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Doutorado em Matemática

Solvability for a Class of Schrödinger Equations with Periodic Potential

por

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Joao Pessoa - PB
Setembro de 2017

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sob orientação do

Prof. Dr. Everaldo Souto de Medeiros

Tese apresentada ao Corpo Docente do Programa Associado de Pós Graduação em Matemática UFPB/UFCG como requisito parcial para obtenção do título de Doutor em Matemática.

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por todo amor de sempre.

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Abstract

In this thesis we study the existence of solutions for a class of semilinear Schrödinger equations of the form

$$-\Delta u + V(x)u = \bar{f}(x,u), \quad x \in \mathbb{R}^N,$$

where $N \geq 2$, the potential V is a 1-periodic continuous function. In dimension $N \geq 3$, we assume that 0 lies in a spectral gap of the Schrödinger operator $\mathcal{S} = -\Delta + V$ and the nonlinearity is from concave and convex type. In dimension $N = 2$, we assume that 0 lies in a spectral gap or on the boundary of a spectral gap of $\mathcal{S}$ and we deal with nonlinearities having exponential growth in the Trudinger-Moser sense. We treat the case where $\bar{f}(x,t)$ is periodic as well as the nonperiodic one. The proofs relies on variational setting, by using linking-type theorems, some Trudinger-Moser inequalities and concentration-compactness principles.

Keywords: Schrödinger Operator, Periodic Potential, Spectral Theory, Linking Theorem, Sublinear Growth, Critical Growth, Trudinger-Moser Inequality.

Resumo

Nesta tese estudamos existência de soluções para uma classe de equações de Schrödinger semilineares da forma

$$-\Delta u + V(x)u = \bar{f}(x,u), \quad x \in \mathbb{R}^N,$$

onde $N \geq 2$, o potencial V é contínuo e 1-periódico. Em dimensão $N \geq 3$, assumimos que 0 localiza-se em algum gap espectral do operador de Schrödinger $\mathcal{S} = -\Delta + V$ e lidamos com não linearidades do tipo côncavo-convexo. Em dimensão $N = 2$, supomos que 0 localiza-se em algum gap espectral ou fronteira de algum gap do operador $\mathcal{S}$ e as não linearidades possuem crescimento exponencial no sentido de Trudinger-Moser. Abordamos os casos em que $\bar{f}(x,t)$ é periódica e não periódica. Nossa abordagem é variacional, utilizamos teoremas de linking, desigualdades do tipo Trudinger-Moser e princípios de concentração de compacidade.

Palavras-chave: Operador de Schrödinger, Potencial Períodico, Teoria Espectral, Teorema de Linking, Crescimento Sublinear, Crescimento Crítico, Desigualdade Trudinger-Moser.

Contents

Notation

We select here some notations used throughout the work.

Spaces

- $L^p(\Omega) = \{\varphi : \Omega \to \mathbb{R},\ \varphi \text{ is Lebesgue mensurable with } \int_\Omega |\varphi(x)|^p dx < \infty\}$, $1 \leq p < \infty$;
- $L^\infty(\Omega) = \{\varphi : \Omega \to \mathbb{R},\ \varphi \text{ is bounded and Lebesgue mensurable}\}$;
- $L^p_{\text{loc}}(\Omega) = \{\varphi : \Omega \to \mathbb{R},\ \varphi\chi_K \in L^p(\Omega) \text{ for every compact set } K \text{ contained in } \Omega\}$, where χ_K denotes the characteristic function of K;
- $H^p(\mathbb{R}^N)$ denotes the usual Sobolev space of p-weak derivatives;
- $C(\Omega)$ denotes the space of continuous real functions in $\Omega \subset \mathbb{R}^N$;
- For an integer $k \geq 1$, $C^k(\Omega)$ denotes the space of k-times continuously differentiable real functions in $\Omega \subset \mathbb{R}^N$;
- $C^\infty(\Omega) = \cap_k C^k(\Omega)$;
- $C_0^\infty(\Omega)$ denotes the space of infinitely differentiable real functions whose support is compact in $\Omega \subset \mathbb{R}^N$;
- $C(X, Y)$ denotes the continuous functions space between X and Y;
- E' denotes the topological dual of the Banach space E.

Norms

- For $1 \leq p \leq \infty$, the standard norm in $L^p(\mathbb{R}^N)$ is denoted by $\|\cdot\|_p$.

Other Notation

- $|A|$ denotes the Lebesgue measure of a set $A \subset \mathbb{R}^N$;

- $supp(\varphi)$ denotes the support of function φ;
- C, C_1, C_2, C_3, $\ldots$ denote positive constants possibly different;
- $C(s)$ denotes constant which depends of s;
- $o_n(1)$ denotes a sequence which converges to 0 as $n \to \infty$;
- $f(t) = O(g(t))$ as $t \to 0$, if and only if, $\lim\limits_{t \to 0} \dfrac{f(t)}{g(t)} \leq C$ for some constant $C > 0$;
- $\rightharpoonup$ denotes weak convergence in a normed space;
- $\to$ denotes strong convergence in a normed space;
- $\hookrightarrow$ denotes continuous embedding;
- $\langle \cdot, \cdot \rangle$ denotes the duality pairing between E and E'.

Introduction

In this thesis we study the existence and multiplicity of weak solutions for a class of nonlinear Schrödinger equations of the form

$$-\Delta u + V(x)u = \bar{f}(x,u), \quad x \in \mathbb{R}^N,$$

where $N \geq 2$ and $V \in C(\mathbb{R}^N, \mathbb{R})$ is a 1-periodic continuous function. We assume that zero lies in a spectral gap or zero is an end point of the continuous spectrum of the Schrödinger operator $\mathcal{S} = -\Delta + V$ in $L^2(\mathbb{R}^N)$. We treat nonlinearities having polynomial growth in $\mathbb{R}^N$, $N \geq 3$, or exponential growth in $\mathbb{R}^2$ in the Trudinger-Moser sense.

In the last decades, Schrödinger's equation has been subject of intense study, see for instance [13, 33, 54–56] and references therein. There are two closely related variants, precisely, the time dependent and time independent one. We quote that the time dependent semilinear Schrödinger equation is given by

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta_x\psi + V(x)\psi - \tilde{f}(x,\psi), \quad x \in \mathbb{R}^N, \tag{1}$$

where m and $\hbar$ are positive constants, $\psi : \mathbb{R}^+ \times \mathbb{R}^N \to \mathbb{C}$, $V \in C(\mathbb{R}^N, \mathbb{R})$, and $\tilde{f} \in C(\mathbb{R}^N \times \mathbb{C}, \mathbb{C})$.

If we assume that $\tilde{f}(x,tz) = f(x,t)z$, $t \in \mathbb{R}$, $z \in \mathbb{C}$, with $|z| = 1$ for some function $f \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$ and we look for *standing wave solutions* to equation (1), i.e., solutions of the form

$$\psi(t,x) = e^{-\frac{i\mathcal{E}t}{\hbar}}v(x), \tag{2}$$

where $\mathcal{E}$ is some real constant and $v : \mathbb{R}^N \to \mathbb{R}$, then applying (2) into (1) we obtain

$$-\frac{\hbar^2}{2m}\Delta v + (V(x) - \mathcal{E})v(x) = f(x,v), \quad x \in \mathbb{R}^N,$$

which is a real elliptic equation for v, the so-called time independent Schrödinger equation. In mathematics, it is not uncommon to normalize $\hbar = 1$ and $m = 1/2$, therefore we can written the time independent semilinear Schrödinger equation as follows

$$-\Delta u + V(x)u = \bar{f}(x,u), \quad x \in \mathbb{R}^N, \tag{3}$$

where $u : \mathbb{R}^N \to \mathbb{R}$, $V \in C(\mathbb{R}^N, \mathbb{R})$ and $\bar{f} : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$. Equation (3) has been extensively studied by means of many methods, among others, topological, numerical and variational. Particularly when it comes to variational method, critical point theory has been applied to various problems in differential equations. Here we apply variational tools in some classes of (3) where the potential $V : \mathbb{R}^N \to \mathbb{R}$ is a continuous and 1-periodic function and the nonlinearity $\bar{f}(x,t)$ satisfies weaker growth conditions than those previously treated by others authors. We apply linking theorems to obtain nontrivial critical points for the energy functional associated to equation (3), denoted by Φ and defined in an appropriated Banach space E.

The location of 0 with respect to the spectrum of the Schrödinger operator $\mathcal{S} = -\Delta + V$ is an indispensable point to observe which type of geometry has the energy functional Φ associated to equation (3). If the spectrum of the operator $\mathcal{S} = -\Delta + V$ lies in positive axis, then mountain-pass type theorems has been applied (see [22]) to study equation (3) under different conditions on $\bar{f}(x,t)$. Here, we suppose the presence of negative spectrum. In our class of potentials, the spectrum is purely continuous, bounded from below and is the union of disjoint closed intervals [56, Theorem XIII.100]. Intervals free of spectrum are called *spectral gaps*, these gaps appear due to the periodicity of the potential, as is well known in solid states physics [7]. In the present work we obtain some existence results for equation (3) when 0 lies in a spectral gap (see Chapters II and III) or on the boundary of a spectral gap (see Chapter IV). In both cases the null function $u \equiv 0$ is a saddle point. Thus, Φ is strongly indefinite and linking theorems are applied. We mention the papers [34, 58, 71] where the authors assumed 0 in a spectral gap of the operator $\mathcal{S} = -\Delta + V$. Our results extend these previous. As far as we know, few papers deal with the case where 0 lies on the boundary of the spectrum. We mention [9, 59, 75, 76]. The polynomial growth in the Sobolev sense is a common factor in all these papers. In this case we establish existence results for nonlinearities having exponential growth.

Relative to the negative and positive parts of the spectrum, the spectral theory provide us a decomposition $E = E^- \oplus E^+$ where the quadratic part of Φ is negative definite in E^- and positive definite in E^+. Each one infinite dimensional due to the nature of the spectrum, see Remark 1.2.13. V. Benci and P. Rabinowitz [12] firstly proved the linking theorems with both spaces infinite dimensional. An abstract result that extends the linking theorem of [12] is due to W. Kryszewski and A. Szulkin [36] where a new degree of Leray-Schauder type is defined by using a suitable topology. This topology has been applied for new generalized linking theorems [37, 46, 60, 68, 75, 76]. We apply the linking theorems obtained in [37] and [60] for our main existence results involving (3).

Since we deal with equations involving functions defined in the whole space, there is a possible loss of compactness. In general, the cause is the invariance of $\mathbb{R}^N$ by the non-compact groups of translations. To overcome this issue, some convergence results are established as well as concentration-compactness principles are applied.

Let us now describe the content of this thesis, divided into four chapters.

In Chapter I we present a short history about the Schrödinger equation and some facts about the periodic case. We briefly discuss the spectral theory applied, necessary for our development. Two main features run through this spectral section: 0 lies in a spectral gap of the Schrödinger operator $\mathcal{S} = -\Delta + V$ or on the boundary of a spectral gap. Furthermore, we establish basic properties of Banach spaces which are domains of energy functional associated with (3). In the last section of Chapter I we present the results of the linking-type used.

In Chapter II we study the existence of solutions for a class of semilinear Schrödinger equations (3) where 0 lies in a spectral gap of the operator $\mathcal{S} = -\Delta + V$ and the nonlinearity $\bar{f}(x,t)$ is a sum of a sublinear and a superlinear term. The combined effect of concave and convex nonlinearities was initially studied by A. Ambrosetti, H. Brezis and G. Cerami [5] on bounded domains. We refer [10, 11, 73] for related results. In this case we assume that the superlinear term satisfies a near condition to (AR) (see below) and a first solution is obtained by means of Ekeland's variational principle. We get a second solution applying a linking-theorem that provides a Cerami sequence for Φ. Its boundedness will be obtained with a restriction in the sublinear power.

The equation studied in this chapter has the form

$$-\Delta u + V(x)u = h(x)g(u) + k(x)f(u), \quad x \in \mathbb{R}^N, \tag{$\mathcal{C}$}$$

where $N \geq 3$, the nonlinearities $g(t)$ and $f(t)$ have sublinear and superlinear growth, respectively, and $h(x)$, $k(x)$ are weight functions. If we denote the spectrum of $\mathcal{S} = -\Delta + V$ by $\sigma(\mathcal{S})$, the following condition on V is assumed

(V_0) $V \in C(\mathbb{R}^N, \mathbb{R})$ is 1-periodic in x_j, $j = 1, 2, ..., N$, and

$$\lambda := \sup[\sigma(\mathcal{S}) \cap (-\infty, 0)] < 0 < \Lambda := \inf[\sigma(\mathcal{S}) \cap (0, \infty)].$$

We suppose the following assumptions on $g(t)$ and $h(x)$:

(g_0) $g(t)$ is continuous and there are $1 < q < 2$ and $C_1, C_2 > 0$ such that

$$|g(t)| \leq C_1|t|^{q-1} \quad \text{and} \quad G(t) \geq C_2|t|^q, \quad \forall\, t \in \mathbb{R};$$

(h_0) $h(x)$ is nonnegative and $h \in L^\sigma(\mathbb{R}^N)$ for some $\frac{2N}{(2-q)N+2q} < \sigma \leq \frac{2}{2-q}$.

We impose the following assumptions on $f(t)$ and $k(x)$:

(f_0) there are $C_0 > 0$ and $2 < p \leq 2^*$ such that $|f(t)| \leq C_0(|t| + |t|^{p-1})$ for all $t \in \mathbb{R}$;

(f_1) $2F(t) \leq f(t)t$ for all $t \in \mathbb{R}$, where $F(t) = \int_0^t f(s)ds$;

(k_0) $k(x)$ is nonnegative and $k \in L^\infty(\mathbb{R}^N)$.

Our first result for equation $(\mathcal{C})$ can be summarized as follows:

Theorem 0.0.1. *Suppose that (V_0), $(g_0), (h_0)$, $(f_0) - (f_1)$ and (k_0) hold. If $h(x)$ is nontrivial then equation $(\mathcal{C})$ admits a nontrivial weak solution u_0 with negative energy.*

To obtain another nontrivial solution for $(\mathcal{C})$, we suppose the following additional hypotheses on $f(t)$ and $k(x)$:

(f_2) $f(t) = o(t)$ as $t \to 0$;

(f_3) there exists $\mu \geq p$ such that $0 < \mu F(t) \leq t f(t)$ for all $t \neq 0$;

(f_4) there exists $0 < \theta \leq p$ such that $\liminf_{t \to 0} F(t)|t|^{-\theta} > 0$;

$(\widehat{k}_0)$ $k(x) > 0$ in $\mathbb{R}^N$ and $k \in L^\infty(\mathbb{R}^N) \cap L^\kappa(\mathbb{R}^N)$ for some $\kappa \geq \frac{2N}{(2-p)N+2p}$ and $2 < p < 2^*$.

In this case, our multiplicity result is summarized as follows.

Theorem 0.0.2 (Subcritical Case)**.** *Assume* (V_0), $(g_0), (h_0)$, (f_0), $(f_2) - (f_4)$ *and* $(\widehat{k}_0)$. *If* $1 < q < p/(p-1) < 2 < p < 2^*$ *and* $0 < \|h\|_\sigma$ *is sufficiently small then equation* $(\mathcal{C})$ *admits two nontrivial weak solutions,* u_0 *with negative energy and another* u_1 *with positive energy.*

Next, we deal with $(\mathcal{C})$ in the critical case. More precisely, we consider the problem

$$-\Delta u + V(x)u = h(x)g(u) + k(x)|u|^{2^*-2}u, \quad x \in \mathbb{R}^N. \tag{$\mathcal{C}_c$}$$

In this situation, we replace condition $(\widehat{k}_0)$ by the assumption

(k_1) $k \in C(\mathbb{R}^N, \mathbb{R})$, $k(x) > 0$ in $\mathbb{R}^N$ and there exists $\nu > 0$ such that

$$\limsup_{|x| \to \infty} |x|^\nu k(x) < \infty.$$

We also establish the existence of two nontrivial solutions, as follows:

Theorem 0.0.3 (Critical Case)**.** *Assume* (V_0), $(g_0), (h_0)$ *and* (k_1). *If* $1 < q < 2^*/(2^*-1)$, $N \geq 4$ *and* $0 < \|h\|_\sigma$ *is sufficiently small then equation* $(\mathcal{C}_c)$ *admits two nontrivial weak solutions.*

In Chapter III we establish two existence results for the equation

$$-\Delta u + V(x)u = f(x,u), \quad x \in \mathbb{R}^2, \tag{$\mathcal{P}_f$}$$

where $f(x,t)$ has subcritical exponential growth in the Trudinger-Moser sense, i.e., for any $\beta > 0$

$$\lim_{|t| \to \infty} \frac{|f(x,t)|}{e^{\beta t^2}} = 0, \quad \text{uniformly in} \quad x \in \mathbb{R}^2. \tag{4}$$

We assume that 0 lies in a spectral gap of the operator $\mathcal{S} = -\Delta + V$ defined in $L^2(\mathbb{R}^2)$. The Theorems presented in this chapter refer to nonlinearities $f(x,t)$ periodic and nonperiodic.

It is well known that the classical Ambrosetti and Rabinowitz [6] superlinear condition, namely, there exists $\Theta > 2$ such that

$$0 < \Theta \bar{F}(x,t) \leq t \bar{f}(x,t), \quad \forall\, x \in \mathbb{R}^2, \quad t \neq 0, \tag{AR}$$

is quite natural to ensure that (PS) sequences are bounded. Furthermore, it can contribute to show that the energy functional Φ has geometric properties required in critical point theorems.

Many efforts has been made to avoid this assumption [39, 41, 47, 60]. In this direction, in the papers [65, 76] the authors supposed the following super-quadratic condition

$$t \mapsto \frac{\bar{f}(x,t)}{|t|} \quad \text{is stricly increasing on} \quad (-\infty, 0) \cup (0, \infty), \quad \text{uniformly in}\, x \in \mathbb{R}^N. \tag{5}$$

The condition (5) was refined in [39] and weaker in [58]. In our results involving nonlinearities with exponential growth, the linking theorem applied produces a (PS) sequence for each functional considered. We assume a consequence of growth condition (5), see condition (f_3) below, as expected, important for boundedness of (PS) sequences obtained.

Precisely, we assume that

(V_0) $V \in C(\mathbb{R}^2, \mathbb{R})$ is 1-periodic in x_j, $j = 1, 2$, and

$$\lambda := \sup[\sigma(\mathcal{S}) \cap (-\infty, 0)] < 0 < \Lambda := \inf[\sigma(\mathcal{S}) \cap (0, \infty)].$$

Setting $F(x,t) = \int_0^t f(x,s)ds$, we suppose that f is continuous, satisfies (4) and the following conditions:

(f_0) there are $\delta > 0$ and $0 < \gamma < \Lambda$ such that $|f(x,t)| \leq \gamma|t|$ for any $|t| < \delta$ and $x \in \mathbb{R}^2$;

(f_1) $2F(x,t) \geq \lambda t^2$ for any $x \in \mathbb{R}^2$, $t \in \mathbb{R}$ and

$$\frac{F(x,t)}{t^2} \to +\infty \quad \text{as} \quad t^2 \to \infty, \quad \text{uniformly in} \quad x \in \mathbb{R}^2;$$

(f_2) $f(x,t)$ is locally bounded in the variable t, that is, for any bounded interval $J \subset \mathbb{R}$, there exists $C > 0$ such that $|f(x,t)| \leq C$ for every $(x,t) \in \mathbb{R}^2 \times J$;

(f_3) there exists $W \in L^1(\mathbb{R}^2)$ such that for all $r \in [0,1]$ it holds

$$2(F(x,t+s) - F(x,t)) \geq (2rs - (r-1)^2 t) f(x,t) - W(x), \quad \forall\, x \in \mathbb{R}^2,\ s,t \in \mathbb{R}.$$

Now, our first existence result for equation $(\mathcal{P}_f)$ can be summarized as follows.

Theorem 0.0.4. *Assume* (V_0) *and* $(f_0) - (f_3)$*. If* $f(x,t)$ *is* 1*-periodic and satisfies* (4) *then equation* $(\mathcal{P}_f)$ *admits a nontrivial weak solution.*

In the case where the nonlinearity $f(x,t)$ is nonperiodic, in addition we will assume that

(f_4) there exist $\alpha_0 > 0$, $R_0 > 0$ and $h \in L^1(B_{R_0}^c)$ such that

$$|F(x,t)| \leq h(x) e^{\alpha_0 t^2}, \quad \forall\, x \in B_{R_0}^c, t \in \mathbb{R}.$$

In this case, our second existence result is the following:

Theorem 0.0.5. *Assume* (V_0) *and* $(f_0)-(f_4)$. *If* $f(x,t)$ *satisfies* (4) *then equation* $(\mathcal{P}_f)$ *admits a nontrivial weak solution.*

Finally, Chapter IV is devoted to study a class of semilinear Schrödinger equations

$$-\Delta u + V(x)u = g(x,u), \quad x \in \mathbb{R}^2, \tag{$\mathcal{P}_g$}$$

where 0 is a right boundary point of the spectrum of the Schrödinger operator $\mathcal{S} = -\Delta + V$. We define the domain of our energy functional as completeness of E with respect to an adequate norm. In fact, a Banach space denoted by $(E_q, \|\cdot\|_{\mathbf{q}})$ and such that $H^1 \subset E_q \subset E$. In order to the energy functional be well-defined we obtain a Trudinger-Moser inequality in the space E_q by using Schwartz symmetrization, among other results. This case is more delicate because E_q is not a Hilbert space and we lost some embeddings.

We assume that the potential $V(x)$ satisfies:

(V_0) $V \in C(\mathbb{R}^2, \mathbb{R})$ is 1-periodic in x_j, $j = 1, 2$;

(V_1) $0 \in \sigma(\mathcal{S})$ and there exists $b > 0$ such that $\sigma(\mathcal{S}) \cap (0, b) = \emptyset$.

We assume that the nonlinearity $g(x,t)$ has exponential subcritical growth at infinity,

$$\lim_{|t|\to\infty} \frac{g(x,t)}{e^{\beta t^2}} = 0 \quad \text{for all} \quad \beta > 0, \tag{6}$$

and satisfies:

(g_0) There are $a > 0$ and $q > 2$ such that

$$2G(x,t) \geq a|t|^q \quad \text{for all} \quad x \in \mathbb{R}^2,\, t \in \mathbb{R}, \text{ where } G(x,t) := \int_0^s g(x,s)ds;$$

(g_1) $g \in C(\mathbb{R}^2 \times \mathbb{R}, \mathbb{R})$ is 1-periodic in x_j for $j = 1, 2$;

(g_2) $g(x,t) = O(|t|^{q-1})$ as $t \to 0$ uniformly in $x \in \mathbb{R}^2$, where $q > 2$ is given in (g_0);

(g_3) $g(x,t)$ is continuous and locally bounded in the variable t, that is, for any bounded interval $J \subset \mathbb{R}$, there exists $C > 0$ such that $|g(x,t)| \leq C$ for every $(x,t) \in \mathbb{R}^2 \times J$;

(g_4) There exists $W \in L^1(\mathbb{R}^2)$ such that for all $x \in \mathbb{R}^2$, $s, t \in \mathbb{R}$ and $r \in [0,1]$ it holds

$$2(G(x,t+s) - G(x,t)) \geq (2rs - (r-1)^2 t)g(x,t) - W(x).$$

Our main result of existence of solution for problem $(\mathcal{P}_g)$ under the above hypotheses can be summarized as follows.

Theorem 0.0.6. *Assume* $(V_0) - (V_1)$ *and* $(g_0) - (g_4)$. *If* $g(x,t)$ *satisfies* (6) *then the problem* $(\mathcal{P}_g)$ *has a nontrivial weak solution. Moreover, if* $\mathcal{M}$ *denotes the collection of the solutions of* $(\mathcal{P}_g)$, *then there is a ground state solution, i.e., a solution of* $(\mathcal{P}_g)$ *that minimizes the energy functional over* $\mathcal{M}$. *Furthermore,* $u \in C^1(\mathbb{R}^2)$ *and* $u(x) \to 0$ *as* $|x| \to \infty$.

Chapter 1

Preliminary Results

Our basic preliminaries, concepts and some results for subsequent chapters are presented. For more details, we refer [26, 27, 50, 56, 64].

1.1 On the Schrödinger Equation

In 1926, the Austrian theoretical physicist Erwin Schrödinger (1887-1961) published four works in the Annalen der Physik journal in which he laid the foundations of Wave Quantum Mechanics. The Schrödinger equation, formulated to describe the quantum state of a system, is celebrated as one of the most important achievements in 20th Century physics. An original interpretation of the physical meaning of the wave function. A consistent theory of microscopic phenomena is the quantum mechanics developed by E. Schrödinger, W. Heisenberg, M. Born, P. Jordan, N. Bohr, W. Pauli, P. Dirac and other scientists.

1.2 The Spectrum of the Periodic Schrödinger Operator

In pure mathematics, the Schrödinger equation is one of the basic equations studied in the field of partial differential equations, and has applications in spectral theory, geometry, integrable systems, among others. There are actually many generalizations and variants of the Schrödinger equation. The Schrödinger equation with periodic potential appears in a natural way, e.g., in the quantum theory of solids [56]. This equation has been studied extensively in recent years and substantial advances have been made both in the theory and in applications.

In this section we present the Schrödinger operator which will be used in the next chapters. The potential $V : \mathbb{R}^N \to \mathbb{R}$ is assumed to be continuous and 1-periodic, i.e., $V(x+T) = V(x)$, for all $T \in \mathbb{Z}^N$. Hereafter we denote by $\mathcal{S} = -\Delta + V$ the self-adjoint operator defined in $L^2(\mathbb{R}^N)$ with domain $H^2(\mathbb{R}^N)$ and $V(x)$ under the above conditions. Precisely, defined via Fourier transform.

By the Plancherel Theorem (see for instance [31]), the Fourier transform $\widehat{\psi}$, of ψ, is a unitary isomorphism on $L^2(\mathbb{R}^N)$. Considering the Sobolev space $H^2(\mathbb{R}^N) = \{\psi \in L^2(\mathbb{R}^N),\ (1 + |p|^2)\,\widehat{\psi} \in L^2(\mathbb{R}^N)\}$, we can define the operator $D = -\Delta$ in $L^2(\mathbb{R}^N)$ acting in $H^2(\mathbb{R}^N)$ by the identity $(-\Delta\psi)\widehat{}\,(p) = |p|^2\widehat{\psi}(p)$.

Let $f : \mathbb{R}^N \to \mathbb{R}$ be a measurable function and let $D(M_f) := \{\psi \in L^2(\mathbb{R}^N),\ f\psi \in L^2(\mathbb{R}^N)\}$. Then, the multiplication operator,

$$M_f : D(M_f) \to L^2(\mathbb{R}^N), \quad M_f(\psi)(p) = f(p)\psi(p),$$

is a self-adjoint operator. As the Fourier transform is a unitary operator, we conclude that $\big(-\Delta,\ H^2(\mathbb{R}^N)\big)$ is a self-adjoint operator. Furthermore, since the potential $V(x)$ is a bounded real function, the multiplication operator M_V is also self-adjoint. Having disposed of this, we can see that $\mathcal{S} = -\Delta + V$ defined in $L^2(\mathbb{R}^N)$ with domain $H^2(\mathbb{R}^N)$ is a self-adjoint operator. For more details we refer the reader to [55].

1.2.1 Elements of Spectral Theory

The literature on spectral theory is very extensive. Listed below are only the basic concepts, properties, as well as some references containing the most fundamental results related to the problems studied in the present work.

Let E be a Hilbert space and let $S : D(S) \subset E \to E$ be a linear operator, where $D(S)$ denotes the domain of S. We denote by $R(S)$ the range of the operator S.

Definition 1.2.1. *We say that $z \in \mathbb{C}$ belongs to the resolvent set of S if there exists the operator $R_z = (S - zId)^{-1}$ which is bounded and $\overline{D(R_z)} = E$. The complement of the resolvent set is called the spectrum, $\sigma(S)$, of S.*

The spectrum of an operator is in fact the disjoint union of sets, which are defined below.

Definition 1.2.2. *The point (or discrete) spectrum of S, $\sigma_p(S)$, consists of all $z \in \mathbb{C}$ such that $R_z = (S - zId)^{-1}$ does not exist.*

Definition 1.2.3. *If $R(S - zId)$ is dense in E and if $S - zId$ has an unbounded inverse, then z is said to belong to the continuous (or essential) spectrum of S, $\sigma_c(S)$.*

Definition 1.2.4. *If $R(S - zId)$ is not dense in E but $S - zId$ has an inverse, bounded or unbounded, then z is said to belong to the residual spectrum of S, $\sigma_r(S)$.*

Thus we have the following decomposition $\sigma(S) = \sigma_p(S) \cup \sigma_c(S) \cup \sigma_r(S)$.

Definition 1.2.5. *The complex number z is called an approximate eigenvalue of S if, for any $\varepsilon > 0$, there exists $u \in D(S)$ such that $\|u\| = 1$ and $\|(S - zId)u\| < \varepsilon$. We denote by $\sigma_a(S)$ the set of all approximate eigenvalues of S and call this set the approximate spectrum of S.*

Below, we have a characterization of this set.

Proposition 1.2.6. *$z \in \sigma_a(S)$ if only if $S - zId$ does not have a bounded inverse.*

As consequence of the above proposition and definitions of $\sigma_p(S)$ and $\sigma_c(S)$ we obtain

$$[\sigma_p(S) \cup \sigma_c(S)] \subset \sigma_a(S) \subset \sigma(S). \tag{1.1}$$

Now we collect some properties of the spectrum of a self-adjoint operator.

Proposition 1.2.7. *Let $S : E \to E$ be a self-adjoint operator. Then $\sigma(T) \subset \mathbb{R}$ and $\sigma_r(T) = \emptyset$.*

The set $\Theta(S) = \{(Su, u) : u \in D(S),\ \|u\| = 1\}$ of complex numbers is called the *numerical range* of the operator S. Thus, this is a subset of real numbers in the case when S is self-adjoint. Moreover, an interval indeed due to its convexity (see [27]). Another interesting fact is the following result.

Proposition 1.2.8 ([27])**.** *Let S be a self-adjoint operator defined in E. Then $\Theta(S)$ is bounded from below if and only if $\sigma(S)$ is bounded from below. Moreover, the lower bounds are equal, i.e.,*

$$\inf\{\lambda : \lambda \in \Theta(S)\} = \inf\{\lambda : \lambda \in \sigma(S)\}.$$

With this we can see that the spectrum of the operator $\mathcal{S} = -\Delta + V$ defined in $L^2(\mathbb{R}^N)$ with domain $H^2(\mathbb{R}^N)$ is bounded from below. Let $\lambda_0 > 0$ such that $V(x) + \lambda_0 > 0$, $\forall\, x \in \mathbb{R}^N$. For any $u \in H^2(\mathbb{R}^N) \setminus \{0\}$, we have

$$(\mathcal{S}u, u)_2 = \int_{\mathbb{R}^N} \left(|\nabla u|^2 + V(x)u^2\right) dx > -\lambda_0 \|u\|_2^2.$$

Thus we obtain

$$\Theta(\mathcal{S}) = \{(\mathcal{S}u, u)_2 : u \in H^2(\mathbb{R}^N),\ \|u\|_2 = 1\} \subset (-\lambda_0, \infty) \quad \text{and so} \quad \sigma(\mathcal{S}) \subset (-\lambda_0, \infty).$$

Furthermore, one can shows that the periodic Schrödinger operator $\mathcal{S}$ has no eigenvalues, in other words, that $\sigma_p(\mathcal{S}) = \emptyset$. A more complete answer about the spectrum of $\mathcal{S}$ is given by the following theorem (see [26], [56]).

Theorem 1.2.9. *Let $\mathcal{S} = -\Delta + V$ the periodic Schrödinger operator defined in Section 1.2. Then $\sigma(\mathcal{S})$ is purely continuous, bounded from below and is the union of disjoint closed intervals.*

We recall that open intervals free of spectrum are called *spectral gaps*.

Let $P : E \to E$ be an orthogonal projector, that is, P is self-adjoint and $P^2 = P$.

Definition 1.2.10. *A family of orthogonal projectors $\{\mathcal{E}(\lambda) : E \to E\}_{\lambda \in \mathbb{R}}$, in a Hilbert space E, is called a resolution of the identity if it satisfies the following conditions:*

(i) $\mathcal{E}(\lambda)\mathcal{E}(\mu) = \mathcal{E}(\min\{\lambda, \mu\})$;

(ii) $\mathcal{E}(-\infty) = 0$ *and* $\mathcal{E}(+\infty) = Id$, *where* $\mathcal{E}(\pm\infty)u := \lim\limits_{\lambda \to \pm\infty} \mathcal{E}(\lambda)u$, $\forall\, u \in E$;

(iii) $\mathcal{E}(\lambda + 0) = \mathcal{E}(\lambda)$, *where* $\mathcal{E}(\lambda + 0)u := \lim\limits_{\mu \to \lambda,\, \mu > \lambda} \mathcal{E}(\mu)u$, $\forall\, u \in E$.

The next result can be found for instance in [50].

Lemma 1.2.11. *Let $\{\mathcal{E}(\lambda) : E \to E\}_{\lambda \in \mathbb{R}}$ be a resolution of the identity. Then, for all $\lambda \in \mathbb{R}$, the operators*

$$\mathcal{E}(\lambda + 0) = \lim_{\mu \to \lambda,\, \mu > \lambda} \mathcal{E}(\mu) \quad \text{and} \quad \mathcal{E}(\lambda - 0) = \lim_{\mu \to \lambda,\, \mu < \lambda} \mathcal{E}(\mu),$$

are well defined when considering the limit for the strong convergence topology.

We can see that the condition (i) it holds if and only if $\lambda \mapsto (\mathcal{E}(\lambda)u, u)$ is non decreasing for each $u \in E$. Moreover, for all $u, v \in E$, the function $\lambda \mapsto (\mathcal{E}(\lambda)u, v)$ is a function of bounded variation. The family $\{\mathcal{E}(\lambda)\}_{\lambda\in\mathbb{R}}$ is also called *decomposition of identity*, *spectral family* or *spectral resolution*. Associate to a spectral family of projectors in a Hilbert space E we have a self-adjoint operator defined in E. The converse is true. For more details, see e.g. [50].

Theorem 1.2.12. *Any self-adjoint operator $S : E \to E$ in a Hilbert space E admits a spectral resolution such that*

$$(Su, v) = \int_{\mathbb{R}} \lambda d(\mathcal{E}(\lambda)u, v), \quad Su = \int_{\mathbb{R}} \lambda d(\mathcal{E}(\lambda)u),$$

where in the right hand we have integrals in the Riemann-Stieltjes sense.

Let $I = (\lambda_1, \lambda_2)$ an interval. By using condition (i) in Definition 1.2.10, we will denote by $\mathcal{E}(I)$ the spectral projector $\mathcal{E}(I) = \mathcal{E}(\lambda_2) - \mathcal{E}(\lambda_1)$.

Remark 1.2.13. *We observe that the continuous spectrum consists of all non-isolated points of $\sigma(S)$ and eigenvalues of infinite multiplicity. Let $\mathcal{E}(I)$ be the spectral projector associated with an interval $I \subset \mathbb{R}$, an equivalent definition for continuous spectrum is given by:*

$$\lambda \in \sigma_c(S) \quad \Leftrightarrow \quad \dim \mathcal{E}(\lambda - \varepsilon, \lambda + \varepsilon)E = \infty, \forall \varepsilon > 0.$$

1.2.2 Zero in a Spectral Gap

Assuming that 0 lies in a spectral gap of the Schrödinger operator $\mathcal{S} = -\Delta + V$ defined in $L^2(\mathbb{R}^N)$, more precisely, under the hypothesis

(V_0) $V \in C(\mathbb{R}^2, \mathbb{R})$ is 1-periodic in x_j, $j = 1, 2$, and

$$\lambda := \sup[\sigma(\mathcal{S}) \cap (-\infty, 0)] < 0 < \Lambda := \inf[\sigma(\mathcal{S}) \cap (0, \infty)],$$

we will find a Banach space $(E, \|\cdot\|)$ on which the energy functional associated to equation (3), $\Phi : E \to \mathbb{R}$ given by

$$\Phi(u) := \int_{\mathbb{R}^2} \left(|\nabla u|^2 + V(x)u^2\right) dx - 2\int_{\mathbb{R}^2} \bar{F}(x, u)dx,$$

is well defined. In order to define the space E we consider the self-adjoint operator $\mathcal{S} = -\Delta + V$ defined in $L^2(\mathbb{R}^N)$ acting in $D(\mathcal{S}) = H^2(\mathbb{R}^N)$. Let $\{\mathcal{E}(\lambda) : L^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N)\}_{\lambda\in\mathbb{R}}$ be the spectral family of $\mathcal{S}$, and $|\mathcal{S}|^{1/2}$ be the square root of $|\mathcal{S}|$. Setting $U = Id - \mathcal{E}(0) - \mathcal{E}(-0)$ we can see that U is unitary and commutes with $\mathcal{S}$, $|\mathcal{S}|$ and $|\mathcal{S}|^{1/2}$. Moreover $\mathcal{S} = U|\mathcal{S}|$ is the polar decomposition of the operator $\mathcal{S}$ (see [35], p. 358).

Let us denote by $E := D(|\mathcal{S}|^{1/2})$ the domain of $|\mathcal{S}|^{1/2}$. It is well known that $\mathcal{E}(\lambda)E \subset E$ for all $\lambda \in \mathbb{R}$. Furthermore, defining

$$E^- := \mathcal{E}(0)E, \quad E^+ := (Id - \mathcal{E}(0))E,$$

$$(u,v) := (|\mathcal{S}|^{1/2}u, |\mathcal{S}|^{1/2}v)_2, \quad \forall\, u,v \in E, \text{ and } \quad \|u\| := \sqrt{(u,u)},$$

where $(\cdot,\cdot)_2$ is the usual inner product in $L^2(\mathbb{R}^N)$, we have the following results (see e.g. [9,68]).

Lemma 1.2.14. *Assume* (V_0). *For any* $u^- \in E^-$, $u^+ \in E^+$, *it holds* $(u^-,u^+)_2 = (u^-,u^+) = 0$. *Moreover,*

$$(\mathcal{S}u,u)_2 = -\|u\|^2 \leq \lambda\|u\|_2^2, \quad \forall\, u \in E^- \tag{1.2}$$

and

$$(\mathcal{S}u,u)_2 = \|u\|^2 \geq \Lambda\|u\|_2^2, \quad \forall\, u \in E^+. \tag{1.3}$$

Proof. Let us first observe that if $u \in E$, then $u \in E^+$ if only if $\mathcal{E}(0)u = 0$. For $u^- \in E^-$ and $u^+ \in E^+$, there are $\tilde{u}^-, \tilde{u}^+ \in E$ such that $u^- = \mathcal{E}(0)\tilde{u}^-$ and $u^+ = [Id - \mathcal{E}(0)]\tilde{u}^+$. Therefore,

$$\begin{aligned}(u^-,u^+)_2 &= (\mathcal{E}(0)\tilde{u}^-, [Id - \mathcal{E}(0)]\tilde{u}^+)_2 \\ &= (\tilde{u}^-, \mathcal{E}(0)[Id - \mathcal{E}(0)]\tilde{u}^+)_2 \\ &= 0\end{aligned}$$

and

$$\begin{aligned}(u^-,u^+) &= (|\mathcal{S}|^{1/2}u^-, |\mathcal{S}|^{1/2}u^+)_2 \\ &= (|\mathcal{S}|^{1/2}\mathcal{E}(0)\tilde{u}^-, |\mathcal{S}|^{1/2}[Id - \mathcal{E}(0)]\tilde{u}^+)_2 \\ &= (|\mathcal{S}|^{1/2}\tilde{u}^-, \mathcal{E}(0)[Id - \mathcal{E}(0)]|\mathcal{S}|^{1/2}\tilde{u}^+)_2 \\ &= 0.\end{aligned}$$

Now, we have

$$\begin{aligned}-\|u^-\|^2 = (\mathcal{S}u^-,u^-)_2 &= \int_{\mathbb{R}} \mu d(\mathcal{E}(\mu)u^-,u^-)_2 \\ &= \int_{-\infty}^{\lambda} \mu d(\mathcal{E}(\mu)u^-,u^-)_2 \\ &\leq \lambda\left([\mathcal{E}(0) - \mathcal{E}(-\infty)]u^-,u^-\right)_2 \\ &= \lambda\|u^-\|_2^2, \quad \forall\, u^- \in E^-.\end{aligned}$$

Similarly, we see that for all $u^+ \in E^+$, one has

$$\begin{aligned}\|u^+\|^2 = (\mathcal{S}u^+,u^+)_2 &= \int_{\mathbb{R}} \mu d(\mathcal{E}(\mu)u^+,u^+)_2 \\ &= \int_{\Lambda}^{\infty} \mu d([\mathcal{E}(\mu) - \mathcal{E}(0)]u^+,u^+)_2 \\ &\geq \Lambda([\mathcal{E}(+\infty) - \mathcal{E}(0)]u^+,u^+)_2 \\ &= \Lambda\|u^+\|_2^2.\end{aligned}$$

□

Lemma 1.2.15. *Assume* (V_0). *Then* $E = E^- \oplus E^+$ *and* $\|\cdot\|$ *is equivalent to* $\|\cdot\|_{H^1}$ *on* E.

Proof. Since $\mathcal{E}(+\infty) = Id$, it follows that for $u \in E$,

$$u = \mathcal{E}(0)u + [\mathcal{E}(+\infty) - \mathcal{E}(0)]u.$$

This together with the previous lemma shows that $E = E^- \oplus E^+$.

Let $u \in C_0^\infty(\mathbb{R}^N)$ and $\lambda_0 > 0$ such that $V(x) + \lambda_0 > 0$ for all $x \in \mathbb{R}^N$. Setting $V_\infty = \sup_{x\in\mathbb{R}^N} |V(x)|$, we have

$$\begin{aligned}
\|u\|^2 &= (|\mathcal{S}|u, u)_2 \\
&= ((\mathcal{S} + \lambda_0)\mathcal{U}u, u)_2 - \lambda_0 (\mathcal{U}u, u)_2 \\
&\leq \|\mathcal{U}(\mathcal{S} + \lambda_0)^{1/2}\|_2 \|(\mathcal{S} + \lambda_0)^{1/2}u\|_2 + \lambda_0 \|\mathcal{U}u\|_2 \|u\|_2 \\
&\leq \|(\mathcal{S} + \lambda_0)^{1/2}u\|_2^2 + \lambda_0 \|u\|_2^2 \\
&\leq (2\lambda_0 + V_\infty)\|u\|_{H^1}^2.
\end{aligned}$$

The proof is completed by showing that $\|u\|_{H^1} \leq C\|u\|$, for some constant $C > 0$. For $u^- \in E^-$ we apply the inequality (1.2) as follows

$$\begin{aligned}
\|u^-\|_{H^1}^2 &\leq \big((\mathcal{S} + \lambda_0 + 1)u^-, u^-\big)_2 \\
&\leq \big(\mathcal{U}|\mathcal{S}|^{1/2}u^-, |\mathcal{S}|^{1/2}u^-\big)_2 + (\lambda_0 + 1)\|u^-\|_2^2 \\
&\leq \||\mathcal{S}|^{1/2}u^-\|_2^2 - (\lambda_0 + 1)/\lambda \|u^-\|^2 \\
&\leq \left(1 - \frac{1}{\lambda}(\lambda_0 + 1)\right) \|u^-\|^2.
\end{aligned}$$

If $u^+ \in E^+$, using (1.3) we get

$$\begin{aligned}
\|u^+\|_{H^1}^2 &\leq \big((\mathcal{S} + \lambda_0 + 1)u^+, u^+\big)_2 \\
&\leq \big(\mathcal{U}|\mathcal{S}|^{1/2}u^+, |\mathcal{S}|^{1/2}u^+\big)_2 + (\lambda_0 + 1)\|u^+\|_2^2 \\
&\leq \||\mathcal{S}|^{1/2}u^+\|_2^2 + (\lambda_0 + 1)/\Lambda \|u^+\|^2 \\
&= \left(\frac{\Lambda + \lambda_0 + 1}{\Lambda}\right) \|u^+\|^2.
\end{aligned}$$

This finishes the proof. □

We have $E = E^- \oplus E^+$ where the bilinear form $B : E \times E \to \mathbb{R}$,

$$B(u, v) = \int_{\mathbb{R}^N} (\nabla u \nabla v + V(x)uv)\, dx,$$

is negative definite and positive definite respectively. This decomposition corresponding to negative part and positive part of the spectrum in real axis. The spaces E^- and E^+ are $\mathcal{S}$-invariants. Moreover, by the previous results, if $u \in E^-$ and $v \in E^+$, then u and v are both orthogonal with respect to $(\cdot, \cdot)_2$ and $(\cdot, \cdot)$. Now, by definition, $|\mathcal{S}|u = \mathcal{S}u$ if $u \in E^+$ and $|\mathcal{S}|u = -\mathcal{S}u$ if $u \in E^-$. Thus, $|\mathcal{S}| : E \to E$ is a positive self-adjoint operator. Therefore, we can define the square root of $|\mathcal{S}|$, which is also a self-adjoint operator. The equality of operators can be verified $\left(|\mathcal{S}|^{\frac{1}{2}}\right)^2 u = |\mathcal{S}|u$, $\forall\, u \in D(\mathcal{S})$.

With this, we obtain

$$
\begin{aligned}
B(u,v) &= \int_{\mathbb{R}^N} (\nabla u \nabla v + V(x)uv)\, dx \\
&= (\mathcal{S}u, v)_2 \\
&= (\mathcal{S}u^- + \mathcal{S}u^+, v)_2 \\
&= (\mathcal{S}u^+, v)_2 + (\mathcal{S}u^-, v)_2 \\
&= (|\mathcal{S}|u^+, v^+)_2 - (|\mathcal{S}|u^-, v^-)_2 \\
&= (|\mathcal{S}|^{\frac{1}{2}}u^+, |\mathcal{S}|^{\frac{1}{2}}v^+)_2 - (|\mathcal{S}|^{\frac{1}{2}}u^-, |\mathcal{S}|^{\frac{1}{2}}v^-)_2 \\
&= (u^+, v^+) - (u^-, v^-) \\
&= (u^+, v) - (u^-, v).
\end{aligned}
$$

In particular,

$$
B(u,u) = \|u^+\|^2 - \|u^-\|^2. \tag{1.4}
$$

Thus, if $u \in E$, we have

$$
\Phi(u) = \frac{1}{2}\left(\|u^+\|^2 - \|u^-\|^2\right) - \int_{\mathbb{R}^N} \bar{F}(x,u)dx.
$$

1.2.3 Zero on the Boundary of a Spectral Gap

In the Chapter IV we treat the case where 0 is a right endpoint of the spectrum of the Schrödinger operator $\mathcal{S}$. This location causes a loss of completeness in the space E^-. We can see this statement proceeds by virtue the following result.

Proposition 1.2.16. *Suppose that $0 \in \sigma(\mathcal{S})$ and there exists $b > 0$ such that $\sigma(\mathcal{S}) \cap (0,b) = \emptyset$. Then norm $\|\cdot\|$ is not equivalent to H^1-norm on E^-.*

Proof. Since $0 \in \sigma(\mathcal{S})$ and $\sigma(\mathcal{S}) = \sigma_c(\mathcal{S}) = \sigma_a(\mathcal{S})$, taking $\varepsilon = 1/n$ in Definition 1.2.5, we obtain a sequence $(u_n) \subset L^2(\mathbb{R}^N) \cap E^-$ such that $\|u_n\|_2 = 1$ and $\|\mathcal{S}u_n\|_2 \to 0$. Therefore, since $H^1(\mathbb{R}^N) \hookrightarrow L^2(\mathbb{R}^N)$ with continuous embedding, there does not exists $C > 0$ satisfying $\|u^-\|_{H^1} \leq C\|u^-\|$ for all $u^- \in E^-$. This completes the proof. □

1.3 The Weak-Strong Topology

In this section we use the same terminology from [9]. Let E be a separable Hilbert space endowed with inner product $(\cdot,\cdot)$ and the associated norm $\|\cdot\|$. Let E^- be a closed subspace of E and let $E^+ := (E^-)^\perp$. On E we define a new norm

$$
\|u\|_\tau := \max\left\{\sum_{k=1}^{\infty} \frac{1}{2^k}|(e_k, u^-)|, \|u^+\|\right\}, \quad u = u^- + u^+ \in E = E^- \oplus E^+,
$$

where $\mathcal{B} = \{e_1, e_2, e_3, \ldots\}$ is a complete orthonormal system in E^-.

For reasons that will become clear later, the topology induced by $\|\cdot\|_\tau$ will be called the *weak-strong topology.* A first indication is the following.

Proposition 1.3.1. *If $(u_n) \subset E^-$ is bounded, then*

$$\|u_n - u\|_\tau \to 0 \quad \Leftrightarrow \quad u_n \rightharpoonup u \quad in \quad E^-.$$

Proof. Suppose that there exists $C > 0$ such that $\|u_n\| \le C$ for all $n \in \mathbb{N}$ and $\|u_n - u\|_\tau \to 0$. Let $v = \sum_{k=1}^{\infty} \alpha_k e_k \in E^-$ and let $\varepsilon > 0$ be given. We define $v_K = \sum_{k=K+1}^{\infty} \alpha_k e_k$ and we take $K > 0$ sufficiently large such that $\|v_K\| < \varepsilon/4C$. Then,

$$|(v_K, u_n - u)| \le 2C\|v_K\| < \varepsilon/2.$$

Now, by taking $n_0 \in \mathbb{N}$ large enough that $\|u_n - u\|_\tau < \varepsilon/(2 \max\limits_{1\le k\le K} 2^k|\alpha_k|)$ for $n > n_0$, we obtain

$$\begin{aligned}
|(v - v_K, u_n - u)| &= \left|\sum_{k=1}^{K} \alpha_k (e_k, u_n - u)\right| \\
&\le \max_{1\le k\le K} 2^k|\alpha_k| \sum_{k=1}^{K} \frac{1}{2^k} |(e_k, u_n - u)| \\
&< \varepsilon/2,
\end{aligned}$$

for $n > n_0$. Therefore we conclude that $|(v, u_n - u)| < \varepsilon$ for all $n > n_0$, i.e., $u_n \rightharpoonup u$ in E^-. Conversely, if $u_n \rightharpoonup u$ in E^- then there is a constant $C > 0$ such that $\|u\| \le C$ and $\|u_n\| \le C$, for all $n \in \mathbb{N}$. For any $\varepsilon > 0$, we take $K > 0$, $n_0 \in \mathbb{N}$ such that $1/2^K < \varepsilon/4C$ and $|(e_k, u_n - u)| < \varepsilon/2$ for $1 \le k \le K$, $n > n_0$. Thus we obtain

$$\begin{aligned}
\|u_n - u\|_\tau &= \sum_{k=1}^{\infty} \frac{1}{2^k}|(e_k, u_n - u)| \\
&= \sum_{k=1}^{K} \frac{1}{2^k}|(e_k, u_n - u)| + \sum_{k=K+1}^{\infty} \frac{1}{2^k}|(e_k, u_n - u)| \\
&\le \frac{\varepsilon}{2}\sum_{k=1}^{K} \frac{1}{2^k} + \sum_{k=K+1}^{\infty} \frac{2C}{2^k} \\
&\le \frac{\varepsilon}{2} + \frac{\varepsilon}{2}.
\end{aligned}$$

This finishes the proof. □

In other words, the above proposition says that on bounded subsets of E^- the topology induced by $\|\cdot\|_\tau$ is equivalent to the weak topology of E^-. In the next chapters we will make use of this topology. In each case, specifying the context in which it is being applied, i.e., the domain of the energy functional.

1.4 Kryszewski-Szulkin's Linking Theorem

In this section we present the linking-theorem proved by W. Kryszewski and A. Szulkin [36] which makes use of the weak-strong topology.

Theorem 1.4.1. *Let E be a Hilbert space and suppose that $\Phi \in C^1(E, \mathbb{R})$ satisfies the following hypotheses:*

(i) $\nabla\Phi$ *is weakly sequentially continuous and there exists a closed separable subspace Y such that Φ is τ-upper semicontinuous, where τ is the weak-strong topology on $E = Y \oplus Y^\perp$;*

(ii) there are constants $\eta > 0$, $\rho > 0$ such that $\Phi|S_\rho \cap Y^\perp \geq \eta$;

(iii) there are $z_0 \in S_1 \cap Y^\perp$ and $R > \rho$ such that $\Phi|\partial Q_R \leq 0$, where $Q_R := \{u = y + sz_0 : y \in Y,\ \|u\| < R,\ s > 0\}$.

Then there exists a sequence (u_n) such that $\nabla\Phi \to 0$ and $\Phi(u_n) \to c$ for some $c \in [\eta, \sup_{\overline{Q}_R} \Phi]$.

Remark 1.4.2. *In Chapter II we will apply a generalization of this theorem obtained by G. Li and A. Szulkin* [37]*. In chapters III and IV we use a variant obtained by M. Schechter and W. Zou* [60]*.*

Chapter 2

On a Schrödinger Equation with Periodic Potential Involving Concave and Convex Nonlinearities

2.1 Introduction and Main Results

This Chapter is concerned to the existence and multiplicity of nontrivial solutions for the following nonlinear stationary Schrödinger equation

$$-\Delta u + V(x)u = h(x)g(u) + k(x)f(u), \quad x \in \mathbb{R}^N, \tag{$\mathcal{C}$}$$

where $N \geq 3$, the potential $V(x) \in C(\mathbb{R}^N, \mathbb{R})$ is 1-periodic, 0 lies in a spectral gap from the spectrum of the Schrödinger operator $\mathcal{S} = -\Delta + V$, the nonlinearities $g(t)$ and $f(t)$ are sublinear and superlinear, respectively, and $h(x)$, $k(x)$ are weight functions satisfying suitable hypotheses. The results obtained in this chapter are the subjects of the paper [45].

The equation ($\mathcal{C}$) with $V(x)$ periodic has been extensively studied in the past years, see for instance [9, 11, 19, 22, 36, 37, 49, 60, 68, 75] and references therein. Problems with combined nonlinearities in bounded domains were first investigated by A. Ambrosetti, H. Brezis an G. Cerami [5] (see also G. Tarantello [69]). Afterwards, many authors have derive a wide number of existence and multiplicity results for elliptic problems involving concave and convex terms in bounded domains, see [5, 11, 30] and references therein. In unbounded domains we refer the works [4, 8, 15, 18, 41, 70] where the authors have studied the existence of solutions in $\mathbb{R}^N$ for some semilinear elliptic equations related to problem ($\mathcal{C}$). Our main aim in this chapter is to consider nonlinearities concave and convex with critical growth. Finally, we refer the reader to the works [19, 20, 60] where the authors have studied problem ($\mathcal{C}$) with periodic potential and periodic nonlinearities with critical growth. Note that, in our hypotheses we are not assuming periodicity conditions on the nonlinearities.

In order to introduce our hypotheses on the potential $V(x)$, let us denote by $\sigma(\mathcal{S})$ the spectrum of the Schrödinger operator $\mathcal{S} = -\Delta + V$ defined in $L^2(\mathbb{R}^N)$. When $V(x)$ is continuous and periodic, it is well known that $\sigma(\mathcal{S})$ is purely continuous, bounded from below and the union of disjoint closed intervals (see [56], Theorem XIII.100). Here, we focus our study in the case where 0 lies in a spectral gap of S. Precisely, we assume the following condition on $V(x)$:

(V_0) $V \in C(\mathbb{R}^N, \mathbb{R})$ is 1-periodic in x_j, $j = 1, 2, ..., N$, and

$$\lambda := \sup[\sigma(\mathcal{S}) \cap (-\infty, 0)] < 0 < \Lambda := \inf[\sigma(\mathcal{S}) \cap (0, \infty)].$$

Setting $G(t) = \int_0^t g(s)ds$, we suppose the following assumptions on $g(t)$ and $h(x)$:

(g_0) $g(t)$ is continuous and there are $1 < q < 2$ and $C_1, C_2 > 0$ such that

$$|g(t)| \leq C_1|t|^{q-1} \quad \text{and} \quad G(t) \geq C_2|t|^q, \quad \forall\, t \in \mathbb{R};$$

(h_0) $h(x)$ is nonnegative and $h \in L^\sigma(\mathbb{R}^N)$ for some $\frac{2N}{(2-q)N+2q} < \sigma \leq \frac{2}{2-q}$.

We impose the following assumptions on $f(t)$ and $k(x)$:

(f_0) there are $C_0 > 0$ and $2 < p \leq 2^*$ such that $|f(t)| \leq C_0(|t| + |t|^{p-1})$ for all $t \in \mathbb{R}$;

(f_1) $2F(t) \leq f(t)t$ for all $t \in \mathbb{R}$, where $F(t) = \int_0^t f(s)ds$;

(k_0) $k(x)$ is nonnegative and $k \in L^\infty(\mathbb{R}^N)$.

Our first result for equation $(\mathcal{C})$ can be summarized as follows:

Theorem 2.1.1. *Suppose that (V_0), $(g_0), (h_0)$, $(f_0) - (f_1)$ and (k_0) hold. If $h(x)$ is nontrivial then equation $(\mathcal{C})$ admits a nontrivial weak solution u_0 with negative energy.*

In order to obtain another nontrivial solution for $(\mathcal{C})$, we suppose the following additional hypotheses on $f(t)$ and $k(x)$:

(f_2) $f(t) = o(t)$ as $t \to 0$;

(f_3) there exists $\mu \geq p$ such that $0 < \mu F(t) \leq tf(t)$ for all $t \neq 0$;

(f_4) there exists $0 < \theta \leq p$ such that $\liminf\limits_{t \to 0} F(t)|t|^{-\theta} > 0$;

$(\widehat{k}_0)$ $k(x) > 0$ in $\mathbb{R}^N$ and $k \in L^\infty(\mathbb{R}^N) \cap L^\kappa(\mathbb{R}^N)$ for some $\kappa \geq \frac{2N}{(2-p)N+2p}$ and $2 < p < 2^*$.

In this case, our multiplicity result is summarized as follows.

Theorem 2.1.2 (Subcritical Case)**.** *Assume (V_0), $(g_0), (h_0)$, (f_0), $(f_2) - (f_4)$ and $(\widehat{k}_0)$. If $1 < q < p/(p-1) < 2 < p < 2^*$ and $0 < \|h\|_\sigma$ is sufficiently small then equation $(\mathcal{C})$ admits two nontrivial weak solutions, u_0 with negative energy and another u_1 with positive energy.*

Next, we deal with $(\mathcal{C})$ in the critical case. More precisely, we consider the problem

$$-\Delta u + V(x)u = h(x)g(u) + k(x)|u|^{2^*-2}u, \quad x \in \mathbb{R}^N. \tag{$\mathcal{C}_c$}$$

In this situation, we replace condition $(\widehat{k}_0)$ by the assumption

(k_1) $k \in C(\mathbb{R}^N, \mathbb{R})$, $k(x) > 0$ in $\mathbb{R}^N$ and there exists $\nu > 0$ such that

$$\limsup_{|x|\to\infty} |x|^\nu k(x) < \infty.$$

We also establish the existence of two nontrivial solutions, as follows:

Theorem 2.1.3 (Critical Case)**.** *Assume* (V_0), (g_0), (h_0) *and* (k_1). *If* $1 < q < 2^*/(2^*-1)$, $N \geq 4$ *and* $0 < \|h\|_\sigma$ *is sufficiently small then equation* $(\mathcal{C}_c)$ *admits two nontrivial weak solutions.*

Remark 2.1.4. *An example of nonlinearity* $g(t)$ *satisfying hypothesis* (g_0) *is given by* $G(t) = (arctg(t) + \pi)|t|^q$. *Indeed, note that* $G(t) \geq \pi|t|^q/2$ *and*

$$g(t) = G'(t) = \frac{1}{1+t^2}|t|^q + q(arctg(t) + \pi)|t|^{q-2}t.$$

Thus $|g(t)| \leq |t|^{q-1}\left[|t|/(1+t^2) + 3\pi q/2\right] \leq C|t|^{q-1}$. *A standard example is* $g(t) = |t|^{q-2}t$.

Remark 2.1.5. *In Theorem 2.1.3 we assume that* $f(t) = |t|^{2^*-2}t$ *for the sake of simplicity. However, our proof of Theorem 2.1.3 holds if we suppose a more general nonlinearity* $f(t)$ *satisfying* (f_0), (f_2), (f_3) *and*

$(\widehat{f}_4)$ $F(t) \geq \dfrac{1}{2^*}|t|^{2^*}$, *for all* $t \in \mathbb{R}$.

In particular, under the hypotheses of Theorem 2.1.3, problem $(\mathcal{C})$ *admits two nontrivial weak solutions if* $f(t) = |t|^{2^*-2}t + C|t|^{p-2}t$ *with* $C > 0$ *and* $2 < p < 2^*$.

Usually, there are at least two ways to get critical points of the energy functional Φ associated to $(\mathcal{C})$, namely, the Ekeland Variational Principle and the minimax approach. Since we are supposing that the potential $V(x)$ is periodic and satisfies (V_0), the quadratic form $B(u,u) := \int_{\mathbb{R}^N}[|\nabla u|^2 + V(x)u^2]dx$ is no longer a norm. In fact, this quadratic form is strongly indefinite (see [9, 68] and references therein) in a subspace of infinite dimension and hence the usual Linking Theorem can not be applied directly. Moreover, roughly speaking, since the Sobolev embedding $H^1(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N)$, for $2 \leq p \leq 2^*$, is not compact, Palais-Smale condition is not valid in general. To overcome these difficulties, we use a version of the Linking Theorem due to G. Li and A. Szulkin [37] to obtain a Cerami sequence at the minimax level. Next, by using some convergence results in combination with the Concentration-Compactness Principle of Lions, we prove that the weak limit of the Cerami sequence is a nontrivial solution of $(\mathcal{C})$. We quote here that the invariance of the energy functional Φ with respect to the $\mathbb{Z}^N$-action on $H^1(\mathbb{R}^N)$, given by $(T * u)(x) = u(T + x)$ with $T \in \mathbb{Z}^N$ and $u \in H^1(\mathbb{R}^N)$, plays an important role in order to obtain nontrivial weak solutions in many papers, see for instance [19,60]. This fact is true if $h(x)$ and $k(x)$ are both periodic. To the authors knowledge, there seems to have very little progress on the existence and multiplicity of solutions for equation $(\mathcal{C})$ with $V(x)$ periodic and $h(x)$, $k(x)$ nonperiodic.

The present Chapter is organized as follows: In Section 2, in order to apply the variational framework, we use spectral theory to obtain a suitable domain for the energy functional associated to the problem. In Section 3, we present the proof of Theorems 2.1.1 by using minimization arguments. In Section 4, we establish the geometry for the energy functional required by the Linking Theorem and we prove Theorem 2.1.2. Finally, in Section 5, by applying the Concentration-Compactness Principle, we prove Theorem 2.1.3.

Throughout this Chapter $H^1(\mathbb{R}^N)$ denotes the Sobolev space endowed with the inner product

$$\langle u, v\rangle = \int_{\mathbb{R}^N} (\nabla u \nabla v + uv)dx, \quad u, v \in H^1(\mathbb{R}^N),$$

and the associated norm is represented by $\|\cdot\|_{H^1}$. As before, we use $\|\cdot\|_p$ to denote the norm of the Lebesgue space $L^p(\mathbb{R}^N)$, $1 \leq p \leq \infty$, and $(\cdot,\cdot)_2$ to represent the inner product in $L^2(\mathbb{R}^N)$. The symbols C, C_i, $i = 0, 1, 2, \ldots$ will denote various constants.

2.2 Variational Setting

In this section, in order to develop a variational approach to study the existence of nontrivial solutions for equation $(\mathcal{C})$, a key step is to identify a suitable function space setting. First, we observe that from (f_0), there are $C_1, C_2 > 0$ such that

$$|F(t)| \leq C_1|t|^2 + C_2|t|^p, \quad \forall\, t \in \mathbb{R}. \tag{2.1}$$

Under the hypothesis (V_0) we will find a Hilbert space E on which the energy functional associated to $(\mathcal{C})$, $\Phi : E \to \mathbb{R}$ given by

$$\Phi(u) := \frac{1}{2}\int_{\mathbb{R}^N} \left(|\nabla u|^2 + V(x)u^2\right) dx - \int_{\mathbb{R}^N} h(x)G(u)dx - \int_{\mathbb{R}^N} k(x)F(u)dx,$$

is well defined. In order to define the space E, we recall that the domain of the self-adjoint operator $\mathcal{S} = -\Delta + V$ defined in $L^2(\mathbb{R}^N)$ is $D(\mathcal{S}) = H^2(\mathbb{R}^N)$. Let $\{\mathcal{E}(\lambda) : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)\}_{\lambda\in\mathbb{R}}$ be the spectral family of $\mathcal{S}$, and $|\mathcal{S}|^{1/2}$ be the square root of $|\mathcal{S}|$. Setting $U = Id - \mathcal{E}(0) - \mathcal{E}(-0)$ we can see that U is unitary and commutes with $\mathcal{S}$, $|\mathcal{S}|$ and $|\mathcal{S}|^{1/2}$. Moreover $\mathcal{S} = U|\mathcal{S}|$ is the polar decomposition of the operator $\mathcal{S}$ (see [35], p. 358).

Next, let us denote by $E := D(|\mathcal{S}|^{1/2})$ the domain of the operator $|\mathcal{S}|^{1/2}$. It is well known that $\mathcal{E}(\lambda)E \subset E$ for all $\lambda \in \mathbb{R}$. Furthermore, defining

$$E^- := \mathcal{E}(0)E, \quad E^+ := (Id - \mathcal{E}(0))E,$$

$$(u,v) := (|\mathcal{S}|^{1/2}u, |\mathcal{S}|^{1/2}v)_2, \quad \forall\, u,v \in E \quad \text{and} \quad \|u\| := \sqrt{(u,u)},$$

we have the following result (see Chapter I).

Lemma 2.2.1. *Assume* (V_0). *Then* $E = E^- \oplus E^+$, *the norm* $\|\cdot\|$ *is equivalent to* $\|\cdot\|_{H^1}$ *on* E *and* $(u^+, u^-) = (u^+, u^-)_2 = 0$ *for any* $u = u^- + u^+ \in E$. *Moreover,*

$$(\mathcal{S}u, u)_2 = -\|u\|^2 \leq \lambda\|u\|_2^2, \quad \forall\, u \in E^- \tag{2.2}$$

and

$$(\mathcal{S}u, u)_2 = \|u\|^2 \geq \Lambda\|u\|_2^2, \quad \forall\, u \in E^+, \tag{2.3}$$

where $\lambda < 0 < \Lambda$ *are defined in hypothesis* (V_0).

Remark 2.2.2. *It follows from Lemma 2.2.1 that* $\|u\|^2 = \|u^-\|^2 + \|u^+\|^2$ *and the embedding* $E \hookrightarrow L^r(\mathbb{R}^N)$ *is continuous for any* $2 \leq r \leq 2^*$.

In view of Lemma 2.2.1, (g_0), (h_0), (k_0) and (2.1) we see that the functional Φ is well defined on E and (see (1.4)) can be rewritten as

$$\Phi(u) = \frac{1}{2}\left(\|u^+\|^2 - \|u^-\|^2\right) - \int_{\mathbb{R}^N} h(x)G(u)dx - \int_{\mathbb{R}^N} k(x)F(u)dx, \quad \forall\, u \in E.$$

Furthermore, combining Remark 2.2.2, (g_0), (h_0), (f_0), (k_0) and standard arguments we have that $\Phi \in C^1(E, \mathbb{R})$ and

$$\langle \Phi'(u), v\rangle = (u^+, v) - (u^-, v) - \int_{\mathbb{R}^N} h(x)g(u)v dx - \int_{\mathbb{R}^N} k(x)f(u)v dx, \quad \forall\, u, v \in E.$$

Thus, critical points of Φ correspond to weak solutions of $(\mathcal{C})$.

2.3 Solution Via Minimization

In this section we will prove the existence of a solution via local minimization arguments. Before proceeding with the proof of Theorem 2.1.1, we need some auxiliary results.

Lemma 2.3.1. *Assume* (g_0), (h_0), (k_0) *and* (f_0). *Then, for any* $\rho > 0$ *it holds*

$$-\infty < c_\rho := \inf_{u \in \overline{B}_\rho} \Phi(u) < 0,$$

where $B_\rho := \{u \in E : \|u\| < \rho\}$.

Proof. Using assumptions (g_0), (h_0) and (k_0) together with (2.1) and Remark 2.2.2, for $u \in \overline{B}_\rho$ we get

$$\int_{\mathbb{R}^N} h(x)G(u)dx + \int_{\mathbb{R}^N} k(x)F(u)dx \leq C_1\|h\|_\sigma\|u\|^q + C_2\|k\|_\infty\|u\|^2 + C_3\|k\|_\infty\|u\|^p.$$

Hence,

$$\begin{aligned}\Phi(u) &\geq -\frac{1}{2}\|u^-\|^2 - C_4\|u\|^q - C_5\|u\|^2 - C_6\|u\|^p \\ &\geq -\frac{1}{2}\rho^2 - C_4\rho^q - C_5\rho^2 - C_6\rho^p\end{aligned}$$

and therefore $c_\rho = \inf_{u\in\overline{\mathrm{B}}_\rho}\Phi(u) > -\infty$. On the other hand, using the fact that $G(t) \geq C_1|t|^q$ together with (2.1), for any $u_0 \in E\backslash\{0\}$ fixed and $t > 0$ we see that

$$\begin{aligned}\Phi(tu_0) &\leq \frac{t^2}{2}\|u_0^+\|^2 - \int_{\mathbb{R}^N} h(x)G(tu_0)dx - \int_{\mathbb{R}^N} k(x)F(tu_0)dx \\ &\leq t^2C_2 - C_3t^q + C_4t^2 + C_5t^p \\ &= t^q\left[t^{2-q}(C_2+C_4) + t^{p-q}C_5 - C_3\right].\end{aligned}$$

Consequently, $\Phi(tu_0) < 0$ for $t > 0$ sufficiently small and this completes the proof. □

To carry forward, we establish the following convergence results.

Lemma 2.3.2. *Suppose that (h_0), (g_0), (k_0) and (f_0) hold. Then the functionals defined by*

$$J_1(u) = \int_{\mathbb{R}^N} h(x)G(u)dx \quad \textit{and} \quad J_2(u) = \int_{\mathbb{R}^N} h(x)g(u)udx$$

are weakly continuous on E. Moreover, if $u_n \rightharpoonup u$ weakly in E then

$$\int_{\mathbb{R}^N} k(x)f(u_n)vdx \to \int_{\mathbb{R}^N} k(x)f(u)vdx, \quad \forall\ v \in E \quad \textit{and} \tag{2.4}$$

and

$$\int_{\mathbb{R}^N} h(x)g(u_n)vdx \to \int_{\mathbb{R}^N} h(x)g(u)vdx, \quad \forall\ v \in E. \tag{2.5}$$

Proof. Consider a sequence $(u_n) \subset E$ such that $u_u \rightharpoonup u$ in E. For any $R > 0$ fixed, we have

$$\begin{aligned}|J_1(u_n) - J_1(u)| &\leq \int_{B_R} h(x)|G(u_n) - G(u)|dx + \int_{|x|\geq R} h(x)|G(u_n) - G(u)|dx \\ &=: I_1(n) + I_2(n).\end{aligned}$$

By the Lebesgue Dominate Convergence Theorem and the compact embedding $E \hookrightarrow L^r(B_R)$, $1 \leq r < 2^*$, we have $I_1(n) = o_n(1)$. Since $(u_n) \subset E$ is bounded and $h \in L^\sigma(\mathbb{R}^N)$, it follows by the Hölder inequality and the continuous embedding $E \hookrightarrow L^r(\mathbb{R}^N)$, $2 \leq r \leq 2^*$, that

$$\begin{aligned}I_2(n) &\leq C_1\left(\int_{|x|\geq R} |h(x)|^\sigma dx\right)^{1/\sigma}\left[\left(\int_{|x|\geq R} |u_n|^{q\sigma'}dx\right)^{1/\sigma'} + \left(\int_{|x|\geq R} |u|^{q\sigma'}dx\right)^{1/\sigma'}\right] \\ &\leq C_2\left(\int_{|x|\geq R} |h(x)|^\sigma dx\right)^{1/\sigma}.\end{aligned}$$

Choosing $R > 0$ sufficiently large we see that $I_2(n) = o_n(1)$. Therefore J_1 is weakly continuous. A similar argument proves that J_2 is weakly continuous. Now, we will prove (2.4). By density we can assume that $v \in C_0^\infty(\mathbb{R}^N)$. If we denote $\Omega = \mathrm{supp}(v)$ then $f(u_n)v \to f(u)v$ almost everywhere in Ω. Moreover, from (f_0) and since, up to a subsequence, $|u_n| \leq \chi_r$ almost everywhere in Ω, with $\chi_r \in L^r(\Omega)$ for $1 \leq r < 2^*$, it follows that

$$|f(u_n)v| \leq C_1|u_n||v| + C_2|u_n|^{p-1}|v| \leq \phi_v \quad \text{almost everywhere in}\, \Omega \subset \mathbb{R}^N,$$

where $\phi_v \in L^1(\Omega)$. Hence, by applying the Lebesgue Dominate Convergence Theorem we obtain the desired result. Similarly, we can prove (2.5). $\square$

Proof of Theorem 2.1.1: Invoking the Ekeland Variational Principle, we obtain a minimizing sequence (u_n) in $\overline{B}_\rho$ such that

$$\Phi(u_n) \to c_\rho \quad \text{and} \quad \Phi'(u_n) \to 0.$$

Since $\|u_n\| \leq \rho$, going to a subsequence if necessary, we can assume that $u_n \rightharpoonup u_0$ weakly in E and $u_n(x) \to u_0(x)$ for almost every in $x \in \mathbb{R}^N$, for some $u_0 \in E$. By Lemma 2.3.2, u_0 is a critical point of Φ, i.e., $\Phi'(u_0) = 0$. Furthermore, we claim that $c_\rho = \Phi(u_0)$. In fact, observe that

$$\Phi(u_n) - \frac{1}{2}\langle \Phi'(u_n), u_n\rangle + \int_{\mathbb{R}^N} h(x)\left[G(u_n) - \frac{g(u_n)u_n}{2}\right] dx = \int_{\mathbb{R}^N} k(x)\left[\frac{f(u_n)u_n}{2} - F(u_n)\right] dx.$$

This, together with Lemma 2.3.2, (f_1) and the Fatou Lemma imply

$$c_\rho + \int_{\mathbb{R}^N} h(x)\left[G(u_0) - \frac{g(u_0)u_0}{2}\right] dx \geq \int_{\mathbb{R}^N} k(x)\left[\frac{f(u_0)u_0}{2} - F(u_0)\right] dx.$$

Consequently,

$$\begin{aligned}\Phi(u_0) \geq c_\rho \geq -\int_{\mathbb{R}^N} h(x)\left[G(u_0) - \frac{g(u_0)u_0}{2}\right] dx + \int_{\mathbb{R}^N} k(x)\left[F(u_0) - \frac{f(u_0)u_0}{2}\right] dx \\ = \Phi(u_0) - \frac{1}{2}\langle \Phi'(u_0), u_0\rangle,\end{aligned}$$

which shows that $c_\rho = \Phi(u_0)$ and this completes the proof of Theorem 2.1.1.

2.4 Linking Geometry

In this section, in order to find a second nontrivial critical point for the functional Φ, we use a Linking Theorem due to G. Li and A. Szulkin [37] (see also [36, 75] for related results). Let E be a real Hilbert space and $\Phi \in C^1(E, \mathbb{R})$. Recall that a sequence $(u_n) \subset E$ is called a Cerami sequence for Φ at the level c ($(C)_c$-sequence for short) if

$$\Phi(u_n) \to c \quad \text{and} \quad (1 + \|u_n\|)\Phi'(u_n) \to 0. \tag{2.6}$$

Suppose that $E = E^- \oplus E^+$ and E^- is separable. For each $u = u^- + u^+$, we can write

$$u^- = \sum_{k=1}^{\infty} c_k(u^-)e_k,$$

where $\mathcal{B} = \{e_1, e_2, \ldots\}$ is dense in E^-. Thus, we can define a new norm in E by setting

$$\|u\|_\tau = \max\left\{\|u^+\|, \sum_{k=1}^{\infty} \frac{|c_k(u^-)|}{2^k}\right\},$$

where $\|\cdot\|$ is the norm in E. One can see that $\|\cdot\|_\tau$ defines a norm in E and satisfies $\|u\|_\tau \leq \|u\|$ for any $u \in E$ (see [36]). For $R > \rho > 0$ and $u_0^+ \in E^+ \setminus \{0\}$, we define

$$S_\rho = \{u^+ \in E^+ : \|u^+\| = \rho\} \quad \text{and} \quad Q_R = \{u = u^- + su_0^+ : s \geq 0, u^- \in E^- \ \text{ and } \ \|u\| < R\}.$$

Next, we consider the following class of applications:

$$\Gamma := \left\{\begin{array}{l} h : [0,1] \times \overline{Q}_R \to E,\ h \text{ is } \tau\text{-continuous. For any } (s_0, u_0) \in [0,1] \times \overline{Q}_R, \\ \text{there is a } \tau\text{-neighborhood } U_{(s_0,u_0)} \text{ such that} \\ \{u - h(s,u) : (s,u) \in U_{(s_0,u_0)} \cap ([0,1] \times \overline{Q}_R)\} \subset E_{fin}, \\ h(0,u) = u,\ \Phi(h(s,u)) \leq \max\{\Phi(u), -1\}, \forall\ s \in [0,1] \text{ and } \forall\, u \in \overline{Q}_R. \end{array}\right\}$$

where E_{fin} denotes various finite-dimensional subspaces of E whose exact dimensions are irrelevant and depend on (s_0, u_0). Notice that $\Gamma \neq \emptyset$ since $I \in \Gamma$, where $I(s,u) = u$ for all $s \in [0,1]$. The Linking Theorem proved in [37] makes use of the class Γ and it is stated as follows:

Theorem 2.4.1. *Let $E = E^- \oplus E^+$ be a separable Hilbert space with E^- orthogonal to E^+ and $\Phi : E \to \mathbb{R}$ given by*

$$\Phi(u) = \frac{1}{2}\left(\|u^+\|^2 - \|u^-\|^2\right) - \psi(u).$$

Suppose that

(i) $\psi \in C^1(E, \mathbb{R})$ is bounded from below, weakly sequentially lower semicontinuous and ψ' is weakly sequentially continuous;

(ii) there exist $u_0 \in E^+ \setminus \{0\}$, $\eta > 0$ and $R > \rho > 0$ such that $\Phi|_{S_\rho} \geq \eta$ and $\Phi|_{\partial Q_R} \leq 0$.

Then there exists a $(C)_{c_1}$-sequence for Φ at the level

$$c_1 := \inf_{h \in \Gamma} \sup_{u \in Q_R} \Phi(h(1,u)). \tag{2.7}$$

Moreover, $c_1 \geq \eta$.

In what follows, the linking structure for the functional Φ associated to $(\mathcal{C})$, required in Theorem 2.4.1, will be proved by deriving some lemmas. Precisely, we will apply Theorem 2.4.1

for the functional $\psi : E \to \mathbb{R}$ given by

$$\psi(u) = \int_{\mathbb{R}^N} h(x)G(u)dx + \int_{\mathbb{R}^N} k(x)F(u)dx.$$

Lemma 2.4.2. *Assume* (h_0), (k_0), (g_0), (f_0) *and* (f_3). *Then the functional* ψ *is bounded below, weakly sequentially lower semicontinuous and* ψ' *is weakly sequentially continuous on* E.

Proof. Clearly we have $\psi(u) \geq 0$ for all $u \in E$ and hence it is bounded below. Furthermore, ψ is weakly sequentially lower semicontinuous by the Fatou Lemma. Now we prove the last statement. Let $(u_n) \subset E$ be such that $u_n \rightharpoonup u$ weakly in E. Invoking Lemma 2.3.2, we get $\langle \psi'(u_n), v\rangle \to \langle \psi'(u), v\rangle$ for all $v \in E$. Therefore, ψ' is weakly sequentially continuous on E. □

Lemma 2.4.3. *Suppose* (V_0), (h_0), (g_0), (k_0), (f_0) *and* (f_2). *If* $\|h\|_\sigma$ *is small enough, then there exist positive constants* η_0 *and* ρ_0 *such that*

$$\Phi(u^+) \geq \eta_0 \quad \textit{for all} \quad u^+ \in E^+ \quad \textit{with} \quad \|u^+\| = \rho_0.$$

Proof. By conditions (g_0) and (h_0), the Hölder inequality and the continuous embedding $E \hookrightarrow L^{\sigma' q}(\mathbb{R}^N)$ we have

$$\int_{\mathbb{R}^N} h(x)G(u^+)dx \leq C_0 \int_{\mathbb{R}^N} h(x)|u^+|^q dx \leq C_0\|h\|_\sigma \|u^+\|^q, \quad \forall\, u^+ \in E^+.$$

Moreover, by (k_0), (f_0) and (f_2) we get

$$\int_{\mathbb{R}^N} k(x)F(u^+)dx \leq \frac{1}{4}\|u^+\|^2 + C_1\|u^+\|^p, \quad \forall\, u^+ \in E^+$$

for some constant $C_1 > 0$. Thus, for any $u^+ \in E^+$ with $\|u^+\| = \rho$, we have

$$\begin{aligned}
\Phi(u^+) &= \frac{1}{2}\|u^+\|^2 - \int_{\mathbb{R}^N} h(x)G(u^+)dx - \int_{\mathbb{R}^N} k(x)F(u^+)dx \\
&\geq \frac{1}{4}\rho^2 - C_0\|h\|_\sigma \rho^q - C_1\rho^p \\
&= \rho^2\left[\frac{1}{4} - C_1\rho^{p-2}\right] - C_0\|h\|_\sigma\rho^q.
\end{aligned}$$

Therefore we can fix $\rho = \rho_0$ so that $\beta_0 := 1/4 - C_1\rho_0^{p-2} > 0$ to conclude that

$$\Phi(u^+) \geq \eta_0 \quad \text{if} \quad \|h\|_\sigma \leq M,$$

where $M = \beta_0\rho_0^{2-q}/(2C_0)$ and $\eta_0 = \beta_0\rho_0^2/2$. □

Lemma 2.4.4. *Assume* (V_0), (h_0), (g_0), $(\widehat{k}_0)$ *and* (f_3). *Fixed* $u_0^+ \in E^+$ *with* $\|u_0^+\| = 1$, *there exists* $R > 0$ *such that*

$$\Phi(u) \leq 0, \quad \forall\, u \in \partial Q_R, \tag{2.8}$$

where

$$Q_R := \{u = u^- + su_0^+ : s \geq 0,\ u^- \in E^- \ \textit{and}\ \ \|u^-\|^2 + s^2 < R^2\}.$$

Proof. If $s = 0$ then from (h_0), (g_0), $(\widehat{k}_0)$ and (f_3) we get

$$\Phi(u) = -\frac{1}{2}\|u^-\|^2 - \int_{\mathbb{R}^N} h(x)G(u^-)dx - \int_{\mathbb{R}^N} k(x)F(u^-)dx \leq 0.$$

Thus, in what follows we assume that $s > 0$. Observe that $u = u^- + su_0^+ \in \partial Q_R$ with $s > 0$ if and only if $\|u^-\|^2 + s^2 = R^2$. Arguing by contradiction, suppose that there are sequences $R_n \to \infty$ and $u_n = u_n^- + s_n u_0^+ \in \partial Q_{R_n}$ such that $\Phi(u_n) > 0$, for all $n \in \mathbb{N}$. If $v_n := u_n/R_n = v_n^- + \tilde{s}_n u_0^+$, we have $\|v_n^-\|^2 + \tilde{s}_n^2 = 1$. Thus, there are renamed subsequences such that $\tilde{s}_n \to \tilde{s}$ in $\mathbb{R}$ and $v_n \rightharpoonup v = v^- + \tilde{s}u_0^+$ in E. Since

$$0 < \frac{1}{R_n^2}\Phi(u_n) = \frac{1}{2}[\tilde{s}_n^2 - \|v_n^-\|^2] - \int_{\mathbb{R}^N} h(x)\frac{G(u_n)}{R_n^2}dx - \int_{\mathbb{R}^N} k(x)\frac{F(u_n)}{R_n^2}dx,$$

we infer that

$$0 \leq \int_{\mathbb{R}^N} k(x)\frac{F(u_n)}{R_n^2}dx < \frac{1}{2}[\tilde{s}_n^2 - (1 - \tilde{s}_n^2)] = \tilde{s}_n^2 - \frac{1}{2} \leq C, \tag{2.9}$$

which implies that $\tilde{s}^2 \geq 1/2$ and consequently $v \not\equiv 0$. Thus, there exists $A \subset \mathbb{R}^N$ with positive measure such that $v \neq 0$ in A. Since $F(t)/t^2 \to \infty$ as $t^2 \to \infty$ and $k(x) > 0$ we have

$$\int_{\mathbb{R}^N} k(x)\frac{F(u_n)}{R_n^2}dx \geq \int_A k(x)\frac{F(u_n)}{u_n^2}v_n^2 dx \to \infty \quad \text{as} \quad n \to \infty,$$

and this contradicts (2.9). □

2.5 (Ce) Sequence

The aim of this section is to show that Cerami sequences for Φ are bounded.

Lemma 2.5.1. *Assume $1 < q < p/(p-1)$. Every $(C)_c$-sequence $(u_n) \subset E$ is bounded in E.*

Proof. Indeed, from (2.6) we get

$$\Phi(u_n) = \frac{1}{2}[\|u_n^+\|^2 - \|u_n^-\|^2] - \int_{\mathbb{R}^N} h(x)G(u_n)dx - \int_{\mathbb{R}^N} k(x)F(u_n)dx = c + o_n(1)$$

and

$$\frac{1}{2}\langle \Phi'(u_n), u_n\rangle = \frac{1}{2}[\|u_n^+\|^2 - \|u_n^-\|^2] - \frac{1}{2}\int_{\mathbb{R}^N} h(x)g(u_n)u_n dx - \frac{1}{2}\int_{\mathbb{R}^N} k(x)f(u_n)u_n dx = o_n(1).$$

This, together with the fact that $|G(u) - g(u)u/2| \leq C|u|^q$ imply that

$$\begin{aligned}\int_{\mathbb{R}^N} k(x)\left[\frac{f(u_n)u_n}{2} - F(u_n)\right]dx &= \int_{\mathbb{R}^N} h(x)\left[G(u_n) - \frac{1}{2}g(u_n)u_n\right]dx + c + o_n(1)\\ &\leq C\|h\|_\sigma\|u_n\|^q + c + o_n(1).\end{aligned} \tag{2.10}$$

By (f_3) there exists $C_1 > 0$ such that $F(t) \geq C_1|t|^\mu$ for $|t| \geq 1$. Since $\mu \geq p$, for $|t| \geq 1$ it follows that $F(t) \geq C_1|t|^p$. On the other hand, by using (f_4) there exists $C_2 > 0$ such that if $|t| \leq 1$ then $F(t) \geq C_2|t|^\theta \geq C_2|t|^p$. Therefore, $F(t) \geq C|t|^p$ for all $t \in \mathbb{R}$ where $C = \min\{C_1, C_2\}$. Thus,

$$\frac{f(t)t}{2} - F(t) \geq \left(\frac{p}{2} - 1\right) F(t) \geq \tilde{C}_1|t|^p, \quad \forall\, t \in \mathbb{R}.$$

Since $k(x) \geq 0$, the above inequality in combination with (2.10) give us

$$\int_{\mathbb{R}^N} k(x)|u_n|^p dx \leq C_1\|u_n\|^q + C_2. \tag{2.11}$$

On the other hand, since $\langle \Phi'(u_n), (u_n^+ - u_n^-)\rangle = o_n(1)$ we infer that

$$\begin{aligned}
\|u_n\|^2 &= \int_{\mathbb{R}^N} h(x)g(u_n)(u_n^+ - u_n^-)dx + \int_{\mathbb{R}^N} k(x)f(u_n)(u_n^+ - u_n^-)dx + o_n(1) \\
&= \int_{\mathbb{R}^N} h(x)g(u_n)(2u_n^+ - u_n)dx + \int_{\mathbb{R}^N} k(x)f(u_n)(2u_n^+ - u_n)dx + o_n(1).
\end{aligned}$$

Using that $f(t)t \geq 0$, we obtain

$$\|u_n\|^2 \leq \int_{\mathbb{R}^N} h(x)g(u_n)(2u_n^+ - u_n)dx + 2\int_{\mathbb{R}^N} k(x)f(u_n)u_n^+ dx + o_n(1). \tag{2.12}$$

Now, we observe that

$$\int_{\mathbb{R}^N} h(x)g(u_n)(2u_n^+ - u_n)dx \leq C\|u_n\|^q,$$

and by (f_0), (f_2), for any $\varepsilon > 0$ there exists $C > 0$ such that $|f(t)| \leq \varepsilon|t| + C|t|^{p-1}$. From this, by (2.11) and since $k \in L^\infty(\mathbb{R}^N)$, we obtain

$$\begin{aligned}
2\int_{\mathbb{R}^N} k(x)f(u_n)u_n^+ dx &\leq C\int_{\mathbb{R}^N} [k(x)]^{(p-1)/p}|u_n|^{p-1}[k(x)]^{1/p}|u_n^+|dx + \varepsilon\int_{\mathbb{R}^N} k(x)|u_n||u_n^+|dx \\
&\leq C\left(\int_{\mathbb{R}^N} k(x)|u_n|^p dx\right)^{(p-1)/p}\left(\int_{\mathbb{R}^N} k(x)|u_n^+|^p dx\right)^{1/p} + \varepsilon\|k\|_\infty\|u_n\|_2\|u_n^+\|_2 \\
&\leq C\left(C_1\|u_n\|^q + C_2\right)^{(p-1)/p}\|k\|_\infty\|u_n\| + \varepsilon\|k\|_\infty\|u_n\|^2.
\end{aligned}$$

This together with (2.12) imply that

$$\|u_n\|^2 \leq C_3(\varepsilon)\left(C_1\|u_n\|^q + C_2\right)^{(p-1)/p}\|u_n\| + C_4(\varepsilon)\|u_n\|^q,$$

for $\varepsilon > 0$ sufficiently small. Since $1 < q < p/(p-1)$ the last inequality implies that (u_n) is bounded in E and the proof is complete. □

2.6 Nontrivial Solution (Subcritical Case)

In the sequel, we establish some convergence results and we prove Theorem 2.1.2.

Lemma 2.6.1. *Assume* (h_0), (g_0), $(\widehat{k}_0)$, (f_0), (f_2) *and* $2 < p < 2^*$. *Let* $(u_n) \subset E$ *be such that* $u_n \rightharpoonup 0$ *in* E. *Then, the following limits hold:*

(i) $\displaystyle\lim_{n\to\infty} \int_{\mathbb{R}^N} h(x)g(u_n)u_n^+ dx = 0;$

(ii) $\displaystyle\lim_{n\to\infty} \int_{\mathbb{R}^N} k(x)f(u_n)u_n^+ dx = 0.$

Proof. To prove item (i), we use the compact embedding $E \hookrightarrow L^r_{loc}(\mathbb{R}^{\mathbb{N}})$, with $r \in [1, 2^*)$, to infer that for any $R > 0$ fixed

$$\begin{aligned}\int_{\mathbb{R}^N} h(x)g(u_n)u_n^+ dx &= \int_{B_R} h(x)g(u_n)u_n^+ dx + \int_{|x|\geq R} h(x)g(u_n)u_n^+ dx \\ &= o_n(1) + \int_{|x|\geq R} h(x)g(u_n)u_n^+ dx.\end{aligned}$$

Since (u_n) is bounded in E, by using the Hölder inequality we get

$$\begin{aligned}\int_{|x|\geq R} h(x)|g(u_n)u_n^+| dx &\leq \int_{|x|\geq R} h(x)|u_n|^{q-1}|u_n^+| dx \\ &= \int_{|x|\geq R} [h(x)]^{\frac{q-1}{q}} |u_n|^{q-1} [h(x)]^{\frac{1}{q}} |u_n^+| dx \\ &\leq C\left(\|h\|_{L^\sigma(|x|\geq R)}\|u_n\|^q\right)^{(q-1)/q}\left(\|h\|_{L^\sigma(|x|\geq R)}\|u_n^+\|^q\right)^{1/q} \\ &\leq C\|h\|_{L^\sigma(|x|\geq R)} \to 0 \quad \text{as} \quad R \to \infty\end{aligned}$$

and item (i) is proved. With respect item (ii), since $|f(t)| \leq \varepsilon|t| + C|t|^{p-1}$ and (u_n) is bounded in E, one has

$$\begin{aligned}\int_{\mathbb{R}^N} k(x)|f(u_n)u_n^+| dx &\leq \varepsilon \int_{\mathbb{R}^N} k(x)|u_n||u_n^+| dx + C\int_{\mathbb{R}^N} k(x)|u_n|^{p-1}|u_n^+| dx \\ &\leq \varepsilon C_1 + o_n(1) + \int_{|x|\geq R} k(x)|u_n|^{p-1}|u_n^+| dx \\ &\leq \varepsilon C_1 + o_n(1) + \left(\|k\|_{L^\kappa(|x|\geq R)}\|u_n\|^p\right)^{(p-1)/p}\left(\|k\|_{L^\kappa(|x|\geq R)}\|u_n^+\|^p\right)^{1/p} \\ &\leq \varepsilon C_1 + o_n(1) + C\|k\|_{L^\kappa(|x|\geq R)}.\end{aligned}$$

Hence, choosing $R > 0$ large enough we obtain the desired result and the proof is complete. $\square$

We also will need the following convergence results.

Lemma 2.6.2. *Assume* $(\widehat{k}_0)$, (f_0) *and* $2 < p < 2^*$. *Then, the functionals*

$$L_1(u) = \int_{\mathbb{R}^N} k(x)F(u)dx \quad \text{and} \quad L_2(u) = \int_{\mathbb{R}^N} k(x)f(u)udx$$

are weakly continuous on E.

Proof. Consider a sequence $(u_n) \subset E$ such that $u_u \rightharpoonup u$ in E. For any $R > 0$ fixed, we have

$$\begin{aligned}|L_1(u_n) - L_1(u)| &\leq \int_{B_R} |k(x)||F(u_n) - F(u)|dx + \int_{|x|\geq R} |k(x)||F(u_n) - F(u)|dx \\ &=: I_1(n) + I_2(n).\end{aligned}$$

By the compact embedding $E \hookrightarrow L^r_{loc}(\mathbb{R}^{\mathbb{N}})$, with $r \in [2, 2^*)$ and Lebesgue Dominate Convergence Theorem, we have $I_1(n) = o_n(1)$. Since $(u_n) \subset E$ is bounded in E and $k \in L^{\kappa}(\mathbb{R}^N)$, it follows by the Hölder inequality that

$$\begin{aligned}I_2(n) &\leq C_1 \left(\int_{|x|\geq R} |k(x)|^{\kappa} dx \right)^{1/\kappa} \left[\left(\int_{|x|\geq R} |u_n|^{p\kappa'} dx \right)^{1/\kappa'} + \left(\int_{|x|\geq R} |u|^{p\kappa'} dx \right)^{1/\kappa'} \right] \\ &\leq C_2 \left(\int_{|x|\geq R} |k(x)|^{\kappa} dx \right)^{1/\kappa}.\end{aligned}$$

Choosing $R > 0$ sufficiently large we get that $I_2(n) = o_n(1)$ and therefore L_1 is weakly continuous. Similar arguments prove that L_2 is also weakly continuous on E and the proof is complete. $\square$

Finalizing the proof of Theorem 2.1.2: Combining Lemmas 3.3.3, 3.3.6, 3.3.7 and Theorem 2.4.1, we obtain a sequence $(u_n) \subset E$ such that

$$\Phi(u_n) \to c_1 \quad \text{and} \quad (1 + \|u_n\|)\Phi'(u_n) \to 0, \tag{2.13}$$

where c_1 is defined in (2.7). By Lemma 2.5.1, it follows that (u_n) is bounded in E and passing to a subsequence we may assume that $u_n \rightharpoonup u_1$ weakly in E. By Lemma 2.3.2 we have that $\Phi'(u_1) = 0$. Furthermore, invoking again Lemmas 2.3.2 and 2.6.2 we conclude that

$$\begin{aligned}0 < c_1 &= \lim_{n\to\infty} \left[\Phi(u_n) - \frac{1}{2}\langle \Phi'(u_n), u_n \rangle \right] \\ &= \lim_{n\to\infty} \int_{\mathbb{R}^N} h(x) \left[\frac{g(u_n)u_n}{2} - G(u_n) \right] dx + \lim_{n\to\infty} \int_{\mathbb{R}^N} k(x) \left[\frac{f(u_n)u_n}{2} - F(u_n) \right] dx \\ &= \int_{\mathbb{R}^N} h(x) \left[\frac{g(u_1)u_1}{2} - G(u_1) \right] dx + \int_{\mathbb{R}^N} k(x) \left[\frac{f(u_1)u_1}{2} - F(u_1) \right] dx \\ &= \Phi(u_1) - \frac{1}{2}\langle \Phi'(u_1), u_1 \rangle = \Phi(u_1),\end{aligned}$$

i.e., the proof of Theorem 2.1.2 is finished.

2.7 Nontrivial Solution (Critical Case)

In this section, we present the proof of Theorem 2.1.3. We will apply a basic estimate and a convergence result. For $\varepsilon > 0$ and $x \in \mathbb{R}^N$, let us consider the modified Talenti function [67]

$$\varphi_\varepsilon(x) = \frac{C_N \psi_r(x) \varepsilon^{(N-2)/2}}{(\varepsilon^2 + |x|^2)^{(N-2)/2}},$$

where $C_N = [N(N-2)]^{(N-2)/4}$ and $\psi_r \in C_0^\infty(\mathbb{R}^N, [0,1])$ with $\psi \equiv 1$ if $|x| \leq r/2$, $\psi_r \equiv 0$ if $|x| \geq r$, for some $r > 0$. For $\varepsilon > 0$, we also consider $Z_\varepsilon := E^- \oplus \mathbb{R}\varphi_\varepsilon^+$ and

$$S = \inf_{E\setminus\{0\}} \frac{\|\nabla u\|_2^2}{\|u\|_{2^*}^2}.$$

The energy functional associated to problem $(\mathcal{C}_c)$ is given by

$$\Phi(u) = \frac{1}{2}(\|u^+\|^2 - \|u^-\|^2) - \int_{\mathbb{R}^N} h(x)G(u)dx - \frac{1}{2^*}\int_{\mathbb{R}^N} k(x)|u|^{2^*}dx.$$

Since $h(x)G(t) \geq 0$, it follows that

$$\Phi(u) \leq I_1(u) := \frac{1}{2}[\|u^+\|^2 - \|u^-\|^2] - \frac{1}{2^*}\int_{\mathbb{R}^N} k(x)|u|^{2^*}dx.$$

Arguing as in the proof of Proposition 4.2 in [19], we have the following estimate:

Lemma 2.7.1 (Minimax Estimate)**.** *If $N \geq 4$ then there exists $\varepsilon_0 > 0$ such that*

$$\sup_{u\in Z_{\varepsilon_0}} I_1(u) < c^* := \frac{S^{N/2}}{N\|k\|_\infty^{(N-2)/2}}.$$

From this estimate, we obtain the following lemma:

Lemma 2.7.2. *Let $(u_n) \subset E$ be a $(C)_c$-sequence for Φ such that $u_n \rightharpoonup u$ in E, with $0 < c < c^*$. Then,*

$$\int_{\mathbb{R}^N} k(x)|u_n|^{2^*}dx \to \int_{\mathbb{R}^N} k(x)|u|^{2^*}dx. \tag{2.14}$$

Proof. Since (u_n) is bounded in E, we can assume that

$$|\nabla u_n|^2 \to \mu \quad \text{and} \quad |u_n|^{2^*} \to \nu \;\; \text{weakly in} \;\; \mathcal{M}^+(\mathbb{R}^N),$$

where $\mathcal{M}^+(\mathbb{R}^N)$ denotes the positive Radon measures over $\mathbb{R}^N$. Invoking the Concentration-Compactness Principle due to Lions [38][Lemma I.1], we obtain a countable set J, $(x_j)_{j\in J} \subset \mathbb{R}^N$ and $(\mu_j)_{j\in J}, (\nu_j)_{j\in J} \subset [0,\infty)$ such that

$$\nu = |u|^{2^*} + \sum_{j\in J}\nu_j\delta_{x_j} \quad \text{and} \qquad \mu \geq |\nabla u|^2 + \sum_{j\in J}\mu_j\delta_{x_j} \quad \text{with} \quad \mu_j \geq S\nu_j^{\frac{2}{2^*}}, \tag{2.15}$$

where δ_{x_j} denotes the Dirac measure concentrated at x_j. We claim that $\nu_j = 0$ for all $j \in J$. In fact, combining (2.15) and Lemma 2.3.2 we get

$$\begin{aligned} c &= \lim_{n\to\infty}\left[\Phi(u_n) - \frac{1}{2}\langle \Phi'(u_n), u_n\rangle\right] \\ &\geq \frac{1}{N}\sum_{j\in J} k(x_j)\nu_j + \frac{1}{N}\int_{\mathbb{R}^N} k(x)|u|^{2^*}dx + \int_{\mathbb{R}^N} h(x)\left[\frac{g(u)u}{2} - G(u)\right]dx \\ &\geq \frac{1}{N}\sum_{j\in J} k(x_j)\nu_j + \frac{1}{N}\int_{\mathbb{R}^N} k(x)|u|^{2^*}dx - C\|h\|_\sigma \|u\|_{2^*}^q, \end{aligned} \tag{2.16}$$

where in the last inequality we have used that $|g(t)t/2 - G(t)| \leq C|t|^q$ and the Hölder inequality. On the other hand, for $\varepsilon > 0$ and $j \in J$ we set $\psi_{\varepsilon,j}(x) := \psi((x - x_j)/\varepsilon)$, $x \in \mathbb{R}^N$, where $\psi \in C_0^\infty(\mathbb{R}^N)$ is such that $\psi \equiv 1$ in $B_1(0)$, $\psi \equiv 0$ in $\mathbb{R}^N \setminus B_2(0)$ and $|\nabla\psi| \leq 2$, with $0 \leq \psi \leq 1$. We observe that $\psi_{\varepsilon,j}u_n \in E$ and $(\psi_{\varepsilon,j}u_n)$ is bounded in E. Thus, we obtain

$$\int_{\mathbb{R}^N} [\nabla(u_n)\nabla(\psi_{\varepsilon,j}u_n) + V(x)u_n^2\psi_{\varepsilon,j}]dx - \int_{\mathbb{R}^N} h(x)g(u_n)u_n\psi_{\varepsilon,j}dx - \int_{\mathbb{R}^N} k(x)|u_n|^{2^*}\psi_{\varepsilon,j}dx = o_n(1).$$

This together with the definitions of μ and ν imply that

$$\int_{\mathbb{R}^N} [\nabla u \nabla\psi_{\varepsilon,j} + V(x)u^2\psi_{\varepsilon,j}]dx - \int_{\mathbb{R}^N} h(x)G(u)\psi_{\varepsilon,j}dx - \int_{\mathbb{R}^N} k(x)\psi_{\varepsilon,j}d\nu + \int_{\mathbb{R}^N} \psi_{\varepsilon,j}d\mu = 0.$$

Now, taking the limit as $\varepsilon \to 0$ we see that $\mu(x_j) \leq k(x_j)\nu_j$. Since $\mu_j \leq \mu(x_j)$ we have

$$S\nu_j^{\frac{2}{2^*}} \leq \mu_j \leq \mu(x_j) \leq k(x_j)\nu_j, \quad \forall\, j \in J.$$

If $\nu_j \neq 0$ for some $j \in J$, the last inequality yields

$$\nu_j \geq \frac{S^{N/2}}{[k(x_j)]^{N/2}}.$$

Since the function $k(x)$ is continuous and bounded, from (2.16) we get

$$c \geq \frac{S^{N/2}}{N\|k\|_\infty^{(N-2)/2}} + \frac{1}{N}\int_{\mathbb{R}^N} k(x)|u|^{2^*}dx - C\|h\|_\sigma\|u\|_{2^*}^q$$

which is a contradiction if $\|h\|_\sigma$ is sufficiently small. Therefore, $\nu_j = 0$ for all $j \in J$ which implies that $\nu = 0$. Thus, by (2.15) we obtain $|u_n|^{2^*} \to |u|^{2^*}$ in $\mathcal{M}^+(\mathbb{R}^N)$. Consequently,

$$u_n \to u \quad \text{in} \quad L^{2^*}_{loc}(\mathbb{R}^N). \tag{2.17}$$

Now, we observe that

$$\left|\int_{\mathbb{R}^N} k(x)(|u_n|^{2^*} - |u|^{2^*})dx\right| \leq \|k\|_\infty \int_{B_R} \left||u_n|^{2^*} - |u|^{2^*}\right| dx + \frac{C}{R^\nu}\int_{|x|\geq R} (|u_n|^{2^*} + |u|^{2^*})dx.$$

Since (u_n) is bounded, choosing $R > 0$ sufficiently large and using the convergence in (2.17), we obtain (2.14) and this completes the proof. □

Proof of Theorem 2.1.3: Considering Q_R with $u_0^+ = \varphi_{\varepsilon_0}$, we see that $\varphi_{\varepsilon_0} \in Q_R \subset Z_{\varepsilon_0}$. Thus, by Lemma 2.7.1

$$c_1 \leq \Phi(\varphi_{\varepsilon_0}) \leq \sup_{u \in Z_{\varepsilon_0}} I_1(u) < c^* := \frac{S^{N/2}}{N\|k\|_\infty^{(N-2)/2}}. \tag{2.18}$$

By Theorem 2.4.1, there exists a Cerami sequence $(u_n) \subset E$ for Φ at the level $c_1 > 0$. By Lemma 2.5.1, (u_n) is bounded in E. Thus, passing to a subsequence, we can assume that $u_n \rightharpoonup u_1$ in E. Invoking Lemma 2.3.2, we see that $\Phi'(u_1) = 0$. On the other hand, combining estimate (2.18) with Lemma 2.7.2 and Lemma 2.3.2 we conclude that

$$\begin{aligned}
0 < c_1 &= \lim_{n\to\infty} \left[\Phi(u_n) - \frac{1}{2}\langle \Phi'(u_n), u_n\rangle\right] \\
&= \lim_{n\to\infty} \int_{\mathbb{R}^N} h(x) \left[\frac{g(u_n)u_n}{2} - G(u_n)\right] dx + \lim_{n\to\infty} \frac{1}{N} \int_{\mathbb{R}^N} k(x)|u_n|^{2^*} dx \\
&= \int_{\mathbb{R}^N} h(x) \left[\frac{g(u_1)u_1}{2} - G(u_1)\right] dx + \frac{1}{N} \int_{\mathbb{R}^N} k(x)|u_1|^{2^*} dx \\
&= \Phi(u_1) - \frac{1}{2}\langle \Phi'(u_1), u_1\rangle = \Phi(u_1),
\end{aligned}$$

and the proof of Theorem 2.1.3 is complete.

Chapter 3

On a Periodic Schrödinger Equation Involving Periodic and Nonperiodic Nonlinearities in $\mathbb{R}^2$

3.1 Introduction and Main Results

In this Chapter we study the equation

$$-\Delta u + V(x)u = f(x,u), \quad x \in \mathbb{R}^2, \tag{$\mathcal{P}_f$}$$

where $f(x,t)$ has exponential growth in the sense of Trudinger-Moser inequality and zero lies in a spectral gap of the Schrödinger operator $\mathcal{S}$. This is the content of the paper [43].

In order to introduce our hypotheses, let us denote by $\sigma(\mathcal{S})$ be the spectrum of the operator $\mathcal{S} = -\Delta + V$ defined in $L^2(\mathbb{R}^2)$. Precisely, we assume that

(V_0) $V \in C(\mathbb{R}^2, \mathbb{R})$ is $1-$periodic in x_j, $j = 1, 2$, and

$$\lambda := \sup[\sigma(\mathcal{S}) \cap (-\infty, 0)] < 0 < \Lambda := \inf[\sigma(\mathcal{S}) \cap (0, \infty)].$$

As we will see in the next section, under the assumption (V_0) the quadratic form $B(u,v) = \int_{\mathbb{R}^2}(\nabla u \nabla v + V(x)uv)dx$ is strongly indefinite, i.e., $H^1(\mathbb{R}^2)$ can be split as a direct sum into two infinite dimensional subspaces $H^1(\mathbb{R}^2) = E^- \oplus E^+$ corresponding to the decomposition of $\sigma(\mathcal{S}) \cap (-\infty, \lambda)$ and $\sigma(\mathcal{S}) \cap (\Lambda, \infty)$. After the Linking Theorem proved by Kryzewski-Szulking in [36], many authors have improved and used this result to obtain critical points of strongly indefinite functionals, see for instance [23, 60, 65, 78] and references therein.

The main purpose in this work is to prove the existence of nontrivial weak solutions to $(\mathcal{P}_f)$ considering zero in a spectral gap of the spectrum of $\mathcal{S} = -\Delta + V$ and $f(x,t)$ with subcritical exponential growth in $\mathbb{R}^2$. Precisely, for any $\beta > 0$

$$\lim_{|t| \to \infty} \frac{|f(x,t)|}{e^{\beta t^2}} = 0, \quad \text{uniformly in} \quad x \in \mathbb{R}^2. \tag{3.1}$$

Setting $F(x,t) = \int_0^t f(x,s)ds$, we suppose that f is continuous and satisfies the following assumptions:

(f_0) there are $\delta > 0$ and $0 < \gamma < \Lambda$ such that $|f(x,t)| \leq \gamma|t|$ for any $|t| < \delta$ and $x \in \mathbb{R}^2$;

(f_1) $2F(x,t) \geq \lambda t^2$ for any $x \in \mathbb{R}^2$, $t \in \mathbb{R}$ and

$$\frac{F(x,t)}{t^2} \to +\infty \quad \text{as} \quad t^2 \to \infty, \quad \text{uniformly in} \quad x \in \mathbb{R}^2.$$

(f_2) $f(x,t)$ is locally bounded in the variable t, that is, for any bounded interval $J \subset \mathbb{R}$, there exists $C > 0$ such that $|f(x,t)| \leq C$ for every $(x,t) \in \mathbb{R}^2 \times J$;

(f_3) there exists $W \in L^1(\mathbb{R}^2)$ such that for all $r \in [0,1]$ it holds

$$2(F(x,t+s) - F(x,t)) \geq (2rs - (r-1)^2 t)f(x,t) - W(x), \quad \forall\, x \in \mathbb{R}^2,\ s,t \in \mathbb{R}.$$

Before stating the main results, we make some remarks on hypothesis (f_3).

Remark 3.1.1. *Hypothesis* (f_3) *appears in a series of paper (see* [59,65] *and references therein). Taking* $r = 0$ *and* $s = -t$ *(respectively,* $s = rw - t$ *) in* (f_3) *we obtain* $H(x,t) := tf(x,t) - 2F(x,t) \geq -W(x)$ *for all* $x \in \mathbb{R}^2$, $t \in \mathbb{R}$*; and for any* $r \in [0,1]$

$$2[F(x,t) - F(x,rw)] - ((r^2+1)t - 2r^2 w)f(x,t) \leq W(x), \quad \forall\, x \in \mathbb{R}^2,\ t,w \in \mathbb{R}. \tag{3.2}$$

Furthermore, choosing $t = r = 0$ *in* (f_3) *we get* $2F(x,s) \geq -W(x)$ *for any* $x \in \mathbb{R}^2, s \in \mathbb{R}$.

Now, our first existence result for equation $(\mathcal{P}_f)$ can be summarized as follows.

Theorem 3.1.2. *Assume* (V_0) *and* $(f_0) - (f_3)$*. If* $f(x,t)$ *is* $1-$*periodic and satisfies* (3.1) *then equation* $(\mathcal{P}_f)$ *admits a nontrivial weak solution.*

A typical example of a nonlinearity $f(x,t)$ satisfying the hypotheses of Theorem 3.1.2 is

$$f(x,t) := a(x)t + b(x)|t|^{p-2}t(e^t - 1), \quad x \in \mathbb{R}^2,\ t \in \mathbb{R},$$

where a, b are continuous and periodic functions satisfying $\lambda \leq a \leq |a| \leq \gamma$, $b \geq 0$ in $\mathbb{R}^2$.

We quote that there are few existence results for the Schrödinger equation $(\mathcal{P}_f)$ in the two dimensional case when the potential V is periodic. In [24] do Ó-Ruf have studied equation $(\mathcal{P}_f)$ when V is periodic and $f(x,t)$ satisfies the Ambrosetti-Rabinowitz condition by using an approach developed by Pankov-Pflüger [51] and Pankov [49] based on an approximation technique of periodic functions and applying the generalized linking theorem due to P. Rabinowitz [53]. As we will see, under the above hypotheses, every Palais-Smale sequence associated with the energy functional is bounded. Furthermore, by the periodicity of V and $f(x,t)$ the energy functional is invariant with respect to the $\mathbb{Z}^2$-action on $H^1(\mathbb{R}^2)$ given by $(T * u)(x) = u(T + x)$ with $T \in \mathbb{Z}^2$

and $u \in H^1(\mathbb{R}^2)$. Consequently, we conclude that the weak limit of this sequence is a nontrivial weak solution of $(\mathcal{P}_f)$ up to translations.

In the case where the nonlinearity $f(x,t)$ is nonperiodic, in addition we will assume that:

(f_4) there exist $\alpha_0 > 0$, $R_0 > 0$ and $h \in L^1(B_{R_0}^c)$ such that

$$|F(x,t)| \leq h(x)e^{\alpha_0 t^2}, \quad \forall\, x \in B_{R_0}^c, t \in \mathbb{R}.$$

In this case, our second existence result is the following:

Theorem 3.1.3. *Assume* (V_0) *and* $(f_0)-(f_4)$*. If* $f(x,t)$ *satisfies* (3.1) *then equation* $(\mathcal{P}_f)$ *admits a nontrivial weak solution.*

Remark 3.1.4. *A typical example of a nonlinearity satisfying the hypotheses of Theorem 3.1.3 is*

$$f(x,t) := a(x)t + b(x)|t|^{p-2}t(e^t - 1), \quad x \in \mathbb{R}^2,\, t \in \mathbb{R},$$

where $\lambda \leq a \leq |a| \leq \gamma$, $0 \leq b \in L^\infty(\mathbb{R}^2)$ *with* $a, b \in L^1(B_1^c)$.

We mention that Theorem 3.1.3 extends some recent results obtained by M. Schechter [58] where the author studied equation $(\mathcal{P}_f)$ with the nonlinearity $f(x,t)$ having subcritical polynomial growth.

This chapter is organized as follows. In Section 2, we use spectral theory to obtain a suitable domain for the energy functional in order to use the variational framework. In Section 3, we establish the geometry for the energy functional required in the linking-theorem to obtain a (PS) sequence. In Section 4 we demonstrate the Theorem 3.1.2. We conclude the chapter in section 5, where we present the proof of Theorem 3.1.3.

3.2 Variational Setting

In this section, in order to develop a variational approach to study the existence of solutions for equation $(\mathcal{P}_f)$, a key step is to identify a suitable function space setting. First we observe that from (3.1), (f_0) and (f_2), for any $\beta > 0$ and $q > 2$ there exists $C_1 > 0$ such that

$$2|F(x,t)| \leq \gamma|t|^2 + C_1|t|^q(e^{\beta t^2} - 1), \quad \forall\, x \in \mathbb{R}^2, t \in \mathbb{R}. \tag{3.3}$$

Under the hypothesis (V_0) we will find a function space E on which the energy functional associated to $(\mathcal{P}_f)$, $\Phi : E \to \mathbb{R}$ given by

$$\Phi(u) := \int_{\mathbb{R}^2} (|\nabla u|^2 + V(x)u^2)dx - 2\int_{\mathbb{R}^2} F(x,u)dx,$$

is well defined. In order to define the space E we consider the self-adjoint operator $\mathcal{S} = -\Delta + V$ defined in $L^2(\mathbb{R}^2)$ acting in $D(\mathcal{S}) = H^2(\mathbb{R}^2)$. Let $\{\mathcal{E}(\lambda) : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)\}_{\lambda \in \mathbb{R}}$ be the spectral family of $\mathcal{S}$, and $|\mathcal{S}|^{1/2}$ be the square root of $|\mathcal{S}|$. Setting $U = Id - \mathcal{E}(0) - \mathcal{E}(-0)$ we can see that U

is unitary and commutes with $\mathcal{S}$, $|\mathcal{S}|$ and $|\mathcal{S}|^{1/2}$. Moreover $\mathcal{S} = U|\mathcal{S}|$ is the polar decomposition of the operator $\mathcal{S}$ (see [35], p. 358).

Let us denote by $E := D(|\mathcal{S}|^{1/2})$ the domain of $|\mathcal{S}|^{1/2}$. It is well known that $\mathcal{E}(\lambda)E \subset E$ for all $\lambda \in \mathbb{R}$. Furthermore, defining

$$E^- := \mathcal{E}(0)E, \quad E^+ := (Id - \mathcal{E}(0))E,$$

$$(u, v) := (|\mathcal{S}|^{1/2}u, |\mathcal{S}|^{1/2}v)_2, \quad \forall\, u, v \in E, \text{ and } \quad \|u\| := \sqrt{(u, u)},$$

where $(\cdot, \cdot)_2$ is the usual inner product in $L^2(\mathbb{R}^2)$, we have the following result (see for instance [9, 68]).

Lemma 3.2.1. *Assume (V_0). Then $E = E^- \oplus E^+$, $\|\cdot\|$ is equivalent to $\|\cdot\|_{H^1}$ on E and for any $u = u^- + u^+ \in E$, it holds $(u^-, u^+) = (u^-, u^+)_2 = 0$. Moreover,*

$$(\mathcal{S}u, u)_2 = -\|u\|^2 \leq \lambda \|u\|_2^2, \quad \forall\, u \in E^- \tag{3.4}$$

and

$$(\mathcal{S}u, u)_2 = \|u\|^2 \geq \Lambda \|u\|_2^2, \quad \forall\, u \in E^+. \tag{3.5}$$

Remark 3.2.2. *It follows from Lemma 3.2.1 that $\|u\|^2 = \|u^-\|^2 + \|u^+\|^2$ and for any $p \in [2, \infty)$ the embedding $E \hookrightarrow L^p(\mathbb{R}^2)$ is continuous.*

The classical Trudinger-Moser inequality asserts that for any $u \in H^1(\mathbb{R}^2)$ and $\beta > 0$ it holds $(e^{\beta u^2} - 1) \in L^1(\mathbb{R}^2)$. Afterward, a uniform inequality has been established by Cao [17] (see also [57]). Namely, if $u \in H^1(\mathbb{R}^2)$ with $\|u\|_{H^1(\mathbb{R}^2)} \leq M$ and $\beta M^2 < 4\pi$, then there exists a constant $C = C(\beta, M) > 0$ such that

$$\int_{\mathbb{R}^2} (e^{\beta u^2} - 1)dx \leq C(\beta, M). \tag{3.6}$$

Since the norms $\|\cdot\|_{H^1}$ and $\|\cdot\|$ are equivalent on E, as a byproduct of (3.6) and the elementary inequality

$$(e^{\beta t^2} - 1)^r \leq (e^{\beta r t^2} - 1), \quad \forall\, t \in \mathbb{R},\, \beta > 0,\, r \geq 1, \tag{3.7}$$

we have the following result.

Lemma 3.2.3. *If $u \in E$ with $\|u\| \leq M$ and $\theta > 0$. Then there exists $C = C(\beta, \theta, M) > 0$ such that*

$$\int_{\mathbb{R}^2} |u|^\theta (e^{\beta u^2} - 1)dx \leq C\|u\|^\theta,$$

for any $\beta(\nu M)^2 < 4\pi$, where $\nu := \sup\limits_{u \in E} \dfrac{\|u\|_{H^1}}{\|u\|}$.

Combining Lemmas 3.2.1 and 3.2.3 we see that the functional Φ is well defined and can be written as

$$\Phi(u) = \|u^+\|^2 - \|u^-\|^2 - 2\int_{\mathbb{R}^2} F(x, u)dx, \quad \forall\, u \in E.$$

Furthermore, using Remark 3.2.2 together with Lemma 3.2.3 and standard arguments we have that $\Phi \in C^1(E, \mathbb{R})$ and

$$\frac{1}{2}\langle \Phi'(u), v\rangle = (u^+, v) - (u^-, v) - \int_{\mathbb{R}^2} f(x,u)v\,dx, \quad \forall\, u, v \in E.$$

Thus, critical points of Φ correspond to weak solutions of $(\mathcal{P}_f)$.

3.3 Linking Geometry

In this section, in order to find critical points of the functional Φ we use a variant weak linking theorem due to Schechter-Zou [60]. Since $E = E^- \oplus E^+$ and E^- is separable, for each $u = u^- + u^+$ we have

$$u^- = \sum_{k=1}^{\infty} c_k(u^-)e_k$$

where $\mathcal{B} = \{e_1, e_2, \ldots\}$ is a complete orthonormal system in E^-. Thus, we can define a new norm in E by setting

$$\|u\|_\tau = \max\left\{\|u^+\|, \sum_{k=1}^{\infty}\frac{|c_k(u^-)|}{2^k}\right\}, \quad \forall\, u \in E.$$

We can see that $\|u\|_\tau \leq \|u\|$ for any $u \in E$ (see [36]). For $R > \rho > 0$ and $u_0^+ \in E^+ \setminus \{0\}$ we define

$$Q_R := \{u = u^- + su_0^+ : s \geq 0, u^- \in E^-, \|u\| < R\}, \quad S_\rho := \{u^+ \in E^+ : \|u^+\| = \rho\}.$$

For a functional $\Phi \in C^1(E, \mathbb{R})$ defined in a Banach space $(E, \|\cdot\|)$ we consider

$$\Gamma := \left\{\begin{array}{l} h : [0,1] \times \bar{Q}_R \to E, h \text{ is } \tau\text{-continuous. For any } (s_0, u_0) \in [0,1] \times \bar{Q}_R, \\ \text{there is a } \tau\text{-neighborhood } U_{(s_0,u_0)} \text{ such that} \\ \{u - h(s,u) : (s,u) \in U_{(s_0,u_0)} \cap ([0,1] \times \bar{Q}_R)\} \subset E_{fin}, \\ h(0,u) = u, \Phi(h(s,u)) \leq \Phi(u), \forall\, u \in \bar{Q}_R \end{array}\right\},$$

where we use E_{fin} to denote various finite-dimensional subspace of E whose exact dimension are irrelevant and depend on (s_0, u_0). We observe that $\Gamma \neq \emptyset$ since $Id \in \Gamma$.

Theorem 3.3.1. *(See [60]) Let E be a Hilbert space with norm $\|\cdot\|$ and $\Phi_\mu : E \to \mathbb{R}$ a family of C^1-functionals of the form:*

$$\Phi_\mu(u) := \mu I(u) - J(u), \quad \mu \in [1,2].$$

Assume that

(a) $I(u) \geq 0$, $\forall\; u \in E$ and $\Phi_1 := \Phi$;

(*b*) $I(u) + |J(u)| \to +\infty$ *as* $\|u\| \to \infty$*;*

(*c*) Φ_μ *is* τ*-upper semicontinuous, maps bounded sets into bounded sets and* Φ'_μ *is weakly sequentially continuous on* E*;*

(*d*) $\sup_{\partial Q_R} \Phi_\mu \leq 0 < \inf_{S_\rho} \Phi_\mu$, $\forall\, \mu \in [1,2]$.

Then for almost all $\mu \in [1,2]$*, there exists a sequence* $(u_n) \subset E$ *such that*

$$\sup_n \|u_n\| < \infty, \quad \Phi'_\mu(u_n) \to 0 \quad \textit{and} \quad \Phi_\mu(u_n) \to c_\mu,$$

where

$$c_\mu := \inf_{h\in\Gamma} \sup_{u\in\bar{Q}_R} \Phi_\mu(h(1,u)).$$

Furthermore, $c_\mu \in [\inf_{S_\rho} \Phi_\mu, \sup_{\bar{Q}_R} \Phi_\mu]$ *and is nondecreasing in* μ*.*

In what follows, we derive in some lemmas the linking structure of Φ_μ required in Theorem 3.3.1. Precisely, we apply Theorem 3.3.1 with

$$I(u) = \|u^+\|^2 \quad \text{and} \quad J(u) = \|u^-\|^2 + 2\int_{\mathbb{R}^2} F(x,u)dx,$$

which clearly satisfies (*a*) in Theorem 3.3.1.

Lemma 3.3.2. *Assume* (f_1) *and* (f_3)*. Let* $(u_n) \subset E$ *be such that* $\|u_n\| \to \infty$ *and* $v_n(x) := u_n(x)/\|u_n\| \to v(x)$ *almost everywhere in* $\mathbb{R}^2$*. The following hold:*

(*i*) *If* $v \not\equiv 0$ *then* $\int_{\mathbb{R}^2} \frac{F(x,u_n)}{\|u_n\|^2}dx \to \infty$ *as* $n \to \infty$*.*

(*ii*) *If* $v \equiv 0$ *then* $\liminf_n \int_{\mathbb{R}^2} \frac{F(x,u_n)}{\|u_n\|^2}dx \geq 0$*.*

Proof. If $v \not\equiv 0$ then there exists $A \subset \mathbb{R}^2$ with positive measure such that $v \neq 0$ in A. Since $2F(x,t) \geq -W(x)$, $F(x,t)/t^2 \to \infty$ as $t^2 \to \infty$ and $W \in L^1(\mathbb{R}^2)$ we have

$$\int_{\mathbb{R}^2} \frac{F(x,u_n)}{\|u_n\|^2}dx \geq \int_A \frac{F(x,u_n)}{u_n^2}v_n^2dx - \frac{1}{2}\int_{\mathbb{R}^2\setminus A} \frac{W(x)}{\|u_n\|^2}dx \to \infty \quad \text{as} \quad n \to \infty.$$

In case that $v \equiv 0$, using that $2F(x,t) \geq -W(x)$ we infer that

$$\int_{\mathbb{R}^2} \frac{F(x,u_n)}{\|u_n\|^2}dx \geq -\frac{1}{2}\int_{\mathbb{R}^2} \frac{W(x)}{\|u_n\|^2}dx \to 0 \quad \text{as} \quad n \to \infty,$$

completing the proof. □

Lemma 3.3.3. *Assume* (f_1) *and* (f_3)*. Then the functional* Φ_μ *satisfies the hypothesis* (*b*) *in Theorem 3.3.1.*

Proof. Since $2F(x,t) \geq -W(x)$ for all $x \in \mathbb{R}^2$ and $t \in \mathbb{R}$, for any $u \in E$ we have

$$I(u) + J(u) \geq \|u^+\|^2 + \|u^-\|^2 - \int_{\mathbb{R}^2} W(x)dx = \|u\|^2 - C \to +\infty \quad \text{as} \quad \|u\| \to +\infty.$$

□

To carry forward, we establish an auxiliary convergence result.

Lemma 3.3.4. *Assume* (f_0), (f_2) *and* (3.1)*. Then for any sequence* $(u_n) \subset E$ *such that* $u_n \rightharpoonup u$ *in* E *we have*

$$\int_{\mathbb{R}^2} f(x,u_n)\varphi dx \to \int_{\mathbb{R}^2} f(x,u)\varphi dx \quad \textit{for any} \quad \varphi \in E.$$

Proof. Initially we consider $\varphi \in C_0^\infty(\mathbb{R}^2)$ and let Ω be the support of φ. Since the embedding $E \hookrightarrow L^r(\Omega)$ is compact for any $r \geq 1$ it follows that $u_n \to u$ strongly in $L^r(\Omega)$ and $u_n(x) \to u(x)$ a.e. in Ω. In particular, $f(x,u_n)\varphi \to f(x,u)\varphi$ a.e. in Ω. From (f_0), (f_2) and (3.1) for any $\beta > 0$ and $\theta \geq 2$ we have

$$|f(x,t)| \leq \gamma|t| + C_1|t|^\theta(e^{\beta t^2} - 1), \quad \forall\, x \in \mathbb{R}^2,\, t \in \mathbb{R}. \tag{3.8}$$

Thus, using the Hölder inequality together with inequality (3.7) we get

$$\int_\Omega |f(x,u_n)\varphi| dx \leq \gamma \|u_n\|_{L^\theta(\Omega)} \|\varphi\|_{L^{\theta'}(\Omega)} + C_1 \|u_n\|_{L^\theta(\Omega)} \left(\int_{\mathbb{R}^2} |u_n|^\theta (e^{\beta \frac{\theta}{\theta-1} u_n^2} - 1) dx \right)^{(\theta-1)/\theta}.$$

Since (u_n) is bounded in E, we can choose $\beta > 0$ sufficiently small and apply Lemma 3.2.3 to obtain

$$\int_\Omega |f(x,u_n)\varphi| dx \leq C \left(\int_\Omega |u_n|^\theta dx \right)^{1/\theta}.$$

On the other hand, there exists $\psi \in L^1(\Omega)$ such that $|u_n| \leq |\psi|$ in Ω. Thus, for each $\varepsilon > 0$, we find a mensurable set $A \subset \Omega$ with $|A| > 0$ sufficiently small such that

$$\int_A |f(x,u_n)\varphi| dx \leq C \left(\int_A |\psi|^\theta dx \right)^{1/\theta} < \varepsilon.$$

Therefore, $(f(x,u_n)\varphi)_n$ is uniformly integrable and by applying the Vitali Theorem we have

$$\int_{\mathbb{R}^2} f(x,u_n)\varphi dx \to \int_{\mathbb{R}^2} f(x,u)\varphi dx \quad \text{for any} \quad \varphi \in C_0^\infty(\mathbb{R}^2). \tag{3.9}$$

On the other hand, using that $C_0^\infty(\mathbb{R}^2)$ is dense in E, for any $\varepsilon > 0$ and $v \in E$ there exists $\varphi \in C_0^\infty(\mathbb{R}^2)$ such that $\|v - \varphi\| \leq \varepsilon$. Using (3.8), (3.9) together with Lemma 3.2.3 and Remark 3.2.2

we obtain

$$\begin{aligned}\int_{\mathbb{R}^2}|(f(x,u_n)-f(x,u))v|dx &\le \int_{\mathbb{R}^2}|f(x,u_n)(v-\varphi)|dx+\int_{\mathbb{R}^2}|(f(x,u_n)-f(x,u))\varphi|dx\\ &\quad+\int_{\mathbb{R}^2}|f(x,u)(\varphi-v)|dx\\ &\le I_1(n)+o_n(1)+C\|\varphi-v\|.\end{aligned}$$

To estimate $I_1(n)$, we use (3.8) with $\theta=2$ to obtain

$$I_1(n)\le\gamma\|u_n\|_2\|v-\varphi\|_2+C_1\|v-\varphi\|_2\left(\int_{\mathbb{R}^2}|u_n|^4(e^{2\beta u_n^2}-1)dx\right)^{1/2}.$$

Since (u_n) is bounded in E, in view of (3.1), we can choose $\beta>0$ sufficiently small and apply Lemma 3.2.3 together with the Sobolev embedding $E\hookrightarrow L^2(\mathbb{R}^2)$ to obtain $I_1(n)\le C_2\varepsilon$, from where we obtain the desired result. $\square$

Lemma 3.3.5. *Assume* (f_0), (f_2) *and* (3.1). *Then for any* $\mu\in[1,2]$ *the functional* Φ_μ *is* τ*-upper semicontinuous and maps bounded sets into bounded sets. Furthermore,* Φ'_μ *is weakly sequentially continuous.*

Proof. Let $(u_n)\subset E$ be such that $\|u_n-u\|_\tau\to0$. Since $\|u_n^--u^-\|_\tau\le\|u_n-u\|_\tau$ and $\|u_n^+-u^+\|\le\|u_n-u\|_\tau$ we have that $\|u_n^--u^-\|_\tau\to0$ and $\|u_n^+-u^+\|\to0$. In particular, (u_n^-) is τ-bounded. Hence up to a subsequence, $u_n^-\rightharpoonup u^-$ in E^- (see Proposition 1.3.1). Thus, $\|u^-\|\le\liminf_n\|u_n^-\|$ and $\|u^+\|=\lim_n\|u_n^+\|$. Since $2F(x,t)\ge-W(x)$ by the Fatou's Lemma

$$\int_{\mathbb{R}^2}F(x,u)dx\le\liminf_n\int_{\mathbb{R}^2}F(x,u_n)dx$$

and consequently

$$\begin{aligned}\Phi_\mu(u)&=\mu\|u^+\|^2-\|u^-\|^2-2\int_{\mathbb{R}^2}F(x,u)dx\\ &\ge\limsup_n\left(\mu\|u_n^+\|^2-\|u_n^-\|^2-2\int_{\mathbb{R}^2}F(x,u_n)dx\right)\\ &=\limsup_n\Phi_\mu(u_n),\end{aligned}$$

proving that Φ_μ is τ-upper semicontinuous. Now consider a bounded sequence $(u_n)\subset E$. Invoking (3.3) together with the embedding $E\hookrightarrow L^2(\mathbb{R}^2)$ and Lemma 3.2.3 we obtain

$$2\int_{\mathbb{R}^2}|F(x,u_n)|dx\le\gamma\int_{\mathbb{R}^2}|u_n|^2dx+C_1\int_{\mathbb{R}^2}|u_n|^\theta(e^{\beta u_n^2}-1)dx\le C.$$

Therefore, $|\Phi_\mu(u_n)|\le3\|u_n\|^2+2\int_{\mathbb{R}^2}|F(x,u_n)|dx\le C$. Finally, suppose that $u_n\rightharpoonup u$ in E. Then for any $\varphi\in E$, $(u_n^+,\varphi)\to(u^+,\varphi)$ and $(u_n^-,\varphi)\to(u^-,\varphi)$. Now, by Lemma 3.3.4 we obtain $\langle\Phi'_\mu(u_n),\varphi\rangle\to\langle\Phi'_\mu(u),\varphi\rangle$ for any $\varphi\in E$ and this completes the proof. $\square$

Lemma 3.3.6. *Assume* (V_0), (f_0) *and* (f_2). *There are positive constants* η *and* ρ *such that, for any* $\mu \in [1,2]$,

$$\Phi_\mu(u^+) \geq \eta \quad \textit{for all} \quad u^+ \in E^+ \quad \textit{with} \quad \|u^+\| = \rho.$$

Proof. Let $\rho > 0$ and $\beta > 0$ such that $\beta(\nu\rho^2) < 4\pi$. If $q > 2$ and $\|u^+\| = \rho$, by Lemma 3.2.3 we have

$$\int_{\mathbb{R}^2} |u^+|^q (e^{\beta(u^+)^2} - 1)dx \leq C\rho^q.$$

This together with inequalities (3.3) and (3.5) imply that

$$2\int_{\mathbb{R}^2} F(x,u^+)dx \leq \gamma\|u^+\|_2^2 + C_1 \int_{\mathbb{R}^2} |u^+|^q(e^{\beta(u^+)^2} - 1)dx \leq \frac{\gamma}{\Lambda}\rho^2 + C_2\rho^q.$$

Thus, for any $\mu \geq 1$ we conclude that

$$\Phi_\mu(u^+) = \mu\|u^+\|^2 - 2\int_{\mathbb{R}^2} F(x,u^+)dx \geq \left(1 - \frac{\gamma}{\Lambda}\right)\rho^2 - C_2\rho^q.$$

Since $q > 2$ and $0 < \gamma < \Lambda$, choosing $\|u^+\| = \rho$ sufficiently small we obtain the desired result. $\square$

The following result is necessary to conclude the Linking geometry.

Lemma 3.3.7. *Assume* (V_0), (f_1) *and* (f_3). *Fixed* $u_0^+ \in E^+$ *with* $\|u_0^+\| = 1$, *there exists* $R > \rho > 0$ *such that for all* $\mu \in [1,2]$,

$$\Phi_\mu(u) \leq 0, \quad \forall\, u \in \partial Q_R,$$

where

$$Q_R := \left\{u = u^- + su_0^+ : \|u^-\|^2 + s^2 \leq R^2, u^- \in E^-, s \geq 0\right\}.$$

Proof. We first observe that if $s = 0$ then from (f_1) and (3.4) we get

$$\Phi_\mu(u) = -\|u^-\|^2 - 2\int_{\mathbb{R}^2} F(x,u^-)dx \leq -\|u^-\|^2 - \lambda\|u^-\|_2^2 \leq 0.$$

Thus, in what follows we assume that $s > 0$. Observe that $u = u^- + su_0^+ \in \partial Q_R$ with $s > 0$ if and only if $\|u^-\|^2 + s^2 = R^2$. Arguing by contradiction, suppose that there are sequences $R_n \to \infty$, $\mu_n \in [1,2]$, $u_n = u_n^- + s_n u_0^+ \in \partial Q_{R_n}$ such that $\Phi_{\mu_n}(u_n) > 0$, $\forall\, n \in \mathbb{N}$. If $v_n := u_n/R_n = v_n^- + \tilde{s}_n u_0^+$, we have $\|v_n^-\|^2 + \tilde{s}_n^2 = 1$. Thus, there are renamed subsequences such that $\mu_n \to \mu$, $\tilde{s}_n \to \tilde{s}$ and $v_n \rightharpoonup v = v^- + \tilde{s}u_0^+$ in E. Since

$$0 < \frac{1}{R_n^2}\Phi_{\mu_n}(u_n) = \mu_n\tilde{s}_n^2 - \|v_n^-\|^2 - 2\int_{\mathbb{R}^2} \frac{F(x,u_n)}{R_n^2}dx,$$

it follows that

$$2\int_{\mathbb{R}^2} \frac{F(x,u_n)}{R_n^2}dx < \mu_n\tilde{s}_n^2 - (1 - \tilde{s}_n^2) \leq C.$$

Then by, (i) of Lemma 3.3.2 $v \equiv 0$. Using again Lemma 3.3.2 we obtain that $0 \leq \mu\tilde{s}^2 - (1 - \tilde{s}^2)$, which implies that $\tilde{s} > 0$ and consequently $v \not\equiv 0$ and this is a contradiction. $\square$

3.4 Nontrivial Solution (Periodic Case)

Before proceeding with the proof of Theorem 3.1.2, we establish some preliminary results. Since the norms $\|\cdot\|_{H^1}$ and $\|\cdot\|$ are equivalent on E, as a direct consequence of the concentration compactness principle of Lions [38] (see also [74, 75]) we have the following result.

Lemma 3.4.1. *Let $r > 0$ and $(u_n) \subset E$ a bounded sequence such that*

$$\sup_{y\in\mathbb{R}^2} \int_{B(y,r)} |u_n|^2 dx \to 0 \quad as \quad n \to \infty,$$

where $B(y, r) \subset \mathbb{R}^2$ denotes the open ball with center y and radius $r > 0$. Then $u_n \to 0$ in $L^p(\mathbb{R}^2)$ for any $p > 2$.

The lemma below will be used to prove the boundedness of a special sequence that will be crucial in the proof of Theorem 3.1.2, see Lemma 3.4.4 below.

Lemma 3.4.2. *Assume (f_3). If $u = u^- + u^+ \in E$ and $r \in [0, 1]$ then*

$$\int_{\mathbb{R}^2} \left[2F(x, u) - 2F(x, ru^+) - \left((r^2 + 1)u - 2r^2u^+\right) f(x, u)\right] dx \le C_1,$$

where the constant C_1 independent of u and r.

Proof. Taking $w = u^+$ and $t = u$ in (3.2) we get

$$2F(x, u) - 2F(x, ru^+) - \left((r^2 + 1)u - 2r^2u^+\right) f(x, u) \le W(x).$$

Now, the result follows by integrating the last inequality and using that $W \in L^1(\mathbb{R}^2)$. $\square$

Lemma 3.4.3. *Assume (f_3). Let $(\mu_n) \subset [1, 2]$ and $u_n = u_n^- + u_n^+ \in E$ such that*

$$\Phi'_{\mu_n}(u_n) \to 0 \quad and \quad \langle\Phi'_{\mu_n}(u_n), u_n\rangle = o_n(1).$$

Then for all $r \in [0, 1]$, there is a constant C independent of n, μ_n and r such that

$$\Phi_{\mu_n}(ru_n^+) + r^2\|u_n^-\|^2 - \Phi_{\mu_n}(u_n) \le C + o_n(1)r^2\|u_n^+\|.$$

Proof. Note that

$$\begin{aligned}\Phi_{\mu_n}(ru_n^+) + r^2\|u_n^-\|^2 - \Phi_{\mu_n}(u_n) =& \mu_n(r^2 - 1)\|u_n^+\|^2 + (r^2 + 1)\|u_n^-\|^2 \\ &+ 2\int_{\mathbb{R}^2} \left[F(x, u_n) - F(x, ru_n^+)\right] dx.\end{aligned} \tag{3.10}$$

Taking $\varphi = (r^2 + 1)u_n^- - (r^2 - 1)u_n^+ = (r^2 + 1)u_n - 2r^2u_n^+$ as a test function we obtain

$$\begin{aligned}\mu_n(r^2 - 1)\|u_n^+\|^2 + (r^2 + 1)\|u_n^-\|^2 =& -\int_{\mathbb{R}^2} \left((r^2 + 1)u_n - 2r^2u_n^+\right) f(x, u_n)dx \\ &- \langle\Phi'_{\mu_n}(u_n), (r^2 + 1)u_n - 2r^2u_n^+\rangle.\end{aligned} \tag{3.11}$$

Combining (3.10), (3.11) and using the previous lemma we get

$$\begin{aligned}\Phi_{\mu_n}(ru^+) + r^2\|u_n^-\|^2 - \Phi_{\mu_n}(u_n) &= \int_{\mathbb{R}^2} \left[2F(x,u_n) - 2F(x,ru_n^+) - \left((r^2+1)u_n - 2r^2u_n^+\right) f(x,u_n)\right] dx \\ &\quad - \langle \Phi'_{\mu_n}(u_n), (r^2+1)u_n - 2r^2u_n^+\rangle \\ &\leq C + o_n(1)r^2\|u_n^+\|,\end{aligned}$$

which completes the proof. □

Lemma 3.4.4. *Suppose that $f(\cdot,t)$ is 1-periodic, (V_0), (f_0)–(f_3) and (3.1) hold. Let $(\mu_n) \subset [1,2]$ and $(u_n) \subset E$ such that*

$$|\Phi_{\mu_n}(u_n)| \leq C, \quad \Phi'_{\mu_n}(u_n) \to 0 \quad \text{and} \quad \langle \Phi'_{\mu_n}(u_n), u_n\rangle = o_n(1).$$

Then, (u_n) has a bounded subsequence in E.

Proof. Suppose by contradiction that $R_n = \|u_n\| \to \infty$ and define $v_n = u_n/R_n$. Then $v_n^+ = u_n^+/R_n$ and $\|v_n^+\| \leq 1$. Passing to a subsequence we may assume that $v_n \rightharpoonup v$ and $v_n^+ \rightharpoonup v^+$ weakly in E. Moreover, $v_n^+ \to v^+$ in $L^2_{\text{loc}}(\mathbb{R}^2)$, $v_n^+(x) \to v^+(x)$ a.e. in $\mathbb{R}^2$. We have two cases to consider:

Case 1: (v_n^+) is vanishing, i.e., there exists $r > 0$ such that

$$\lim_{n\to+\infty} \sup_{y\in\mathbb{R}^2} \int_{B(y,r)} |v_n^+|^2 dx = 0.$$

According to Lemma 3.4.1 we have that $\|v_n^+\|_q \to 0$ for any $q > 2$ because (v_n^+) is bounded in E. Since $|\Phi_{\mu_n}(u_n)| \leq C$, by Lemma 3.4.3 we see that

$$\Phi_{\mu_n}(r_nu_n^+) + r_n^2\|u_n^-\|^2 \leq C + o_n(1)r_n^2\|u_n^+\|.$$

Taking $r_n = s/R_n$ with $s > 0$ fixed we get

$$\Phi_{\mu_n}(sv_n^+) + s^2\|v_n^-\|^2 \leq C + o_n(1)s^2. \tag{3.12}$$

On the other hand,

$$\begin{aligned}\Phi_{\mu_n}(sv_n^+) + s^2\|v_n^-\|^2 &= \mu_n s^2\|v_n^+\|^2 + s^2\|v_n^-\|^2 - 2\int_{\mathbb{R}^2} F(x,sv_n^+)dx \\ &\geq s^2 - 2\int_{\mathbb{R}^2} F(x,sv_n^+)dx.\end{aligned}$$

Now using (3.5) and applying the Hölder inequality together with Lemma 3.2.3 we obtain

$$\begin{aligned} 2\int_{\mathbb{R}^2} F(x, sv_n^+)dx &\leq \frac{\gamma}{\Lambda}s^2\|v_n^+\|^2 + C_1 s^q \|v_n^+\|_q \left(\int_{\mathbb{R}^2} |v_n^+|^q (e^{\beta\frac{q}{q-1}s^2(v_n^+)^2} - 1)dx\right)^{(q-1)/q} \\ &\leq \frac{\gamma}{\Lambda}s^2 + C_2 s^q o_n(1), \end{aligned}$$

for $\beta > 0$ sufficiently small. Consequently,

$$\Phi_{\mu_n}(sv_n^+) + s^2\|v_n^-\|^2 \geq \left(1 - \frac{\gamma}{\Lambda}\right)s^2 - C_2 s^q o_n(1),$$

which contradicts inequality (3.12) if s and n are sufficiently large.
Case 2: (v_n^+) is non-vanishing, i.e.,there is a sequence $(y_n) \subset \mathbb{Z}^2$ such that

$$\int_{B(y_n,r)} |v_n^+|^2 dx \geq \alpha > 0. \tag{3.13}$$

Defining $u_n'(x) := u_n(x-y_n)$ and $w_n(x) := u_n'(x)/\|u_n'\|$ we see that $\|u_n'\| = \|u_n\|$,$\|w_n\| = \|v_n\| = 1$ and $w_n(x) = v_n(x-y_n)$. Thus, passing to a subsequence we can assume that $w_n \rightharpoonup w$, $w_n^+ \rightharpoonup w^+$ in E, strongly in $L^2_{\text{loc}}(\mathbb{R}^2)$ and a.e. in $\mathbb{R}^2$. From (3.13) we have

$$\int_{B(0,r)} |w_n^+|^2 dx \geq \frac{\alpha}{2} > 0,$$

which implies that $w^+ \neq 0$ and so $w \neq 0$. According to Lemma 3.3.2

$$\int_{\mathbb{R}^2} \frac{F(x, u_n')}{\|u_n'\|^2} dx \to \infty \quad \text{as} \quad n \to \infty.$$

Since V and $f(\cdot, t)$ are $1-$periodic we have $\Phi_{\mu_n}(u_n) = \Phi_{\mu_n}(u_n')$. Thus,

$$2\int_{\mathbb{R}^2} \frac{F(x, u_n')}{\|u_n'\|^2} dx = \mu_n\|v_n^+\|^2 - \|v_n^-\|^2 - \frac{\Phi(u_n')}{\|u_n'\|^2} \leq C.$$

Now, taking $n \to \infty$ we obtain a contradiction and this complete the proof. □

Proof of Theorem 3.1.2: By applying Theorem 3.3.1, there exists a sequence $(\mu_n) \subset (1, 2]$, with $\mu_n \to 1$, such that it is possible to find a sequence $(u_m^n) \subset E$ verifying

$$\sup_m \|u_m^n\| < \infty, \quad \Phi'_{\mu_n}(u_m^n) \to 0 \quad \text{and} \quad \Phi_{\mu_n}(u_m^n) \to c_{\mu_n}, \quad \text{as} \quad m \to +\infty$$

where

$$c_{\mu_n} := \inf_{h\in\Gamma} \sup_{u\in\bar{Q}_R} \Phi_{\mu_n}(h(1, u)).$$

From this, for each $n \in \mathbb{N}$, there exists $m_n \in \mathbb{N}$ such that

$$|\Phi_{\mu_n}(u_{m_n}^n) - c_{\mu_n}| \leq \frac{1}{n}, \quad \left|\langle \Phi'_{\mu_n}(u_{m_n}^n), u_{m_n}^n\rangle\right| \leq \frac{1}{n} \quad \text{and} \quad \|\Phi'_{\mu_n}(u_{m_n}^n)\| \leq \frac{1}{n}, \quad \forall n \in \mathbb{N}.$$

In what follows, we denote $u^n_{m_n}$ by u_n, hence we can rewritten the above limits of the following way

$$|\Phi_{\mu_n}(u_n) - c_{\mu_n}| \leq \frac{1}{n}, \quad \left|\langle \Phi'_{\mu_n}(u_n), u_n\rangle\right| \leq \frac{1}{n} \quad \text{and} \quad \|\Phi'_{\mu_n}(u_n)\| \leq \frac{1}{n}, \quad \forall n \in \mathbb{N}.$$

Since $0 < c_1 \leq c_{\mu_n} \leq c_2$ for all $n \in \mathbb{N}$, without loss of generality we can assume that

$$0 < \frac{c_{\mu_1}}{2} \leq \Phi_{\mu_n}(u_n) \leq c_2 + \frac{1}{n} \quad \forall n \in \mathbb{N}.$$

According to Lemma 3.4.4, after a renamed subsequence $u_n \rightharpoonup u$ weakly in E, $u_n \to u$ strongly in $L^q_{\text{loc}}(\mathbb{R}^2)$ for $q > 2$ and a.e. in $\mathbb{R}^2$. Since for any $\varphi \in E$

$$o_n(1) = \frac{1}{2}\langle \Phi'_{\mu_n}(u_n), \varphi\rangle = \mu_n(u_n^+, \varphi) - (u_n^-, \varphi) - \int_{\mathbb{R}^2} f(x, u_n)\varphi dx, \tag{3.14}$$

taking the limit and using Lemma 3.3.4 we get $\Phi'(u) = 0$. We claim that $u \not\equiv 0$. Indeed, from (3.1), (f_0), (f_2),(3.5), the Hölder inequality and Lemma 3.2.3 we get

$$\begin{aligned}\int_{\mathbb{R}^2} f(x, u_n)u_n^+ dx &\leq \frac{\gamma}{\Lambda}\|u_n^+\|^2 + C_1\|u_n^+\|_q \left(\int_{\mathbb{R}^2} |u_n|^q (e^{\beta\frac{q}{q-1}u_n^2} - 1)dx\right)^{(q-1)/q} \\ &\leq \frac{\gamma}{\Lambda}\|u_n^+\|^2 + C_2\|u_n^+\|_q.\end{aligned} \tag{3.15}$$

Then, choosing $\varphi = u_n^+$ in (3.14) and using (3.15), we obtain

$$\mu_n\|u_n^+\|^2 = \int_{\mathbb{R}^2} f(x, u_n)u_n^+ dx + o_n(1) \leq \frac{\gamma}{\Lambda}\|u_n^+\|^2 + C_1\|u_n^+\|_q + o_n(1),$$

which implies that

$$\left(1 - \frac{\gamma}{\Lambda}\right)\|u_n^+\|^2 \leq C_1\|u_n^+\|_q + o_n(1). \tag{3.16}$$

On the other hand, by (3.4), (f_1), Lemma 3.2.1 and (3.5) we have

$$0 < \frac{c_1}{2} \leq \Phi_{\mu_n}(u_n) \leq \mu_n\|u_n^+\|^2 + \lambda\|u_n^-\|_2^2 - \lambda\|u_n\|_2^2 \leq \left(2 - \frac{\lambda}{\Lambda}\right)\|u_n^+\|^2. \tag{3.17}$$

If, for $r > 0$ fixed,

$$\sup_{y\in\mathbb{R}^2} \int_{B(y,r)} |u_n^+|^2 dx \to 0 \quad \text{as} \quad n \to \infty,$$

then by Lemma 3.4.1 we get that $u_n^+ \to 0$ in $L^q(\mathbb{R}^2)$. From (3.16)-(3.17) we get a contradiction. Consequently, there exists a sequence $(y_n) \subset \mathbb{Z}^2$ such that

$$\int_{B(y_n,r)} |u_n^+|^2 dx \geq \alpha > 0.$$

Thus, using that $V, f(\cdot, t)$ are periodic and defining $w_n(x) := u_n(x + y_n)$ we have

$$\sup_n \|w_n\| < \infty, \quad 0 < \frac{c_1}{2} \leq \Phi_{\mu_n}(w_n) \leq c_2 + \frac{1}{n}, \quad \|\Phi'_{\mu_n}(w_n)\| \leq \frac{1}{n}, \quad \forall n \in \mathbb{N}$$

and

$$\int_{B(y_n,r)} |w_n^+|^2 dx \geq \alpha > 0, \quad \forall n \in \mathbb{N}. \tag{3.18}$$

By the continuity of the projection map, we may assume that $w_n^+ \rightharpoonup w^+$ in E^+. Furthermore, using the compact embedding $E^+ \hookrightarrow L^2_{loc}(\mathbb{R}^2)$ together with (3.18) we have that $w^+ \neq 0$. Consequently w is a nontrivial critical point of Φ. □

3.5 Nontrivial Solution (Nonperiodic Case)

This section is devoted to the proof of Theorem 3.1.3. We quote that in this section the nonlinearity $f(x,t)$ is not assumed to be periodic and therefore we cannot use the Lions Lemma. Before to present the proof of Theorem 3.1.3, we establish some auxiliary results. We start with the following convergence lemma (see [78] for related results).

Lemma 3.5.1. *Assume hypotheses* (3.1) *and* (f_4). *Then the functional* $L(u) = \int_{\mathbb{R}^2} F(x,u)dx$ *is weakly continuous on* E *for* $\beta > 0$ *small.*

Proof. By condition (3.1), given $\varepsilon > 0$ there exists $r > 0$ such that for all $\beta > 0$

$$|F(x,t)| \leq \varepsilon(e^{\beta t^2} - 1), \quad \forall x \in \mathbb{R}^2, \quad |t| \geq r.$$

Now considering the continuous function $w_r : \mathbb{R} \to \mathbb{R}$ defined by

$$w_r(t) = \begin{cases} r, & t \geq r \\ t, & |t| \leq r \\ -r, & t \leq -r, \end{cases}$$

we see that

$$|F(x,t) - F(x,w_r(t))| \leq 2\varepsilon(e^{\beta t^2} - 1), \quad \forall\, x \in \mathbb{R}^2,\, |t| \geq r. \tag{3.19}$$

Let $u_n \rightharpoonup u$ in E. According to definition of w_r we have

$$\begin{aligned} |L(u_n) - L(u)| \leq & \int_{|u_n|>r} |F(x,u_n) - F(x,w_r(u_n))|dx \\ & + \int_{\mathbb{R}^2} |F(x,w_r(u_n)) - F(x,w_r(u))|dx \\ & + \int_{|u|>r} |F(x,w_r(u)) - F(x,u)|dx \\ & =: I_1 + I_2 + I_3. \end{aligned}$$

Since $(u_n) \subset E$ is bounded, using estimate (3.19) together with Lemma 3.2.3, for $\beta > 0$ small we obtain

$$I_1 \leq 2\varepsilon \int_{\mathbb{R}^2} (e^{\beta u_n^2} - 1)dx \leq \varepsilon C,$$

for some $C > 0$. An analogous estimate holds to I_3. To estimate I_2 we observe that $F(x, w_r(u_n)) - F(x, w_r(u)) \to 0$ a.e. in $\mathbb{R}^2$ and

$$\begin{aligned} I_2 &= \int_{\mathbb{R}^2} |F(x, w_r(u_n)) - F(x, w_r(u))| dx \\ &\leq \int_{|x| \leq R_0} |F(x, w_r(u_n)) - F(x, w_r(u))| dx + \int_{|x| > R_0} |F(x, w_r(u_n)) - F(x, w_r(u))| dx. \end{aligned} \tag{3.20}$$

Now, observe that the first integral in the right-hand side of (3.20) converges to zero in view of the compact embedding $E \hookrightarrow L^1_{\text{loc}}(\mathbb{R}^2)$. To estimate the second integral in (3.20), since $|w_r(t)| \leq r$ by (f_4) we see that

$$|F(x, w_r(u_n) - F(x, w_r(u)| \leq 2h(x) e^{\alpha_0 r^2} = C(r) h(x), \quad \forall t \in \mathbb{R},\ x \in B^c_{R_0}.$$

Since $h \in L^1(B^c_{R_0})$ the result follows by applying the Lebesgue Dominate Convergence Theorem. $\square$

Remark 3.5.2. *An analogous argument can be used to show that the functional* $\widetilde{L} : E \to \mathbb{R}$ *given by* $\widetilde{L}(u) = \int_{\mathbb{R}^2} f(x, u) u dx$ *is weakly continuous.*

Proposition 3.5.3. *Suppose* (V_0) *and* $(f_0) - (f_4)$ *are satisfied. For any* $\mu \in [1, 2]$, *there are sequences* $(\mu_n) \subset [1, 2]$, $\mu_n \to \mu$ *and* $(u_n) \subset E \setminus \{0\}$ *such that*

$$0 < c_1 \leq \Phi_{\mu_n}(u_n) = c_{\mu_n} \leq c_2 \quad \textit{and} \quad \Phi'_{\mu_n}(u_n) = 0,$$

where c_1, c_2 *are the minimax levels defined in Theorem 3.3.1 with* $\mu = 1$ *and* $\mu = 2$, *respectively.*

Proof. For almost everywhere $\mu \in [1, 2]$, in view of Lemmas 3.3.3, 3.3.6, 3.3.7, 3.3.5 we can apply Theorem 3.3.1 to obtain a sequence $(u_n) \subset E$ such that

$$\sup_n \|u_n\| < \infty, \quad \Phi'_\mu(u_n) \to 0 \quad \text{and} \quad \Phi_\mu(u_n) \to c_\mu > 0.$$

For any $v \in E$ we have

$$\frac{1}{2} \langle \Phi'_\mu(u_n), v \rangle = \mu(u_n^+, v) - (u_n^-, v) - \int_{\mathbb{R}^2} f(x, u_n) v dx \to 0 \quad \text{as} \quad n \to \infty.$$

Passing to a subsequence we may assume that $u_n \rightharpoonup u_\mu$ weakly in E, $u_n \to u_\mu$ strongly in $L^q_{\text{loc}}(\mathbb{R}^2)$ and a.e. in $\mathbb{R}^2$. Therefore,

$$\frac{1}{2} \langle \Phi'_\mu(u_\mu), v \rangle = \mu(u_\mu^+, v) - (u_\mu^-, v) - \int_{\mathbb{R}^2} f(x, u_\mu) v dx = 0, \quad \forall\, v \in E,$$

i.e., $\Phi'_\mu(u_\mu) = 0$. Let $H(x, t) = t f(x, t) - 2F(x, t)$. Since $\int H(x, u_n) dx = \Phi_\mu(u_n) - \frac{1}{2} \langle \Phi'_\mu(u_n), u_n \rangle$ and (u_n) is bounded, it follows that $\int H(x, u_n) dx \longrightarrow c_\mu$. On the other hand, according to

Lemma 3.5.1 and Remark 3.5.2, $\int H(x,u_n)dx \to \int H(x,u_\mu)dx$. Therefore,

$$\Phi_\mu(u_\mu) = \int_{\mathbb{R}^2} H(x,u_\mu)dx = \lim_n \int_{\mathbb{R}^2} H(x,u_n) = c_\mu \geq c_1 > 0,$$

which implies that $u_\mu \not\equiv 0$. From the first step, for any $\mu \in [1,2]$ there are sequences $\mu_n \to \mu$ and $(u_{\mu_n}) \subset E \setminus \{0\}$ such that $\Phi_{\mu_n}(u_{\mu_n}) = c_{\mu_n}$ and $\Phi'_{\mu_n}(u_{\mu_n}) = 0$. Thus the result follows by taking $u_n = u_{\mu_n}$ and $n \in \mathbb{N}$ sufficiently large such that $\Phi_{\mu_n}(u_{\mu_n}) \geq \eta > 0$. $\square$

Lemma 3.5.4. *Assume* (V_0), $(f_1)-(f_4)$ *and* (3.1). *Let* $1 \leq \mu_n \leq 2$ *and* $(u_n) \subset E$ *such that* $|\Phi_{\mu_n}(u_n)| \leq C$ *and* $\Phi'_{\mu_n}(u_n) = 0$. *Then* (u_n) *has a bounded subsequence in* E.

Proof. Suppose by contradiction that $R_n = \|u_n\| \to \infty$ and define $v_n = u_n/R_n$. Then $v_n^+ = u_n^+/R_n$ and $\|v_n^+\| \leq 1$. Passing to a subsequence we may assume that $v_n \rightharpoonup v$ and $v_n^+ \rightharpoonup v^+$ weakly in E. Moreover, $v_n \to v$ and $v_n^+ \to v^+$ in $L^2_{\text{loc}}(\mathbb{R}^2)$, $v_n^+(x) \to v^+(x)$ a.e in $\mathbb{R}^2$. Now we consider two cases.

Case 1: Assume that $v^+ \neq 0$. In this case, $v \neq 0$ and invoking Lema 3.2.3 we see that

$$\int_{\mathbb{R}^2} \frac{F(x,u_n)}{\|u_n\|^2}\, dx \to +\infty \quad \text{as} \quad n \to +\infty.$$

On the other hand, we have

$$2\int_{\mathbb{R}^2} \frac{F(x,u_n)}{\|u_n\|^2}\, dx \leq \mu_n \|v_n^+\|^2 - \frac{\Phi_{\mu_n}(u_n)}{\|u_n\|^2} \leq C,$$

which is a contradiction.

Case 2: Assume that $v^+ = 0$. For $s > 0$ fixed define $r_n := s/R_n \to 0$. Since $|\Phi_{\mu_n}(u_n)| \leq C$, by Lemma 3.4.3 we conclude that

$$\Phi_{\mu_n}(r_n u_n^+) + r_n^2 \|u_n^-\|^2 \leq C'.$$

Now, using Lemma 3.5.1 together with the fact that $\mu_n \geq 1$ we infer that

$$\begin{aligned} C' \geq \Phi_{\mu_n}(r_n u_n^+) + r_n^2\|u_n^-\|^2 &= \mu_n s^2 \|v_n^+\|^2 + s^2\|v_n^-\|^2 - 2\int_{\mathbb{R}^2} F(x, s v_n^+)dx \\ &\geq s^2 - 2\int_{\mathbb{R}^2} F(x, s v_n^+)dx \\ &\to s^2. \end{aligned}$$

Thus, we have a contradiction if $s > 0$ is sufficiently large and this concludes the proof. $\square$

Proof of Theorem 3.1.3: Consider a sequence $(\mu_n) \subset (1,2]$ such that $\mu_n \to 1$. According to

Proposition 3.5.3, there exists a sequence $(u_n) \subset E$ such that

$$0 < \eta \leq \Phi_{\mu_n}(u_n) = c_{\mu_n} \quad \text{and} \quad \Phi'_{\mu_n}(u_n) = 0.$$

By Lemma 3.5.4, after a renamed subsequence $u_n \rightharpoonup u$, $u_n^\pm \rightharpoonup u^\pm$ weakly in E, strongly in $L^q_{\text{loc}}(\mathbb{R}^2)$ and a.e. in $\mathbb{R}^2$. Since for any $\varphi \in C_0^\infty(\mathbb{R}^2)$

$$0 = \frac{1}{2}\langle \Phi'_{\mu_n}(u_n), \varphi \rangle = \mu_n(u_n^+, \varphi) - (u_n^-, \varphi) - \int_{\mathbb{R}^2} f(x, u_n)\varphi dx,$$

taking the limit and using Lemma 3.3.4 we obtain $\Phi'(u) = 0$. Proceeding as in the proof of Proposition 3.5.3 we see that $u \not\equiv 0$ and this completes the proof. □

Chapter 4

A Semilinear Schrödinger Equation with Zero on the Boundary of the Spectrum and Exponential Growth

4.1 Introduction and Main Result

Chapter IV is devoted to existence of weak solution for the following semilinear Schrödinger equation

$$-\Delta u + V(x)u = g(x,u), \quad x \in \mathbb{R}^2, \tag{$\mathcal{P}_g$}$$

where 0 is a right boundary point of the spectrum of Schrödinger operator $\mathcal{S} = -\Delta + V$ and $g(x,t)$ has exponential growth. We emphasize that this work is the content of the paper [44]. Further investigations and developments for equation $(\mathcal{P}_g)$ have been carried out depending on the location of 0 with respect to $\sigma(\mathcal{S})$. Let us remember them:

Case 1 : If $0 < \inf \sigma(\mathcal{S})$. In this case, Coti-Zelati and Rabinowitz [22] proved that $(\mathcal{P}_g)$ has infinitely many solutions provided that the nonlinear term $g(x,t)$ satisfies some suitable growth condition as the well known Ambrosetti-Rabinowitz condition.

Case 2 : If 0 *lies in a gap of the spectrum* $\sigma(\mathcal{S})$. When the primitive of $g(x,t)$ is strictly convex Alama and Li [1], [2], Buffoni et al. [16] and Jeanjean [34] found solutions using a reduction method to solve the problem by applying the mountain-pass theorem. Troestler and Willem [71] proved that $(\mathcal{P}_g)$ has a nontrivial solution without the convexity hypothesis on G, they require assumptions on $g(x,t)$ which implies that the associated functional Φ is of class C^2. Under conditions weaker than those, W. Kryszewski and A. Szulkin [36] proved and applied a generalized linking theorem which requires the construction of a new degree theory in order to handle the lack of compactness in this problem. This approach has been simplified by A. Pankov and K. Pflüger in [51] by using the approximation technique with periodic functions. In the papers [58] and [60] the authors established a variant and generalized weak linking theorem and obtained solution for the Schrödinger equation $(\mathcal{P}_g)$ when the nonlinearity has subcritical and critical growth with respect to Sobolev's embeddings. See also [40, 43] for related results.

Case 3 : If 0 *lies in the interior of* $\sigma(\mathcal{S})$. There exist only some bifurcation results [29, 63].

Case 4 : If 0 *is a boundary point of a gap of* $\sigma(\mathcal{S})$. Bartsch and Ding [9] obtained a nontrivial solution to $(\mathcal{P}_g)$ assuming that 0 is a right endpoint of $\sigma(\mathcal{S})$, among others, the (AR) condition and the lower bound estimate:

(g_0) There are $a > 0$ and $q > 2$ such that

$$2G(x,t) \geq a|t|^q \quad \text{for all} \quad x \in \mathbb{R}^2, t \in \mathbb{R}.$$

In [75] Willem and Zou relaxed condition (AR), developed the so-called monotonicity trick for strongly indefinite problems and established weak linking results. Recently M. Yang et al. [76] obtain a nontrivial weak solution for problem $(\mathcal{P}_g)$ replacing condition (AR) by a general super-quadratic condition, to namely (see [65])

$$t \mapsto \frac{g(x,t)}{|t|} \quad \text{is strictly increasing on} \quad (-\infty, 0) \cup (0, \infty). \tag{4.1}$$

We observe that assumption (4.1) implies the statement

$$2G(x, t+s) - 2G(x,t) - (2rs - (r-1)^2 t) g(x,t) \geq 0, \ \forall\, x \in \mathbb{R}^2,\ s, t \in \mathbb{R}, r \in [0,1]. \tag{4.2}$$

We point out that M. Schechter [59] assumed the conditions (g_0) and (4.2) and proved the existence of ground state solution. In all those papers with zero on the boundary of $\sigma(\mathcal{S})$ dealt only with polynomial subcritical case. To the authors' knowledge, there are few papers treating problem $(\mathcal{P}_g)$ with V periodic, $0 \notin \sigma(\mathcal{S})$ and $g(x,t)$ has exponential growth (see for instance [3, 24]). Thus, our result generalize many works in the line of the papers [46, 59, 60, 68] for nonlinearity involving exponential growth and $0 \in \sigma(\mathcal{S})$.

In the sequel, throughout this Chapter, we assume that the potential V satisfies:

(V_0) $V : \mathbb{R}^2 \to \mathbb{R}$ is a continuous and 1-periodic function;

(V_1) $0 \in \sigma(\mathcal{S})$ and there exists $b > 0$ such that $\sigma(\mathcal{S}) \cap (0, b) = \emptyset$, where $\sigma(\mathcal{S})$ denotes the spectrum of the operator $\mathcal{S} = -\Delta + V$ in $L^2(\mathbb{R}^2)$.

In addition to condition (g_0) we assume that $g(x,t)$ has exponential subcritical growth at infinity,

$$\lim_{|t| \to \infty} \frac{g(x,t)}{e^{\beta t^2}} = 0 \quad \text{for all} \quad \beta > 0, \tag{4.3}$$

and satisfies:

(g_1) $g \in C(\mathbb{R}^2 \times \mathbb{R}, \mathbb{R})$ and is 1-periodic in x_j for $j = 1, 2$.

(g_2) $g(x,t) = O(|t|^{q-1})$ as $t \to 0$ uniformly in $x \in \mathbb{R}^2$, where $q > 2$ is given in (g_0).

(g_3) $g(x,t)$ is locally bounded in the variable t, that is, for any bounded interval $J \subset \mathbb{R}$, there exists $C > 0$ such that $|g(x,t)| \leq C$ for every $(x,t) \in \mathbb{R}^2 \times J$.

(g_4) There exists $W \in L^1(\mathbb{R}^2)$ such that for all $x \in \mathbb{R}^2$, $s, t \in \mathbb{R}$ and $r \in [0,1]$ it holds

$$2(G(x,t+s) - G(x,t)) - (2rs - (r-1)^2 t)g(x,t) \geq -W(x).$$

Remark 4.1.1. *Taking $r = 0$ and $s = -t$ (respectively, $s = rw - t$) in (g_4) we obtain:*

$$H(x,t) := tg(x,t) - 2G(x,t) \geq -W(x), \quad \forall\, x \in \mathbb{R}^2, t \in \mathbb{R}, \tag{4.4}$$

and

$$2G(x,t) - 2G(x,rw) - ((r^2+1)t - 2r^2 w)g(x,t) \leq W(x), \quad \forall\, x \in \mathbb{R}^2,\ t, w \in \mathbb{R},\ r \in [0,1]. \tag{4.5}$$

Furthermore, taking $t = s = 0$ in (g_4) we get $W \geq 0$ in $\mathbb{R}^2$ and so (g_4) implies (4.2).

Remark 4.1.2. *We observe that if the potential V satisfies assumption (V_0), replacing case necessary $V(x)$ by $V(x) + const$, it was shown in Stuart* [63] *that V satisfies assumption (V_1). We also quote that a typical example of a nonlinearity satisfying our assumptions is*

$$g(x,t) = a(x)|t|^{q-2}t + b(x)|t|^{p-2}t(e^t - 1), \quad x \in \mathbb{R}^2,\, t \in \mathbb{R},$$

where $2 < q \leq p$ and $a(x) \geq a_0 > 0$, $b(x) \geq 0$ are periodic.

Our main result of existence of solution for problem $(\mathcal{P}_g)$ under the above hypotheses can be summarized as follows.

Theorem 4.1.3. *Assume $(V_0) - (V_1)$ and $(g_0) - (g_4)$. If $g(x,t)$ satisfies* (4.3) *then the problem $(\mathcal{P}_g)$ has a nontrivial weak solution. Moreover, if $\mathcal{M}$ denotes the collection of the solutions of $(\mathcal{P}_g)$, then there is a ground state solution, i.e., a solution of $(\mathcal{P}_g)$ that minimizes the functional energy over $\mathcal{M}$. Furthermore, $u \in C^1(\mathbb{R}^2)$ and $u(x) \to 0 \quad as \quad |x| \to \infty$.*

The present Chapter is organized as follows. In the next section we formulate our problem in a variational setting and we also prove a Trudinger-Moser inequality for our variational framework. In Section 3, we establish some geometric properties of the energy functional, which are required for the application of the linking-type theorem used and in Section 4 we prove that (PS) sequences are bounded. Finally, in Section 5, we conclude the chapter with the proof of Theorem 4.1.3.

4.2 Variational Setting

In this section we will construct the domain for our energy functional, a reflexive Banach space $(E_q, \|\cdot\|_{\mathbf{q}})$, where we can apply the same linking theorem used in the previous chapter. For this application, we need to establish a Trudinger-Moser inequality in the space E_q, what will be done by using Schwarz symmetrization.

First we observe that from (4.3) and $(g_2)-(g_3)$, for any $\beta>0$, there exists $C_1, C_2>0$ such that

$$G(x,t)\leq C_1|t|^q+C_2|t|^q(e^{\beta t^2}-1),\quad \forall\, x\in\mathbb{R}^2, t\in\mathbb{R}. \tag{4.6}$$

Under the hypotheses $(V_0)-(V_1)$, we will find a function space E_q on which the energy functional associated to $(\mathcal{P}_g)$, $\Phi: E_q\to\mathbb{R}$, given by

$$\Phi(u)=\int_{\mathbb{R}^2}(|\nabla u|^2+V(x)u^2)dx-2\int_{\mathbb{R}^2}G(x,u)dx$$

is well defined. Moreover $\Phi\in C^1(E_q,\mathbb{R})$ and for any $u,v\in E_q$

$$\frac{1}{2}\langle\Phi'(u),v\rangle=\int_{\mathbb{R}^2}(\nabla u\nabla v+V(x)uv)dx-\int_{\mathbb{R}^2}g(x,u)vdx.$$

Thus, critical points of Φ correspond to weak solutions to $(\mathcal{P}_g)$. In order to find the function space E_q, let $\mathcal{S}=-\Delta+V$ be the self-adjoint operator defined in $L^2(\mathbb{R}^2)$ with domain $D(\mathcal{S})=H^2(\mathbb{R}^2)$. Let $\{\mathcal{E}(\lambda)\}$, $-\infty\leq\lambda\leq+\infty$ be the spectral family of S, and $|\mathcal{S}|^{1/2}$ be the square root of $|\mathcal{S}|$. Set $U=I-\mathcal{E}(0)-\mathcal{E}(-0)$. Then U is unitary, commutes with $\mathcal{S}$, $|\mathcal{S}|$ and $|\mathcal{S}|^{1/2}$, moreover $\mathcal{S}=U|\mathcal{S}|$ is the polar decomposition of the operator $\mathcal{S}$ (see [35], p. 358).

As in the previous chapter, denote by $E=D(|\mathcal{S}|^{1/2})$, the domain of $|\mathcal{S}|^{1/2}$, then $\mathcal{E}(\lambda)E\subset E$ for all $\lambda\in\mathbb{R}$. Under the hypothesis (V_0) one can see that $E=D(|\mathcal{S}|^{1/2})=H^1(\mathbb{R}^2)$. Furthermore, if we define

$$E^-:=\mathcal{E}(0)E,\quad E^+:=[\mathcal{E}(\infty)-\mathcal{E}(0)]E,$$

$$(u,v)=(|\mathcal{S}|^{1/2}u,|\mathcal{S}|^{1/2}v)_2,\quad \forall\, u,v\in E, \text{ and } \quad \|u\|=\sqrt{(u,u)},$$

we have the following result (see Chapter I).

Lemma 4.2.1. *Assume that (V_0) holds. Then $\|\cdot\|$ is equivalent to $\|\cdot\|_{H^1}$ on E^+. Moreover, $E=E^-\oplus E^+$ and for any $u=u^-+u^+\in E$ it holds*

$$(u^-,u^+)=(u^-,u^+)_2=0.$$

However, as proved in Lemma 1.2.16, $\|\cdot\|$ is not equivalent to $\|\cdot\|_{H^1}$ on E^- because $0\in\sigma(\mathcal{S})$. Thus, we need introduce another norm in E^- by setting

$$\|u\|_-=(\|u\|^2+\|u\|_q^2)^{1/2},\quad u\in E^-,$$

where $q>2$. Let E_q^- be the completion of E^- with respect to $\|\cdot\|_-$. Then E_q^- is separable and reflexive. Moreover, the following embedding holds (see [9], Lemma 2.1)

$$E_q^-\hookrightarrow H^1_{\mathrm{loc}}(\mathbb{R}^2). \tag{4.7}$$

Since E^+ is a closed subspace of $H^1(\mathbb{R}^2)$ we can define

$$E_q := E_q^- + E^+.$$

Furthermore, it is easy to see that E_q is a reflexive Banach space as endowed with the norm (see [9])

$$\|u\|_{\mathbf{q}} = (\|u^-\|_-^2 + \|u^+\|^2)^{1/2}.$$

To the proof of the next lemma we refer to [9, 68].

Lemma 4.2.2. *The norm $\|\cdot\|_{\mathbf{q}}$ in E_q is invariant after translations in $\mathbb{Z}^2$ and the embeddings*

$$E_q \hookrightarrow H^1_{loc}(\mathbb{R}^2) \quad \textit{and} \quad E_q \hookrightarrow L^p(\mathbb{R}^2), \quad \forall\, p \geq q > 2,$$

are continuous.

We recall that the Trudinger-Moser inequality for unbounded domains established by Cao in [17], asserts that for any $u \in H^1(\mathbb{R}^2)$ and $\alpha > 0$ it holds $(e^{\alpha u^2} - 1) \in L^1(\mathbb{R}^2)$. Moreover, if $\|u\|_{H^1} \leq M$ and $\alpha < 4\pi$, then there exists a constant $C = C(\alpha, M) > 0$ such that

$$\int_{\mathbb{R}^2} (e^{\alpha u^2} - 1)dx \leq C(\alpha, M). \tag{4.8}$$

Since E_q is not immersed in $L^2(\mathbb{R}^2)$ is natural to consider the Young function $\Psi_\beta(t) = |t|^\theta(e^{\beta t^2} - 1)$, $\theta > q - 2$, for the Trudinger-Moser inequality. In view of Lemma 4.2.2, let S be the best constant of the embedding $E_q \hookrightarrow H^1(B_1)$, i.e.,

$$\frac{1}{S} = \inf_{u \in E_q \setminus \{0\}} \frac{\|u\|_{\mathbf{q}}}{\|u\|_{H^1(B_1)}}.$$

For our variational framework we will establish the following version of the Trudinger-Moser inequality in the space E_q.

Theorem 4.2.3 (Trudinger-Moser)**.** *For any $u \in E_q$, $\beta > 0$ and $\theta > q - 2$,*

$$|u|^\theta(e^{\beta u^2} - 1) \in L^1(\mathbb{R}^2).$$

Moreover, if $\|u\|_{\mathbf{q}} \leq M$ then there exists a constant $C = C(\beta, \theta, M) > 0$ such that

$$\int_{\mathbb{R}^2} |u|^\theta(e^{\beta u^2} - 1)dx \leq C, \tag{4.9}$$

for any $\beta > 0$ such that $\beta(SM)^2 < 4\pi$.

Proof. Let u^* be the symmetrization of u, then it is well known that u^* depends on $|x|$ only and is nonnegative decreasing function of $|x|$. Furthermore,

$$\begin{aligned}\int_{\mathbb{R}^2} |u|^\theta (e^{\beta u^2} - 1)dx &= \int_{\mathbb{R}^2} |u^*|^\theta (e^{\beta |u^*|^2} - 1)dx \\ &= \int_{B_1} |u^*|^\theta (e^{\beta |u^*|^2} - 1)dx + \int_{B_1^c} |u^*|^\theta (e^{\beta |u^*|^2} - 1)dx \\ &:= I_1 + I_2.\end{aligned}$$

To estimate I_1, since $E_q \hookrightarrow L^p(\mathbb{R}^2)$ for any $p \geq q > 2$ and

$$\int_{\mathbb{R}^2} |u|^p dx = \int_{\mathbb{R}^2} |u^*|^p dx,$$

invoking the Radial Lemma (see [14], Lemma A.IV), for $u^* \in L^p(\mathbb{R}^2)$ radially decreasing we have

$$|u^*(x)| \leq C_1 \|u^*\|_p |x|^{-2/p}, \quad |x| \neq 0. \tag{4.10}$$

Now we set

$$v(r) = \begin{cases} u^*(r) - u^*(1), & 0 \leq r \leq 1 \\ 0, & r \geq 1. \end{cases}$$

For each $\varepsilon > 0$ by the Young inequality, Lemma 4.2.2 and (4.10) we obtain

$$|u^*(r)|^2 \leq (1+\varepsilon)v^2(r) + (1 + C(\varepsilon))|u^*(1)|^2 \leq (1+\varepsilon)v^2(r) + C(\varepsilon, M).$$

Thus, for any $\gamma > 1$ we get

$$\int_{B_1} e^{\gamma\beta|u^*|^2} dx \leq e^{\gamma\beta C(\varepsilon,M)} \int_{B_1} e^{\gamma\beta(1+\varepsilon)v^2} dx. \tag{4.11}$$

Since $E_q \hookrightarrow H^1(B_1)$ and $\|u\|_{\mathbf{q}} \leq M$, by the Pólya-Szegö inequality we get

$$\int_{B_1} |\nabla v|^2 dx = \int_{B_1} |\nabla u^*|^2 dx \leq \int_{B_1} |\nabla u|^2 dx \leq (SM)^2.$$

Using that $\beta(SM)^2 < 4\pi$ we can choose $\varepsilon > 0$ sufficiently small and $\gamma > 1$ near to 1 such that $\gamma\beta(1+\varepsilon)(SM)^2 \leq 4\pi$. Since $v \in H_0^1(B_1)$, we can invoke the Trudinger-Moser inequality in the ball B_1 to obtain $C_2 > 0$ such that

$$\int_{B_1} e^{\gamma\beta(1+\varepsilon)v^2} dx = \int_{B_1} e^{\gamma\beta(1+\varepsilon)(SM)^2(\frac{v}{SM})^2} dx \leq C_2.$$

This, together with the Hölder inequality and (4.11) imply

$$I_1 \leq \left(\int_{B_1} |u^*|^{\theta\gamma'} dx\right)^{1/\gamma'} \left(\int_{B_1} e^{\gamma\beta|u^*|^2} dx\right)^{1/\gamma} \leq \|u^*\|^\theta_{\theta\gamma'} e^{\beta C(\varepsilon,M)} (C_2)^{1/\gamma} \leq C_3, \tag{4.12}$$

provided that $\|u\|_{\mathbf{q}} \leq M$, $\frac{1}{\gamma} + \frac{1}{\gamma'} = 1$ and $\theta\gamma' \geq q$ (which is possible since $\gamma \to 1^+$ if and only if $\gamma' \to \infty$). Now we will estimate I_2. To this end, we fix $2 < q \le p < 2+\theta$. Using (4.10) it follows from the Monotone Convergence Theorem that

$$I_2 = \int_{|x|\ge 1} |u^*|^\theta \sum_{k=1}^{\infty} \frac{(\beta u^2)^k}{k!} dx \le C_1^\theta \|u^*\|_p^\theta \sum_{k=1}^{\infty} \frac{(\beta C_1^2 \|u^*\|_p^2)^k}{k!} \int_{|x|\ge 1} |x|^{-\frac{4k+2\theta}{p}} dx$$
$$\le \frac{p\pi C_1^\theta \|u^*\|_p^\theta}{2+\theta-p} \sum_{k=1}^{\infty} \frac{(\beta C_1^2 \|u^*\|_p^2)^k}{k!}.$$

Since $\|u^*\|_p = \|u\|_p \le C_2 \|u\|_{\mathbf{q}} \le C_2 M$ we get

$$I_2 \le \frac{p\pi}{2+\theta-p} (C_1 C_2 M)^\theta \sum_{k=1}^{\infty} \frac{(\beta (C_1 C_2 M)^2)^k}{k!} = \frac{p\pi}{2+\theta-p} (C_1 C_2 M)^\theta [e^{\beta (C_1 C_2 M)^2} - 1]. \quad (4.13)$$

From estimates (4.12) and (4.13) we conclude the proof. □

As a byproduct of the proof of Theorem 4.2.3 we can prove the next corollary. It will be useful throughout the paper.

Corollary 4.2.4. *If $u \in E_q$, $\beta > 0$, $\theta > q-2$ and $\|u\|_{\mathbf{q}} \le M$ with $\beta (SM)^2 < 4\pi$, then there exists $C = C(\beta, \theta, M) > 0$ such that*

$$\int_{\mathbb{R}^2} |u|^\theta (e^{\beta u^2} - 1) dx \le C \|u\|_{\mathbf{q}}^\theta.$$

Invoking Corollary 4.2.4 and inequality (4.6) we conclude that the energy functional Φ associated to $(\mathcal{P}_g)$ is well defined. Furthermore,

$$\Phi(u) = \|u^+\|^2 - \|u^-\|^2 - 2\int_{\mathbb{R}^2} G(x,u)dx, \quad \forall\, u = u^- + u^+ \in E_q = E_q^- + E^+$$

and

$$\frac{1}{2}\langle \Phi'(u), v\rangle = (u^+, v) - (u^-, v) - \int_{\mathbb{R}^2} g(x,u) v dx, \quad \forall\, u, v \in E_q.$$

4.3 Linking Geometry

In this section, in order to find critical points of the functional Φ we will use one more time the linking theorem due to Schechter-Zou [60] (see also [66, 75] for related results). With the conditions of the theorem satisfied, we obtain a (PS) sequence for our energy functional. For the convenience of the reader we will define the τ-topology in E_q and we will present the linking theorem in this context. Since $E_q = E_q^- + E^+$ and E_q^- is separable, for each $u = u^- + u^+$ we have

$$u^- = \sum_{k=1}^{\infty} c_k(u^-) e_k$$

where $\mathcal{B} = \{e_1, e_2, \ldots\}$ is dense in E_q^-. Thus we can introduce a new norm in E_q by setting

$$\|u\|_\tau = \max\left\{\|u^+\|_{\mathbf{q}}, \sum_{k=1}^{\infty} \frac{|c_k(u^-)|}{2^k}\right\}. \tag{4.14}$$

One can see that $\|\cdot\|_\tau$ satisfies $\|u\|_\tau \leq \|u\|_{\mathbf{q}}$ for any $u \in E_q$ (see [36]). For $R > \rho > 0$ and $u_0^+ \in E^+ \setminus \{0\}$ we define

$$Q_R := \{u = u^- + su_0^+ : s \geq 0, u^- \in E^-, \|u\|_{\mathbf{q}} < R\}, \quad S_\rho := \{u^+ \in E^+ : \|u^+\|_{\mathbf{q}} = \rho\}.$$

For a functional $\Phi \in C^1(E_q, \mathbb{R})$ defined in a Banach space E_q we consider

$$\Gamma := \left\{\begin{array}{l} h : [0,1] \times \bar{Q} \to E_q, h \text{ is } \tau\text{-continuous. For any } (s_0, u_0) \in [0,1] \times \bar{Q}, \\ \text{there is a } \tau\text{-neighborhood } U_{(s_0,u_0)} \text{ such that} \\ \{u - h(s,u) : (s,u) \in U_{(s_0,u_0)} \cap ([0,1] \times \bar{Q})\} \subset E_{fin}, \\ h(0,u) = u, \Phi(h(s,u)) \leq \Phi(u), \forall\, u \in \bar{Q}. \end{array}\right\}$$

where we use E_{fin} to denote various finite-dimensional subspace of E_q whose exact dimension are irrelevant and depend on (s_0, u_0). Note that $\Gamma \neq \emptyset$ since $id \in \Gamma$.

Theorem 4.3.1. [60] *Suppose that a family of C^1-functionals $(\Phi_\mu : E_q \to \mathbb{R})_\mu$ has the form*

$$\Phi_\mu(u) := \mu I(u) - J(u), \quad \mu \in [1,2].$$

Assume that

a) $I(u) \geq 0$, $\forall\, u \in E_q$ and $\Phi_1 := \Phi$;

b) $I(u) + |J(u)| \to \infty$ as $\|u\|_{\mathbf{q}} \to \infty$;

c) Φ_μ is τ-upper semicontinuous, maps bounded sets to bounded sets and Φ'_μ is weakly sequentially continuous on E_q;

d) $\sup_{\partial Q} \Phi_\mu \leq 0 < \inf_{S_\rho} \Phi_\mu$, $\forall\, \mu \in [1,2]$.

Then for almost all $\mu \in [1,2]$, there exists a sequence $(u_n) \subset E_q$ such that

$$\sup_n \|u_n\|_{\mathbf{q}} < \infty, \quad \Phi'_\mu(u_n) \to 0, \quad \Phi_\mu(u_n) \to c_\mu,$$

where

$$c_\mu := \inf_{h \in \Gamma} \sup_{u \in \bar{Q}} \Phi_\mu(h(1,u)).$$

Furthermore, $c_\mu \in [\inf_{S_\rho} \Phi_\mu, \sup_{\bar{Q}} \Phi_\mu]$ and is nondecreasing in μ.

In what follows, we obtain the linking structure of Φ_μ required in Theorem 4.3.1. Precisely, we apply Theorem 4.3.1 with

$$I(u) = \|u^+\|^2 = \|u^+\|^2_{\mathbf{q}} \quad \text{and} \quad J(u) = \|u^-\|^2 + 2\int_{\mathbb{R}^2} G(x,u)dx,$$

that is, $\Phi_\mu : E_q \to \mathbb{R}$,

$$\Phi_\mu(u) = \mu\|u^+\|^2 - \|u^-\|^2 - 2\int_{\mathbb{R}^2} G(x,u)dx, \quad \mu \in [1,2],$$

which clearly satisfies $a)$ in Theorem 4.3.1.

Lemma 4.3.2. *Assume (g_0). Then the functional Φ_μ satisfies condition $b)$ in Theorem 4.3.1. In fact, we have $I(u) + J(u) \to \infty$ as $\|u\|_{\mathbf{q}} \to \infty$.*

Proof. For any $u \in E_q$, we write $u = u^- + u^+$, with $u^- \in E_q^-$ and $u^+ \in E^+$. Since $\|u\|^2_{\mathbf{q}} = \|u^-\|^2_- + \|u^+\|^2$, if $\|u\|_{\mathbf{q}} \to \infty$ then $\|u^-\|_{\mathbf{q}} = \|u^-\|_- \to \infty$ or $\|u^+\|_{\mathbf{q}} = \|u^+\| \to \infty$. From (g_0) we get

$$J(u) = \|u^-\|^2 + 2\int_{\mathbb{R}^2} G(x,u)dx \geq \|u^-\|^2 + a\|u\|^q_q \geq 0.$$

Thus, $I(u) + J(u) \geq I(u) = \|u^+\|^2 \to \infty$ if $\|u^+\|_{\mathbf{q}} \to \infty$. Now suppose that there exists a sequence $(u_n) \subset E_q$ such that $\|u_n\|_{\mathbf{q}} \to \infty$, $\|u_n^+\|_{\mathbf{q}} \leq C$ and $I(u_n) + J(u_n) \leq C$. Thus,

$$\begin{aligned} C \geq I(u_n) + J(u_n) &= \|u_n^+\|^2 + \|u_n^-\|^2 + 2\int_{\mathbb{R}^2} G(x,u_n)dx \\ &\geq \|u_n\|^2_{\mathbf{q}} - \|u_n^-\|^2_q + a\|u_n\|^q_q \\ &\geq a\|u_n\|^q_q, \end{aligned} \tag{4.15}$$

which implies that $\|u_n\|_q \leq C$. Since $\|u_n^+\|_q \leq C\|u_n^+\|_{\mathbf{q}}$ we have

$$\|u_n^-\|_q \leq \|u_n\|_q + \|u_n^+\|_q \leq C.$$

This together with (4.15) imply that $(\|u_n\|_{\mathbf{q}})_n$ is bounded and this is a contradiction. □

To carry forward, we establish an auxiliary convergence result.

Lemma 4.3.3. *Assume* (4.3) *and* (g_2). *Then for any sequence $(u_n) \subset E_q$ such that $u_n \rightharpoonup u$ in E_q we have*

$$\int_{\mathbb{R}^2} g(x,u_n)\varphi dx \to \int_{\mathbb{R}^2} g(x,u)\varphi dx, \quad \textit{for any} \quad \varphi \in C_0^\infty(\mathbb{R}^2).$$

Proof. Let $\Omega = \text{supp}(\varphi)$. Since the embedding $E_q \hookrightarrow L^r(\Omega)$ is compact for any $r \geq 1$ it follows that $u_n \to u$ strongly in $L^r(\Omega)$ and $u_n(x) \to u(x)$ a.e. in Ω. In particular, $g(x,u_n)\varphi \to g(x,u)\varphi$ a.e. in Ω. From (4.3) and $(g_2) - (g_3)$ we can find $C_1, C_2 > 0$ such that

$$|g(x,t)| \leq C_1|t|^{q-1} + C_2|t|^q(e^{\beta t^2} - 1), \quad \forall\,(x,t) \in \mathbb{R}^2 \times \mathbb{R}. \tag{4.16}$$

Thus

$$\int_\Omega |g(x,u_n)\varphi|dx \le C_1\int_\Omega |u_n|^{q-1}|\varphi|dx + C_2\int_\Omega |u_n|^{q-1}|u_n|(e^{\beta u_n^2}-1)|\varphi|dx.$$

Using the Hölder inequality and invoking the elementary inequality

$$(e^{\beta t^2}-1)^r \le (e^{\beta r t^2}-1), \quad \forall\, t\in\mathbb{R},\ \beta>0,\ r\ge 1, \tag{4.17}$$

we get

$$\int_\Omega |g(x,u_n)\varphi|dx \le C\|\varphi\|_\infty \left(\int_\Omega |u_n|^q dx\right)^{\frac{q-1}{q}} \left[1+\left(\int_{\mathbb{R}^2}|u_n|^q(e^{\beta q u_n^2}-1)dx\right)^{\frac{1}{q}}\right].$$

Since (u_n) is bounded, choosing $\beta>0$ sufficiently small, by Theorem 4.2.3 we get

$$\int_\Omega |g(x,u_n)\varphi|dx \le C\left(\int_\Omega |u_n|^q dx\right)^{\frac{q-1}{q}}.$$

On the other hand, there exists $\psi\in L^1(\Omega)$ such that $|u_n|\le|\psi|$ in Ω. Thus, for each $\varepsilon>0$, we can choose a mensurable set $A\subset\Omega$ with $|A|$ sufficiently small such that

$$\int_A |g(x,u_n)\varphi|dx \le C\left(\int_A |\psi|^q dx\right)^{\frac{q-1}{q}} < \varepsilon.$$

Therefore, $(g(x,u_n)\varphi)_n$ is uniformly integrable and the result follows by applying the Vitali's Theorem. □

In the proof of next result we will use that the topology induced by the norm $\|\cdot\|_\tau$ defined in (4.14) is equivalent to the weak topology of E^- on bounded subsets (see Chapter 1). More precisely, if $(u_n)\subset E^-$ is bounded then

$$\|u_n-u\|_\tau\to 0 \quad\Longleftrightarrow\quad u_n\rightharpoonup u\in E^-. \tag{4.18}$$

Lemma 4.3.4. *Assume hypotheses* (4.3), (g_0), (g_2) *and* (g_3). *Then for each* $\mu\in[1,2]$ *the functional* Φ_μ *is* τ*-upper semicontinuous and maps bounded sets into bounded sets. Furthermore,* Φ'_μ *is weakly sequentially continuous on* E_q.

Proof. Let $(u_n)\subset E_q$ be such that $\|u_n-u\|_\tau\to 0$. From (4.18) we have $\|u_n^- - u^-\|_\tau\to 0$ and $\|u_n^+ - u^+\|\to 0$. Consequently $(\|u_n^-\|)_n$ is bounded, hence up to a subsequence, $u_n^-\rightharpoonup u^-$ in E^-. Thus,

$$\|u^-\|\le\liminf_n\|u_n^-\| \quad\text{and}\quad \|u^+\|=\lim_n\|u_n^+\|.$$

Since $G(x,t)\ge 0$, by the Fatou's lemma

$$\int_{\mathbb{R}^2} G(x,u)dx \le \liminf_n \int_{\mathbb{R}^2} G(x,u_n)dx.$$

Therefore,

$$\begin{aligned}\Phi_\mu(u) &= \mu\|u^+\|^2 - \|u^-\|^2 - 2\int_{\mathbb{R}^2} G(x,u)dx \\ &\geq \limsup_n \left(\mu\|u_n^+\|^2 - \|u_n^-\|^2 - 2\int_{\mathbb{R}^2} G(x,u_n)dx\right) \\ &= \limsup_n \Phi_\mu(u_n),\end{aligned}$$

that is, Φ_μ is τ-upper semicontinuous. Now, let $(u_n) \subset E_q$ such that $\|u_n\|_{\mathbf{q}} \leq C_1$. In particular, $\|u_n^-\| \leq C_1$ e $\|u_n^+\| \leq C_1$. Hence,

$$\Phi_\mu(u_n) = \mu\|u_n^+\|^2 - \|u_n^-\|^2 - 2\int_{\mathbb{R}^2} G(x,u_n)dx \leq \mu\|u_n^+\|^2 \leq C_2. \tag{4.19}$$

On the other hand, invoking inequality (4.6), the embedding $E_q \hookrightarrow L^q(\mathbb{R}^2)$ and Corollary 4.2.4 we have

$$2\int_{\mathbb{R}^2} G(x,u_n)dx \leq C_3\int_{\mathbb{R}^2}|u_n|^q dx + C_4\int_{\mathbb{R}^2}|u_n|^q(e^{\beta u_n^2}-1)dx \leq C_5\|u_n\|_{\mathbf{q}}^q \leq C_6.$$

Thus, we get

$$\Phi_\mu(u_n) \geq -\|u_n^-\|^2 - 2\int_{\mathbb{R}^2} G(x,u_n)dx \geq -C_7.$$

This, together with (4.19) implies that $(|\Phi_\mu(u_n)|)_n$ is bounded. Finally, suppose that $u_n \rightharpoonup u = u^- + u^+$ in E_q. Then for any $\varphi \in C_0^\infty(\mathbb{R}^2)$, $(u_n^+,\varphi) \to (u^+,\varphi)$ and $(u_n^-,\varphi) \to (u^-,\varphi)$. Invoking Lemma 4.3.3 we obtain

$$\langle \Phi_\mu'(u_n),\varphi\rangle \to \langle \Phi_\mu'(u),\varphi\rangle, \quad \forall\, \varphi \in C_0^\infty(\mathbb{R}^2),$$

and this conclude the proof. □

Lemma 4.3.5. *Assume* (4.3) *and* $(g_2)-(g_3)$*. Then there are positive constants* η *and* ρ *such that, for any* $1 \leq \mu \leq 2$,

$$\Phi_\mu(u^+) \geq \eta \quad \textit{for all} \quad u^+ \in E^+ \quad \textit{with} \quad \|u^+\|_{\mathbf{q}} = \rho.$$

Proof. Let $\rho > 0$ and $\beta > 0$ such that $\beta\rho^2 S^2 < 4\pi$. If $\|u^+\|_{\mathbf{q}} = \rho$ by Corollary 4.2.4 we get

$$\int_{\mathbb{R}^2}|u^+|^q(e^{\beta(u^+)^2}-1)dx \leq C\rho^q.$$

This together with inequality (4.6) and the embedding $E_q \hookrightarrow L^q(\mathbb{R}^2)$ imply

$$\int_{\mathbb{R}^2} G(x,u^+)dx \leq C_1\|u^+\|_q^q + C_2\int_{\mathbb{R}^2}|u^+|^q(e^{\beta(u^+)^2}-1)dx \leq C_3\rho^q.$$

Thus, we conclude that

$$\Phi_\mu(u^+) = \mu I(u^+) - 2\int_{\mathbb{R}^2} G(x,u^+)dx \geq \rho^2 - C\rho^q.$$

Since $q > 2$, choosing $\|u^+\|_{\mathbf{q}} = \rho$ sufficiently small we obtain the desired result. $\square$

Lemma 4.3.6. *Assume* (g_0)*. Fixed* $u_0^+ \in E^+\backslash\{0\}$*, there exists* $R > 0$ *such that for all* $1 \leq \mu \leq 2$

$$\Phi_\mu(u) \leq 0, \quad \forall\, u \in \partial Q_R, \tag{4.20}$$

where

$$Q_R := \{u = u^- + su_0^+ : s^2\|u_0^+\|^2 + \|u^-\|_-^2 \leq R^2, u^- \in E_q^-, s \geq 0\}.$$

Proof. First we observe that if $s = 0$ then $\Phi_\mu(u) = -\|u^-\|^2 - 2\int G(x,u^-)dx \leq 0$ by (g_0). Thus, in what follows we assume that $s > 0$. Note that $u = u^- + su_0^+ \in \partial Q_R$ with $s > 0$ if and only if

$$s^2\|u_0^+\|^2 + \|u^-\|^2 + \|u^-\|_q^2 = R^2. \tag{4.21}$$

If

$$(\mu+1)s^2\|u_0^+\|^2 + \|u^-\|_q^2 \leq R^2,$$

using that $G(x,u) \geq 0$ together with (4.21) we obtain

$$\begin{aligned}\Phi_\mu(u) &= \mu s^2\|u_0^+\|^2 - \|u^-\|^2 - 2\int_{\mathbb{R}^2} G(x,u)dx \\ &\leq \mu s^2\|u_0^+\|^2 + s^2\|u_0^+\|^2 + \|u^-\|_q^2 - R^2 \\ &= (\mu+1)s^2\|u_0^+\|^2 + \|u^-\|_q^2 - R^2 \leq 0.\end{aligned}$$

On the other hand, if

$$(\mu+1)s^2\|u_0^+\|^2 + \|u^-\|_q^2 \geq R^2, \tag{4.22}$$

and (4.20) does not holds, that is, there exists a sequence $R_n \to +\infty$, $u_n = u_n^- + s_n u_0^+ \in \partial Q_{R_n}$ such that $\Phi_\mu(u_n) > 0$. We consider two cases, to namely:

Case 1: Suppose that $s_n/R_n \to 0$. From (4.22) we get

$$(\mu+1)\left(\frac{s_n}{R_n}\right)^2\|u_0^+\|^2 + \left\|\frac{u_n^-}{R_n}\right\|_q^2 \geq 1,$$

which implies that $\left\|\frac{u_n^-}{R_n}\right\|_q^2 \geq 1/2$ for n large. Since $\frac{s_n}{R_n} \to 0$ we obtain

$$\left\|\frac{u_n^-}{R_n} + \frac{s_n}{R_n}u_0^+\right\|_q \geq \left\|\frac{u_n^-}{R_n}\right\|_q - \left\|\frac{s_n}{R_n}u_0^+\right\|_q \geq \frac{1}{4}.$$

This together with (g_0) imply for n sufficiently large that

$$
\begin{aligned}
0 < \frac{1}{R_n^q}\Phi_\mu(u_n) &\leq \frac{(\mu+1)}{R_n^{q-2}}\left(\frac{s_n}{R_n}\right)^2 \|u_0^+\|^2 - a\left\|\frac{u_n^-}{R_n} + \frac{s_n}{R_n}u_0^+\right\|_q^q \\
&\leq \frac{a}{8^q} - \frac{a}{4^q},
\end{aligned}
$$

which is a contradiction.

Case 2: Suppose that $s_n/R_n \geq c_1 > 0$. From (4.21) we have

$$
\|u_0^+\|^2 + \left\|\frac{u_n^-}{s_n}\right\|_-^2 = \left(\frac{R_n}{s_n}\right)^2 \leq \frac{1}{c_1^2}.
$$

Since E_q^- is reflexive there exists $w \in E_q^-$ such that $\frac{u_n^-}{s_n} \rightharpoonup w$ in E_q^-, $\frac{u_n^-}{s_n} \to w$ in $L^q_{\text{loc}}(\mathbb{R}^2)$ and $\frac{u_n^-}{s_n} \to w$ almost everywhere in $\mathbb{R}^2$. We claim that there exists $c > 0$ such that

$$
\| s_n u_0^+ \|_q \leq c \| u_n^- + s_n u_0^+ \|_q \,. \tag{4.23}
$$

Indeed, otherwise, after take a subsequence we have

$$
\|s_n u_0^+\|_q \geq n\|u_n^- + s_n u_0^+\|_q.
$$

Thus,

$$
\frac{1}{n} \geq \frac{1}{\|u_0^+\|_q}\left\|\frac{u_n^-}{s_n} + u_0^+\right\|_q .
$$

Consequently $\frac{u_n^-}{s_n} \to -u_0^+$ in $L^q(\mathbb{R}^2)$. Therefore $w = -u_0^+$. Since $\left(\frac{u_n^-}{s_n}, u_0^+\right) = 0$ for all $n \in \mathbb{N}$ and the functional $\zeta(u) = (u, u_0^+)$ belongs to E' (the dual of E) we get

$$
0 = \lim_{n\to\infty}\left(\frac{u_n^-}{s_n}, u_0^+\right) = (-u_0^+, u_0^+) = -\|u_0^+\|^2,
$$

which is a contradiction. Therefore (4.23) holds and using (f_0) together with (4.21) we obtain

$$
\begin{aligned}
0 < \Phi_\mu(u_n) &\leq (\mu+1)s_n^2\|u_0^+\|^2 + \|u_n^-\|_q^2 - R_n^2 - a\|u_n^- + s_n u_0^+\|_q^q \\
&\leq (\mu+1)s_n^2\|u_0^+\|^2 - Cs_n^q\|u_0^+\|_q^q.
\end{aligned}
$$

Since $s_n \to \infty$ and $q > 2$ we get another contradiction. $\square$

4.4 (PS) Sequence

We observe that for almost everywhere $\mu \in [1, 2]$, Theorem 4.3.1 provide a (PS) sequence, $(u_n)_n \subset E_q$, for Φ_μ, such that

$$
\sup_n \|u_n\|_{\mathbf{q}} < \infty, \quad \Phi'_\mu(u_n) \to 0 \quad \text{and} \quad \Phi_\mu(u_n) \to c_\mu.
$$

Furthermore, $c_\mu \in [\inf_{S_\rho} \Phi_\mu, \sup_{\overline{Q}_R} \Phi_\mu]$.

We will make use of the following version of the Lions Lemma whose proof can be found in [75], Lemma 3.3.

Lemma 4.4.1. *Let $r > 0$ and $(u_n) \subset E_q$ be bounded. If*

$$\sup_{y\in\mathbb{R}^2} \int_{B(y,r)} |u_n|^2 dx \to 0 \quad as \quad n \to \infty,$$

where $B(y,r) \subset \mathbb{R}^2$ denotes the open ball with center y and radius $r > 0$, then $u_n \to 0$ in $L^t(\mathbb{R}^2)$ for $t > q$. Particularly, if $(u_n) \subset E^+$, then $u_n \to 0$ in $L^t(\mathbb{R}^2)$ for $t > 2$.

Proposition 4.4.2. *Suppose (V_0), (g_0), (g_2), (g_3) and (g_4) are satisfied. For almost everywhere $\mu \in [1,2]$, there is a sequence $(u_n) \subset E_q \setminus \{0\}$ and a constant $\eta > 0$ such that*

$$\eta \leq \Phi_\mu(u_n) \leq c_\mu \quad and \quad \Phi'_{\mu_n}(u_n) = 0.$$

Proof. In view of Lemmas 4.3.2, 4.3.5, 4.3.6, 4.3.4 and Theorem 4.3.1, for almost everywhere $\mu \in [1,2]$, there exists a sequence $(u_n) \subset E_q$ such that

$$\sup_n \|u_n\|_{\mathbf{q}} < \infty, \quad \Phi'_\mu(u_n) \to 0 \quad \text{and} \quad \Phi_\mu(u_n) \to c_\mu \geq \inf_{S_\rho} \Phi_\mu \geq \eta > 0.$$

Hence, for any $v \in E_q$ we have

$$\frac{1}{2}\langle \Phi'_\mu(u_n), v\rangle = \mu(u_n^+, v) - (u_n^-, v) - \int_{\mathbb{R}^2} g(x,u_n)v dx \to 0 \quad \text{as} \quad n \to \infty.$$

Since E_q is a reflexive Banach space, there is a renamed subsequence of (u_n) such that $u_n \rightharpoonup u_\mu$ weakly in E_q, strongly in $L^q_{\text{loc}}(\mathbb{R}^2)$ and almost everywhere in $\mathbb{R}^2$. Therefore, by Lemma 4.3.3 and density arguments as used in Lemma 3.3.4 we conclude that

$$\frac{1}{2}\langle \Phi'_\mu(u_\mu), v\rangle = \mu(u_\mu^+, v) - (u_\mu^-, v) - \int_{\mathbb{R}^2} g(x,u_\mu)v dx = 0, \quad \forall\, v \in E_q,$$

i.e., $\Phi'_\mu(u_\mu) = 0$. Let $H(x,t) := tg(x,t) - 2G(x,t)$ and observe that

$$\int H(x,u_n)dx = \Phi_\mu(u_n) - \frac{1}{2}\langle \Phi'_\mu(u_n), u_n\rangle \to c_\mu,$$

since $(u_n) \subset E_q$ is bounded. Using that $H(x,u_n) \to H(x,u_\mu)$ a.e. in $\mathbb{R}^2$ and $H(x,u_n) \geq -W(x)$ for all $x \in \mathbb{R}^2$, by the Fatou's lemma

$$c_\mu = \lim_n \int_{\mathbb{R}^2} H(x,u_n)dx \geq \int_{\mathbb{R}^2} H(x,u_\mu)dx = \Phi_\mu(u_\mu).$$

It remains to show that $u_\mu \neq 0$ (up to translations). If we prove that $u_\mu^+ \neq 0$, the assertion follows. By Theorem 4.3.1 and Lemma 4.3.5, $c_\mu > 0$. Moreover, by (g_0) we have

$$\mu\|u_n^+\|^2 - \|u_n^-\|^2 = \Phi_\mu(u_n) + 2\int_{\mathbb{R}^2} G(x,u_n)dx \geq \Phi_\mu(u_n),$$

which implies that

$$\liminf_{n\to+\infty}(\mu\|u_n^+\|^2 - \|u_n^-\|^2) \geq c_\mu > 0. \tag{4.24}$$

Fixed $r>0$, if $\sup_{y\in\mathbb{R}^2}\int_{B(y,r)} |u_n^+|^2dx \to 0$ as $n\to\infty$, then $\|u_n^+\|_q \to 0$ by Lemma 4.4.1. Now note that

$$\mu\|u_n^+\|^2 - \|u_n^-\|^2 \leq \mu\|u_n^+\|^2 = \frac{1}{2}\langle \Phi_\mu'(u_n), u_n^+\rangle + \int_{\mathbb{R}^2} g(x,u_n)u_n^+dx.$$

It follows from (4.6) that

$$\begin{aligned}\int_{\mathbb{R}^2} g(x,u_n)u_n^+dx &\leq C_1\int_{\mathbb{R}^2}|u_n|^{q-1}|u_n^+|dx + C_2\int_{\mathbb{R}^2}|u_n|^{q-1}(e^{\beta u_n^2}-1)|u_n^+|dx \\ &\leq \|u_n^+\|_q\left[C_1\left(\int_{\mathbb{R}^2}|u_n|^qdx\right)^{\frac{q-1}{q}} + C_2\left(\int_{\mathbb{R}^2}|u_n|^q(e^{\beta\frac{q}{q-1}u_n^2}-1)dx\right)^{\frac{q-1}{q}}\right].\end{aligned}$$

Since (u_n) is bounded in E_q, we can use the embedding $E_q \hookrightarrow L^q(\mathbb{R}^2)$ and Corollary 4.2.4 with $\beta>0$ sufficiently small to obtain

$$C_1\left(\int_{\mathbb{R}^2}|u_n|^qdx\right)^{\frac{q-1}{q}} + C_2\left(\int_{\mathbb{R}^2}|u_n|^q(e^{\beta\frac{q}{q-1}u_n^2}-1)dx\right)^{\frac{q-1}{q}} \leq C.$$

Hence

$$\mu\|u_n^+\|^2 - \|u_n^-\|^2 \leq \frac{1}{2}\langle \Phi_\mu'(u_n), u_n^+\rangle + C\|u_n^+\|_q,$$

which implies that

$$\limsup_{n\to+\infty}(\mu\|u_n^+\|^2 - \|u_n^-\|^2) \leq 0,$$

which contradicts (4.24). Thus there is a sequence $(y_n)\subset\mathbb{Z}^2$ and a renamed subsequence $(u_n^+)\subset E^+$ such that

$$\int_{B(0,r)}|w_n^+|^2dx = \int_{B(y_n,r)}|u_n^+|^2dx \geq \alpha > 0 \tag{4.25}$$

where $w_n(x) = u_n(x+y_n)$. Since $V(x)$ and $g(x,t)$ are 1-periodic we have that

$$\sup_n \|w_n\|_{\mathbf{q}} < \infty, \quad \Phi_\mu(w_n)\to c_\mu \quad \text{and} \quad \Phi_\mu'(w_n)\to 0.$$

We may assume that $w_n \rightharpoonup w$ and $w_n^+ \rightharpoonup w^+$ in E_q. Furthermore, using the compact embedding $E^+ \hookrightarrow L^2_{\text{loc}}(\mathbb{R}^2)$ together with (4.25) we obtain that $w^+\neq 0$. Consequently w is a nontrivial critical point of Φ_μ. $\square$

In what follows we will show that every approximated (PS) sequence is bounded. To this end we make use of the following auxiliary results.

Lemma 4.4.3. *Suppose that (g_4) is satisfied. If $u = u^- + u^+ \in E_q = E_q^- + E^+$ and $r \in [0,1]$, then*

$$\int_{\mathbb{R}^2} \left[2G(x,u) - 2G(x,ru^+) - \left((r^2+1)u - 2r^2u^+\right) g(x,u)\right] dx \le C,$$

where the constant C does not depend on u, u^+, r.

Proof. Taking $w = u^+$ and $t = u$ in (4.5) we get

$$2G(x,u) - 2G(x,ru^+) - \left((r^2+1)u - 2r^2u^+\right) g(x,u) \le W(x).$$

The desired result follows by integrating the last inequality and using that $W \in L^1(\mathbb{R}^2)$. $\square$

Lemma 4.4.4. *Suppose hypothesis (g_4). Let $(\mu_n)_n \subset [1,2]$ and $u_n = u_n^- + u_n^+ \in E_q$, where $u_n^- \in E_q^-$, $u_n^+ \in E^+$, such that $\Phi'_{\mu_n}(u_n) \to 0$ and $\langle \Phi'_{\mu_n}(u_n), u_n\rangle = o_n(1)$. Then for all $r \in [0,1]$, there is a constant C independent of n, μ_n and r such that*

$$\Phi_{\mu_n}(ru_n^+) + r^2\|u_n^-\|^2 - \Phi_{\mu_n}(u_n) \le C + o_n(1)r^2\|u_n^+\|.$$

Proof. In fact,

$$\begin{aligned}
\Phi_{\mu_n}(ru_n^+) + r^2\|u_n^-\|^2 - \Phi_{\mu_n}(u_n) &= \mu_n r^2\|u_n^+\|^2 - 2\int_{\mathbb{R}^2} G(x,ru_n^+)dx + r^2\|u_n^-\|^2 - \mu_n\|u_n^+\|^2 \\
&\quad + \|u_n^-\|^2 + 2\int_{\mathbb{R}^2} G(x,u_n)dx \\
&= \int_{\mathbb{R}^2} [2G(x,u_n) - 2G(x,ru_n^+)]dx + \mu_n(r^2-1)\|u_n^+\|^2 + (r^2+1)\|u_n^-\|^2.
\end{aligned}$$

Taking $\varphi = (r^2+1)u_n^- - (r^2-1)u_n^+ = (r^2+1)u_n - 2r^2u_n^+$ as a test function we obtain

$$\begin{aligned}
\mu_n(r^2-1)\|u_n^+\|^2 + (r^2+1)\|u_n^-\|^2 &= -\int_{\mathbb{R}^2} \left((r^2+1)u_n - 2r^2u_n^+\right) g(x,u_n)dx \\
&\quad - \langle \Phi'_{\mu_n}(u_n), (r^2+1)u_n - 2r^2u_n^+\rangle.
\end{aligned}$$

Thus,

$$\begin{aligned}
\Phi_{\mu_n}(ru_n^+) + r^2\|u_n^-\|^2 - \Phi_{\mu_n}(u_n) &= \int_{\mathbb{R}^2} \left[2G(x,u_n) - 2G(x,ru_n^+) - \left((r^2+1)u_n - 2r^2u_n^+\right) g(x,u_n)\right] dx \\
&\quad - \langle \Phi'_{\mu_n}(u_n), (r^2+1)u_n - 2r^2u_n^+\rangle \\
&\le C + o_n(1)r^2\|u_n^+\|,
\end{aligned}$$

by the previous lemma. $\square$

Lemma 4.4.5. *Assume (V_0), (4.3) and (g_0)-(g_4). Let $1 \le \mu_n \le 2$ and $(u_n) \subset E_q$ satisfying $|\Phi_{\mu_n}(u_n)| \le C$, $\Phi'_{\mu_n}(u_n) \to 0$ and $\langle \Phi'_{\mu_n}(u_n), u_n\rangle = o_n(1)$.*
Then (u_n) has a bounded subsequence in E_q.

Proof. Let $u_n = u_n^- + u_n^+$ satisfying the hypotheses of lemma. Since $\langle \Phi'_{\mu_n}(u_n), u_n\rangle = o_n(1)$, by assumptions (g_0) and (g_4) (see (4.4)) we get

$$\begin{aligned}\|u_n^+\|^2 - \|u_n^-\|^2 &= \int_{\mathbb{R}^2} g(x,u_n)u_n dx + o_n(1)\\ &\geq 2\int_{\mathbb{R}^2} G(x,u_n)dx - C + o_n(1)\\ &\geq a\|u_n\|_q^q - C + o_n(1).\end{aligned}$$

In particular, $a\|u_n\|_q^q \leq \|u_n^+\|^2 + C$. This together with the triangle inequality and continuous embedding $E_q \hookrightarrow L^q(\mathbb{R}^2)$ imply that

$$\|u_n^-\|_q \leq \|u_n\|_q + \|u_n^+\|_q \leq \|u_n\|_q + C\|u_n^+\|_{\mathbf{q}} = \|u_n\|_q + C\|u_n^+\| \leq C_1\|u_n^+\|^{2/q} + C\|u_n^+\| + C_2.$$

Thus, it suffices to prove that $(\|u_n\|)_n$ is bounded. Suppose that $R_n = \|u_n\| \to \infty$ and let $v_n = u_n/R_n = v_n^- + v_n^+$. Since $\|v_n^+\| \leq 1$ there exists a subsequence still denoted by (v_n^+) such that $v_n^+ \rightharpoonup v$ weakly in E^+, $v_n^+ \to v$ strongly in $L^2_{\text{loc}}(\mathbb{R}^2)$ and almost everywhere in $\mathbb{R}^2$. There are two cases to consider:

Case 1:

$$\sup_{y\in\mathbb{R}^2}\int_{B(y,r)} |v_n^+|^2 dx \to 0 \quad \text{as} \quad n\to\infty.$$

In this case, by Lemma 4.4.1 $v_n^+ \to 0$ in $L^q(\mathbb{R}^2)$. By Lemma 4.4.4

$$\Phi_{\mu_n}(r_n u_n^+) + r_n^2\|u_n^-\|^2 - \Phi_{\mu_n}(u_n) \leq C + o_n(1)r_n^2\|u_n^+\|.$$

Taking $r_n = s/R_n$, for $s > 0$ to be choose later we have

$$\Phi_{\mu_n}(sv_n^+) + s^2\|v_n^-\|^2 \leq C + o_n(1)s^2. \tag{4.26}$$

On the other hand,

$$\begin{aligned}\Phi_{\mu_n}(sv_n^+) + s^2\|v_n^-\|^2 &= \mu_n s^2\|v_n^+\|^2 + s^2\|v_n^-\|^2 - 2\int_{\mathbb{R}^2} G(x,sv_n^+)dx\\ &\geq s^2\|v_n\|^2 - 2\int_{\mathbb{R}^2} G(x,sv_n^+)dx \qquad (4.27)\\ &= s^2 - 2\int_{\mathbb{R}^2} G(x,sv_n^+)dx.\end{aligned}$$

Now using inequalities (4.6) and (4.17) together with Corollary 4.2.4 we get

$$\begin{aligned}\int_{\mathbb{R}^2} G(x,sv_n^+)dx &\leq C_1 s^q\int_{\mathbb{R}^2}|v_n^+|^q dx + C_2 s^q\int_{\mathbb{R}^2}|v_n^+|^{q-1}(e^{\beta s^2(v_n^+)^2}-1)|v_n^+|dx\\ &\leq C_1 s^q\|v_n^+\|_q^q + C_2 s^q\|v_n^+\|_q\left(\int_{\mathbb{R}^2}|v_n^+|^q(e^{\beta\frac{q}{q-1}s^2(v_n^+)^2}-1)dx\right)^{\frac{q-1}{q}}\\ &\leq C_1 s^q o_n(1) + C_3 s^q o_n(1),\end{aligned}$$

for $\beta > 0$ sufficiently small. Therefore we obtain

$$\Phi_{\mu_n}(sv_n^+) + s^2\|v_n^-\|^2 \geq s^2 - s^q o_n(1),$$

which contradicts inequality (4.26) for n and s sufficiently large.

Case 2: There is a sequence $(y_n) \subset \mathbb{Z}^2$ and a renamed subsequence (v_n^+) such that, for all $n \in \mathbb{N}$,

$$\int_{B(y_n,r)} |v_n^+|^2 dx \geq \alpha > 0. \tag{4.28}$$

Let $u_n'(x) := u_n(x - y_n)$ and using that $V(x)$, $g(x,t)$ are 1-periodic we have $\Phi_{\mu_n}(u_n) = \Phi_{\mu_n}(u_n')$. Consequently

$$2\int_{\mathbb{R}^2} \frac{G(x,u_n')}{R_n^2} dx = \mu_n\|v_n^+\|^2 - \|v_n^-\|^2 - \frac{1}{R_n^2}\Phi_{\mu_n}(u_n) \leq C. \tag{4.29}$$

We claim that

$$\int_{\mathbb{R}^2} \frac{G(x,u_n')}{R_n^2} dx \to \infty \quad \text{as} \quad n \to \infty.$$

Indeed, defining $w_n(x) := v_n(x - y_n)$ we have $\|w_n\| = \|v_n\| = 1$. Thus there exists a subsequence of $(w_n) \subset E$ such that $w_n \rightharpoonup w$, $w_n^+ \rightharpoonup w^+$ in E, strongly in $L^2_{\text{loc}}(\mathbb{R}^2)$ and a.e. in $\mathbb{R}^2$. It follows from (4.28) that

$$\int_{B(0,r)} |w_n^+|^2 dx \geq \frac{\alpha}{2} > 0,$$

which implies that $w^+ \neq 0$ and hence $w \neq 0$. Now consider a subset $A \subset \mathbb{R}^2$ with $|A| > 0$ where $w \neq 0$. Since $|u_n'(x)| = |w_n(x)|\|u_n\| \to \infty$ as $n \to \infty$, invoking (g_0) we have $G(x,t)/t^2 \to \infty$ as $t \to \infty$. Thus,

$$\int_{\mathbb{R}^2} \frac{G(x,u_n')}{R_n^2} dx \geq \int_A \frac{G(x,u_n')}{|u_n'|^2}|w_n|^2 dx \to \infty,$$

proving the claim. Now taking the limit in (4.29) as $n \to \infty$ we obtain a contradiction and this concludes the proof. $\square$

4.5 Ground State Solution

Now we are ready to present the proof of Theorem 4.1.3. By applying Theorem 4.3.1, there exists a sequence $(\mu_n)_n \subset (1,2]$, with $\mu_n \to 1$, for which is possible to find a sequence $(u_m^n)_m \subset E_q \setminus \{0\}$ verifying

$$\sup_m \|u_m^n\| < \infty, \quad \Phi_{\mu_n}'(u_m^n) \to 0 \quad \text{and} \quad \Phi_{\mu_n}(u_m^n) \to c_{\mu_n} \quad \text{as} \quad m \to \infty,$$

where $c_{\mu_n} = \inf\limits_{h\in\Gamma} \sup\limits_{\overline{Q}_R} \Phi_{\mu_n}(h(1,u))$. From this, for each $n \in \mathbb{N}$, there exists $m_n \in \mathbb{N}$ such that

$$|\Phi_{\mu_n}(u_{m_n}^n) - c_{\mu_n}| \leq \frac{1}{n}, \quad |\langle \Phi_{\mu_n}'(u_{m_n}^n), u_{m_n}^n\rangle| \leq \frac{1}{n} \quad \text{and} \quad \|\Phi_{\mu_n}'(u_{m_n}^n)\| \leq \frac{1}{n}, \quad \forall\, n \in \mathbb{N}.$$

Hereafter, we denote $u^n_{m_n}$ by u_n, hence we can rewritten the above limits as follows

$$|\Phi_{\mu_n}(u_n) - c_{\mu_n}| \le \frac{1}{n}, \quad |\langle \Phi'_{\mu_n}(u_n), u_n\rangle| \le \frac{1}{n} \quad \text{and} \quad \|\Phi'_{\mu_n}(u_n)\| \le \frac{1}{n}, \quad \forall\, n \in \mathbb{N}.$$

Since $0 < c_1 \le c_{\mu_n} \le c_2$ for all $n \in \mathbb{N}$, we can assume that $0 < \frac{c_1}{2} \le \Phi_{\mu_n}(u_n) \le c_2 + \frac{1}{n}$.

By Lemma 4.4.5, after a renamed subsequence $u_n \rightharpoonup u$ weakly in E_q, strongly in $L^q_{\text{loc}}(\mathbb{R}^2)$ and a.e. in $\mathbb{R}^2$. Since,

$$0 = \frac{1}{2}\langle \Phi'_{\mu_n}(u_n), \varphi\rangle = \mu_n(u_n^+, \varphi) - (u_n^-, \varphi) - \int_{\mathbb{R}^2} g(x, u_n)\varphi dx, \quad \forall\, \varphi \in C_0^\infty(\mathbb{R}^2), \tag{4.30}$$

taking the limit and using Lemma 4.3.3 together with density arguments as in Lemma 3.3.4 we get $\Phi'(u) = 0$. We claim that $u \not\equiv 0$. Indeed, taking $\varphi = u_n^+$ in (4.30) and combining the inequalities (4.16) and (4.17) we get

$$\begin{aligned}
\mu_n\|u_n^+\|^2 &= \int_{\mathbb{R}^2} g(x, u_n)u_n^+ dx \\
&\le C_1 \int_{\mathbb{R}^2} |u_n|^{q-1}|u_n^+| dx + C_2 \int_{\mathbb{R}^2} |u_n|^{q-1}(e^{\beta u_n^2} - 1)|u_n^+| dx \\
&\le \left(\int_{\mathbb{R}^2} |u_n^+|^q dx\right)^{\frac{1}{q}} \left[C_1 \left(\int_{\mathbb{R}^2} |u_n|^q dx\right)^{\frac{q-1}{q}} + C_2 \left(\int_{\mathbb{R}^2} |u_n|^q (e^{\beta \frac{q}{q-1} u_n^2} - 1) dx\right)^{\frac{q-1}{q}}\right] \\
&= \|u_n^+\|_q \left[C_1 \|u_n\|_q^{q-1} + C_2 \left(\int_{\mathbb{R}^2} |u_n|^q (e^{\beta \frac{q}{q-1} u_n^2} - 1) dx\right)^{\frac{q-1}{q}}\right].
\end{aligned}$$

Since (u_n) is bounded, we can choose $\beta > 0$ sufficiently small and invoking Corollary 4.2.4 to obtain

$$\|u_n\|_q^{q-1} + C_2 \left(\int_{\mathbb{R}^2} |u_n|^q (e^{\beta \frac{q}{q-1} u_n^2} - 1) dx\right)^{\frac{q-1}{q}} \le C.$$

As a consequence we get

$$\frac{c_1}{2} \le \Phi_{\mu_n}(u_n) = \mu_n\|u_n^+\|^2 - \|u_n^-\|^2 - \int_{\mathbb{R}^2} G(x, u_n) dx \le \mu_n \|u_n^+\|^2 \le C\|u_n^+\|_q. \tag{4.31}$$

If for $r > 0$ fixed,

$$\sup_{y \in \mathbb{R}^2} \int_{B(y,r)} |u_n^+|^2 dx \to 0, \quad \text{as} \quad n \to \infty,$$

then by Lemma 4.4.1 we get $u_n^+ \to 0$ in $L^q(\mathbb{R}^2)$ and this contradicts (4.31). Consequently this does not occur, that is, there exists a sequence $(y_n) \subset \mathbb{Z}^2$ and $\alpha > 0$ such that

$$\int_{B(y_n,r)} |u_n^+|^2 dx \ge \alpha > 0.$$

Now proceeding as in the end of the proof of Proposition 4.4.2, after to translation we have that $u^+ \not\equiv 0$ and hence $u \not\equiv 0$.

To finish the proof we recall that $\mathcal{M}$ denotes the set of solutions of $(\mathcal{P}_g)$. Observe that if

$u \in \mathcal{M}$ then by (4.4)

$$\Phi(u) = \Phi(u) - \langle \Phi'(u), u \rangle = \int_{\mathbb{R}^2} H(x,u)dx \geq -\int_{\mathbb{R}^2} W(x)dx.$$

Consequently, the number

$$\beta := \inf_{u \in \mathcal{M}} \Phi(u)$$

is well defined. Now consider a sequence $(u_n) \subset \mathcal{M}$ such that $\Phi(u_n) \to \beta$. By Lemma 4.4.5, the sequence (u_n) is bounded in E_q. Thus, after a renamed subsequence we may assume that $u_n \rightharpoonup u$ in E_q, $u_n \to u$ in $L^q_{\text{loc}}(\mathbb{R}^2)$ and a.e in $\mathbb{R}^2$. Hence we have

$$0 = \frac{1}{2}\langle \Phi'(u_n), \varphi \rangle = (u_n^+, \varphi) - (u_n^-, \varphi) - \int_{\mathbb{R}^2} f(x, u_n)\varphi dx,$$

and passing the limit we get $\Phi'(u) = 0$, i.e., $u \in \mathcal{M}$. On the other hand,

$$\Phi(u_n) = \Phi(u_n) - \frac{1}{2}\langle \Phi'(u_n), \varphi \rangle = \int_{\mathbb{R}^2} H(x, u_n)dx.$$

By the Fatou's Lemma we have

$$\begin{aligned}
\Phi(u) + \int_{\mathbb{R}^2} W(x)dx &= \int_{\mathbb{R}^2} [H(x,u) + W(x)]dx \\
&\leq \liminf \int_{\mathbb{R}^2} [H(x,u_n) + W(x)]dx \\
&= \beta + \int_{\mathbb{R}^2} W(x)dx \\
&\leq \Phi(u) + \int_{\mathbb{R}^2} W(x)dx.
\end{aligned}$$

Therefore, $\beta = \Phi(u)$. To complete the proof, we observe that by the first step we obtain a weak solution $u \in E_q$ of

$$-\Delta u = -V(x)u + g(x,u), \quad x \in \mathbb{R}^2.$$

Invoking Lemma 4.2.2 we obtain that $u \in H^1_{\text{loc}}(\mathbb{R}^2)$. Moreover, $f(x) = -V(x)u + g(x,u) \in L^p(\mathbb{R}^2)$ for any $p \geq q > 2$. Using L^p-regularity theory we obtain $u \in C^1(\mathbb{R}^2)$. Using the Harnack inequality (see [33], Theorem 8.17) for $p > q$ we get

$$\|u\|_{L^\infty B(y,1)} \leq C\|u\|_{L^p(B(y,2))}, \quad \forall\, y \in \mathbb{R}^2, \tag{4.32}$$

where $C > 0$ is a constant independent of $y \in \mathbb{R}^2$. Now we fix $\varepsilon > 0$. Since $u \in L^p(\mathbb{R}^2)$ we have $\lim_{R\to+\infty} \int_{|x|\geq R} |u|^p dx = 0$. Thus we can take $R > 0$ sufficiently large such that $\int_{|x|\geq R} |u|^p dx < \varepsilon$. Then for $y \in \mathbb{R}^2$ with $|y| = R + 2$ we have

$$\|u\|_{L^\infty(B(y,1))} \leq C_2\varepsilon,$$

by (4.32). Since $\varepsilon > 0$ is arbitrary we conclude that $|u(x)| \to 0$ as $|x| \to \infty$ and this completes the proof of Theorem 4.1.3.

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