Universidade Federal da Paraíba
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Degenerations of classical square matrices and their determinantal structure

por

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João Pessoa – PB
Março/2017

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sob orientação do

Prof. Dr. Aron Simis

Tese apresentada ao Corpo Docente do Programa em Associação de Pós Graduação em Matemática UFPB/UFCG, como requisito parcial para obtenção do título de Doutor em Matemática.

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Aos meus pais,
aos meus irmãos
e à Laura

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A Deus.

Aos meus pais, pelo amor e dedicação.

Resumo

Nesta tese nós estudamos certas degenerações/especializações da matriz quadrada genérica sobre um corpo k de característica zero juntamente com suas principais estruturas subjacentes, tais como o determinante da matriz, o ideal gerado por suas derivadas parciais, o mapa polar definido por essas derivadas, a matriz Hessiana e o ideal dos menores submáximos da matriz. Os tipos de degenerações da matriz quadrada genérica consideradas aqui são: (1) degeneração por "clonagem" (repetição de uma variável); (2) substituição de um subconjunto de entradas por zeros, em uma disposição estratégica; (3) outras degenerações dos tipos acima partindo de certas especializaçẽs da matriz quadrada genérica, tais como a matriz genérica simétrica e a matriz quadrada genérica de Hankel. O foco em todas essas degenerações é nos invariantes descritos acima, com destaque para o comportamento homaloide do determinante da matriz. Para tal, empregamos ferramentas provenientes álgebra comutativa, com ênfase na teoria de ideais e na teoria de sizigia.

Palavras-chave: Matriz genérica, matriz simétrica, matriz de Hankel, matriz Hessiana, determinante homaloide, ideal gradiente, posto linear.

Abstract

In this thesis we study certain degenerations/specializations of the generic square matrix over a field k of characteristic zero along its main related structures, such the determinant of the matrix, the ideal generated by its partial derivative, the polar map defined by these derivatives, the Hessian matrix and the ideal of submaximal minors of the matrix. The degeneration types of the generic square matrix considered here are: (1) degeneration by "cloning" (repeating) a variable; (2) replacing a subset of entries by zeros, in a strategic layout; (3) further degeneration of the above types starting from certain specializations of the generic square matrix, such as the generic symmetric matrix and the generic square Hankel matrix. The focus in all these degenerations is in the invariants described above, highlighting on the homaloidal behavior of the matrix determinant. For this, we employ tools coming from commutative algebra, with emphasis on ideal theory and syzygy theory.

Keywords: Generic matrix, symmetric matrix,Hankel matrix, Hessian matrix , homaloidal determinant, gradient ideal, linear rank.

Contents

Introduction

Let $\mathbb{P}^n = \mathbb{P}^n_k$ denote the projective space over a field k. The subject of the Cremona transformations of $\mathbb{P}^n$ is a classical chapter of algebraic geometry, yet the classification of such maps is presently poorly understood. Indeed, the group of Cremona transformations of $\mathbb{P}^n$ is only reasonably understood for $n \leq 2$ and depends on results that have been proved only recently. An important class of Cremona maps of $\mathbb{P}^n$ arises from the so-called polar maps, i.e., rational maps whose coordinates are the partial derivatives of a homogeneous polynomial f in the homogeneous coordinate ring $R = k[x_0, \ldots, x_n]$ of $\mathbb{P}^n$. A homogeneous polynomial $f \in R$ for which the polar map is a Cremona map is called *homaloidal* – though more often this designation applies to the corresponding hypersurface rather than to f itself. This terminology stems from an older terminology for certain plane linear systems ("homaloidal nets").

In the projective plane, a smooth conic, the union of three distinct non-concurrent lines and the union of a smooth conic with one of its tangent lines are the only reduced homaloidal curves. This result has been established by Dolgachev in [11] and has thereafter given several different proofs. It is worth emphasizing that the core of Dolgachev's result is the fact the degree of a homaloidal polynomial in $k[x_0, x_1, x_2]$ is at most 3. Alas, for $n \geq 3$ there is no counterpart to this result. In fact, recently families of irreducible homaloidal hypersurfaces of degree d in projective space $\mathbb{P}^n$, for any $n \geq 3$ and any $d \geq 2n - 3$ have been produced in [5]. These examples are strongly based on the theory of normal scrolls and their particular projections. While the existence of these families shows that there are plenty of homaloidal polynomials around, including polynomials whose degree is not bounded in terms of the embedding dimension, it does not make the structure of the Cremona group any easier to grasp. In fact, other families have been described afterwards (see [23], [32]) and still the group nature of Cremona transformations is largely untouched, even for $n = 3$.

Perhaps because the structure of the Cremona group is so intricate and rich, little attention has been given in the past to the commutative algebra lying on the foreground of the subject. To our knowledge, the first incursions bringing out sufficiently organized ideas in this direction are [9], [14], [24], [34], [36], to mention a few. A thorough examination of the base ideal of a plane Cremona map was first given in [19].

We are mainly interested in the search of irreducible homaloidal polynomials in the environment consisting of determinants of square matrices with homogeneous entries of the same degree. In this context, [5] introduced an infinite family of determinantal homaloidal hypersurfaces based on a certain degeneration of a generic Hankel matrix.

Recently, [28] and [29] considered structured square matrices whose entries are indeterminates over a field k and looked at the corresponding determinants from the viewpoint of homaloidness. Inspired by the latter results our main object in this thesis is that of the determinant of a square matrix with entries which are either variables in a polynomial ring over a field k or zeros. In other words, we look at degenerations/specializations of classical square matrices. The term "classical" refers to the generic and symmetric generic matrices and the generic Hankel matrix. Of course, they are all specializations of the generic model, yet their study may use distinctive methods. The goal of this work is to understand the effect of such degeneration/specialization on the properties of the underlying ideal theoretic structures. Some of the prominent gadgets envisaged are the full determinant of the matrix, its gradient (Jacobian) ideal, the associated polar map and its image, and the ideal of submaximal minors.

The degeneration types of the generic square matrix considered here are: (1) degeneration by "cloning" (repeating) a variable; (2) replacing a subset of entries by zeros, in a strategic layout; (3) further degeneration of certain specializations of the generic matrix, such as those of the generic symmetric square matrix and the generic square Hankel matrix. A minuscule part is dedicated to the case of a so called r-leap generic catalecticant matrix, mainly in the way of suggesting the multiple possibilities in sight.

By and large we consider this work as an overture towards the problems envisaged, with the hope that a lot more be dealt with in the near future. The colorful and varied situations in which the present degenerations appear constitute a true source of problems in commutative algebra and algebraic geometry.

As in [29], we will assume that the base field has characteristic zero, because the study of a polar map in characteristic zero is primevally driven by the properties of the Hessian determinant $H(f)$ of f, the reason being the classically known criterion for the dominance of the polar map in terms of the non-vanishing of the corresponding Hessian determinant.

A more detailed description of the contents of this thesis goes as follows.

The first chapter explains the required background from commutative algebra incident to homaloidal maps.

The second chapter focuses on degenerations of generic square matrix and consists of two sections. In the first section we deal with the degeneration referred to as (*diagonal*) *cloning*. Here we show that the determinant f of the diagonally cloned matrix is homaloidal. For this, we first prove that the Jacobian ideal J has maximal linear rank and that the Hessian determinant $H(f)$ of f does not vanish. We then move on to the ideal I of the submaximal minors. It will be a Gorenstein ideal of codimension 4, a fairly immediate consequence of specialization. Yet, showing it is a prime ideal required a result of D. Eisenbud drawn upon the 2-generic property of the generic matrix – we believe that R/I is actually a normal ring, but only cared to prove it in the case $m = 3$. It turns out that I is the minimal primary component of J and the latter defines a double structure on the variety $V(I)$ with a unique embedded component, which is a linear subspace of codimension $4m - 5$. An additional result is that the rational map defined by the submaximal minors is birational onto its image. We give the explicit form of the image through its defining equation, a determinantal expression of degree $m - 1$. From the purely algebraic side, the bearing is to the proof that the ideal J is not a reduction of its minimal component I.

In the subsequent section we replace generic entries by zeros in a strategic position to be explained in the text. For any given $1 \leq r \leq m - 2$, the degenerated matrix will acquire $\binom{r+1}{2}$ zeros. We prove that the ideal J still has maximal linear rank. However, the Hessian determinant vanishes and, in fact, the image of the polar map is proved to have dimension $m^2 - r(r + 1) - 1$. Moreover, its homogeneous coordinate ring is a ladder determinantal Gorenstein ring. In the sequel, as in the previous section, our drive is the nature of the ideal I of submaximal minors. It will be still Gorenstein ideal of codimension 4, but is not anymore prime for all values of r. Using the result of D. Eisenbud mentioned before we conclude that the bound $\binom{r+1}{2} \leq m - 3$ implies the primeness of I. Others algebraic results come naturally while trying to uncover the nature of the relationship between the three ideals J, I and $J : I$. The main geometric result of this section is that the submaximal minors define a birational map onto its image and the latter is a cone over the polar variety of f with vertex cut by $\binom{r+1}{2}$ coordinate hyperplanes.

In the third chapter we study in parallel degenerations of the generic symmetric matrix. We first look at cloning degenerations that preserve the symmetric structure, in which case there are two natural possibilities: cloning along the main diagonal or else along the anti-diagonals. In this work one considers only the first of these possibilities. At the other end, we study the degeneration by one single zero, obtaining parallel results to the generic case. The results for both degeneration setups follow the same pattern but parts of the proofs may differ – such as is the case of conveying the primeness of the ideal of submaximal minors. We close the chapter with a remark on other prospective degenerations preserving the symmetry.

The fourth chapter is entirely dedicated to degenerations of the generic Hankel matrix, most emphatically those obtained by zeros. Because of its importance throughout areas of mathematics other than algebra and the fact that they are the extreme case of symmetric matrices, Hankel matrices seem like a good choice for the chapter. Besides, it takes up the paradigm of the generic case developed in [29] in such a way as to throw some light back at the generic case at least in questions of ideal theoretic nature.

Here we first develop the preliminaries of properties of Hankel matrices that do not depend on degeneration. For example, a major tool is the celebrated Gruson–Peskine size-independent lemma. Of some usefulness is also the behavior of the gradient ideal of the Hankel determinant under homomorphisms. Quite surprisingly, as much as in the case of the so-called subHankel matrices considered in [5], the Hessian of any degeneration by zeros does not vanish – the relative surprise coming from the fact that the degeneration of either the generic or the generic symmetric matrices have vanishing Hessians. By and large, however, the subHankel case is atypical. Thus, for example, we show that for all other cases of the $m \times m$ Hankel matrix degenerations by r zeros the ideal of submaximal minors is prime and further so is the ideal of the $(m-2)$-minors if $r \leq m-4$. This outcome rests strongly on the 1-generic property of the generic Hankel matrix via a result of D. Eisenbud. The classical result about generic Hankel maximal minors tells us that their defining polynomial relations are generated by Grassmann–Plücker relations and the latter define a Cohen–Macaulay ideal. Moreover, the entire presentation ideal of the corresponding Rees algebra is of fiber type and Cohen–Macaulay as well. For the case of degeneration by zeros there appear also polynomial relations of degree 3 as minimal generators – a phenomenon as yet not fully understood. We conjecture that the ideal of polynomial relations is still Cohen–Macaulay and the presentation ideal of the corresponding Rees algebra is of fiber type and Cohen–Macaulay, just as in the generic case.

We conclude the thesis with a very short chapter on the potential results for the cloning degeneration of the so-called catalecticants matrices with a leap possibly higher than 1. Even in the generic case the ideal theoretic behavior of these matrices is largely unknown (see [29]). The examples displayed here suggest a sufficiently complex situation justifying tackling it elsewhere.

Chapter 1

Preliminaries

The aim of this chapter is to introduce the algebraic tool required throughout. The overall objective is understand the effect of particular specializations of the classical square matrices on properties of the underlying ideal theoretic structures. For this, we need recap some notions and tools from ideal theory in birational maps, as well recall the ideal theoretic structures of the classical square matrices.

1.1 Recap of ideal theory

In this section we review a few basic concepts from commutative ring theory that will be used throughout this work.

Let R denote a commutative Noetherian ring and let $I \subset R$ stand for an ideal. Let $\mathcal{S}_R(I)$ and $\mathcal{R}_R(I)$ denote the symmetric and the Rees algebra of I, respectively. The literature on this is quite extensive (see [21], [38] for general guidance). Recall that there is structural graded R-algebra surjective homomorphism $\mathcal{S}_R(I) \twoheadrightarrow \mathcal{R}_R(I)$. One says that the ideal I is of *linear type* if this map is injective.

Let $(R, \mathfrak{m})$ denote a Notherian local ring and its maximal ideal (respectively, a standard graded ring over a field and its irrelevant ideal). For an ideal $I \subset \mathfrak{m}$ (respectively, a homogeneous ideal $I \subset \mathfrak{m}$), the *special fiber* of I is the ring $\mathcal{R}(I)/\mathfrak{m}\mathcal{R}(I)$. Note that this is an algebra over the residue field of R.

The (Krull) dimension of this algebra is called the *analytic spread* of I and is denoted $\ell(I)$.

Quite generally, given $J \subset I$ ideals in a ring R, J is said to be a *reduction* of I if there exists an integer $n \geq 0$ such that $I^{n+1} = JI^n$. Obviously, any ideal is a reduction of itself, but one is interested in "minimal" possible reductions.

Note that if $JI^n = I^{n+1}$, then for all positive integers m, $I^{m+n} = JI^{m+n-1} = \cdots = J^m I^n$. Thus, if $J \subset I$ is a reduction, there exists an integer n such that for all $m \geq 1$, $I^{m+n} \subset J^m$. In particular, an ideal shares the same radical with all its reductions. Therefore, they share the same set of minimal primes and have the same codimension.

A reduction J of I is called *minimal* if no ideal strictly contained in J is a reduction of I. The *reduction number* of I with respect to a reduction J is the minimum integer n such that $JI^n = I^{n+1}$. It is denoted by $\mathrm{red}_J(I)$. The (absolute) *reduction number* of I is defined as $\mathrm{red}(I) = \min\{\mathrm{red}_J(I) \mid J \subset I \text{ is a minimal reduction of } I\}$.

If $R/\mathfrak{m}$ is infinite, then every minimal reduction of I is minimally generated by exactly $\ell(I)$ elements. In particular, every reduction of I contains a reduction generated by $\ell(I)$ elements.

In this context, the following invariants are related in the case of $(R, \mathfrak{m})$:

$$\mathrm{ht}(I) \leq \ell(I) \leq \min\{\mu(I), \dim(R)\},$$

where $\mu(I)$ stands for the minimal number of generators of I. If the rightmost inequality turns out to be an equality, one says that I has *maximal analytic spread*. By and large, the ideals considered in this work will have $\dim R \leq \mu(I)$, hence being of maximal analytic spread means in this case that $\ell(I) = \dim R$.

Suppose now that R is a standard graded over a field k and I is generated by $n+1$ forms of a given degree s. In this case, I is more precisely given by means of a free graded presentation

$$R(-(s+1))^{\ell} \,\oplus\, \sum_{j\geq 2} R(-(s+j)) \xrightarrow{\varphi} R(-s)^{n+1} \longrightarrow I \longrightarrow 0$$

for suitable shifts $-(s+j)$ and rank ℓ. Of much interest in this work is the value of ℓ, so let us state in which for. We call the image of $R(-(s+1))^{\ell}$ by φ the *linear part of* φ – often denoted φ_1. One says that the rank of φ_1 is the *linear rank of* φ (or of I for that matter) and that φ has *maximal liner rank* provided its linear rank is n (=rank(φ)). Clearly, the latter condition is trivially satisfied if $\varphi = \varphi_1$, in which case I is said to have *linear presentation* (or is *linearly presented*).

Note that φ is a graded matrix whose columns generate the (first) *syzygy module of* I (corresponding to the given choice of generators) and a *syzyzy* of I is an element of this module – that is, a linear relation with coefficients in R on the chosen generators. In this context, φ_1 can be taken as the submatrix of φ whose entries are linear forms of the standard graded ring R. Thus, the linear rank is the rank of the matrix of the linear syzygies.

Recall the notion of the initial ideal of a polynomial ideal over a field. For this one has to introduce a monomial order in the polynomial ring. Given such a monomial order, if $f \in R$ we denote by $\text{in}(f)$ the initial term of f and by $\text{in}(I)$ the ideal generated by the initial terms of the elements of I – this ideal is called the *initial ideal* of I. The following result in this regard is very useful: $\dim R/I = \dim R/\text{in}(I)$.

A subset I' of the ideal I is called a *Gröbner basis* of I if $\text{in}(I)$ is generated by the initial terms of the elements of I'. For the general theory of monomial ideals and Gröbner basis we refer to [20].

For a Noetherian local ring $(R, \mathfrak{m})$, the depth of R (the maximum length of a regular sequence in the maximal ideal of R) is at most $\dim R$. The ring $(R, \mathfrak{m})$ is called (local) *Cohen–Macaulay ring* if its depth is equal to its dimension. More generally, a Noetherian ring is called *Cohen–Macaulay ring* if all of its localizations at maximal ideals are (local) Cohen–Macaulay rings .

The Gorenstein rings are particular examples of Cohen-Macaulay rings. A n-dimensional Noetherian local ring $(R, \mathfrak{m})$ is said a (local) *Gorenstein ring* if it is a Cohen-Macaulay ring and $\dim_k \text{Ext}^n_R(R/\mathfrak{m}, R) = 1$, i.e., it is Cohen-Macaulay of type 1. A Noetherian ring is a *Gorenstein ring* if its localization at every maximal ideal is a (local) Gorenstein ring.

As a final point, when a cyclic R-module R/I is Cohen–Macaulay (respectively, Gorenstein) we by abuse say that the ideal I is Cohen–Macaulay (respectively, Gorenstein).

1.2 Recap of classical matrices

In this section we recall results and properties of some classical matrices. Throughout for a matrix $\mathcal{M}$ the notation $I_r(\mathcal{M})$ denotes the ideal generated by the r-minors of $\mathcal{M}$. We started, defining the following notion introduced in [16]:

Definition 1.2.1. Let $\mathcal{M}$ denote a $m \times n$ matrix of linear forms ($m \leq n$). We say that $\mathcal{M}$ is *l-generic* for some integer $1 \leq l \leq m$ if even after arbitrary invertible row and column operations, any l of the linear forms $\mathcal{M}_{i,j}$ in $\mathcal{M}$ are linearly independent.

It was proved in [16] that the $m \times n$ generic matrix (whose entries are distinct variables in a polynomial ring) is m-generic, in particular, this matrix is l-generic for any $1 \leq l \leq m$. The $m \times n$ generic matrix is a extreme case of a $m \times n$ *generic catalecticant*, whose definition is given below:

Definition 1.2.2. Let $m \geq 2$ and $1 \leq r \leq m+1$ be given integers. Let $R = [x_1, \ldots, x_s]$ be a polynomial ring with $s = (m-1)r + n$. The *r-leap $m \times n$ generic catalecticant* is the matrix

$$
\mathcal{C}_{m,r} = \begin{pmatrix} x_1 & x_2 & x_3 & \cdots & x_n \\ x_{r+1} & x_{r+2} & x_{r+3} & \cdots & x_{r+n} \\ x_{2r+1} & x_{2r+2} & x_{2r+3} & \cdots & x_{2r+n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_{(m-1)r+1} & x_{(m-1)r+2} & x_{(m-1)r+3} & \cdots & x_{(m-1)r+n} \end{pmatrix} \tag{1.1}
$$

The extreme values $r = 1$ and $r = m$ yield, respectively, the generic Hankel matrix and the generic matrix.

A crucial property proved in [16] is that the generic Hankel matrix of arbitrary size $m \times n$ is 1-generic. Using this property, it has been proved in [33] that the generic catalecticant matrix of arbitrary size and leap is 1-generic. This notion implies, in particular, that the determinant of a square such a matrix is irreducible (cf. the next theorem). Although the generic symmetric matrix is not an extreme case of a catalecticant, it is also 1-generic, which has been proven in [16, Proposition 4.4].

The following is a result of Eisenbud ([16, Theorem 2.1]) relating the property l-generic of the matrix and the primeness of the r-minors ideal, for certain values of r. With an appropriate language adaptation of the original notation, the part of the result we need reads as follows:

Proposition 1.2.3. *One is given integers $1 \leq w \leq v$. Let $\mathcal{G}$ denote the $w \times v$ generic matrix over a ground field. Let $\mathcal{M}'$ denote a $w \times v$ matrix of linear forms in the entries of $\mathcal{G}$ and let further $\mathcal{M}$ denote a $w \times v$ matrix of linear forms in the entries of $\mathcal{M}'$. Let there be given an integer $k \geq 1$ such that $\mathcal{M}'$ is a $(w-k)$-generic matrix and such that the vector space spanned by the entries of $\mathcal{M}$ has codimension at most $k-1$ in the vector space spanned by the entries of $\mathcal{M}'$. Then the ideal $I_{k+1}(\mathcal{M})$ is prime.*

It is known that the ideals of k-minors of the generic and generic symmetric matrices are prime ideals of codimension $(m-k+1)(m-k+1)$ and $\binom{m-k+2}{2}$ (see [13] and [27], respectively). In particular, the ideal of submaximal minors of a square generic matrix (respectively, a square generic symmetric matrix) is a prime ideal of codimension 4 (respectively, a prime ideal of codimension 3), regardless of the size of the matrix. An important property of the latter ideals is that they are (linearly presented) of linear type (see [25] for the generic case and [26] for the generic symmetric case).

In turn, the result [16, Proposition 4.3] ensures that the ideal of k-minors of the $m \times m$ generic Hankel matrix $\mathcal{H}_m$ is also prime and of codimension $2m - 2k + 1$. The proof of this result uses an important property of the Hankel matrix first made explicit in the work of Gruson and Peskine [18] (for yet another proof of this property see [8]), namely:

Theorem 1.2.4. *Consider the generic Hankel matrix of arbitrary size which we write as follows:*

$$\mathcal{H}_{j,2m-j} := \begin{pmatrix} x_1 & x_2 & \dots & x_{2m-j} \\ x_2 & x_3 & \dots & x_{m+1} \\ \vdots & \vdots & \dots & \vdots \\ x_j & x_{m+1} & \dots & x_{2m-1} \end{pmatrix},$$

where $j < 2m$. Then $I_t(\mathcal{H}_{j,2m-j}) = I_t(\mathcal{H}_{t,2m-t})$ for all $t \leq j \leq 2m - t$.

This property allows to reduce to the case of maximal minors. In this case, one may use the fact that the Hankel matrix specializes to the well-known specialization using only $2m-2k+1$ variables. In particular, in the case of the square Hankel matrix $\mathcal{H}_m$, its ideal of submaximal minors coincides with the ideal of maximal minors of $\mathcal{H}_{m-1,,m+1}$.

Several properties relating $I_{m-1}(\mathcal{H}_m)$ to the *gradient ideal* J generated by partial derivatives of $\det(\mathcal{H}_m)$ are proved in [29]. One of these is that $I_{m-1}(\mathcal{H}_m)$ is the minimal component of the primary decomposition of J. In [28] it was proved that the *linear rank* of J is 3 and conjectured that J is of linear type.

The following result, originally proved in [17] and independently obtained in [[28], Proposition 5.3.1], plays a role in looking at degenerations by zeros:

Proposition 1.2.5. *Let $\mathcal{M}$ denote a square matrix over $R = k[x_0, \dots, x_n]$ satisfying the following requirements:*

- *Every entry of $\mathcal{M}$ is either 0 or x_i for some $i = 1, \dots, n$;*
- *Any variable x_i appears at most once on every row or column.*

Let $f := \det(\mathcal{M})$. Then, for each $i = 0, \dots, n$, the partial derivative of f with respect to x_i is the sum of the (signed) cofactors of the entry x_i, in all its appearances as an entry of $\mathcal{M}$.

1.3 Homaloidal polynomials

Let k be an arbitrary field. For the purpose of the full geometric picture we may assume k to be algebraically closed. We denote by $\mathbb{P}^n = \mathbb{P}^n_k$ the n-th projective space, where we naturally assume throughout that $n \geq 1$.

A rational map $\mathcal{F} : \mathbb{P}^n \dashrightarrow \mathbb{P}^m$ is defined by $m+1$ forms $\mathbf{f} = \{f_0, \dots, f_m\} \subset R := k[\mathbf{x}] = k[x_0, \dots, x_n]$ of the same degree $d \geq 1$, not all null. We often write $\mathcal{F} = (f_0 : \cdots : f_m)$ to underscore the projective setup. Any rational map an without lost of generality be brought to satisfy the condition that $\gcd\{f_0, \ldots, f_m\} = 1$ (in the geometric terminology, $\mathcal{F}$ *has no fixed part*). The common degree d of the forms f_j is the *degree* of $\mathcal{F}$ and the ideal $I_{\mathcal{F}} = (f_0, \ldots, f_m)$ is called the *base ideal* of $\mathcal{F}$.

The *image* of $\mathcal{F}$ is the projective subvariety $W \subset \mathbb{P}^m$ whose homogeneous coordinate ring is the k-subalgebra $k[\mathbf{f}] \subset R$ after degree renormalization. Write $S := k[\mathbf{f}] \simeq k[\mathbf{y}]/I(W)$, where $I(W) \subset k[\mathbf{y}] = k[y_0, \ldots, y_m]$ is the homogeneous defining ideal of the image in the embedding $W \subset \mathbb{P}^m$.

We say that $\mathcal{F}$ is *birational onto the image* if there is a rational backwards $\mathbb{P}^m \dashrightarrow \mathbb{P}^n$ such that the residue class $\mathbf{g} = \{g_0, \ldots, g_n\} \subset S$ of a set of defining coordinates do not simultaneously vanish and satisfy the relations

$$(g_0(\mathbf{f}) : \cdots : g_n(\mathbf{f})) = (x_0 : \cdots : x_n),\ (f_0(\mathbf{g}) : \cdots : f_m(\mathbf{g})) = (y_0 : \cdots : y_m)$$

When $m = n$ and $\mathcal{F}$ is a birational map of $\mathbb{P}^n$, we say that $\mathcal{F}$ is a *Cremona map*. An important class of Cremona maps of $\mathbb{P}^n$ comes off the so-called *polar maps*, that is, rational maps whose coordinates are the partial derivatives of a homogeneous polynomial f in the ring $R = k[x_0, \ldots, x_n]$. More precisely:

Definition 1.3.1. Let $f \in k[\mathbf{x}] = k[x_0, \ldots, x_n]$ be a square homogeneous polynomial of degree $d \geq 2$. let

$$I = \left(\frac{\partial f}{\partial x_0}, \ldots, \frac{\partial f}{\partial x_n}\right) \subset k[\mathbf{x}]$$

the so called *gradient* ideal of f. The rational map $\mathcal{P}_f = \left(\frac{\partial f}{\partial x_0} : \cdots : \frac{\partial f}{\partial x_n}\right)$ is called *the polar map* defined by f. If $\mathcal{P}_f$ is birational one says that f is *homaloidal*.

We note that the image of this map is the subvariety on the target whose homogeneous coordinate ring is given by the k-subalgebra $k[\partial f/\partial x_0, \ldots, \partial f/\partial x_n] \subset k[\mathbf{x}]$. We call this variety of the *polar variety*.

A notable parallel construction is that of the *dual variety* $V(f)^*$ to the hypersurface $V(f)$. The homogeneous coordinate ring of its embedding in the usual dual coordinates can here be dealt with through the k-subalgebra

$$\frac{k[\partial f/\partial x_0, \ldots, \partial f/\partial x_n]}{(f) \cap k[\partial f/\partial x_0, \ldots, \partial f/\partial x_n]} \subset \frac{k[\mathbf{x}]}{(f)}.$$

The following birationality criterion will be very useful in this work:

Theorem 1.3.2. [[12], Theorem 3.2] *Let $\mathcal{F} : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ be a rational map given by $n+1$ forms $\mathbf{f} = \{f_0, \ldots, f_n\}$ of a fixed degree. If $\dim(k[\mathbf{f}]) = n+1$ and the linear rank of the base ideal $I_{\mathcal{F}}$ is n (maximal possible) then $\mathcal{F}$ is birational onto its image.*

It is a classical result in characteristic zero that $\dim(k[\mathbf{f}]) = n+1$ coincides with the rank of the Jacobian matrix of $\mathbf{f} = \{f_0, \ldots, f_n\}$. Assuming that the ground field has characteristic zero, if the Hessian determinant $H(f)$ does not vanish and the linear rank of the gradient ideal of f is maximal, then f is homaloidal.

It has been noted in [29] that the Hessian determinants of the generic and symmetric matrices are nonzero and that the determinants of these matrices are homaloidal polynomials.

Chapter 2

Degenerations of the generic square matrix

Throughout this chapter all matrices will have as entries either variables in a polynomial ring over a field or zeros, viewed as particular degenerations of the generic square matrix. Our goal is to understand the effect of such degenerations on the properties of underlying ideal theoretic structures such as the full determinant of the matrix, its gradient ideal, the associated polar map and its image, and the ideal of submaximal minors.

The degeneration types considered are: (1) degeneration by "cloning" (repeating a variable); (2) replacing a subset of entries with zeros, in a strategic layout;

2.1 Degeneration by cloning

More broadly, let $A = (a_{i,j})_{1\leq i,j\leq m}$ denote a $m \times m$ matrix where $a_{i,j}$ is either a variable on a ground polynomial ring $R = k[\mathbf{x}]$ over a field k or $a_{i,j} = 0$. Among the simplest specializations is going modulo a binomial of the shape $a_{i,j} - a_{i',j'}$, where $a_{i,j} \neq a_{i',j'}$ and $a_{i',j'} \neq 0$. The idea is to replace a certain nonzero entry $a_{i',j'}$ (variable) by a different entry $a_{i,j}$ (possibly zero), keeping $a_{i,j}$ as it was – somewhat like cloning a variable, but keeping the mold. This has the effect of dropping the number of times a variable appears as an entry and often also dropping the total number of variables. It also seems natural to expect that the new cloning position should matter as far as the finer properties of the ideals are concerned.

The main object of this section is the behavior of the generic square matrix under this sort of cloning degeneration. We will use the following notation for the generic square matrix:

$$\mathcal{G} := \begin{pmatrix} x_{1,1} & x_{1,2} & \ldots & x_{1,m-1} & x_{1,m} \\ x_{2,1} & x_{2,2} & \ldots & x_{2,m-1} & x_{2,m} \\ \vdots & \vdots & \ldots & \vdots & \vdots \\ x_{m-1,1} & x_{m-1,2} & \ldots & x_{m-1,m-1} & x_{m-1,m} \\ x_{m,1} & x_{m,2} & \ldots & x_{m,m-1} & x_{m,m} \end{pmatrix}, \tag{2.1}$$

where the entries are independent variables over a field k.

Now, we distinguish essentially two sorts of cloning: the one that replaces an entry $x_{i',j'}$ by another entry $x_{i,j}$ such that $i \neq i'$ and $j \neq j'$, and the one in which this replacement has either $i = i'$ or $j = j'$.

In the situation of the second kind of cloning, by an obvious elementary operation and renaming of variables (which is possible since the original matrix is generic), one can assume that the matrix is the result of replacing a variable by zero on a generic matrix. Such a procedure is recurrent, letting several entries being replaced by zeros. The resulting matrix along with its main properties will be studied in the Section 2.2.

Therefore, this section will deal exclusively with the first kind of cloning – which, for emphasis, could be refereed to as *diagonal cloning*. Up to elementary row/column operations and renaming of variables, we assume once for all that the diagonally cloned matrix has the shape

$$\mathcal{GC} := \begin{pmatrix} x_{1,1} & x_{1,2} & \ldots & x_{1,m-1} & x_{1,m} \\ x_{2,1} & x_{2,2} & \ldots & x_{2,m-1} & x_{2,m} \\ \vdots & \vdots & \ldots & \vdots & \vdots \\ x_{m-1,1} & x_{m-1,2} & \ldots & x_{m-1,m-1} & x_{m-1,m} \\ x_{m,1} & x_{m,2} & \ldots & x_{m,m-1} & x_{m-1,m-1} \end{pmatrix}, \tag{2.2}$$

where the entry $x_{m-1,m-1}$ has been cloned as the (m,m)-entry of the $m \times m$ generic matrix. Beyond a mere expression, the cloning imagery will remind us of a close interchange between properties associated to one or the other copy of the same variable in its position as an entry of the matrix.

2.1.1 Polar behavior

Throughout we set $f := \det(\mathcal{GC})$ and let $J = J_f$ denote the ideal generated by the partial derivatives of f with respect to the variables of R, the polynomial ring in the entries of $\mathcal{GC}$ over a ground field k. For convenience we call J the *gradient ideal* of f – wishfully to distinguish it from the widely accepted terminology *Jacobian ideal* when thinking of the partial derivatives modulo f.

Sticking to a more geometric terminology, we let the term *polar* be associated with the behavior of the gradient ideal as defining a rational map and the geometry of this map.

Theorem 2.1.1. *Consider the diagonally cloned matrix as in* (2.2). *One has:*

(i) *f is irreducible.*

(ii) *The Hessian determinant $H(f)$ does not vanish.*

(iii) *The linear rank of the gradient ideal of f is $m^2 - 2$ (maximum possible).*

(iv) *f is homaloidal.*

Proof. (i) We induct on m, the initial step of the induction being subsumed in the general step.

By the Laplace expansion along the first row, one sees that

$$f = x_{1,1}\Delta_{1,1} + g,$$

where $\Delta_{1,1}$ is the determinant of the $(m-1) \times (m-1)$ cloned generic matrix obtained from $\mathcal{GC}$ by omitting the first row and the first column. Note that both $\Delta_{1,1}$ and g belong to the subring $k[x_{1,2}, \ldots, x_{m,m-1}]$. Thus, in order to show that f is irreducible it suffices to prove that it is a primitive polynomial (of degree 1) in $k[x_{1,2}, \ldots, x_{m,m-1}][x_{1,1}]$.

Now, on one hand, $\Delta_{1,1}$ is irreducible by the inductive hypothesis. Therefore, it is enough to see that $\Delta_{1,1}$ is not factor of g. For this, one verifies their initial terms in the revlex monomial order: $\mathrm{in}(\Delta_{1,1}) = x_{2,m-1} \ldots x_{m-1,2}x_{m-1,m-1}$ and $\mathrm{in}(g) = \mathrm{in}(f) = x_{1,m-1} \cdots x_{m-1,1}x_{m-1,m-1}$.

An alternative more sophisticated argument is to use that the ideal P of submaximal minors has codimension 4, as shown independently in the Theorem 2.1.2 (i) below. Since $P = (J, \Delta_{m,m})$, as pointed out in the proof of the latter proposition, then J has codimension at least 3. Therefore, the ring $R/(f)$ is locally regular in codimension one, so it must be normal. But f is homogeneous, hence irreducible.

(ii) Set $\mathbf{v} := \{x_{1,1}, x_{2,2}, x_{3,3}, \ldots, x_{m-1,m-1}\}$ for the set of variables along the main diagonal. We argue by a specialization procedure, namely, consider the ring endomorphism φ of R by mapping any variable in $\mathbf{v}$ to itself and by mapping any variable off $\mathbf{v}$ to zero. Clearly, it suffices to show that by applying φ to the entries of the Hessian matrix $\mathcal{H}(f)$ the resulting matrix has a nonzero determinant.

Note that the partial derivative of f with respect to any $x_{i,i} \in \mathbf{v}$ coincides with the (signed) cofactor of $x_{i,i}$, for $i \leq m-2$, while for $i = m-1$ it is the sum of the respective (signed) cofactors of $x_{i,i}$ corresponding to its two appearances.

By expanding each such a cofactor according to the Leibniz rule it is clear that it has a unique (nonzero) term whose support lies in $\mathbf{v}$ and, moreover, the remaining terms have degree at least 2 in the variables off $\mathbf{v}$. Observe that in the two cofactors of $x_{m-1,m-1}$ the terms whose support lies in $\mathbf{v}$ coincide.

Now, for $x_{i,j} \notin \mathbf{v}$, without exception, the corresponding partial derivative coincides with the (signed) cofactor. By a similar token, the Leibniz expansion of this cofactor has no term whose support lies in $\mathbf{v}$ and has exactly one nonzero term of degree 1 in the variables off $\mathbf{v}$.

By the preceding observation, applying φ to any second partial derivative of f will return zero or a monomial supported on the variables in $\mathbf{v}$. Thus, the entries of the specialized Hessian matrix of f, which we will denote $\mathcal{H}'$, are zeros or monomials supported on the variables in $\mathbf{v}$.

To see that the determinant of this matrix $\mathcal{H}'$ is nonzero, consider the Jacobian matrix of the set of partial derivatives $\{f_v \,|\, v \in \mathbf{v}\}$ with respect to the variables in $\mathbf{v}$. Let M_0 denote the specialization of this Jacobian matrix by φ, considered as a corresponding submatrix of $\mathcal{H}'$. Up to permutation of rows and columns of $\mathcal{H}'$, we may write

$$\mathcal{H}' = \begin{pmatrix} M_0 & N \\ P & M_1 \end{pmatrix},$$

for suitable M_1. Now, by the way the second partial derivatives of f specialize via φ, as explained above, one must have $N = P = 0$. Therefore, $\det(\mathcal{H}') = \det(M_0)\det(M_1)$, so it remains to prove the nonvanishing of these two subdeterminants.

Now the first block is the Hessian matrix of the form

$$g := \left(\prod_{i=1}^{m-2} x_{i,i}\right) x_{m-1,m-1}^2.$$

This is the product of the generators of the k-subalgebra

$$k[x_{1,1}, \ldots, x_{m-2,m-2}, x_{m-1,m-1}^2] \subset k[x_{1,1}, \ldots, x_{m-2,m-2}, x_{m-1,m-1}].$$

Clearly these generators are algebraically independent over k, hence the subalgebra is isomorphic to a polynomial ring itself. Then g becomes the product of the variables of a polynomial ring over k. This is a classical homaloidal polynomial, hence we are done for the first matrix block.

As for the second block, by construction it has exactly one nonzero entry on each row and each column. Therefore, it has a nonzero determinant. $\square$

(iii) Let $f_{i,j}$ denote the $x_{i,j}$-derivative of f and let $\Delta_{j,i}$ stand for the (signed) cofactor of the (i,j)-entry of the matrix $\mathcal{GC}$. Note that, $\Delta_{m-1,m-1}$ (respectively, $\Delta_{m,m}$)

is the cofactor of $x_{m-1,m-1}$ in the $(m-1,m-1)$-entry of $\mathcal{GC}$ (respectively, in the (m,m)-entry of $\mathcal{GC}$) and $\Delta_{j,i}$ is the cofactor of $x_{i,j}$ on $\mathcal{GC}$, for any (i,j) other than $(m-1,m-1)$ and (m,m).

The classical Cauchy cofactor formula

$$\mathcal{GC}\cdot \mathrm{adj}(\mathcal{GC}) = \mathrm{adj}(\mathcal{GC})\cdot \mathcal{GC} = \det(\mathcal{GC})\,\mathbb{I}_m \tag{2.3}$$

yields by expansion a set of linear relations involving the (signed) cofactors of $\mathcal{GC}$:

$$\sum_{j=1}^{m} x_{i,j}\Delta_{j,k} = 0, \text{ for } 1\leq i\leq m-1 \text{ and } 1\leq k\leq m-2 \ (k\neq i); \tag{2.4}$$

$$\sum_{j=1}^{m-1} x_{m,j}\Delta_{j,k} + x_{m-1,m-1}\Delta_{m,k} = 0, \text{ for } 1\leq k\leq m-2; \tag{2.5}$$

$$\sum_{j=1}^{m} x_{i,j}\Delta_{j,i} = \sum_{j=1}^{m} x_{i+1,j}\Delta_{j,i+1}, \text{ for } 1\leq i\leq m-3; \tag{2.6}$$

$$\sum_{i=1}^{m} x_{i,k}\Delta_{j,i} = 0, \text{ for } 1\leq j\leq m-3 \text{ and } j<k\leq j+2; \tag{2.7}$$

$$\sum_{i=1}^{m} x_{i,m-1}\Delta_{m-2,i} = 0; \tag{2.8}$$

$$\sum_{i=1}^{m-1} x_{i,m}\Delta_{m-2,i} + x_{m-1,m-1}\Delta_{m-2,m} = 0. \tag{2.9}$$

Since $f_{i,j} = \Delta_{j,i}$ for every $(i,j)\neq(m-1,m-1)$ and the above relations do not involve $\Delta_{m-1,m-1}$ or $\Delta_{m,m}$, then they give linear syzygies of the partial derivatives of f.

In addition, (2.3) yields the following linear relations:

$$\sum_{j=1}^{m-1} x_{m-1,j}\Delta_{j,m} + x_{m-1,m}\Delta_{m,m} = 0; \tag{2.10}$$

$$\sum_{i=1}^{m-2} x_{i,m}\Delta_{m-1,i} + x_{m-1,m}\Delta_{m-1,m-1} + x_{m-1,m-1}\Delta_{m-1,m} = 0; \tag{2.11}$$

$$\sum_{i=1}^{m-1} x_{i,m-1}\Delta_{m,i} + x_{m,m-1}\Delta_{m,m} = 0; \tag{2.12}$$

$$\sum_{j=1}^{m-2} x_{m,j}\Delta_{j,m-1} + x_{m,m-1}\Delta_{m-1,m-1} + x_{m-1,m-1}\Delta_{m,m-1} = 0; \tag{2.13}$$

$$\sum_{j=1,\, j\neq m-1}^{m} x_{m-1,j}\Delta_{j,m-1} + x_{m-1,m-1}\Delta_{m-1,m-1} = \sum_{j=1}^{m} x_{m-2,j}\Delta_{j,m-2}; \tag{2.14}$$

$$\sum_{j=1}^{m-1} x_{m,j}\Delta_{j,m} + x_{m-1,m-1}\Delta_{m,m} = \sum_{j=1}^{m} x_{m-2,j}\Delta_{j,m-2}. \tag{2.15}$$

As $f_{m-1,m-1} = \Delta_{m-1,m-1} + \Delta_{m,m}$, adding (2.10) to (2.11), (2.12) to (2.13) and (2.14) to (2.15), respectively, outputs three new linear syzygies of the partial derivatives of f. Thus one has a total of $(m-2)(m-1)+(m-3)+2(m-2)+3 = m^2-2$ linear syzygies of J.

It remains to show that these are independent. For this we order the set of partial derivatives $f_{i,j}$ in accordance with the following ordered list of the entries $x_{i,j}$:

$$x_{1,1}, x_{1,2}, \ldots, x_{1,m} \rightsquigarrow x_{2,1}, x_{2,2}, \ldots, x_{2,m} \rightsquigarrow \ldots \rightsquigarrow x_{m-2,1}, x_{m-2,2}, \ldots, x_{m-2,m},$$
$$x_{m-1,1}, x_{m,1} \rightsquigarrow x_{m-1,2}, x_{m,2} \rightsquigarrow \ldots \rightsquigarrow x_{m-1,m-1}, x_{m,m-1} \rightsquigarrow x_{m-1,m}.$$

Here we traverse the entries along the matrix rows, left to right, starting with the first row and stopping prior to the row having $x_{m-1,m-1}$ as an entry; then start traversing the last two rows along its columns top to bottom, until exhausting all variables.

We now claim that, the above sets of linear relations can be grouped into the following block matrix of linear syzygies:

$$\left(\begin{array}{cccc|ccccc|ccc}
\varphi_1 & & & & & & & & & & & \\
0 & \varphi_2 & \ldots & & & & & & & & & \\
\vdots & \vdots & \ddots & \vdots & & & & & & & & \\
0 & 0 & \ldots & \varphi_{m-2} & & & & & & & & \\
\hline
0_2^{m-1} & 0_2^m & \ldots & 0_2^m & \varphi_2^3 & & & & & & & \\
0_2^{m-1} & 0_2^m & \ldots & 0_2^m & 0_2^2 & \varphi_3^4 & & & & & & \\
\vdots & \vdots & \ldots & \vdots & \vdots & \vdots & \ddots & & & & & \\
0_2^{m-1} & 0_2^m & \ldots & 0_2^m & 0_2^2 & 0_2^2 & \ldots & \varphi_{(m-2)}^{(m-1)} & & & & \\
0_2^{m-1} & 0_2^m & \ldots & 0_2^m & 0_2^2 & 0_2^2 & \ldots & 0_2^2 & \varphi_{(m-1)}^m & & & \\
\hline
0_1^{m-1} & 0_1^m & \ldots & 0_1^m & 0_1^2 & 0_1^2 & \ldots & 0_1^2 & 0_1^2 & x_{m-1,m} & x_{m,m-1} & x_{m-1,m-1} \\
0_1^{m-1} & 0_1^m & \ldots & 0_1^m & 0_1^2 & 0_1^2 & \ldots & 0_1^2 & 0_1^2 & 2x_{m-1,m-1} & 0 & x_{m,m-1} \\
0_1^{m-1} & 0_1^m & \ldots & 0_1^m & 0_1^2 & 0_1^2 & \ldots & 0_1^2 & 0_1^2 & 0 & 2x_{m-1,m-1} & x_{m-1,m}
\end{array}\right)$$

We next explain the blocks of the above matrix:

- φ_1 is the matrix obtained from the transpose $\mathcal{GC}^t$ of $\mathcal{GC}$ by omitting the first column;

- $\varphi_2, \ldots, \varphi_{m-2}$ are each a copy of $\mathcal{GC}^t$ (up to column permutation);

- $\varphi_r^{r+1} = \begin{pmatrix} x_{m-1,r} & x_{m-1,r+1} \\ x_{m,r} & x_{m,r+1)} \end{pmatrix}$, for $r = 2, \ldots, m-2$; $\varphi_{(m-1)}^{m} = \begin{pmatrix} x_{m-1,m-1} & x_{m-1,m} \\ x_{m,m-1} & x_{m-1,m-1} \end{pmatrix}$;
- Each 0 under φ_1 is a zero block of the size $m \times (m-1)$ and each 0 under φ_i is a zero block of the size $m \times m$ for $i = 2, \ldots, m-3$;
- 0_r^c denotes a zero block of size $r \times c$, for $r = 1, 2$ and $c = 2, m-1, m$.

Next we justify why these blocks make up (linear) syzygies.

First, as already observed, the relations (2.4) through (2.15) yield linear syzygies of the partial derivatives of f. Setting $k = 1$ in the relations (2.4) and (2.5) they can be written as $\sum_{j=1}^{m} x_{i,j} f_{1,j} = 0$ for all $i = 2, \ldots, m-1$ and $\sum_{j=1}^{m-1} x_{m,j} f_{1,j} + x_{m-1,m-1} f_{1,m} = 0$, respectively. Ordering the set of partial derivatives $f_{i,j}$ as explained before, the coefficients of these relations form the first matrix above

$$\varphi_1 := \begin{pmatrix} x_{2,1} & x_{3,1} & \cdots & x_{m-1,1} & x_{m,1} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ x_{2,m-1} & x_{3,m-1} & \cdots & x_{m-1,m-1} & x_{m,m-1} \\ x_{2,m} & x_{3,m} & \cdots & x_{m-1,m} & x_{m-1,m-1} \end{pmatrix}$$

Note that φ_1 coincides indeed with the submatrix of $\mathcal{GC}^t$ obtained by omitting its first column.

Getting φ_k, for $k = 2, \ldots, m-2$, is similar, namely, use again relations (2.4) and (2.5) retrieving a submatrix of $\mathcal{GC}^t$ excluding the kth column and replacing it with an extra column that comes from relation (2.6) taking $i = k-1$.

Continuing, for each $r = 2, \ldots, m-2$ the block φ_r^{r+1} comes from the relation (2.7) (setting $j = r-1$) and φ_{m-1}^{m} comes from the relations (2.8) and (2.9). Finally, the lower right corner 3×3 block of the matrix of linear syzygies comes from the three last relations obtained by adding (2.10) to (2.11), (2.12) to (2.13) and (2.14) to (2.15).

This proves the claim about the large matrix above. Counting through the sizes of the various blocks, one sees that this matrix is $(m^2-1) \times (m^2-2)$. Omitting its first row obtains a block-diagonal submatrix of size $(m^2-2) \times (m^2-2)$, where each block has nonzero determinant. Thus, the linear rank of J attains the maximum.

(iv) By (ii) the polar map of f is dominant. Since the linear rank is maximum by (iii), one can apply the Theorem 1.3.2 to conclude that f is homaloidal. □

2.1.2 Primality

In this part we study the nature of the ideal of submaximal minors (cofactors) of $\mathcal{GC}$. As previously, J denotes the gradient ideal of $f = \det(\mathcal{GC})$.

Theorem 2.1.2. *Consider the matrix* $\mathcal{GC}$ *as in* (2.2), *with* $m \geq 3$. *Let* $P := I_{m-1}(\mathcal{GC})$ *denote its ideal of* $(m-1)$*-minors. Then*

(i) P *is a Gorenstein prime ideal of codimension* 4.

(ii) J *has codimension* 4 *and* P *is the minimal primary component of* J *in* R.

(iii) J *defines a double structure on the variety defined by* P, *with a unique embedded component and the latter is a linear space of codimension* $4m-5$.

(iv) *Letting* $\mathbb{D}_{i,j}$ *denote the cofactor of the* (i,j)*-entry of the generic matrix* $(y_{i,j})_{1\leq i,j\leq m}$, *the* $(m-1)$*-minors* $\mathbf{\Delta} = \{\Delta_{i,j}\}$ *of* $\mathcal{GC}$ *define a birational map* $\mathbb{P}^{m^2-2} \dashrightarrow \mathbb{P}^{m^2-1}$ *onto a hypersurface of degree* $m-1$ *with defining equation* $\mathbb{D}_{m,m} - \mathbb{D}_{m-1,m-1}$ *and inverse map defined by the linear system spanned by* $\widetilde{\mathbb{D}} := \{\mathbb{D}_{i,j} \,|\, (i,j) \neq (m,m)\}$ *modulo* $\mathbb{D}_{m,m} - \mathbb{D}_{m-1,m-1}$.

(v) J *is not a reduction of* P.

Proof. (i) Let $\mathcal{P}$ denote the ideal of submaximal minors of the fully generic matrix (2.1). The linear form $x_{m,m} - x_{m-1,m-1}$ is regular on the corresponding polynomial ambient and also modulo $\mathcal{P}$ as the latter is prime and generated in degree $m-1 \geq 2$. Since $\mathcal{P}$ is a Gorenstein ideal of codimension 4 by a well-known result ("Scandinavian complex"), then so is P.

In order to prove primality, we first consider the case $m = 3$ which seems to require a direct intervention. We will show more, namely, that R/P is normal – and, hence a domain as P is a homogeneous ideal. Since R/P is a Gorenstein ring, it suffices to show that R/P is locally regular in codimension one. For this consider the Jacobian matrix of P:

$$\begin{pmatrix}
x_{2,2} & -x_{2,1} & 0 & -x_{1,2} & x_{1,1} & 0 & 0 & 0 \\
x_{2,3} & 0 & -x_{2,1} & -x_{1,3} & 0 & x_{1,1} & 0 & 0 \\
0 & x_{2,3} & -x_{2,2} & 0 & -x_{1,3} & x_{1,2} & 0 & 0 \\
x_{3,2} & -x_{3,1} & 0 & 0 & 0 & 0 & -x_{1,2} & x_{1,1} \\
x_{2,2} & 0 & -x_{3,1} & 0 & x_{1,1} & 0 & -x_{1,3} & 0 \\
0 & x_{2,2} & -x_{3,2} & 0 & x_{1,2} & 0 & 0 & -x_{1,3} \\
0 & 0 & 0 & x_{32} & -x_{3,1} & 0 & -x_{2,2} & x_{21} \\
0 & 0 & 0 & x_{2,2} & x_{2,1} & -x_{3,1} & -x_{2,3} & 0 \\
0 & 0 & 0 & 0 & 2x_{2,2} & -x_{3,2} & 0 & -x_{2,3}
\end{pmatrix}.$$

Direct inspection yields that the following pure powers are (up to sign) 4-minors of this matrix: $x_{1,3}^4$, $x_{2,1}^4$, $x_{2,2}^4$, $x_{2,3}^4$, $x_{3,1}^4$ and $x_{3,2}^4$. Therefore, the ideal of 4-minors of the

Jacobian matrix has codimension at least $6 = 4 + 2$, thus ensuring that R/P satisfies the condition (R_1).

For $m \geq 4$ we will apply the Theorem 1.2.3 in the case where $\mathcal{M}' = \mathcal{G}$ is an $m \times m$ generic matrix and $\mathcal{M} = \mathcal{GC}$ is the cloned generic matrix as in the statement. In addition, we take $k = m - 2$, so $k + 1 = m - 1$ is the size of the submaximal minors. Since $m \geq 4$ and the vector space codimension in the theorem is now 1, one has $1 \leq m - 3 = k - 1$ as required. Finally, the $m \times m$ generic matrix is m-generic as explained in [16, Examples, p. 548]; in particular, it is $2 = m - (m-2)$-generic. The theorem applies to give that the ideal $P = I_{m-1}(\mathcal{GC})$ is prime.

(ii) By item (i), P is a prime ideal of codimension 4. We first show that $\operatorname{codim}(J : P) > 4$, which ensures that the radical of the unmixed part of J has no primes of codimension < 4 and coincides with P – in particular, J will turn out to have codimension 4 as stated.

For this note that $P = (J, \Delta_{m,m})$, where $\Delta_{m,m}$ denotes the cofactor of the (m,m)-entry of $\mathcal{GC}$. From the Cauchy cofactor formula we read the following relations:

$$\begin{aligned} \sum_{j=1}^{m} x_{k,j}\Delta_{j,m} &= 0, \text{ for } k = 1, \ldots, m-1; \\ \sum_{j=1}^{m-1} x_{m,j}\Delta_{j,m} + x_{m-1,m-1}\Delta_{m,m} &= \sum_{j=1}^{m} x_{1,j}\Delta_{j,1}; \\ \sum_{i=1}^{m} x_{i,k}\Delta_{m,i} &= 0, \text{ for } k = 1, \ldots, m-1. \end{aligned}$$

Since the partial derivative $f_{i,j}$ of f with respect to the variable $x_{i,j}$ is the (signed) cofactor $\Delta_{j,i}$, with the single exception of the partial derivative with respect to the variable $x_{m-1,m-1}$, we have that the entries of the m-th column and the m-th row all belong to the ideal $J : \Delta_{m,m} = J : P$. In particular, the codimension of $J : P$ is at least 5, as needed.

In addition, since P has codimension 4 then $J : P \not\subset P$. Picking an element $a \in J : P \setminus P$ shows that $P_P \subset J_P$. Therefore P is the unmixed part of J.

To prove that P is actually the entire minimal primary component of J we argue as follows. In addition, also note that $P = (J, \Delta_{m-1,m-1})$, where $\Delta_{m-1,m-1}$ denotes the cofactor of the $(m-1,m-1)$-entry of $\mathcal{GC}$. From the cofactor identity we read the following relations:

$$\sum_{j=1, j\neq m-1}^{m} x_{k,j}\Delta_{j,m-1} + x_{k,m-1}\Delta_{m-1,m-1} = 0, \text{ for } k = 1, \ldots, m, \ (k \neq m-1);$$

$$\begin{aligned}\sum_{j=1,j\neq m-1}^{m} x_{m-1,j}\Delta_{j,m-1} + x_{m-1,m-1}\Delta_{m-1,m-1} &= \sum_{j=1}^{m} x_{1,j}\Delta_{j,1};\\ \sum_{i=1,i\neq m-1}^{m} x_{i,k}\Delta_{m-1,i} + x_{m-1,k}\Delta_{m-1,m-1} &= 0, \text{ for } k=1,\ldots,m\ (k\neq m-1);\end{aligned}$$

Then as above we have that the entries of the $(m-1)$-th column and the $(m-1)$-th row belong to the ideal $J:\Delta_{m-1,m-1} = J:P$.

From this, the variables of the last two rows and of the last two columns of $\mathcal{GC}$ multiply P into J. As is clear that P is contained in the ideal generated by these variables it follows that $P^2 \subset J$ (of course, this much could eventually be verified by inspection). Therefore, the radical of J – i.e., the radical of the minimal primary part of J – is P.

(iii) By (ii), P is the minimal component of a primary decomposition of J. We claim that $J:P$ is generated by the $4m-5$ entries of $\mathcal{GC}$ off the upper left submatrix of size $(m-2)\times(m-2)$. Let I denote the ideal generated by these entries.

As seen in the previous item, $I \subset J:P$. We now prove the reverse inclusion by writing $I = I' + I''$ as sum of two prime ideals, where I' (respectively, I'') is the ideal generated by the variables on the $(m-1)$-th row and on the $(m-1)$-th column of $\mathcal{GC}$ (respectively, by the variables on the m-th row and on the m-th column of $\mathcal{GC}$). Observe that the cofactors $\Delta_{i,j} \in I''$ for all $(i,j) \neq (m,m)$ and $\Delta_{i,j} \in I'$ for all $(i,j) \neq (m-1,m-1)$. Clearly, then $\Delta_{m,m} \notin I''$ and $\Delta_{m-1,m-1} \notin I'$.

Let $b \in J:P = J:\Delta_{m,m}$, say,

$$\begin{aligned} b\,\Delta_{m,m} &= \sum_{(i,j)\neq(m-1,m-1)} a_{i,j} f_{i,j} + a f_{m-1,m-1}\\ &= \sum_{(i,j)\neq(m-1,m-1)} a_{i,j}\Delta_{j,i} + a(\Delta_{m-1,m-1} + \Delta_{m,m}) \end{aligned} \tag{2.16}$$

for certain $a_{i,j}, a \in R$. Then

$$(b-a)\Delta_{m,m} = \sum_{(i,j)\neq(m-1,m-1)} a_{i,j}\Delta_{j,i} + a\Delta_{m-1,m-1} \in I''.$$

Since I'' is a prime ideal and $\Delta_{m,m} \notin I''$, we have $c := b-a \in I''$. Substituting for $a = b-c$ in (2.16) gives

$$(-b+c)\Delta_{m-1,m-1} = \sum_{(i,j)\neq(m-1,m-1)} a_{i,j}\Delta_{j,i} - c\Delta_{m,m} \in I'.$$

By a similar token, since $\Delta_{m-1,m-1} \notin I'$, then $-b+c \in I'$. Therefore

$$b = c - (-b+c) \in I'' + I' = I,$$

as required.

In particular, $J : P$ is a prime ideal and is the only embedded prime of J. As pointed out, $P \subset J : P$, hence $P^2 \subset J$. Therefore, J defines a double structure on the irreducible variety defined by P, with a unique embedded component – the latter being a linear variety of codimension $4m-5$.

(iv) By Theorem 2.1.1 (ii), the polar map is dominant, i.e., the partial derivatives of f generate a subalgebra of maximum dimension ($= m^2-1$). Since $J \subset P$ is an inclusion in the same degree, the subalgebra generated by the submaximal minors has dimension m^2-1 as well. On the other hand, since P is a specialization from the generic case, it is linearly presented. Therefore, the minors define a birational map (see Theorem 1.3.2) onto a hypersurface. To get the defining equation of the latter we proceed as follows.

Write $\Delta_{j,i}$ for the cofactor of the (i,j)-entry of $\mathcal{GC}$. It suffices to show that $\mathbb{D}_{m,m} - \mathbb{D}_{m-1,m-1}$ belongs to the kernel of the k-algebra map

$$\psi : k[y_{i,j} \,|\, 1 \leq i,j \leq m] \to k[\underline{\Delta}] = k[\Delta_{i,j} \,|\, 1 \leq i,j \leq m],$$

as it is clearly an irreducible polynomial.

Consider the following well-known matrix identity

$$\mathrm{adj}(\mathrm{adj}(\mathcal{GC})) = f^{m-2} \cdot \mathcal{GC}, \tag{2.17}$$

where $\mathrm{adj}(M)$ denotes the transpose of the matrix of cofactors of a square matrix M. Looking at the $(m-1,m-1)$-entry and the (m,m)-entry of the right-hand side matrix we obviously see the same element, namely, $f^{m-2}x_{m-1,m-1}$.

As to the entries of the matrix on the left-hand side, for any (k,l), the (k,l)-entry is $\mathbb{D}_{l,k}(\underline{\Delta})$. Indeed, the (k,l)-entry of $\mathrm{adj}(\mathrm{adj}(\mathcal{GC}))$ is the cofactor of the entry $\Delta_{l,k}$ in the matrix $\mathrm{adj}(\mathcal{GC})$. Clearly, this cofactor is the (l,k)-cofactor $\mathbb{D}_{l,k}$ of the generic matrix $(y_{i,j})_{1\leq i,j\leq m}$ evaluated at $\underline{\Delta}$.

Therefore, we get $(\mathbb{D}_{m,m} - \mathbb{D}_{m-1,m-1})(\underline{\Delta}) = 0$, as required.

Finally, by the same token, from (2.17) one deduces that the inverse map has coordinates $\widetilde{\mathbb{D}} := \{\mathbb{D}_{i,j} \,|\, (i,j) \neq (m,m)\}$ modulo $\mathbb{D}_{m,m} - \mathbb{D}_{m-1,m-1}$.

(v) It follows from (iv) that the reduction number of a minimal reduction of P is $m-2$. Thus, to conclude, it suffices to prove that $P^{m-1} \not\subset JP^{m-2}$.

We will show that $\Delta_{m,m}^{m-1} \in P^{m-1}$ does not belong to JP^{m-2}.

Recall from previous passages that J is generated by the cofactors

$$\Delta_{l,h}, \text{ with } (l,h) \neq (m-1, m-1), (l,h) \neq (m,m)$$

and the additional form $\Delta_{m,m} + \Delta_{m-1,m-1}$.

If $\Delta_{m,m}^{m-1} \in JP^{m-2}$, we can write

$$\Delta_{m,m}^{m-1} = \sum_{\substack{(l,h)\neq(m-1,m-1)\\(l,h)\neq(m,m)}} \Delta_{l,h} Q_{l,h}(\underline{\Delta}) + (\Delta_{m,m} + \Delta_{m-1,m-1}) Q(\underline{\Delta}) \tag{2.18}$$

where $Q_{l,h}(\underline{\Delta})$ and $Q(\underline{\Delta})$ are homogeneous polynomial expressions of degree $m-2$ in the set

$$\underline{\Delta} = \{\Delta_{i,j} \,|\, 1 \leq i, j \leq m\}$$

of the cofactors (generators of P).

Clearly, this gives a polynomial relation of degree $m-1$ on the generators of P, so the corresponding form of degree $m-1$ in $k[y_{i,j}|\ 1 \leq i,j \leq m]$ is a scalar multiple of the defining equation $\boldsymbol{H} := \mathbb{D}_{m,m} - \mathbb{D}_{m-1,m-1}$ obtained in the previous item. Note that $\mathbf{H}$ contains only squarefree terms. We now argue that such a relation is impossible.

Namely, observe that the sum

$$\sum_{\substack{(l,h)\neq(m-1,m-1)\\(l,h)\neq(m,m)}} \Delta_{l,h} Q_{l,h}(\underline{\Delta})$$

does not contain any nonzero terms of the form $\alpha\Delta_{m,m}^{m-1}$ or $\beta\Delta_{m-1,m-1}\Delta_{m,m}^{m-2}$. In addition, if these two terms appear in $(\Delta_{m,m} + \Delta_{m-1,m-1})Q(\underline{\Delta})$ they must have the same scalar coefficient, say, $c \in k$. Bring the first of these to the left-hand side of (2.18) to get a polynomial relation of P having a nonzero term $(1-c)y_{m-1,m-1}^{m-1}$. If $c \neq 1$, this is a contradiction due to the squarefree nature of $\mathbf{H}$.

On the other hand, if $c = 1$ then we still have a polynomial relation of P having a nonzero term $y_{m-1,m-1}y_{m,m}^{m-2}$. Now, if $m > 3$ this is again a contradiction vis-à-vis the nature of $\mathbf{H}$ as the nonzero terms of the latter are squarefree monomials of degree $m-1 > 3-1 = 2$. Finally, if $m = 3$ a direct checking shows that the monomial $y_{m-1,m-1}y_{m,m}$ cannot be the support of a nonzero term in $\boldsymbol{H}$. This concludes the statement. $\square$

2.2 Degeneration by zeros

In this section we look at the cloning degeneration where an entry is cloned along the same row or column of the original generic matrix. As mentioned before, up to elementary operations of rows and/or columns the resulting matrix has a zero entry. A glimpse of this first status has been tackled in [29, Proposition 4.9 (a)].

This procedure can be repeated to add more zeros. Aiming at a uniform treatment of all these cases, we will fix integers m, r with $1 \leq r \leq m-2$ and consider the following degeneration of the $m \times m$ generic matrix:

$$\begin{pmatrix} x_{1,1} & \cdots & x_{1,m-r} & x_{1,m-r+1} & x_{1,m-r+2} & \cdots & x_{1,m-1} & x_{1,m} \\ \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ x_{m-r,1} & \cdots & x_{m-r,m-r} & x_{m-r,m-r+1} & x_{m-r,m-r+2} & \cdots & x_{m-r,m-1} & x_{m-r,m} \\ x_{m-r+1,1} & \cdots & x_{m-r+1,m-r} & x_{m-r+1,m-r+1} & x_{m-r+1,m-r+2} & \cdots & x_{m-r+1,m-1} & 0 \\ x_{m-r+2,1} & \cdots & x_{m-r+2,m-r} & x_{m-r+2,m-r+1} & x_{m-r+2,m-r+2} & \cdots & 0 & 0 \\ \vdots & \cdots & \vdots & \vdots & \vdots & \iddots & \vdots & \vdots \\ x_{m-1,1} & \cdots & x_{m-1,m-r} & x_{m-1,m-r+1} & 0 & \cdots & 0 & 0 \\ x_{m,1} & \cdots & x_{m,m-r} & 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \tag{2.19}$$

Assuming m is fixed in the context, let us denote the above matrix by $\mathcal{DG}(r)$.

2.2.1 Polar behavior

Theorem 2.2.1. *Let $R = k[\mathbf{x}]$ denote the polynomial ring in the nonzero entries of $\mathcal{DG}(r)$, with $1 \leq r \leq m-2$, let $f := \det(\mathcal{DG}(r))$ and let $J \subset R$ denote the gradient ideal of f. Then:*

(i) *f is irreducible.*

(ii) *J has maximal linear rank.*

(iii) *The homogeneous coordinate ring of the polar variety of f in $\mathbb{P}^{m^2-\binom{r+1}{2}-1}$ is a Gorenstein ladder determinantal ring of dimension $m^2 - r(r+1)$; in particular, the analytic spread of J is $m^2 - r(r+1)$.*

Proof. (i) Expanding the determinant by Laplace along the first row, we can write $f = x_{1,1}\Delta_{1,1} + g$, where $\Delta_{1,1}$ is the cofactor of $x_{1,1}$ on $\mathcal{DG}(r)$. Clearly, both $\Delta_{1,1}$ and g belong to the polynomial subring omitting the variable $x_{1,1}$. Thus, in order to show that f is irreducible it suffices to prove that it is a primitive polynomial (of degree 1) in $k[x_{1,2}, \ldots, x_{m,m-r}][x_{1,1}]$. In order words, we need to check that no irreducible factor of $\Delta_{1,1}$ is a factor of g.

We induct on $m \geq r+2$. If $m = r+2$ then $\Delta_{1,1} = x_{2,m}x_{3,m-1}\cdots x_{m-1,3}x_{m,2}$, while the initial term of g in the revlex monomial order is

$$\mathrm{in}(g) = \mathrm{in}(f) = x_{1,m}x_{2,m-1}\cdots x_{m,1}.$$

Thus, assume that $m > r+2$. By the inductive step, $\Delta_{1,1}$ is irreducible being the determinant of an $(m-1)\times(m-1)$ matrix of the same kind (same r). But $\deg(\Delta_{1,1}) = \deg(g)-1$. Therefore, it suffices to show that $\Delta_{1,1}$ is not a factor of g. Supposing it were, we would get that f is multiple of $\Delta_{1,1}$ by a linear factor – this is clearly impossible.

Once more, an alternative argument is to use that the ideal J has codimension 4, as will be shown independently in Theorem 2.2.7 (ii). Therefore, the ring $R/(f)$ is locally regular in codimension at least one, so it must be normal. But f is homogeneous, hence irreducible.

(ii) The proof is similar to the one of Theorem 2.1.1 (iii), but there is a numerical diversion and, besides, the cases where $r > m-r-1$ and $r \leq m-r-1$ keep slight differences.

Let $f_{i,j}$ denote the $x_{i,j}$-derivative of f and let $\Delta_{j,i}$ stand for the (signed) cofactor of $x_{i,j}$ on $\mathcal{DG}(r)$. We first assume that $r > m-r-1$. The Cauchy cofactor formula

$$\mathcal{DG}(r)\cdot \mathrm{adj}(\mathcal{DG}(r)) = \mathrm{adj}(\mathcal{DG}(r))\cdot\mathcal{DG}(r) = \det(\mathcal{DG}(r))\mathbb{I}_m$$

yields by expansion the following three blocks of linear relations involving the (signed) cofactors of $\mathcal{DG}(r)$:

$$\begin{cases} \sum_{j=1}^{m} x_{i,j}\Delta_{j,k} = 0 & \text{for } 1\leq i\leq m-r,\ 1\leq k\leq m-r\ (k\neq i)\\ \sum_{j=1}^{m-l} x_{m-r+l,j}\Delta_{j,k} = 0 & \text{for } 1\leq l\leq r,\ 1\leq k\leq m-r\\ \sum_{j=1}^{m} x_{i,j}\Delta_{j,i} - \sum_{j=1}^{m} x_{i+1,j}\Delta_{j,i+1} = 0 & \text{for } 1\leq i\leq m-r-1 \end{cases} \tag{2.20}$$

with m^2-rm-1 such relations;

$$\begin{cases} \sum_{i=1}^{m} x_{i,j}\Delta_{k,i} = 0 & \text{for } 1\leq j\leq m-r,\ 1\leq k\leq m-r\ (k\neq j)\\ \sum_{i=1}^{m-l} x_{i,m-r+l}\Delta_{k,i} = 0 & \text{for } 1\leq l\leq 2r-m+1,\ 1\leq k\leq m-r \end{cases} \tag{2.21}$$

with $(m-r)(m-r-1)+(m-r)(2r-m+1) = r(m-r)$ such relations; and

$$\begin{cases} \sum_{i=1}^{m-l} x_{i,m-r+l}\Delta_{m-r+l,i} - \sum_{j=1}^{m} x_{i,1}\Delta_{1,i} = 0 & \text{for } 1 \leq l \leq r-1 \\ \sum_{i=1}^{m-k} x_{i,m-r+k}\Delta_{m-r+l,i} = 0 & \text{for } 1 \leq l \leq r-2,\ l+1 \leq k \leq r-1 \end{cases} \tag{2.22}$$

with $r(r-1)/2$ such relations.

Similarly, when $r \leq m-r-1$, the classical Cauchy cofactor formula outputs by expansion three blocks of linear relations involving the (signed) cofactors of $\mathcal{DG}(r)$. Here, the first and third blocks are, respectively, exactly as the above ones, while the second one requires a modification due to the inequality reversal; namely, we get

$$\begin{cases} \sum_{i=1}^{m} x_{i,j}\Delta_{k,i} = 0 & \text{for } 1 \leq j \leq r+1,\ 1 \leq k \leq r\ (k \neq j) \\ \sum_{i=1}^{m} x_{i,j}\Delta_{k,i} = 0 & \text{for } 1 \leq j \leq r,\ r+1 \leq k \leq m-r, \end{cases} \tag{2.23}$$

with $r(m-r)$ such relations (as before).

Since $f_{i,j}$ coincides with the (signed) cofactor $\Delta_{j,i}$, any of the above relations gives a linear syzygy of the partial derivatives of f. Thus one has a total of $m^2 - rm - 1 + r(m-r) + r(r-1)/2 = m^2 - \binom{r+1}{2} - 1$ linear syzygies of J.

It remains to show that these are independent.

For this, we adopt the same strategy as in the proof of Theorem 2.1.1 (iii), whereby we list the partial derivatives according to the following ordering of the nonzero entries: we traverse the first row from left to right, then the second row in the same way, and so on until we reach the last row with no zero entry; thereafter we start from the first row having a zero and travel along the columns, from left to right, on each column from top to bottom, till we all nonzero entries are counted.

Thus, the desired ordering is depicted in the following scheme, where we once more used arrows for easy reading:

$$x_{1,1}, x_{1,2}, \ldots, x_{1,m} \rightsquigarrow x_{2,1}, x_{2,2}, \ldots, x_{2,m} \rightsquigarrow \ldots \rightsquigarrow x_{m-r,1}, x_{m-r,2} \ldots, x_{m-r,m} \rightsquigarrow$$
$$x_{m-r+1,1}, \ldots, x_{m,1} \rightsquigarrow x_{m-r+1,2}, \ldots, x_{m,2} \rightsquigarrow \ldots \rightsquigarrow x_{m-r+1,m-r}, x_{m-r+2,m-r}, \ldots,$$
$$x_{m,m-r} \rightsquigarrow x_{m-r+1,m-r+1}, \ldots, x_{m-1,m-r+1} \rightsquigarrow x_{m-r+1,m-r+2}, \ldots, x_{m-2,m-r+2} \rightsquigarrow$$
$$\ldots \rightsquigarrow x_{m-r+1,m-2}, x_{m-r+2,m-2} \rightsquigarrow x_{m-r+1,m-1}.$$

With this ordering the above linear relations translate into linear syzygies collected in the following block matrix

$$\mathcal{M} = \left(\begin{array}{cccc|cccc|cccc}
\varphi_1 & \dots & & & & & & & & & & \\
0 & \varphi_2 & \dots & & & & & & & & & \\
\vdots & \vdots & \ddots & \vdots & & & & & & & & \\
0 & 0 & \dots & \varphi_{m-r} & & & & & & & & \\
\hline
0_r^{m-1} & 0_r^m & \dots & 0_r^m & \varphi_r^1 & & & & & & & \\
0_r^{m-1} & 0_r^m & \dots & 0_r^m & 0_r^r & \varphi_r^2 & & & & & & \\
\vdots & \vdots & \dots & \vdots & \vdots & \vdots & \ddots & & & & & \\
0_r^{m-1} & 0_r^m & \dots & 0_r^m & 0_r^r & 0_r^r & \dots & \varphi_r^{(m-r)} & & & & \\
\hline
0_{r-1}^{m-1} & 0_{r-1}^m & \dots & 0_{r-1}^m & 0_{r-1}^r & 0_{r-1}^r & \dots & 0_{r-1}^r & \Phi_1 & & & \\
0_{r-2}^{m-1} & 0_{r-2}^m & \dots & 0_{r-2}^m & 0_{r-2}^r & 0_{r-2}^r & \dots & 0_{r-2}^r & 0_{r-2}^{r-1} & \Phi_2 & & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \\
0_1^{m-1} & 0_1^m & \dots & 0_1^m & 0_1^r & 0_1^r & \dots & 0_1^r & 0_1^{r-1} & 0_1^{r-2} & \dots & \Phi_{r-1}
\end{array}\right),$$

where:

- φ_1 is the matrix obtained from the transpose $\mathcal{DG}(r)^t$ of $\mathcal{DG}(r)$ by omitting the first column
- $\varphi_2, \dots, \varphi_{m-r}$ are each a copy of $\mathcal{DG}(r)^t$ (up to column permutation);
- When $r > m - r - 1$, φ_r^i is the $r \times r$ minor omitting the i-th column for $i = 1, \dots, m - r$ of the following submatrix of $\mathcal{DG}(r)$:

$$\left(\begin{array}{ccccc}
x_{m-r+1,1} & \dots & x_{m-r+1,m-r} & x_{m-r+1,m-r+1} & x_{m-r+1,m-r+2} \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
x_{m-(2r-m+1),1} & \dots & x_{m-(2r-m+1),m-r} & x_{m-(2r-m+1),m-r+1} & x_{m-(2r-m+1),m-r+2} \\
x_{m-(2r-m+1)+1,1} & \dots & x_{m-(2r-m+1)+1,m-r} & x_{m-(2r-m+1)+1,m-r+1} & x_{m-(2r-m+1)+1,m-r+2} \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
x_{m-1,1} & \dots & x_{m-1,m-r} & x_{m-2,m-r+1} & 0 \\
x_{m,1} & \dots & x_{m,m-r} & 0 & 0
\end{array}\right.$$

$$
\begin{pmatrix}
\cdots & x_{m-r+1,r+1} \\
\vdots & \vdots \\
\cdots & x_{m-(2r-m+1),r+1} \\
\cdots & 0 \\
\vdots & \vdots \\
\cdots & 0 \\
\cdots & 0
\end{pmatrix}.
$$

When $r \leq m-r-1$, φ_r^i is the $r \times r$ minor obtained from the following submatrix of $\mathcal{DG}(r)$

$$
\begin{pmatrix}
x_{m-r+1,1} & \cdots & x_{m-r+1,r} & x_{m-r+1,r+1} \\
\vdots & \vdots & \vdots & \vdots \\
{}_{m,1} & \cdots & x_{m,r} & x_{m,r+1}
\end{pmatrix}
$$

by omitting the i-th column for $i = 1, \ldots, r$ and by omitting the $(r+1)$-th column for $i = r+1, \ldots, m-r$.

- Each 0 under φ_1 is a zero block of the size $m \times (m-1)$ and each 0 under φ_i is a zero block of the size $m \times m$ for $i = 2, \ldots, m-r-1$;
- 0_l^c denotes a zero block of size $l \times c$.
- Φ_i is the $(r-i) \times (r-i)$ submatrix of $\mathcal{DG}(r)$ described bellow:

$$
\Phi_i = \begin{pmatrix}
x_{m-r+1,m-r+i} & x_{m-r+1,m-r+i+1} & \cdots & x_{m-r+1,m-1} \\
x_{m-r+2,m-r+i} & x_{m-r+2,m-r+i+1} & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots \\
x_{m-i,m-r+i} & 0 & \cdots & 0
\end{pmatrix}
$$

Next we justify why these blocks make up (linear) syzygies. As already explained, the relations in (2.20), (2.21) and (2.22) yield linear syzygies of the partial derivatives of f. Setting $k = 1$ in the first two relations of (2.20), the latter can be written as $\sum_{j=1}^{m} x_{i,j} f_{1,j} = 0$, for $i = 2, \ldots, m-r$, and $\sum_{j=1}^{m-l} x_{m-r+l,j} f_{1,j} = 0$, for all $l = 1, \ldots, r$. Ordering the set of partial derivatives $f_{i,j}$ as explained before the coefficients of these relations form the first matrix above:

$$\varphi_1 := \begin{pmatrix} x_{2,1} & \cdots & x_{m-r,1} & x_{m-r+1,1} & x_{m-r+2,1} & \cdots & x_{m-1,1} & x_{m,1} \\ \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ x_{2,m-r} & \cdots & x_{m-r,m-r} & x_{m-r+1,m-r} & x_{m-r+2,m-r} & \cdots & x_{m-1,m-r} & x_{m,m-r} \\ x_{2,m-r+1} & \cdots & x_{m-r,m-r+1} & x_{m-r+1,m-r+1} & x_{m-r+2,m-r+1} & \cdots & x_{m-1,m-r+1} & 0 \\ x_{2,m-r+2} & \cdots & x_{m-r,m-r+2} & x_{m-r+1,m-r+2} & x_{m-r+2,m-r+2} & \cdots & 0 & 0 \\ \vdots & \cdots & \vdots & \vdots & \vdots & \iddots & \vdots & \vdots \\ x_{2,m-1} & \cdots & x_{m-r,m-1} & x_{m-r+1,m-1} & 0 & \cdots & 0 & 0 \\ x_{2,m} & \cdots & x_{m-r,m} & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

Note that φ_1 coincides indeed with the submatrix of $\mathcal{DG}(r)^t$ obtained by omitting its first column.

Getting φ_k, for $k = 2, \ldots, m-r$, is similar, namely, we use again the first two relations in the block (2.20) retrieving the submatrix of $\mathcal{DG}(r)^t$ excluding the kth column and replacing it with an extra column that comes from the last relation in (2.20) by taking $i = k-1$.

Continuing, for each $i = 1, \ldots, m-r$ the block φ_r^i comes from the relations in the blocks (2.21), if $r > m-r-1$, or (2.23), if $r \le m-r-1$, by setting $k = i$. Finally, for each $i = 1, \ldots, r-1$, the block Φ_i comes from the relations in (2.22) by setting $l = i$.

This proves the claim about the large matrix above. Counting through the sizes of the various blocks, one sees that this matrix is $(m^2 - \binom{r+1}{2}) \times (m^2 - \binom{r+1}{2} - 1)$. Omitting its first row obtains a square block-diagonal submatrix where each block has nonzero determinant. Thus, the linear rank of J attains the maximum.

(iii) Note that the polar map can be thought as the map of $\mathbb{P}^{m^2-\binom{r+1}{2}-1}$ to itself defined by the partial derivatives of f. As such, the polar variety will be described in terms of defining equations in the original $\mathbf{x}$-variables.

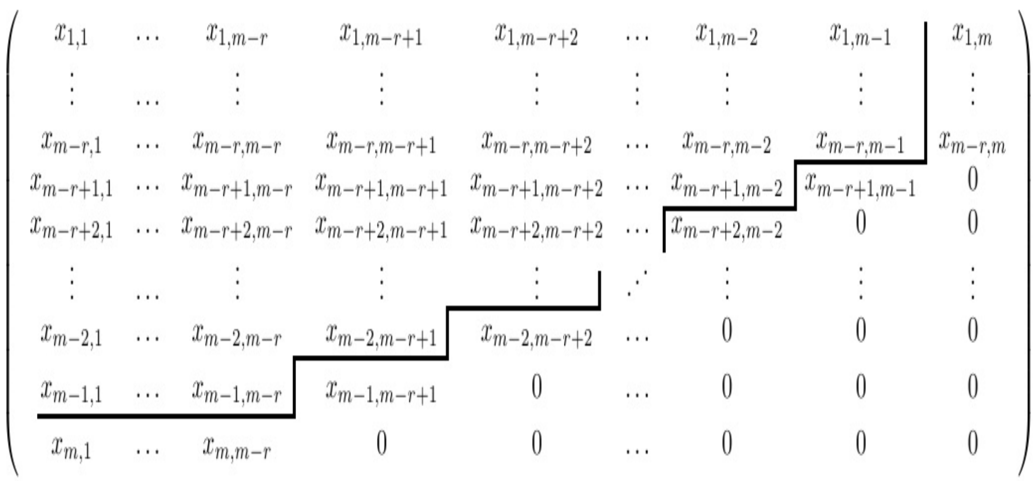

Figure 2.1: stair-like polygonal.

Let $\mathcal{L} = \mathcal{L}(m, r)$ denote the set of variables in $\mathcal{DG}(r)$ lying to the left and above the stair-like polygonal in Figure 2.1 above and let $I_{m-r}(\mathcal{L})$ stand for the ideal generated by the $(m-r) \times (m-r)$ minors of $\mathcal{DG}(r)$ involving only the variables in $\mathcal{L}$.

Since $\mathcal{L}$ can be completed to a fully generic matrix of size $(m-1) \times (m-1)$, the ring $K[\mathcal{L}]/I_{m-r}(\mathcal{L})$ is one of the so-called ladder determinantal rings.

Claim: The homogeneous defining ideal of the image of the polar map of f contains the ideal $I_{m-r}(\mathcal{L})$.

Let $x_{i,j}$ denote a nonzero entry of $\mathcal{DG}(r)$. Since the nonzero entries of the matrix are independent variables, it follows easily from the Laplace expansion along the ith row that the $x_{i,j}$-derivative $f_{i,j}$ of f coincides with the (signed) cofactor of $x_{i,j}$, heretofore denoted $\Delta_{j,i}$.

Given integers $1 \leq i_1 < i_2 < \ldots < i_{m-r} \leq m-1$, consider the following submatrix of the adjoint matrix:

$$F = \begin{pmatrix} \Delta_{i_1,1} & \Delta_{i_1,2} & \Delta_{i_1,3} & \cdots & \Delta_{i_1,m-i_{m-r}+(m-r-1)} \\ \Delta_{i_2,1} & \Delta_{i_2,2} & \Delta_{i_2,3} & \cdots & \Delta_{i_2,m-i_{m-r}+(m-r-1)} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \Delta_{i_{m-r},1} & \Delta_{i_{m-r},2} & \Delta_{i_{m-r},3} & \cdots & \Delta_{i_{m-r},m-i_{m-r}+(m-r-1)} \end{pmatrix}.$$

Letting

$$C = \begin{pmatrix} x_{1,i_{m-r}+1} & x_{1,i_{m-r}+2} & \cdots & x_{1,m-1} & x_{1,m} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ x_{m-r,i_{m-r}+1} & x_{m-r,i_{m-r}+2} & \cdots & x_{m-r,m-1} & x_{m-r,m} \\ x_{m-r+1,i_{m-r}+1} & x_{m-r+1,i_{m-r}+2} & \cdots & x_{m-r+1,m-1} & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ x_{m-i_{m-r}+(m-r-2),i_{m-r}+1} & x_{m-i_{m-r}+(m-r-2),i_{m-r}+2} & \cdots & 0 & 0 \\ x_{m-i_{m-r}+(m-r-1),i_{m-r}+1} & 0 & \cdots & 0 & 0 \end{pmatrix},$$

the cofactor identity $\operatorname{adj}(\mathcal{DG}(r)) \cdot \mathcal{DG}(r) = \det(\mathcal{DG}(r))\mathbb{I}_m$ yields the relation

$$F \cdot C = 0.$$

Since the columns of C are linearly independent, it follows that the rank of F is at most $m - i_{m-r} + (m-r-1) - (m - i_{m-r}) = (m-r) - 1$. In other words, the maximal minors of the following matrix

$$\begin{pmatrix} x_{i_1,1} & x_{i_1,2} & x_{i_1,3} & \cdots & x_{i_1,m-i_{m-r}+(m-r-1)} \\ x_{i_2,1} & x_{i_2,2} & x_{i_2,3} & \cdots & x_{i_2,m-i_{m-r}+(m-r-1)} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ x_{i_{m-r},1} & x_{i_{m-r},2} & x_{i_{m-r},3} & \cdots & x_{i_{m-r},m-i_{m-r}+(m-r-1)} \end{pmatrix}.$$

all vanish on the partial derivatives of f, thus proving the claim.

Claim: The codimension of the ideal $I_{m-r}(\mathcal{L}(m,r))$ is at least $\binom{r+1}{2}$.

For this we use induction with the following inductive hypothesis: suppose that for any $(m-1)\times(m-1)$ matrix of the form $\mathcal{DG}(r-1))$, the ideal $I_{m-1-(r-1)}(\mathcal{L}(m-1,r-1))$ has codimension $\binom{r}{2}$. Note that $m-1-(r-1) = m-r$, hence the size of the inner minors does not change in the inductive step.

To construct a suitable inductive precedent, let $\widetilde{\mathcal{L}}$ the set of variables that are to the left and above the stair-like polygonal in Figure 2.2 and denote $I_{m-r}(\widetilde{\mathcal{L}})$ the ideal generated by the $(m-r)\times(m-r)$ minors of $\mathcal{DG}(r)$ involving only the variables in $\widetilde{\mathcal{L}}$. Note that $\widetilde{\mathcal{L}}$ is of the form $\mathcal{L}(m-1,r-1)$ relative to a matrix of the form $\mathcal{DG}(r-1))$. Clearly, $I_{m-r}(\widetilde{\mathcal{L}})$ it too is a ladder determinantal ideal on a suitable $(m-2)\times(m-2)$ generic matrix; in particular, it is a Cohen-Macaulay prime ideal (see [31] for primeness and [22] for Cohen–Macaulayness). By the inductive hypothesis, the codimension of $I_{m-r}(\widetilde{\mathcal{L}})$ is at least $\binom{r}{2}$.

$$\begin{pmatrix} x_{1,1} & \cdots & x_{1,m-r} & x_{1,m-r+1} & x_{1,m-r+2} & \cdots & x_{1,m-2} & x_{1,m-1} \\ \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ x_{m-r,1} & \cdots & x_{m-r,m-r} & x_{m-r,m-r+1} & x_{m-r,m-r+2} & \cdots & x_{m-r,m-2} & x_{m-r,m-1} \\ x_{m-r+1,1} & \cdots & x_{m-r+1,m-r} & x_{m-r+1,m-r+1} & x_{m-r+1,m-r+2} & \cdots & x_{m-r+1,m-2} & x_{m-r+1,m-1} \\ x_{m-r+2,1} & \cdots & x_{m-r+2,m-r} & x_{m-r+2,m-r+1} & x_{m-r+2,m-r+2} & \cdots & x_{m-r+1,m-2} & 0 \\ \vdots & \cdots & \vdots & \vdots & \vdots & \iddots & \vdots & \vdots \\ x_{m-2,1} & \cdots & x_{m-2,m-r} & x_{m-2,m-r+1} & x_{m-2,m-r+1} & \cdots & 0 & 0 \\ x_{m-1,1} & \cdots & x_{m-1,m-r} & x_{m-1,m-r+1} & 0 & \cdots & 0 & 0 \end{pmatrix}$$

Figure 2.2: Sub-stair-like inductive.

Note that $\widetilde{\mathcal{L}}$ is a subset of $\mathcal{L}$, hence there is a natural ring surjection

$$S := \frac{k[\mathcal{L}]}{I_{m-r}(\widetilde{\mathcal{L}})\,k[\mathcal{L}]} = \frac{k[\widetilde{\mathcal{L}}]}{I_{m-r}(\widetilde{\mathcal{L}})}[\mathcal{L}\setminus\widetilde{\mathcal{L}}] \twoheadrightarrow \frac{k[\mathcal{L}]}{I_{m-r}(\mathcal{L})}$$

To complete the statement, since $\binom{r}{2}+r=\binom{r+1}{2}$, it suffices to exhibit r elements of $I_{m-r}(\mathcal{L})$ forming a regular sequence on the ring $S := k[\mathcal{L}]/I_{m-r}(\widetilde{\mathcal{L}})k[\mathcal{L}]$.

Consider the matrices

$$\left(\begin{array}{cccc|c} x_{1,1} & \dots & x_{1,m-r-1} & & x_{1,m-i} \\ \vdots & \vdots & \vdots & & \vdots \\ x_{m-r-1,1} & \dots & x_{m-r-1,m-r-1} & & x_{m-r-1,m-i} \\ \hline x_{m-r-1+i,1} & \dots & x_{m-r-1+i,m-r-1} & & x_{m-r-1+i,m-i} \end{array}\right) \tag{2.24}$$

for $i=1,\ldots,r$. Let $\Delta_i \in I_{m-r}(\mathcal{L})$ denote the determinant of the above matrix, for $i=1,\ldots,r$.

The claim is that $\Delta := \{\Delta_1,\ldots,\Delta_r\}$ is a regular sequence on S.

Let δ denote the $(m-r-1)$-minor in the upper left corner of (2.24). Clearly, δ is a regular element on S as its defining ideal is a prime ideal generated in degrees $m-r$. Therefore, it suffices to show that the localized sequence

$$\Delta_\delta := \{(\Delta_1)_\delta,\ldots,(\Delta_r)_\delta\}$$

is a regular on S_δ. On the other hand, since S is Cohen-Macaulay, it is suffices to show that $\dim\, S_\delta/\Delta_\delta S_\delta = \dim\, S_\delta - r$.

Write $\mathbf{X}' := \{x_{m-r,m-1}, x_{m-r+1,m-2},\ldots,x_{m-2,m-r+1},x_{m-1,m-r}\}$. Note that, for every $i=1,\ldots,r$, one has $(\Delta_i)_\delta = x_{m-r-1+i,m-i} + (1/\delta)\Gamma_i$, with $x_{m-r-1+i,m-i}\in \mathbf{X}'$ and $\Gamma_i \in k[\mathcal{L}\setminus \mathbf{X}']$. The association $x_{m-r-1+i,m-i}\mapsto -(1/\delta)\Gamma_i$ therefore defines a ring homomorphism

$$k[\mathcal{L}]_\delta/(\Delta_\delta) = (k[X'][\mathcal{L}\setminus X'])_\delta/(\Delta_\delta) \simeq k[\mathcal{L}\setminus X'])_\delta$$

This entails a ring isomorphism

$$\frac{S_\delta}{\Delta_\delta S_\delta} \simeq \frac{k[\mathcal{L}\setminus \mathbf{X}']_\delta}{(I_{m-r}(\widetilde{\mathcal{L}}))k[\mathcal{L}\setminus \mathbf{X}']_\delta}.$$

Thus, $\dim\, S_\delta/\Delta_\delta S_\delta = \dim k[\mathcal{L}]_\delta - r - \operatorname{codim}(I_{m-r}(\widetilde{\mathcal{L}}))_\delta = \dim\, S_\delta - r$

Therefore, $\operatorname{codim}(I_{m-r}(\mathcal{L}))$ is at least $\operatorname{codim}(I_{m-r}(\widetilde{\mathcal{L}})) + r = \binom{r+1}{2}$.

In order to show that $I_{m-r}(\mathcal{L})$ is the homogeneous defining ideal of the polar variety it suffices to show that the latter has codimension at most $\binom{r+1}{2}$. Since the dimension of the homogeneous coordinate ring of the polar variety coincides with the rank of the Hessian matrix of f, it now suffices to show that the latter is at least $\dim R - \binom{r+1}{2} = m^2 - \binom{r+1}{2} - \binom{r+1}{2} = m^2 - r(r+1)$.

For this, we proceed along the same line of the proof of Theorem 2.1.1 (ii).

Namely, set $X := \{x_{i,j} \mid i+j = r+2, r+3, \ldots, 2m-r\}$ and consider the set of partial derivatives of f with respect to the variables in X. Let M denote the Jacobian matrix of these partial derivatives with respect to the variables in X. Observe that M is a submatrix of size $(m^2 - r(r+1)) \times (m^2 - r(r+1))$ of the Hessian matrix. We will show that $\det(M) \neq 0$.

Set $\mathbf{v} := \{x_{1,m}, x_{2,m-1}, \ldots, x_{m,1}\} \subset X$ is the set of variables along the main anti-diagonal of $\mathcal{DG}(r)$.

As already pointed out, the partial derivative of f with respect to any $x_{i,j} \in X$ coincides with the (signed) cofactor of $x_{i,j}$. By expanding according to the Leibniz rule one can check that the cofactor of a variable in the set $\mathbf{v}$ has the unique (nonzero) term whose support lies in $\mathbf{v}$. Similarly, the cofactor of a variable outside $\mathbf{v}$ has no term whose support lies in $\mathbf{v}$ and has exactly one (nonzero) term of degree 1 in the variables off $\mathbf{v}$. In fact, if $i+j \neq m+1$, one finds

$$\begin{aligned}\Delta_{j,i} &= x_{m+1-j,m+1-i}(x_{1,m} \cdots \widehat{x_{i,m-i+1}} \cdots \widehat{x_{m-j+1,j}} \cdots x_{m,1}) \\ &\quad + \text{terms of degree at least 2 off } \mathbf{v},\end{aligned}$$

where the term inside the parenthesis has support in $\mathbf{v}$.

Consider the ring endomorphism φ of R that maps any variable in $\mathbf{v}$ to itself and any variable off $\mathbf{v}$ to zero. By the preceding observation, applying the map to any second partial derivative of f involving only the variables of X will return zero or a monomial supported on the variables in $\mathbf{v}$. Let $\widetilde{M}$ denote the resulting specialized matrix of M. Thus, any of its entries is either zero or a monomial supported on the variables in $\mathbf{v}$.

We will show that $\det(\widetilde{M})$ is nonzero. For this, consider the Jacobian matrix of the set of partial derivatives $\{f_v : v \in \mathbf{v}\}$ with respect to the variables in $\mathbf{v}$. Let M_0 denote the specialization of this Jacobian matrix by φ considered as a corresponding submatrix of $\widetilde{M}$. Up to permutation of rows and columns of $\widetilde{M}$, we may write

$$\widetilde{M} = \begin{pmatrix} M_0 & N_0 \\ N_1 & M_1 \end{pmatrix},$$

for suitable M_1. Now, by the way the second partial derivatives of f specialize via φ as explained above, one must have $N_0 = N_1 = 0$. Therefore, $\det(\widetilde{M}) = \det(M_0)\det(M_1)$, so it remains to prove the nonvanishing of these two subdeterminants. Now the first block is the Hessian matrix of the form g being taken as the product of the entries in the main anti-diagonal of the matrix $\mathcal{DG}(r)$. By a similar argument used in the proof of Theorem 2.1.1 (ii), one has that g is a well-known homaloidal polynomial, hence we are done for the first matrix block. As for the second block, by construction it has exactly one nonzero entry on each row and each column. Therefore, it has a nonzero determinant.

To conclude the assertion of this item it remains to argue that the ladder determinantal ring in question is Gorenstein. For this we use the criterion in [6, Theorem, (b) p. 120]. By the latter, we only need to see that the inner corners of the ladder depicted in Figure 2.1 have indices (a, b) satisfying the equality $a + b = m - 1 + (m - r) - 1 = 2m - r - 2$, where the ladder is a structure in an $(m - 1) \times (m - 1)$ matrix.

This completes the proof of this item. The supplementary assertion on the analytic spread of J is clear since the dimension of the latter equals the dimension of the k-subalgebra generated by the partial derivatives. □

Remark 2.2.2. The result of (iii) for the case where $r = 1$ has appeared earlier in [30].

2.2.2 Primality

We will need the following lemmas.

The first is a non-generic version of [3, Theorem 10.16 (b)]:

Lemma 2.2.3. *Let M be a square matrix with entries either variables over a field k or zeros, such that* $\det(M) \neq 0$. *Let R denote the polynomial ring over k on the nonzero entries of M and let $S \subset R$ denote the k-subalgebra generated by the submaximal minors. Then the extension $S \subset R$ is algebraic at the level of the respective fields of fractions.*

The proof is the same as the one given in [3, Theorem 10.16 (b)].

The second lemma was communicated to us by Aldo Conca, as a particular case of a more general setup:

Lemma 2.2.4. *Let $\mathcal{G}$ denote a generic square matrix. Then the submaximal minors of M are a Gröbner base in the reverse lexicographic order and the initial ideal of any minor is the product of its entries along the main anti-diagonal.*

This result is the counterpart of the classical result in the case of the lexicographic order, where the initial ideals are the products of the entries along the main diagonals. In both versions, the chosen term order should respect the rows and columns of M.

The content of the third lemma does not seem to have been noted before:

Lemma 2.2.5. *Let $\mathcal{G}$ denote a generic $m \times m$ matrix and let $\mathcal{X}$ denote the set of entries none of which belongs to the main anti-diagonal of a submaximal minor. Then $\mathcal{X}$ is a regular sequence modulo the ideal generated by the submaximal minors in the polynomial ring of the entries of $\mathcal{G}$ over a field k.*

Proof. As for easy visualization, $\mathcal{X}$ is the set of bulleted entries below (for $m \geq 6$):

$$\begin{pmatrix} \bullet & \bullet & \bullet & \cdots & \bullet & \bullet & \bullet & x_{1,m-1} & x_{1,m} \\ \bullet & \bullet & \bullet & \cdots & \bullet & \bullet & x_{2,m-2} & x_{2,m-1} & x_{2,m} \\ \bullet & \bullet & \bullet & \cdots & \bullet & x_{3,m-3} & x_{3,m-2} & x_{3,m-1} & \bullet \\ \bullet & \bullet & \bullet & \cdots & x_{4,m-4} & x_{4,m-3} & x_{4,m-2} & \bullet & \bullet \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots \\ \bullet & x_{m-2,2} & x_{m-2,3} & \cdots & x_{m-2,m-4} & \bullet & \bullet & \bullet & \bullet \\ x_{m-1,1} & x_{m-1,2} & x_{m-1,3} & \cdots & \bullet & \bullet & \bullet & \bullet & \bullet \\ x_{m,1} & x_{m,2} & \bullet & \cdots & \bullet & \bullet & \bullet & \bullet & \bullet \end{pmatrix}$$

(A similar picture can be depicted for $m \leq 5$).

Clearly, the cardinality of $\mathcal{X}$ is $2\binom{m-1}{2} = (m-1)(m-2)$. Fix an ordering of the elements $\{a_1, \ldots, a_{(m-1)(m-2)}\}$ of $\mathcal{X}$. By Lemma 2.2.4 and the assumption that every a_i avoids the initial ideal of any submaximal minor, it follows that the initial ideal of the ideal $(a_1, \ldots, a_i, \mathcal{P})$ is $(a_1, \ldots, a_i, \mathrm{in}(\mathcal{P}))$. Clearly, a_{i+1} is not a zero divisor modulo the latter ideal, and hence, by a well known procedure, it is neither a zero divisor modulo $(a_1, \ldots, a_i, \mathcal{P})$. $\square$

In the subsequent parts we will relate the gradient ideal $J \subset R$ of the determinant of the matrix $\mathcal{DG}(r)$ in (2.19) to the ideal $I_{m-1}(\mathcal{DG}(r)) \subset R$ of its submaximal minors. As an easy preliminary, we observe that, for any nonzero entry of $x_{i,j}$ of $\mathcal{DG}(r)$, since the nonzero entries of the matrix are independent variables, it follows easily from the Laplace expansion along the ith row that the $x_{i,j}$-derivative $f_{i,j}$ of f coincides with the (signed) cofactor of $x_{i,j}$. In particular, one has $J \subset I_{m-1}(\mathcal{DG}(r))$ throughout the entire subsequent discussion and understanding the conductor $J : I_{m-1}(\mathcal{DG}(r))$ will be crucial.

Proposition 2.2.6. *Let $\mathcal{DG}(r)$ as in (2.19) denote our basic degenerate matrix, with $1 \leq r \leq m-2$. For every $0 \leq j \leq m$, consider the submatrices M_j and N_j of $\mathcal{DG}(r)$ consisting of its last j columns and the its last j rows, respectively. Write $I := I_{m-1}(\mathcal{DG}(r)) \subset R$ for the ideal of $(m-1)$-minors of $\mathcal{DG}(r)$ and J for the gradient ideal of $f := \det(\mathcal{DG}(r))$. Then $I_j(N_j) \cdot I_{r-j}(M_{r-j}) \subset J : I$ for every $0 \leq j \leq r$.*

Proof. (A) For a fixed $1 \leq j \leq r$, we write the matrices $\mathcal{DG}(m-2)$ and its adjoint $\mathrm{adj}(\mathcal{DG}(m-2))$ in the following block form:

$$\mathcal{DG}(m-2) = \begin{pmatrix} \widetilde{N}_j \\ N_j \end{pmatrix}, \quad \mathrm{adj}(\mathcal{DG}(m-2)) = \left(\begin{array}{c|c} \Theta_{1,j} & \Theta_{2,j} \\ \hline \Theta_{3,j} & \Theta_{4,j} \end{array}\right); \tag{2.25}$$

where $\Theta_{1,j}, \Theta_{2,j}, \Theta_{3,j}, \Theta_{4,j}$ stand for submatrices of sizes $(j+m-r) \times (m-j)$, $(j+m-$

$r) \times j, (r-j) \times (m-j)$ and $(r-j) \times j,$ *respectively* Thus, we have

$$\operatorname{adj}(\mathcal{DG}(r)) \cdot \mathcal{DG}(r) = \begin{pmatrix} \Theta_{1,j}\widetilde{N}_j + \Theta_{2,j}N_j \\ \Theta_{3,j}\widetilde{N}_j + \Theta_{4,j}N_j \end{pmatrix} = f \cdot \mathbb{I}_m. \tag{2.26}$$

with $\mathbb{I}_m$ denoting the identity matrix of order m. Since f belongs to J, then $I_1(\Theta_{1,j}\widetilde{N}_j + \Theta_{2,j}N_j) \subset J$. On the other hand, the entries of $\Theta_{1,j}$ are cofactors of the entries on the upper left corner of $\mathcal{DG}(r)$, hence belong to J as well. Therefore $I_1(\Theta_{2,j}N_j) \subset J$ as well. From this by an easy argument it follows that

$$I_1(\Theta_{2,j})I_j(N_j) \subset J \tag{2.27}$$

and, for even more reason,

$$\boxed{I_1(\Theta_{2,j})I_j(N_j) \cdot I_{r-j}(M_{r-j}) \subset J} \tag{2.28}$$

Similarly, writing

$$\mathcal{DG}(r) = \left(\widetilde{M}_{r-j} \;\middle|\; M_{r-j} \right)$$

we have:

$$\mathcal{DG}(r) \cdot \operatorname{adj}(\mathcal{DG}(r)) = \left(\widetilde{M}_{r-j}\Theta_{1,j} + M_{r-j}\Theta_{3,j} \;\middle|\; \widetilde{M}_{r-j}\Theta_{2,j} + M_{r-j}\Theta_{4,j} \right) = f \cdot \mathrm{Id}_m.$$

An entirely analogous reasoning leads to the inclusion $I_1(\Theta_{3,j})I_{r-j}(M_{r-j}) \subset J$, and for even more reason

$$\boxed{I_1(\Theta_{3,j})I_j(N_j) \cdot I_{r-j}(M_{r-j}) \subset J} \tag{2.29}$$

Arguing now with the second block $\widetilde{M}_{r-j}\Theta_{2,j} + M_{r-j}\Theta_{4,j}$, again $I_1(\widetilde{M}_{r-j}\Theta_{2,j} + M_{r-j}\Theta_{4,j}) \subset J$, and hence for each $\delta \in I_j(N_j)$, also $I_1(\delta\widetilde{M}_{r-j}\Theta_{2,j} + \delta M_{r-j}\Theta_{4,j}) \subset J$. But, by (2.27), the entries of $\delta\widetilde{M}_{r-j}\Theta_{2,j}$ belong to J. Thus, the entries de $\delta M_{r-j}\Theta_{4,j}$ belong to J and consequently

$$\boxed{I_1(\Theta_{4,j})I_j(N_j) \cdot I_{r-j}(M_{r-j}) \subset J}. \tag{2.30}$$

It follows from (2.28), (2.29) and (2.30) that

$$(I_1(\Theta_{2,j}) + I_1(\Theta_{3,j}) + I_1(\Theta_{4,j})I_j(N_j) \cdot I_{r-j}(M_{r-j})) \tag{2.31}$$

Since also $I_1(\Theta_{1,j}) \subset J$, we have

$$(I_1(\Theta_{1,j}) + I_1(\Theta_{2,j}) + I_1(\Theta_{3,j}) + I_1(\Theta_{4,j})I_j(N_j) \cdot I_{r-j}(M_{r-j})). \tag{2.32}$$

From this equality it obtains

$$I_j(N_j) \cdot I_{r-j}(M_{r-j})I \subset J \tag{2.33}$$

because

$$I = I_{m-1}(\mathcal{DG}(r)) = I_1(\mathrm{adj}(\mathcal{DG}(r))) = I_1(\Theta_{1,j}) + I_1(\Theta_{2,j}) + I_1(\Theta_{3,j}) + I_1(\Theta_{4,j}).$$

This establishes the assertion above – we note that it contains as a special case (with $j = 0$) the inclusion $I_r(M_r) + I_r(N_r) \subset J : I$. □

Theorem 2.2.7. *Consider the matrix $\mathcal{DG}(r)$ as in (2.19), with $1 \leq r \leq m-2$. Let $I := I_{m-1}(\mathcal{DG}(r))$ denote its ideal of $(m-1)$-minors and J the gradient ideal of $f := \det(\mathcal{DG}(r))$. Then*

(i) *I is a Gorenstein ideal of codimension 4 and maximal analytic spread.*

(ii) *The $(m-1)$-minors of $\mathcal{DG}(r)$ define a birational map $\mathbb{P}^{m^2-\binom{r+1}{2}-1} \dashrightarrow \mathbb{P}^{m^2-1}$ onto a cone over the polar variety of f with vertex cut by $\binom{r+1}{2}$ coordinate hyperplanes.*

(iii) *The conductor $J : I$ has codimension at least $2(m-r) \geq 4$; in particular, J has codimension 4.*

(iv) *If $r \leq m-3$ then I is contained in the unmixed part of J; in particular, if R/J is Cohen–Macaulay then $r = m-2$.*

(v) *If, moreover, $\binom{r+1}{2} \leq m-3$, then I is a prime ideal; in particular, it coincides with the unmixed part of J.*

Proof. (i) The analytic spread follows from Lemma 2.2.3.

The remaining assertions follow from Lemma 2.2.5, which shows that I is a specialization of the ideal of generic submaximal minors, provided we argue that the set

$$\{x_{m,m-r+1},\ x_{m,m-r+2},\ x_{m-1,m-r+2},\ \ldots,\ x_{m,m},\ x_{m-1,m},\ \ldots,\ x_{m-r+1,m}\}$$

of variables on the voided entry places of the generic $m \times m$ matrix

$$\begin{pmatrix}
x_{1,1} & \cdots & x_{1,m-r} & x_{1,m-r+1} & x_{1,m-r+2} & \cdots & x_{1,m-1} & x_{1,m} \\
\vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\
x_{m-r,1} & \cdots & x_{m-r,m-r} & x_{m-r,m-r+1} & x_{m-r,m-r+2} & \cdots & x_{m-r,m-1} & x_{m-r,m} \\
x_{m-r+1,1} & \cdots & x_{m-r+1,m-r} & x_{m-r+1,m-r+1} & x_{m-r+1,m-r+2} & \cdots & x_{m-r+1,m-1} & \\
x_{m-r+2,1} & \cdots & x_{m-r+2,m-r} & x_{m-r+2,m-r+1} & x_{m-r+2,m-r+2} & \cdots & & \\
\vdots & \cdots & \vdots & \vdots & \vdots & \iddots & \vdots & \vdots \\
x_{m-1,1} & \cdots & x_{m-1,m-r} & x_{m-1,m-r+1} & & \cdots & & \\
x_{m,1} & \cdots & x_{m,m-r} & & & \cdots & &
\end{pmatrix}$$

is a subset of $\mathcal{X}$ as in the lemma. But this is immediate because of the assumption $r \leq m-2$.

(ii) Using the same principles as in the proof of Theorem 2.1.2 (iv), one has that the map is birational onto its image. It remains to argue that the image is a cone over the polar variety, the latter as described in Theorem 2.2.1 (iii).

To see this note the homogeneous inclusion $T := k[J_{m-1}] \subset T' := k[I_{m-1}]$ of k-algebras which are domains, where I_{m-1} is minimally generated by the generators of J and by $\binom{r+1}{2}$ additional generators, say, $f_1, \ldots, f_s$, where $s = \binom{r+1}{2}$, that is, $T' = T[f_1, \ldots, f_s]$. On the other hand, by previous item one has $\dim T' = m^2 - \binom{r+1}{2}$ and by Theorem 2.2.1(ii) one has $\dim T = m^2 - r(r+1)$. Therefore, $\text{tr.deg}_{k(T)} k(T)(f_1, \ldots, f_s) = \dim T' - \dim T = \binom{r+1}{2} = s$, where $k(T)$ denotes the field of fractions of T. This means that $f_1, \ldots, f_s$ are algebraically independent over $k(T)$ and, a fortiori, over T. This shows that T' is a polynomial ring over T in $\binom{r+1}{2}$ indeterminates. Geometrically, the image of the map defined by the $(m-1)$-minors is a cone over the polar image with vertex cut by $\binom{r+1}{2}$ independent hyperplanes.

(iii) We will use Proposition 2.2.6, to get the required bound for the codimension of the conductor $J : I$ – the assertion that J has codimension 4 is then ensured as $J \subset I$ and I has codimension 4 by item (i).

For that we need some intermediate results.

Claim 1. For every $1 \leq j \leq r$, both $I_j(M_j)$ and $I_j(N_j)$ have codimension $m - r$.

By the clear symmetry, it suffices to consider $I_j(M_j)$. Note that M_j has $r - j + 1$ null rows, so its ideal of j-minors coincides with the ideal of j-minors of its $(m - (r - j + 1)) \times j$ submatrix M'_j with no null rows. Clearly, this ideal of (maximal) minors has codimension at most $(m - (r - j + 1)) - j + 1 = m - r$. Now, the matrix M'_j specializes to the well-known diagonal specialization using only $m - r$ variables – by definition, the latter is the specialization of a suitable Hankel matrix via the ring homomorphism mapping to zero the variables of the upper left and lower right corner except the last variables of first column and the first variable of the last column. This ensures that M'_j has codimension at least $m-r$.

Claim 2. For every $1 \leq j \leq r-1$, the respective sets of nonzero entries of N_j and M_{r-j+1} are disjoint. In particular, the codimension of $I_j(N_j) + I_{r-j+1}(M_{r-j+1})$ is $2(m-r)$.

The disjointness assertion is clear by inspection and the codimension follows for the previous claim.

To proceed, we envisage the following chains of inclusions

$$I_1(N_1) \supset I_2(N_2) \supset \ldots \supset I_{r-1}(N_{r-1}) \supset I_r(N_r) \tag{2.34}$$

and

$$I_1(M_1) \supset I_2(M_2) \supset \ldots \supset I_{r-1}(M_{r-1}) \supset I_r(M_r). \tag{2.35}$$

Let P denote a prime ideal containing the conductor $J : I$. By Proposition 2.2.6 one has the inclusion $I_1(N) \cdot I_{r-1}(M_{r-1}) \subset P$. Thus,

(A_1) either $I_1(N_1 \subset P$, or else

(B_1) $I_1(N_1) \not\subset J : I$ but $I_{r-1}(M_{r-1}) \subset P$.

If (A_1) is the case, then $I_1(N_1)+I_r(M_r) \subset P$, because $I_r(M_r) \subset J : I$ again by by Proposition 2.2.6 (with $j = 0$). By Claim 2 above, we then see that the codimension of $J : I$ is at least $2(m-r)$.

If (B_1) takes place then we consider the inclusion $I_2(N_2) \cdot I_{r-2}(M_{r-2}) \subset J : I \subset P$ by Proposition 2.2.6. The latter in turn gives rise to two possibilities according to which

(A_2) either $I_2(N_2 \subset P$, or else

(B_2) $I_2(N_2) \not\subset P$ but $I_{r-2}(M_{r-2}) \subset P$.

Again, if (A_2) is the case then $I_2(N_2) + I_{r-1}(M_{r-1}) \subset P$ since $I_{r-1}(M_{r-1}) \subset P$ by hypothesis. Once more, by Claim 2, the codimension of P is at least $2(m-r)$.

If intead (B_2) occurs then we step up to the inclusion $I_3(N_3) \cdot I_{r-3}(M_{r-3}) \subset P$ and repeat the argument. Proceeding in this way, we may eventually find an index $1 \leq j \leq r-1$ such that the first alternative (A_j) holds, in which case we are through always by Claim 2. Otherwise, we must be facing the situation where $I_j(N_j) \not\subset P$ for every $1 \leq j \leq r-1$. In particular, $I_{r-1}(N_{r-1}) \not\subset P$ and $I_1(M_1) \subset P$. Thus, $I_r(N_r)+I_1(M_1) \subset P$, and once more by Claim 2, P has codimension at least $2(m-r)$. This concludes the proof of this item.

The assertion that J has codimension 4 is then ensured as $J \subset I$ and I has codimension 4 by item (i). $\square$

(iv) As I is a Cohen-Macaulay ideal containing J, its associated primes are all of codimension 4. In particular, they are minimal primes ideals of R/J. Now, any minimal prime of J which does not contain I must contain the ideal $J : I$. By (iii), if $r \leq m-3$ then $J : I$ has codimension at least $2(m-3) \geq 6$. This implies I and J^{un} coincide up to radical. We now claim that $I \subset J^{\text{un}}$. For this let P_i denote the associated primes of R/I and let Q_i denote the P_i-primary component of J, for $i = 1, \ldots, r$. Suppose $a \in I \setminus J^{\text{un}}$, say, $a \notin Q_1$. Given any $c \in J : I$ one has $a \cdot c \in J \subset J^{\text{un}} \subset Q_1$. Therefore, $c \in \sqrt{Q_1} = P_1$ and hence $J : I \subset P_1$, forcing $J : I$ to have codimension at most 4. This is a contradiction because $\operatorname{codim}(J : I) \geq 6$.

The first assertion on the Cohen–Macaulayness of R/J is clear since then J is already unmixed, hence $J = I$ which is impossible since $r \geq 1$.

(v) We will apply Proposition 1.2.3 in the case where $\mathcal{M}' = \mathcal{G}$ is an $m \times m$ generic matrix and $\mathcal{M} = \mathcal{DG}(r)$ is the degenerated generic matrix as in the statement. In addition, we take $k = m-2$, so $k+1 = m-1$ is the size of the submaximal minors. Observe that the vector space spanned by the entries of $\mathcal{M}$ has codimension $\binom{r+1}{2}$ in the vector space spanned by the entries of $\mathcal{M}'$. Since the $m \times m$ generic matrix is $2 = m - (m-2)$-generic (it is m-generic as explained in [16, Examples, p. 548]), the theorem ensures that if $\binom{r+1}{2} \leq k - 1 = m - 3$ then $I_{m-1}(\mathcal{DG})$ is prime.

Since in particular $r \leq m-3$ then item (iv) says that $I \subset J^{\text{un}}$. But I is prime, hence $I = J^{\text{un}}$. $\square$

Remark 2.2.8. The statement of item (i) in Theorem 2.2.7 depends not only the number of the entries forming a regular sequence on $\mathcal{P}$ but also their mutual position. Thus, for example, if more than r of the entries belong to one same column or row it may happen that I has codimension strictly less than 4.

Chapter 3

Degenerations of the generic symmetric matrix

In this chapter we deal with particular degenerations of the $m \times m$ generic symmetric matrix:

$$\mathcal{S} := \begin{pmatrix} x_{1,1} & x_{1,2} & \ldots & x_{1,m-1} & x_{1,m} \\ x_{1,2} & x_{2,2} & \ldots & x_{2,m-1} & x_{2,m} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ x_{1,m-1} & x_{2,m-1} & \ldots & x_{m-1,m-1} & x_{m-1,m} \\ x_{1,m} & x_{2,m} & \ldots & x_{m-1,m} & x_{m,m} \end{pmatrix}. \tag{3.1}$$

It is known that the submaximal minors of $\mathcal{S}$ is a prime ideal of codimension 3. Moreover, this ideal is of linear type and linearly presented. As a consequence, $\det(\mathcal{S})$ is homaloidal (see [29] for more details).

In this chapter, the goal is to understand the effect of certain degenerations of $\mathcal{S}$ on its main related structure, such as the determinant of the matrix, the ideal generated by its partial derivatives, the polar map defined by these derivatives, the Hessian matrix and the ideal of the submaximal minors of the matrix.

As in the previous chapter, we first study the degeneration by cloning finding similar results to those obtained in the fully generic case. We emphasize that cloning a variable in the case of symmetric matrices requires additional care, since , for example, symmetry is not preserved by elementary operations. Of course these will preserve homaloidness if it happens to be the case, but it has the inconvenient of trading us out of the class of symmetric matrices.

Finally, we consider the degenerations of $\mathcal{S}$ consisting of replacing a variable in the main diagonal by zero. An analogue of this study to the generic case was handled in [29].

3.1 Degeneration by cloning

We distinguish two sorts of cloning provided the symmetry is preserved: in the one, a variable of the main diagonal is cloned along the main diagonal and in the other an variable $x_{i,j}$ off de main diagonal, such that $i+j$ is even, is cloned in the position $(\frac{i+j}{2}, \frac{i+j}{2})$ on the main diagonal. The situation of the second kind is largely conjectural, while in this section we deal mainly with the first kind of cloning – which, for emphasis, could be refereed as *diagonal cloning*.

By suitable permutation of rows/columns, without affecting symmetry, we move the variable $x_{i,j}$ and its clone to the bottom right of the cloned matrix preserving the symmetry. Thus, we may assume that the entry $x_{m,m}$ is replaced by $x_{m-1,m-1}$, so that the cloned matrix has the form

$$\mathcal{SC} := \begin{pmatrix} x_{1,1} & x_{1,2} & x_{1,3} & \cdots & x_{1,m-2} & x_{1,m-1} & x_{1,m} \\ x_{1,2} & x_{2,2} & x_{2,3} & \cdots & x_{2,m-2} & x_{2,m-1} & x_{2,m} \\ x_{1,3} & x_{2,3} & x_{3,3} & \cdots & x_{3,m-2} & x_{3,m-1} & x_{3,m} \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots \\ x_{1,m-2} & x_{2,m-2} & x_{3,m-2} & \cdots & x_{m-2,m-2} & x_{m-2,m-1} & x_{m-2,m} \\ x_{1,m-1} & x_{2,m-1} & x_{3,m-1} & \cdots & x_{m-2,m-1} & x_{m-1,m-1} & x_{m-1,m} \\ x_{1,m} & x_{2,m} & x_{3,m} & \cdots & x_{m-2,m} & x_{m-1,m} & x_{m-1,m-1} \end{pmatrix}. \quad (3.2)$$

3.1.1 Polar behavior

Throughout we set $f := \det(\mathcal{SC})$ and let $J = J_f$ denote the ideal generated by the partial derivatives of f with respect to the variables of R, the polynomial ring in the entries of $\mathcal{GC}$ over a ground field k.

As in the previous chapter sticking to a more geometric terminology, we let the term *polar* be associated with the behavior of the gradient ideal as defining a rational map and the geometry of this map.

Theorem 3.1.1. *Consider the diagonally cloned matrix as in* (3.2)*. One has:*

(i) *J is a codimension 3 ideal contained in $I_{m-1}(\mathcal{SC})$, the ideal of submaximal minors of $\mathcal{SC}$.*

(ii) *f is irreducible.*

(iii) *The Hessian determinant $H(f)$ does not vanish.*

(iv) *The linear rank of the gradient ideal of f is $\binom{m+1}{2} - 2$ (maximum possible).*

(v) f *is homaloidal.*

Proof. Let $f_{i,j}$ denote the $x_{i,j}$-derivative of f and let $\Delta_{j,i}$ stand for the (signed) cofactor of the (i,j)-entry of $\mathcal{SC}$. Observe that, by symmetry, one has $\Delta_{i,j} = \Delta_{j,i}$.

(i) Note that $f_{i,i} = \Delta_{i,i}$, for $i = 1, \ldots, m-2$, while $f_{m-1,m-1}$ is the sum of the respective (signed) cofactors of $x_{m-1,m-1}$ corresponding to its two appearances (see Theorem 1.2.5). Moreover, the partial derivatives relative to the variables off the main diagonal will be the corresponding (signed) cofactor multiplied by 2. This ensures that $J \subset I_{m-1}(\mathcal{SC})$. Since $\mathcal{SC}$ is a specialization of the $m \times m$ generic symmetric matrix, the codimension of $I_{m-1}(\mathcal{SC})$ is 3. Therefore, the codimension of J is at most 3.

To show that the codimension of J is exactly 3 we consider the initial ideal of J in the reverse lexicographic order. For $m \geq 5$, direct inspection shows that for m odd one has

$$\begin{aligned} \mathrm{in}(f_{1,1}) &= x_{2,m}^2 \cdot x_{3,m-1}^2 \cdots x_{\lfloor\frac{m+2}{2}\rfloor,\lfloor\frac{m+2}{2}\rfloor+1}^2 \\ \mathrm{in}(f_{1,m}) &= 2x_{1,m} \cdot x_{2,m-1}^2 \cdots x_{\frac{m+1}{2}-1,\frac{m+1}{2}+1}^2 \cdot x_{\frac{m+1}{2},\frac{m+1}{2}} \\ \mathrm{in}(f_{m-1,m-1}) &= x_{1,m-1}^2 \cdot x_{2,m-2}^2 \cdots x_{\lfloor\frac{m}{2}\rfloor,\lfloor\frac{m}{2}\rfloor+1}^2 \end{aligned}$$

while for m even it obtains

$$\begin{aligned} \mathrm{in}(f_{1,1}) &= x_{2,m}^2 \cdot x_{3,m-1}^2 \cdots x_{\frac{m+2}{2}-1,\frac{m+2}{2}+1}^2 \cdot x_{\frac{m+2}{2},\frac{m+2}{2}} \\ \mathrm{in}(f_{1,m}) &= 2x_{1,m} \cdot x_{2,m-1}^2 \cdots x_{\lfloor\frac{m+1}{2}\rfloor,\lfloor\frac{m+1}{2}\rfloor+1}^2 \\ \mathrm{in}(f_{m-1,m-1}) &= x_{1,m-1}^2 \cdot x_{2,m-2}^2 \cdots x_{\frac{m}{2},\frac{m}{2}}^2 \end{aligned}$$

Since there are no common variables among the three terms in each bloc, $\mathrm{in}(J)$ has codimension at least 3.

For $m = 3$, an easy verification shows that the monomials $x_{1,2}^2, x_{2,2}^2$ and $x_{1,3}^3$ belong to $\mathrm{in}(J)$. For $m = 4$, which is the hardest case, we resort to a calculation with [1] to find a a minimal set of generators of $\mathrm{in}(J)$:

$$\begin{gathered} x_{2,3}^2x_{3,3} \quad x_{1,4}x_{2,3}x_{2,4} \quad x_{1,4}x_{23}^2 \quad x_{1,4}^2x_{2,3} \quad x_{1,4}^2x_{2,2}x_{3,4} \quad x_{1,4}^2x_{2,2}x_{3,3} \quad x_{1,4}^2x_{2,2}x_{2,4} \\ x_{1,4}^3x_{2,2} \quad x_{1,3}x_{2,3}x_{3,3} \quad x_{1,3}x_{1,4}x_{2,3} \quad x_{1,3}x_{1,4}x_{2,2} \quad x_{1,3}^2x_{3,3} \quad x_{1,3}^2x_{2,2} \end{gathered}$$

It suffices to observe that there are not two variables which divide all these monomials. Equivalently, one can observe that the log matrix of these monomials

$$\log(\mathrm{in}(J)) = \begin{pmatrix} 0&0&0&0&0&2&0&1&0\\ 0&0&0&1&0&1&1&0&0\\ 0&0&0&1&0&2&0&0&0\\ 0&0&0&2&0&1&0&0&0\\ 0&0&0&2&1&0&0&0&1\\ 0&0&0&2&1&0&0&1&0\\ 0&0&0&2&1&0&1&0&0\\ 0&0&0&3&1&0&0&0&0\\ 0&0&1&0&0&1&0&1&0\\ 0&0&1&1&0&1&0&0&0\\ 0&0&1&1&1&0&0&0&0\\ 0&0&2&0&0&0&0&1&0\\ 0&0&2&0&1&0&0&0&0 \end{pmatrix}$$

is such that any submatrix of the above matrix consisting of two columns has a null row.

(ii) Since $\operatorname{codim} J = 3$ by (i), then $R/(f)$ satisfies the property (R_1) of Serre and hence it is normal. Since f is homogeneous, $R/(f)$ is a domain.

An alternative argument goes by induction on the size $m (m \geq 3)$ of the matrix, as follows. We omit the initial step as it is contained in the inductive step with a slight adaptation.

By Laplace expansion along the first row, one sees that

$$f = x_{1,1}\Delta_{1,1} + g,$$

where $\Delta_{1,1}$ is the determinant of the $(m-1)\times(m-1)$ cloned symmetric matrix obtained from $\mathcal{SC}$ by omitting the first row and the first column. Note that both $\Delta_{1,1}$ and g belong to the subring $A := k[x_{1,2}, \ldots, x_{1,m}, x_{2,2}, \ldots, x_{2,m}, \ldots, x_{m-1,m-1}, x_{m-1,m}]$.

To show that f is irreducible it suffices to prove that it is primitive as a polynomial in the ring $A[x_{1,1}]$. As such, since it has degree 1, it is enough to show that $\gcd(\Delta_{1,1}, g) = 1$. We will argue via initial ideals assuming the revlex monomial order. In the generic symmetric case $\mathrm{in}(f)$ is well-known to be the product of the entries along the main subdiagonal of $\mathcal{SC}$ – same will work here since cloning at the rightmost corner will not affect this result. In particular, $\mathrm{in}(f)$ is not a term of $x_{1,1}\Delta_{1,1}$, hence it is necessarily the initial term of g as well, that is, $\mathrm{in}(g) = \mathrm{in}(f)$.

Let $m = 3$. It suffices to prove that g itself is irreducible. A straightforward calculation yields $g = -(x_{1,2}^2 + x_{1,3}^2)x_{2,2} + 2x_{1,2}x_{1,3}x_{2,3}$. Clearly, as a polynomial in $x_{2,2}$ its coefficients have no proper common factor. By the same token, g is irreducible.

Let $m \geq 4$. This time around $\Delta_{1,1}$ is irreducible by the inductive hypothesis. Therefore, it suffices to show that neither $x_{1,1}$ nor $\mathrm{in}(\Delta_{1,1})$ divides $\mathrm{in}(g) = \mathrm{in}(f)$, which is pretty clear as well.

(iii) The proof is essentially the same as that of Theorem 2.1.1(ii), with easy adaptations. We argue by a specialization procedure, namely, consider the ring endomorphism φ of R by mapping any variable in $\mathbf{v}$ to itself and by mapping any variable off $\mathbf{v}$ to zero, where $\mathbf{v} := \{x_{1,1}, x_{2,2}, x_{3,3}, \ldots, x_{m-1,m-1}\}$ is the set of variables along the main diagonal. Clearly, it suffices to show that by applying φ to the entries of the Hessian matrix $\mathcal{H}(f)$ of f the resulting matrix, which we denote $\mathcal{H}'$, has a nonzero determinant.

Recall that the partial derivative of f with respect to any $x_{i,i} \in \mathbf{v}$ coincides with the (signed) cofactor of $x_{i,i}$, for $i = 1, \ldots, m-2$, while for $i = m-1$ it is the sum of the respective (signed) cofactors of $x_{i,i}$ corresponding to its two appearances.

By expanding each such a cofactor according to the Leibniz rule it is clear that it has a unique (nonzero) term whose support lies in $\mathbf{v}$ and, moreover, the remaining terms have degree at least 2 in the variables off $\mathbf{v}$. Observe that in the two cofactors of $x_{m-1,m-1}$ the terms supported in the variables of $\mathbf{v}$ coincide.

Now, for $x_{i,j} \notin \mathbf{v}$, the corresponding partial derivative is the sum of the respective (signed) cofactors of $x_{i,j}$ corresponding to its two appearances (see Theorem 1.2.5). Since these cofactors coincide, one has $f_{i,j} = 2\Delta_{i,j}$, where $\Delta_{i,j}$ is the (signed) cofactor of the (i,j)-entry. Expanding $\Delta_{i,j}$ according to the Leibniz rule yields :

$$\Delta_{ij} = x_{i,j}(x_{1,1} \ldots \widehat{x_{i,i}} \ldots \widehat{x_{j,j}} \ldots x_{m-1,m-1}x_{m-1,m-1}) + (\text{terms of degree at least 2 off } \mathbf{v}).$$

So, $\Delta_{i,j}$ has no term whose support lies in $\mathbf{v}$ and has exactly one nonzero term of degree 1 in the variables off $\mathbf{v}$.

By the preceding observation, applying φ to any second partial derivative of f will return zero or a monomial supported on the variables in $\mathbf{v}$. Thus, the entries of the specialized Hessian $\mathcal{H}'$ are zeros or monomials supported on the variables in $\mathbf{v}$.

To see that the determinant of the matrix $\mathcal{H}'$ is nonzero, consider the Jacobian matrix of the set of partial derivatives $\{f_v \,|\, v \in \mathbf{v}\}$ with respect to the variables in $\mathbf{v}$. Let M_0 denote the specialization of this Jacobian matrix by φ, considered as a corresponding submatrix of $\mathcal{H}'$. Up to permutation of rows and columns of $\mathcal{H}'$, we may write

$$\mathcal{H}' = \begin{pmatrix} M_0 & N \\ P & M_1 \end{pmatrix},$$

where M_1 has exactly one nonzero entry on each row and each column. Now, by the way the second partial derivatives of f specialize via φ, as explained above, one must have $N = P = 0$. Therefore, $\det(M) = \det(M_0)\det(M_1)$, so it remains to prove the nonvanishing of these two subdeterminants.

Now the first block is the Hessian matrix of the form

$$g := \left(\prod_{i=1}^{m-2} x_{i,i}\right) x_{m-1.m-1}^2.$$

As argued in the Theorem 2.1.1(ii) g is a classical homaloidal polynomial, hence we are done for the first matrix block.

As for the second block, by construction it has exactly one nonzero entry on each row and each column. Therefore, it has a nonzero determinant.

(iv) Using the Cauchy cofactor equality

$$\mathcal{SC} \cdot \mathrm{adj}(\mathcal{SC}) = \det(\mathcal{SC})\, \mathbb{I}_m \tag{3.3}$$

we derive the following set of linear linear relations involving the cofactors of $\mathcal{SC}$:

$$\sum_{j=1}^{m} \widehat{x_{i,j}} \Delta_{j,1} = 0, \text{ for } 2 \leq i \leq m-1; \tag{3.4}$$

$$\sum_{j=1}^{m} \widehat{x_{i,j}} \Delta_{j,k} = 0, \text{ for } 2 \leq k \leq m-2 \text{ and } k-1 \leq i \leq m-1 \ (k \neq i); \tag{3.5}$$

$$\sum_{j=1}^{m-1} \widehat{x_{m,j}} \Delta_{j,k} + x_{m-1,m-1}\Delta_{m,k} = 0, \text{ for } 1 \leq k \leq m-2; \tag{3.6}$$

where $\widehat{x_{i,j}} = x_{i,j}$ if $i \leq j$ and $\widehat{x_{i,j}} = x_{j,i}$ if $i \geq j$.

Note that the partial derivative of f with respect to the variables off the main diagonal will be the corresponding (signed) cofactor multiplied by 2, while the partial derivatives $f_{i,i}$ is the the (signed) cofactor $\Delta_{i,i}$ for all $i \neq m-1$. Thus, any of the above relations gives a linear syzygy of the partial derivatives of f.

In addition, 3.3 yields the following linear relations:

$$\sum_{i=1}^{m-1} x_{i,m-1}\Delta_{i,m} + x_{m-1,m}\Delta_{m,m} = 0 \tag{3.7}$$

$$\sum_{i=1}^{m-2} x_{i,m}\Delta_{i,m-1} + x_{m-1,m}\Delta_{m-1,m-1} + x_{m-1,m-1}\Delta_{m,m-1} = 0 \tag{3.8}$$

$$\sum_{i=1}^{m-2} x_{i,m-1}\Delta_{i,m-1} + x_{m-1,m-1}\Delta_{m-1,m-1} + x_{m-1,m}\Delta_{m,m-1} = \sum_{j=1}^{m} x_{1,j}\Delta_{j,1} \tag{3.9}$$

$$\sum_{j=1}^{m-1} x_{j,m}\Delta_{j,m} + x_{m-1,m-1}\Delta_{m,m} = \sum_{j=1}^{m} x_{1,j}\Delta_{j,1}. \tag{3.10}$$

As $f_{m-1,m-1} = \Delta_{m-1,m-1} + \Delta_{m,m}$, adding (3.7) to (3.8) and (3.9) to (3.10) outputs two additional linear syzygies of the partial derivatives of f. Thus one has counted a total of $(m-1)+(m-1)+(m-2)+\ldots+3+2 = \binom{m+1}{2} - 2$ linear syzygies of J.

We need to show that these are independent. For this we order the set of partial derivatives $f_{i,j}$ in accordance with the following ordered list of the entries $x_{i,j}$:

$$x_{1,1}, x_{1,2}, \ldots, x_{1,m} \rightsquigarrow x_{2,2}, \ldots, x_{2,m} \rightsquigarrow \ldots \rightsquigarrow x_{m-2,m-2}, x_{m-2,m-1} x_{m-2,m} \rightsquigarrow x_{m-1,m-1}, x_{m-1,m}.$$

Here we traverse the entries along the matrix rows, left to right, until exhausting all variables.

We now claim that, the above sets of linear relations can be grouped into the following block matrix of linear syzygies:

$$\left(\begin{array}{cccccc|cc}
\varphi_1 & \cdots & & & & & & \\
0^{m-1}_{m-1} & \varphi_2 & \cdots & & & & & \\
0^{m-1}_{m-2} & 0^{m-1}_{m-2} & \varphi_3 & & & & & \\
\vdots & \vdots & \vdots & \ddots & & & & \\
0^{m-1}_{4} & 0^{m-1}_{4} & 0^{m-2}_{4} & \cdots & \varphi_{m-3} & & & \\
0^{m-1}_{3} & 0^{m-1}_{3} & 0^{m-2}_{3} & \cdots & 0^{4}_{3} & \varphi_{m-2} & & \\
\hline
0^{m-1}_{1} & 0^{m-1}_{1} & 0^{m-2}_{1} & \cdots & 0^{4}_{1} & 0^{3}_{1} & x_{m-1,m} & x_{m-1,m-1} \\
0^{m-1}_{1} & 0^{m-1}_{1} & 0^{m-2}_{1} & \cdots & 0^{4}_{1} & 0^{3}_{1} & x_{m-1,m-1} & x_{m-1,m}
\end{array}\right),$$

where:

- φ_1 is the matrix obtained from $(\mathcal{SC})^t$ obtained by multiplying the first row by 2 and omitting the first column.
- φ_2 is the matrix obtained from $(\mathcal{SC})^t$ by multiplying the second row by 2 and omitting the second column and the first row .
- For $l = 3, \ldots, m-2$, φ_l is the matrix obtained from $(\mathcal{SC})^t$ by multiplying the lth row by 2 and omitting the columns $1, \ldots, l-2, l$ and the rows $1, \ldots, l-1$.
- 0^c_r denotes a zero block of size $r \times c$.

Next we justify why these blocks make up (linear) syzygies.

First, as already observed, the relations (3.4) through (3.10) yield linear syzygies of the partial derivatives of f.

Since $\Delta_{i,i} = f_{i,i}$, for all $i \neq m-1$, and $\Delta_{j,i} = 1/2 f_{i,j}$, for all $i \neq j$, the relation (3.4) can be written as

$$\widehat{x_{i,1}} 2 f_{1,1} + \sum_{j=2}^{m} \widehat{x_{i,j}} f_{1,j} = 0,$$

for $i = 2, \ldots, m-1$. Moreover, setting $k = 1$ in the relation (3.6) they can be written as

$$\widehat{x_{m,1}} 2 f_{1,1} + \sum_{j=2}^{m-1} \widehat{x_{m,j}} f_{1,j} + x_{m-1,m-1} f_{1,m} = 0$$

Ordering the set of partial derivatives $f_{i,j}$ as explained before, the coefficients of these relations form the first matrix above

$$\varphi_1 := \begin{pmatrix} 2\widehat{x_{2,1}} & 2\widehat{x_{3,1}} & \cdots & 2\widehat{x_{m-1,1}} & 2\widehat{x_{m,1}} \\ \widehat{x_{2,2}} & \widehat{x_{3,2}} & \cdots & \widehat{x_{m-1,2}} & \widehat{x_{m,2}} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ \widehat{x_{2,m-1}} & \widehat{x_{3,m-1}} & \cdots & \widehat{x_{m-1,m-1}} & \widehat{x_{m,m-1}} \\ \widehat{x_{2,m}} & \widehat{x_{3,m}} & \cdots & \widehat{x_{m-1,m}} & x_{m-1,m-1} \end{pmatrix} = \begin{pmatrix} 2x_{1,2} & 2x_{1,3} & \cdots & 2x_{1,m-1} & 2x_{1,m} \\ x_{2,2} & x_{2,3} & \cdots & x_{2,m-1} & x_{2,m} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ x_{2,m-1} & x_{3,m-1} & \cdots & x_{m-1,m-1} & x_{m-1,m} \\ x_{2,m} & x_{3,m} & \cdots & x_{m-1,m} & x_{m-1,m-1} \end{pmatrix}.$$

Note that φ_1 coincides indeed with the submatrix of $\mathcal{GC}^t$ obtained by multiplying the first row by 2 and omitting the first column.

Continuing, for each $l = 2, \ldots, m-2$ the block φ_l comes from the relation (3.5) and (3.6) (setting $k = l$). Finally, the lower right corner 2×2 block of the matrix of linear syzygies comes from the two last relations obtained by adding (3.7) to (3.8) and (3.9) to (3.10).

This proves the claim about the large matrix above. Counting through the sizes of the various blocks, one sees that this matrix is $(\binom{m+1}{2} - 1) \times (\binom{m+1}{2} - 2)$. Omitting its first row obtains a block-diagonal submatrix of size $(\binom{m+1}{2} - 2) \times (\binom{m+1}{2} - 2)$, where each block has nonzero determinant. Thus, the linear rank of J attains the maximum.

(v) By (iii) the polar map of f is dominant. Since the linear rank is maximum by (iv), one can apply Theorem 1.3.2 to conclude that f is homaloidal. $\square$

Remark 3.1.2. The idea used in the proof that the Hessian determinants of the cloned generic and symmetric matrices are nonzero can be used to prove that Hessian determinants of the respective generic matrices are nonzero as well – of course, this outcome is a well-known result.

Conjecture 3.1.3. *If $i \neq j$ and $i + j$ is even then under the cloning $x_{i,j} \rightsquigarrow x_{\frac{i+j}{2}, \frac{i+j}{2}}$ the determinant of the resulting matrix is not homaloidal.*

3.1.2 Primality

In this part we study the nature of the ideal of submaximal minors of $\mathcal{SC}$. As previously, J denotes the gradient ideal of $f = \det(\mathcal{SC})$. Below we show that $P =$

$I_{m-1}(\mathcal{SC})$ is the minimal component of the primary decomposition of J in R – a result analogue to Theorem 2.1.2.

In the previous chapter, showing the primeness of the ideal of submaximal minors we required a result of Eisenbud drawn upon the 2-generic property of the generic matrix. Since the generic symmetric matrix is not 2-generic, we can not use the same argument. So, we will show that $R/I_{m-1}(\mathcal{SC})$ is normal – and, hence a domain since $I_{m-1}(\mathcal{SC})$ is a homogeneous ideal.

Proposition 3.1.4. *Consider the matrix $\mathcal{SC}$ as in (3.2), with $m \geq 4$. Let $P := I_{m-1(\mathcal{SC})}$ denote its ideal of submaximal minors. Then*

(i) *P is a Cohen-Macaulay prime ideal of codimension 3.*

(ii) *P is the minimal component of the primary decomposition of J in R.*

(iii) *J defines a double structure on the variety defined by P, with a unique embedded component and the latter is a linear space of codimension $2(m-1)$.*

(iv) *The $(m-1)$-minors of $\mathcal{SC}$ define a birational map $\mathbb{P}^{\binom{m+1}{2}-2} \dashrightarrow \mathbb{P}^{\binom{m+1}{2}-1}$ onto a hypersurface of degree $m-1$ with defining equation $\mathbb{D}_{m,m} - \mathbb{D}_{m-1,m-1}$, where $\mathbb{D}_{m,m}$ and $\mathbb{D}_{m-1,m-1}$ denote the cofactors of $y_{m,m}$ and $y_{m-1,m-1}$, respectively, in the $m \times m$ generic symmetric matrix $(y_{i,j})_{1\leq i\leq j\leq m}$.*

Proof. We may assume that the resulting matrix has the shape as in (3.2).

(i) The easiest part is that the ideal P is a Cohen-Macaulay ideal of codimension 3 since it specializes from the symmetric generic case.

As P is homogeneous, to prove that P is prime it suffices to show that it is a normal ideal. It certainly satisfies Serre's property (S_2), since it is a Cohen-Macaulay ideal. Therefore, it remains to prove that it satisfies condition (R_1). For this, let θ denote the Jacobian matrix of the generators of P with respect to all variables in sight. The goal is to show that $\operatorname{codim}(I_3(\theta) + P) \geq 3 + 2$.

We will argue via initial ideal assuming the revlex monomial order induced by the ordering the variables in the sequence in which they appear in the matrix respecting the rows. In the generic symmetric case, the initial of a minor of size $(m-2) \times (m-2)$ or $(m-1) \times (m-1)$ is well-known to be the product of the entries along the its main anti-diagonal. Cloning at the rightmost corner will not affect this result, for $m \geq 6$. Indeed, Let M denote a $(m-2)$-minor or a $(m-1)$-minor of $\mathcal{SC}$ and let D denote the product of the entries along the its main anti-diagonal. Observe that each variable $x_{i,j}$ in the anti-diagonal of M satisfies $x_{i,j} \geq x_{(m+2)/2,(m+2)/2}$ (if m is even) or $x_{i,j} \geq x_{(m+3)/2,(m+3)/2}$ (if m is odd). So, when $m \geq 6$ one has $x_{i,j} > x_{m-1,m-1}$ since that $x_{(m+2)/2,(m+2)/2} > x_{m-1,m-1}$ (if m is even) or $x_{(m+3)/2,(m+3)/2} > x_{m-1,m-1}$ (if m is odd).

This ensures that, in the revlex monomial order, any monomial involving the cloned variable $x_{m-1,m-1}$ is smaller than D and, therefore, D is leading term of $\det(M)$.

Next, we will show that $\operatorname{codim}(I_3(\theta)+P) \geq 3+2$ for $m \geq 6$. We consider the following submatrices of θ:

$$\theta_1 := \begin{pmatrix} \partial\Delta_{2,2}/\partial x_{1,1} & \partial\Delta_{1,2}/\partial x_{1,1} & \partial\Delta_{1,1}/\partial x_{1,1} \\ \partial\Delta_{2,2}/\partial x_{1,2} & \partial\Delta_{1,2}/\partial x_{1,2} & \partial\Delta_{1,1}/\partial x_{1,2} \\ \partial\Delta_{2,2}/\partial x_{2,2} & \partial\Delta_{1,2}/\partial x_{2,2} & \partial\Delta_{1,1}/\partial x_{2,2} \end{pmatrix}$$

$$\theta_2 := \begin{pmatrix} \partial\Delta_{m,m}/\partial x_{1,1} & \partial\Delta_{1,m}/\partial x_{1,1} & \partial\Delta_{1,1}/\partial x_{1,1} \\ \partial\Delta_{m,m}/\partial x_{1,m} & \partial\Delta_{1,m}/\partial x_{1,m} & \partial\Delta_{1,1}/\partial x_{1,m} \\ \partial\Delta_{m,m}/\partial x_{m-1,m-1} & \partial\Delta_{1,m}/\partial x_{m-1,m-1} & \partial\Delta_{1,1}/\partial x_{m-1,m-1} \end{pmatrix}$$

and

$$\theta_3 := \begin{pmatrix} \partial\Delta_{m-1,m}/\partial x_{m-2,m} & \partial\Delta_{m-1,m-1}/\partial x_{m-2,m} & \partial\Delta_{m-2,m}/\partial x_{m-2,m} \\ \partial\Delta_{m-1,m}/\partial x_{m-1,m-1} & \partial\Delta_{m-1,m-1}/\partial x_{m-1,m-1} & \partial\Delta_{m-2,m}/\partial x_{m-1,m-1} \\ \partial\Delta_{m-1,m}/\partial x_{m-1,m} & \partial\Delta_{m-1,m-1}/\partial x_{m-1,m} & \partial\Delta_{m-2,m}/\partial x_{m-1,m} \end{pmatrix}$$

A close inspection of the cofactors $\Delta_{1,1}$, $\Delta_{1,2}$, $\Delta_{2,2}, \Delta_{1,m}$, $\Delta_{m,m}$ give us that the partial $\partial\Delta_{1,2}/\partial x_{1,1}$, $\partial\Delta_{1,1}/\partial x_{1,1}$, $\partial\Delta_{1,1}/\partial x_{1,2}$ $\partial\Delta_{1,m}/\partial x_{1,1}$ and $\partial\Delta_{1,1}/\partial x_{1,m}$ are zeros. By Proposition 1.2.5, the partial derivative of $\Delta_{i,j}$ with respect to $x_{k,l}$ is the sum of the (signed) cofactors on $\Delta_{i,j}$ of the entry $x_{k,l}$, in all its appearances as an entry of $\Delta_{i,j}$. Thus, the partial derivatives $\partial\Delta_{2,2}/\partial x_{1,1}$, $\partial\Delta_{1,2}/\partial x_{1,2}$ and $\partial\Delta_{1,1}/\partial x_{2,2}$ coincide, up to a sign, with the determinant of the matrix

$$M_1 = \begin{pmatrix} x_{3,3} & \dots & x_{3,m-2} & x_{3,m-1} & x_{3,m} \\ \dots & \vdots & \vdots & \vdots & \vdots \\ x_{3,m-2} & \dots & x_{m-2,m-2} & x_{m-2,m-1} & x_{m-2,m} \\ x_{3,m-1} & \dots & x_{m-2,m-1} & x_{m-1,m-1} & x_{m-1,m} \\ x_{3,m} & \dots & x_{m-2,m} & x_{m-1,m} & x_{m-1,m-1} \end{pmatrix}$$

obtained of $\mathcal{SC}$ by omitting the two first columns and the two first rows. Moreover, $\partial\Delta_{m,m}/\partial x_{1,1} = \det(M_2)$, $\partial\Delta_{1,m}/\partial x_{1,m} = -\det(M_2)$ and $\partial\Delta_{1,1}/\partial x_{m-1,m-1} = \det(M_2) + \det(M_3)$, where

$$M_2 = \begin{pmatrix} x_{2,2} & \dots & x_{2,m-2} & x_{2,m-1} \\ \vdots & \vdots & \vdots & \vdots \\ x_{2,m-2} & \dots & x_{m-2,m-2} & x_{m-2,m-1} \\ x_{2,m-1} & \dots & x_{m-2,m-1} & x_{m-1,m-1} \end{pmatrix} \text{ and } M_3 = \begin{pmatrix} x_{2,2} & \dots & x_{2,m-2} & x_{2,m} \\ \vdots & \vdots & \vdots & \vdots \\ x_{2,m-2} & \dots & x_{m-2,m-2} & x_{m-2,m} \\ x_{2,m} & \dots & x_{m-2,m} & x_{m-1,m-1} \end{pmatrix}$$

are obtained of $\mathcal{SC}$ by omitting the first column and the first row; and the mth (respectively, the $(m-1)$-th) column and mth (respectively, the $(m-1)$-th) row. Thus,

$$\det(\theta_1) = \det\begin{pmatrix} \det(M_1) & 0 & 0 \\ \partial\Delta_{2,2}/\partial x_{1,2} & -\det(M_1) & 0 \\ \partial\Delta_{2,2}/\partial x_{2,2} & \partial\Delta_{1,2}/\partial x_{2,2} & \det(M_1) \end{pmatrix} = -\det(M_1)^3$$

and

$$\det(\theta_2) = \det\begin{pmatrix} \det(M_2) & 0 & 0 \\ \partial\Delta_{m,m}/\partial x_{1,m} & -\det(M_2) & 0 \\ \partial\Delta_{m,m}/\partial x_{m,m} & \partial\Delta_{1,m}/\partial x_{m,m} & \det(M_2)+\det(M_3) \end{pmatrix}$$
$$= -\det(M_2)^3 - \det(M_2)^2\det(M_3)$$

Turning our attention to the cofactors $\Delta_{m-1,m}$, $\Delta_{m-1,m-1}$ and $\Delta_{m-2,m}$ we observe that the partial derivatives $\Delta_{m-1,m}/\partial x_{m-1,m-1}$ and $\partial\Delta_{m-1,m-1}/\partial x_{m-1,m}$ are zeros and by Proposition 1.2.5, we have

$$\partial\Delta_{m-1,m}/\partial x_{m-2,m} = \det(M_4) = \partial\Delta_{m-2,m}/\partial x_{m-1,m};$$

$$\partial\Delta_{m-1,m}/\partial x_{m-1,m} = -\det(M_5) = -\partial\Delta_{m-1,m-1}/\partial x_{m-1,m-1};$$

$$\partial\Delta_{m-1,m-1}/\partial x_{m-2,m} = 2\partial\Delta_{m-2,m}/\partial x_{m-1,m-1} = -2\det(M_6);$$

$$\partial\Delta_{m-2,m}/\partial x_{m-2,m} = -\det(M_7);$$

where

$$M_4 = \begin{pmatrix} x_{1,1} & \cdots & x_{1,m-3} & x_{1,m-2} \\ \vdots & \vdots & \vdots & \vdots \\ x_{1,m-3} & \cdots & x_{m-3,m-3} & x_{m-3,m-2} \\ x_{1,m-1} & \cdots & x_{m-3,m-1} & x_{m-2,m-1} \end{pmatrix}, \quad M_5 = \begin{pmatrix} x_{1,1} & \cdots & x_{1,m-3} & x_{1,m-2} \\ \vdots & \vdots & \vdots & \vdots \\ x_{1,m-3} & \cdots & x_{m-3,m-3} & x_{m-3,m-2} \\ x_{1,m-2} & \cdots & x_{m-3,m-2} & x_{m-2,m-2} \end{pmatrix},$$

$$M_6 = \begin{pmatrix} x_{1,1} & \cdots & x_{1,m-3} & x_{1,m-2} \\ \vdots & \vdots & \vdots & \vdots \\ x_{1,m-3} & \cdots & x_{m-3,m-3} & x_{m-3,m-2} \\ x_{1,m} & \cdots & x_{m-3,m} & x_{m-2,m} \end{pmatrix} \text{ and } M_7 = \begin{pmatrix} x_{1,1} & \cdots & x_{1,m-3} & x_{1,m-1} \\ \vdots & \vdots & \vdots & \vdots \\ x_{1,m-3} & \cdots & x_{m-3,m-3} & x_{m-3,m-1} \\ x_{1,m-1} & \cdots & x_{m-3,m-1} & x_{m-1,m-1} \end{pmatrix}$$

Thus,

$$\det(\theta_3) = \begin{pmatrix} \det(M_4) & -2\det(M_6) & -\det(M_7) \\ 0 & \det(M_5) & -\det(M_6) \\ -\det(M_5) & 0 & \det(M_4) \end{pmatrix}$$
$$= \det(M_4)^2 \det(M_5) - \det(M_5)^2 \det(M_7) - 2\det(M_6)^2 \det(M_5) \tag{3.11}$$

We claim that the ideal $(\Delta_{1,1},\ \Delta_{m,m},\ \det(\theta_1),\ \det(\theta_2),\ \det(\theta_3)) \subset I_3(\theta) + P$ has codimension 5. For this, we consider the initial ideal of $(\Delta_{1,1},\ \Delta_{m,m},\ \det(\theta_1),\ \det(\theta_2),\ \det(\theta_3))$ in the reverse lexicographic order. We saw that, when $m \geq 6$, the initial of an $(m-2)$-minor or an $(m-1)$-minor still is the product of the entries along its main anti-diagonal. Thus, $\mathrm{in}(\Delta_{1,1}) = -\prod_{i+j=m+2} x_{i,j}$, $\mathrm{in}(\Delta_{m,m}) = -\prod_{i+j=m} x_{i,j}$ and $\mathrm{in}(\det(\theta_1)) = (\mathrm{in}(\det(M_1)))^3 = -\left(\prod_{i+j=m+3} x_{i,j}\right)^3$.

Now,

$$\mathrm{in}(\det(\theta_2)) = -\mathrm{in}(\det(M_2)^2) \cdot \max\left\{\mathrm{in}(\det(M_2)), \mathrm{in}(\det(M_3))\right\}.$$

Observing that

$$\mathrm{in}(\det(M_2)) = -\prod_{i+j=m+1, i\neq 1,2} x_{i,j} x_{2,m-1}^2 \quad \text{and} \quad \mathrm{in}(\det(M_3)) = -\prod_{i+j=m+1, i\neq 1,2} x_{i,j} x_{2,m}^2$$

and since in the revlex monomial order $x_{2,m}$ is smaller than $x_{2,m-1}$, we conclude that $\mathrm{in}(\det(\theta_2)) = (\prod_{i+j=m+1, i\neq 1} x_{i,j})^3$.

Finally, we consider

$$\det(\theta_3) = -2\det(M_6)^2 \det(M_5) + \det(M_4)^2 \det(M_5) - \det(M_5)^2 \det(M_7).$$

Let D denote the product of all variables along the main diagonal of M_5 excluding the variables $x_{1,m-2}$, that is, $D = \prod_{i+j=m-1, i\neq 1} x_{i,j}$. We observe that

- $\mathrm{in}(\det(M_4)) = -x_{1,m-2}x_{1,m-1}D$;
- $\mathrm{in}(\det(M_5)) = -x_{1,m-2}^2 D$;
- $\mathrm{in}(\det(M_6)) = -x_{1,m-2}x_{1,m}D$;
- $\mathrm{in}(\det(M_7)) = -x_{1,m-1}^2 D$.

Thus,

$$\mathrm{in}(-2\det(M_6)^2 \det(M_5)) = 2x_{1,m-2}^4 x_{1,m}^2 D^3$$

and

$$\operatorname{in}(\det(M_4))^2\det(M_5)) = -x_{1,m-2}^4 x_{1,m-1}^2 D^3 = (\det(M_5))^2\det(M_7).$$

Consequently, $\operatorname{in}(\det(M_4)^2\det(M_5) - \det(M_5)^2\det(M_7)) < x_{1,m-2}^4 x_{1,m-1}^2 D^3$ and

$$\operatorname{in}(\det(\theta_3)) \leq \max\left\{2x_{1,m-2}^4 x_{1,m}^2 D^3,\ \operatorname{in}(\det(M_4)^2\det(M_5) - \det(M_5)^2\det(M_7))\right\}.$$

As the variable $x_{1,m}$ does not appears in the submatrices M_4, M_5 and M_6 of $\mathcal{DS}$, we have that the polynomial $(\det(M_4))^2\det(M_5) - (\det(M_5))^2\det(M_7)$ does note have the monomials $x_{1,m-2}^4 x_{1,m-1} x_{1,m} D^3$ or $x_{1,m-2}^4 x_{1,m}^2 D^3$. This ensures that $\operatorname{in}(\det(\theta_3)) = 2x_{1,m-2}^4 x_{1,m}^2 D^3$.

Since there are no common variables among the monomials

$$\operatorname{in}(\Delta_{1,1}) = -\prod_{i+j=m+2} x_{i,j}, \quad \operatorname{in}(\Delta_{m,m}) = -\prod_{i+j=m} x_{i,j}, \quad \operatorname{in}(\det(\theta_1)) = -\left(\prod_{i+j=m+3} x_{i,j}\right)^3$$

,

$$\operatorname{in}(\det(\theta_2)) = \left(\prod_{i+j=m+1, i\neq 1} x_{i,j}\right)^3 \quad \text{and} \quad \operatorname{in}(\det(\theta_3)) = 2x_{1,m-2}^4 x_{1,m}^2 D^3,$$

it follows that $(\Delta_{1,1}, \Delta_{m,m}, \det(\theta_1), \det(\theta_2), \det(\theta_3))$ contained in $I_3(\theta) + P$ has codimension 5 and hence $I_3(\theta) + P$ has codimension at least 5. Therefore, P is normal.

For $m = 4$, an easy verification shows that the monomials $x_{3,3}^6$, $x_{2,4}^6$, $x_{2,3}^6$, $x_{1,4}^6$ and $x_{1,3}^6$ belong to $\operatorname{in}(I_3(\theta))$. For $m = 5$, which is the more complicated case, we resort to a calculation with [3] to find the following monomials in the initial ideal of $I_3(\theta) + P$: $x_{1,3}^6 x_{2,2}^3 x_{4,5}$, $x_{1,4}^2 x_{2,3}^2$, $x_{1,5}^3 x_{2,4}^6$, $x_{2,5}^6 x_{3,3}^3$ and $x_{3,4}^6 x_{4,4}^3$. So, for $m = 4, 5$ we have $\operatorname{codim}(I_3(\theta) + P) \geq 3 + 2$ and thereby P is prime ideal.

(ii) By item (i), P is a prime ideal of codimension 3. As in the proof of the analogous result for the cloned generic matrix, we first show that $\operatorname{codim}(J : P) > 3$, which ensures that the radical of the unmixed part of J has no primes of codimension < 3 and coincides with P. Here, the partial derivatives, except $f_{m-1,m-1}$ which coincides with the sum of cofactors $\Delta_{m-1,m-1} + \Delta_{m,m}$, are scalar multiple of the cofactors and so we can write $P = (J, \Delta_{m,m})$ and $P = (J, \Delta_{m-1,m-1})$ as in the Theorem 2.1.2 and both ways of writing P will be needed to complete $\operatorname{codim}(J : P) > 3$.

Since $f_{m-1,m-1} = \Delta_{m-1,m-1} + \Delta_{m,m}$ and the others partial derivatives $f_{i,j}$ coincide with $2\Delta_{i,j}$ (when $i \neq j$) or $\Delta_{i,j}$ (when $i = j$), we can write $P = (J, \Delta_{m,m})$ and $P = (J, \Delta_{m-1,m-1})$, where $\Delta_{m,m}$ and $\Delta_{m-1,m-1}$ denote the cofactors of the entries in the positions (m, m) and $(m-1, m-1)$, respectively. In particular $J : P = J : \Delta_{m,m}$ and $J : P = J : \Delta_{m-1,m-1}$.

From the cofactor identity yields the following relations:

$$\begin{aligned}\sum_{j=1}^{m} \widehat{x_{k,j}}\Delta_{j,m} &= 0, \text{ for } k=1,\ldots,m-1\\ \sum_{i=1}^{m-1} x_{i,m}\Delta_{i,m} + x_{m-1,m-1}\Delta_{m,m} &= \sum_{j=1}^{m} x_{1,j}\Delta_{1,j};\end{aligned}$$

where $\widehat{x_{i,j}} = x_{i,j}$ if $i \leq j$ and $\widehat{x_{i,j}} = x_{j,i}$ if $i \geq j$. Since $f_{i,j} = \Delta_{i,j}$ or $1/2 f_{i,j} = \Delta_{i,j}$ for $(i,j) \neq (m-1,m-1)$, the above relations give us that the entries of the m-th column of $\mathcal{SC}$ belong to the ideal $J : \Delta_{m,m} = J : P$.

In addition, from the cofactor identity we read the following relations:

$$\begin{aligned}\sum_{j=1, j\neq m-1}^{m} \widehat{x_{k,j}}\Delta_{j,m-1} + \widehat{x_{k,m-1}}\Delta_{m-1,m-1} &= 0, \text{ for } k=1,\ldots,m, \ (k \neq m-1);\\ \sum_{j=1, j\neq m-1}^{m} \widehat{x_{m-1,j}}\Delta_{j,m-1} + x_{m-1,m-1}\Delta_{m-1,m-1} &= \sum_{j=1}^{m} x_{1,j}\Delta_{j,1};\end{aligned}$$

Then as above we have that the entries of the $(m-1)$-th column of $\mathcal{SC}$ belong to the ideal $J : \Delta_{m-1,m-1} = J : P$.

From this, the variables of the two rightmost columns of $\mathcal{SC}$ conduct P into J. In, particular, the codimension of $J : P$ is at least 4, as needed.

In addition, since P has codimension 3 then $J : P \not\subset P$. Picking a element $a \in J : P \setminus P$ shows that $P_P \subset J_P$. Therefore P is the unmixed part of J.

To conclude that P is the minimal primary component of J, we observe that, by symmetry, the entries of the two rightmost columns, are the entries of the two last rows. As is clear that P is contained in the ideal generated by these variables it follows that $P^2 \subset J$. Therefore, the radical of J – i.e., the radical of the minimal primary part of J– is P.

(iii) The proof is essentially the same as that of Theorem 2.1.2(iii). By (ii), P is the minimal component of a primary decomposition of J and yields the inclusion $I \subset J : P$, where I denotes the ideal generated by the $2(m-1)$ variables of the two rightmost columns of $\mathcal{SC}$. Here, the claim is that $J : P = I$.

By symmetry, the entries in the last column are the same in the last row. We denote by I'' the prime ideal generated by these variables. Clearly, the cofactor $\Delta_{m,m}$ does not belong to I'' and $\Delta_{i,j} \in I''$ for all $(i,j) \neq (m,m)$, since $\Delta_{i,j}$ have a row or column made up of elements I''. Similarly, we observe that $\Delta_{m-1,m-1}$ does not belong to I' and $\Delta_{i,j} \in I'$ for all $(i,j) \neq (m-1, m-1)$, where I' denotes the prime ideal generated by the entries in the $(m-1)$-th column.

Writting $I = I' + I''$, we argue as in the Theorem 2.1.2(iii) that if $b \in J : P$ then $b \in I$.

In particular, $J : P$ is a prime ideal and is the only embedded prime of J. As pointed out, $P \subset J : P$, hence $P^2 \subset J$. Therefore, J defines a double structure on the irreducible variety defined by P, with a unique embedded component – the latter being a linear variety of codimension $2(m-1)$.

(iv) The proof is analogous as in the Theorem 2.1.2(iv). The difference is that, in this case, the cofactor matrix is symmetric, so the defining equation of the image of the birational map defined by the minors comes from a generic symmetric matrix $(y_{i,j})_{1\leq i,j\leq m}$ with $y_{i,j} = y_{j,i}$.

By Theorem 3.1.1 (a)(v) the partial derivatives of f generate a subalgebra of maximum dimension $(= \binom{m+1}{2} - 1)$. Since $J \subset P$ is an inclusion in the same degree, the subalgebra generated by the submaximal minors has dimension $\binom{m+1}{2} - 1$. On the other hand, since P is a specialization from the symmetric case, it is linearly presented. Therefore, the minors define a birational map (see Theorem 1.3.2) onto a hypersurface.

To get the defining equation of the latter we proceed as in the Theorem 2.1.2(iv).

Write $\Delta_{i,j}$ for the cofactor of the (i,j)-entry of $\mathcal{SC}$. Observe that $\Delta_{i,j} = \Delta_{j,i}$ and so the matrix $\mathrm{adj}(\mathcal{SC}) = (\Delta_{i,j})_{1\leq i,j\leq m}$ is symmetric. It suffices to show that $\mathbb{D}_{m,m} - \mathbb{D}_{m-1,m-1}$ belongs to the kernel of the k-algebra map

$$\psi : k[y_{i,j} \,|\, 1 \leq i \leq j \leq m] \to k[\underline{\Delta}] = k[\Delta_{i,j} \,|\, 1 \leq i \leq j \leq m],$$

as it is clearly an irreducible polynomial.

Consider the following well-known matrix identity

$$\mathrm{adj}(\mathrm{adj}(\mathcal{SC})) = f^{m-2} \cdot \mathcal{SC}, \tag{3.12}$$

where $\mathrm{adj}(M)$ denotes the matrix of cofactors of a square matrix M. Looking at the $(m-1, m-1)$-entry and the (m,m)-entry of the right-hand side matrix we obviously see the same element, namely, $f^{m-2}x_{m-1,m-1}$. Note that the corresponding entries on the left-hand side matrix are $\mathbb{D}_{m-1,m-1}(\underline{\Delta})$ and $\mathbb{D}_{m,m}(\underline{\Delta})$, respectively. Indeed, the $(m-1, m-1)$-entry of $\mathrm{adj}(\mathrm{adj}(\mathcal{SC}))$ is the cofactor of the entry $\Delta_{m-1,m-1}$ in the symmetric matrix $\mathrm{adj}(\mathcal{SC}) = (\Delta_{i,j})_{1\leq i,j\leq m}$. Clearly, this cofactor is the (m,m)-cofactor $\mathbb{D}_{m-1,m-1}$ of the generic symmetric matrix $(y_{i,j})_{1\leq i,j\leq m}$ evaluated in $\underline{\Delta}$. Similarly, we see that the $(m-1, m-1)$-entry of $\mathrm{adj}(\mathrm{adj}(\mathcal{GC}))$ is $\mathbb{D}_{m,m}(\underline{\Delta})$. Therefore, we get

$(\mathbb{D}_{m,m} - \mathbb{D}_{m-1,m-1})(\underline{\Delta}) = 0$, as required.

(v) We proceed as in the Theorem 2.1.2(v) controlling the fact that in this case the terms of $\boldsymbol{H} := \mathbb{D}_{m,m} - \mathbb{D}_{m-1,m-1}$ are note square free.

It follows from (iv) that the reduction number of a minimal reduction of P is $m-2$. Thus, to conclude, it suffices to prove that $P^{m-1} \not\subset JP^{m-2}$. For this we will show that $\Delta_{m,m}^{m-1} \in P^{m-1}$ does not belong to JP^{m-2}.

Recall from previous passages that J is generated by scalar multiples of the cofactors $\Delta_{l,h}$ $(1 \leq l \leq h \leq m)$, with $(l,h) \neq (m-1, m-1)$ and $(l,h) \neq (m,m)$, and the additional form $\Delta_{m,m} + \Delta_{m-1,m-1}$.

If $\Delta_{m,m}^{m-1} \in JP^{m-2}$, we can write

$$\Delta_{m,m}^{m-1} = \sum_{\substack{1 \leq l \leq h \leq m \\ (l,h) \neq (m-1,m-1),(m,m)}} \Delta_{l,h} Q_{l,h}(\underline{\Delta}) + (\Delta_{m,m} + \Delta_{m-1,m-1}) Q(\underline{\Delta}) \tag{3.13}$$

where $Q_{l,h}(\underline{\Delta})$ and $Q(\underline{\Delta})$ are homogeneous polynomial expressions of degree $m-2$ in the set

$$\underline{\Delta} = \{\Delta_{i,j} \,|\, 1 \leq i \leq j \leq m\}$$

of the cofactors (generators of P).

Clearly, this gives a polynomial relation of degree $m-1$ on the generators of P, so the corresponding form of degree $m-1$ in $k[y_{i,j} | 1 \leq i \leq j \leq m]$ is a scalar multiple of the defining equation $\boldsymbol{H} := \mathbb{D}_{m,m} - \mathbb{D}_{m-1,m-1}$ obtained in the previous item. Recall that $\boldsymbol{H}$ is the sum of two cofactor of an generic symmetric matrix $\mathcal{S}$ in the variables $y_{i,j}$, with $1 \leq i \leq j \leq m$. Since the variable $y_{m,m}$ appears only one time in $\mathcal{S}$, we have that each term of $\mathbf{H}$ has degree at most 1 in the variable $y_{m,m}$. We now argue that such a relation is impossible.

Namely, observe that the sum

$$\sum_{\substack{(l,h) \neq (m-1,m-1) \\ (l,h) \neq (m,m)}} \Delta_{l,h} Q_{l,h}(\underline{\Delta})$$

does not contain any nonzero terms of the form $\alpha \Delta_{m,m}^{m-1}$ or $\beta \Delta_{m-1,m-1} \Delta_{m,m}^{m-2}$. In addition, if these two terms appear in $(\Delta_{m,m} + \Delta_{m-1,m-1})Q(\underline{\Delta})$ they must have the same scalar coefficient, say, $c \in k$. Bring the first of these to the left-hand side of (3.13) to get a polynomial relation of P having a nonzero term $(1-c)y_{m-1,m-1}^{m-1}$. If $c \neq 1$, this is a contradiction because an term of $\mathbf{H}$ has degree at most 1 in the variable $y_{m,m}$.

On the other hand, if $c = 1$ then we still have a polynomial relation of P having a nonzero term $y_{m-1,m-1}y_{m,m}^{m-2}$. Now, if $m > 3$ this is again a contradiction due the nature of $\mathbf{H}$ as the nonzero term of the latter has degree at most 1 in the variable $y_{m,m}$. Finally, if $m = 3$ a direct checking shows that the monomial $y_{m-1,m-1}y_{m,m}$ cannot be the support of a nonzero term in $\boldsymbol{H}$. This concludes the statement. □

Remark 3.1.5. One attempt to prove the condition (R_1) would be to show that the singular locus of the determinantal variety defined by $I_{m-1}(\mathcal{SC})$ is set-theoretic defined by the immediately lower minors, just as happens in the symmetric generic case. In classical cases, this is proved by a localization argument, which allow us conclude $I_{m-1}(U) = I_{m-2}(\widetilde{U})$, where U is a $m \times m$ matrix of indeterminates and $\widetilde{U}$ is an $(m-1)\times(m-1)$ submatrix obtained of U after suitable elementary row and column transformations. The point is to use the inductive hypothesis in the matrix $\widetilde{U}$ which can be regarded as a matrix of the same type as U with one size down. We try to adapt the argument unsuccessful, because after elementary row and column operations, the cloned entries become different entries in $\widetilde{\mathcal{SC}}$, that is, we can not see $\widetilde{\mathcal{SC}}$ as a cloned symmetric matrix.

3.2 Degeneration by a single zero

We consider the simplest degeneration of the $m \times m$ generic symmetric matrix, consisting of replacing a variable in the main diagonal by zero. This is an analogue of the result in [29, Proposition 4.9] (with subsequent corrections in [30]).

Since the replaced variable is on the main diagonal then by suitable permutations of rows and columns, we can assume that the matrix has the following shape:

$$\mathcal{DS} := \left(\begin{array}{cccc|c} x_{1,1} & x_{1,2} & \dots & x_{1,m-1} & \mathbf{x_{1,m}} \\ x_{1,2} & x_{2,2} & \dots & x_{2,m-1} & \mathbf{x_{2,m}} \\ \vdots & \vdots & \dots & \vdots & \vdots \\ x_{1,m-1} & x_{2,m-1} & \dots & x_{m-1,m-1} & \mathbf{x_{m-1,m}} \\ \hline \mathbf{x_{1,m}} & \mathbf{x_{2,m}} & \dots & \mathbf{x_{m-1,m}} & 0 \end{array}\right)$$

The following overture has many aspects in common with the analogous generic degeneration displayed in Theorem 2.2.1 and Theorem 2.2.7:

Proposition 3.2.1. *Let $R = k[x_{i,j} \,|\, 1 \le i \le j \le m]$, let $f := \det(\mathcal{DS})$ and let $J \subset R$ denote the gradient ideal of f. Then:*

(i) *f is irreducible.*

(ii) *J has maximal linear rank.*

(iii) *The image of the polar map of f is a hypersurface defined by the determinant of a symmetric matrix.*

(iv) *If $m \geq 4$ then $I = I_{m-1}(\mathcal{DS})$ is a Cohen-Macaulay ideal of codimension 3 and J has codimension* 3; *in addition, I is a radical ideal and the minimal primes of R/J are the associated primes of R/I and the ideal generated by the entries of the last column of $\mathcal{DS}$.*

(v) *If $m \geq 5$ then I is the unmixed part of the gradient ideal J.*

Proof.

(i) We first show that f is irreducible - although this fact will not play a major role in the subsequent arguments.

The proof that f is irreducible is the same as in the Theorem 3.1.1 (ii), with small adjustments. As is well-known, in the generic symmetric case $\text{in}(f)$ in the revlex order is the product of the entries along the main anti-diagonal of $\mathcal{S}$. It is quite clear that this fact won't be affected by the present degeneration. As for the first step inductive (m=3), a straightforward calculation yields $f = x_{1,1}(-x_{2,3}^2) + g$, where $g = -x_{1,3}^2 x_{2,2} + 2x_{1,2}x_{1,3}x_{2,3}$. Observing that $\text{in}(g) = \text{in}(f) = -x_{1,3}^2 x_{2,2}$ and $-x_{2,3}^2$ do not have common factors, we conclude that f is a primitive polynomial in the ring $k[x_{1,2}, x_{1,3}, x_{2,2}, x_{2,3}][x_{1,1}]$, thus it is a irreducible polynomial.

The rest of the proof stays pretty much the same as in Theorem 3.1.1 (ii).

(ii) We argue as in the Theorem 3.1.1 (iv), observing that, in this case, all the partial derivatives of f will be (signed) cofactors, up to a scalar, and this simplifies the argument.

Let $\Delta_{j,i}$ stand for the (signed) cofactor of the (i,j)-entry of $\mathcal{DS}$ and let $f_{i,j}$ denote the $x_{i,j}$-derivative of f. Observe that, by symmetry, one has $\Delta_{j,i} = \Delta_{i,j}$. Using the Cauchy cofactor equality

$$\mathcal{DS}\,\text{adj}(\mathcal{DS}) = \text{adj}(\mathcal{DS})\,\mathcal{DS} = \det(\mathcal{DS})\mathbf{1}_n,$$

one obtains the following set of linear relations involving the cofactors of $\mathcal{DS}$:

$$\sum_{j=1}^{m} \widehat{x_{i,j}}\Delta_{j,1} = 0, \text{ for } 2 \leq i \leq m-1; \tag{3.14}$$

$$\sum_{j=1}^{m-1} \widehat{x_{m,j}}\Delta_{j,k} = 0, \text{ for } 1 \leq k \leq m-1; \tag{3.15}$$

$$\sum_{j=1}^{m} \widehat{x_{i,j}} \Delta_{j,k} = 0, \text{ for } 2 \leq k \leq m-1 \text{ and } k-1 \leq i \leq m-1 \ (k \neq i); \tag{3.16}$$

where $\widehat{x_{i,j}} = x_{i,j}$ if $i \leq j$ and $\widehat{x_{i,j}} = x_{j,i}$ if $i \geq j$.

Note that the partial derivatives of f with respect to the variables off the main diagonal will be the corresponding (signed) cofactor multiplied by 2, while the partial derivatives $f_{i,i}$ is the the (signed) cofactor $\Delta_{i,i}$ for all $i \neq m$. Thus, any of the above relations gives a linear syzygy of the partial derivatives of f and one has counted a total of $(m-2)+(m-1)+(m-2)(m-1)/2+ = \binom{m+1}{2} - 2$ linear syzygies of J.

We need to show that these are independent. For this we order the set of partial derivatives $f_{i,j}$ in accordance with the following ordered list of the entries $x_{i,j}$:

$$x_{1,1}, x_{1,2}, \ldots, x_{1,m} \rightsquigarrow x_{2,2}, \ldots, x_{2,m} \rightsquigarrow \ldots \rightsquigarrow x_{m-2,m-2}, x_{m-2,m-1} x_{m-2,m} \rightsquigarrow x_{m-1,m-1}, x_{m-1,m}.$$

Here we traverse the entries along the matrix rows, left to right, until exhausting all variables.

We now claim that, the above sets of linear relations can be grouped into the following block matrix of linear syzygies:

$$\begin{pmatrix} \varphi_1 & \ldots & & & & \\ 0_{m-1}^{m-1} & \varphi_2 & \ldots & & & \\ 0_{m-2}^{m-1} & 0_{m-2}^{m-1} & \varphi_3 & \ldots & & \\ 0_{m-3}^{m-1} & 0_{m-3}^{m-1} & 0_{m-3}^{m-2} & \varphi_4 & \ldots & \\ \vdots & \vdots & \vdots & \vdots & \ddots & \\ 0_2^{m-1} & 0_2^{m-1} & 0_2^{m-2} & 0_2^{m-3} & \ldots & \varphi_{m-1} \end{pmatrix},$$

where:

- φ_1 is the matrix obtained from $(\mathcal{DS})^t$ by multiplying the first row by 2 and omitting the first column.

- φ_2 is the matrix obtained from $(\mathcal{DS})^t$ by multiplying the second row by 2 and omitting the second column and the first row.

- For $l = 3, \ldots, m-1$, φ_l is the matrix obtained from $(\mathcal{DS})^t$ by multiplying the lth row by 2 and omitting the rows $1, \ldots, l-1$ and the columns $1, \ldots, l-2, l$.

- 0_r^c denotes a zero block of size $r \times c$.

Next we justify why these blocks make up (linear) syzygies.

First, as already observed, the relations (3.14), (3.15) and (3.16) yield linear syzygies of the partial derivatives of f. Since $\Delta_{i,i} = f_{i,i}$ and $\Delta_{j,i} = 1/2 f_{i,j}$, for all $i \neq j$, the relation (3.14) can be written as

$$\widehat{x_{i,1}} 2 f_{1,1} + \sum_{j=2}^{m} \widehat{x_{i,j}} f_{1,j} = 0,$$

for $i = 2, \ldots, m-1$. Moreover, setting $k = 1$ in the relation (3.15) it can be written as

$$\widehat{x_{m,1}} 2 f_{1,1} + \sum_{j=2}^{m-1} \widehat{x_{m,j}} f_{1,j} = 0$$

Ordering the set of partial derivatives $f_{i,j}$ as explained before, the coefficients of these relations form the first matrix above

$$\varphi_1 := \begin{pmatrix} 2\widehat{x_{2,1}} & 2\widehat{x_{3,1}} & \cdots & 2\widehat{x_{m-1,1}} & 2\widehat{x_{m,1}} \\ \widehat{x_{2,2}} & \widehat{x_{3,2}} & \cdots & \widehat{x_{m-1,2}} & \widehat{x_{m,2}} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ \widehat{x_{2,m-1}} & \widehat{x_{3,m-1}} & \cdots & \widehat{x_{m-1,m-1}} & \widehat{x_{m,m-1}} \\ \widehat{x_{2,m}} & \widehat{x_{3,m}} & \cdots & \widehat{x_{m-1,m}} & 0 \end{pmatrix} = \begin{pmatrix} 2x_{1,2} & 2x_{1,3} & \cdots & 2x_{1,m-1} & 2x_{1,m} \\ x_{2,2} & x_{2,3} & \cdots & x_{2,m-1} & x_{2,m} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ x_{2,m-1} & x_{3,m-1} & \cdots & x_{m-1,m-1} & x_{m-1,m} \\ x_{2,m} & x_{3,m} & \cdots & x_{m-1,m} & 0 \end{pmatrix}$$

Note that φ_1 coincides indeed with the submatrix of $\mathcal{GC}^t$ obtained by multiplying the first row by 2 and omitting the first column.

Continuing, for each $l = 2, \ldots, m-1$ the block φ_l comes from the relation (3.15) and (3.16) (setting $k = l$).

This proves the claim about the large matrix above. Counting through the sizes of the various blocks, one sees that this matrix is $(\binom{m+1}{2} - 1) \times (\binom{m+1}{2} - 2)$. Omitting its first row obtains a block-diagonal submatrix of size $(\binom{m+1}{2} - 2) \times (\binom{m+1}{2} - 2)$, where each block has nonzero determinant. Thus, the linear rank of J attains the maximum.

(iii) We claim that the homogeneous defining ideal of the image of the polar map of f contains the determinant of the following symmetric matrix:

$$\widetilde{\mathbf{X}} = \begin{pmatrix} x_{1,1} & (1/2)x_{1,2} & \ldots & (1/2)x_{1,m-1} \\ (1/2)x_{1,2} & x_{2,2} & \ldots & (1/2)x_{2,m-1} \\ \vdots & \vdots & \ldots & \vdots \\ (1/2)x_{1,m-1} & (1/2)x_{2,m-1} & \ldots & x_{m-1,m-1} \end{pmatrix}$$

Note that this gives that only $\binom{m}{2}$ of the partial derivatives will really come in the relation, in fact those partial derivatives with respect to the variables in the $(m-1) \times (m-1)$ initial submatrix of the original matrix $\mathcal{DS}$.

Write $\mathbf{X}$ for the initial $(m-1) \times (m-1)$ submatrix in $\mathcal{DS}$ and $\mathbf{F}$ for the $(m-1) \times$

$(m-1)$ (transposed) matrix of the cofactors $\Delta_{j,i}(1 \leq i, j \leq m-1)$. Also denote $\mathbf{Y}$ the row vector of the variables $x_{1,m}, x_{2,m}, \ldots, x_{m-1,m}$. Then $\mathcal{DS}$ and its adjoint matrix–i.e., the transposed matrix of its cofactors– have the respective shapes

$$\mathcal{DS} = \begin{pmatrix} \mathbf{X} & \mathbf{Y}^t \\ \mathbf{Y} & 0 \end{pmatrix}, \quad \mathrm{adj}(\mathcal{DS}) = \begin{pmatrix} \mathbf{F} & * \\ * & \det(\mathbf{X}) \end{pmatrix}$$

By the usual way of multiplying out the two matrices, which yields a diagonal matrix, we derive the relation

$$\mathbf{F} \cdot \mathbf{Y}^t = 0.$$

This means that the matrix $\mathbf{F}$ has rank at most $m-2$, in other words, $\det(\mathbf{X})$ vanishes on the cofactors $\Delta_{j,i}$. Since the partial derivative of f with respect $x_{i,i}$ coincides with the (signed) cofactor of $x_{i,i}$ for $1 \leq i \leq m-1$ and the partial derivative $f_{i,j}$ of f with respect $x_{i,j}$ is the (signed) cofactor of the (i, j)-entry multiplied by 2 for all $i < j$, one has

$$0 = \det(\mathbf{F}) = \det \begin{pmatrix} f_{1,1} & (1/2)f_{1,2} & \ldots & (1/2)f_{1,m-1} \\ (1/2)f_{1,2} & f_{2,2} & \ldots & (1/2)f_{2,m-1} \\ \vdots & \vdots & \ldots & \vdots \\ (1/2)f_{1,m-1} & (1/2)f_{2,m-1} & \ldots & f_{m-1,m-1} \end{pmatrix}.$$

This means that $\det(\widetilde{X})$ vanishes on the partial derivatives $f_{i,j}$ $(1 \leq i \leq j \leq m-1)$ and hence belongs to the homogeneous defining ideal of the polar variety.

In order to complete the proof of this item it suffices to show that the ideal defining the polar variety is principal, in other words, that the Hessian determinant has rank $\binom{m+1}{2} - 2$ (one less than the maximum possible rank).

Let $\mathbf{v} := \{x_{i,j} | i + j = m + 1 \text{ for } 1 \leq i \leq j \leq m\}$ for the set of the entries along the principal anti-diagonal of $\mathcal{DS}$. We argue by a specialization procedure, namely, consider the ring endomorphism φ of R by mapping any variable in $\mathbf{v}$ to itself and by mapping any variable off $\mathbf{v}$ to zero. Clearly, its suffices to show that by applying φ to the entries of $H(f)$ the resulting matrix has rank $\binom{m+1}{2} - 2$.

For further use, denote $\widehat{x_{i,j}} = x_{i,j}$ if $i \leq j$ and $\widehat{x_{i,j}} = x_{j,i}$ if $i > j$.

We know that the partial derivative of f with respect $x_{i,i}$ coincides with the (signed) cofactor of $x_{i,i}$ for $1 \leq i \leq m-1$ and the partial derivative $f_{i,j}$ of f with respect $x_{i,j}$ is the (signed) cofactor of the (i, j)-entry multiplied by 2 for all $i < j$. Therefore, we may deal with (signed) cofactors instead of partial derivatives.

By expanding of the Leibniz rule it is verified easily that the cofactor of a variable in the set $\mathbf{v}$ has the unique (nonzero) term whose support lies in $\mathbf{v}$ and the remaining terms have degree at least 2 in the variables off $\mathbf{v}$. Similarly, the cofactor of a variable outside $\mathbf{v}$ has no term whose support lies in $\mathbf{v}$ and has exactly one (nonzero) term of degree 1 in the variables off $\mathbf{v}$– with one single exception, that of the cofactor of $x_{1,1}$, the latter having no term of degree 1 in the variables off $\mathbf{v}$.

By the preceding observation, applying the map φ to any second partial derivative of f will return zero or a monomial supported on the variables in $\mathbf{v}$. Thus, the entries of the specialized Hessian matrix of f are zeros or monomials supported on the variables in$\mathbf{v}$.

By the special shape of the derivative $f_{1,1}$ of f with respect the variable $x_{1,1}$ observed above, it follows that the first row of the specialized Hessian matrix of f is null, and so is its first column (the Hessian matrix is symmetric).

We then consider the $(\binom{m+1}{2} - 2) \times (\binom{m+1}{2} - 2)$ submatrix M of the specialized Hessian matrix of f omitting the first row and the first column. We will show that $\det(M) \neq 0$.

To see that the determinant of this matrix M is nonzero, consider the Jacobian matrix of the set of partial derivatives $\{f_v \,|\, v \in \mathbf{v}\}$ with respect to the variables in $\mathbf{v}$.

Let M_0 denote the specialization of this Jacobian matrix by φ, considered as a corresponding submatrix of M. Up to permutation of rows and columns of M, we may write

$$M = \begin{pmatrix} M_0 & N \\ P & M_1 \end{pmatrix},$$

for suitable M_1. Now, by the way the second partial derivatives of f specialize via φ, as explained above, one must have $N = P = 0$. Therefore, $\det(M) = \det(M_0)\det(M_1)$, so it remains to prove the nonvanishing of these two subdeterminants.

Now the first block is the Hessian matrix of the form g being taken as the product of the entries in the main anti-diagonal of the matrix $\mathcal{DS}$, i.e.,

$$g := x_{1,m}^2 x_{2,m-1}^2 \cdots x^2_{\frac{m+1}{2}-1,\frac{m+1}{2}+1} x_{\frac{m+1}{2},\frac{m+1}{2}},$$

when m is odd or

$$g := x_{1,m}^2 x_{2,m-1}^2 \cdots x^2_{\frac{m}{2},\frac{m}{2}+1},$$

when m is even.

By a similar argument used in Theorem 2.1.1 one has that g is a classical homaloidal polynomial, hence we are done for the first matrix block.

As for the second block, by construction it has exactly one nonzero entry on each row and each column. Therefore, it has a nonzero determinant.

(iv) The easiest part is that the ideal I is a Cohen-Macaulay ideal of codimension 3 since it specializes from the symmetric generic case.

Thus, J has codimension at most 3 since clearly $J \subset I$.

In order to show that J has codimension at least 3, we resort to its initial ideal in the revlex monomial order. Note that

$$\begin{aligned}
\mathrm{in}(f_{1,1}) &= x_{2,m}^2 \cdot x_{3,m-1}^2 \cdots x^2_{\lfloor \frac{m+2}{2} \rfloor, \lfloor \frac{m+2}{2} \rfloor+1} \\
\mathrm{in}(f_{1,m}) &= 2x_{1,m} \cdot x_{2,m-1}^2 \cdots x^2_{\frac{m+1}{2}-1, \frac{m+1}{2}+1} \cdot x_{\frac{m+1}{2}, \frac{m+1}{2}}
\end{aligned}$$

when m is odd, and

$$\begin{aligned}
\mathrm{in}(f_{1,1}) &= x_{2,m}^2 \cdot x_{3,m-1}^2 \cdots x^2_{\frac{m+2}{2}-1, \frac{m+2}{2}+1} \cdot x_{\frac{m+2}{2}, \frac{m+2}{2}} \\
\mathrm{in}(f_{1,m}) &= 2x_{1,m} \cdot x_{2,m-1}^2 \cdots x^2_{\lfloor \frac{m+1}{2} \rfloor, \lfloor \frac{m+1}{2} \rfloor+1}
\end{aligned}$$

when m is even.

On the other hand, one has:

$$\mathrm{in}(\Delta_{m,m}) = x_{1,m-1}^2 \cdot x_{2,m-2}^2 \cdots x^2_{\frac{m-1}{2}, \frac{m-1}{2}+1}$$

when m is odd, and

$$\mathrm{in}(\Delta_{m,m}) = x_{1,m-1}^2 \cdot x_{2,m-2}^2 \cdots x^2_{\frac{m}{2}-1, \frac{m}{2}+1} \cdot x_{\frac{m}{2}, \frac{m}{2}},$$

when m is even. Therefore, provided $x_{m-1,m}\mathrm{in}(\Delta_{m,m}) \subset J$, we obtain a regular sequence in J of length 3, which will prove our contention.

For this, we read out the cofactor formula

$$\mathcal{S}\,\mathrm{adj}(\mathcal{S}) = \mathrm{adj}(\mathcal{S})\,\mathcal{S} = \det(\mathcal{S})\mathbf{1}_n,$$

where $\mathcal{S}$ denotes the generic symmetric matrix. It yields the following relations:

$$\sum_{i=1}^{m-1} x_{i,m-1}\mathbb{D}_{i,m} + x_{m-1,m}\mathbb{D}_{m,m} = 0$$

where $\mathbb{D}_{j,i}$ denotes the cofactor of the (i,j)-entry of $\mathcal{S}$. Clearly, the image of $\mathbb{D}_{j,i}$ is $\Delta_{j,i}$ under the surjective homomorphism

$$\begin{array}{rcl} k[x_{1,1},\dots,x_{m-1,m},x_{m,m}] & \longrightarrow & k[x_{1,1},\dots,x_{m-1,m}] \\ x_{i,j} & \mapsto & x_{i,j} \text{ for } (i,j)\neq(m,m) \\ x_{m,m} & \mapsto & 0. \end{array}$$

Thus,

$$\sum_{i=1}^{m-1} x_{i,m-1}\Delta_{i,m} + x_{m-1,m}\Delta_{m,m} = 0.$$

As $\Delta_{i,m} = 1/2 f_{i,j}$, for each $i = 1,\dots,m-1$, this ensures that $x_{m-1,m}\Delta_{m,m} \in J$ as was to be shown.

Since R/I is Cohen-Macaulay, hence satisfies the property (S_1), in order to prove that I is radical it suffices to show that R/I satisfies the property (R_0). Let θ denote the Jacobian matrix of the generators of I with respect to all variables in sight. The goal is to show that $\operatorname{codim}(I_3(\theta)+I) \geq 3+1 = 4$. We consider the following submatrix of θ:

$$\theta_1 := \begin{pmatrix} \partial\Delta_{2,2}/\partial x_{1,1} & \partial\Delta_{1,2}/\partial x_{1,1} & \partial\Delta_{1,1}/\partial x_{1,1} \\ \partial\Delta_{2,2}/\partial x_{1,2} & \partial\Delta_{1,2}/\partial x_{1,2} & \partial\Delta_{1,1}/\partial x_{1,2} \\ \partial\Delta_{2,2}/\partial x_{2,2} & \partial\Delta_{1,2}/\partial x_{2,2} & \partial\Delta_{1,1}/\partial x_{2,2} \end{pmatrix}$$

A close inspection of the cofactors $\Delta_{1,1}$, $\Delta_{1,2}$, $\Delta_{2,2}$ yields

$$\partial\Delta_{1,2}/\partial x_{1,1} = \partial\Delta_{1,1}/\partial x_{1,1} = \partial\Delta_{1,1}/\partial x_{1,2} = 0.$$

By Proposition 1.2.5, the partial derivative of $\Delta_{i,j}$ with respect to $x_{k,l}$ is the sum of the (signed) cofactors of the entry $x_{k,l}$, in all its appearances as an entry of $\Delta_{i,j}$. Thus, the partial derivatives $\partial\Delta_{2,2}/\partial x_{1,1}$, $\partial\Delta_{1,2}/\partial x_{1,2}$ and $\partial\Delta_{1,1}/\partial x_{2,2}$ coincide, up to a sign, with the determinant of the matrix

$$M_1 = \begin{pmatrix} x_{3,3} & \dots & x_{3,m-2} & x_{3,m-1} & x_{3,m} \\ \dots & \vdots & \vdots & \vdots & \vdots \\ x_{3,m-2} & \dots & x_{m-2,m-2} & x_{m-2,m-1} & x_{m-2,m} \\ x_{3,m-1} & \dots & x_{m-2,m-1} & x_{m-1,m-1} & x_{m-1,m} \\ x_{3,m} & \dots & x_{m-2,m} & x_{m-1,m} & 0 \end{pmatrix}$$

obtained from $\mathcal{SD}$ by omitting the first two columns and the first two rows. Thus,

$$\det(\theta_1) = \det \begin{pmatrix} \pm\det(M_1) & 0 & 0 \\ \partial\Delta_{2,2}/\partial x_{1,2} & \pm\det(M_1) & 0 \\ \partial\Delta_{2,2}/\partial x_{2,2} & \partial\Delta_{1,2}/\partial x_{2,2} & \pm\det(M_1) \end{pmatrix} = \pm(\det(M_1))^3$$

We claim that the ideal $(\Delta_{1,1},\ f,\ \Delta_{m,m}, \det(\theta_1))$ has codimension 4. For this, we consider the initial ideal of $(f_{1,1},\ f,\ \Delta_{m,m}, \det(\theta_1))$ in the revlex order. We observe that the initial term of an $(m-2)$-minor or an $(m-1)$-minor still is the product of the entries along its main anti-diagonal. Thus, $\mathrm{in}(\Delta_{1,1}) = \prod_{i+j=m+2} x_{i,j},\ \mathrm{in}(\Delta_{m,m}) = \prod_{i+j=m} x_{i,j}$, $\mathrm{in}(f) = \prod_{i+j=m+1} x_{i,j}$ and $\mathrm{in}(\det(\theta_1)) = \pm(\mathrm{in}(\det(M_1)))^3 = \pm\left(\prod_{i+j=m+3} x_{i,j}\right)^3$. Since these monomials do not have common variables, it follows that $(\Delta_{1,1},\ f,\ \Delta_{m,m}, \det(\theta_1))$ has codimension 4. As this ideal is contained in $I_3(\theta) + I$, we are done.

To close the remaining statements, we look at the minimal primes of R/J. As R/I is a Cohen-Macaulay ring its associated primes are all of codimension 3. In particular, they are minimal prime ideals of R/J. Now, if Q is a minimal prime of J which does not contain I then $J : I \subset Q$. Therefore, it will suffice to show that $J : I$ is itself a prime ideal. Observe that the entries of the last column each conducts $\Delta_{m,m}$ into J. Indeed, once again the cofactor formula

$$\mathcal{S}\,\mathrm{adj}(\mathcal{S}) = \mathrm{adj}(\mathcal{S})\,\mathcal{S} = \det(\mathcal{S})\mathbf{1}_n,$$

yields the following relations:

$$\sum_{i=1}^{m-1} \widehat{x_{i,k}}\mathbb{D}_{i,m} + x_{k,m}\mathbb{D}_{m,m} = 0,$$

for all $k = 1, \ldots, m-2$, where $\mathbb{D}_{j,i}$ denotes the cofactor of the (i,j)-entry of $\mathcal{S}$ and $\widehat{x_{i,k}} = x_{i,k}$, if $i \leq k$ or $\widehat{x_{i,k}} = x_{k,i}$ if $k < i$. Since $\mathbb{D}_{j,i}$ maps down to $\Delta_{j,i}$, one has

$$\sum_{i=1}^{m-1} \widehat{x_{i,k}}\Delta_{i,m} + x_{k,m}\Delta_{m,m} = 0.$$

From this, one sees that $(x_{1,m},\ x_{2,m}, \ldots, x_{m-1,m}) \subset J : \Delta_{m,m} = J : I$ since that $I = (J, \Delta_{m,m})$. The other inclusion is obvious since $J \subset (x_{1,m},\ x_{2,m}, \ldots, x_{m-1,m})$. Therefore

$$J : I = (J : \Delta_{m,m}) = (x_{1,m},\ x_{2,m}, \ldots, x_{m-1,m})\,,$$

showing in particular that $J : I$ is a prime ideal.

(v) By the previous item $J : I$ is a minimal prime ideal of codimension $m - 1$ of R/J, the remaining minimal associated primes being the associated prime ideals of R/I. When $m \geq 5$, codim $(J : I) > 3$ and this ensures that I is the radical of J^{un}.

We now claim that $I \subset J^{\text{un}}$. For this let P_i denote the associated primes of R/I and let Q_i denote the P_i-primary component of J, for $i = 1, \ldots, r$. Suppose $a \in I \setminus J^{\text{un}}$, say, $a \notin Q_1$. Given any $c \in J : I$ one has $a \cdot c \in J \subset J^{\text{un}} \subset Q_1$. Therefore, $c \in \sqrt{Q_1} = P_1$ and hence $J : I \subset P_1$, forcing $J : I$ to have codimension at most 3. This is a contradiction because codim $(J : I) > 3$.

It now follows that $I \subset J^{\text{un}} \subset \sqrt{J^{\text{un}}} = \sqrt{I} = I$ and therefore, I is the unmixed part of J. □

Conjecture 3.2.2. *If $m \geq 4$, then I is a prime ideal. In particular, the minimal associated primes of R/J are $I_{m-1}(\mathcal{DS})$ and the ideal generated by the entries of the last column of $\mathcal{DS}$.*

Computational experiment indicates that R/I satisfies (R_1) for $m = 4, 5$ and satisfies (R_2) for $m \geq 6$. In any case it looks reasonable to expect that for $m >> 0$ the singular locus of R/I is set-theoretically defined by $R/I_{m-2}(\mathcal{DS})$.

3.2.1 Remarks on further degenerations

A natural question arises as to whether one can develop a parallel theory to the generic situation for the degenerations by zeros $\mathcal{DG}[r]$ – i.e., an ideal theory in the case of a similarly defined $\mathcal{DS}[r]$.

Preliminary evidence shows that although some similarity is maintained, as a whole the expected behavior may vary quite a bit.

An additional sort of degeneration consists in further stepwise symmetrizing $\mathcal{DS}[r]$ until reaching the generic Hankel matrix. These degenerations can be so organized as to assume that on each step one is getting closer to the Hankel matrix by symmetrizing the entries along the anti-diagonals starting from the left and choosing a definite way of ordering the symmetrization action along each anti-diagonal, such as for example:

$$\mathcal{DS}[2] = \begin{pmatrix} x_{1,1} & x_{1,2} & x_{1,3} & x_{1,4} \\ x_{1,2} & x_{2,2} & x_{2,3} & x_{2,4} \\ x_{1,3} & x_{2,3} & x_{3,3} & 0 \\ x_{1,4} & x_{2,4} & 0 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} x_{1,1} & x_{1,2} & x_{1,3} & x_{1,4} \\ x_{1,2} & x_{1,3} & x_{2,3} & x_{2,4} \\ x_{1,3} & x_{2,3} & x_{3,3} & 0 \\ x_{1,4} & x_{2,4} & 0 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} x_{1,1} & x_{1,2} & x_{1,3} & x_{1,4} \\ x_{1,2} & x_{1,3} & x_{1,4} & x_{2,4} \\ x_{1,3} & x_{1,4} & x_{3,3} & 0 \\ x_{1,4} & x_{2,4} & 0 & 0 \end{pmatrix}$$

$$\rightsquigarrow \begin{pmatrix} x_{1,1} & x_{1,2} & x_{1,3} & x_{1,4} \\ x_{1,2} & x_{1,3} & x_{1,4} & x_{1,5} \\ x_{1,3} & x_{1,4} & x_{1,5} & 0 \\ x_{1,4} & x_{1,5} & 0 & 0 \end{pmatrix}$$

A major question is the behavior of the Hessian of the corresponding determinant along this sort of symmetrization sequence. Namely, we know that the Hessian of the first determinant $\det(\mathcal{DS}[r])$ $(r > 0)$ vanishes and the last one of the Hankel matrix does not vanish (as will be shown in the next chapter). There arise two questions: the first asks what is the first term in the sequence such that the corresponding Hessian does not vanish; the second asks whether after that first terms all the subsequent ones have non-vanishing Hessian.

The question of degenerating the generic Hankel matrix by zeros is our endeavor in the next chapter. That is to say, we will focus on the last term along the above symmetrization degeneration.

A comprehensive consideration of the above questions will hopefully be tackled in future work.

Chapter 4

Degeneration of the generic Hankel matrix

The generic square Hankel matrix of size $m \times m$ is the symmetric matrix

$$\mathcal{H}_m := \begin{pmatrix} x_1 & x_2 & \dots & x_m \\ x_2 & x_3 & \dots & x_{m+1} \\ \vdots & \vdots & \dots & \vdots \\ x_m & x_{m+1} & \dots & x_{2m-1} \end{pmatrix} \tag{4.1}$$

Note the *total symmetry* enjoyed by these matrices, in the sense that every anti-diagonal involves a unique variable.

This part is inspired in two main sources: [29] and [33].

In the first of these sources a thorough study is made of the ideal theoretic and homological natures of the generic square Hankel determinant and its associated objects. In addition there is a good deal of results on the so-called sub-Hankel matrices, a one single case of degeneration of the generic Hankel matrix largely considered in [5] for its geometric significance.

The second source above deals with Hankel matrices of Hilbert–Burch format and its degenerations, as particular cases of more general catalecticant matrices. In it a strong use is made of the notion of a 1-generic matrix explored by Eisenbud and Harris (see [15], [16]). The main character here is the ideal of maximal minors, which is a codimension 2 perfect ideal from its very inception.

A common thread between both sources is the need to understand the properties of the corresponding Jacobian ideals – in the first case, the gradient ideal of the square determinant and the Jacobian ideal of the ideal os submaximal minors, in the second the Jacobian ideal of the codimension 2 perfect ideal of maximal minors. In this chapter we develop the in-between aspects of the above approaches, namely, we deal with various degenerations of the square generic Hankel matrix obtained by setting a few strategic entries (variables) to zero.

Throughout this chapter all matrices will have as entries either variables in a polynomial ring over a field or zeros, viewed as particular specializations of the square Hankel matrix. The prevailing tone is to grasp the effect of such specializations on the properties of the underlying ideal theoretic structures.

We will have need to consider generic Hankel matrices which are not necessarily square, namely:

$$\mathcal{H}_{j,2m-j} := \begin{pmatrix} x_1 & x_2 & \dots & x_{2m-j} \\ x_2 & x_3 & \dots & x_{m+1} \\ \vdots & \vdots & \dots & \vdots \\ x_j & x_{m+1} & \dots & x_{2m-1} \end{pmatrix}, \tag{4.2}$$

where $j < 2m$. The square case has $j = m$.

We denote $R' := k[x_1, \dots, x_{2m-1}]$ the polynomial ring on the entries of the generic Hankel square matrix.

4.1 Structure preserving degenerations

The degeneration one has in mind is induced by a ring homomorphism of the polynomial ambient ring R to a polynomial ring on a subset of these variables by fixing certain variables and mapping others to 0; this has moreover the advantage that the total symmetry of the generic square Hankel matrix along its anti-diagonals is preserved, i.e., if an entry of the matrix is thus replaced by 0 then every entry along its anti-diagonal also gets replaced by 0.

Other Hankel matrix degenerations not induced by ring homomorphisms will not be considered in this work. Thus, for example, the following matrix degenerations will be out of our study:

$$\begin{pmatrix} x_1 & 0 \\ x_2 & x_3 \end{pmatrix}, \quad \begin{pmatrix} x_1 & x_2 & 0 \\ x_2 & x_3 & x_4 \\ 0 & x_4 & x_5 \end{pmatrix}.$$

Here the first one is not even symmetric, while the second one is symmetric but not totally symmetric.

Let $x_{s_1}, x_{s_2}, \dots, x_{s_t}$ $(1 \leq s_1 < \dots < s_t \leq 2m-1)$ be an ordered subset of the variables and consider the ring endomorphism of R' defined as follows:

$$\Phi(x_i) = \begin{cases} 0 & \text{if } i \in \{s_1, \ldots, s_t\} \\ x_i & \text{if } i \notin \{s_1, \ldots, s_t\} \end{cases}$$

Note that $\Phi(R') \subset R'$ is itself a polynomial ring on the variables off $\{x_{s_1}, x_{s_2}, \ldots, x_{s_t}\}$.

Applying Φ to the entries of $\mathcal{H}_{j,2m-j}$ (notation as in (4.2)) yields the matrix

$$\mathcal{D}_\Phi(\mathcal{H}_{j,2m-j}) := \begin{pmatrix} \Phi(x_1) & \Phi(x_2) & \ldots & \Phi(x_{m-1}) & \Phi(x_{2m-j}) \\ \Phi(x_2) & \Phi(x_3) & \ldots & \Phi(x_{m-2}) & \Phi(x_{2m-j+1}) \\ \vdots & \vdots & \ldots & \vdots & \\ \Phi(x_{j-1}) & \Phi(x_j) & \ldots & \Phi(x_{2m-3}) & \Phi(x_{2m-2}) \\ \Phi(x_j) & \Phi(x_{j+1}) & \ldots & \Phi(x_{2m-2}) & \Phi(x_{2m-1}) \end{pmatrix}$$

Note that in the square case $\mathcal{H}_m = \mathcal{H}_{m,m}$, $\mathcal{D}_\Phi(\mathcal{H}_m)$ will preserve the total symmetry along sub-diagonals. In this case, up to a change of projective coordinates in the source and the target, $\mathcal{D}_\Phi(\mathcal{H}_m)$ will coincide with some degeneration step along a certain degeneration sequence to be subsequently looked at.

It is quite clear that taking minors commute with ring homomorphisms. However, for referencing convenience we isolate this fact:

Lemma 4.1.1. *With the above notation, one has* $I_r(\mathcal{D}_\Phi(\mathcal{H}_{j,2m-j})) = \Phi(I_r(\mathcal{H}_{j,2m-j}))$, *for all* $1 \leq r \leq j$.

Proof. The proof consists in applying Φ to the expression of a determinant as a sum of (signed) terms in the entries of the matrix. □

Mapping down partial derivatives is a little trickier. The following result relates the gradient ideal of the square Hankel matrix $\mathcal{H}_m$ and the gradient ideal of any of its degenerations.

Lemma 4.1.2. *Let* $g := \det(\mathcal{H}_m)$ *and* $g_i := \partial g/\partial x_i$. *Letting* Φ *be as above, one has*

$$J(\mathcal{D}_\Phi(\mathcal{H}_m)) = \Phi(\{g_1, \ldots, g_{2m-1}\} \setminus \{g_{s_1}, \ldots, g_{s_t}\}),$$

where $J(\mathcal{D}_\Phi(\mathcal{H}_m))$ *denote the gradient ideal of* $f := \det(\mathcal{D}_\Phi(\mathcal{H}_m))$.

Proof. Write $g = x_{s_1}h_1 + x_{s_2}h_2 + \cdots + x_{s_k}h_k + \tilde{g}$, where $\tilde{g}$ does not involve $x_{s_1}, x_{s_2}, \ldots, x_{s_t}$. Then

$$g_i = \sum_{j=1}^{t} \frac{\partial h_j}{\partial x_i} x_{s_j} + \frac{\partial \tilde{g}}{\partial x_i},$$

for all $i \notin \{s_1, \ldots, s_t\}$.

Clearly, $\Phi(g) = \tilde{g}$. By Lemma 4.1.1, $\Phi(g) = \det(\mathcal{D}_\Phi(\mathcal{H}_m)) = f$. Then

$$\Phi(g_i) = \frac{\partial \tilde{g}}{\partial x_i} = \frac{\partial \Phi(g)}{\partial x_i} = \frac{\partial f}{\partial x_i} \; \forall \; i \notin \{s_1, \ldots, s_t\}.$$

Therefore, $J(\mathcal{D}_\Phi(\mathcal{H}_m)) = \{\partial f/\partial x_i\}_{i \notin \{s_1,\ldots,s_t\}} = \Phi\left(\{g_i\}_{i \notin \{s_1,\ldots,s_t\}}\right)$. □

The next corollary extends Proposition 1.2.5 to further degenerations; in particular, it does not require that a null entry appear at most once on every row or column.

Corollary 4.1.3. *Let $\mathcal{D}_\Phi(\mathcal{H}_m)$ denote a degeneration as above of the generic square Hankel matrix $\mathcal{H}_m$. Then the partial derivative f_k of $f := \det(\mathcal{D}\Phi(\mathcal{H}_m))$ with respect to $x_k \in \Phi(R')$ is the sum of the (signed) cofactors of all appearances of x_k on $\mathcal{D}_\Phi(\mathcal{H}_m)$.*

Proof. Observe that if x_k appears on $\mathcal{D}_\Phi(\mathcal{H}_m)$ then it does so in the same spot as it appears on $\mathcal{H}_m$. Let a_{ij} denote the entry x_k of $\mathcal{H}_m$ on the ith row and jth column and consider its (signed) cofactor $C_{ij} = (-1)^{i+j}\det(\Delta_{ij})$. From Lemma 4.1.1, $\Phi(C_{ij}) = (-1)^{i+j}\det(\mathcal{D}_\Phi(\Delta_{\mathrm{ij}}))$ is the (signed) cofactor of the entry $\Phi(a_{ij}) = a_{ij}$ of the matrix $\mathcal{D}_\Phi(\mathcal{H}_m)$, while from Lemma 4.1.2, $f_k = \Phi(g_k)$, where g_k denotes the partial derivative of the Hankel determinant with respect to x_k. Since $\mathcal{H}_m$ has no null entries and each variable appears at most once on every row or column, by Proposition 1.2.5 g_k is the sum of the respective (signed) cofactors of all appearances of x_k on $\mathcal{H}_m$. As Φ maps cofactors of the entry a_{ij} of the matrix $\mathcal{H}_m$ to cofactors of the entry a_{ij} of the matrix $\mathcal{D}_\Phi(\mathcal{H}_m)$, the required result is clear. □

Another useful consequence is the following adapted version of the result of Gruson–Peskine:

Corollary 4.1.4. *Given $t \geq 1$ such that $t \leq j \leq 2m - t$, then*

$$I_t(\mathcal{D}_\Phi(\mathcal{H}_{j,2m-j})) = I_t(\mathcal{D}_\Phi(\mathcal{H}_{t,2m-t}).$$

Proof. By Lemma 1.2.4 we have $I_t(\mathcal{H}_{j,2m-j}) = I_t(\mathcal{H}_{t,2m-t})$, for all $t \leq j \leq 2m - t$. It follows from Lemma 4.1.2 that

$$I_t(\mathcal{D}_\Phi(\mathcal{H}_{j,2m-j})) = \Phi(I_t(\mathcal{H}_{j,2m-j})) = \Phi(I_t(\mathcal{H}_{t,2m-t})) = I_t(\mathcal{D}_\Phi(\mathcal{H}_{t,2m-t}).$$

□

4.2 Degeneration by zeros: determinants

Henceforth, pretty much as in previous chapters, we will focus on the following thread of degenerations of the generic square Hankel matrix:

$$\begin{pmatrix} x_1 & x_2 & \dots & x_{m-1} & x_m \\ x_2 & x_3 & \dots & x_m & x_{m+1} \\ \vdots & \vdots & \dots & \vdots & \\ x_{m-1} & x_m & \dots & x_{2m-3} & x_{2m-2} \\ x_m & x_{m+1} & \dots & x_{2m-2} & 0 \end{pmatrix}, \begin{pmatrix} x_1 & x_2 & \dots & x_{m-2} & x_{m-1} & x_m \\ x_2 & x_3 & \dots & x_{m-1} & x_m & x_{m+1} \\ \vdots & \vdots & \dots & \dots & \vdots & \vdots \\ x_{m-2} & x_{m-1} & \dots & x_{2m-5} & x_{2m-4} & x_{2m-3} \\ x_{m-1} & x_m & \dots & x_{2m-4} & x_{2m-3} & 0 \\ x_m & x_{m+1} & \dots & x_{2m-3} & 0 & 0 \end{pmatrix},$$

$$\dots, \begin{pmatrix} x_1 & x_2 & x_3 & \dots & x_{m-1} & x_m \\ x_2 & x_3 & x_4 & \dots & x_m & x_{m+1} \\ x_3 & x_4 & x_5 & \dots & x_{m+1} & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ x_{m-2} & x_{m-1} & x_m & \dots & 0 & 0 \\ x_{m-1} & x_m & x_{m+1} & \dots & 0 & 0 \\ x_m & x_{m+1} & 0 & \dots & 0 & 0 \end{pmatrix}$$

We will denote by $\mathcal{H}_m[r]$ a Hankel degeneration as above, where r denotes the number of zeros on the last column. The last matrix in the above sequence ($r = m - 2$) was called *sub-Hankel* in [5] (see also [28], [29]).

This notation will also be used when the Hankel matrix is not necessarily square, namely, $\mathcal{H}_{j,2m-j}[r]$. Note that in each such r-degeneration the base ring is the polynomial ring $k[x_1, \dots, x_{2m-r-1}]$. If no confusion arises, when r is fixed in the discussion, we will denote this ring simply by R.

The subsequent subsections will address some questions regarding $f := \det(\mathcal{H}_m[r])$, the gradient ideal $J \subset R$ of f and the ideal of sub-maximal minors $I_{m-1}(\mathcal{H}_m[r]) \subset R$.

4.2.1 The Hankel determinant

The proof of the following proposition is inspired from an elementary fact observed in the case of the sub-Hankel in [5, Remark 4.6 (c)], sufficiently generalized to the general case of a Hankel degeneration.

Actually, the observation will work for the generic Hankel matrix itself, thus avoiding drawing upon the general result about this matrix being 1-generic [16].

Proposition 4.2.1. *Let $\mathcal{H}_m[r]$ denote a Hankel degeneration as above, of order $m \times m$ with $m \geq 3$ and $r \geq 0$ zeros on its rightmost column. Let R denote the polynomial ring on the* *distinct non zero entries of the matrix. Then*

(i) $\det(\mathcal{H}_m[r]) \neq 0$.

(ii) $\det(\mathcal{H}_m[r]) \in R$ *is irreducible if and only if* $r \leq m-2$.

Proof. Set $f := \det(\mathcal{H}_m[r])$.

(i) There are many elementary ways of verifying the non-vanishing of f. Perhaps an easy one is to see that f has a unique nonzero pure term in x_m, namely, the product of the entries along the main anti-diagonal.

(ii) The "only if" part is obvious since the determinant would then be a power of x_m or zero.

For the reverse implication we will induct on m. The initial step of the induction will be subsumed in the general step.

By the Laplace expansion along the first row, since x_1 only appears once and on the first row, one sees that $f = x_1 f_1 + g$, where f_1 is the determinant of the Hankel degeneration $\mathcal{H}_{m-1}[r]$ obtained by omitting the first row and the first column of the original Hankel degeneration, and both f_1 and g belong to the subring $k[x_2, \ldots, x_{2m-1-r}]$.

To show that f is irreducible it suffices to prove that it is a primitive polynomial (of degree 1) in $k[x_2, \ldots, x_{2m-1-r}][x_1]$. Now, on one hand, f_1 is irreducible by the inductive hypothesis. Therefore, it is enough to see that f_1 is not a factor of g. For this, one verifies their initial terms in the revlex monomial order: $\mathrm{in}(f_1) = x_{m+1}^{m-1}$ and $\mathrm{in}(g) = \mathrm{in}(f) = x_m^m$. □

Since f is homogeneous, an alternative argument for the case $r \leq m-3$ consists in showing that $R/(f)$ is normal. Since $R/(f)$ is a hypersurface ring, it suffices to prove that it is locally regular in codimension one. By Proposition 4.4.1 below, proved independently, the gradient ideal J has codimension $3 = 1+2$ provided $r \leq m-3$. This proves that f is irreducible when $r \leq m-3$.

4.2.2 The Hankel Hessian

In this subsection, we will show that the Hessian $H(f)$ of $f = \det(\mathcal{H}_m[r])$ does not vanish for arbitrary m and r. We observe that the extreme case $m - r = 2$ (sub-Hankel) has been proved in [5] by showing that the corresponding Hessian is a power of x_{m+1} up to a nonzero coefficient from k.

The following result deals with the case $m - r \geq 3$, for any $r \geq 0$. In particular it gives another proof of the non-vanishing of the Hessian of the generic Hankel determinant ([28, Proposition 3.3.11]).

Theorem 4.2.2. *Let* $f = \det(\mathcal{H}_m[r])$. *If* $m - r \geq 3$, *the Hessian* $H(f)$ *does not vanish.*

Proof. The method is analogous to the one used in the case of the generic and the generic symmetric matrices. We consider the ring endomorphism φ of R mapping any variable in

$\mathbf{v} := \{x_1, x_{m-r-1}, x_{2m-r-1}\}$ to itself and mapping any variable off $\mathbf{v}$ to zero. We will show that by applying φ to the entries of $H(f)$ the resulting matrix $H(f)(\mathbf{v})$ has non-vanishing determinant.

For visualization we depict the matrix $\mathcal{H}_m[r]$ for arbitrary $r \leq m-3$:

$$\left(\begin{array}{ccccc|ccccc}
x_1 & x_2 & \cdots & x_{m-r-2} & x_{m-r-1} & x_{m-r} & x_{m-r+1} & \cdots & x_{m-1} & x_m \\
x_2 & x_3 & \cdots & x_{m-r-1} & x_{m-r} & x_{m-r+1} & x_{m-r+2} & \cdots & x_m & x_{m+1} \\
\vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\
x_{m-r-1} & x_{m-r} & \cdots & x_{2m-2r-4} & x_{2m-2r-3} & x_{2m-2r-2} & x_{2m-2r-1} & \cdots & x_{2m-r-3} & x_{2m-r-2} \\
\hline
x_{m-r} & x_{m-r+1} & \cdots & x_{2m-2r-3} & x_{2m-2r-2} & x_{2m-2r-1} & x_{2m-2r} & \cdots & x_{2m-r-2} & x_{2m-r-1} \\
x_{m-r+1} & x_{m-r+2} & \cdots & x_{2m-2r-2} & x_{2m-2r-1} & x_{2m-2r} & x_{2m-2r+1} & \cdots & x_{2m-r-1} & 0 \\
\vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \iddots & \vdots & \vdots \\
x_{m-1} & x_m & \cdots & x_{2m-r-4} & x_{2m-r-3} & x_{2m-r-2} & x_{2m-r-1} & \cdots & 0 & 0 \\
x_m & x_{m+1} & \cdots & x_{2m-r-3} & x_{2m-r-2} & x_{2m-r-1} & 0 & \cdots & 0 & 0
\end{array}\right)$$

In order to precisely locate an entry in the matrix it is convenient to reset the indices to double indices:

$$z_{i,j} = \begin{cases} x_{i+j-1}, & \text{if } i+j \leq 2m-r, \\ 0, & \text{if } i+j > 2m-r. \end{cases}$$

As earlier, the main principle is to isolate terms of the partial derivatives of f that have in their support a product of at least two variables off $\mathbf{v}$, since such terms will produce variables off $\mathbf{v}$ in the entries of $H(f)$ and hence will vanish thereof. To avoid tediously repeating the expression "terms of degree at least 2 off $\mathbf{v}$" in the sense just explained, we replace it by capital T.

Recall that the partial derivatives of f are sums of (signed) cofactors (Corollary 4.1.3). More precisely, for $k = 1, \ldots, 2m-r-1$, we have $f_k = \sum_{i+j=k+1} M_{i,j}$, where $M_{i,j} = (-1)^{i+j}\det(C_{i,j})$ is the (signed) cofactor of the (i,j)-entry.

Let us pick up the shape of a partial derivative $f_k = f_{i+j-1}$ of f as we go through the various relevant intervals for the sum $i+j$:

(a) $i+j \leq m-r$

Expanding the $C_{i,j}$ according to Laplace rule along its first $m-r-1$ rows yields

$$C_{i,j} = D_{i,j} x_{2m-r-1}^{r+1} + T,$$

where $D_{i,j}$ is the cofactor of the (i,j)-entry of the submatrix:

$$D = \begin{pmatrix} x_1 & x_2 & \cdots & x_{m-r-1} \\ x_2 & x_3 & \cdots & x_{m-r} \\ \vdots & \vdots & \cdots & \vdots \\ x_{m-r-1} & x_{m-r} & \cdots & x_{2m-2r-3} \end{pmatrix}. \tag{4.3}$$

As for the expansion of $D_{i,j}$, we find when $i+j<m-r$:

$$D_{i,j}=z_{m-r-i,m-r-j}x_{m-r-1}^{m-r-3}+T=x_{2m-2r-(i+j)-1}x_{m-r-1}^{m-r-3}+T.$$

Then, for $1\leq k<m-r-1$ and provided $i+j<m-r$, one gets

$$\begin{aligned} f_k &= \pm\sum_{i+j=k+1} D_{i,j}\,x_{2m-r-1}^{r+1}+T=\pm\sum_{i+j=k+1} x_{2m-2r-(i+j)-1}\,x_{m-r-1}^{m-r-3}\,x_{2m-r-1}^{r+1}+T \\ &= \pm k\cdot x_{2m-2r-(k+1)-1}\,x_{m-r-1}^{m-r-3}\,x_{2m-r-1}^{r+1}+T. \end{aligned}$$

Thus,

$$\varphi\left(\frac{\partial^2 f_k}{\partial x_l\partial x_k}\right)=\begin{cases}\pm k\cdot x_{m-r-1}^{m-r-3}x_{2m-r-1}^{r+1}, & \text{if } l=2m-2r-(k+1)-1,\\ 0, \quad \text{if } l\neq 2m-2r-(k+1)-1.\end{cases}$$

As for $k+1=i+j=m-r$, one has

$$f_k=f_{m-r-1}=\pm\sum_{i+j=m-r} D_{i,j}x_{2m-r-1}^{r+1}+T.$$

If $i=1$ or $j=1$, expanding the minor $D_{i,j}$ gives $D_{i,j}=x_{m-r-1}^{m-r-2}+T$, whereas if $i\neq 1$ or $j\neq 1$ then necessarily we have $m-r\geq 4$ and expanding yields

$$D_{i,j}=x_{m-r-1}^{m-r-2}+x_1\,z_{m-r-1,m-r-1}\,x_{m-r-1}^{m-r-4}+T=x_{m-r-1}^{m-r-2}+x_{2m-2r-3}\,x_{m-r-1}^{m-r-4}+T.$$

Substituting above obtains:

$$f_{m-r-1}=\pm(m-r-1)\,x_{m-r-1}^{m-r-2}\,x_{2m-r-1}^{r+1}\pm(m-r-3)\,x_1\,x_{2m-2r-3}\,x_{m-r-1}^{m-r-4}\,x_{2m-r-1}^{r+1}+T.$$

Upon applying φ yields

$$\varphi\left(\frac{\partial^2 f_{m-r-1}}{\partial x_l\partial x_{m-r-1}}\right)=\begin{cases}\pm(m-r-1)(m-r-2)x_{m-r-1}^{m-r-3}x_{2m-r-1}^{r+1}, & \text{if } l=m-r-1,\\ \pm(m-r-1)(r+1)x_{m-r-1}^{m-r-2}x_{2m-r-1}^{r}, & \text{if } l=2m-r-1.\end{cases}$$

Note that the above discussion gives us the first $(m-r-1)$ columns of $H(f)(\mathbf{v})$ and, by symmetry, its first $(m-r-1)$ rows. In particular, columns $m-r,\ldots,2m-2r-3$ of $H(f)(\mathbf{v})$ are partly obtained. For the closing argument at the end of the proof this knowledge so far suffices. For this reason, in the sequel we move all the way to the interval $2m-2r-1\leq i+j\leq 2m-r-1$, which will give us the shape of columns $2m-2r-2,\ldots,2m-r-1$ of $H(f)(\mathbf{v})$.

(b) $2m-2r-1\leq i+j<2m-r-1$

First note that if $i>m-r-1$ and $j>m-r-1$, then the cofactor $M_{i,j}$ have neither terms supported on $\mathbf{v}$ nor degree 1 terms off $\mathbf{v}$.

Indeed, consider the matrix $C_{i,j}$ obtained by omitting the ith row and the jth column of $\mathcal{H}_m(r)$, so $M_{i,j} = (-1)^{i+j}\det(C_{i,j})$. Observe that $C_{i,j}$ misses the entry x_{2m-r-1} originally sitting on the $(i, 2m-r-i)$th and $(2m-r-j, j)$th places on $\mathcal{H}_m(r)$. Therefore, $C_{i,j}$ has only $(m-2)$ columns with some entry in $\mathbf{v}$. Since $\deg M_{i,j} = m-1$, it cannot have any term supported on $\mathbf{v}$, and moreover, any of its degree 1 terms off $\mathbf{v}$ must involve the $(m-r-1)$ variables x_{m-r-1}, the $(r-1)$ variables x_{2m-r-1} on the matrix $C_{i,j}$ and the entry $z_{2m-r-j,2m-r-i}$ of $\mathcal{H}_m(r)$. But this entry is zero, since $(2m-r-j)+(2m-r-i) > 2m-r$ when $2m-2r-1 \leq i+j < 2m-r-1$.

Thus, we are left with the following possibilities: (i) $i < m-r-1$ or $j < m-r-1$

By symmetry, it suffices to consider the case where $i < m-r-1$. Then $C_{i,j}$ misses the entries x_{m-r-1} and x_{2m-r-1} originally in the $(i, m-r-i)$th and $(2m-r-j, j)$th places on $\mathcal{H}_m(r)$, respectively. Clearly, $M_{i,j}$ does not have terms supported in $\mathbf{v}$. Any term of $M_{i,j}$ involving x_1 cannot simultaneously involve variables of $\mathbf{v}$ in places $(m-r-1, 1)$ and $(1, m-r-1)$, and hence ought to have degree at least two in the variables off $\mathbf{v}$. Then one has

$$M_{i,j} = \pm x_{m-r-1}^{m-r-2} x_{2m-r-1}^{r} x_{3m-2r-(i+j)-1} + T.$$

(ii) $i = m-r-1$ ou $j = m-r-1$

Again, by symmetry it suffices to argue the case where $i = m-r-1$. A similar argument as above concerning places $(m-r-1, 1)$ and $(2m-r-j, j)$ of $\mathcal{H}_m(r)$ will do the job and one gets

$$M_{i,j} = \pm x_{m-r-1}^{m-r-2} x_{2m-r-1}^{r} x_{2m-r-j} \pm \quad x_1 x_{m-r-1}^{m-r-3} x_{2m-r-1}^{r} x_{3m-2r-j-2} + T.$$

A count of these cofactors give that, for each $l = 2m-2r-2, \ldots, 2m-r-2$, one has

$$f_l = \pm 2x_1 x_{m-r-1}^{m-r-3} x_{2m-r-1}^{r} x_{4m-3r-l-4} \pm c x_{m-r-1}^{m-r-2} x_{2m-r-1}^{r} x_{3m-2r-l-2} + T$$

Therefore

$$\varphi\left(\frac{\partial^2 f}{\partial x_k \partial x_l}\right) = \begin{cases} \pm 2\mathbf{q} = \pm 2x_1 x_{m-r-1}^{m-r-3} x_{2m-r-1}^{r}, & \text{if } k = 4m-3r-l-4, \\ 0, \quad \text{if } k > 4m-3r-l-4. \end{cases}$$

(c) $i + j = 2m - r - 1$

Here expanding along the first $m-r-1$ rows, one has $M_{i,j} = D \cdot x_{2m-r-1}^{r} + T$, where

$$D = \begin{pmatrix} x_1 & x_2 & \dots & x_{m-r-1} \\ x_2 & x_3 & \dots & x_{m-r} \\ \vdots & \vdots & \dots & \vdots \\ x_{m-r-1} & x_{m-r} & \dots & x_{2m-2r-3} \end{pmatrix}.$$

Thus,

$$M_{i,j} = \pm x_1 x_{m-r-1}^{m-r-3} x_{2m-r-1}^{r} x_{2m-2r-3} \pm x_{m-r-1}^{m-r-1} x_{2m-r-1}^{r} \; + T$$

and therefore,

$$f_{2m-r-1} = \pm(r+1) x_1 x_{m-r-1}^{m-r-3} x_{2m-r-1}^{r} x_{2m-2r-3} \pm (r+1) x_{m-r-1}^{m-r-1} x_{2m-r-1}^{r} \; + T,$$

Clearly,

$$\varphi\left(\frac{\partial^2 f}{\partial x_{2m-r-1} \partial x_{2m-r-1}}\right) = \pm\, (r+1) r x_{m-r-1}^{m-r-1} x_{2m-r-1}^{r-1}$$

.

We now see that upon degenerating the Hessian matrix $H(f)$ as originally indicated, one obtains a matrix in the following form:

$$H(f)(\mathbf{v}) = \left(\begin{array}{c|c} A & B^t \\ \hline B & A' \end{array}\right)$$

where the leftmost stack $\frac{A}{B}$ has the following shape:

$$\left(\begin{array}{ccccccccc}
0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & \pm\mathbf{p} \\
0 & 0 & \dots & 0 & 0 & 0 & \dots & \pm 2\mathbf{p} & 0 \\
\vdots & \vdots & \dots & \vdots & \vdots & \vdots & \iddots & \vdots & \vdots \\
0 & 0 & \dots & 0 & 0 & \pm(m-r-2)\mathbf{p} & \dots & 0 & 0 \\
0 & 0 & \dots & 0 & \pm(m-r-1)(m-r-2)\mathbf{p} & 0 & \dots & 0 & * \\
0 & 0 & \dots & \pm(m-r-2)\mathbf{p} & 0 & * & \dots & * & * \\
\vdots & \vdots & \iddots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\
0 & \pm 2\mathbf{p} & \dots & 0 & 0 & * & \dots & * & * \\
\pm\mathbf{p} & 0 & \dots & 0 & * & * & \dots & * & * \\
\hline
0 & 0 & \dots & 0 & 0 & * & \dots & * & * \\
\vdots & \vdots & \iddots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\
0 & 0 & \dots & 0 & 0 & * & \dots & * & * \\
0 & 0 & \dots & 0 & \pm(m-r-1)(r+1)x_{m-r-1}^{m-r-2}x_{2m-r-1}^{r} & * & \dots & * & *
\end{array}\right)$$

where A and B are $(2m-2r-3) \times (2m-2r-3)$ and $(r+2) \times (2m-2r-3)$ matrices, respectively. This part follows from the result in item (a) and the symmetry of $H(f)(\mathbf{v})$.

As for the matrix A', its shape follows from items (b) and (c):

$$A' = \begin{pmatrix} * & * & \dots & \pm 2\mathbf{q} & 0 \\ \vdots & \vdots & \iddots & \vdots & \vdots \\ * & \pm 2\mathbf{q} & \dots & 0 & 0 \\ \pm 2\mathbf{q} & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & \pm(r+1)r x_{m-r-1}^{m-r-1} x_{2m-r-1}^{r-1} \end{pmatrix},$$

with $\mathbf{p} = x_{m-r-1}^{m-r-3} x_{2m-r-1}^{r+1}$ and $\mathbf{q} = x_1 x_{m-r-1}^{m-r-3} x_{2m-2r-1}^{r}$.

Now expand the above determinant along the first $2m-2r-3$ rows. Note that the complementary minor to a $(2m-2r-3)$-minor of the first $2m-2r-3$ rows and avoiding the first $m-r-2$ columns vanishes as any of its columns is null. At the other end, the collection of non-vanishing minors of the first $(2m-2r-3)$ rows and involving the first $m-r-2$ columns consists of A itself and the following matrix X

$$\begin{pmatrix} 0 & 0 & \dots & 0 & 0 & 0 & 0 & \pm\mathbf{p} & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & \pm 2\mathbf{p} & 0 & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots & \iddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & \pm(m-r-2)\mathbf{p} & \dots & 0 & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 & & \pm(m-r-1)(r+1) x_{m-r-1}^{m-r-2} x_{2m-r-1}^{r} \\ 0 & 0 & \dots & \pm(m-r-2)\mathbf{p} & * & \dots & * & * & * \\ \vdots & \vdots & \iddots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots \\ 0 & \pm 2\mathbf{p} & \dots & 0 & * & \dots & * & * & * \\ \pm\mathbf{p} & 0 & \dots & 0 & * & \dots & * & * & * \end{pmatrix}$$

obtained upon replacing the $(m-r-1)$th column of A with the last column of B^t (i.e., the transpose of the last row of B). Their complementary matrices are, respectively, A' and

$$X' = \begin{pmatrix} * & * & * & \dots & \pm 2\mathbf{q} \\ \vdots & \vdots & \vdots & \iddots & \vdots \\ * & * & \pm 2\mathbf{q} & \dots & 0 \\ * & \pm 2\mathbf{q} & 0 & \dots & 0 \\ (m-r-1)(r+1) x_{m-r-1}^{m-r-2} x_{2m-r-1}^{r} & 0 & 0 & \dots & 0 \end{pmatrix}$$

Hence we obtain $\det(H(f)(\mathbf{v})) = \pm\det(A)\det(A') \pm \det(X)\det(X')$. Expanding the various determinants in this expression gives

$$\begin{aligned} \det(H(f)(\mathbf{v})) &= 2^{r+2}(r+1)(m-r-1)!\,\mathbf{p}^{2m-2r-4}\mathbf{q}^{r+1} \\ &\quad \cdot \left(\pm r(m-r-2)\mathbf{p} x_{m-r-1}^{m-r-1} x_{2m-r-1}^{r-1} \pm (m-r-1)(r+1) x_{m-r-1}^{2m-2r-4} x_{2m-r-1}^{2r}\right). \end{aligned}$$

Since $\mathbf{p} = x_{m-r-1}^{m-r-3} x_{2m-r-1}^{r+1}$, this expression is a nonzero monomial. $\square$

An interesting question in general is whether f is a factor of its Hessian determinant $H(f)$ with multiplicity ≥ 1. If this is the case, then f is said in addition to have the *expected* *multiplicity* (according to Segre) if its multiplicity as a factor of $H(f)$ is $m^2 - 2 - \dim V(f)^* - 1 = m^2 - 3 - \dim V(f)^* = \operatorname{codim}(V(f)^*) - 1$, where $V(f)^*$ denotes the dual variety to the hypersurface $V(f)$ (see [5]).

Conjecture 4.2.3. *Let* $0 \leq r \leq m-3$. *Then* f *is a factor of its Hessian with the expected multiplicity, namely,* $\operatorname{codim}(V(f)^*) - 1 = m - r - 2$*; in particular,* $\dim V(f)^* = 2m - r - 2 - (m - r - 1) = m - 1$.

Even for the fully generic Hankel matrix (i.e., $r = 0$) this seems to be unknown (cf. [29, Examples 3.5 and 3.6]). Note that in the subHankel degeneration ($r = m - 2$) f is not a factor of its Hessian (cf. [5, Theorem 4.4 (iii)]).

4.3 Degeneration by zeros: ideal of submaximal minors

4.3.1 Primality and codimension

With the notation as in the end of Section 4.1, let $I_d(\mathcal{H}_m[r])$ denote the ideal of d-minors of the matrix $\mathcal{H}_m[r]$.

Proposition 4.3.1. *Assume that* $r \leq m-2$*, one has:*

(i) $I_{m-1}(\mathcal{H}_m[r])$ *has codimension* 3 *and* $I_{m-2}(\mathcal{H}_m[r])$ *has codimension* 5.

(ii) $I_{m-1}(\mathcal{H}_m[r])$ *is a prime ideal if and only if* $r \leq m-3$.

(ii) $I_{m-2}(\mathcal{H}_m[r])$ *is a prime ideal if and only if* $r \leq m-4$.

Proof. (i) By Corollary 4.1.4, the ideal $I_{m-1}(\mathcal{H}_m[r])$ is generated by the maximal minors of $\mathcal{H}_{m-1,m+1}[r]$. Clearly, then its codimension is at most $m+1-(m-1)+1 = 3$. On the other hand, direct inspection of the latter matrix shows that the powers

$$x_{m-1}^{m-1},\, x_m^{m-1},\, x_{m+1}^{m-1}$$

belong to the initial ideal of $I_{m-1}(\mathcal{H}_m[r])$ in the revlex monomial order. Therefore $I_{m-1}(\mathcal{H}_m[r])$ is an ideal of codimension 3.

By a similar token, by Corollary 4.1.4, the ideal $I_{m-2}(\mathcal{H}_m[r])$ is generated by the maximal minors of $\mathcal{H}_{m-2,m+2}[r]$. Then its codimension is at most $m+2-(m-2)+1 = 5$. Direct inspection as before gives that

$$x_{m-2}^{m-2},\, x_{m-1}^{m-2},\, x_m^{m-2},\, x_{m+1}^{m-2},\, x_{m+2}^{m-2}$$

belong to the initial ideal of $I_{m-2}(\mathcal{H}_m[r])$ in the revlex monomial order. Therefore $I_{m-2}(\mathcal{H}_m[r])$ is an ideal of codimension 5.

(ii) The "only if" assertion is clear by considering the minor on the bottom-right corner of the matrix.

For the reverse implication we proceed as follows. Once again, by Corollary 4.1.4, the ideal $I_{m-1}(\mathcal{H}_{m-1,m+1}[r]))$ is the image of the ideal

$$(x_{2m-r}, \ldots, x_{2m-1}, I_{m-1}(\mathcal{H}_{m-1,m+1})),$$

under the map carrying the r variables $x_{2m-r}, \ldots, x_{2m-1}$ on $\mathcal{H}_{m-1,m+1}$ map to zero. On the other hand, it is known that the generic Hankel matrix is 1-generic (see [16, Proposition 4.3]). Therefore, by the result of [15, Theorem 1 (ii)] the ideal $(x_{i_1}, \ldots, x_{i_s}, I_{m-1}(\mathcal{H}_{m-1,m+1}))$ is prime whenever $s \leq m-3$. Taking $s := 2m-1-(2m-r)+1 = r$, we are through by hypothesis. Therefore, its image is a prime ideal.

(iii) The "only if" assertion is clear by considering the minor on the bottom-right corner of the matrix.

For the reverse implication we proceed in a similar way as in (ii) trading $\mathcal{H}_{m-2,m+2})$ for $\mathcal{H}_{m-1,m+1})$ and applying [15, Theorem 1 (ii)] again. □

Remark 4.3.2. Clearly, similar results will hold for the ideal of minors of even lower order. It is interesting to note that, even for $r \leq m-3$, the ring $R/I_{m-1}(\mathcal{H}_m[r])$ is not always normal, a property that may require $m >> r$.

4.3.2 The special fiber and the analytic spread

The following result is a non-generic version of [3, Theorem 10.16 (b)], with the same proof.

Lemma 4.3.3. *Let M be a square matrix with entries either variables over a field k or zeros, such that $\det(M) \neq 0$. Let R denote the polynomial ring over k on the nonzero entries of M and let $S \subset R$ denote the k-subalgebra generated by the submaximal minors. Then the extension $S \subset R$ is algebraic at the level of the respective fields of fractions.*

Recall from Section 1.1 that the analytic spread $\ell(I)$ of an ideal $I \subset R$ is at most $\min\{\mu(I), \dim R\}$, where $\mu(I)$ denotes the minimal number of generators of I. The ideal is said to have *maximal* analytic spread when this upper bound is attained.

As an immediate consequence of the lemma and of Proposition 4.2.1 (i), one has:

Proposition 4.3.4. *With the notation of the previous subsection, the ideal $I_{m-1}(\mathcal{H}_m[r]) \subset R$ has maximal analytic spread.*

Proof. The analytic spread of an ideal is also the dimension of the central fiber algebra of its Rees algebra. In this case, the ideal is a homogeneous ideal of the polynomial ring R generated in one single degree. Thus, this algebra is isomorphic to the k-subalgebra $S \subset R$ generated by the minors. Therefore, Lemma 2.2.3 is applicable. $\square$

Note that the above lemma is slightly stronger since it actually tells us that the linear system spanned by the minors has vector dimension $\dim R$, hence in particular the minimal number of generators of $I_{m-1}(\mathcal{H}_m[r])$ (not just the original number of minors) is at least $\dim R$.

Question 4.3.5. What are the defining equations of the special fiber of $I_{m-1}(\mathcal{H}_m[r])$? If $m-1 \geq r+2$ then these minors are the maximal minors of $\mathcal{H}_{m-1,m+1}[r]$ in an obvious notation, with the generic number of minors. Therefore, among its minimal relations there are the Plücker relations. In the Hankel case (i.e., $r=0$) these generate the ideal of relations, but for arbitrary r minimal cubic relations show up, perhaps inheriting the nature of non-maximal minors in the generic case (see [4]). Are these degenerated "shape relations" from the ones explained in the latter reference?

Conjecture 4.3.6. *The Rees algebra of the ideal $I_{m-1}(\mathcal{H}_m[r])$ is Cohen–Macaulay and of fiber type.*

Again the evidence comes out of computational verification when $m = 3, 4, 5$.

4.4 Degeneration by zeros: gradient ideal

4.4.1 The codimension

Proposition 4.4.1. *Let $J \subset R$ denote the gradient ideal of $\det(\mathcal{H}_m[r])$, where $m - r \geq 2$. Then*

$$\operatorname{codim}(J) = \begin{cases} 2 & \text{if } m-r=2 \\ 3 & \text{otherwise.} \end{cases}$$

Proof. By Corollary 4.1.3, J is contained in $I_{m-1}(\mathcal{H}_m[r])$, for every degeneration step. The latter has codimension 3 by Proposition 4.3.1. Therefore, J has codimension at most 3 (this also follows from the fact that the generic Hankel matrix has codimension 3).

The case where $m - r = 2$ is easily checked and, in any case, sufficiently studied in [5] and [29]. Thus, we assume that $m - r \geq 3$ and induct on $m - r$.

When $m - r = 3$ we proceed as follows.

Consider the initial ideal $\operatorname{in}(J)$ of J in the revlex monomial order. As is easily seen, x_m^{m-1} and x_{m+1}^{m-1} belong to $\operatorname{in}(J)$. We will show that $x_{m+2}^{2m-3}x_{m-1}^{m-1}$ belongs to the ideal $\operatorname{in}(J)$, which will imply that codimension of $\operatorname{in}(J)$ is at least 3.

Claim: $x_{m+2}^{2m-3} \in J : P$

Letting $\Delta_{i,j}$ denote the (signed) cofactor of the (j,i)-entry of $\mathcal{H}_m[r]$, we are to show that $x_{m+2}^{2m-3}\Delta_{i,j} \in J$ for any $1 \le i \le j \le m$. (Observe that, by symmetry, $\Delta_{i,j} = \Delta_{j,i}$.) We prove the claim by showing that for any $k \in \{3,\ldots,m\}$ one has $x_{m+2}^{2k-3}\Delta_{i,j} \in J$ for all $1 \le i \le j \le k$ and then set $k = m$.

We induct on k. The initial step will actually be included in the inductive step, but we choose to make it explicit anyway. For it, consider the following 3×3 submatrix of the adjoint matrix of $\mathcal{H}_m(r)$:

$$\begin{pmatrix} \Delta_{1,1} & \Delta_{1,2} & \Delta_{1,3} \\ \Delta_{2,1} & \Delta_{2,2} & \Delta_{2,3} \\ \Delta_{3,1} & \Delta_{3,2} & \Delta_{3,3} \end{pmatrix}$$

The cofactor formula yields the following relations:

$$x_m\Delta_{1,j} + x_{m+1}\Delta_{2,j} + x_{m+2}\Delta_{3,j} = 0 \text{ for all } j = 1,\ldots,3 \tag{4.4}$$

Since $f_1 = \Delta_{1,1}$ e $f_2 = 2\Delta_{2,1}$ are elements of J, it follows from above relation with $j = 1$ that $x_{m+2}\Delta_{3,1} \in J$ and consequently

$$x_{m+2}\Delta_{2,2} = x_{m+2}f_3 - 2x_{m+2}\Delta_{3,1} \in J$$

because $f_3 = 2\Delta_{3,1} + \Delta_{2,2} \in J$.

Taking $j = 2$ in the relation (4.4) and multiplying by x_{m+2} we obtain

$$x_m x_{m+2}\Delta_{1,2} + x_{m+1}x_{m+2}\Delta_{2,2} + x_{m+2}^2\Delta_{3,2} = 0.$$

Since $\Delta_{1,2}$ e $x_{m+2}\Delta_{2,2}$ belong to J we obtain $x_{m+2}^2\Delta_{3,2} \in J$. Finally, taking $j = 3$ in the relation (4.4) and multiplying by x_{m+2}^2 we conclude that $x_{m+2}^3\Delta_{3,3}$ belongs to J.

For the inductive step, suppose that $x_{m+2}^{2k-3}\Delta_{i,j} \in J$ for $1 \le i \le j \le k$ and consider similarly the following submatrix of the adjoint matrix of $\mathcal{H}_m(r)$:

$$\left(\begin{array}{ccc|c} \Delta_{1,1} & \cdots & \Delta_{1,k} & \Delta_{1,k+1} \\ \vdots & \cdots & \vdots & \vdots \\ \Delta_{k,1} & \cdots & \Delta_{k,k} & \Delta_{k,k+1} \\ \hline \Delta_{k+1,1} & \cdots & \Delta_{k+1,k} & \Delta_{k+1,k+1} \end{array}\right)$$

We will show that $x_{m+2}^{2(k+1)-3}\Delta_{i,j} \in J$ for all $1 \le i \le j \le k+1$. Again the cofactor formula yields the following relations:

$$x_{m-k+2}\Delta_{1,j} + \cdots + x_{m+1}\Delta_{k,j} + x_{m+2}\Delta_{k+1,j} = 0, \text{ for all } 1 \le j \le k+1. \tag{4.5}$$

For $j \neq k+1$ we multiply this equality by x_{m+2}^{2k-3} obtaining the following expression:

$$x_{m-k+2}x_{m+2}^{2k-3}\Delta_{1,j} + \cdots + x_{m+1}x_{m+2}^{2k-3}\Delta_{k,j} + x_{m+2}^{2k-3+1}\Delta_{k+1,j} = 0, \text{ for all } j = 1, \ldots, k.$$

Since, by hypothesis, $x_{m+2}^{2k-3}\Delta_{i,j} \in J$ for all $i = 1, \ldots, k$, this expression give us

$$x_{m+2}^{2k-3+1}\Delta_{k+1,j} \in J \text{ for all } j = 1, \ldots, k$$

.

For $j = k+1$, since $x_{m+2}^{2k-3+1}\Delta_{i,j} \in J$ and $\Delta_{k+1,j} = \Delta_{j,k+1}$ for all $i = 1, \ldots, k$, we multiply the equality (4.5) by x_{m+2}^{2k-3+1} obtaining that

$$x_{m+2}^{2k-3+2}\Delta_{k+1,k+1} \in J.$$

This takes care of the claim.

In particular, it follows from the above claim that $x_{m+2}^{2m-3}\Delta_{m,m} \in J$. A direct inspection shows that $\text{in}(x_{m+2}^{2m-3}\Delta_{m,m}) = x_{m+2}^{2m-3}x_{m-1}^{m-1}$ and this ensures that the initial ideal of J has codimension 3. Thus, we are through with the case where $m - r = 3$.

For the inductive step, note that the ascending induction step from $m-r$ to $m-r+1 = m-(r-1)$ corresponds to a descending induction step from r to $r-1$. Thus, we are given the matrix $\mathcal{H}_m[r-1]$, with $r-1 \leq m-4$, and the corresponding gradient ideal $J \subset R = k[x_1, \ldots, x_{2m-r}]$ and assume by induction that the corresponding gradient ideal $J' \subset R' = k[x_1, \ldots, x_{2m-r-1}]$ of $\det(\mathcal{H}_m[r])$ has codimension 3. Since the latter matrix is a degeneration of the former by setting $x_{2m-r} \mapsto 0$, the principle in Lemma 4.1.2 applies to show that $(J', x_{2m-r}) \subset (J, x_{2m-r})$, hence the codimension of J is at least that of J'.

Remark 4.4.2. It is possible that the variables that are set to zero form a regular sequence modulo the gradient ideal of the generic Hankel matrix $\mathcal{H}_m$. Alas, this is not a great help since the partial derivatives specialize in the way explained in Lemma 4.1.2, thus messing up the subsequent specialization steps beyond the first one - and, indeed, some of these variables will be zero-divisors on the degenerated gradient ideal. Even for $r = 1$, one would have to know the associated primes of the generic gradient ideal besides the ideal of submaximal minors (in [29, Corollary 3.17 (i)] it is conjectured that $I_{m-2}(\mathcal{H}_m)$ is the only other associated prime).

4.4.2 The associated primes

In this part we will suppose throughout that $r \leq m-3$. The case where $r = m-2$ has been thoroughly dissected in [5] and [29].

Conjecture 4.4.3. *Let $J \subset R$ denote the gradient ideal of the determinant of $\mathcal{H}_m[r]$ and let Q denote the ideal generated by the $m-r$ nonzero variables of its last column. Assume that*

$1 \leq r \leq m-3$. *Then:*

(i) *The minimal primes of* R/J *are* Q *and* $P := I_{m-1}(\mathcal{H}_m[r])$. *In particular,* J *is not a reduction of* P (*as otherwise* $\sqrt{J} = \sqrt{P}$) *and, moreover, the minimal component of* J *coincides with its unmixed component if and only if* $r = m-3$.

(ii) *The minimal component of* J *is its radical – hence, it is* $P \cap Q$ *by* (i) . *In particular, one has*

$$e(R/J) = \begin{cases} e(R/P) & \text{if } r \leq m-4 \\ e(R/P)+1 & \text{if } r = m-3. \end{cases}$$

(iii) *The embedded associated primes of* R/J *are* (x_{m-1}, Q) (*in codimension* $m-r+1$) *and* $\sqrt{I_{m-2}(\mathcal{H}_m[r])}$ (*in codimension* 5).

That P is prime of codimension 3 has been proved in Proposition 4.3.1 (ii). Since J has codimension 3 (Proposition 4.4.1), P is a minimal prime thereof. It is also clear that Q is a minimal prime of J when $r = m-3$, hence one can assume that $r \leq m-4$ in the item (ii) of the above conjectural theorem.

4.4.3 Linear behavior

The linear rank

It has been proved in [28] that for $\mathcal{H}_m$ (Hankel) the linear rank of J is 3, while in [5] the linear rank of $\mathcal{H}_m[m-2]$ (sub-Hankel) was shown to be maximal possible ($= m$).

Problem: what is the linear rank of J in the intermediary degeneration steps?

Computational evidence points to linear rank 2 in any intermediary step.

The linear type property

Problem: Is J an ideal of linear type for $r \leq m-3$?

A weaker expectation is that the partial derivatives be analytically independent forms.

Again one has some meager computational evidence for an affirmative answer. Note that even the fully generic Hankel case is open ([29, Corollary 3.17 (iv)]) but the subHankel case has been affirmatively settled ([29, Theorem 4.8]),

If the answer to the second problem above is affirmative and if the linear rank of J is small as predicted in the first problem above, then one can conclude that $f = \det(\mathcal{H}_m[r])$ is not homaloidal for $r \leq m-3$. For the moment this is an open question due to the lack of alternative approaches.

Chapter 5

Degenerations of a square generic catalecticant

Here one skims over the case of arbitrary catalecticants – these are classical specializations of the fully generic matrix which are not symmetric (except for the very extreme case of a Hankel catalecticant).

In a precise way, given two integers $1 \leq r \leq m$, the *r-leap* $m \times m$ *generic catalecticant* is, as in (1.1), the matrix

$$\mathcal{C}_{m,r} = \begin{pmatrix} x_1 & x_2 & x_3 & \cdots & x_m \\ x_{r+1} & x_{r+2} & x_{r+3} & \cdots & x_{m+r} \\ x_{2r+1} & x_{2r+2} & x_{2r+3} & \cdots & x_{m+2r} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_{(m-1)r+1} & x_{(m-1)r+2} & x_{(m-1)r+3} & \cdots & x_{(m-1)r+m} \end{pmatrix}$$

Note that the corresponding determinant will have low degree ($= n$) as compared to the dimension of the ring and still involve all variables. The extreme values $r = 1$ and $r = m$ yield, respectively, the ordinary Hankel matrix and the fully generic matrix.

5.1 Degeneration by cloning

Next is the behavior of cloning for small values of m. We will leave out the case of the Hankel matrix as it has been sufficiently dealt with in a previous chapter. The computation has been done with [1].

5.1.1 $m = 3$

Here there is only one catalecticant which is neither fully generic nor Hankel, namely:

$$\mathcal{C}_3 = \begin{pmatrix} x_1 & x_2 & x_3 \\ x_3 & x_4 & x_5 \\ x_5 & x_6 & x_7 \end{pmatrix}$$

(i) Cloning x_4 into x_7 (along the main diagonal).

The ideal J is of linear type; in particular, the Hessian does not vanish. However, there is only one minimal linear syzygy. Therefore, the polar map is not a Cremona map. This is in flagrant contrast to the cloning of the same nature on the fully generic matrix.

(ii) Cloning x_6 into x_7 (along last row).

The ideal J is of linear type; in particular, the Hessian does not vanish. However, the linear rank of J is only 3. Therefore, the polar map is not a Cremona map.

(iii) Cloning x_3 or x_5 into x_7 (along last column).

The ideal J has maximal linear rank (= 5) and the Hessian determinant is nonzero. Therefore, the polar map is a Cremona map.

(iv) Cloning x_2 into x_5 (along an upper diagonal).

In this cloning the original number of variables in the catalecticant stays the same. The ideal J has maximal linear rank (= 5) and the Hessian determinant is nonzero. Therefore, the polar map is a Cremona map.

(v) Cloning x_3 into x_6 (along a lower diagonal).

The ideal J is of linear type; in particular, the Hessian does not vanish. However, the linear rank of J is only 2. Therefore, the polar map is not a Cremona map.

Remark 5.1.1. Considering the degeneration of the above 2-leap catalecticant by replacing x_7 by 0, we get as in the Hankel case (next section) a homaloidal determinant. Thus, contrarily to the fully generic case, where this degeneration has vanishing Hessian, one cannot reduce cases (iii) and (iv) above to this degeneration case by row/column operations.

5.1.2 $m = 4$

Here one has two catalecticants which are neither fully generic nor Hankel, namely:

$$\mathcal{C}_{4,2} = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ x_3 & x_4 & x_5 & x_6 \\ x_5 & x_6 & x_7 & x_8 \\ x_7 & x_8 & x_9 & x_{10} \end{pmatrix}$$

and

$$\mathcal{C}_{4,3} = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ x_4 & x_5 & x_6 & x_7 \\ x_7 & x_8 & x_9 & x_{10} \\ x_{10} & x_{11} & x_{12} & x_{13} \end{pmatrix}$$

It is not at all clear which of these is a best analogue of the unique previous 3×3 catalecticant. In any case, pursuing the possible analogues of the clonings in that simpler situation yields by and large: for $\mathcal{C}_{4,2}$, the ideal J is of linear type – hence the Hessian is non-vanishing – but the linear rank is not maximal; for $\mathcal{C}_{4,3}$, the Hessian determinant is null.

Here is what we expect for the 2-catalecticant matrix $C_{m,2}$, for any $m \geq 4$.

Consider $I = I_{m-1}(C_{m,2})$ the ideal of submaximal minors of $C_{m,2}$. By the same principle of Gruson-Peskine I is contained in the ideal P of the maximal minors of an $(m-1) \times (m+2)$ 2-catalecticant.

First, P is a prime ideal of codimension 4. Indeed, since all catalecticant of arbitrary size is 1-generic the statement follows from [16].

The following properties should be within reach, although we have not pursued along them any further.

1. $I \subset P$ has codimension 4
2. P is Cohen-Macaulay and I is a Gorenstein
3. $Q := I : P \subset I_{m-2}(C_{m,2})$ is generated in degree $m-2$ and is a smooth prime ideal with linear resolution
4. $I = P \cap Q$. Consequently $e(R/I) = e(R/P) + e(R/Q)$.

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