1.2.12** *$\Phi_W$ is coercive on $\mathcal{M}$, that is, $\Phi_W(u) \to +\infty$ as $||u|| \to +\infty$ and $u \in \mathcal{M}$.*

The Lemma 1.2.4 permits to consider a function

$$m : E^+ \setminus \{0\} \to \mathcal{M} \quad \text{where} \quad m(u) \in \hat{E}(u) \cap \mathcal{M}, \quad \forall u \in E^+ \setminus \{0\}. \tag{2.25}$$

The above function will be crucial in our approach. Next, we establish its continuity.

**Lemma 1.2.13** *The function $m$ is continuous.*

**Proof.** Suppose $u_n \to u$ in $E^+ \setminus \{0\}$. Since

$$\frac{u_n}{||u_n||} \to \frac{u}{||u||}, \quad m\left(\frac{u_n}{||u_n||}\right) = m(u_n) \quad \text{and} \quad m\left(\frac{u}{||u||}\right) = m(u),$$

without loss of generality, we may assume that $||u_n|| = ||u|| = 1$.

There are $t_n, t \in [0, +\infty)$ and $v_n, v \in E^-$ such that

$$m(u_n) = t_n u_n + v_n \quad \text{and} \quad m(u) = tu + v.$$

Note that $K := \{u_n\}_{n\in\mathbb{N}} \cup \{u\}$ is a compact set. Thereby, by Lemma 1.2.2, there exists $R > 0$ such that $\Phi_W(w) \leq 0$ in $E(z) \setminus B_R(0)$ for all $z \in K$. Hence,

$$0 < \Phi_W(m(u_n)) = \sup_{\widehat{E}(u_n)} \Phi_W = \sup_{\widehat{E}(u_n)\cap B_R(0)} \Phi_W \leq \sup_{w\in\widehat{E}(u_n)\cap B_R(0)} \frac{1}{2}||w^+||^2 \leq \frac{1}{2}R^2,$$

showing that $(\Phi_W(m(u_n)))$ is a bounded sequence, and so, by Corollary 1.2.12, $(m(u_n))$ is a bounded sequence. The boundedness of $(m(u_n))$ implies that $(t_n)$ and $(v_n)$ are also bounded. Then, for some subsequence (not renamed),

$$t_n \to t_0 \quad \text{in} \quad \mathbb{R}, \; v_n \rightharpoonup v_0 \quad \text{in} \quad E^- \quad \text{and} \quad m(u_n) \rightharpoonup t_0 u + v_0 \quad \text{in} \quad E^-. \tag{2.26}$$

Recalling that $\Phi_W(m(u_n)) \geq \Phi_W(tu_n + v)$, we obtain

$$\liminf_{n\to+\infty} \Phi_W(m(u_n)) \geq \Phi_W(m(u)).$$

Thus, the Fatou's Lemma combined with the weakly lower semicontinuous of the norm gives

$$\begin{aligned}
\Phi_W(m(u)) \quad & \leq \liminf_{n\to+\infty} \Phi_W(m(u_n)) \leq \limsup_{n\to+\infty} \Phi_W(m(u_n)) \\
& \limsup_{n\to+\infty} \Big[\tfrac{1}{2}t_n^2||u_n||^2 - \tfrac{1}{2}||v_n||^2 - \tfrac{1}{2}\textstyle\int W(x)m(u_n)^2dx \\
& -\tfrac{1}{q+1}\textstyle\int h(x)|m(u_n)|^{q+1}dx - \tfrac{1}{2^*}\int k(x)|m(u_n)|^{2^*}dx\Big] \\
& \leq \tfrac{1}{2}t_0^2 - \tfrac{1}{2}||v_0||^2 - -\frac{1}{2}\int W(x)|t_0u + v_0|^2dx \\
& -\tfrac{1}{q+1}\textstyle\int h(x)|t_0u + v_0|^{q+1}dx - \tfrac{1}{2^*}\int k(x)|t_0u + v_0|^{2^*}dx \\
& = \Phi_W(t_0u + v_0) \leq \Phi_W(m(u)),
\end{aligned}$$

implying that

$$\lim_{n\to+\infty} ||v_n|| = ||v_0|| \quad \text{and} \quad \Phi_W(t_0u+v_0) = \Phi_W(m(u)). \tag{2.27}$$

From (2.26) and (2.27), $v_n \to v_0$ in $E^-$. Now, the Lemma 1.2.1 together with (2.27) guarantees that $t_0u+v_0 = m(u)$. Consequently,

$$m(u_n) = t_nu_n + v_n \to t_0u + v_0 = m(u),$$

finishing the proof. ■

Hereafter, we consider the functional $\hat{\Psi} : E^+ \setminus \{0\} \to \mathbb{R}$ defined by $\hat{\Psi}(u) := \Phi_W(m(u))$. We know that $\hat{\Psi}$ is continuous by previous lemma. In the sequel, we denote by $\Psi : S^+ \to \mathbb{R}$ the restriction of $\hat{\Psi}$ to $S^+ = B_1(0) \cap E^+$.

The next three results establish some important properties involving the functionals $\Psi$ and $\hat{\Psi}$ and their proofs follow as in [45].

**Lemma 1.2.14** *$\hat{\Psi} \in C^1(E^+ \setminus \{0\}, \mathbb{R})$, and*

$$\hat{\Psi}'(y)z = \frac{||m(y)^+||}{||y||}\Phi_W'(m(y))z, \ \forall y,z \in E^+, \ y \neq 0. \tag{2.28}$$

**Corollary 1.2.15** *The following assertions hold:*

*(a)* $\Psi \in C^1(S^+)$, *and*

$$\Psi'(y)z = ||m(y)^+||\Phi_W'(m(y))z, \ \text{for } z \in T_yS^+.$$

*(b)* $(w_n)$ *is a (PS)$_c$ sequence for* $\Psi$ *if and only if* $(m(w_n))$ *is a (PS)$_c$ sequence for* $\Phi_W$.

*(c) If* $\gamma_W = \inf_{\mathcal{M}} \Phi_W$ *is attained by* $u \in \mathcal{M}$, *then* $\Phi_W'(u) = 0$.

**Proposition 1.2.16** *There exists a (PS)$_{\gamma_W}$ sequence for* $\Phi_W$.

Our next lemma will be used to prove the existence of ground state solution for the periodic case.

**Lemma 1.2.17** *Let $(u_n)$ be a (PS)$_c$ sequence for the functional $\Phi$ given in (2.6) with $c \neq 0$. Then, there are $r, \epsilon > 0$ and $(y_n)$ in $\mathbb{Z}^N$ satisfying*

$$\limsup_{n\in\mathbb{N}} \int_{B_r(y_n)} |u_n|^{2^*} dx \geq \epsilon. \tag{2.29}$$

*In addition, if $c \in (-\infty, S^{N/2}|k_0|_\infty^{\frac{2-N}{2}}/N) \setminus \{0\}$, the sequence $v_n = u_n(\cdot - y_n)$ is also a $(PS)_c$ sequence for $\Phi$, and for some subsequence, $v_n \rightharpoonup v$ in $H^1(\mathbb{R}^N)$ with $v \neq 0$.*

**Proof.** By Corollary 1.2.11, the sequence $(u_n)$ is bounded in $H^1(\mathbb{R}^N)$. Arguing by contradiction, we suppose that

$$\limsup_{n\to+\infty} \sup_{y\in\mathbb{R}^N} \int_{B_R(y)} |u_n|^{2^*}\,dx = 0,$$

for some $R > 0$. Applying [43, Lemma 2.1], it follows that $u_n \to 0$ in $L^{2^*}(\mathbb{R}^N)$, and so, by interpolation on the Lebesgue spaces, $u_n \to 0$ in $L^p(\mathbb{R}^N)$ for all $p \in (2, 2^*]$. As

$$\Phi'(u_n)(u_n^-) = -||u_n^-||^2 - \int h_0(x)|u_n|^{q-1}u_n u_n^- dx - \int k_0(x)|u_n|^{2^*-2}u_n u_n^- dx,$$

we deduce that $u_n^- \to 0$ in $H^1(\mathbb{R}^N)$. By a similar argument $u_n^+ \to 0$ in $H^1(\mathbb{R}^N)$. Hence

$$u_n \to 0 \text{ in } \quad H^1(\mathbb{R}^N).$$

Thereby, by continuity of $\Phi$, $c = \lim \Phi(u_n) = \Phi(0) = 0$, which is absurd. Thus, there are $(z_n) \subset \mathbb{R}^N$ and $\eta > 0$ satisfying

$$\int_{B_R(z_n)} |u_n^+|^{2^*} dx \geq \eta > 0, \quad \forall n \in \mathbb{N}.$$

Recalling that for each $n \in \mathbb{N}$ there is $y_n \in \mathbb{Z}^N$ such that

$$B_R(z_n) \subset B_{R+\sqrt{N}}(y_n),$$

we have

$$\int_{B_{R+\sqrt{N}}(y_n)} |u_n^+|^{2^*} dx \geq \eta > 0, \quad \forall n \in \mathbb{N},$$

finishing the proof of (2.29).

Now, assume $c \in (-\infty, S^{N/2}|k_0|_\infty^{\frac{2-N}{2}}/N) \setminus \{0\}$ and set $v_n := u_n(\cdot - y_n)$. By a simple computation, we see that $(v_n)$ is also a $(PS)_c$ sequence for $\Phi$ with

$$\limsup_{n\to+\infty} \int_{B_r(0)} |v_n^+|^{2^*}\,dx \geq \epsilon. \tag{2.30}$$

By Corollary 1.2.12, $(v_n)$ is bounded, and so, for some subsequence ( still denoted by $(v_n)$ ), $v_n \rightharpoonup v$ in $H^1(\mathbb{R}^N)$ for some $v \in H^1(\mathbb{R}^N)$. Suppose by contradiction $v = 0$ and assume that

$$|\nabla v_n|^2 \rightharpoonup \mu \qquad \text{and} \qquad |v_n|^{2^*} dx \rightharpoonup \nu \text{ in } \mathcal{M}^+(\mathbb{R}^N). \tag{2.31}$$

By Concentration-Compactness Principle II due to Lions [29], there exist a countable set $\mathcal{J}$, $(x_j)_{j\in\mathcal{J}} \subset \mathbb{R}^N$ and $(\mu_j)_{j\in\mathcal{J}}, (\nu_j)_{j\in\mathcal{J}} \subset [0,+\infty)$ such that

$$\nu = \sum_{j\in\mathcal{J}} \nu_j \delta_{x_j} \quad \mu \quad \geq \sum_{j\in\mathcal{J}} \mu_j \delta_{x_j} \quad \text{with} \quad \mu_j \geq S\nu_j^{\frac{2}{2^*}}. \tag{2.32}$$

Now, our goal is showing that $\nu_j = 0$ for all $j \in \mathcal{J}$. First of all, note that

$$c = \lim_{n\to+\infty} \left[\Phi(v_n) - \frac{1}{2}\Phi'(v_n)v_n\right] \geq \frac{1}{N}\sum_{j\in\mathcal{J}} k_0(x_j)\nu_j. \tag{2.33}$$

On the other hand, setting $\psi_\epsilon(x) := \psi((x-x_j)/\epsilon), \forall\ x \in \mathbb{R}^N, \forall\ \epsilon > 0$, where $\psi \in C_c^\infty(\mathbb{R}^N)$ is such that $\psi \equiv 1$ in $B_1(0)$, $\psi \equiv 0$ in $\mathbb{R}^N \setminus B_2(0)$ and $|\nabla\psi| \leq 2$, with $0 \leq \psi \leq 1$, we have that $\psi_\epsilon v_n \in H^1(\mathbb{R}^N)$ and $(\psi_\epsilon v_n)$ is bounded in $H^1(\mathbb{R}^N)$. So

$$\Phi'(v_n)(\psi_\epsilon v_n) \to 0$$

or equivalently

$$\int \nabla v_n \nabla(\psi_\epsilon v_n)\, dx + \int V(x)\psi_\epsilon v_n^2\, dx - \int h_0(x)\psi_\epsilon |v_n|^{q+1} dx - \int k_0(x)|v_n|^{2^*}\psi_\epsilon dx \to 0.$$

By using the definition of $\nu$ and $\mu$ together with the last limit, we derive

$$\int \nabla v(\nabla\psi_\epsilon)v\, dx + \int V(x)\psi_\epsilon v^2\, dx - \int h_0(x)\psi_\epsilon |v|^{q+1} dx + \int \psi_\epsilon d\mu - \int k_0\psi_\epsilon d\nu = 0.$$

Now, taking the limit $\epsilon \to 0$, we find

$$\mu(x_j) = k_0(x_j)\nu_j.$$

By (2.32), $\mu_j \leq \mu(x_j)$. Then,

$$S\nu_j^{2/(2^*)} = \mu_j \leq \mu(x_j) = k_0(x_j)\nu_j.$$

If $\nu_j \neq 0$, the last inequality gives

$$\nu_j \geq \frac{S^{N/2}}{|k_0|_\infty^{\frac{N-2}{2}}}. \tag{2.34}$$

Thereby, by (2.33) and (2.34), if there exists $j \in \mathcal{J}$ such that $\nu_j \neq 0$, we would have

$$c \geq \frac{S^{N/2}}{N|k_0|_\infty^{\frac{N-2}{2}}}$$

which is absurd. Hence $\nu_j = 0$ for all $j \in \mathcal{J}$, so $\nu \equiv 0$, and by (2.31), $|v_n|^{2^*} \rightharpoonup 0$ in $\mathcal{M}^+(\mathbb{R}^N)$. Consequently $v_n \to 0$ in $L^{2^*}_{loc}(\mathbb{R}^N)$ which contradicts (2.30), showing that $v \neq 0$. ■

### 1.2.2 Proof of Theorem 1.1.1: The case $N \geq 3$.

The proof will be divided into two cases, more precisely, the Periodic Case and the Asymptotically Periodic Case.

**1- The Periodic Case:**

**Proof.** From Proposition 1.2.16, there exists a $(PS)_\gamma$ sequence $(u_n)$ for $\Phi$, where $\gamma$ was given in (2.8). By Lemma 1.2.17, passing to a subsequence if necessary, $u_n \rightharpoonup u \neq 0$ and $u \in H^1(\mathbb{R}^N)$ is a solution of problem (2.7), and so, $\Phi(u) \geq \gamma$. On the other hand

$$\gamma = \lim_{n\to+\infty}\left[\Phi(u_n) - \frac{1}{2}\Phi'(u_n)(u_n)\right] = \liminf_{n\to+\infty}\left[\left(\frac{1}{2} - \frac{1}{q+1}\right)\int h(x)|u_n|^{q+1}dx\right.$$
$$\left. + \left(\frac{1}{2} - \frac{1}{2^*}\right)\int k(x)|u_n|^{2^*}dx\right] \geq \left[\left(\frac{1}{2} - \frac{1}{q+1}\right)\int h(x)|u|^{q+1}dx + \right.$$
$$\left. + \left(\frac{1}{2} - \frac{1}{2^*}\right)\int k(x)|u|^{2^*}dx\right] = \Phi(u) - \frac{1}{2}\Phi'(u)u = \Phi(u).$$

From this, $u \in H^1(\mathbb{R}^N)$ is a ground state solution for the problem (2.7). ■

**2- Asymptotically Periodic Case**

**Proof.** From definition of $\Phi_W$ and $\Phi$, we have the inequality

$$\gamma_W \leq \gamma.$$

Next, our analysis will be divide into two cases, more precisely, $\gamma_W = \gamma$ and $\gamma_W < \gamma$.

Assume firstly $\gamma_W = \gamma$. Let $u \in H^1(\mathbb{R}^N)$ be a ground state solution of (2.7) for the periodic case and $v \in \widehat{E}(u)$ such that

$$\Phi_W(v) = \sup_{\widehat{E}(u)} \Phi_W.$$

Then,

$$\gamma_W \leq \Phi_W(v) \leq \Phi(v) \leq \Phi(u) = \gamma = \gamma_W,$$

implying that $\Phi_W(v) = \gamma_W$ with $v \in \mathcal{M}$. By Corollary 1.2.15, part (c), we deduce that $v$ is a ground state solution of (2.1).

Now, assume $\gamma_W < \gamma$ and let $(u_n)$ be a $(PS)_{\gamma_W}$ sequence for $\Phi_W$ given by Proposition 1.2.16. By Lemma 1.2.10, $(u_n)$ is a bounded sequence, then for some subsequence (still denoted by $(u_n)$) $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^N)$. We claim that $u \neq 0$. Indeed, if $u = 0$ it is easy to see that

$$\int W(x)u_n^2 dx \to 0 \ \text{ and } \ \sup_{\|\psi\|\leq 1}\left|\int W(x)u_n\psi dx\right| \to 0.$$

In addiction, by $(C_1)$, we also have

$$\int h^*(x)|u_n|^{q+1}dx \to 0 \quad \text{and} \quad \sup_{\|\psi\|\leq 1}\left|\int h^*(x)|u_n|^{q-1}u\psi dx\right| \to 0.$$

Arguing as in Lemma 1.2.17, we derive that $u_n \to 0$ in $L^{2^*}_{loc}(\mathbb{R}^N)$, and so,

$$\int k^*(x)|u_n|^{2^*}dx \to 0 \quad \text{and} \quad \sup_{\|\psi\|\leq 1}\left|\int k^*(x)|u_n|^{2^*-2}u_n\psi dx\right| \to 0.$$

Hence

$$\Phi_W(u_n) \to \gamma_W \ \text{ and } \ ||\Phi'_W(u_n)|| \to 0,$$

that is, $(u_n)$ is a $(PS)_{\gamma_W}$ sequence for $\Phi_W$. By Proposition 1.2.9,

$$\gamma_W < \frac{S^{N/2}}{N|k_0|_\infty^{\frac{N-2}{2}}}.$$

Then, Proposition 1.2.17 guarantees the existence of $(y_n) \subset \mathbb{Z}^N$ such that $v_n := u_n(\cdot - y_n) \rightharpoonup v \neq 0$ in $H^1(\mathbb{R}^N)$ and $\Phi'(v) = 0$. Consequently

$$\begin{aligned}\gamma_W &= \lim_{n\to+\infty}\Phi_W(u_n) = \lim_{n\to+\infty}\Phi(u_n)\\ &= \lim_{n\to+\infty}\Phi(v_n) = \lim_{n\to+\infty}\left[\Phi(v_n) - \tfrac{1}{2}\Phi'(v_n)v_n\right]\\ &\geq \Phi(v) - \tfrac{1}{2}\Phi'(v)v = \Phi(v) \geq \gamma\end{aligned}$$

which is absurd, proving that $u \neq 0$. Now, we repeat the same argument explored in the periodic case to conclude that $u$ is a ground state solution of (2.1). ■

## 1.3 The case $N = 2$

In this section we are going to show the existence of ground state solution for the following indefinite problem

$$\begin{cases} -\Delta u + (V(x) - W(x))u = f(x,u), & \text{in } \mathbb{R}^2,\\ u \in H^1(\mathbb{R}^2), \end{cases} \tag{3.35}$$

by assuming $(V_1), (V_2), (W_1), (W_2)$ and $(f_1) - (f_6)$. Since we will work with exponential critical growth, in the next subsection we recall some facts involving this type of growth.

### 1.3.1 Results involving exponential critical growth

The exponential critical growth on $f$ is motivated by the following estimates proved by Trudinger [53] and Moser [34].

**Lemma 1.3.1 (Trudinger-Moser inequality for bounded domains)** *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain. Given any $u \in H_0^1(\Omega)$, we have*

$$\int_\Omega e^{\alpha|u|^2} dx < \infty, \quad \textit{for every } \ \alpha > 0.$$

*Moreover, there exists a positive constant $C = C(|\Omega|)$ such that*

$$\sup_{||u|| \leq 1} \int_\Omega e^{\alpha|u|^2} dx \leq C, \quad \textit{for all } \ \alpha \leq 4\pi,$$

The next result is a version of the Trudinger-Moser inequality for whole $\mathbb{R}^2$, and its proof can be found in Cao [13] ( see also Ruf [44] ).

**Lemma 1.3.2 (Trudinger-Moser inequality for unbounded domains)** *For all $u \in H^1(\mathbb{R}^2)$, we have*

$$\int \left(e^{\alpha|u|^2} - 1\right) dx < \infty, \quad \textit{for every } \ \alpha > 0.$$

*Moreover, if $|\nabla u|_2^2 \leq 1$, $|u|_2 \leq M < \infty$ and $\alpha < 4\pi$, then there exists a positive constant $C = C(M, \alpha)$ such that*

$$\int \left(e^{\alpha|u|^2} - 1\right) dx \leq C,$$

The Trudinger-Moser inequalities will be strongly utilized throughout this section in order to deduce important estimates. The reader can find more recent results involving this inequality in [15], [23], [24], [33] and references therein

In the sequel, we state some technical lemmas found in [4] and [18], which will be essential to carry out the proof of our results.

**Lemma 1.3.3** *Let $\alpha > 0$ and $t \geq 1$. Then, for every $\beta > t$, there exists a constant $C = C(\beta, t) > 0$ such that*

$$\left(e^{4\pi|s|^2} - 1\right)^t \leq C\left(e^{\beta 4\pi|s|^2} - 1\right), \quad \forall s \in \mathbb{R}.$$

**Lemma 1.3.4** *Let $(u_n)$ be a sequence such that $u_n(x) \to u(x)$ a.e. in $\mathbb{R}^2$ and $(f(x, u_n)u_n)$ is bounded in $L^1(\mathbb{R}^2)$. Then, $f(x, u_n) \to f(x, u)$ in $L^1(B_R(0))$ for all $R > 0$, and so,*

$$\int f(x, u_n)\phi \, dx \to \int f(x, u)\phi \, dx, \quad \forall \phi \in C_0^\infty(\mathbb{R}^2).$$

### 1.3.2 Technical Lemmas

In this subsection we have used the same notations of Section 2, however we will recall some of them for the convenience of the reader. In what follows, we denote by $\Phi_W : H^1(\mathbb{R}^2) \to \mathbb{R}$ the energy functional given by

$$\Phi_W(u) := \frac{1}{2}B(u,u) - \frac{1}{2}\int W(x)|u|^2 dx - \int F(x,u)dx,$$

where $B : H^1(\mathbb{R}^2) \times H^1(\mathbb{R}^2) \to \mathbb{R}$ is the bilinear form

$$B(u,v) := \int (\nabla u \nabla v + V(x)uv)dx, \quad \forall\ u,v \in H^1(\mathbb{R}^2).$$

It is well known that $\Phi_W \in C^1(H^1(\mathbb{R}^2),\mathbb{R})$ with

$$\Phi'_W(u)v = B(u,v) - \int W(x)uvdx - \int f(x,u)vdx, \quad \forall u,v \in H^1(\mathbb{R}^2).$$

Therefore critical points of $\Phi_W$ are solutions of (3.35). Moreover, we can rewrite the functional $\Phi_W$ of the form

$$\Phi_W(u) = \frac{1}{2}\|u^+\|^2 - \frac{1}{2}\|u^-\|^2 - \frac{1}{2}\int W(x)|u|^2\, dx - \int F(x,u)\, dx,$$

In what follows, we also consider the $C^1$-functional $\Phi : H^1(\mathbb{R}^2) \to \mathbb{R}$

$$\Phi(u) := \frac{1}{2}B(u,u) - \int F_0(x,u)dx$$

or equivalently

$$\Phi(u) = \frac{1}{2}\|u^+\|^2 - \frac{1}{2}\|u^-\|^2 - \int F_0(x,u)\, dx,$$

whose the critical points are weak solutions of periodic problem

$$\begin{cases} -\Delta u + V(x) = F_0(x,u), \quad \text{in } \mathbb{R}^2, \\ u \in H^1(\mathbb{R}^2) \end{cases} \tag{3.36}$$

As in Section 2, we will consider the sets

$$\mathcal{M} := \{u \in H^1(\mathbb{R}^2) \setminus E^-\ ;\ \Phi'_W(u)u = 0 \text{ and } \Phi'_W(u)v = 0, \forall\ v \in E^-\},$$

$$E(u) := E^- \oplus \mathbb{R}u \quad \text{and} \quad \hat{E}(u) := E^- \oplus [0,+\infty)u$$

Hence

$$E(u) = E^- \oplus \mathbb{R}u^+ \quad \text{and} \quad \hat{E}(u) = E^- \oplus [0,+\infty)u^+.$$

Moreover, we fix the real numbers

$$\gamma_W := \inf_{\mathcal{M}} \Phi_W \quad \text{and} \quad \gamma := \inf_{\mathcal{M}} \Phi.$$

**Lemma 1.3.5** *If $u \in \mathcal{M}$ and $w = su + v$ where $s \geq 1$ and $v \in E^-$ such that $w \neq 0$, then*

$$\Phi_W(u + w) < \Phi_W(u)$$

**Proof.** The proof follows as in Lemma 1.2.1. ■

**Lemma 1.3.6** *Let $\mathcal{K} \subset E^+ \setminus \{0\}$ be a compact subset, then there exists $R > 0$ such that $\Phi_W(w) \leq 0,\ \forall\ w \in E(u) \setminus B_R(0)$ and $u \in \mathcal{K}$.*

**Proof.** We repeat the argument used in the proof from [45, Lemma 2.2] ■

**Lemma 1.3.7** *For all $u \in H^1(\mathbb{R}^2)$, the functional $\Phi_W|_{E(u)}$ is weakly upper semicontinuous.*

**Proof.** See proof of Lemma 1.2.3. ■

**Lemma 1.3.8** *For all $u \in H^1(\mathbb{R}^2) \setminus E^-$, $\mathcal{M} \cap \hat{E}(u)$ is a singleton set and the element of this set is the unique global maximum of $\Phi_W|_{\hat{E}(u)}$*

**Proof.** See proof of Lemma 1.2.4. ■

In the proof of next lemma the fact that $f$ has an exponential critical growth brings some difficulty and we will do its proof.

**Lemma 1.3.9** *There exists $\rho > 0$ such that $\inf\limits_{B_\rho(0)\cap E^+} \Phi_W > 0$.*

**Proof.** Given $p > 2$ and $\epsilon > 0$, there is $C_\epsilon > 0$ such that

$$|F(x,t)| \leq \epsilon |t|^2 + C_\epsilon |t|^p (e^{4\pi t^2} - 1), \quad \forall (x,t) \in \mathbb{R}^2 \times \mathbb{R}.$$

Then, for all $u \in E^+$, the Lemmas 1.3.2 and 1.3.3 lead to

$$\begin{aligned} \Phi_W(u) = \ & \tfrac{1}{2}||u||^2 - \tfrac{1}{2}\textstyle\int W(x)|u|^2 dx - \int F(x,u)dx \\ & \geq \tfrac{1}{2}||u||^2 - \tfrac{\Theta}{2}\textstyle\int |u|^2 dx - \epsilon \int |u|^2 dx - C_\epsilon \int |u|^p (e^{4\pi u^2} - 1)dx \\ & = \tfrac{1}{2}||u||^2 - \tfrac{\Theta}{2\Lambda}|||u||^2 - \tfrac{\epsilon}{\Lambda}||u||^2 - C_\epsilon |u|^p_{2p} \left(\textstyle\int (e^{8\pi u^2} - 1)dx\right)^{\frac{1}{2}} \\ & \geq \left[\tfrac{1}{2}\left(1 - \tfrac{\Theta}{\Lambda}\right) - \tfrac{\epsilon}{\Lambda}\right] ||u||^2 - C||u||^p \left(\textstyle\int (e^{8\pi u^2} - 1)dx\right)^{\frac{1}{2}}. \end{aligned}$$

By Lemma 1.3.2, if $\rho < \frac{\sqrt{3}}{2\sqrt{2}}$,

$$\sup_{\|u\|=\rho} \int (e^{8\pi u^2} - 1)dx \leq \sup_{\|v\|\leq 1} \int (e^{3\pi u^2} - 1)dx = C < \infty.$$

So,

$$\Phi_W(u) \geq \left[\frac{1}{2}\left(1 - \frac{\Theta}{\overline{\Lambda}}\right) - \frac{\epsilon}{\overline{\Lambda}}\right] ||u||^2 - C||u||^p.$$

Hence, decreasing $\rho$ if necessary and fixing $\epsilon$ small enough, we get

$$\Phi_W(u) \geq \left[\frac{1}{2}\left(1 - \frac{\Theta}{\overline{\Lambda}}\right) - \frac{\epsilon}{\overline{\Lambda}}\right] \rho^2 - C\rho^p = \beta > 0.$$

■

**Lemma 1.3.10** *The real number $\gamma_W$ is positive. In addition, if $u \in \mathcal{M}$ then $||u^+|| \geq \max\{||u^-||, \sqrt{2\gamma_W}\}$.*

**Proof.** See proof of Lemma 1.2.6 ■

The next lemma shows that $(PS)$ sequences of $\Phi_W$ are bounded, as we are working with the exponential critical growth the arguments explored in Section 2 does not work in this case and a new proof must be done.

**Lemma 1.3.11** *If $(u_n)$ is a sequence such that*

$$\Phi_W(u_n) \leq d, \quad \pm\Phi'_W(u_n)u_n^{\pm} \leq d||u_n|| \quad \text{and} \quad -\Phi'_W(u_n)u_n \leq d$$

*for some $d > 0$, then $(u_n)$ is bounded in $H^1(\mathbb{R}^2)$ and $(f(u_n)u_n)$ is bounded in $L^1(\mathbb{R}^2)$.*

**Proof.** First of all, note that

$$\left(\frac{1}{2} - \frac{1}{\theta}\right) \int f(x, u_n)u_n dx \leq \Phi_W(u_n) - \frac{1}{2}\Phi'_W(u_n)u_n \leq 2d.$$

Hence, $\left(\int f(x, u_n)u_n dx\right)$ is bounded. Recalling that $f(x, t)t \geq 0$ for all $t \in \mathbb{R}$ and $x \in \mathbb{R}^2$, it follows that $(f(x, u_n)u_n)$ is bounded in $L^1(\mathbb{R}^2)$. On the other hand, we know that

$$||u_n^+||^2 \leq d||u_n^+|| + \int f(x, u_n)u_n^+ dx + \int W(x)u_n u_n^+ dx$$

and so,

$$||u_n^+||^2 \leq d||u_n^+|| + \left(\int f(x, u_n)v_n dx\right) ||u_n^+||_{H^1(\mathbb{R}^2)} + \int W(x)u_n u_n^+ dx \qquad (3.37)$$

where $v_n := \frac{u_n^+}{||u_n^+||_{H^1(\mathbb{R}^2)}}$.

**Claim 1.3.12** $\left(\int f(x, u_n)v_n dx\right)$ *is a bounded sequence.*

Indeed, by a direct computation, there exists $K > 0$ such that

$$|f(x,t)| \leq \Gamma e^{1/4} \text{ implies } |f(x,t)|^2 \leq K f(x,t)t, \quad \text{uniformly in} \quad x. \tag{3.38}$$

Moreover, by [17, Lemma 2.11],

$$rs \leq (e^{r^2} - 1) + s(log^+ s)^{1/2} + \frac{1}{4} s^2 \chi_{[0,e^{1/4}]}(s) \quad \forall r, s \geq 0. \tag{3.39}$$

Now, the Lemma 1.3.2 combined with the above inequalities for $r = |v_n|$ and $s = \frac{1}{\Gamma}|f(u_n)|$ leads to

$$\begin{aligned}
&\left|\int f(x,u_n)v_n dx\right| \leq \Gamma \int \tfrac{1}{\Gamma}|f(u_n)||v_n| dx \leq \Gamma \int (e^{v_n^2} - 1)dx + \\
&+ \int |f(x,u_n)| \left(log^+ \left(\tfrac{1}{\Gamma}|f(x,u_n)|\right)\right)^{1/2} dx + \\
&\tfrac{1}{4\Gamma} \int |f(x,u_n)|^2 \chi_{[0,e^{1/4}]} \left(\tfrac{1}{\Gamma}|f(x,u_n)|\right) dx \leq \\
&\Gamma T + \int |f(x,u_n)| \left(log^+ \left(e^{4\pi u_n^2}\right)\right)^{1/2} dx + \tfrac{1}{4\Gamma} \int_{|f(x,u_n)| \leq \Gamma e^{1/4}} |f(x,u_n)|^2 dx \leq \\
&\Gamma T + \int |f(x,u_n)||u_n|\sqrt{4\pi} dx + \tfrac{1}{4\Gamma} \int_{|f(x,u_n)| \leq \Gamma e^{1/4}} K f(x,u_n)u_n dx.
\end{aligned}$$

As $(f(x,u_n)u_n)$ is bounded in $L^1(\mathbb{R}^2)$, the last inequality yields $\left(\int f(x,u_n)v_n dx\right)$ is bounded. Consequently, there exists $A_0 > 0$ satisfying

$$\left|\int f(x,u_n)v_n dx\right| \leq A_0 \quad \forall n \in \mathbb{N}.$$

Thereby, by (3.37),

$$||u_n^+||^2 \leq d||u_n^+|| + A_0||u_n^+||_{H^1(\mathbb{R}^N)} + \int W(x)u_n u_n^+ dx. \tag{3.40}$$

Analogously, there is $B_0 > 0$ such that

$$||u_n^-||^2 \leq d||u_n^-|| + B_0||u_n^-||_{H^1(\mathbb{R}^N)} - \int W(x)u_n u_n^- dx. \tag{3.41}$$

The inequalities (3.40) and (3.41) combine to give

$$\begin{aligned}
&||u_n||^2 \leq C||u_n|| + C||u_n|| + \int W(x)(u_n u_n^+ - u_n u_n^-)dx = 2C||u_n|| + \\
&+ \int W(x)((u_n^+)^2 - (u_n^-)^2)dx \leq 2C||u_n|| + \int W(x)(u_n^+)^2 dx \leq 2C||u_n|| + \tfrac{\Theta}{\Lambda}||u_n^+||^2
\end{aligned}$$

for some $C > 0$. Hence,

$$\left(1 - \frac{\Theta}{\Lambda}\right)||u_n||^2 \leq 2\widetilde{C}||u_n||,$$

from where it follows that $(u_n)$ is bounded. ■

As a byproduct of the last lemma we have the corollary below

**Corollary 1.3.13** *$\Phi_W$ is coercive on $\mathcal{M}$, that is, $\Phi_W(u) \to +\infty$ as $||u|| \to +\infty,\ u \in \mathcal{M}$.*

As in Section 2, the Lemma 1.3.8 permits to define a function

$$m : E^+ \setminus \{0\} \to \mathcal{M} \ \text{ where } \ m(u) \in \hat{E}(u) \cap \mathcal{M} \ \ \forall u \in E^+ \setminus \{0\}.$$

Now, we invite the reader to observe that the same approach used in Section 2 works to guarantee that the proposition below holds

**Proposition 1.3.14** *There exists a $(PS)_{\gamma_W}$ sequence for $\Phi_W$.*

Our next proposition is crucial when $f$ has an exponential critical growth.

**Proposition 1.3.15** *Fixed $\widetilde{A} \in (0, 1/a)$ , there is $\lambda^* > 0$ such that $\gamma_W < \frac{\widetilde{A}^2}{2}$ for $\inf_{\mathbb{R}^2} D(x) > \lambda^*$, where $a$ was given in (2.5).*

**Proof.** Let $u \in E^+$ with $u \neq 0$ and set

$$h_D(s) := As^2 - \lambda B s^q,$$

where

$$\lambda = \inf_{x\in\mathbb{R}^2} D(x), \quad A = \frac{1}{2}||u||^2 \quad \text{and} \quad B = \xi \int |u|^q dx,$$

with $\xi$ given in Lemma 1.2.7. Then, a straightforward computation leads to

$$\max_{s\geq 0} h_D(s) = \left(A - \frac{2A}{q}\right)\left(\sqrt[q-2]{\frac{2A}{qB\lambda}}\right)^2.$$

Thereby, by $(f_6)$ and Lemma 1.2.7,

$$\begin{aligned}
c &\leq \sup_{\substack{s\in[0,+\infty)\\ v\in E^-}} \Phi_W(su+v) = \sup_{\substack{||su+v||\leq r\\ s\geq s_0,\, v\in E^-}} \Phi_W(su+v)\\
&\leq \sup_{\substack{||su+v||\leq r\\ s\geq s_0,\, v\in E^-}} \left[\tfrac{1}{2}s^2||u||^2 - \textstyle\int F(x, su+v)dx\right]\\
&\leq \sup_{\substack{||su+v||\leq r\\ s\geq s_0,\, v\in E^-}} \left[\tfrac{1}{2}s^2||u||^2 - \lambda \textstyle\int |su+v|^q dx\right]\\
&\leq \sup_{\substack{||su+v||\leq r\\ s\geq s_0,\, v\in E^-}} \left[\tfrac{1}{2}s^2||u||^2 - \lambda\xi s^q \textstyle\int |u|^q dx\right]\\
&= \sup_{\substack{||su+v||\leq r\\ s\geq s_0,\, v\in E^-}} h_D(s)\\
&\leq \max_{s\geq 0} h_D(s) = \left(A - \tfrac{2A}{q}\right)\left(\sqrt[q-2]{\tfrac{2A}{qB\lambda}}\right)^2.
\end{aligned}$$

From the last inequality there is $\lambda^* > 0$ such that

$$\left(A - \frac{2A}{q}\right)\left(\sqrt[q-2]{\frac{2A}{qB\lambda}}\right)^2 < \frac{\widetilde{A}^2}{2}, \quad \forall \lambda \geq \lambda^*,$$

finishing the proof. ■

**Proposition 1.3.16** *Fix* $\inf_{x\in\mathbb{R}^2} D(x) \geq \lambda^*$ *and* $r > 0$. *Then, there exist a sequence* $(y_n) \subset \mathbb{R}^2$ *and* $\eta > 0$ *such that*

$$\int_{B_r(y_n)} |u_n^+|^2 dx \geq \eta > 0, \quad \forall n \in \mathbb{N}.$$

*Moreover, increasing* $r$ *if necessary, the sequence* $(y_n)$ *can be chosen in* $\mathbb{Z}^2$.

**Proof.** Suppose by contradiction that the lemma does not hold for some $r > 0$. Then, by a lemma due to Lions [28],

$$u_n^+ \to 0 \ \text{ in } L^p(\mathbb{R}^2), \ \forall\, p \in (2, +\infty).$$

Define $w_n := \widetilde{A}\frac{u_n^+}{||u_n^+||}$. Since $u_n \in \mathcal{M}$ for all $n \in \mathbb{N}$, from Lemma 1.3.10 we have $\liminf_{n\in\mathbb{N}} ||u_n^+|| > 0$, and so,

$$w_n \to 0 \ \text{ in } L^p(\mathbb{R}^2), \ \forall\, p \in (2, +\infty).$$

On the other hand, we also know that

$$||w_n||_{H^1(\mathbb{R}^2)} = \widetilde{A}\frac{||u_n^+||_{H^1(\mathbb{R}^2)}}{||u_n^+||} \leq \widetilde{A}a\frac{||u_n^+||}{||u_n^+||} = \widetilde{A}a < 1.$$

As $w_n \in \widehat{E}(u_n)$ and $u_n \in \mathcal{M}$, we derive that

$$\Phi(u_n) \geq \Phi(w_n) = \frac{1}{2}\widetilde{A}^2 - \int F(x, w_n)dx. \tag{3.42}$$

By [2, Proposition 2.3], we have $\int F(x, w_n)dx \to 0$. Therefore, passing to the limit in (3.42) as $n \to +\infty$, we obtain

$$\gamma_W \geq \frac{\widetilde{A}^2}{2},$$

which contradicts the Proposition 1.3.15. Thus, there are $(z_n) \subset \mathbb{R}^2$ and $\eta > 0$ such that

$$\int_{B_r(z_n)} |u_n^+|^2 dx \geq \eta > 0, \quad \forall n \in \mathbb{N}.$$

Now, we repeat the same idea explored in Lemma 1.2.17 to conclude the proof. ■

### 1.3.3 Proof of Theorem 1.1.1: The case $N = 2$.

As in Section 2, the proof will be divided into two cases, the Periodic Case and the Asymptotically Periodic Case.

### 1.3.4 Periodic Case

**Proof.** First of all, we recall there is a $(PS)_{\gamma_W}$ sequence $(u_n)$ for $\Phi$ which must be bounded. Thus, there is $u \in H^1(\mathbb{R}^2)$ such that for some subsequence of $(u_n)$, still denoted by itself, we have

$$u_n \rightharpoonup u \quad \text{in} \quad H^1(\mathbb{R}^2)$$

and

$$u_n(x) \to u(x) \quad \text{a.e. in} \quad \mathbb{R}^2.$$

Moreover, by Lemma 1.3.11 the sequence $(f(x, u_n)u_n)$ is bounded in $L^1(\mathbb{R}^2)$. Therefore, by Lemma 1.3.4,

$$\Phi'(u)\phi = 0, \quad \forall \phi \in C_0^\infty(\mathbb{R}^2).$$

If we combine the Lemma 1.3.2 with the density of $C_0^\infty(\mathbb{R}^2)$ in $H^1(\mathbb{R}^2)$, we see that $u$ is a critical point of $\Phi$, that is,

$$\Phi'(u)v = 0, \quad \forall v \in H^1(\mathbb{R}^2).$$

Moreover, by Fatou's Lemma, we also have

$$\Phi(u) \leq \gamma.$$

If $u \neq 0$, we must have

$$\Phi(u) \geq \gamma,$$

showing that $\Phi(u) = \gamma$, and so, $u$ is a ground state solution.

If $u = 0$, we can apply Lemma 1.3.16 to get a sequence $(y_n) \subset \mathbb{Z}^2$ and real numbers $r, \eta > 0$ verifying

$$\int_{B_r(y_n)} |u_n^+|^2 dx \geq \eta > 0, \quad \forall n \in \mathbb{N}.$$

Setting $v_n(x) = u_n(x + y_n)$, a direct computation gives that $(v_n)$ is also a $(PS)_\gamma$ for $\Phi$. Moreover, for some subsequence, there is $v \in H^1(\mathbb{R}^2)$ such that

$$v_n \rightharpoonup v \quad \text{in} \quad H^1(\mathbb{R}^2) \quad \text{and} \quad \int_{B_r(0)} |v^+|^2 dx \geq \eta > 0,$$

showing that $v \neq 0$. Therefore, arguing as above, $v$ is a ground state solution for $\Phi$. ■

### 1.3.5 The Asymptotically Periodic Case

**Proof.** First of all, we recall that $\Phi_W \leq \Phi$, and so, $\gamma_W \leq \gamma$. As in Section 2, we will consider the cases $\gamma_W = \gamma$ and $\gamma_W < \gamma$. The first one follows as in Section 2, and we will omit its proof.

In what follows, we are considering $\gamma_W < \gamma$ and $(u_n)$ be a $(PS)_{\gamma_W}$ sequence for $\Phi_W$ which was given in Lemma 1.3.14. The sequence $(u_n)$ is bounded by Lemma 1.3.11. Thus, there is $u \in H^1(\mathbb{R}^2)$ and a subsequence of $(u_n)$, still denoted by itself, such that $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^2)$. Suppose by contradiction $u = 0$. Repeating the arguments explored in the case $N \geq 3$, we have

$$\int W(x)|u_n|^2 dx \to 0 \quad \text{and} \quad \sup_{||\psi|| \leq 1} \left| \int W(x) u_n \psi dx \right| \to 0.$$

From $(f_1)$, given $\epsilon > 0$ and $\beta > 0$ such that

$$\beta < \frac{2\pi}{\sup_{n \in \mathbb{N}} ||u_n||^2},$$

it must exist $\eta > 0$ satisfying

$$|f^*(x,s)| \leq \epsilon(e^{\beta s^2} - 1) \ \text{ for } \ |t| \geq \eta \quad \text{and} \quad x \in \mathbb{R}^2 \setminus B_\eta(0).$$

Therefore, by Lemma 1.3.2

$$\begin{aligned}&\textstyle\int_{[|x| \geq \eta] \cap [|u_n| \geq \eta]} |f^*(x,u_n)||\psi| dx \leq \int_{[|x| \geq \eta] \cap [|u_n| \geq \eta]} \epsilon |e^{\beta u_n^2} - 1||\psi| dx \leq \\ &\leq \epsilon \left( \int_{\mathbb{R}^2} |e^{\beta u_n^2} - 1|^2 dx \right)^{1/2} \left( \int_{\mathbb{R}^2} |\psi|^2 dx \right)^{1/2} dx \leq \epsilon K ||\psi||_{H^1(\mathbb{R}^2)}.\end{aligned}$$

On the other hand, fixing $R$ large enough

$$\begin{aligned}\int_{[|x| \geq R] \cap [|u_n| \leq \eta]} |f^*(x,u_n)||\psi| dx &\leq C \int_{|x| \geq R} H(x)|\psi| dx \\ &\leq \left( \int_{|x| \geq R} |H(x)|^2 dx \right)^{1/2} \left( \int_{\mathbb{R}^2} |\psi|^2 dx \right)^{1/2} \\ &\leq \epsilon C ||\psi||_{H^1(\mathbb{R}^2)}.\end{aligned}$$

Thus,

$$\sup_{\|\psi\|\leq 1}\left|\int_{|x|\geq\eta} f^*(x,u_n)\psi\,dx\right| \leq \epsilon(C+K)||\psi||_{H^1(\mathbb{R}^2)}.$$

Now, as $f^*$ has a subcritical growth and $u_n \to 0$ in $L^2(B_\eta(0))$, we have that

$$\sup_{\|\psi\|\leq 1}\left|\int_{|x|\leq\eta} f^*(x,u_n)\psi\,dx\right| \to 0.$$

Thus,

$$\sup_{\|\psi\|\leq 1}\left|\int_{\mathbb{R}^2} f^*(x,u_n)\psi\,dx\right| \to 0.$$

A similar argument works to prove that

$$0 \leq \int F^*(x,u_n)dx \leq \int f^*(x,u_n)u_n dx \to 0.$$

The above limits yield

$$\Phi(u_n) \to \gamma_W \quad \text{and} \quad ||\Phi'(u_n)|| \to 0.$$

Arguing as in the periodic case, without loss of generality, we can assume that

$$u_n \rightharpoonup u \quad \text{in} \quad H^1(\mathbb{R}^2), u \neq 0 \ \text{ and } \ \Phi'(u) = 0.$$

Thus, $\Phi(u) \geq \gamma$. On the other hand, by Fatou's Lemma,

$$\Phi(u) \leq \liminf_{n\to+\infty} \Phi(u_n) = \gamma_W,$$

which is absurd, because we are supposing $\gamma_W < \gamma$. Thereby, $u \neq 0$ and since $(f(x,u_n)u_n)$ is bounded in $L^1(\mathbb{R}^2)$, we can conclude that $u$ is a ground state solution of $\Phi_W$. ■

# Capítulo 2

# Existência e concentração de soluções de energia mínima para uma classe de problemas variacionais indefinidos

## Existence and concentration of ground state solution for a class of indefinite variational problems

CLAUDIANOR O. ALVES and GEILSON F. GERMANO


## Abstract

In this paper we study the existence and concentration of solution for a class of strongly indefinite problem like

$$\begin{cases} -\Delta u + V(x)u = A(\epsilon x)f(u) \quad \text{in} \quad \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \end{cases} \qquad (P)_\epsilon$$

where $N \geq 1$, $\epsilon$ is a positive parameter, $f : \mathbb{R} \to \mathbb{R}$ is a continuous function with subcritical growth and $V, A : \mathbb{R}^N \to \mathbb{R}$ are continuous functions verifying some technical conditions. Here $V$ is a $\mathbb{Z}^N$-periodic function, $0 \notin \sigma(-\Delta + V)$, the spectrum of $-\Delta + V$, and

$$0 < \inf_{x \in \mathbb{R}^N} A(x) \leq \lim_{|x| \to +\infty} A(x) < \sup_{x \in \mathbb{R}^N} A(x).$$

**Mathematics Subject Classifications (2010):** 35B40, 35J2, 47A10 .

**Keywords:** concentration of solutions, variational methods, indefinite strongly functional.

## 2.1 Introduction

This paper concerns with the existence and concentration of ground state solution for the semilinear Schrödinger equation

$$\begin{cases} -\Delta u + V(x)u = A(\epsilon x)f(u) \quad \text{in} \quad \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \end{cases} \tag{$(P)_\epsilon$}$$

where $N \geq 1$, $\epsilon$ is a positive parameter, $f : \mathbb{R} \to \mathbb{R}$ is a continuous function with subcritical growth and $V, A : \mathbb{R} \to \mathbb{R}$ are continuous functions verifying some technical conditions.

In whole this paper, $V$ is $\mathbb{Z}^N$-periodic with

$$0 \notin \sigma(-\Delta + V), \quad \text{the spectrum of} \quad -\Delta + V, \tag{$V_1$}$$

which becomes the problem strongly indefinite. Related to the function $A$, we assume that it is a continuous function satisfying

$$0 < A_0 = \inf_{x \in \mathbb{R}^N} A(x) \leq A_\infty = \lim_{|x| \to +\infty} A(x) < \sup_{x \in \mathbb{R}^N} A(x). \tag{$A_1$}$$

The present article has as first motivation some recent articles that have studied the existence of ground state solution for related problems with $(P)_\epsilon$, more precisely for strongly indefinite problems of the type

$$\begin{cases} -\Delta u + V(x)u = f(x,u), \quad \text{in} \quad \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N). \end{cases} \tag{$P_1$}$$

In [27], Kryszewski and Szulkin have studied the existence of ground state solution for $(P_1)$ by supposing the condition $(V_1)$. Related to the function $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$, they assumed that $f$ is continuous, $\mathbb{Z}^N$-periodic in $x$ with

$$|f(x,t)| \leq c(|t|^{q-1} + |t|^{p-1}), \quad \forall t \in \mathbb{R} \quad \text{and} \quad x \in \mathbb{R}^N \tag{$h_1$}$$

and

$$0 < \alpha F(x,t) \leq t f(x,t) \quad \forall (x,t) \in \mathbb{R}^N \times \mathbb{R}^*, \quad F(x,t) = \int_0^t f(x,s)\,ds \qquad (h_2)$$

for some $c > 0$, $\alpha > 2$ and $2 < q < p < 2^*$ where $2^* = \frac{2N}{N-2}$ if $N \geq 3$ and $2^* = +\infty$ if $N = 1, 2$. The above hypotheses guarantee that the energy functional associated with $(P_1)$ given by

$$J(u) = \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)|u|^2\,dx) - \int_{\mathbb{R}^N} F(x,u)\,dx, \ \forall u \in H^1(\mathbb{R}^N),$$

is well defined and belongs to $C^1(H^1(\mathbb{R}^N), \mathbb{R})$. By $(V_1)$, there is an equivalent inner product $\langle \ , \ \rangle$ in $H^1(\mathbb{R}^N)$ such that

$$J(u) = \frac{1}{2}\|u^+\|^2 - \frac{1}{2}\|u^-\|^2 - \int_{\mathbb{R}^N} F(x,u)\,dx,$$

where $\|u\| = \sqrt{\langle u, u\rangle}$ and $H^1(\mathbb{R}^N) = E^+ \oplus E^-$ corresponds to the spectral decomposition of $-\Delta + V$ with respect to the positive and negative part of the spectrum with $u = u^+ + u^-$, where $u^+ \in E^+$ and $u^- \in E^-$. In order to show the existence of solution for $(P_1)$, Kryszewski and Szulkin introduced a new and interesting generalized link theorem. In [31], Li and Szulkin have improved this generalized link theorem to prove the existence of solution for a class of strongly indefinite problem with $f$ being asymptotically linear at infinity.

The Link theorems above mentioned have been used in a lot of papers, we would like to cite Chabrowski and Szulkin [14], do Ó and Ruf [17], Furtado and Marchi [20], Tang [51, 52] and their references.

Pankov and Pflüger [39] also have considered the existence of solution for problem $(P_1)$ with the same conditions considered in [27], however the approach is based on an approximation technique of periodic function together with the linking theorem due to Rabinowitz [40]. After, Pankov [38] has studied the existence of solution for problems of the type

$$\begin{cases} -\Delta u + V(x)u = \pm f(x,u), \quad \text{in} \quad \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \end{cases} \qquad (P_2)$$

by supposing $(V_1)$, $(h_1) - (h_2)$ and employing the same approach explored in [39]. In [38] and [39], the existence of ground state solution has been established by supposing that $f$ is $C^1$ and there is $\theta \in (0,1)$ such that

$$0 < t^{-1} f(x,t) \leq \theta f'_t(x,t), \quad \forall t \neq 0 \quad \text{and} \quad x \in \mathbb{R}^N. \qquad (h_3)$$

However, in [38], Pankov has found a ground state solution by minimizing the energy functional $J$ on the set

$$\mathcal{O} = \left\{ u \in H^1(\mathbb{R}^N) \setminus E^- \ ; \ J'(u)u = 0 \text{ and } J'(u)v = 0, \forall \ v \in E^- \right\}.$$

The reader is invited to see that if $J$ is strongly definite, that is, when $E^- = \{0\}$, the set $\mathcal{O}$ is exactly the Nehari manifold associated with $J$. Hereafter, we say that $u_0 \in H^1(\mathbb{R}^N)$ is a *ground state solution* if

$$J'(u_0) = 0, \quad u_0 \in \mathcal{O} \quad \text{and} \quad J(u_0) = \inf_{w \in \mathcal{O}} J(w).$$

In [45], Szulkin and Weth have established the existence of ground state solution for problem $(P_1)$ by completing the study made in [38], in the sense that, they also minimize the energy functional on $\mathcal{O}$, however they have used more weaker conditions on $f$, for example $f$ is continuous, $\mathbb{Z}^N$-periodic in $x$ and satisfies

$$|f(x,t)| \leq C(1 + |t|^{p-1}), \ \ \forall t \in \mathbb{R} \quad \text{and} \quad x \in \mathbb{R}^N \tag{$h_4$}$$

for some $C > 0$ and $p \in (2, 2^*)$.

$$f(x,t) = o(t) \ \text{ uniformly in } \ x \text{ as } |t| \to 0. \tag{$h_5$}$$

$$F(x,t)/|t|^2 \to +\infty \ \text{ uniformly in } \ x \text{ as } |t| \to +\infty, \tag{$h_6$}$$

and

$$t \mapsto f(x,t)/|t| \ \text{ is strictly increasing on } \ \mathbb{R} \setminus \{0\}. \tag{$h_7$}$$

The same approach has been used by Zhang, Xu and Zhang [60, 61] to study a class of indefinite and asymptotically periodic problem.

After a review bibliography, we have observed that there are no papers involving strongly indefinite problem whose the nonlinearity is of the form

$$f(x,t) = A(\epsilon x) f(t), \quad \forall x \in \mathbb{R}^N \quad \text{and} \quad \forall t \in \mathbb{R},$$

with $A$ verifying the condition $(A_1)$ and $\epsilon > 0$. The motivation to consider this type of nonlinearity comes from many studies involving the existence and concentration of standing-wave solutions for the nonlinear Schrödinger equation

$$i\epsilon \frac{\partial \Psi}{\partial t} = -\epsilon^2 \Delta \Psi + (V(x) + E)\Psi - f(\Psi) \ \text{ for all } \ x \in \mathbb{R}^N, \tag{$NLS$}$$

where $N \geq 1$, $\epsilon > 0$ is a parameter and $V, f$ are continuous functions verifying some conditions. This class of equation is one of the main objects of the quantum physics, because it appears in problems that involve nonlinear optics, plasma physics and condensed matter physics.

Knowledge of the solutions for the elliptic equation like

$$\begin{cases} -\epsilon^2 \Delta u + V(x)u = f(u) \text{ in } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \end{cases} \tag{$(S)_\epsilon$}$$

or equivalently

$$\begin{cases} -\Delta u + V(\epsilon x)u = f(u) \text{ in } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \end{cases} \tag{$(S')_\epsilon$,}$$

has a great importance in the study of standing-wave solutions of $(NLS)$. In recent years, the existence and concentration of positive solutions for general semilinear elliptic equations $(S)_\epsilon$ have been extensively studied, see for example, Floer and Weinstein [19], Oh [36, 37], Rabinowitz [42], Wang [54], Ambrosetti and Malchiodi [11], Ambrosetti, Badiale and Cingolani [12], del Pino and Felmer [16] and their references.

In some of the above mentioned papers, the existence, multiplicity and concentration of positive solutions have been obtained in connection with the geometry of the potential $V$ by supposing that

$$\inf(\sigma(-\Delta + V)) > 0.$$

By using the above condition, we have that the problem is strongly definite, which permits to show, in some cases, that the energy functional satisfies the mountain pass geometry and that the mountain pass level is a critical level. In some papers it was proved that the maximum points of the solutions are close to the set

$$\mathcal{V} = \left\{ x \in \mathbb{R}^N \;:\; V(x) = \min_{z \in \mathbb{R}^N} V(z) \right\},$$

when $\epsilon$ is small enough. Moreover, in a lot of problems, the multiplicity of solutions is associated with the topology richness of $\mathcal{V}$.

In [42], by a mountain pass argument, Rabinowitz proved the existence of positive solutions of $(S)_\epsilon$, for $\epsilon > 0$ small, whenever

$$\liminf_{|x| \to \infty} V(x) > \inf_{x \in \mathbb{R}^N} V(x) = V_0 > 0.$$

Later Wang [54] showed that these solutions concentrate at global minimum points of $V$ as $\epsilon$ tends to 0.

In [16], del Pino and Felmer have found solutions that concentrate around local minimum of $V$ by introducing of a penalization method. More precisely, they assume that

$$V(x) \geq \inf_{z \in \mathbb{R}^N} V(z) = V_0 > 0 \text{ for all } \; x \in \mathbb{R}^N$$

and there is an open and bounded set $\Omega \subset \mathbb{R}^N$ such that

$$\inf_{x \in \Omega} V(x) < \min_{x \in \partial\Omega} V(x).$$

Here, we intend to study the existence of standing-wave solutions for $(NSL)$ by supposing $h = 1$ and $g$ be a function of the type

$$g(x,t) = A(\epsilon x) f(t),$$

where $\epsilon$ is a positive number with $V, A$ satisfying the conditions $(V_1)$ and $(A_1)$ respectively. More precisely, we will prove the existence of ground state solution $u_\epsilon$ for $(P)_\epsilon$ when $\epsilon$ is small enough. After, we study the concentration of the maximum points of $|u_\epsilon|$ with related to the set of maximum points of $A$. We would like point out that one of the main difficulties is the loss of the mountain pass geometry, because we are working with a strongly indefinite problem. Then, if $I_\epsilon$ denotes the energy functional associated with $(P)_\epsilon$, we were taken to do a careful study involving the behavior of number $c_\epsilon$ given by

$$c_\epsilon = \inf_{u \in \mathcal{M}_\epsilon} I_\epsilon(u) \tag{1.1}$$

where

$$\mathcal{M}_\epsilon = \left\{ u \in H^1(\mathbb{R}^N) \setminus E^- \; ; \; I'_\epsilon(u)u = 0 \text{ and } I'_\epsilon(u)v = 0, \forall \; v \in E^- \right\}. \tag{1.2}$$

The understanding of the behavior of $c_\epsilon$ is a key point in our approach to show the existence and concentration of ground state solution when $\epsilon$ is small enough.

Hereafter, $f : \mathbb{R} \to \mathbb{R}$ is a continuous function that verifies the following assumptions:

$(f_1)$ $\dfrac{f(t)}{t} \to 0$ as $t \to 0$.

$(f_2)$ $\limsup_{|t|\to+\infty} \frac{|f(t)|}{|t|^q} < +\infty$ for some $q \in (1, 2^* - 1)$.

$(f_3)$ $t \mapsto f(t)/t$ is increasing on $(0, +\infty)$ and decreasing on $(-\infty, 0)$.

$(f_4)$ (Ambrosetti-Rabinowitz) There exists $\theta > 2$ such that

$$0 < \theta F(t) \leq f(t)t, \ \forall \ t \neq 0$$

where $F(t) := \int_0^t f(s)ds$.

Our main theorem is the following

**Theorem 2.1.1** *Suppose that $(V_1), (A_1)$ and $(f_1)-(f_4)$ hold. Then, there exists $\epsilon_0 > 0$ such that $(P)_\epsilon$ has a ground state solution $u_\epsilon$ for all $\epsilon \in (0, \epsilon_0)$. Moreover, if $x_\epsilon \in \mathbb{R}^N$ denotes a global maximum point of $|u_\epsilon|$, then*

$$\lim_{\epsilon\to 0} A(\epsilon x_\epsilon) = \sup_{x\in\mathbb{R}^N} A(x).$$

The plan of the paper is as follows: In Section 2 we do a study involving the autonomous problem. In Section 3 we show the existence of ground state solution for $\epsilon$ small, while in Section 4 we establish the concentration phenomena.

**Notation.** In this paper, we use the following notations:

- $o_n(1)$ denotes a sequence that converges to zero.
- If $g$ is a mensurable function, the integral $\int_{\mathbb{R}^N} g(x)\, dx$ will be denoted by $\int g(x)\, dx$.
- $B_R(z)$ denotes the open ball with center $z$ and radius $R$ in $\mathbb{R}^N$.
- The usual norms in $H^1(\mathbb{R}^N)$ and $L^p(\mathbb{R}^N)$ will be denoted by $\| \ \|_{H^1(\mathbb{R}^N)}$ and $| \ |_p$ respectively.
- For each $u \in H^1(\mathbb{R}^N)$, the equality $u = u^+ + u^-$ yields $u^+ \in E^+$ and $u^- \in E^-$.

## 2.2 Some results involving the autonomous problem.

Consider the following autonomous problem

$$\begin{cases} -\Delta u + V(x)u = \lambda f(u) & \text{in} \quad \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \end{cases} \qquad (AP)_\lambda$$

where $\lambda > 0$ and $V, f$ verify the conditions $(V_1)$ and $(f_1)-(f_4)$ respectively. Associated with $(AP)_\lambda$ we have the energy functional $J_\lambda : H^1(\mathbb{R}^N) \to \mathbb{R}$ given by

$$J_\lambda(u) = \frac{1}{2}\int (|\nabla u|^2 + V(x)|u|^2\,dx) - \lambda \int F(u)\,dx,$$

or equivalently

$$J_\lambda(u) = \frac{1}{2}\|u^+\|^2 - \frac{1}{2}\|u^-\|^2 - \lambda \int F(u)\,dx.$$

In what follows, let us denote by $d_\lambda$ the real number defined by

$$d_\lambda = \inf_{u \in \mathcal{N}_\lambda} J_\lambda(u); \tag{2.3}$$

where

$$\mathcal{N}_\lambda = \left\{u \in H^1(\mathbb{R}^N) \setminus E^- \ ;\ J'_\lambda(u)u = 0 \text{ and } J'_\lambda(u)v = 0, \forall\ v \in E^-\right\}. \tag{2.4}$$

Moreover, for each $u \in H^1(\mathbb{R}^N)$, the sets $E(u)$ and $\hat{E}(u)$ designate

$$E(u) = E^- \oplus \mathbb{R}u \quad \text{and} \quad \hat{E}(u) = E^- \oplus [0, +\infty)u. \tag{2.5}$$

The reader is invited to observe that $E(u)$ and $\hat{E}(u)$ are independent of $\lambda$, more precisely they depend on only of the operator $-\Delta + V$. This remark is very important because these sets will be used in the next sections.

In [45], Szulkin and Weth have proved that for each $\lambda > 0$, the problem $(AP)_\lambda$ possesses a ground state solution $u_\lambda \in H^1(\mathbb{R}^N)$, that is,

$$u_\lambda \in \mathcal{N}_\lambda, \quad J_\lambda(u_\lambda) = d_\lambda \quad \text{and} \quad J'_\lambda(u) = 0.$$

In the above mentioned paper, the authors also proved that

$$0 < d_\lambda = \inf_{u \in E^+ \setminus \{0\}} \max_{v \in \widehat{E}(u)} J_\lambda(u). \tag{2.6}$$

Moreover, an interesting and important fact is that for each $u \in H^1(\mathbb{R}^N) \setminus E^-$, $\mathcal{N}_\lambda \cap \hat{E}(u)$ is a singleton set and the element of this set is the unique global maximum of $J_\lambda|_{\hat{E}(u)}$, that is, there are $t^* \geq 0$ and $v^* \in E^-$ such that

$$J_\lambda(t^*u + v^*) = \max_{w \in \widehat{E}(u)} J_\lambda(w). \tag{2.7}$$

The next two lemmas will be used in the study of the behavior of $d_\lambda$ and $c_\epsilon$.

**Lemma 2.2.1** *For all $u = u^+ + u^- \in H^1(\mathbb{R}^N)$ and $y \in \mathbb{Z}^N$, if $u_y(x) := u(x+y)$ then $u_y \in H^1(\mathbb{R}^N)$ with $u_y^+(x) = u^+(x+y)$ and $u_y^-(x) = u^-(x+y)$.*

**Proof.** Define

$$\begin{array}{rccc} T: & H^1(\mathbb{R}^N) & \to & H^1(\mathbb{R}^N) \\ & u & \mapsto & u_y \end{array}$$

such that $u_y(x) := u(x+y)$ for all $x \in \mathbb{R}^N$. A direct computation gives $T(E^+) \subset E^+$ and $T(E^-) \subset E^-$. Consequently,

$$u(x+y) = u^+(x+y) + u^-(x+y)$$

or equivalently

$$T(u) = T(u^+) + T(u^-).$$

Since $T(u^+) \in E^+$ and $T(u^-) \in E^-$, we derive that $T(u)^+ = T(u^+)$ and $T(u)^- = T(u^-)$, obtaining the desired result. ■

The next lemma is a weak version of [45, Lemma 2.5].

**Lemma 2.2.2** *Let $\mathcal{V} \subset E^+ \setminus \{0\}$ be a bounded set with $0 \notin \overline{\mathcal{V}}^{\sigma(H^1(\mathbb{R}^N), H^1(\mathbb{R}^N)')}$, $W \in C(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ with $\inf_{x \in \mathbb{R}^N} W(x) = W_0 > 0$ and $F : \mathbb{R} \to \mathbb{R}$ be a continuous function verifying*

*(i)* $\frac{F(t)}{t^2} \to +\infty$ *as* $|t| \to +\infty$.

*(ii)* $F(t) \geq 0$ *for all* $t \in \mathbb{R}$.

*For the functional $\varphi : H^1(\mathbb{R}^N) \to \mathbb{R} \cup \{-\infty\}$ given by*

$$\varphi(u) = \frac{1}{2}||u^+||^2 - \frac{1}{2}||u^-||^2 - \int W(x)F(u)dx,$$

*there exists $R > 0$ such that $\varphi(u) < 0$ on $\widehat{E}(u) \setminus B_R(0)$, for all $u \in \mathcal{V}$.*

**Proof.** Suppose by contradiction that there exist $(u_n) \subset \mathcal{V}$ and $(w_n) \subset \widehat{E}(u_n) \setminus B_n(0)$ with $\varphi(w_n) \geq 0$. First of all, note that $\varphi(w_n) \geq 0$ implies that

$$0 \leq \int W(x)F(w_n)\,dx < +\infty, \quad \text{for all } n \in \mathbb{N}.$$

As $||w_n|| \to +\infty$, we set $v_n := \frac{w_n}{||w_n||} \in \widehat{E}(u_n)$. Then, there is $s_n \geq 0$ such that

$$v_n = s_n u_n + v_n^-.$$

Consequently $w_n = ||w_n|| s_n u_n + ||w_n|| v_n^-$ and

$$0 \leq \frac{\varphi(w_n)}{||w_n||^2} = \frac{1}{2} s_n^2 ||u_n||^2 - \frac{1}{2} ||v_n^-||^2 - \int \frac{W(x)F(w_n)}{||w_n||^2} dx. \tag{2.8}$$

From this, $s_n u_n \not\to 0$. In fact, otherwise, $s_n ||u_n|| \to 0$ leads to

$$0 \leq \frac{1}{2} ||v_n^-||^2 + \int \frac{W(x)F(w_n)}{||w_n||^2} dx \leq \frac{1}{2} s_n^2 ||u_n||^2 \to 0.$$

Therefore $v_n^- \to 0$ and $v_n = s_n u_n + v_n^- \to 0$, which is absurd, because $||v_n|| = 1$ for all $n \in \mathbb{N}$. Thereby, $s_n u_n \not\to 0$. As $(u_n)$ is bounded, we have $s_n \not\to 0$. On the other hand, since $0 \notin \overline{\mathcal{V}}^{\sigma(H^1(\mathbb{R}^N), H^1(\mathbb{R}^N)')}$, it follows that $u_n \not\rightharpoonup 0$, and so, $u_n \not\to 0$. Since $s_n^2 ||u_n||^2 \leq ||v_n||^2 = 1$, we conclude that $s_n \not\to +\infty$. Thus, for some subsequence, $s_n \to s \neq 0$, $u_n \rightharpoonup u \neq 0$ and

$$v_n = s_n u_n + v_n^- \rightharpoonup v = su + v^- \neq 0.$$

So, by Fatou's Lemma,

$$\int \frac{W(x)F(w_n)}{||w_n||^2} dx \geq \int \frac{W(x)F(w_n)}{|w_n|^2} |v_n|^2 dx \geq \int_{[v \neq 0]} \frac{W(x)F(w_n)}{|w_n|^2} |v_n|^2 dx \to +\infty,$$

contradicting (2.8). ■

After the above commentaries we are ready to prove the main result this section.

**Proposition 2.2.3** *The function $\lambda \mapsto d_\lambda$ is decreasing and continuous on $(0, +\infty)$.*

**Proof.** In the sequel, $u_\lambda$ and $u_\mu$ denote a ground state solution for $J_\lambda$ and $J_\mu$ respectively. Note that if $\lambda > \mu$, then

$$J_\mu(u) - J_\lambda(u) = (\lambda - \mu) \int F(u)\, dx \geq 0, \quad \forall u \in H^1(\mathbb{R}^N).$$

Hence

$$d_\lambda = \inf_{u \in E^+ \setminus \{0\}} \max_{v \in \widehat{E}(u)} J_\lambda(u) \leq \inf_{u \in E^+ \setminus \{0\}} \max_{v \in \widehat{E}(u)} J_\mu(u) = d_\mu,$$

showing that $\lambda \mapsto d_\lambda$ is monotone non-creasing. We claim that $d_\lambda < d_\mu$. Indeed, suppose $d_\lambda = d_\mu$ and let $t_\mu \geq 0$ and $v_\mu \in E^-$ satisfying

$$J_\lambda(t_\mu u_\mu + v_\mu) = \max_{u \in \widehat{E}(u_\mu)} J_\lambda(u). \quad \text{(see (2.7))}$$

Therefore,

$$\begin{aligned} d_\lambda & \leq J_\lambda(t_\mu u_\mu + v_\mu) = (\mu - \lambda) \int F(t_\mu u_\mu + v_\mu)\, dx + J_\mu(t_\mu u_\mu + v_\mu) \\ & \leq (\mu - \lambda) \int F(t_\mu u_\mu + v_\mu)\, dx + J_\mu(u_\mu) \\ & = (\mu - \lambda) \int F(t_\mu u_\mu + v_\mu)\, dx + d_\mu. \end{aligned}$$

As $d_\lambda = d_\mu$, it follows that

$$(\mu - \lambda) \int F(t_\mu u_\mu + v_\mu)\, dx \geq 0.$$

By using the fact that $\lambda > \mu$ and $(f_4)$, we get $t_\mu u_\mu + v_\mu = 0$ a.e. in $\mathbb{R}^N$, and so, $d_\lambda \leq J_\lambda(t_\mu u_\mu + v_\mu) = 0$, contradicting (2.6) . From this, the function $\lambda \mapsto d_\lambda$ is injective and decreasing.

Now we are going to prove the continuity of $\lambda \mapsto d_\lambda$. To this end, we will divide into two steps our arguments:

**Step 1:** Let $(\lambda_n)$ be a sequence with $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n \to \lambda$. Our goal is to prove that $\lim_{n\to+\infty} d_{\lambda_n} = d_\lambda$. Since $\lambda \mapsto d_\lambda$ is decreasing then $d_\lambda \leq d_{\lambda_n}$, $\forall\, n \in \mathbb{N}$. For each $n \in \mathbb{N}$, let us fix $t_n \geq 0$ and $v_n \in E^-$ verifying

$$J_{\lambda_n}(t_n u_\lambda + v_n) = \max_{u \in \widehat{E}(u_\lambda)} J_{\lambda_n}(u).$$

From Lemma 2.2.2, there exists $R > 0$ such that $J_{\lambda_1}(u) \leq 0$ for all $u \in \widehat{E}(u_\lambda) \setminus B_R(0)$. Recalling that $J_{\lambda_n} \leq J_{\lambda_1}$, we have

$$J_{\lambda_n}(u) \leq 0,\ \forall\, u \in \widehat{E}(u_\lambda) \setminus B_R(0) \quad \text{and} \quad \forall\, n \in \mathbb{N}. \tag{2.9}$$

On the other hand $J_{\lambda_n}(t_n u_\lambda + v_n) = \max_{u \in \widehat{E}(u_\lambda)} J_{\lambda_n}(u) \geq d_{\lambda_n} \geq d_\lambda > 0$, i. e.,

$$J_{\lambda_n}(t_n u_\lambda + v_n) > 0, \quad \forall\, n \in \mathbb{N}. \tag{2.10}$$

By (2.9) and (2.10), $||t_n u_\lambda + v_n|| \leq R$ for all $n \in \mathbb{N}$. Then, $(t_n u_\lambda + v_n)$ is bounded in $H^1(\mathbb{R}^N)$ and

$$\begin{aligned} d_{\lambda_n} & \leq J_{\lambda_n}(t_n u_\lambda + v_n) \\ & = (\lambda - \lambda_n) \int F(t_n u_\lambda + v_n) dx + J_\lambda(t_n u_\lambda + v_n) \\ & \leq (\lambda - \lambda_n) \int F(t_n u_\lambda + v_n) dx + J_\lambda(u_\lambda) = o_n(1) + d_\lambda. \end{aligned}$$

From this,

$$d_{\lambda_n} \leq o_n(1) + d_\lambda \quad \text{and} \quad d_\lambda \leq d_{\lambda_n}, \quad \forall n \in \mathbb{N},$$

implying that $\lim_{n\to+\infty} d_{\lambda_n} = d_\lambda$.

**Step 2:** Let $(\lambda_n)$ be a sequence with $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_n \to \lambda$. Our goal is to prove $\lim_{n\to+\infty} d_{\lambda_n} = d_\lambda$. Since $\lambda \mapsto d_\lambda$ is decreasing then $d_{\lambda_1} \leq d_{\lambda_n} \leq d_\lambda$, for all $n \in \mathbb{N}$. From [45], for each $n \in \mathbb{N}$ let $u_n$ be a ground state solution of $(AP)_{\lambda_n}$, $t_n \geq 0$ and $v_n \in E^-$ verifying

$$J_\lambda(t_n u_n + v_n) = \max_{u\in\widehat{E}(u_n)} J_\lambda(u).$$

Our next goal is to show that $(u_n)$ is bounded. Inspired by [45, Proposition 2.7], suppose by contradiction that $||u_n|| \to +\infty$ and let $w_n := \frac{u_n}{||u_n||}$. As $||u_n^+|| \geq ||u_n^-||$, then $||w_n^+||^2 \geq ||w_n^-||^2$. Using the equality $||w_n^+||^2 + ||w_n^-||^2 = ||w_n||^2 = 1$, we derive $||w_n^+||^2 \geq 1/2, \ \forall\, n \in \mathbb{N}$. Consequently there exist $(y_n) \subset \mathbb{Z}^N$ and $r, \eta > 0$ such that

$$\int_{B_r(y_n)} |w_n^+(x)|^2 dx \geq \eta, \quad \forall\, n \in \mathbb{N}. \tag{2.11}$$

Otherwise, we can apply Lions [30, Lemma I.1] to conclude that $w_n^+ \to 0$ in $L^p(\mathbb{R}^N)$ for $p \in (2, 2^*)$. Then, $\int F(sw_n^+)dx \to 0$ for each $s > 0$ and

$$\begin{aligned} d_\lambda \geq \ & d_{\lambda_n} = J_{\lambda_n}(u_n) \geq J_{\lambda_n}(sw_n^+) = \tfrac{1}{2}s^2||w_n^+||^2 - \lambda_n \textstyle\int F(sw_n^+)dx \\ & \geq \tfrac{s^2}{4} - \lambda_n \textstyle\int F(sw_n^+)dx \to \tfrac{s^2}{4}, \end{aligned}$$

which is absurd because $s$ is arbitrary, showing (2.11). Now, we set

$$\widetilde{u}_n(x) := u_n(x + y_n) \quad \text{and} \quad \widetilde{w}_n(x) := w_n(x + y_n).$$

By Lemma 2.2.1, $\widetilde{w}_n^+(x) = w_n^+(x + y_n)$. Moreover, by (2.11), $\widetilde{w}_n \rightharpoonup w$ with $w^+ \neq 0$, because $\widetilde{w}_n^+ \rightharpoonup w^+$. Since $\widetilde{u}_n = \widetilde{w}_n||u_n||$, it follows that $|\widetilde{u}_n(x)| \to +\infty$ for each $x \in \mathbb{R}^N$ with $w(x) \neq 0$. Therefore, by Fatou's Lemma,

$$\int \frac{F(\widetilde{u}_n)}{|\widetilde{u}_n|^2}|\widetilde{w}_n|^2 dx \to +\infty.$$

Hence

$$\begin{aligned} 0 \ \ & \leq \tfrac{J_{\lambda_n}(u_n)}{||u_n||^2} = \tfrac{1}{2}||w_n^+||^2 - \tfrac{1}{2}||w_n^-||^2 - \lambda_n \textstyle\int \tfrac{F(u_n)}{|u_n|^2}|w_n|^2 dx \\ & = \tfrac{1}{2}||w_n^+||^2 - \tfrac{1}{2}||w_n^-||^2 - \lambda_n \textstyle\int \tfrac{F(\widetilde{u}_n)}{|\widetilde{u}_n|^2}|\widetilde{w}_n|^2 dx \to -\infty \end{aligned}$$

obtaining a contradiction. This proves that $(u_n)$ is bounded.

Now, we are ready to prove that $\lim_{n\to+\infty} d_{\lambda_n} = d_\lambda$. First of all, there exists $\eta > 0$ such that

$$\max_{y\in\mathbb{R}^N} \int_{B_1(y)} |u_n^+(x)|^2 dx \geq \eta, \quad \forall\, n \in \mathbb{N}. \tag{2.12}$$

Otherwise, Lions [30, Lemma I.1] ensures that $u_n^+ \to 0$ in $L^p(\mathbb{R}^N)$, $\forall\, p \in (2, 2^*)$. Then, by $(f_1) - (f_2)$, $\int f(u_n)u_n^+ dx \to 0$. Now, combining this limit with the equality below

$$0 = J'_{\lambda_n}(u_n)u_n^+ = ||u_n^+||^2 - \lambda_n \int f(u_n)u_n^+ dx = ||u_n^+||^2 + o_n(1),$$

we derive $||u_n^+|| \to 0$, contradicting the inequality $||u_n^+|| \geq \sqrt{2d_{\lambda_n}} \geq \sqrt{2d_{\lambda_1}}$ for all $n \in \mathbb{N}$. This proves (2.12), and so, there exist $(y_n) \subset \mathbb{Z}^N$ and $r > 0$ such that

$$\int_{B_r(y_n)} |u_n^+(x)|^2 dx \geq \eta$$

Defining $\widetilde{u}_n(x) := u_n(x + y_n)$, we have that $(\widetilde{u}_n)$ is bounded and $\widetilde{u}_{n_j}^+ \not\to 0$ as $n_j \to +\infty$ for any subsequence. Fixing $\mathcal{V} := \{\widetilde{u}_n^+\}_{n\in\mathbb{N}} \subset E^+ \setminus \{0\}$, it follows that $\mathcal{V}$ is bounded and $0 \notin \overline{\mathcal{V}}^{\sigma(H^1(\mathbb{R}^N), H^1(\mathbb{R}^N)')}$. Thus, by Lemma 2.2.2, there exists $R > 0$ such that

$$J_\lambda(w) < 0 \text{ for } w \in E(u) \setminus B_R(0), \ \forall\, u \in \mathcal{V}. \tag{2.13}$$

On the other hand, if $\widetilde{v}_n(x) := v_n(x + y_n)$, we have

$$J_\lambda(t_n\widetilde{u}_n + \widetilde{v}_n) = J_\lambda(t_n u_n + v_n) = \max_{u\in\widehat{E}(u_n)} J_\lambda(u) \geq d_\lambda > 0, \ \forall\, n \in \mathbb{N}. \tag{2.14}$$

By (2.13) and (2.14), it follows that $||t_n\widetilde{u}_n + \widetilde{v}_n|| \leq R$, for all $n \in \mathbb{N}$. Therefore $||t_n u_n + v_n|| \leq R$, for all $n \in \mathbb{N}$, that is, $(t_n u_n + v_n)$ is bounded. Finally,

$$\begin{aligned} d_\lambda &\leq J_\lambda(t_n u_n + v_n) \\ &= (\lambda_n - \lambda) \textstyle\int F(t_n u_n + v_n)dx + J_{\lambda_n}(t_n u_n + v_n) \\ &\leq o_n(1) + J_{\lambda_n}(u_n) = o_n + d_{\lambda_n}, \end{aligned}$$

that is,

$$d_\lambda \leq o_n(1) + d_{\lambda_n}, \ \forall\, n \in \mathbb{N}.$$

Since $d_\lambda \geq d_{\lambda_n}$ for all $n \in \mathbb{N}$, we have $\lim\limits_{n\to+\infty} d_{\lambda_n} = d_\lambda$, finishing the proof. ■

## 2.3 Existence of ground state for problem $(P)_\epsilon$.

In this section our main goal is proving that $c_\epsilon$ given in (1.1) is a critical level for $I_\epsilon$ when $\epsilon$ small enough. Hereafter, for each $\epsilon \geq 0$, we denote by $I_\epsilon : H^1(\mathbb{R}^N) \to \mathbb{R}$ the energy functional associated with $(P)_\epsilon$ given by

$$I_\epsilon(u) = \frac{1}{2}\int (|\nabla u|^2 + V(x)|u|^2\, dx) - \int A(\epsilon x)F(u)\, dx,$$

or equivalently

$$I_\epsilon(u) = \frac{1}{2}\|u^+\|^2 - \frac{1}{2}\|u^-\|^2 - \int A(\epsilon x)F(u)\,dx.$$

Here, it is very important to observe that by using the notations explored in Section 2, we derive that

$$c_0 = d_{A(0)}, \quad I_0 = J_{A(0)} \quad \text{and} \quad \mathcal{M}_0 = \mathcal{N}_{A(0)}.$$

From now on, without loss of generality we assume that

$$A(0) = \sup_{x\in\mathbb{R}^N} A(x). \tag{3.15}$$

The same idea explored in [45, Lemma 2.4] gives

$$0 < c_\epsilon = \inf_{u\in E^+\setminus\{0\}} \max_{v\in\widehat{E}(u)} I_\epsilon(u). \tag{3.16}$$

Moreover, the Lemma 2.2.2 permits to argue as in [45, Lemma 2.6] to prove that for each $u \in H^1(\mathbb{R}^N)\setminus E^-$, $\mathcal{M}_\epsilon \cap \hat{E}(u)$ is a singleton set and the element of this set is the unique global maximum of $I_\epsilon|_{\hat{E}(u)}$, that is, there are $t_* \geq 0$ and $v_* \in E^-$ such that

$$I_\epsilon(t_*u + v_*) = \max_{w\in\widehat{E}(u)} I_\epsilon(w). \tag{3.17}$$

Our first lemma shows an important relation between $c_\epsilon$ and $c_0$.

**Lemma 2.3.1** *It occurs the limit* $\lim\limits_{\epsilon\to 0} c_\epsilon = c_0$.

**Proof.** Consider $\epsilon_n \to 0$ with $\epsilon_n > 0$. Our goal is to prove that $c_{\epsilon_n} \to c_0$. First of all, note that $c_0 \leq c_{\epsilon_n}$ for all $n \in \mathbb{N}$, which leads to $c_0 \leq \liminf_{n\to+\infty} c_{\epsilon_n}$. On the other hand, by (3.17), if $w_0 \in H^1(\mathbb{R}^N)$ is a ground state solution of $(P)_0$, there are $t_n \in [0,+\infty)$ and $v_n \in E^-$ such that $t_n w_0^+ + v_n \in \mathcal{M}_{\epsilon_n}$, implying that

$$I_{\epsilon_n}(t_n w_0^+ + v_n) \geq c_{\epsilon_n} > 0, \quad \forall\, n \in \mathbb{N}.$$

As in the previous section, $(t_n w_0^+ + v_n)$ is bounded. Thus, without loss of generality, we can consider that $t_n \to t_0$ and $v_n \rightharpoonup v$ in $H^1(\mathbb{R}^N)$. Note that

$$c_{\epsilon_n} \leq I_{\epsilon_n}(t_n w_0^+ + v_n) = \frac{1}{2}t_n^2||w_0^+||^2 - \frac{1}{2}||v_n||^2 - \int A(\epsilon_n x)F(t_n w_0^+ + v_n)dx.$$

Hence, since the norm is weakly lower semicontinous, the Fatou's Lemma gives

$$\begin{aligned}\limsup_{n\to+\infty} c_{\epsilon_n} &\leq \limsup_{n\to+\infty}\left(\tfrac{1}{2}t_n^2||w_0^+||^2 - \tfrac{1}{2}||v_n||^2\right) + \\ &\limsup_{n\to+\infty}\left(-\int A(\epsilon_n x)F(t_n w_0^+ + v_n)dx\right) \\ &\leq \tfrac{1}{2}t_0^2||w_0^+||^2 - \tfrac{1}{2}||v||^2 - \int A(0)F(t_0 w_0^+ + v)dx \\ &= I_0(t_0 w_0^+ + v) \leq I_0(w_0) = c_0.\end{aligned}$$

From this, $\lim_{n\to+\infty} c_{\epsilon_n} = c_0$. ■

As an immediate consequence of the last lemma we have the corollary below

**Corollary 2.3.2** *There exists $\epsilon_0 > 0$ such that for all $\epsilon \in (0, \epsilon_0)$ yields $c_\epsilon < d_{A_\infty}$, where $A_\infty = \lim_{|x|\to+\infty} A(x)$.*

**Proof.** By condition $(A_1)$, $A(0) > A_\infty$, then the Proposition 2.2.3 ensures that $d_{A(0)} < d_{A_\infty}$, or equivalently, $c_0 < d_{A_\infty}$. Now it is enough to apply the Lemma 2.3.1 to get the desired result. ■

As a byproduct of the proof of Lemma 2.3.1, we also have the following result, which can be useful for related problems.

**Lemma 2.3.3** *Let $(t_n) \subset [0, +\infty)$ and $(v_n) \subset E^-$ the sequences defined in the proof of Lemma 2.3.1. Then, for some subsequence,*

$$t_n \to 1 \quad \textit{and} \quad v_n \to w_0^-.$$

*Hence, $t_n w_0^+ + v_n \to w_0$ in $H^1(\mathbb{R}^N)$.*

**Proof.** Note that in the proof of Lemma 2.3.1, we find that

$$\liminf_{n\to+\infty} ||v_n||^2 = ||v||^2.$$

Then for some subsequence $\lim_{n\to+\infty} ||v_n|| = ||v||$, and so, $v_n \to v$. Furthermore, from the previous lemma $I_0(w_0) = I_0(t_0 w_0^+ + v)$, where $w_0 \in \mathcal{M}_0$. Hence $t_0 w_0^+ + v = w_0$, from where it follows that $t_0 = 1$ and $v = w_0^-$. Thereby, $t_n \to 1$ and $v_n \to w_0^-$. ■

Our next result is related to the [45, Proposition 2.7], however as in the present paper $A$ is not periodic, we cannot repeat the same arguments explored in that paper, then some adjustments are necessary in the proof to get the same result.

**Proposition 2.3.4** *$I_\epsilon$ is coercive on $\mathcal{M}_\epsilon$.*

**Proof.** Suppose that there exists $(u_n) \subset \mathcal{M}_\epsilon$ verifying

$$I_\epsilon(u_n) \leq d \quad \text{and} \quad ||u_n|| \to +\infty,$$

for some $d \in \mathbb{R}$. Setting $v_n := \frac{u_n}{||u_n||}$, it follows that $||v_n^+|| \geq ||v_n^-||$ and $||v_n^+||^2 \geq \frac{1}{2}$. On the other hand, there exist $(y_n) \subset \mathbb{Z}^N$ and $r, \eta > 0$ such that,

$$\int_{B_r(y_n)} |v_n^+|^2 dx > \eta, \quad \forall n \in \mathbb{N}. \tag{3.18}$$

In fact, suppose by contradiction that (3.18) does not hold. Then, applying again Lions [30, Lemma I.1], $v_n^+ \to 0$ in $L^p(\mathbb{R}^N)$ for all $p \in (2, 2^*)$. Hence, by $(f_1) - (f_2)$, $\int F(sv_n^+)dx \to 0$ for all $s > 0$. Thereby,

$$\begin{aligned} d \ &\geq I_\epsilon(u_n) \geq I_\epsilon(sv_n^+) = \tfrac{1}{2}s^2||v_n^+||^2 - \textstyle\int A(\epsilon x)F(sv_n^+)dx \geq \\ &\geq \tfrac{s^2}{4} - \textstyle\int A(0)F(sv_n^+)dx \to \tfrac{s^2}{4}, \end{aligned}$$

which absurd, because $s$ is arbitrary. This shows that (3.18) is valid.

Fixing $\widetilde{u}_n(x) := u_n(x + y_n)$ and $\widetilde{v}_n(x) := v_n(x + y_n)$, by Lemma 2.2.1, we have $\widetilde{v}_n^+(x) := v_n^+(x + y_n)$ and $\widetilde{u}_n = \widetilde{v}_n||u_n||$. Since $v_n \rightharpoonup v$, by (3.18), $v \neq 0$. Then, $\widetilde{u}_n(x) \to +\infty$ when $v(x) \neq 0$. By using the Fatou's Lemma, we get

$$\int \frac{F(u_n)}{||u_n||^2} dx \geq \int \frac{F(u_n)}{|u_n|^2}|v_n|^2 dx = \int \frac{F(\widetilde{u}_n)}{|\widetilde{u}_n|^2}|\widetilde{v}_n|^2 dx \geq \int_{[v\neq 0]} \frac{F(\widetilde{u}_n)}{|\widetilde{u}_n|^2}|\widetilde{v}_n|^2 dx \to +\infty.$$

The above limit yields

$$\begin{aligned} 0 \ &\leq \tfrac{I_\epsilon(u_n)}{||u_n||^2} = \tfrac{1}{2}||v_n^+||^2 - \tfrac{1}{2}||v_n^-||^2 - \textstyle\int A(\epsilon x)\tfrac{F(u_n)}{||u_n||^2}dx \\ &\leq \tfrac{1}{2} - A_0 \textstyle\int \tfrac{F(u_n)}{||u_n||^2}dx \to -\infty, \end{aligned}$$

obtaining a new absurd. ■

Now, we can repeat the same arguments found in [45, see proof of Theorem 1.1] to guarantee the existence of a $(PS)$ sequence $(u_n) \subset \mathcal{M}_\epsilon$ associated with $c_\epsilon$, that is,

$$I_\epsilon(u_n) \to c_\epsilon \quad \text{and} \quad I'_\epsilon(u_n) \to 0.$$

**Theorem 2.3.5** *The problem $(P)_\epsilon$ has a ground state solution for all $\epsilon \in (0, \epsilon_0)$, where $\epsilon_0 > 0$ was given in Corollary 2.3.2.*

**Proof.** First of all, the fact that $(u_n) \subset \mathcal{M}_\epsilon$ leads to

$$0 = I'_\epsilon(u_n)u_n^+ = ||u_n^+||^2 - \int A(\epsilon x)f(u_n)u_n^+ dx \geq 2c_\epsilon - \int A(\epsilon x)f(u_n)u_n^+ dx.$$

Therefore $\int A(\epsilon x)f(u_n)u_n^+dx \not\to 0$. Since $(u_n)$ is bounded, by Lions [30, Lemma I.1], there exist $\eta, \delta > 0$ and $(z_n) \subset \mathbb{Z}^N$ such that

$$\int_{B_\delta(z_n)} |u_n^+|^2 dx > \eta, \quad \forall n \in \mathbb{N}.$$

**Claim 2.3.6** *$(z_n)$ is a bounded sequence.*

If $(z_n)$ is unbounded, for some subsequence, we must have $|z_n| \to +\infty$. Fixing $w_n(x) := u_n(x + z_n)$, we derive $w_n \rightharpoonup w \neq 0$. Now, for each $\phi \in C_0^\infty(\mathbb{R}^N)$,

$$\begin{aligned} o_n &= I'_\epsilon(u_n)\phi(\cdot - z_n) = B(u_n, \phi(\cdot - z_n)) - \int A(\epsilon x)f(u_n)\phi(\cdot - z_n)dx \\ &= B(w_n, \phi) - \int A(\epsilon x + \epsilon z_n)f(w_n)\phi dx, \end{aligned}$$

where

$$B(u, v) = \int (\nabla u \nabla v + V(x)uv)\, dx, \quad \forall u, v \in H^1(\mathbb{R}^N).$$

Taking the limit $n \to +\infty$, we obtain

$$0 = B(w, \phi) - \int A_\infty f(w)\phi dx = J'_{A_\infty}(w)\phi, \ \forall\ \phi \in C_0^\infty(\mathbb{R}^N).$$

Now, the density of $C_0^\infty(\mathbb{R}^N)$ in $H^1(\mathbb{R}^N)$ gives

$$0 = B(w, v) - \int A_\infty f(w)v dx = J'_{A_\infty}(w)v, \ \forall\ v \in H^1(\mathbb{R}^N).$$

The last equality says that $w$ is a nontrivial solution of $(AP)_{A_\infty}$. From characterization of $d_{A_\infty}$ and Fatou's Lemma,

$$\begin{aligned} d_{A_\infty} &\leq J_{A_\infty}(w) = J_{A_\infty}(w) - \tfrac{1}{2}J'_{A_\infty}(w)w = \int A_\infty \left(\tfrac{1}{2}f(w)w - F(w)\right) dx \\ &\leq \liminf_{n\to+\infty} \int A(\epsilon x + \epsilon z_n) \left(\tfrac{1}{2}f(w_n)w_n - F(w_n)\right) dx \\ &= \liminf_{n\to+\infty} \int A(\epsilon x) \left(\tfrac{1}{2}f(u_n)u_n - F(u_n)\right) dx \\ &= \liminf_{n\to+\infty} \left(I_\epsilon(u_n) - \tfrac{1}{2}I'_\epsilon(u_n)u_n\right) = c_\epsilon \end{aligned}$$

that is

$$d_{A_\infty} \leq c_\epsilon, \quad \forall \epsilon > 0.$$

On the other hand, by Corollary 2.3.2, $c_\epsilon < d_{A_\infty}$ when $\epsilon < \epsilon_0$, which is absurd. Therefore, $(z_n)$ is bounded.

As $(z_n)$ is bounded, there exists $r > 0$ such that $B_\delta(z_n) \subset B_r(0)$ for all $n \in \mathbb{N}$. Then,

$$\int_{B_r(0)} |u_n^+|^2 dx \geq \int_{B_\delta(z_n)} |u_n^+|^2 dx > \eta, \quad \forall n \in \mathbb{N}.$$

From this, $u_n \rightharpoonup u$ with $u \neq 0$. Now, it is enough to repeat the arguments found [5, page 23] to conclude that $u$ is a ground state solution for $(P)_\epsilon$. ■

## 2.4 Concentration of the solutions

In this section, we denote by $u_\epsilon$ the ground state solution obtained in Section 3. Our main goal is to show that if $x_\epsilon$ is a maximum point of $|u_\epsilon|$, then

$$\lim_{\epsilon \to 0} A(\epsilon x_\epsilon) = A(0).$$

Of a more precise way, we have proved that if $\epsilon_n \to 0$, for some subsequence, $\epsilon_n x_{\epsilon_n} \to x_0$ for some $x_0 \in \mathcal{A}$ where

$$\mathcal{A} = \{z \in \mathbb{R}^N \,:\, A(z) = A(0)\}.$$

In what follows, we set $(\epsilon_n) \subset (0, \epsilon_0)$ with $\epsilon_n \to 0$, $I_n = I_{\epsilon_n}$, $c_n := c_{\epsilon_n}$ and $u_n = u_{\epsilon_n}$, that is,

$$I_n'(u_n) = 0 \quad \text{and} \quad I_n(u_n) = c_n.$$

By $(A_1)$, $c_n \geq c_0 > 0$ for all $n \in \mathbb{N}$.

Next, we will prove some technical lemmas that are crucial to get the concentration of the solutions.

**Lemma 2.4.1** *The sequence $(u_n)$ is bounded.*

**Proof.** The proof follows as in Proposition 2.3.4. ■

**Lemma 2.4.2** *There exist $(y_n) \subset \mathbb{Z}^N$ and $R$, $\eta > 0$ verifying*

$$\int_{B_R(y_n)} |u_n^+|^2 dx \geq \eta, \quad \forall n \in \mathbb{N}.$$

**Proof.** If the lemma does not hold, by Lions [30, Lemma I.1], $u_n^+ \to 0$ in $L^p(\mathbb{R}^N)$ for all $p \in (2, 2^*)$. Therefore $||u_n^+||^2 = \int A(\epsilon_n x) f(u_n) u_n^+ dx \to 0$. On the other hand, from [45, Lemma 2.4], we know that $||u_n^+|| \geq \sqrt{2c_n} \geq \sqrt{2c_0}$, which contradicts the last limit. ■

In the sequel, $v_n(x) := u_n(x + y_n)$ for all $x \in \mathbb{R}^N$. Thus, for some subsequence, $v_n \rightharpoonup v \neq 0$.

**Lemma 2.4.3** *The sequence $(\epsilon_n y_n)$ is bounded in $\mathbb{R}^N$. Furthermore, if for a subsequence $\epsilon_n y_n \to z$, then $z \in \mathcal{A}$ and $I_0'(v) = 0$.*

**Proof.** First of all, we will prove the boundedness of the sequence $(\epsilon_n y_n)$. Arguing by contradiction, suppose that for some subsequence $|\epsilon_n y_n| \to +\infty$. Since $u_n$ is a ground state solution for $(P)_{\epsilon_n}$,

$$\int (\nabla u_n \nabla \phi(x - y_n) + V(x) u_n \phi(x - y_n)) dx = \int A(\epsilon_n x) f(u_n) \phi(x - y_n) dx,$$

for all $\phi \in C_0^\infty(\mathbb{R}^N)$. Hence, by a change variable,

$$\int (\nabla v_n \nabla \phi + V(x) v_n \phi) dx = \int A(\epsilon_n x + \epsilon_n y_n) f(v_n) \phi dx$$

for all $\phi \in C_0(\mathbb{R}^N)$. Now, taking the limit as $n \to +\infty$, we find

$$\int (\nabla v \nabla \phi \, dx + V(x) v \phi) dx = \int A_\infty f(v) \phi dx$$

for all $\phi \in C_0(\mathbb{R}^N)$. This combined with the density of $C_0^\infty(\mathbb{R}^N)$ in $H^1(\mathbb{R}^N)$ gives

$$\int (\nabla v \nabla \psi + V(x) v \psi) dx = \int A_\infty f(v) \psi dx, \quad \forall \psi \in H^1(\mathbb{R}^N).$$

Then $v$ is a nontrivial solution of $(AP)_{A_\infty}$, and so, $v \in \mathcal{M}_{A_\infty}$. By Fatou's lemma,

$$\begin{aligned} d_{A_\infty} &\leq J_{A_\infty}(v) = J_{A_\infty}(v) - \tfrac{1}{2} J'_{A_\infty}(v) v = \textstyle\int A_\infty \left(\tfrac{1}{2} f(v) v - F(v)\right) dx \\ &\leq \liminf_{n \to +\infty} \textstyle\int A(\epsilon x + \epsilon_n y_n) \left(\tfrac{1}{2} f(v_n) v_n - F(v_n)\right) dx \\ &= \liminf_{n \to +\infty} \textstyle\int A(\epsilon_n x) \left(\tfrac{1}{2} f(u_n) u_n - F(u_n)\right) dx \\ &= \liminf_{n \to +\infty} \left(I_n(u_n) - \tfrac{1}{2} I'_n(u_n) u_n\right) \\ &= \liminf_{n \to +\infty} I_n(u_n) = \lim_{n \in \mathbb{N}} c_n = c_0 < d_{A_\infty}, \end{aligned}$$

obtaining a contradiction. Consequently $(\epsilon_n y_n)$ is bounded, and we can assume that $\epsilon_n y_n \to z$. The same argument works to prove that

$$\int (\nabla v \nabla \psi + V(x) v \psi) dx = \int A(z) f(v) \psi dx, \quad \forall \psi \in H^1(\mathbb{R}^N).$$

Hence $v$ is a nontrivial solution of $(AP)_{A(z)}$, and so, $v \in \mathcal{M}_{A(z)}$. The previous arguments lead to $d_{A(z)} \leq c_0 = d_{A(0)}$. Then the monotonicity of $\lambda \to d_\lambda$ implies that $A(0) \leq A(z)$. As $A(0) \geq A(z)$, it follows that $A(0) = A(z)$, showing that $z \in \mathcal{A}$. ■

From now on, we are considering that $\epsilon_n y_n \to z$ with $z \in \mathcal{A}$, i.e., $A(z) = A(0)$. Here, it is very important to observe that

$$J_{A(z)} = J_{A(0)} = I_0 \quad \text{and} \quad I'_0(v) = 0.$$

By growth condition on $f$, we know that for each $\tau > 0$ there exists $\delta := \delta_\tau \in (0,1)$ such that

$$\frac{|f(t)|}{|t|} < \tau, \quad \forall t \in (-\delta, \delta).$$

In what follows, we set $g_\tau(t) := \chi_\delta(t)f(t)$ and $j_\tau(t) := \tilde{\chi}_\delta(t)f(t)$, where $\chi_\delta$ is the characteristic function on $(-\delta, \delta)$ and $\tilde{\chi}_\delta(t) = 1 - \chi_\delta(t)$.

**Lemma 2.4.4** *For each $\tau > 0$, there is $c_\tau > 0$ such that*

$$|g_\tau(t)| \leq \tau|t| \quad \textit{and} \quad |j_\tau(t)|^r \leq c_\tau t f(t), \quad \forall t \in \mathbb{R},$$

*where $r = \frac{q+1}{q}$ with $q$ given in $(f_2)$.*

**Proof.** By using the definition of $g_\tau$, it is obvious that above inequality involving the function $g_\tau$ holds.

In order to prove the second inequality, note that $[-1, -\delta] \cup [\delta, 1] \subset \mathbb{R}$ is compact set, then there exists $\widetilde{c_\tau} > 0$ such that

$$\frac{|f(t)|^{r-1}}{|t|} \leq \widetilde{c_\tau}, \quad \forall t \in [-1, -\delta] \cup [\delta, 1],$$

consequently

$$|j_\tau(t)|^{r-1} \leq \widetilde{c_\tau}|t|, \quad \forall t \in [-1, -\delta] \cup [\delta, 1].$$

On the other hand, there exists $\widetilde{b_\tau} > 0$ verifying

$$|f(t)| \leq \tau|t| + \widetilde{b_\tau}|t|^q, \ \forall \ t \in \mathbb{R}.$$

Thus, there exist $A_\tau, B_\tau, \widehat{c_\tau} > 0$ such that

$$|j_\tau(t)|^{r-1} = |f(t)|^{r-1} \leq A_\tau|t|^{r-1} + B_\tau|t|^{(r-1)q} = A_\tau|t|^{r-1} + B_\tau|t| \leq \widehat{c_\tau}|t|, \quad \forall |t| > 1.$$

From this,

$$|j_\tau(t)|^{r-1} \leq c_\tau|t|, \quad \forall t \in \mathbb{R},$$

for some $c_\tau > 0$. Thereby,

$$|j_\tau(t)|^r \leq c_\tau|t||j_\tau(t)| \leq c_\tau t f(t), \quad \forall t \in \mathbb{R},$$

finishing the proof. ■

The last lemma permit us to prove an important convergence involving the sequence $(v_n)$.

**Proposition 2.4.5** *The sequence $(v_n)$ converges strongly to $v$ in $H^1(\mathbb{R}^N)$.*

**Proof.** First of all, note that

$$\begin{aligned} c_0 \leq \quad & I_0(v) = I_0(v) - \tfrac{1}{2}I_0'(v)v = \int A(0)\left(\tfrac{1}{2}f(v)v - F(v)\right)dx \\ & = \int A(z)\left(\tfrac{1}{2}f(v)v - F(v)\right)dx \\ & \leq \liminf_{n\to+\infty} \int A(\epsilon_n x + \epsilon_n y_n)\left(\tfrac{1}{2}f(v_n)v_n - F(v_n)\right)dx \\ & \leq \limsup_{n\to+\infty} \int A(\epsilon_n x + \epsilon_n y_n)\left(\tfrac{1}{2}f(v_n)v_n - F(v_n)\right)dx \\ & = \limsup_{n\to+\infty} \int A(\epsilon_n x)\left(\tfrac{1}{2}f(u_n)u_n - F(u_n)\right)dx \\ & = \limsup_{n\to+\infty} \left(I_n(u_n) - \tfrac{1}{2}I_n'(u_n)u_n\right) \\ & = \lim_{n\to+\infty} c_n = c_0. \end{aligned}$$

Therefore

$$\lim_{n\to+\infty} \int A(\epsilon_n x + \epsilon_n y_n)\left(\frac{1}{2}f(v_n)v_n - F(v_n)\right)dx = \int A(z)\left(\frac{1}{2}f(v)v - F(v)\right)dx.$$

Since

$$A(\epsilon_n x + \epsilon_n y_n)\left(\frac{1}{2}f(v_n)v_n - F(v_n)\right) \geq 0, \quad \forall n \in \mathbb{N},$$

and supposing that

$$v_n(x) \to v(x) \quad \text{a.e. in} \quad \mathbb{R}^N,$$

we deduce that

$$A(\epsilon_n x + \epsilon_n y_n)\left(\frac{1}{2}f(v_n)v_n - F(v_n)\right) \to A(z)\left(\frac{1}{2}f(v)v - F(v)\right) \text{ in } L^1(\mathbb{R}^N).$$

Thus, for some subsequence, there exists $H \in L^1(\mathbb{R}^N)$ such that

$$A_0\left(\frac{1}{2}f(v_n)v_n - F(v_n)\right) \leq A(\epsilon_n x + \epsilon_n y_n)\left(\frac{1}{2}f(v_n)v_n - F(v_n)\right) \leq H \quad \text{a.e. in} \quad \mathbb{R}^N$$

for all $n \in \mathbb{N}$. Then, by $(f_4)$,

$$A_0\left(\frac{1}{2} - \frac{1}{\theta}\right) f(v_n)v_n \leq H, \quad \forall n \in \mathbb{N}.$$

Consequently there exists $c > 0$ such that

$$f(v_n)v_n \leq c\,H, \quad \forall n \in \mathbb{N}.$$

In what follows, we set

$$Q_n := f(v_n)v_n^+ - f(v)v^+.$$

Our goal is to prove that

$$\int |Q_n| dx \to 0.$$

First of all, as $f$ has subcritical growth,

$$\int_{B_R(0)} |Q_n| dx \to 0, \quad \forall R > 0. \tag{4.19}$$

On the other hand, for each $\tau > 0$, we can fix $R$ large enough a such way that

$$\int_{B_R(0)^c} |f(v)v^+| dx < \tau.$$

**Claim 2.4.6** *Increasing $R$ if necessary, we also have*

$$\int_{B_R(0)^c} |f(v_n)v_n^+| dx < 2\Theta\tau, \quad \forall n \in \mathbb{N}$$

*where*

$$\Theta := \sup_{n\in\mathbb{N}} \left\{ \left( \int |v_n^+|^{q+1} dx \right)^{\frac{1}{q+1}}, \int |v_n v_n^+| dx \right\}.$$

In fact, for each $\tau > 0$, the Lemma 2.4.4 ensures the existence of $c_\tau > 0$ such that

$$|j_\tau(t)|^r \leq c_\tau t f(t), \quad \text{where} \quad r = \frac{q+1}{q}.$$

From Lemma 2.4.4,

$$\int_{B_R(0)^c} |f(v_n)v_n^+| dx = \int_{B_R(0)^c} |g_\tau(v_n)||v_n^+| dx + \int_{B_R(0)^c} |j_\tau(v_n)||v_n^+| dx \leq$$

$$\leq \tau \int_{B_R(0)^c} |v_n||v_n^+| dx + \left( \int_{B_R(0)^c} |j_\tau(v_n)|^r dx \right)^{1/r} \left( \int_{B_R(0)^c} |v_n^+|^{q+1} dx \right)^{1/(q+1)}$$

$$\leq \tau\Theta + \left( \int_{B_R(0)^c} c_\tau f(v_n) v_n dx \right)^{1/r} \Theta \leq \tau\Theta + c_\tau \left( \int_{B_R(0)^c} cH \ dx \right)^{1/r} \Theta.$$

Now, increasing $R$ if necessary, a such way that

$$c_\tau \left( \int_{B_R(0)^c} cH \ dx \right)^{1/r} < \tau$$

we get

$$\int_{B_R(0)^c} |f(v_n)v_n^+| dx \leq 2\tau\Theta,$$

proving the claim. From (4.19) and Claim 2.4.6,

$$\int |Q_n| \, dx \to 0.$$

Therefore

$$f(v_n)v_n^+ \to f(v)v^+ \text{ in } L^1(\mathbb{R}^N).$$

Analogously,

$$f(v_n)v_n^- \to f(v)v^- \text{ in } L^1(\mathbb{R}^N).$$

Since $I_n'(u_n)u_n^+ = 0$, it follows that

$$||v_n^+||^2 = \int A(\epsilon_n x + \epsilon_n y_n) f(v_n) v_n^+ dx \to \int A(z) f(v) v^+ dx = ||v^+||^2,$$

showing that $v_n^+ \to v^+$ in $H^1(\mathbb{R}^N)$, because $v_n^+ \rightharpoonup v^+$ in $H^1(\mathbb{R}^N)$. Likewise $v_n^- \to v^-$ in $H^1(\mathbb{R}^N)$. Thereby $v_n = v_n^+ + v_n^- \to v^+ + v^- = v$ in $H^1(\mathbb{R}^N)$, finishing the proof. ■

**Corollary 2.4.7** $||v_n||_{L^\infty(\mathbb{R}^N)} \not\to 0$.

**Proof.** If $||v_n||_{L^\infty(\mathbb{R}^N)} \to 0$, by Proposition 2.4.5, we must have $v = 0$, which is absurd. ■

**Lemma 2.4.8** *For all $n \in \mathbb{N}$, $v_n \in C(\mathbb{R}^N)$. Furthermore, there exist a continuous function $P : \mathbb{R} \to \mathbb{R}$ with $P(0) = 0$ and $K > 0$ such that*

$$||v_n||_{C(\overline{B_1(z)})} \leq K \cdot P\left(||v_n||_{L^{2^*}(B_2(z))}\right),$$

*for all $n \in \mathbb{N}$ and for all $z \in \mathbb{R}^N$.*

**Proof.** Since $u_n$ is solution of $(P)_{\epsilon_n}$, $v_n$ is a solution of

$$\begin{cases} -\Delta v_n + V(x)v_n = A(\epsilon_n x + \epsilon_n y_n) f(v_n) & \text{in} \quad \mathbb{R}^N, \\ v_n \in H^1(\mathbb{R}^N). \end{cases}$$

Setting $\Psi_n(x,t) := A(\epsilon_n x + \epsilon_n y_n) f(t)$, it is easy to check that there exists $C > 0$, independently of $n \in \mathbb{N}$, verifying

$$\Psi_n(x,t) \leq C(|t| + |t|^q), \quad \forall x \in \mathbb{R}^N \quad \text{and} \quad \forall t \in \mathbb{R}.$$

Moreover, for each $R > 0$ and $z \in \mathbb{R}^N$, we have that $u \in L^s(B_2(z))$ with $s \geq q$, $\Psi_n(\cdot, u(\cdot)) \in L^{s/q}(B_2(z))$ and there exist $C_s = C(s) > 0$, independent of $z$, such that

$$||\Psi_n(\cdot, u(\cdot))||_{L^{s/q}(B_2(z))} \leq C_s(||u||_{L^{s/q}(B_2(z))} + ||u||^q_{L^s(B_2(z))}), \quad \forall n \in \mathbb{N}.$$

Here we have used the fact that $A$ is a bounded function. Now, recalling that potential $V$ is also a bounded function, we can proceed in the same manner as in [41, Proposition 2.15] to get the desired result. ■

As a byproduct of the last lemma we have the corollary below

**Corollary 2.4.9** *Given $\delta > 0$, there exists $R := R_\delta > 0$ such that $|v_n(x)| \leq \delta$ for all $x \in \mathbb{R}^N \setminus B_R(0)$, that is,* $\lim_{|x|\to+\infty} v_n(x) = 0$ *uniformly in* $\mathbb{N}$.

**Proof.** Since $v_n \to v$ in $H^1(\mathbb{R}^N)$, given $\tau > 0$ there are $R > 0$ such that

$$\|v_n\|_{L^{2^*}(B_2(z))} < \tau, \quad \text{for all } |z| \geq R \quad \text{and} \quad n \in \mathbb{N}.$$

As $P$ is a continuous function and $P(0) = 0$, given $\beta > 0$, there is $\tau > 0$ such that

$$|P(t)| < \beta/K, \quad \text{for} \quad |t| < \tau.$$

Hence, by Lemma 2.4.8,

$$||v_n||_{C(\overline{B_1(z)})} < \beta \quad \text{for} \quad |z| \geq R \quad \text{and} \quad n \in \mathbb{N}.$$

This proves the corollary. ■

Finally we are ready to show the concentration.

**Concentration of the solutions:**

From Corollary 2.4.9, there is $z_n \in \mathbb{R}^N$ such that $|v_n(z_n)| = \max_{x\in\mathbb{R}^N} |v_n(x)|$. Now, applying Corollary 2.4.7, there exists $\delta > 0$ such that $|v_n(z_n)| \geq \delta$ for all $n \in \mathbb{N}$, implying that $(z_n)$ is bounded. Therefore if $\xi_n := z_n + y_n$, it follows that

$$|u_n(\xi_n)| = \max_{x\in\mathbb{R}^N} |u_n(x)|$$

and

$$\epsilon_n \xi_n = \epsilon_n z_n + \epsilon_n y_n \to 0 + z = z$$

with $z \in \mathcal{A}$, finishing the study of the concentration phenomena.

# Capítulo 3

# Existência e fenômeno de concentração para uma classe de problemas variacionais indefinidos com crescimento crítico

## Existence and concentration phenomena for a class of indefinite variational problems with critical growth

CLAUDIANOR O. ALVES and GEILSON F. GERMANO


## Abstract

In this paper we are interested to prove the existence and concentration of ground state solution for the following class of problems

$$-\Delta u + V(x)u = A(\epsilon x)f(u), \quad x \in \mathbb{R}^N, \tag{$(P)_\epsilon$}$$

where $N \geq 2$, $\epsilon > 0$, $A : \mathbb{R}^N \to \mathbb{R}$ is a continuous function that satisfies

$$0 < \inf_{x \in \mathbb{R}^N} A(x) \leq \lim_{|x| \to +\infty} A(x) < \sup_{x \in \mathbb{R}^N} A(x) = A(0), \tag{$A$}$$

$f : \mathbb{R} \to \mathbb{R}$ is a continuous function having critical growth, $V : \mathbb{R}^N \to \mathbb{R}$ is a continuous $\mathbb{Z}^N$–periodic with $0 \notin \sigma(\Delta + V)$. By using variational methods, we prove the existence

of solution for $\epsilon$ small enough. After that, we show that the maximum points of the solutions concentrate around of a maximum point of $A$.

**Mathematics Subject Classifications (2010):** 35B40, 35J2, 47A10 .

**Keywords:** concentration of solutions, variational methods, indefinite strongly functional, critical growth.

## 3.1 Introduction

This paper concerns with the existence and concentration of ground state solution for the semilinear Schrödinger equation

$$\begin{cases} -\Delta u + V(x)u = A(\epsilon x)f(u), & x \in \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \end{cases} \tag{$(P)_\epsilon$}$$

where $N \geq 2$, $\epsilon$ is a positive parameter, $f : \mathbb{R} \to \mathbb{R}$ is a continuous function with critical growth and $V, A : \mathbb{R} \to \mathbb{R}$ are continuous functions verifying some technical conditions.

In whole this paper, $V$ is $\mathbb{Z}^N$-periodic with

$$0 \notin \sigma(-\Delta + V), \quad \text{the spectrum of} \quad -\Delta + V, \tag{$V$}$$

which becomes the problem strongly indefinite. Related to the function $A$, we assume that it is a continuous function satisfying

$$0 < A_0 = \inf_{x \in \mathbb{R}^N} A(x) \leq \lim_{|x| \to +\infty} A(x) = A_\infty < \sup_{x \in \mathbb{R}^N} A(x). \tag{$A$}$$

The present article has as first motivation some recent articles that have studied the existence of ground state solution for related problems with $(P)_\epsilon$, more precisely for strongly indefinite problems of the type

$$\begin{cases} -\Delta u + V(x)u = f(x, u), & x \in \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N). \end{cases} \tag{$P_1$}$$

In [27], Kryszewski and Szulkin have studied the existence of ground state solution for $(P_1)$ by supposing the condition $(V)$. Related to the function $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$, they assumed that $f$ is continuous, $\mathbb{Z}^N$-periodic in $x$ with

$$|f(x, t)| \leq c(|t|^{q-1} + |t|^{p-1}), \quad \forall t \in \mathbb{R} \quad \text{and} \quad x \in \mathbb{R}^N \tag{$h_1$}$$

and

$$0 < \alpha F(x,t) \le t f(x,t) \quad \forall (x,t) \in \mathbb{R}^N \times \mathbb{R}^*, \quad F(x,t) = \int_0^t f(x,s)\, ds \qquad (h_2)$$

for some $c > 0$, $\alpha > 2$ and $2 < q < p < 2^*$ where $2^* = \frac{2N}{N-2}$ if $N \ge 3$ and $2^* = +\infty$ if $N = 1, 2$. The above hypotheses guarantee that the energy functional associated with $(P_1)$ given by

$$J(u) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)|u|^2)\, dx - \int_{\mathbb{R}^N} F(x,u)\, dx, \ \forall u \in H^1(\mathbb{R}^N),$$

is well defined and belongs to $C^1(H^1(\mathbb{R}^N), \mathbb{R})$. By $(V)$, there is an equivalent inner product $\langle \ , \ \rangle$ in $H^1(\mathbb{R}^N)$ such that

$$J(u) = \frac{1}{2}\|u^+\|^2 - \frac{1}{2}\|u^-\|^2 - \int_{\mathbb{R}^N} F(x,u)\, dx,$$

where $\|u\| = \sqrt{\langle u, u\rangle}$ and $H^1(\mathbb{R}^N) = E^+ \oplus E^-$ corresponds to the spectral decomposition of $-\Delta + V$ with respect to the positive and negative part of the spectrum with $u = u^+ + u^-$, where $u^+ \in E^+$ and $u^- \in E^-$. In order to show the existence of solution for $(P_1)$, Kryszewski and Szulkin introduced a new and interesting generalized link theorem. In [31], Li and Szulkin have improved this generalized link theorem to prove the existence of solution for a class of strongly indefinite problem with $f$ being asymptotically linear at infinity.

The link theorems above mentioned have been used in a lot of papers, we would like to cite Chabrowski and Szulkin [14], do Ó and Ruf [17], Furtado and Marchi [20], Tang [51, 52] and their references.

Pankov and Pflüger [39] also have considered the existence of solution for problem $(P_1)$ with the same conditions considered in [27], however the approach is based on an approximation technique of periodic function together with the linking theorem due to Rabinowitz [40]. After, Pankov [38] has studied the existence of solution for problems of the type

$$\begin{cases} -\Delta u + V(x)u = \pm f(x,u), & x \in \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \end{cases} \qquad (P_2)$$

by supposing $(V)$, $(h_1) - (h_2)$ and employing the same approach explored in [39]. In [38] and [39], the existence of ground state solution has been established by supposing that $f$ is $C^1$ and there is $\theta \in (0,1)$ such that

$$0 < t^{-1} f(x,t) \le \theta f_t'(x,t), \quad \forall t \ne 0 \quad \text{and} \quad x \in \mathbb{R}^N. \qquad (h_3)$$

However, in [38], Pankov has found a ground state solution by minimizing the energy functional $J$ on the set

$$\mathcal{O}=\left\{u\in H^1(\mathbb{R}^N)\setminus E^-\ ;\ J'(u)u=0 \text{ and } J'(u)v=0,\forall\ v\in E^-\right\}.$$

The reader is invited to see that if $J$ is strongly definite, that is, when $E^-=\{0\}$, the set $\mathcal{O}$ is exactly the Nehari manifold associated with $J$. Hereafter, we say that $u_0\in H^1(\mathbb{R}^N)$ is a *ground state solution* if

$$J'(u_0)=0,\quad u_0\in\mathcal{O}\quad\text{and}\quad J(u_0)=\inf_{w\in\mathcal{O}}J(w).$$

In [45], Szulkin and Weth have established the existence of ground state solution for problem $(P_1)$ by completing the study made in [38], in the sense that, they also minimize the energy functional on $\mathcal{O}$, however they have used more weaker conditions on $f$, for example $f$ is continuous, $\mathbb{Z}^N$-periodic in $x$ and satisfies

$$|f(x,t)|\leq C(1+|t|^{p-1}),\ \ \forall t\in\mathbb{R}\quad\text{and}\quad x\in\mathbb{R}^N \tag{$h_4$}$$

for some $C>0$ and $p\in(2,2^*)$.

$$f(x,t)=o(t)\ \text{ uniformly in }\ x\text{ as }|t|\to 0. \tag{$h_5$}$$

$$F(x,t)/|t|^2\to+\infty\ \text{ uniformly in }\ x\text{ as }|t|\to+\infty, \tag{$h_6$}$$

and

$$t\mapsto f(x,t)/|t|\ \text{ is strictly increasing on }\ \mathbb{R}\setminus\{0\}. \tag{$h_7$}$$

The same approach has been used by Zhang, Xu and Zhang [60, 61] to study a class of indefinite and asymptotically periodic problem.

In [5], Alves and Germano have studied the existence of ground state solution for problem $(P_1)$ by supposing the $f$ has a critical growth for $N\geq 2$, while in [6] the authors have established the existence and concentration of solution for problem $(P)_\epsilon$ by supposing that $f$ has a subcritical growth and $V,A$ verify the conditions $(V)$ and $(A)$ respectively.

Motivated by results found [5, 6], in the present paper we intend to study the existence and concentration of solution for problem $(P)_\epsilon$ for the case where function $f$ has a critical growth. Since the critical growth brings a lost of compactness, we have established new estimates for the problem. Here, the concentration phenomena is very subtle, because we need to be careful to prove some estimates involving the $L^\infty$ norm of the solutions for $\epsilon$ small enough, for more details see Section 2.2 for $N \geq 3$, and Section 3.3 for $N = 2$. Moreover of the conditions $(V)$ and $(A)$ on the functions $V$ and $A$ respectively, we are supposing the following conditions on $f$:

**The Case $N \geq 3$:**

In this case $f : \mathbb{R} \to \mathbb{R}$ is of the form

$$(f_0) \qquad f(t) = \xi |t|^{q-1} t + |t|^{2^*-2} t, \quad \forall t \in \mathbb{R};$$

with $\xi > 0, q \in (2, 2^*)$ and $2^* = 2N/N - 2$.

**The Case $N = 2$:**

In this case $f : \mathbb{R} \to \mathbb{R}$ is a continuous function that satisfies

$(f_1)$ $\frac{f(t)}{t} \to 0$ as $t \to 0$ ;

$(f_2)$ The function $t \mapsto \frac{f(t)}{t}$ is increasing on $(0, +\infty)$ and decreasing on $(-\infty, 0)$;

$(f_3)$ There exists $\theta > 2$ such that

$$0 < \theta F(t) \leq f(t)t, \quad \forall t \in \mathbb{R} \setminus \{0\}$$

where

$$F(t) := \int_0^t f(s) ds;$$

$(f_4)$ There exists $\Gamma > 0$ such that $|f(t)| \leq \Gamma e^{4\pi t^2}$ for all $t \in \mathbb{R}$;

$(f_5)$ There exist $\tau > 0$ and $q > 2$ such that $F(t) \geq \tau |t|^q$ for all $t \in \mathbb{R}$.

The condition $(f_4)$ says that $f$ can have an exponential critical growth. Here, we recall that a function $f$ has an exponential critical growth, if there is $\alpha_0 > 0$ such that

$$\lim_{|t| \to +\infty} \frac{|f(t)|}{e^{\alpha |t|^2}} = 0, \ \ \forall \alpha > \alpha_0, \lim_{|t| \to +\infty} \frac{|f(t)|}{e^{\alpha |t|^2}} = +\infty, \ \ \forall \alpha < \alpha_0.$$

Our main theorem is the following

**Theorem 3.1.1** *Assume* $(V), (A)$, $(f_0)$ *for* $N \geq 3$, $(f_1) - (f_5)$ *for* $N = 2$. *Then, there exist* $\tau_0, \xi_0, \epsilon_0 > 0$ *such that* $(P)_\epsilon$ *has a ground state solution* $u_\epsilon$ *for all* $\epsilon \in (0, \epsilon_0)$, *with* $\xi \geq \xi_0$ *if* $N = 3$ *and* $\tau \geq \tau_0$ *if* $N = 2$. *Moreover, if* $x_\epsilon \in \mathbb{R}^N$ *denotes a global maximum point of* $|u_\epsilon|$, *then*

$$\lim_{\epsilon \to 0} A(\epsilon x_\epsilon) = \sup_{x \in \mathbb{R}^N} A(x).$$

In the proof of Theorem 3.1.1, we will use variational methods to get a critical point for the energy function $I_\epsilon : H^1(\mathbb{R}^N) \to \mathbb{R}$ given by

$$I_\epsilon(u) = \frac{1}{2}B(u,u) - \int_{\mathbb{R}^N} A(\epsilon x)F(u)dx,$$

where $B : H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N) \to \mathbb{R}$ is the bilinear form

$$B(u,v) = \int_{\mathbb{R}^N} (\nabla u \nabla v + V(x)uv)\, dx, \quad \forall u, v \in H^1(\mathbb{R}^N). \tag{1.1}$$

It is well known that $I_\epsilon \in C^1(H^1(\mathbb{R}^N), \mathbb{R})$ with

$$I_\epsilon'(u)v = B(u,v) - \int_{\mathbb{R}^N} A(\epsilon x)f(u)v dx, \quad \forall u, v \in H^1(\mathbb{R}^N).$$

Consequently, critical points of $I_\epsilon$ are precisely the weak solutions of $(P)_\epsilon$.

Note that the bilinear form $B$ is not positive definite, therefore it does not induce a norm. As in [45], there is an inner product $\langle\ ,\ \rangle$ in $H^1(\mathbb{R}^N)$ such that

$$I_\epsilon(u) = \frac{1}{2}\|u^+\|^2 - \frac{1}{2}\|u^-\|^2 - \int_{\mathbb{R}^N} A(\epsilon x)F(u)\, dx, \tag{1.2}$$

where $\|u\| = \sqrt{\langle u, u\rangle}$ and $H^1(\mathbb{R}^N) = E^+ \oplus E^-$ corresponds to the spectral decomposition of $-\Delta + V$ with respect to the positive and negative part of the spectrum with $u = u^+ + u^-$, where $u^+ \in E^+$ and $u^- \in E^-$. It is well known that $B$ is positive definite on $E^+$, $B$ is negative definite on $E^-$ and the norm $\|\ \|$ is an equivalent norm to the usual norm in $H^1(\mathbb{R}^N)$, that is, there are $a, b > 0$ such that

$$b||u|| \leq ||u||_{H^1(\mathbb{R}^N)} \leq a||u||, \quad \forall\, u \in H^1(\mathbb{R}^N). \tag{1.3}$$

From now on, for each $u \in H^1(\mathbb{R}^N)$, $\hat{E}(u)$ designates the set

$$\hat{E}(u) = E^- \oplus [0, +\infty)u. \tag{1.4}$$

The plan of the paper is as follows: In Section 2 we will study the existence and concentration of solution for $N \geq 3$, while in Section 3 we will focus our attention to dimension $N = 2$.

**Notation:** In this paper, we use the following notations:

- The usual norms in $H^1(\mathbb{R}^N)$ and $L^p(\mathbb{R}^N)$ will be denoted by $\| \quad \|_{H^1(\mathbb{R}^N)}$ and $| \quad |_p$ respectively.

- $C$ denotes (possible different) any positive constant.

- $B_R(z)$ denotes the open ball with center $z$ and radius $R$ in $\mathbb{R}^N$.

- We say that $u_n \to u$ in $L^p_{loc}(\mathbb{R}^N)$ when

$$u_n \to u \quad \text{in} \quad L^p(B_R(0)), \quad \forall R > 0.$$

- If $g$ is a mensurable function, the integral $\int_{\mathbb{R}^N} g(x)\,dx$ will be denoted by $\int g(x)\,dx$.

- We denote $\delta_x$ the Dirac measure.

- If $\varphi \in C_c^\infty(\mathbb{R}^N)$, the set $\overline{\{x \in \mathbb{R}^N \ ; \ \varphi(x) \neq 0\}}$ will be denoted by $supp\varphi$.

## 3.2 The case $N \geq 3$.

We begin this section by studying the case where $A$ is a constant function. More precisely, we consider the following autonomous problem

$$\begin{cases} -\Delta u + V(x)u = \lambda f(u), & x \in \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \end{cases} \tag{$(AP)_\lambda$}$$

with $\lambda \in [A_0, +\infty)$ and $f : \mathbb{R} \to \mathbb{R}$ being of the form

$$f(t) = \xi|t|^{q-1}t + |t|^{2^*-2}t \quad \forall t \in \mathbb{R};$$

with $\xi > 0, q \in (2, 2^*)$ and $2^* = 2N/N-2$.

Associated with $(AP)_\lambda$, we have the energy functional $J_\lambda : H^1(\mathbb{R}^N) \to \mathbb{R}$ given by

$$J_\lambda(u) = \frac{1}{2}\int (|\nabla u|^2 + V(x)|u|^2)\,dx - \lambda \int F(u)\,dx,$$

or equivalently

$$J_\lambda(u) = \frac{1}{2}\|u^+\|^2 - \frac{1}{2}\|u^-\|^2 - \lambda \int F(u)\,dx.$$

In what follows, let us denote by $d_\lambda$ the real number defined by

$$d_\lambda = \inf_{u \in \mathcal{N}_\lambda} J_\lambda(u); \tag{2.5}$$

where

$$\mathcal{N}_\lambda = \left\{ u \in H^1(\mathbb{R}^N) \setminus E^- \ ; \ J'_\lambda(u)u = 0 \text{ and } J'_\lambda(u)v = 0, \forall\, v \in E^- \right\}. \tag{2.6}$$

In [5], Alves and Germano have proved that for each $\lambda \in [A_0, +\infty)$, the problem $(AP)_\lambda$ possesses a ground state solution $u_\lambda \in H^1(\mathbb{R}^N)$, that is,

$$u_\lambda \in \mathcal{N}_\lambda, \quad J_\lambda(u_\lambda) = d_\lambda \quad \text{and} \quad J'_\lambda(u) = 0.$$

A key point to prove the existence of the ground state $u_\lambda$ are the following informations involving $d_\lambda$:

$$0 < d_\lambda = \inf_{u \in E^+ \setminus \{0\}} \max_{v \in \widehat{E}(u)} J_\lambda(u) \tag{2.7}$$

and

$$d_\lambda < \frac{1}{N} \frac{S^{N/2}}{\lambda^{\frac{N-2}{2}}}, \quad \forall \lambda > A_0. \tag{2.8}$$

Here, we would like to point out that (2.8) holds for $N = 3$ if $\xi$ is large enough, while for $N \geq 4$ there is no restriction on $\xi$. This fact justifies why $\xi$ must be large for $N = 3$ in Theorem 3.1.1.

An interesting and important fact is that for each $u \in H^1(\mathbb{R}^N) \setminus E^-$, $\mathcal{N}_\lambda \cap \hat{E}(u)$ is a singleton set and the element of this set is the unique global maximum of $J_\lambda|_{\hat{E}(u)}$, that is, there are $t^* \geq 0$ and $v^* \in E^-$ such that

$$J_\lambda(t^* u + v^*) = \max_{w \in \widehat{E}(u)} J_\lambda(w). \tag{2.9}$$

After the above commentaries we are ready to prove an important result involving the function $\lambda \mapsto d_\lambda$.

**Proposition 3.2.1** *The function $\lambda \mapsto d_\lambda$ is decreasing and continuous on $[A_0, +\infty)$.*

**Proof.** From [6, Proposition 2.3], the function $\lambda \mapsto d_\lambda$ is decreasing, and if $\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq ... \leq \lambda_n \to \lambda$ then $\lim_n d_{\lambda_n} = d_\lambda$. It suffices to check that $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq ... \geq \lambda_n \to \lambda$ implies $\lim_n d_{\lambda_n} = d_\lambda$. Let $u_n$ be a ground state solution of $(AP)_{\lambda_n}$, $t_n > 0$ and $v_n \in E^-$ verifying

$$J_\lambda(t_n u_n + v_n) = \max_{\widehat{E}(u_n)} J_\lambda.$$

Our goal is to show that $(u_n)$ is bounded in $H^1(\mathbb{R}^N)$. First of all, note that

$$\left(\frac{1}{2} - \frac{1}{q}\right) \int f(u_n) u_n dx \leq \int \left(\frac{1}{2} f(u_n) u_n - F(u_n)\right) dx = \tag{2.10}$$

$$= \frac{1}{\lambda_n}\left(J_{\lambda_n}(u_n) - \frac{1}{2}J'_{\lambda_n}(u_n)u_n\right) = \frac{1}{\lambda_n}J_{\lambda_n}(u_n) = \frac{1}{\lambda_n}d_{\lambda_n} \le \frac{1}{\lambda}d_\lambda,$$

which proves the boundedness of $\left(\int f(u_n)u_n dx\right)$. Fixing $g(t) = \chi_{[-1,1]}(t)f(t)$ and $l(t) = \chi_{[-1,1]^c}(t)f(t)$, we have that

$$g(t) + l(t) = f(t), \quad \forall t \in \mathbb{R}.$$

From definition of $g$ and $l$, there exists $k > 0$ such that

$$|g(t)|^r \le ktf(t) \text{ and } |l(t)|^s \le ktf(t), \quad \forall t \in \mathbb{R},$$

where $r := \frac{q+1}{q}$ and $s := \frac{2^*}{2^*-1}$. Thus,

$$\left|\int f(u_n)u_n^+ dx\right| \le \int |g(u_n)u_n^+|dx + \int |l(u_n)u_n^+|dx \le$$

$$\le \left(\int |g(u_n)|^r dx\right)^{1/r} |u_n^+|_{q+1} + \left(\int |l(u_n)|^s dx\right)^{1/s} |u_n^+|_{2^*} \le$$

$$\le C\left(\int f(u_n)u_n dx\right)^{1/r} ||u_n^+|| + C\left(\int f(u_n)u_n dx\right)^{1/s} ||u_n^+|| \le C||u_n||.$$

Suppose by contradiction that $||u_n|| \to +\infty$. Then

$$\int \frac{f(u_n)u_n^+}{||u_n||^2}dx \to 0.$$

On the other hand, the equality

$$0 = \frac{J'_{\lambda_n}(u_n)u_n^+}{||u_n||^2} = \frac{||u_n^+||^2}{||u_n||^2} - \lambda_n \int \frac{f(u_n)u_n^+}{||u_n||^2}dx$$

leads to

$$\frac{||u_n^+||^2}{||u_n||^2} \to 0.$$

As $u_n \in \mathcal{N}_{\lambda_n}$, it follows that $\|u_n^-\| \le \|u_n^+\|$, and thus,

$$1 = \frac{||u_n^+||^2}{||u_n||^2} + \frac{||u_n^-||^2}{||u_n||^2} \le 2\frac{||u_n^+||^2}{||u_n||^2} \to 0,$$

a contradiction. This shows the boundedness of $(u_n)$. We claim that there are $(y_n) \subset \mathbb{Z}^N$ and $r, \eta > 0$ such that

$$\int_{B_r(y_n)} |u_n|^{2^*} dx > \eta, \quad \forall n \in \mathbb{N}. \tag{2.11}$$

Arguing by contradiction, if the inequality does not occur, from [43, Lemma 2.1], $u_n \to 0$ in $L^p(\mathbb{R}^N)$ for all $p \in (2, 2^*]$, and so, $\int f(u_n)u_n^+ dx \to 0$. This together with the equality below

$$0 = J'_{\lambda_n}(u_n)u_n^+ = ||u_n^+||^2 - \lambda_n \int f(u_n)u_n^+ dx.$$

gives $||u_n^+|| \to 0$, which is a contradiction because $||u_n|| \geq \sqrt{2d_{\lambda_n}} \geq \sqrt{2d_{\lambda_1}}$. Thereby (2.11) follows.

Define $\widetilde{u}_n(x) := u_n(x + y_n)$. By [6, Lemma 2.1], $\widetilde{u}_n^+(x) = u_n^+(x + y_n)$ and $(\widetilde{u}_n)$ is bounded in $H^1(\mathbb{R}^N)$. In the sequel, let us assume that for some subsequence $\widetilde{u}_n \rightharpoonup u$ in $H^1(\mathbb{R}^N)$. Our goal is to show that $u \neq 0$. Inspired by [5, Lemma 2.17], let us suppose by contradiction $u = 0$ and

$$|\nabla \widetilde{u}_n|^2 \rightharpoonup \mu, \quad |\widetilde{u}_n|^{2^*} \rightharpoonup \nu \ \text{ in } \mathcal{M}^+(\mathbb{R}^N).$$

By Concentration-Compactness Principle due to Lions [29], there exist a countable set J, $(x_i)_{i\in \mathrm{J}} \subset \mathbb{R}^N$ and $(\mu_i)_{i\in \mathrm{J}}, (\nu_i)_{i\in \mathrm{J}} \subset [0, +\infty)$ such that

$$\nu = \sum_{i\in \mathrm{J}} \nu_i \delta_{x_i}, \ \mu \geq \sum_{i \in \mathrm{J}} \mu_i \delta_{x_i}, \quad \text{and } \mu_i = S\nu_i^{2/2^*}.$$

We will prove that $\nu_i = 0$ for all $i \in$J. Suppose there exists $i \in$J such that $\nu_i \neq 0$. Then,

$$\begin{aligned} d_\lambda \ &\geq \lim_n d_{\lambda_n} = \lim_n \left( J_{\lambda_n}(u_n) - \frac{1}{2} J'_{\lambda_n}(u_n)u_n \right) \\ &\geq \lim_n \lambda_n \left( \frac{1}{2} - \frac{1}{2^*} \right) \int |u_n|^{2^*} dx \\ &= \lim_n \frac{\lambda_n}{N} \int |\widetilde{u}_n|^{2^*} dx = \frac{\lambda}{N} \sum_{j\in \mathrm{J}} \nu_j, \end{aligned}$$

which means

$$d_\lambda \geq \frac{\lambda}{N} \sum_{j \in \mathrm{J}} \nu_j. \tag{2.12}$$

Let $\varphi_\delta(x) := \varphi\left(\frac{x - x_i}{\delta}\right)$ for all $x \in \mathbb{R}^N$ and $\delta > 0$, where $\varphi \in C_c^\infty(\mathbb{R}^N)$ is such that $\varphi \equiv 1$ on $B_1(0)$, $\varphi \equiv 0$ on $\mathbb{R}^N \setminus B_2(0)$, $0 \leq \varphi \leq 1$ and $|\nabla\varphi| \leq 2$. Consequently $(\varphi_\delta \widetilde{u}_n)$ is bounded in $H^1(\mathbb{R}^N)$ and

$$J'_{\lambda_n}(\widetilde{u}_n)(\varphi_\delta \widetilde{u}_n) = 0,$$

that is,

$$\int \nabla \widetilde{u}_n \nabla(\varphi_\delta \widetilde{u}_n) dx + \int V(x)\varphi_\delta \widetilde{u}_n^2 dx = \lambda_n \xi \int |\widetilde{u}_n|^{q+1} \varphi_\delta dx + \lambda_n \int |\widetilde{u}_n|^{2^*} \varphi_\delta dx.$$

Passing to the limit as $n \to +\infty$,

$$\int \varphi_\delta d\mu = \lambda \int \varphi_\delta d\nu$$

Now, taking the limit $\delta \to 0$,

$$\mu(x_i) = \lambda\nu_i.$$

From the fact that $\mu(x_i) \geq \mu_i$, we derive

$$S\nu_i^{2/2^*} = \mu_i \leq \mu(x_i) = \lambda\nu_i,$$

and so

$$S^{N/2} \leq \lambda^{N/2}\nu_i.$$

Consequently,

$$\frac{\lambda}{N}\nu_i \geq \frac{1}{N}\frac{S^{N/2}}{\lambda^{\frac{N-2}{2}}}. \tag{2.13}$$

From (2.12) and (2.13),

$$d_\lambda \geq \frac{1}{N}\frac{S^{N/2}}{\lambda^{\frac{N-2}{2}}},$$

contrary to (2.8). From this, $\nu_i = 0$ for all $i \in$J and $\widetilde{u}_n \to 0$ in $L^{2^*}_{loc}(\mathbb{R}^N)$, which contradicts (2.11). This permit us to conclude that $u \neq 0$.

**Claim 3.2.2** *If* $u^+ = 0$*, then* $u^- = 0$.

In fact, if $u^+ = 0$,

$$\int f(u)u^- dx = \int f(u)u^+ dx + \int f(u)u^- dx = \int f(u)u dx \geq 0.$$

On the other hand, letting $n \to +\infty$ in the equality below

$$0 = J_{\lambda_n}(\widetilde{u}_n)u^- = B(\widetilde{u}_n, u^-) - \lambda_n \int f(\widetilde{u}_n)u^- dx$$

we find

$$-||u^-||^2 = B(u, u^-) = \lambda \int f(u)u^- dx \geq 0,$$

thereby showing that $u^- = 0$.

The Claim 3.2.2 implies that $u^+ \neq 0$, because $u \neq 0$ and $u = u^+ + u^-$. Define $\mathcal{V} := \{\widetilde{u}_n^+\}_{n\in\mathbb{N}}$. Since $\widetilde{u}_n^+ \rightharpoonup u^+ \neq 0$, then $0 \notin \overline{\mathcal{V}}^{\sigma(H^1(\mathbb{R}^N),H^1(\mathbb{R}^N)')}$ and $\mathcal{V}$ is bounded in $H^1(\mathbb{R}^N)$. Applying [6, Lemma 2.2], there exists $R > 0$ such that

$$J_\lambda \leq 0 \text{ on } \widehat{E}(u) \setminus B_R(0), \quad \text{for all } u \in \mathcal{V}. \tag{2.14}$$

Setting $\widetilde{v}_n(x) := v_n(x + y_n)$,

$$J_\lambda(t_n\widetilde{u}_n + \widetilde{v}_n) = J_\lambda(t_n u_n + v_n) \geq d_\lambda > 0. \tag{2.15}$$

By (2.14) and (2.15), $||t_n\widetilde{u}_n + \widetilde{v}_n|| \leq R$ for all $n \in \mathbb{N}$. As $||t_n u_n + v_n|| = ||t_n\widetilde{u}_n + \widetilde{v}_n||$, $(t_n u_n + v_n)$ is also bounded in $H^1(\mathbb{R}^N)$ and

$$\begin{aligned} d_\lambda \ &\leq J_\lambda(t_n u_n + v_n) = (\lambda_n - \lambda)\int F(t_n u_n + v_n)dx + J_{\lambda_n}(t_n u_n + v_n) \leq \\ &\leq o_n + J_{\lambda_n}(u_n) = o_n + d_{\lambda_n} \leq o_n + d_\lambda, \end{aligned}$$

from where it follows that $\lim\limits_{n} d_{\lambda_n} = d_\lambda$. ■

### 3.2.1 Existence of ground state for problem $(P)_\epsilon$.

In the sequel, we fix

$$\mathcal{M}_\epsilon := \{u \in H^1(\mathbb{R}^N) \setminus E^- \ ; \ I'_\epsilon(u)u = I'_\epsilon(u)v = 0, \text{ for all } v \in E^-\}$$

and

$$c_\epsilon = \inf_{\mathcal{M}_\epsilon} I_\epsilon.$$

By using the same arguments found in [5], it follows that $c_\epsilon > 0$, and for each $u \in H^1(\mathbb{R}^N) \setminus E^-$, there exist $t \geq 0$ and $v \in E^-$ verifying

$$I_\epsilon(tu + v) = \max_{\widehat{E}(u)} I_\epsilon \ \text{ and } \ \{tu + v\} = \mathcal{M}_\epsilon \cap \widehat{E}(u).$$

The same idea of [5, Lemma 2.6] proves that

$$||u^+||^2 \geq 2c_\epsilon, \quad \text{for all } u \in \mathcal{M}_\epsilon \quad \text{and} \quad \epsilon > 0. \tag{2.16}$$

In what follows, without loss of generality we assume that

$$A(0) = \max_{x \in \mathbb{R}^N} A(x).$$

Our first result in this section establishes an important relation involving the levels $c_\epsilon$ and $c_0$.

**Lemma 3.2.3** *The limit* $\lim_{\epsilon \to 0} c_\epsilon = c_0$ *holds. Moreover, let* $w_0$ *be a ground state solution of the problem* $(P)_0$, $t_\epsilon \geq 0$ *and* $v_\epsilon \in E^-$ *such that* $t_\epsilon w_0 + v_\epsilon \in \mathcal{M}_\epsilon$. *Then*

$$t_\epsilon \to 1 \quad \textit{and} \quad v_\epsilon \to 0 \quad \textit{as} \quad \epsilon \to 0.$$

**Proof.** See [6, Lemmas 3.1 and 3.3]. ■

**Corollary 3.2.4** *There exists* $\epsilon_0 > 0$ *such that*

$$c_\epsilon < d_{A_\infty} \quad \textit{and} \quad c_\epsilon < \frac{S^{N/2}}{NA(0)^{\frac{N-2}{2}}}, \qquad \forall \epsilon \in (0, \epsilon_0).$$

**Proof.** Since $c_0 < d_{A_\infty}$ and

$$c_0 < \frac{S^{N/2}}{NA(0)^{\frac{N-2}{2}}}, \quad \text{(see (2.8))}$$

the corollary is an immediate consequence of Lemma 3.2.3. ■

The next result is essential to show the existence of ground state solution of $(P)_\epsilon$ for $\epsilon$ small enough. Since it follows as in [5, Proposition 2.16], we omit its proof.

**Proposition 3.2.5** *There exists a bounded sequence* $(u_n) \subset \mathcal{M}_\epsilon$ *such that* $(u_n)$ *is* $(PS)_{c_\epsilon}$ *for* $I_\epsilon$.

The following result is the main result this section

**Theorem 3.2.6** *The problem* $(P)_\epsilon$ *has a ground state solution for all* $\epsilon \in (0, \epsilon_0)$, *where* $\epsilon_0 > 0$ *was given in Corollary 3.2.4.*

**Proof.** Let $(u_n) \subset \mathcal{M}_\epsilon$ be the $(PS)_{c_\epsilon}$ sequence for $I_\epsilon$ given in Proposition 3.2.5. Then, there exist $(z_n) \subset \mathbb{Z}^N$ and $\eta, r > 0$ such that

$$\int_{B_r(z_n)} |u_n|^{2^*} dx > \eta, \quad \forall n \in \mathbb{N}. \tag{2.17}$$

In fact, otherwise, by [43, Lemma 2.1], $u_n \to 0$ in $L^p(\mathbb{R}^N)$ for all $p \in (2, 2^*]$. Then,

$$||u_n^+||^2 = \int A(\epsilon x) f(u_n) u_n^+ dx \to 0,$$

which is a contradiction with (2.16), and (2.17) is proved.

**Claim 3.2.7** *The sequence* $(z_n)$ *is bounded in* $\mathbb{R}^N$.

Arguing by contradiction, suppose $|z_n| \to +\infty$ and define $w_n(x) := u_n(x + z_n)$. Then $(w_n)$ is bounded, and for some subsequence, $w_n \rightharpoonup w$ in $H^1(\mathbb{R}^N)$. Our goal is to prove that $w \neq 0$. Suppose $w = 0$ and

$$|\nabla w_n|^2 \rightharpoonup \mu, \quad |w_n|^{2^*} \rightharpoonup \nu, \quad \text{in } \mathcal{M}^+(\mathbb{R}^N).$$

By Concentration-Compactness Principle due to Lions [29], there exist a countable set J, $(x_i)_{i\in J}\subset\mathbb{R}^N$ and $(\mu_i)_{i\in J},(\nu_i)_{i\in J}\subset[0,+\infty)$ satisfying

$$\nu=\sum_{i\in J}\nu_i\delta_{x_i},\quad \mu\geq\sum_{i\in J}\mu_i\delta_{x_i},\quad\text{and}\quad \mu_i=S\nu_i^{2/2^*}.$$

Next, we are going to prove that $\nu_i=0$ for all $i\in$J. Suppose that there exists $i\in$J such that $\nu_i\neq 0$. Note that

$$\begin{aligned}
c_\epsilon&=\lim_n\left(I_\epsilon(u_n)-\frac{1}{2}I'_\epsilon(u_n)u_n\right)\geq\frac{1}{N}\lim_n\int A(\epsilon x)|u_n|^{2^*}dx=\\
&=\tfrac{1}{N}\lim_n\int A(\epsilon x+\epsilon z_n)|w_n|^{2^*}dx\geq\frac{1}{N}\lim_n\int_{B_\delta(x_i)}A(\epsilon x+\epsilon z_n)|w_n|^{2^*}dx=\\
&=\tfrac{1}{N}\lim_n\int_{B_\delta(x_i)}(A(\epsilon x+\epsilon z_n)-A_\infty)|w_n|^{2^*}dx+\frac{1}{N}\lim_n\int_{B_\delta(x_i)}A_\infty|w_n|^{2^*}dx\geq\\
&\geq\tfrac{1}{N}\int A_\infty\varphi_{\delta/2}(x)d\nu,
\end{aligned}$$

where $\varphi_\delta(x)=\varphi\left(\frac{x-x_i}{\delta}\right)$, and $\varphi\in C_c^\infty(\mathbb{R}^N)$ satisfies $0\leq\varphi\leq 1$, $|\nabla\varphi|\leq 2$, $\varphi\equiv 1$ on $B_1(0)$ and $\varphi\equiv 0$ on $\mathbb{R}^N\setminus B_2(0)$.

By Dominated Convergence Theorem,

$$\lim_{\delta\to 0}\int A_\infty\varphi_{\delta/2}(x)d\nu=A_\infty\nu_i,$$

thus

$$c_\epsilon\geq\frac{1}{N}A_\infty\nu_i. \tag{2.18}$$

On the other hand, by a simple calculus, $(\varphi_\delta w_n)$ is bounded in $H^1(\mathbb{R}^N)$. Setting $\varphi_{\delta,n}(x):=\varphi_t(x-z_n)$,

$$||\varphi_{\delta,n}u_n||=||\varphi_\delta w_n||,\quad\forall n\in\mathbb{N}$$

and so,

$$I'_\epsilon(u_n)(\varphi_{\delta,n}u_n)\to 0,$$

or equivalently

$$\begin{aligned}
&\textstyle\int|\nabla w_n|^2\varphi_\delta dx+\int(\nabla w_n\nabla\varphi_\delta)w_ndx+\int V(x)\varphi_\delta w_n^2dx-\\
&\textstyle-\int A(\epsilon x+\epsilon z_n)|w_n|^{q+1}\varphi_\delta dx-\int A(\epsilon x+\epsilon z_n)|w_n|^{2^*}\varphi_\delta dx\to 0
\end{aligned}$$

Taking the limit $n\to+\infty$, and after $\delta\to 0$, we obtain

$$\mu(x_i)=A_\infty\nu_i.$$

Since $S\nu_i^{2/2^*} \leq \mu(x_i)$, it follows that

$$S^{N/2} \leq A_\infty^{\frac{N}{2}}\nu_i \leq A(0)^{\frac{N-2}{2}}A_\infty\nu_i. \tag{2.19}$$

By (2.18) and (2.19),

$$c_\epsilon \geq \frac{S^{N/2}}{NA(0)^{\frac{N-2}{2}}}$$

contrary to Corollary 3.2.4. Consequently $\nu_i = 0$ for all $i \in$J, which means $w_n \to 0$ in $L^{2^*}_{loc}(\mathbb{R}^N)$, contrary to (2.17). From this, $w \neq 0$.

Now, consider $\psi \in H^1(\mathbb{R}^N)$ and $\psi_n(x) := \psi(x+z_n)$. Then,

$$o_n(1) = I'_\epsilon(u_n)\psi_n = B(u_n,\psi_n) - \int A(\epsilon x)f(u_n)\psi_n dx$$

or equivalently

$$o_n = B(w_n,\psi) - \int A(\epsilon x + \epsilon z_n)f(w_n)\psi dx.$$

Taking the limit $n \to +\infty$, $J'_{A_\infty}(w)\psi = 0$. As $\psi \in H^1(\mathbb{R}^N)$ is arbitrary, $w$ is a critical point of $J_{A_\infty}$, and thus, by Fatou's Lemma

$$\begin{aligned} d_{A_\infty} &\leq J_{A_\infty}(w) = J_{A_\infty}(w) - \tfrac{1}{2}J'_{A_\infty}(w)w \\ &= \textstyle\int A_\infty \left(\tfrac{1}{2}f(w)w - F(w)\right)dx \\ &\leq \liminf_n \int A(\epsilon x + \epsilon z_n)\left(\frac{1}{2}f(w_n)w_n - F(x,w_n)\right)dx \\ &= \liminf_n \int A(\epsilon x)\left(\frac{1}{2}f(u_n)u_n - F(u_n)\right)dx \\ &= \lim_n \left(I_\epsilon(u_n) - \frac{1}{2}I'_\epsilon(u_n)u_n\right) = c_\epsilon < d_{A_\infty}, \end{aligned}$$

which is impossible. Thereby $(z_n)$ is bounded in $\mathbb{R}^N$, and the claim follows.

Consider $R > 0$ such that $B_r(z_n) \subset B_R(0)$. By (2.17),

$$\int_{B_R(0)} |u_n|^{2^*}dx > \eta, \quad \forall n \in \mathbb{N}.$$

By considering that $u_n \rightharpoonup u$ and proceeding as in Claim 3.2.7, $u \neq 0$. Since $u$ is a nontrivial critical point for $I_\epsilon$, we must have $I_\epsilon(u) \geq c_\epsilon$. On the other hand, by Fatou's Lemma,

$$\begin{aligned} c_\epsilon &= \lim_n \left(I_\epsilon(u_n) - \frac{1}{2}I'_\epsilon(u_n)u_n\right) = \lim_n \int A(\epsilon x)\left(\frac{1}{2}f(u_n)u_n - F(u_n)\right)dx \\ &\geq \textstyle\int A(\epsilon x)\left(\tfrac{1}{2}f(u)u - F(u)\right)dx = I_\epsilon(u) - \tfrac{1}{2}I'_\epsilon(u)u = I_\epsilon(u). \end{aligned}$$

This proves that $u$ is a ground state solution of $(P)_\epsilon$ for all $\epsilon \in (0,\epsilon_0)$. ■

### 3.2.2 Concentration of the solutions.

In what follows, we consider the set

$$\mathcal{A} := \{z \in \mathbb{R}^N \ ; \ A(z) = A(0)\},$$

and a sequence $(\epsilon_n) \subset (0, \epsilon_0)$ with $\epsilon_n \to 0$ as $n \to +\infty$. Moreover, we fix $u_n \in H^1(\mathbb{R}^N)$ satisfying

$$I_n(u_n) = c_n \quad \text{and} \quad I'_n(u_n) = 0,$$

where $I_n := I_{\epsilon_n}$ and $c_n := c_{\epsilon_n}$. Using the same arguments explored in [5, Lemma 2.6],

$$||u_n^+||^2 \geq 2c_n \geq 2c_0, \quad \forall n \in \mathbb{N}. \tag{2.20}$$

**Lemma 3.2.8** *The sequence $(u_n)$ is bounded in $H^1(\mathbb{R}^N)$.*

**Proof.** See [5, Lemma 2.10]. ■

**Lemma 3.2.9** *There exist $(y_n) \subset \mathbb{Z}^N$ and $r, \eta > 0$ such that*

$$\int_{B_r(y_n)} |u_n|^{2^*} dx > \eta, \quad \forall n \in \mathbb{N}.$$

**Proof.** Suppose the lemma were false. Then, by [43, Lemma 2.1], $u_n \to 0$ in $L^p(\mathbb{R}^N)$ for all $p \in (2, 2^*]$, and so,

$$\int A(\epsilon_n x) f(u_n) u_n^+ dx \to 0.$$

As $I'_n(u_n)u_n^+ = 0$, it follows that $||u_n^+||^2 \to 0$, a contradiction. This proves the lemma. ■

In the sequel, we fix $v_n(x) := u_n(x+y_n)$ for all $x \in \mathbb{R}^N$ and for all $n \in \mathbb{N}$. Thereby, for some subsequence, we can assume that $v_n \rightharpoonup v$ in $H^1(\mathbb{R}^2)$. It is very important to point out that only one of the cases below holds for some subsequence:

$$\epsilon_n y_n \to z \in \mathbb{R}^N$$

or

$$|\epsilon_n y_n| \to +\infty.$$

For this reason, we will consider a subsequence of $(\epsilon_n)$ such that one of the above conditions holds. Have this in mind, let us denote

$$A_z := \begin{cases} A(z), & \text{if the condition (1) holds} \\ A_\infty, & \text{if the condition (2) holds.} \end{cases}$$

Since $A$ is continuous, it follows that $|A(\epsilon_n x + \epsilon_n y_n) - A_z| \to 0$ uniformly with respect to $x$ on bounded Borel sets $B \subset \mathbb{R}^N$. Consequently

$$\lim \int_B A(\epsilon_n x + \epsilon_n y_n)|v_n|^{2^*}\varphi dx = \lim \int_B A_z |v_n|^{2^*}\varphi dx, \tag{2.21}$$

for each $\varphi \in L^\infty(\mathbb{R}^N)$.

By using (2.21) and applying the same idea of Claim 3.2.7, we see that $v \neq 0$.

**Lemma 3.2.10** *The sequence $(\epsilon_n y_n)$ is bounded in $\mathbb{R}^N$. Moreover, $J'_{A(0)}(v) = 0$ and if $\epsilon_n y_n \to z \in \mathbb{R}^N$, then $z \in \mathcal{A}$.*

**Proof.** First of all, we will prove that $(\epsilon_n y_n)$ is bounded. Suppose that $|\epsilon_n y_n| \to +\infty$. Consider $\psi \in C_c^\infty(\mathbb{R}^N)$ and $\psi_n(x) := \psi(x - y_n)$. Since $I'_n(u_n)\psi_n = 0$ for all $n \in \mathbb{N}$, then

$$\int \nabla u_n \nabla \psi_n + V(x) u_n \psi_n dx = \int A(\epsilon_n x) f(u_n)\psi_n dx,$$

or equivalently

$$\int \nabla v_n \nabla \psi + V(x) v_n \psi dx = \int A(\epsilon_n x + \epsilon_n y_n) f(v_n)\psi dx.$$

Taking the limit $n \to +\infty$, we derive

$$\int \nabla v \nabla \psi + V(x) v \psi dx = \int A_\infty f(v)\psi dx,$$

thereby showing that $J'_{A_\infty}(v) = 0$. As $v \neq 0$, the Fatou's Lemma yields

$$\begin{aligned} d_{A_\infty} &\leq J_{A_\infty}(v) = J_{A_\infty}(v) - \tfrac{1}{2}J'_{A_\infty}(v)v = \textstyle\int A_\infty \left(\tfrac{1}{2}f(v)v - F(v)\right) dx \\ &\leq \liminf_n \int A(\epsilon_n x + \epsilon_n y_n)\left(\frac{1}{2}f(v_n)v_n - F(v_n)\right) dx \\ &= \liminf_n \int A(\epsilon_n x)\left(\frac{1}{2}f(u_n)u_n - F(u_n)\right) dx \\ &= \liminf_n \left(I_n(u_n) - \frac{1}{2}I'_n(u_n)u_n\right) = \lim_n c_n = c_0, \end{aligned}$$

which is absurd, because $c_0 < d_{A_\infty}$. This completes the proof that $(\epsilon_n y_n)$ is bounded in $\mathbb{R}^N$. Now suppose $\epsilon_n y_n \to z \in \mathbb{R}^N$. Arguing as above, we find

$$\int \nabla v \nabla \psi + V(x) v \psi dx = \int A(z) f(v)\psi dx, \quad \psi \in C_c^\infty(\mathbb{R}^N),$$

and so $J'_{A(z)}(v) = 0$. Hence,

$$d_{A(z)} \leq J_{A(z)}(v) - \frac{1}{2}J'_{A(z)}(v)v \leq \liminf_n \left(I_n(u_n) - \frac{1}{2}I'_n(u_n)u_n\right) = c_0 = d_{A(0)}.$$

Since $\lambda \mapsto d_\lambda$ is decreasing and $d_{A(z)} \leq d_{A(0)}$, we must have $A(0) \leq A(z)$. From the fact that $A(0) = \max_{x\in\mathbb{R}^N} A(x)$, we obtain $A(0) = A(z)$, or equivalently, $z \in \mathcal{A}$. Moreover, we also have $J'_{A(0)}(v) = J'_{A(z)}(v) = 0$. ■

From now on we consider $\epsilon_n y_n \to z$ with $z \in \mathcal{A}$. Our goal is to prove that $v_n \to v$ in $H^1(\mathbb{R}^N)$ and $v_n(x) \to 0$ as $|x| \to +\infty$ uniformly in $n$. Have this in mind, we need of the following estimate

**Proposition 3.2.11** *There exists $h \in L^1(\mathbb{R}^N)$ and a subsequence of $(v_n)$ such that*

$$|f(v_n(x))v_n(x)| \leq h(x), \quad \forall x \in \mathbb{R}^N \quad \textit{and} \quad n \in \mathbb{N}.$$

**Proof.** Note that, by Fatou's Lemma,

$$\begin{aligned}
d_{A(0)} &\leq J_{A(0)}(v) = J_{A(0)}(v) - \tfrac{1}{2}J'_{A(0)}(v)v \\
&= \textstyle\int A(0)\left(\tfrac{1}{2}f(v)v - F(v)\right)dx \\
&= \textstyle\int A(z)\left(\tfrac{1}{2}f(v)v - F(v)\right)dx \\
&\leq \liminf_n \int A(\epsilon_n x + \epsilon_n y_n)\left(\frac{1}{2}f(v_n)v_n - F(v_n)\right)dx \\
&\leq \limsup_n \int A(\epsilon_n x + \epsilon_n y_n)\left(\frac{1}{2}f(v_n)v_n - F(v_n)\right)dx \\
&= \limsup_n \int A(\epsilon_n x)\left(\frac{1}{2}f(u_n)u_n - F(u_n)\right)dx \\
&= \limsup_n \left(I_n(u_n) - \frac{1}{2}I'_n(u_n)u_n\right) = \lim_n c_n = c_0 = d_{A(0)},
\end{aligned}$$

from where it follows that

$$\lim_n \int A(\epsilon_n x + \epsilon_n y_n)\left(\frac{1}{2}f(v_n)v_n - F(v_n)\right)dx = \int A(z)\left(\frac{1}{2}f(v)v - F(v)\right)dx.$$

Since

$$A(\epsilon_n x + \epsilon_n y_n)\left(\frac{1}{2}f(v_n)v_n - F(v_n)\right) \geq 0$$

and

$$A(\epsilon_n x + \epsilon_n y_n)\left(\frac{1}{2}f(v_n)v_n - F(v_n)\right) \to A(z)\left(\frac{1}{2}f(v)v - F(v)\right) \quad \text{a.e. in} \quad \mathbb{R}^N,$$

we can ensure that

$$A(\epsilon_n x + \epsilon_n y_n)\left(\frac{1}{2}f(v_n)v_n - F(v_n)\right) \to A(z)\left(\frac{1}{2}f(v)v - F(v)\right) \quad \text{in} \quad L^1(\mathbb{R}^N).$$

Thereby, there exists $\widetilde{h} \in L^1(\mathbb{R}^N)$ such that, for some subsequence,

$$A(\epsilon_n x + \epsilon_n y_n)\left(\frac{1}{2}f(v_n)v_n - F(v_n)\right) \leq \widetilde{h}(x), \quad \forall n \in \mathbb{N}.$$

As
$$\left(\frac{1}{2}-\frac{1}{q+1}\right)\left(\inf_{\mathbb{R}^N} A\right) f(v_n)v_n \leq A(\epsilon_n x+\epsilon_n y_n)\left(\frac{1}{2}f(v_n)v_n - F(v_n)\right),$$
we get the desired result. ■

An immediate consequence of the last proposition is the following corollary

**Corollary 3.2.12** *$v_n \to v$ in $L^{2^*}(\mathbb{R}^N)$.*

**Proof.** The result follows because $|v_n|^{2^*} \leq f(v_n)v_n$ for all $n \in \mathbb{N}$ and $v_n(x) \to v(x)$ a.e. in $\mathbb{R}^N$.

■

Our next result establishes a key estimate involving the $L^\infty$ norm on balls for the sequence $(v_n)$. To this end, we fix $v_{n,+} = \max\{0, v_n\}$ and $v_{n,-} = \max\{0, -v_n\}$.

**Lemma 3.2.13** *There exist $R > 0$ and $C > 0$ such that*
$$|v_n|_{L^\infty(B_R(x))} \leq C|v_n|_{L^{2^*}(B_{2R}(x))}, \quad \forall n \in \mathbb{N} \quad \text{and} \quad \forall x \in \mathbb{R}^N \tag{2.22}$$
*Hence, as $(v_n)$ is a bounded sequence in $L^{2^*}(\mathbb{R}^N)$, $v_n \in L^\infty(\mathbb{R}^N)$ and there is $C > 0$ such that*
$$|v_n|_\infty \leq C, \quad \forall n \in \mathbb{N}. \tag{2.23}$$

**Proof.** It suffices to check that
$$|v_{n,+}|_{L^\infty(B_R(x))} \leq C|v_{n,+}|_{L^{2^*}(B_{2R}(x))},$$
for all $n \in \mathbb{N}$ and $x \in \mathbb{R}^N$, because similar reasoning proves
$$|v_{n,-}|_{L^\infty(B_R(x))} \leq C|v_{n,-}|_{L^{2^*}(B_{2R}(x))},$$
for all $n \in \mathbb{N}$ and $x \in \mathbb{R}^N$. To begin with, we recall that there exist $c_1, c_2 > 0$ satisfying
$$|f(t)| \leq c_1|t| + c_2|t|^{2^*-1}, \ \ \text{for all } t \in \mathbb{R} \tag{2.24}$$
and that $v_n$ is a solution for the problem
$$\begin{cases} -\Delta v_n + V(x)v_n = A(\epsilon_n x + \epsilon_n y_n)f(v_n) & \text{in} \quad \mathbb{R}^N, \\ v_n \in H^1(\mathbb{R}^N). \end{cases}$$

We consider $\eta \in C_c^\infty(\mathbb{R}^N)$, $L > 0$ and $\beta > 1$ arbitrary, and define $z_{L,n} := \eta^2 v_{L,n}^{2(\beta-1)} v_{n,+}$ and $w_{L,n} := \eta v_{n,+} v_{L,n}^{\beta-1}$ where $v_{L,n} = \min\{v_{n,+}, L\}$. Applying $z_{L,n}$ as a test function, we find

$$\int \eta^2 v_{L,n}^{2(\beta-1)} |\nabla v_{n,+}|^2 dx \leq |A|_\infty \int |f(v_n)| \eta^2 v_{L,n}^{2(\beta-1)} v_{n,+} dx - \tag{2.25}$$

$$- \int V(x) v_n v_{L,n}^{2(\beta-1)} \eta^2 v_{n,+} dx - 2 \int (\nabla v_n \nabla \eta) \eta v_{L,n}^{2(\beta-1)} v_{n,+} dx.$$

Since

$$\left| \int v_{L,n}^{2(\beta-1)} (v_{n,+} \nabla \eta)(\eta \nabla v_n) dx \right| \leq C \int v_{L,n}^{2(\beta-1)} v_{n,+}^2 |\nabla \eta|^2 dx + \tag{2.26}$$

$$+ \frac{1}{4} \int v_{L,n}^{2(\beta-1)} \eta^2 |\nabla v_{n,+}|^2 dx,$$

combining (2.24), (2.25) and (2.26), we obtain

$$\int \eta^2 v_{L,n}^{2(\beta-1)} |\nabla v_{n,+}|^2 dx \leq C \int |v_{n,+}|^2 \eta^2 v_{L,n}^{2(\beta-1)} dx + \tag{2.27}$$

$$+ C \int |v_n|^{2^*} \eta^2 v_{L,n}^{2(\beta-1)} dx + C \int v_{L,n}^{2(\beta-1)} v_{n,+}^2 |\nabla \eta|^2 dx$$

where $C > 0$ is independently of $\beta > 1, \eta \in C_c^\infty(\mathbb{R}^N)$ and $L > 0$.

On the other hand, since $H^1(\mathbb{R}^N) \hookrightarrow D^{1,2}(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N)$,

$$|w_{L,n}|_{2^*}^2 \leq C \int |\nabla w_{L,n}|^2 dx \leq C \int |\nabla \eta|^2 v_{L,n}^{2(\beta-1)} v_{n,+}^2 dx + \tag{2.28}$$

$$C \int \eta^2 v_{L,n}^{2(\beta-1)} |\nabla v_{n,+}|^2 dx + C \int \eta^2 |\nabla v_{L,n}^{(\beta-1)}|^2 v_{n,+}^2 dx,$$

and thus

$$|w_{L,n}|_{2^*}^2 \leq C\beta^2 \left( \int |\nabla \eta|^2 v_{L,n}^{2(\beta-1)} v_{n,+}^2 dx + \int \eta^2 v_{L,n}^{2(\beta-1)} |\nabla v_{n,+}|^2 dx \right). \tag{2.29}$$

Then, from (2.27) and (2.29),

$$|w_{L,n}|_{2^*}^2 \leq C\beta^2 \left( \int |v_{n,+}|^2 \eta^2 v_{L,n}^{2(\beta-1)} dx + \right. \tag{2.30}$$

$$\left. + \int |v_n|^{2^*} \eta^2 v_{L,n}^{2(\beta-1)} dx + \int v_{L,n}^{2(\beta-1)} v_{n,+}^2 |\nabla \eta|^2 dx \right),$$

where $C > 0$ is independently of $n \in \mathbb{N}$, $\beta > 1$, $L > 0$ and $\eta \in C_c^\infty(\mathbb{R}^N)$.

**Claim 3.2.14** *There exists $R > 0$ such that*

$$\sup_{n \in \mathbb{N}, x \in \mathbb{R}^N} \int_{B_{3R}(x)} v_{n,+}^{\frac{2^{*2}}{2}} dx < +\infty.$$

In fact, fix $\beta_0 := \frac{2^*}{2}$. By using the limit $v_n \to v$ in $L^{2^*}(\mathbb{R}^N)$, we can fix $R > 0$ sufficiently small verifying

$$C\beta_0^2 \left( \int_{B_{4R}(x)} v_{n,+}^{2^*} dx \right)^{\frac{2^*-2}{2}} < \frac{1}{2}, \quad \text{for all } n \in \mathbb{N} \text{ and } x \in \mathbb{R}^N, \tag{2.31}$$

where $C$ is given in (2.30). On the other hand, consider $\eta_x \in C_c^\infty(\mathbb{R}^N, [0,1])$ such that $\eta_x \equiv 1$ on $B_{3R}(x)$, $\eta_x \equiv 0$ on $\mathbb{R}^N \setminus B_{4R}(x)$ and $x \mapsto ||\nabla \eta_x||_\infty$ is a constant function. Then,

$$\int v_{n,+}^{2^*} \eta_x^2 v_{L,n}^{2(\beta_0-1)} = \int v_{n,+}^{2^*} \eta_x^2 v_{L,n}^{2^*-2} = \int_{B_{4R}(x)} \left( v_{n,+}^2 \eta_x^2 v_{L,n}^{2^*-2} \right) v_{n,+}^{2^*-2} dx \leq$$

$$\leq \left( \int \left( v_{n,+} \eta_x v_{L,n}^{\frac{2^*-2}{2}} \right)^{2^*} dx \right)^{\frac{2}{2^*}} \left( \int_{B_{4R}(x)} v_{n,+}^{2^*} dx \right)^{\frac{2^*-2}{2}} \leq \frac{1}{2C\beta_0^2} |w_{L,n}|_{2^*}^2$$

Applying (2.30) with $\eta = \eta_x$ and $\beta = \beta_0$, we get

$$|w_{L,n}|_{2^*}^2 \leq C\beta_0^2 \left( \int \eta_x^2 v_{n,+}^{2^*} dx + \frac{1}{2C\beta_0} |w_{L,n}|_{2^*}^2 + \int v_{n,+}^{2^*} |\nabla \eta_x|^2 dx \right),$$

which leads to

$$|w_{L,n}|_{2^*}^2 \leq C\beta_0^2 \left( 1 + ||\nabla \eta_x||_\infty \right) \int v_{n,+}^{2^*} dx.$$

By using Fatou's Lemma for $L \to +\infty$, we obtain

$$\left( \int_{B_{3R}(x)} v_{n,+}^{\frac{2^{*2}}{2}} dx \right)^{\frac{2}{2^*}} \leq C\beta_0^2 \int v_{n,+}^{2^*} dx$$

for all $n \in \mathbb{N}$ and for all $x \in \mathbb{R}^N$. This proves Claim 3.2.14.

In what follows, we fix $R > 0$ as in Claim 3.2.14, $r_m := \frac{2R}{2^m}$,

$$t := \frac{2^{*2}}{2(2^* - 2)} \quad \text{and} \quad \chi := \frac{2^*(t-1)}{2t} > 1.$$

**Claim 3.2.15** *Consider $\beta > 1$ arbitrary such that $v_{n,+} \in L^{\beta \frac{2^*}{\chi}}(B_{R+r_m}(x))$ for all $n \in \mathbb{N}$ and for some $m \in \mathbb{N}$. Then*

$$|v_{n,+}|_{L^{2^*\beta}(B_{R+r_{m+1}}(x))} \leq C^{1/\beta} \beta^{1/2\beta} (1 + 4^m)^{1/2\beta} |v_{n,+}|_{L^{2^* \frac{\beta}{\chi}}(B_{R+r_m}(x))} \tag{2.32}$$

*where $C > 0$ is independently of $n, m \in \mathbb{N}$, $\beta > 1$ and $x \in \mathbb{R}^N$.*

In fact, since $2^* \frac{\beta}{\chi} = \beta \frac{2t}{t-1}$, $v_{n,+} \in L^{\frac{2\beta t}{t-1}}(B_{R+r_m}(x))$ for all $n \in \mathbb{N}$. Consider $\eta_{x,m} \in C_c^\infty(\mathbb{R}^N, [0,1])$ such that $\eta_{x,m} \equiv 1$ in $B_{R+r_{m+1}}(x)$, $\eta_{x,m} \equiv 0$ in $\mathbb{R}^N \setminus B_{R+r_m}(x)$ and $|\eta_{x,m}|_\infty < \frac{2}{r_{m+1}}$. Using $\eta = \eta_{x,m}$ in (2.30),

$$|w_{L,n}|_{2^*}^2 \leq C\beta^2 \left( \int_{B_{R+r_m}(x)} |v_{n,+}|^{2\beta} dx + \int_{B_{R+r_m}(x)} v_{n,+}^{2^*-2} v_{n,+}^{2\beta} dx + \right.$$

$$+\left(\frac{2}{r_{m+1}}\right)^2\int_{B_{R+r_m}(x)} v_{n,+}^{2\beta}dx\right) \leq C\beta^2\left((1+4^m)\int_{B_{R+r_m}(x)} v_{n,+}^{2\beta}dx+\right.$$

$$+\int_{B_{R+r_m}(x)} v_{n,+}^{2^*-2}v_{n,+}^{2\beta}dx\right) \leq C\beta^2\left((1+4^m)\left(\int_{B_{3R}(0)} 1dx\right)^{1/t}\right. .$$

$$\cdot\left(\int_{B_{R+r_m}(x)} v_{n,+}^{2\beta t/(t-1)}dx\right)^{(t-1)/t} + \left(\int_{B_{3R}(x)} v_{n,+}^{(2^*-2)t}dx\right)^{1/t} .$$

$$\left.\left(\int_{B_{R+r_m}(x)} v_{n,+}^{2\beta t/(t-1)}dx\right)^{(t-1)/t}\right) \leq$$

$$\leq C\beta^2\left((1+4^m)\left(\int_{B_{R+r_m}(x)} v_{n,+}^{2\beta t/(t-1)}dx\right)^{(t-1)/t}\right).$$

Thus

$$|w_{L,n}|_{2^*}^2 \leq C\beta^2(1+4^m)|v_{n,+}|^{2\beta}_{L^{2\beta t/(t-1)}(B_{R+r_m}(x))}.$$

Applying Fatou's Lemma as $L \to +\infty$ we get (2.32). Consequently, by induction,

$$|v_{n,+}|_{L^{2^*\chi^m}(B_{R+r_{m+1}}(x))} \leq C^{\sum_{i=1}^m \frac{1}{\chi^i}}\chi^{\sum_{i=1}^m \frac{i}{2\chi^i}}\prod_{i=1}^m(1+4^i)^{\frac{1}{2\chi^i}}|v_{n,+}|_{L^{2^*}(B_{2R}(x))} \tag{2.33}$$

Since $\left(\sum_{i=1}^m \frac{1}{\chi^i}\right)_m$ and $\left(\sum_{i=1}^m \frac{i}{\chi^i}\right)_m$ are convergent because $\chi > 1$, and that

$$\prod_{i=1}^m(1+4^i)^{\frac{1}{2\chi^i}} = 4^{\sum_{i=1}^m \frac{log_4(1+4^i)}{2\chi^i}} \leq 4^{\sum_{i=1}^m \frac{log_4(4^{i+1})}{2\chi^i}} = 4^{\sum_{i=1}^m \frac{i+1}{2\chi^i}},$$

there exists $C > 0$ independently of $n, m \in \mathbb{N}$ and $x \in \mathbb{R}^N$ such that

$$|v_{n,+}|_{L^{2^*\chi^m}(B_R(x))} \leq C|v_{n,+}|_{L^{2^*}(B_{2R}(x))}.$$

Now (2.22) follows by taking the limite of $m \to +\infty$. ■

**Corollary 3.2.16** *For each $\delta > 0$ there exist $R > 0$ such that $|v_n(x)| \leq \delta$ for all $x \in \mathbb{R}^N \setminus B_R(0)$ and $n \in \mathbb{N}$.*

**Proof.** By Lemma 3.2.13,

$$|v_n|_{L^\infty(B_R(x))} \leq C|v_n|_{L^{2^*}(B_{2R}(x))}, \quad \text{for all} \quad n \in \mathbb{N} \quad \text{and} \quad x \in \mathbb{R}^N.$$

This fact combined with the limit $v_n \to v$ in $L^{2^*}(\mathbb{R}^N)$ proves the result. ■

**Concentration of the solutions:**

As $v \neq 0$, we must have $|v_n|_{L^\infty(\mathbb{R}^N)} \not\to 0$. Hence, we can assume that $|v_n|_{L^\infty(\mathbb{R}^N)} > \delta$ for any $\delta > 0$ and $n \in \mathbb{N}$. In what follows, we fix $z_n \in \mathbb{R}^N$ verifying

$$|v_n(z_n)| = \max_{x \in \mathbb{R}^N} |v_n(x)|.$$

Since $v_n(x) = u_n(x + y_n)$, the point $x_n := z_n + y_n$ satisfies

$$|u_n(x_n)| = \max_{x \in \mathbb{R}^N} |u_n(x)|.$$

From Corollary 3.2.16, $(z_n)$ is bounded in $\mathbb{R}^N$, then

$$\epsilon_n x_n = \epsilon_n z_n + \epsilon_n y_n \to z \in \mathcal{A}.$$

and

$$\lim_n A(\epsilon_n x_n) = A(z) = A(0).$$

## 3.3 The case $N = 2$.

In this section we will consider the case where $f$ has an exponential critical growth. For this type of function, it is well known that Trundiger-Moser type inequalities are key points to apply variational methods. In the present paper we will use a Trudinger-Moser type inequality for whole $\mathbb{R}^2$ due to Cao [13] ( see also Ruf [44] ).

**Lemma 3.3.1 (Trudinger-Moser inequality for unbounded domains)** *For all $u \in H^1(\mathbb{R}^2)$, we have*

$$\int \left(e^{\alpha|u|^2} - 1\right) dx < \infty, \quad \textit{for every} \ \ \alpha > 0.$$

*Moreover, if $|\nabla u|_2^2 \leq 1$, $|u|_2 \leq M < \infty$ and $\alpha < 4\pi$, then there exists a positive constant $C = C(M, \alpha)$ such that*

$$\int \left(e^{\alpha|u|^2} - 1\right) dx \leq C.$$

The reader can find other Trundiger-Moser type inequalities in [15], [23], [24], [33] and references therein

As in the previous section, firstly we need to study the autonomous case.

### 3.3.1 A result involving the autonomous problem.

We consider the problem

$$\begin{cases} -\Delta u + V(x)u = \lambda f(u), \quad x \in \mathbb{R}^2, \\ u \in H^1(\mathbb{R}^2), \end{cases} \qquad (AP)_\lambda^{exp}$$

where $f : \mathbb{R} \to \mathbb{R}$ satisfies $(f_1)-(f_5)$. Associated with this problem, we have the energy function $J_\lambda : H^1(\mathbb{R}^2) \to \mathbb{R}$ given by

$$J_\lambda(u) = \frac{1}{2}||u^+||^2 - \frac{1}{2}||u^-||^2 - \lambda \int F(u)dx.$$

It is well known that $J_\lambda \in C^1(H^1(\mathbb{R}^2), \mathbb{R})$ with

$$J'_\lambda(u)v = B(u,v) - \lambda \int f(u)vdx, \quad \forall u, v \in H^1(\mathbb{R}^2).$$

In the sequel,

$$\mathcal{N}_\lambda = \{u \in H^1(\mathbb{R}^2) \setminus E^- \ ; \ J'_\lambda(u)u = J'_\lambda(u)v = 0, \forall\ v \in E^-\}$$

and

$$d_\lambda = \inf_{\mathcal{N}_\lambda} J_\lambda.$$

In [5], Alves and Germano have proved that there exists a constant $\tau_0 > 0$ such that $(AP)_\lambda^{exp}$ has a ground state solution if

$$\lambda \geq A(0) \quad \text{and} \quad \tau \geq \tau_0, \tag{3.34}$$

where $\tau$ was fixed in $(f_5)$. More precisely, it has been shown that for $\lambda \geq A(0)$ and $\tau \geq \tau_0$, there exists $u_\lambda \in H^1(\mathbb{R}^2)$ verifying

$$J'_\lambda(u_\lambda) = 0 \text{ and } J_\lambda(u_\lambda) = d_\lambda$$

with

$$d_\lambda < \frac{\widetilde{A}^2}{2} \tag{3.35}$$

where $\widetilde{A} < 1/a$ and $a$ was given in (1.3). This restriction on $\tau$ has been mentioned in Theorem 3.1.1, and it will be assume in whole this section.

Moreover, the authors have proved that for all $u \in H^1(\mathbb{R}^2) \setminus E^-$ the set $\mathcal{N}_\lambda \cap \widehat{E}(u)$ is a singleton set and the element of this set is the unique global maximum of $J_\lambda|_{\widehat{E}(u)}$, which means precisely that there exist uniquely $t^* \geq 0$ and $v^* \in E^-$ such that

$$J_\lambda(t^*u + v^*) = \max_{w \in \widehat{E}(u)} J_\lambda(w) \quad \text{and} \quad \{t^*u + v^*\} = \mathcal{N}_\lambda \cap \widehat{E}(u)$$

As in the case $N \geq 3$, we begin by studying the behavior of the function $\lambda \mapsto d_\lambda$.

**Proposition 3.3.2** *The function $\lambda \mapsto d_\lambda$ is decreasing and continuous on $[A_0, +\infty)$.*

**Proof.** The monotonicity of $\lambda \mapsto d_\lambda$ and some details of the proof are analogous to Proposition 3.2.1 and [6, Proposition 2.3]. In order to get the limit $\lim_n d_{\lambda_n} = d_\lambda$, it suffices to consider $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_n \to \lambda$. Let $u_n$ be a ground state solution of the problem $(AP)^{exp}_{\lambda_n}$. Let $t_n \geq 0$ and $v_n \in E^-$ such that $t_n u_n + v_n \in \mathcal{N}_\lambda$. Consequently

$$J_\lambda(t_n u_n + v_n) = \max_{\widehat{E}(u_n)} J_\lambda \geq d_\lambda,$$

and the same ideas explored in Proposition 3.2.1 remain valid to show that $\left(\int f(u_n)u_n dx\right)$ is bounded in $\mathbb{R}$. Now, arguing as in [5, Lemma 3.11], we see that $(u_n)$ is bounded in $H^1(\mathbb{R}^2)$.

Note that there exist $(y_n)$ in $\mathbb{Z}^2$, $r, \eta > 0$ such that

$$\int_{B_r(y_n)} |u_n^+|^2 dx > \eta, \quad \forall n \in \mathbb{N}. \tag{3.36}$$

Otherwise, $u_n^+ \to 0$ in $L^p(\mathbb{R}^2)$ for all $p > 2$. Defining $w_n(x) := \widetilde{A}\frac{u_n^+(x)}{||u_n||}$ where $\widetilde{A}$ was given in (3.35), we have

$$||w_n||_{H^1(\mathbb{R}^2)} \leq \widetilde{A}a < 1, \quad \forall n \in \mathbb{N}.$$

This fact permits to repeat the same approach found in [2, Proposition 2.3] to get the limit

$$\int F(w_n)dx \to 0.$$

As $w_n \in \widehat{E}(u_n)$ and $u_n \in \mathcal{N}_{\lambda_n}$, it follows that

$$d_\lambda \geq d_{\lambda_n} = J_{\lambda_n}(u_n) \geq J_{\lambda_n}(w_n) = \frac{\widetilde{A}}{2} - \lambda_n \int F(w_n)dx.$$

Passing to the limit as $n \to +\infty$ we obtain $d_\lambda \geq \widetilde{A}/2$, which contradicts (3.35), and (3.36) holds. If $\widetilde{u}_n(x) := u_n(x + y_n)$, then $\widetilde{u}_n^+(x) := u_n^+(x + y_n)$, and by (3.36), $\widetilde{u}_n^+ \rightharpoonup u \neq 0$. This implies that $\mathcal{V} := \{\widetilde{u}_n^+\}_{n\in\mathbb{N}}$ satisfies $0 \notin \overline{\mathcal{V}}^{\sigma(H^1(\mathbb{R}^2), H^1(\mathbb{R}^2)')}$ and $\mathcal{V}$ is bounded in $H^1(\mathbb{R}^2)$. We proceed as in Proposition 3.2.1 to conclude $(t_n u_n + v_n)$ is bounded and $d_{\lambda_n} \leq d_\lambda + o_n$. This finishes the proof. ■

### 3.3.2 Existence of ground state for problem $(P)_\epsilon$.

The three first results this section follow as in the case $N \geq 3$, then we will omit their proofs.

**Lemma 3.3.3** *The limit* $\lim_{\epsilon\to 0} c_\epsilon = c_0$ *holds. Moreover, if* $w_0$ *is a ground state solution of the problem* $(P)_0$ *and let* $t_\epsilon \geq 0$ *and* $v_\epsilon \in E^-$ *such that* $t_\epsilon w_0 + v_\epsilon \in \mathcal{M}_\epsilon$. *Then*

$$t_\epsilon \to 1 \quad \textit{and} \quad v_\epsilon \to 0$$

*as* $\epsilon \to 0$.

**Corollary 3.3.4** *There exists* $\epsilon_0 > 0$ *such that*

$$c_\epsilon < d_{A_\infty} \quad \textit{and} \quad c_\epsilon < \frac{\widetilde{A}^2}{2}, \quad \textit{for all} \quad \epsilon \in (0, \epsilon_0).$$

**Proposition 3.3.5** *There exists a bounded sequence* $(u_n) \subset \mathcal{M}_\epsilon$ *such that* $(u_n)$ *is* $(PS)_{c_\epsilon}$ *for* $I_\epsilon$.

Now we are ready to prove the existence of solution for $\epsilon$ small enough.

**Theorem 3.3.6** *Problem* $(P)_\epsilon$ *has a ground state solution for* $\epsilon \in (0, \epsilon_0)$ .

**Proof.** To begin with, we claim that there are $(z_n) \subset \mathbb{Z}^2$ and $r, \eta > 0$ such that

$$\int_{B_r(z_n)} |u_n^+|^2 dx > \eta, \quad \forall n \in \mathbb{N}. \tag{3.37}$$

In fact, if the claim does not hold, we must have $u_n^+ \to 0$ in $L^p(\mathbb{R}^2)$ for all $p \in (2, +\infty)$. Since $u_n \in \mathcal{M}_\epsilon$, by (2.16), $||u_n^+||^2 \geq 2c_\epsilon \geq 2c_0$. Setting $\widetilde{w}_n(x) := \widetilde{A}\frac{u_n^+}{||u_n^+||}$ and arguing as in Proposition 3.3.2, we find $c_\epsilon \geq \frac{\widetilde{A}^2}{2}$, which is a contradiction. Therefore (3.37) holds.

**Claim 3.3.7** $(z_n)$ *is bounded in* $\mathbb{R}^2$.

Suppose $|z_n| \to +\infty$ and define $w_n(x) := u_n(x + z_n)$. From (3.37), we can suppose that $w_n \rightharpoonup w \neq 0$ in $H^1(\mathbb{R}^2)$. As it was done in (2.10), $\left(\int f(w_n)w_n dx\right)$ is bounded in $L^1(\mathbb{R}^2)$. By [18, Lemma 2.1],

$$f(w_n) \to f(w) \quad \text{in } L^1(B),$$

for all $B \subset \mathbb{R}^2$ bounded Borel set. Now, we repeat the same idea explored in Claim 3.2.7 to deduce that $w$ is a critical point of $J_{A_\infty}$ with $d_{A_\infty} \leq c_\epsilon$, which is absurd. This proves the Claim 3.3.7.

To conclude the proof we proceed as in Theorem 3.2.6 to prove that the weak limit of $(u_n)$ is a ground state solution for $I_\epsilon$. ■

### 3.3.3 Concentration of the solutions.

In this section we fix $\epsilon_n \to 0$ with $\epsilon_n \in (0, \epsilon_0)$ for all $n \in \mathbb{N}$. By results of the previous section, for each $n \in \mathbb{N}$ there exists $u_n$ in $H^1(\mathbb{R}^2)$ such that

$$I_n(u_n) = c_n \quad \text{and} \quad I_n'(u_n) = 0,$$

with the notation $I_n := I_{\epsilon_n}$ and $c_n := c_{\epsilon_n}$.

**Lemma 3.3.8** *The sequence $(u_n)$ is bounded in $H^1(\mathbb{R}^2)$.*

**Proof.** See proof of [5, Lemma 3.11]. ■

**Lemma 3.3.9** *There are $r, \eta > 0$ and $(y_n) \subset \mathbb{Z}^2$ such that*

$$\int_{B_r(y_n)} |u_n^+|^2 dx > \eta. \tag{3.38}$$

**Proof.** See proof of (3.37). ■

From now on, we set $v_n(x) := u_n(x + y_n)$. Then, by (3.38), $v_n \rightharpoonup v \neq 0$ in $H^1(\mathbb{R}^2)$ for some subsequence.

**Lemma 3.3.10** *The sequence $(\epsilon_n y_n)$ is bounded in $\mathbb{R}^2$. Moreover, $I_0(v) = 0$ and if $\epsilon_n y_n \to z \in \mathbb{R}^2$ then $z \in \mathcal{A}$ or equivalently $A(z) = A(0)$.*

**Proof.** As in the previous section, $(f(u_n)u_n)$ is bounded in $L^1(\mathbb{R}^2)$. Then, by [18, Lemma 2.1],

$$f(u_n) \to f(u) \ \text{ in } \ L^1(B),$$

for all bounded Borel set $B \subset \mathbb{R}^2$ . The above limit permits to repeat the same arguments explored in Lemma 3.2.10. ■

Our next proposition follows with the same idea explored in Proposition 3.2.11, then we omit its proof.

**Proposition 3.3.11** *There exists $h \in L^1(\mathbb{R}^2)$ and a subsequence of $(v_n)$ such that*

$$|f(v_n(x))v_n(x)| \leq h(x), \quad \textit{for all} \quad x \in \mathbb{R}^2 \quad \textit{and} \quad n \in \mathbb{N}.$$

As an immediate consequence of the last lemma, we have the following corollary

**Corollary 3.3.12** *$v_n \to v$ in $L^q(\mathbb{R}^2)$ where $q$ was given in $(f_5)$.*

**Proof.** It suffices to note that $f(v_n)v_n \geq \theta F(v_n) \geq \theta\tau|v_n|^q$, for all $n \in \mathbb{N}$ and $v_n(x) \to v(x)$ a.e in $\mathbb{R}^N$. ■

The next lemma have been motivated by an inequality found [17, Lemma 2.11], however it is a little different, because we need to adapt it to our problem.

**Lemma 3.3.13** *For all* $t, s \geq 0$ *and* $\beta \in (0,1]$,

$$ts \leq \begin{cases} 4(e^{t^2}-1)(ln^+ s) + s(ln^+ s)^{1/2}, & \text{if } s > e^{1/4} \\ e^{1/4} t s^{\beta}, & \text{if } s \in [0, e^{1/4}]. \end{cases}$$

**Proof.** From [17, Lemma 2.11], if $s > e^{1/4}$ then $ln^+ s > 1/4$ and

$$ts \leq (e^{t^2}-1) + s(ln^+ s)^{1/2} \leq 4(e^{t^2}-1)(ln^+ s) + s(ln^+ s)^{1/2}.$$

For $s \in [0,1)$, we have $ts \leq ts^{\beta} \leq e^{1/4}ts^{\beta}$, and if $s \in [1, e^{1/4}]$, then $ts \leq te^{1/4} \leq e^{1/4}ts^{\beta}$. This proves the inequality. ■

**Proposition 3.3.14** $v_n \to v$ *in* $H^1(\mathbb{R}^2)$.

**Proof.** To begin with, by $(f_1)$, there exists $K > 0$ such that

$$|f(t)| \leq \Gamma e^{1/4} \implies |f(t)|^2 \leq K f(t)t.$$

On the other hand,

$$\left(|f(v_n)|\chi_{[0,e^{1/4}]}\left(\frac{1}{\Gamma}|f(v_n)|\right)\right)^2 = |f(v_n)|^2\chi_{[0,\Gamma e^{1/4}]}(|f(v_n)|) \leq$$

$$\leq K f(v_n)v_n \leq Kh \in L^1(\mathbb{R}^2).$$

Thus, there exists $\widetilde{h} \in L^2(\mathbb{R}^2)$ such that

$$|f(v_n)|\chi_{[0,e^{1/4}]}\left(\frac{1}{\Gamma}|f(v_n)|\right) \leq \widetilde{h}, \quad \forall n \in \mathbb{N}.$$

In what follows, fixing $\alpha > 0$ such that $\frac{\alpha^2 q}{q-1}\sup_{n\in\mathbb{N}}\|v_n^+\|^2_{H^1(\mathbb{R}^2)} < 1$, the Lemma 3.3.1 guarantees that

$$b_n := (e^{\alpha^2|v_n^+|^2}-1) \in L^{\frac{q}{q-1}}(\mathbb{R}^2) \quad \text{and} \quad |b_n|_{\frac{q}{q-1}} \leq C$$

for all $n \in \mathbb{N}$ and some $C > 0$. Applying the Lemma 3.3.13 for $t = \alpha|v_n^+|$, $s = \frac{1}{\Gamma}|f(v_n)|$ and $\beta = 1$, we obtain

$$|f(v_n)v_n^+| = \frac{\Gamma}{\alpha}\frac{|f(v_n)|}{\Gamma}\alpha|v_n^+| \leq \frac{\Gamma}{\alpha}4(e^{\alpha^2|v_n^+|^2}-1)\left(ln^+\left(\frac{1}{\Gamma}|f(v_n)|\right)\right) +$$

$$+\frac{1}{\alpha}|f(v_n)|\left(ln^+\left(\frac{1}{\Gamma}|f(v_n)|\right)\right)^{1/2}+e^{1/4}|v_n^+||f(v_n)|\chi_{[0,e^{1/4}]}\left(\frac{1}{\Gamma}f(v_n)\right)\leq$$

$$\leq\frac{16\Gamma\pi}{\alpha}b_n|v_n|^2+\frac{\sqrt{4\pi}}{\alpha}f(v_n)v_n+e^{1/4}|v_n^+|\widetilde{h}.$$

Since $b_n \rightharpoonup b$ in $L^{\frac{q}{q-1}}(\mathbb{R}^2)$ and $v_n \to v$ in $L^q(\mathbb{R}^2)$, we have that $(b_n|v_n|^2)$ is strongly convergent in $L^1(\mathbb{R}^2)$. Here, we have used the fact that $b_n|v_n|^2 \geq 0$ and $v_n(x) \to v(x)$ a.e in $\mathbb{R}^N$. Analogously $(|v_n^+|\widetilde{h})$ converges in $L^1(\mathbb{R}^2)$. Consequently there is $H_1 \in L^1(\mathbb{R}^2)$ such that, for some subsequence,

$$|f(v_n)v_n^+| \leq H, \quad \forall n \in \mathbb{N}.$$

The same argument works to show that there exists $H_2 \in L^1(\mathbb{R}^2)$ such that, for some subsequence,

$$|f(v_n)v_n^-| \leq H_2, \quad \forall n \in \mathbb{N}.$$

As an consequence of the above information,

$$f(v_n)v_n^+ \to f(v)v^+ \quad \text{and} \quad f(v_n)v_n^- \to f(v_n)v^- \quad \text{in} \quad L^1(\mathbb{R}^2).$$

Now, recalling that $I_0'(v) = I_n'(v_n)v_n^+ = I_n'(v_n)v_n^- = 0, v_n^+ \rightharpoonup v^+$, and $v_n^- \rightharpoonup v^-$ in $H^1(\mathbb{R}^2)$, we get the desired result. ■

**Lemma 3.3.15** *For all $n \in \mathbb{N}$, $v_n \in C(\mathbb{R}^2)$. Moreover, there exist $G \in L^3(\mathbb{R}^2)$, $C > 0$ independently of $x \in \mathbb{R}^2$ and $n \in \mathbb{N}$ such that*

$$||v_n||_{C(\overline{B_1(x)})} \leq C|G|_{L^3(B_2(x))}, \quad \textit{for all} \quad n \in \mathbb{N} \quad \textit{and} \quad x \in \mathbb{R}^2.$$

*Hence, there exists $C > 0$ such that $|v_n|_{L^\infty(\mathbb{R}^2)} \leq C$ and*

$$|v_n(x)| \to 0 \ \textit{as} \ |x| \to +\infty, \quad \textit{uniformly in} \quad n \in \mathbb{N}.$$

**Proof.** We know that there are $C_1, C_2 > 0$ such that

$$|f(t)| \leq C_1|t| + C_2(e^{5\pi t^2} - 1) \quad \forall t \in \mathbb{R}.$$

By Proposition 3.3.14, there exists $H \in H^1(\mathbb{R}^2)$ such that $|v_n(x)| \leq H(x)$ for all $n \in \mathbb{N}$ and $x \in \mathbb{R}^2$. Setting

$$G := (||V||_\infty + A(0)C_1)H + A(0)C_2(e^{5\pi H^2} - 1) \in L^3(\mathbb{R}^2)$$

it follows that

$$|A(\epsilon_n x + \epsilon_n y_n)f(v_n) - V(x)v_n| \leq G(x), \quad \text{for all} \quad n \in \mathbb{N} \quad \text{and} \quad x \in \mathbb{R}^2.$$

Since

$$\begin{cases} -\Delta v_n + V(x)v_n = A(\epsilon_n x + \epsilon_n y_n)f(v_n), & \text{in} \quad \mathbb{R}^2, \\ v_n \in H^1(\mathbb{R}^2) \end{cases}$$

From [21, Theorems 9.11 and 9.13], there exists $C_3 > 0$ independently of $x \in \mathbb{R}^2$ and $n \in \mathbb{N}$ such that $v_n \in W^{2,3}(B_2(x))$ and

$$||v_n||_{W^{2,3}(B_2(x))} \leq C_3|G|_{L^3(B_2(x))}, \quad \text{for all } n \in \mathbb{N}. \tag{3.39}$$

On the other hand, from continuous embedding $W^{2,3}(B_2(x)) \hookrightarrow C(\overline{B_1(x)})$, there is $C_4 > 0$ independently of $x \in \mathbb{R}^2$ such that

$$||u||_{C(\overline{B_1(x)})} \leq C_4||u||_{W^{2,3}(B_2(x))}, \quad \text{for all } u \in W^{2,3}(B_2(x)). \tag{3.40}$$

The result follows from (3.39) and (3.40). ■

**Concentration of the solutions:**

The proof of the concentration follows with the same idea explored in the case $N \geq 3$, then we omit its proof.

# Apêndices

# Apêndice A

# Decomposição Espectral

The main goal this section is to prove the following abstract theorem, which follows by using some results found in functional analysis.

**Theorem A.1** *Let $(H, \langle , \rangle)$ be a Hilbert space and $A : H \to H$ be a bounded and linear symmetric operator such that $0 \notin \sigma(A)$, or equivalently, $A$ is a bijection. Then there exist $E^+, E^- \subset H$ closed subspaces such that the bilinear form*

$$\begin{array}{rccl} B & : & H \times H & \to & \mathbb{R} \\ & & (u,v) & \mapsto & \langle Au, v\rangle \end{array} \tag{0.1}$$

*is definite positive on $E^+$ and definite negative on $E^-$, with $(E^-)^\perp = E^+$ and $(E^+)^\perp = E^-$ and the orthogonality associated with the bilinear form $B$ coincides with the orthogonality of the usual scalar product of $H$. Moreover there exists a scalar product $\langle \cdot, \cdot \rangle_A$ such that its norm $||.||_A$ is equivalently to original norm of Hilbert space $H$ and $E^+$ is orthogonal to $E^-$ and such that*

$$B(u,u) = ||u||_A^2, \ \forall u \in E^+ \quad \textit{and} \quad B(u,u) = -||u||_A^2, \ \forall u \in E^-.$$

*Moreover, if $P_+$ and $P_-$ are the linear projections on $E^+$ and $E^-$, then $P_+$ and $P_-$ commute with $A$, i.e., $AP_+ = P_+A$ and $AP_- = P_-A$*

**Proof.** First of all, note that $A^2$ is definite positive. In fact, for all $x \in H$, we have

$$\langle A^2 x, x\rangle = \langle Ax, Ax\rangle \geq (||A^-||^{-1})^2 ||x||^2,$$

where $\| \ \|$ is the norm associated with the scalar product $\langle \, , \rangle$. Therefore, from [26, Theorem 9.4-2, Theorem 9.8-1(b)] there exists a unique definite positive and continuous operator $C : H \to H$ such that $C^2 = A^2$ and $AC = CA$. Setting

$$A^+ = \frac{1}{2}(A + C), \quad \text{and} \quad A^- = \frac{1}{2}(C - A)$$

it follows that $A^+$ and $A^-$ are symmetric operators and

$$A = A^+ - A^-, \qquad \text{and} \qquad C = A^+ + A^-.$$

In what follows, we fix $E^- := kerA^+$ and $E^+ := (E^-)^\perp$, where this orthogonality is associated with the usual scalar product of Hilbert space $H$.

In the sequel, we will divide the proof into several steps.

**Step A.2** *$P_+$ and $P_-$ commute with $A$ and $C$.*

Indeed, since $I = P_+ + P_-$, it suffices to check that $P_-A = AP_-$ and $P_-C = CP_-$. First of all, note that $A(E^-) \subset E^-$. In fact, if $x \in E^-$ then

$$A^+(Ax) = \frac{1}{2}(A + C)Ax = A\left(\frac{1}{2}(A + C)x\right) = A(A^+x) = 0,$$

then $A(E^-) \subset E^-$. Note that for all $x \in H$ and for all $y \in E^-$,

$$\langle Ax - \underbrace{AP_-x}_{\in E^-}, y\rangle = \langle x - P_-x, \underbrace{Ay}_{\in E^-}\rangle = 0.$$

Therefore $P_-(Ax) = A(P_-x)$, and so, $AP_- = P_-A$. Analogously $CP_- = P_-C$. This proves the Step A.2.

**Step A.3** *$A(E^-) = E^-$ and there exists $\alpha > 0$ such that*

$$B(x, x) \leq -\alpha||x||^2, \quad \forall x \in E^-. \tag{0.2}$$

*Moreover $Ax = -A^-x$ for all $x \in E^-$.*

Note that $A(E^-) = A(P_-(H)) = P_-A(H) = P_-(H) = E^-$. Then, $A|_{E^-} : E^- \to E^-$ is a bijective continuous linear operator. As $A$ is symmetric,

$$M := \sup_{\substack{x \in E^-, \\ ||x|| = 1}} \langle Ax, x\rangle \in \sigma(A|_{E^-}). \tag{0.3}$$

On the other hand, for $x \in E^-$, $Ax + Cx = 2A^+x = 0$ that yields $\langle Ax, x\rangle = -C(x, x) \leq 0$, which gives $M \leq 0$. Since $A|_{E^-}$ is bijection, we must have $0 \notin \sigma(A|_{E^-})$. Thus $M \neq 0$, or equivalently $M < 0$. Fixing $\alpha := -M$, by (0.3),

$$B(x, x) = \langle Ax, x\rangle \leq -\alpha||x||^2, \quad \forall x \in E^-.$$

To prove the last part, it is enough to note that for all $x \in E^-$, $A^+x = 0$, and thus,

$$Ax = A^+x - A^-x = -A^-x,$$

which concludes the claim.

**Step A.4** *$A(E^+) = E^+$, $A : E^+ \to E^+$ is a bijection, and there exists $\beta > 0$ such that*

$$B(x,x) \geq \beta ||x||^2 \tag{0.4}$$

*for all $x \in E^+$. Moreover $Ax = A^+x$ for all $x \in E^+$.*

In fact, note that $A(E^+) = A(P_+(H)) = P_+(A(H)) = P_+(H) = E^+$. Therefore $A : E^+ \to E^+$ is bijection. From equality below

$$A^+ \circ A^- = \left[\frac{1}{2}(A + C)\right] \circ \left[\frac{1}{2}(C - A)\right] = \frac{1}{4}(C^2 - CA + AC - A^2) = 0.$$

From this, $A^+(A^-(H)) = \{0\}$, from where it follows that $A^-(H) \subset E^-$. On the other hand, for $x \in E^+$,

$$||A^-x||^2 = \langle A^-x, A^-x\rangle = \langle x, A^-(A^-x)\rangle = 0,$$

which leads to $A^-x = 0$ and $Ax = A^+x$. The inequality (0.4) follows as in (0.2).

**Step A.5** *$E^+ = kerA^-$.*

In fact, from Step A.4, if $x \in E^+$ then $A^+x = Ax$ and $x \in kerA^-$. Suppose that $x \in kerA^-$ and let $y \in E^-$, then

$$\langle x, Ay\rangle = \langle Ax, y\rangle = \langle A^+x - A^-x, y\rangle = \langle A^+x, y\rangle = \langle x, A^+y\rangle = 0$$

because $E^- = kerA^+$. Since, from Step A.3, we conclude that $\langle x, w\rangle = 0$ for all $w \in E^-$, or equivalently $x \in E^+$.

Note that, as it was done in proof of Step A.4, we have $A^+ \circ A^- = A^- \circ A^+ = 0$. Therefore $A^+(H) \subset kerA^-$ and $A^-(H) \subset kerA^+$, that is,

$$A^+(H) \subset E^+ \quad \text{and} \quad A^-(H) \subset E^-.$$

Moreover,

$$\{x \in H \ ; \ B(x,y) = 0, \ \text{ for all } y \in E^-\} = E^+.$$

Indeed, if $x \in E^+$ then $B(x,y) = \langle Ax, y\rangle = 0$ for all $y \in E^-$. On the other hand, if $x \in H$ verifies $B(x,y) = 0$ for all $y \in E^-$, then

$$\langle x, y\rangle = \langle x, A(A^{-1}(y))\rangle = \langle Ax, A^{-1}y\rangle = B(x, A^{-1}y) = 0,$$

implying that $x \in E^+$.

In what follows, we define on $H$ the bilinear form

$$\langle \cdot, \cdot\rangle_A : H \times H \to \mathbb{R}$$

by

$$\langle x, y\rangle_A = \langle AP_+(x), P_+y\rangle - \langle AP_-x, P_-y\rangle.$$

Then

$$\langle x, y\rangle_A := \langle A^+x, y\rangle + \langle A^-x, y\rangle \quad \forall x, y \in H. \tag{0.5}$$

Note that

$$\langle AP_+x, P_+y\rangle = \langle \underbrace{A^+P_+x}_{\in E^+} - \underbrace{A^-P_+x}_{\in E^-}, \underbrace{P_+y}_{\in E^+}\rangle = \langle A^+P_+x, y\rangle =$$

$$= \langle P_+x, \underbrace{A^+y}_{\in E^+}\rangle = \langle x, A^+y\rangle = \langle A^+x, y\rangle.$$

Analogously $\langle AP_-x, P_-y\rangle = \langle -A^-x, y\rangle$, which proves (0.5).

From the above study, it follows that $\langle \cdot, \cdot\rangle_A$ is a scalar product on $H$. Hereafter, we denotes by $||x||_A$ the norm associated with the inner product, that is,

$$||x||_A := \sqrt{\langle x, x\rangle_A}.$$

Next, we will prove that the scalar product $||\ ||_A$ is equivalent to norm of $H$. First of all, note that for all $x \in H$

$$\langle x, x\rangle_A = \langle AP_+x, P_+x\rangle - \langle AP_-x, P_-x\rangle \geq \beta||P_+x||^2 + \alpha||P_-x||^2$$

$$= (\beta + \alpha)(||P_+x||^2 + ||P_-x||^2) = (\beta + \alpha)||x||^2.$$

On the other hand, from (0.5),

$$\langle x, x\rangle_A \leq (||A^+|| + ||A^-||)||x||^2,$$

finishing the proof that $||.||_A$ is equivalent to norm of $H$.

By the above analysis,

$$B(x,x) = \langle Ax, x\rangle = \langle P_+Ax, P_+x\rangle - \underbrace{\langle P_-Ax, P_-x\rangle}_{=0} = ||x||_A^2, \quad \forall x \in E^+$$

and

$$B(x,x) = \langle Ax, x\rangle = -\underbrace{\langle P_+Ax, P_+x\rangle}_{=0} + \langle P_-Ax, P_-x\rangle = -||x||_A^2, \quad \forall x \in E^-.$$

■

**Corollary A.6** *Moreover, if $T : H \to H$ is a linear isomorphism such that $\langle Tx, Tu\rangle_H = \langle x, y\rangle_H$ and $B(Tx, Ty) = B(x, y)$ for all $x, y \in H$, then $T(E^+) = E^+$ and $T(E^-) = E^-$.*

**Proof.** First of all, our goal is to prove that $A$ and $C$ commute with $T$. Note that for all $x, y \in H$

$$\langle Ax, y\rangle = B(x, y) = B(Tx, Ty) = \langle ATx, Ty\rangle = \langle T^{-1}ATx, y\rangle.$$

Therefore $T^{-1} \circ A \circ T = A$, or equivalently, $A \circ T = T \circ A$. On the other hand,

$$\langle T^{-1}CTx, x\rangle = \langle CTx, Tx\rangle \geq 0, \quad \forall x \in H. \tag{0.6}$$

Moreover

$$\langle T^{-1}CTx, y\rangle = \langle CTx, Ty\rangle = \langle CTy, Tx\rangle = \langle T^{-1}CTy, x\rangle \tag{0.7}$$

for all $x, y \in H$. Since

$$(T^{-1}CT)^2 = T^{-1}C^2T = T^{-1}A^2T = T^{-1}TA^2 = A^2$$

then $S := T^{-1} \circ C \circ T$ is definite positive, symmetric and $S^2 = A^2$. Consequently, by uniqueness of $C$, we must have $S = C$, or equivalently $T \circ C = C \circ T$. Now our goal is to prove that $T(E^+) = E^+$. Note that $E^+ = kerA^- = \{x \in H\ ;\ Ax = Cx\}$. Since $T^{-1}$ is bijective and commutes with $A$ and $C$, we have

$$T(E^+) = T\{x \in H\ ;\ Ax = Cx\} = \{x \in H;\ AT^{-1}x = CT^{-1}x\} =$$

$$= \{x \in H\ ;\ T^{-1}Ax = T^{-1}Cx\} = \{x \in H\ ;\ Ax = Cx\} = E^+.$$

The same argument works to show that $T(E^-) = E^-$. This proves the claim and finishes the proof of the corollary. ■

## A.1 Construction of the operator $A : H \to H$.

Hereafter, we assume that $V : \mathbb{R}^N \to \mathbb{R}$ is continuous and $\mathbb{Z}^N$–periodic. Since $0 \notin \sigma(-\Delta + V)$, the operator

$$\begin{array}{rcccl} -\Delta + V & : & H^2(\mathbb{R}^N) \subset L^2(\mathbb{R}^N) & \to & L^2(\mathbb{R}^N) \\ & & u & \mapsto & -\Delta u + Vu \end{array}$$

is a continuous bijection and $(-\Delta + V)^{-1}$ is continuous with relation to topology of $L^2(\mathbb{R}^N)$. Note that $-\Delta + V$ is also continuous in the usual norm of $H^2(\mathbb{R}^N)$, because

$$\int |-\Delta u + V(x)u|^2 dx \leq \int (4|\Delta u|^2 + 4V(x)^2|u|^2) dx \leq$$

$$\leq C \int (\sum_{i=1}^{N} |u_{x_i x_i}|^2 + |u|^2) dx \leq C||u||^2_{H^2(\mathbb{R}^N)}$$

for all $u \in H^2(\mathbb{R}^N)$. Hence,

$$|(-\Delta + V)u|_{L^2(\mathbb{R}^N)} \leq C||u||_{L^2(\mathbb{R}^N)}, \quad \forall u \in H^2(\mathbb{R}^N).$$

Defining $Q : H^1(\mathbb{R}^N) \to \mathbb{R}$ by

$$Q(u) := \frac{1}{2} \int (|\nabla u|^2 + V(x)|u|^2) dx,$$

we have that $Q \in C^1(H^1(\mathbb{R}^N), \mathbb{R})$ and

$$Q'(u)v = \int (\nabla u \nabla v + V(x)uv) dx, \quad \forall u, v \in H^1(\mathbb{R}^N). \tag{1.8}$$

Then, by Riesz's Theorem, there exists $A : H^1(\mathbb{R}^N) \to H^1(\mathbb{R}^N)$ such that

$$Q'(u)v = \langle Au, v \rangle_{H^1(\mathbb{R}^N)}, \quad \forall u, v \in H^1(\mathbb{R}^N). \tag{1.9}$$

From (1.8) and (1.9), $A$ is linear, symmetric and continuous.

**Proposition A.1** $0 \notin \sigma(A)$, *or equivalently,* $A$ *is bijective with* $A^{-1} : H^1(\mathbb{R}^N) \to H^1(\mathbb{R}^N)$ *being continuous.*

**Proof.** Our first goal is to prove that $A$ is injective. Indeed, if $Au = 0$, then

$$\langle Au, v \rangle = 0, \quad \text{for all } v \in H^1(\mathbb{R}^N).$$

that is,

$$\int (\nabla u \nabla v + V(x)uv)dx = 0, \quad \text{for all } v \in H^1(\mathbb{R}^N).$$

Thus $u$ is a solution of

$$\begin{cases} -\Delta u + V(x)u = 0, & \text{in} \quad \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N) \end{cases}$$

From [21, Theorem 9.9], $u \in H^2(\mathbb{R}^N)$ and

$$\int (-\Delta u + V(x)u)v dx = 0, \quad \forall v \in H^1(\mathbb{R}^N).$$

Therefore $-\Delta u + V(x)u = 0$ a.e. in $\mathbb{R}^N$. Since $(-\Delta u + V(x)u) = 0$ and $-\Delta + V$ is injective, we must have $u = 0$, by proving that $A$ is injective. Let us to prove that $A$ is subjective. Consider $w \in H^1(\mathbb{R}^N)$ and $(w_n)_n$ be a sequence in $C_c^\infty(\mathbb{R}^N)$ verifying

$$w_n \rightharpoonup w \quad \text{in} \quad H^1(\mathbb{R}^N).$$

By regularity theory, there exists $(u_n)_n$ in $H^2(\mathbb{R}^N)$ such that

$$(-\Delta + V)u_n = -\Delta w_n + w_n, \quad \forall n \in \mathbb{N},$$

because $(-\Delta + V)$ is subjective and $-\Delta w_n + w_n \in L^2(\mathbb{R}^N)$. Our goal is to prove that $(||u_n||_{L^2(\mathbb{R}^N)})$ is bounded. To see why, consider $\varphi \in L^2(\mathbb{R}^N)$ such that $||\varphi||_{L^2(\mathbb{R}^N)} \leq 1$. Setting $L := -\Delta + V$, we have

$$\int u_n \varphi dx = \int u_n L L^{-1}(\varphi) dx = \int L u_n L^{-1}(\varphi) dx = \tag{1.10}$$

$$= \int (-\Delta u_n + V(x)u_n)(L^{-1}\varphi)dx = \int (-\Delta w_n + w_n)L^{-1}(\varphi)dx.$$

On the other hand,

$$LL^{-1}\varphi = \varphi, \quad \text{or equivalently,} \ -\Delta L^{-1}(\varphi) + L^{-1}\varphi = \varphi - V(x)L^{-1}\varphi + L^{-1}\varphi.$$

Therefore, by [21, Theorem 9.9], there exists $C > 0$ independently of $\varphi$ such that

$$||L^{-1}\varphi||_{H^2(\mathbb{R}^N)} \leq C|\varphi - V(x)L^{-1}\varphi + L^{-1}\varphi|_{L^2(\mathbb{R}^N)} \leq \tag{1.11}$$

$$\leq C|\varphi|_{L^2(\mathbb{R}^N)} + (||V||_\infty + 1)|L^{-1}\varphi|_{L^2(\mathbb{R}^N)} \leq C + (||V||_\infty + 1)||L^{-1}||.$$

Thus, from (1.10) and (1.11),

$$\int u_n\varphi = \int \nabla w_n \nabla L^{-1}\varphi + w_n L^{-1}\varphi dx = \langle w_n, L^{-1}\varphi\rangle_{H^1(\mathbb{R}^N)} \leq$$

$$\leq ||w_n||_{H^1(\mathbb{R}^N)}||L^{-1}\varphi||_{H^1(\mathbb{R}^N)} \leq M$$

where $M > 0$ is independently of $n \in \mathbb{N}$ and $\varphi \in C_c^\infty(\mathbb{R}^N)$ satisfying $||\varphi||_{L^2(\mathbb{R}^N)} \leq 1$. Consequently

$$\sup_{\substack{\varphi \in L^2(\mathbb{R}^N), \\ ||\varphi||_{L^2(\mathbb{R}^N)} \leq 1}} \int u_n\varphi dx \leq M, \forall n \in \mathbb{N}$$

implying that $(u_n)$ is bounded in $L^2(\mathbb{R}^N)$. On the other hand,

$$\int |\nabla u_n|^2 - ||V||_\infty u_n^2 dx \leq \int |\nabla u_n|^2 + V(x)u_n^2 dx = \int u_n(-\Delta u_n + V(x)u_n)dx \leq$$

$$\leq \int u_n L u_n dx \leq M.|Lu_n|_{L^2(\mathbb{R}^N)} \leq M.||L||.|u_n|_{L^2(\mathbb{R}^N)} \leq M^2||L||.$$

Then $(|\nabla u_n|)$ is bounded in $L^2(\mathbb{R}^N)$, and so, $(u_n)$ is bounded in $H^1(\mathbb{R}^N)$. Consequently there exists $u \in H^1(\mathbb{R}^N)$ such that, after passing to subsequence,

$$u_n \rightharpoonup u \quad \text{in} \quad H^1(\mathbb{R}^N).$$

Note that for all $v \in H^2(\mathbb{R}^N)$,

$$\langle u_n, Lv\rangle_{L^2(\mathbb{R}^N)} = \int(\nabla u_n \nabla v + V(x)u_n v)dx = \langle Lu_n, v\rangle_{L^2(\mathbb{R}^N)} =$$

$$= \langle -\Delta w_n + w_n, v\rangle_{L^2(\mathbb{R}^N)} = \int(\nabla w_n \nabla v + w_n v)dx = \langle w_n, v\rangle_{H^1(\mathbb{R}^N)}$$

passing to the limit as $n \to +\infty$,

$$\langle u, Lv\rangle_{L^2(\mathbb{R}^N)} = \langle w, v\rangle_{H^1(\mathbb{R}^N)}, \quad \text{for all } v \in H^2(\mathbb{R}^N)$$

or equivalently

$$\langle Au, v\rangle_{H^1(\mathbb{R}^N)} = \langle w, v\rangle_{H^1(\mathbb{R}^N)}, \quad \text{for all } v \in H^2(\mathbb{R}^N).$$

This implies that $Au = w$, showing that $A$ is subjective. ■

# Referências Bibliográficas

[1] C. O. Alves, *Existência de solução positiva de equações elípticas não-lineares variacionais em* $\mathbb{R}^N$, Doct. dissertation, UnB, 1996.

[2] C. O. Alves, J. M. B. do Ó and O.H. Miyagaki. *On nonlinear pertubations of a periodic elliptic problem in* $\mathbb{R}^2$ *involving critical growth.* Nonlinear Analysis 45 (2001) 849-863.

[3] C. O. Alves, P. C. Carrião and O. H. Miyagaki, *Nonlinear perturbations of a periodic elliptic problem with critical growth*, J. Math. Anal. Appl. 260 (2001) 133-146.

[4] C. O. Alves and L.R. de Freitas, *Multiplicity of nonradial solutions for a class of quasilinear equations on annulus with exponential critical growth*, Topol. Methods in Nonlinear Anal. 39 (2012), 243-262.

[5] C. O. Alves and G. F. Germano, *Ground state solution for a class of indefinite variational problems with critical growth*, ArXiv:1704.01385v1[math.AP]

[6] C. O. Alves and G. F. Germano, *Existence and concentration of solution of ground state solution for a class of indefinite variational problems*,ArXiv:1801.06872v1[math.AP].

[7] C. O. Alves and G. F. Germano, *Existence and concentration phenomena for a class of indefinite variational problems with critical growth*, ArXiv:1801.08138v1[math.AP].

[8] C. O. Alves and M. A. S. Souto, *On existence and concentration behavior of ground state solutions for a class of problems with critical growth.* Comm. Pure and Applied Analysis, 1 (2002), 417-431.

[9] C. O. Alves and S.H.M. Soares, *On the location and profile of spike-layer nodal solutions to nonlinear Schrödinger equations.* J. Math. Anal. Appl. 296 (2004), 563 - 577.

[10] C. O. Alves and S.H.M. Soares, *Nodal solutions for singularly perturbed equations with critical exponential growth*, J. Differential Equations 234 (2007), 464 - 484.

[11] A. Ambrosetti and A. Malchiodi, *Concentration phenomena for for NLS: recent results and new perspectives. Perspectives in nonlinear partial differential equations*, Contemp. Math., 446, Amer. Math. Soc., Providence, RI, (2007) 19-30.

[12] A. Ambrosetti, M. Badiale and S. Cingolani, *Semiclassical states of nonlinear Schrödinger equations,* Arch. Rational Mech. Anal. 140 (1997), 285-300.

[13] D.M. Cao, *Nontrivial solution of semilinear elliptic equation with critical exponent in* $\mathbb{R}^2$, Comm. Partial Differential Equations 17 (1992), 407-435.

[14] J. Chabrowski and A. Szulkin, *On a semilinear Schrödinger equation with critical Sobolev exponent.* Proc. Amer. Math. Soc. 130 (2001) 85-93.

[15] D. Cassani and C. Tarsi, *A Moser-type inequality in Lorentz-Sobolev spaces for unbounded domains in* $\mathbb{R}^N$, Asymptotic Anal. **64**, 29-51 (2009).

[16] M. del Pino and P.L. Felmer, *Local mountain pass for semilinear elliptic problems in unbounded domains,* Calc. Var. Partial Differential Equations 4, no. 2, (1996) 121-137.

[17] J. M. B. do Ó and B. Ruf. *On a Schrödinger equation with periodic potential and critical growth in* $\mathbb{R}^2$ . Nonlinear Differential Equations Appl. 13 (2006) 167-192.

[18] D.G. de Figueiredo, O.H. Miyagaki and B. Ruf, Elliptic equations in $\mathbb{R}^2$ with nonlinearities in the critical growth range, Calc. Var. Partial Differential Equations **3**, 139-153 (1995).

[19] A. Floer and A. Weinstein, *Nonspreading wave packets for the cubic Schrödinger equation with a bounded potential,* J. Funct. Anal. 69, no. 3, (1986) 397-408.

[20] M.F. Furtado and R. Marchi, *Existence of solutions to asymptotically periodic Schrödinger equations*, Electronic Journal of Differential Equations, Vol. 2017 (2017), No. 15, 1-7.

[21] D. Gilbarg and N.S. Trundiger, *Elliptic partial differential equation of second order. $2^{nd}$ edition.* Grundlehren 224, Springer, Berlin-Heidelberg-New York-Tokyo (1983).

[22] C. Gui, *Existence of multi-bump solutions for nonlinear Schrödinger equations via variational method*, Comm.PDE, 21 (1996), 787-820.

[23] S. Ibrahim, N. Masmoudi and K. Nakanishi, *Trudinger-Moser inequality on the whole plane with the exact growth condition*, J. Eur. Math. Soc. **17**(4)(2015), 819-835 .

[24] M. Ishiwata, *Existence and nonexistence of maximizers for variational problems associated with Trudinger-Moser type inequalities in $\mathbb{R}^N$*, Mathematische Annalen **351**(2011), 781-804 .

[25] L. Jeanjean, *Solutions in spectral gaps for a nonlinear equation of Schrödinger type,* J. Differential Equations 112 (1994), 53-80.

[26] E. Kreyszig, *Introductory Functional Analysis With Applications. New York: Wiley, 1989.*

[27] W. Kryszewski and A. Szulkin, *Generalized linking theorem with an application to a semilinear Schrödinger equation.* Adv. Differential Equations 3 (1998), 441-472.

[28] P.L. Lions, *The concentration-compactness principle in the calculus of variations. The locally compact case. Part I.*, Ann. Inst. Henri Poincaré, Anal. Non Linéaire 1 (1984), 109-145.

[29] P. L. Lions, *The concentration-compactness principle in the calculus of variations. The limit case,* Rev. Mat. Iberoamericana 1 (1985), 145-201.

[30] P.L. Lions, *The concentration-compactness principle in the calculus of variations. The locally compact case. Part II.*, Ann. Inst. Henri Poincaré, Anal. Non Linéaire 4 (1984), 223-283.

[31] G.B. Li and A. Szulkin, *An asymptotically periodic Schrödinger equation with indefinite linear part.* Commun. Contemp. Math, 2002, 4: 763-776

[32] X. Lin and X. Tang, *An asymptotically periodic and asymptotically linear Schrödinger equation with indefinite linear part*, Computers & Mathematics with Applications 70 (2015), 726-736.

[33] N. Masmoudi and F. Sani, *Trudinger-Moser Inequalities with the Exact Growth Condition in $\mathbb{R}^N$ and Applications*, J. Comm. in Partial Diff. Eqns **40**(2015), 1404-1440 .

[34] J. Moser, *A sharp form of an inequality by N. Trudinger*, Indiana Univ. Math. J. 20 (1971), 1077-1092.

[35] E. S. Noussair and J. Wei, *On the location of spikes and profile of nodal solutions for a singularly pertubed Neumann problem*, Comm. Partial Differential Equations, 13 (1998), 793-816.

[36] Y.G. Oh, *Existence of semi-classical bound states of nonlinear Schrödinger equations with potentials of the class $(V)_\alpha$,* Comm. Partial Differential Equations 13, no. 12, (1988) 1499-1519.

[37] Y.G. Oh, *On positive multi-lump bound states of nonlinear Schrödinger equations under multiple well potential*, Comm. Math. Phys. 131, no. 2, (1990) 223-253.

[38] A. Pankov, *Periodic nonlinear Schrödinger equation with application to photonic crystals,* Milan J. Math. 73 (2005) 259-287.

[39] A. Pankov and K. Pflüger, *On a semilinear Schrödinger equation with periodic potential*, Nonl. Anal. TMA 33 (1998), 593-609.

[40] P.H. Rabinowitz, *Minimax methods in critical point theory with applications to differential equations. AMS Reg. Conf. Ser. Math. 65 (1986)*

[41] P.H. Rabinowitz, *A note on semilinear elliptic equation on* $\mathbb{R}^n$*, Nonlinear Analysis: A Tribute in Honour of G. Prodi*, Quad. Scu. Norm. Super. Pisa, (1991), 307-318.

[42] P.H. Rabinowitz, On a class of nonlinear Schrödinger equations, Z. Angew. Math. Phys. 43, no. 2, (1992) 270-291.

[43] M. Ramos, Z-Q. Wang and M. Willem, *Positive solutions for elliptic equations with critical growth in unbounded domains,* Calculus of Variations and Differential Equations, Chapman & Hall/CRC Press, Boca Raton, 2000, pp. 192-199.

[44] B. Ruf, *A sharp Trudinger-Moser type inequality for unbounded domains in* $\mathbb{R}^2$, Journal of Functional Analysis 219 (2005) 340-367.

[45] A. Szulkin and T. Weth, *Ground state solutions for some indefinite variational problems,* J. Funct. Anal. 257 (2009) 3802-3822

[46] A. Szulkin and T. Weth, *The method of Nehari manifold.* Handb. Nonconvex Anal. Appl. 2010; 597-632.

[47] M. Schechter, *Linking Methods in Critical Point Theory,* Birkhäuser, Boston, 1999.

[48] M. Schechter, *Nonlinear Schrödinger operators with zero in the spectrum,* Z. Angew. Math. Phys. 66 (2015), no. 2125-2141.

[49] M. Schechter and W. Zou, *Weak linking theorems and Schrödinger equations with critical Sobolev exponent,* ESAIM Control Optim. Calc. Var. 9 (2003) 601-619 (electronic)

[50] E. Schrödinger, *An Undulatory Theory of the Mechanics of Atoms and Molecules.* Phys. Rev. 1926, 28, 1049-1070.

[51] X.H. Tang, *Non-Nehari manifold method for asymptotically periodic Schrödinger equations*, Science China Mathematics 58 (2015), 715-728.

[52] X.H. Tang, *New super-quadratic conditions for asymptotically periodic Schrödinger equation*, Preprint

[53] N. Trudinger, *On imbedding into Orlicz space and some applications*, J. Math. Mech. 17 (1967), 473-484.

[54] X. Wang, *On concentration of positive bound states of nonlinear Schrödinger equations,* Comm. Math. Phys., **53** (1993), 229-244.

[55] X. F. Wang, B. Zeng, *On concentration of positive bound states of nonlinear Schroedinger equations with competing potential functions*, SIAM J. Math. Anal., 28 (1997), 633-655.

[56] M. Willem, *Minimax Theorems,* Birkhauser, 1996.

[57] M. Willem and W. Zou, *On a Schrödinger equation with periodic potential and spectrum point zero,* Indiana Univ. Math. J. 52 (2003), 109-132

[58] M. Yang, *Ground state solutions for a periodic Schrödinger equation with superlinear nonlinearities,* Nonlinear Anal. 72 (5) (2010) 2620-2627.

[59] V. Coti Zelati, *A (short) Introduction to Critical Point Theory*, Second School on Nonlinear Functional Analysis and Applications to Differential Equations (21 April - 9 May 1997).

[60] H. Zhang, J. Xu and F. Zhang, *Ground state solutions asymptotically periodic Schrödinger equations with indefinite linear part*, Mathematical Methods in the Applied Sciences 38 (2015), 113-122.

[61] H. Zhang, J. Xu and F. Zhang, *On a class of semilinear Schrödinger equation with indefinite linear part*, J. Math. Anal. Appl. 414 (2014), 710-724.