Universidade Federal da Paraíba
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Programa Associado de Pós-Graduação em Matemática
Doutorado em Matemática

Estimativas de Carleman para uma classe de problemas parabólicos degenerados e aplicações à controlabilidade multi-objetivo

por

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Tese apresentada ao Corpo Docente do Programa Associado de Pós-Graduação em Matemática - UFPB/UFCG, como requisito parcial para obtenção do título de Doutor em Matemática.

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Resumo

Neste trabalho apresentamos estimativas de Carleman para uma classe de problemas parabólicos degenerados sobre um quadrado (no caso bidimensional) ou sobre um intervalo limitado (no caso unidimensional). Consideramos um operador diferencial que degenera apenas em uma parte da fronteira. Provamos resultados de existência, unicidade e estimativas de energia via teoria do semigrupo. Em seguida usamos funções peso adequadas para obter estimativas de Carleman e, como aplicações, resultados de controlabilidade multi-objetivo.

Palavras-chave: Controle nulo de Stackelberg-Nash; Desigualdade de Carleman; Observabilidade; Equações com coeficiente degenerados.

Abstract

This work presents Carleman estimates to a class of degenerate parabolic problems over a square (in the two dimensional case) or a bounded interval (in the one dimensional case). We consider a differential operator that degenerate only in a part of the boundary. Using semigroup theory, we prove well posedness results. Then, using suitables weight functions, we prove Carleman estimates and, as application, results on multi-objective controllability.

Keywords: Stackelberg-Nash null controllability; Carleman estimates; Observability; Degenerate equations.

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O primeiro trabalho científico é um grande desafio. Dado o desenvolvimento da ciência nos dias atuais, não é nada fácil extrapolar seus limites. Mas quando se tem uma gama de agentes apoiando, guiando o estudante, a meta deixa de parecer tão distante. Por ter me conduzido até aqui, não ter me deixado cair uma queda que eu não pudesse me levantar, me ajudado a sempre se erguer, agradeço ao Criador da vida: Deus. Por toda a paciência e por terem acreditado no meu trabalho, agradeço aos meus orientadores, Fagner e Enrique. Me sinto muito honrado por ter tido a oportunidade de trabalhar com estes pesquisadores tão talentosos.

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"Se o conhecimento pode criar problemas, não é através da ignorância que podemos solucioná-los."

Isaac Asimov

Dedicatória

Para Elon Lages Lima, o grande escritor de livros matemáticos brasileiros.

Sumário

Introdução

0.1 Controlabilidade nula de equações parabólicas degeneradas

Com o desenvolvimento do Calculo Diferencial e Integral em meados do século XVII, a ciência provocou um grande impacto na sociedade. Desde então, um sem número de fenômenos foram analisados, com resultados que permitiram avanços tecnológicos sem precedentes na história da humanidade. Dentre as principais ferramentas responsáveis por tamanho progresso, destacam-se as equações diferenciais. Capazes de modelar diversos problemas, a resolução dessas equações permitiam prever o comportamento futuro de várias variáveis. Infelizmente (ou felizmente) muitas destas equações são tão difíceis de se resolver, que mesmo 4 séculos depois do inicio do Cálculo, ainda existem inúmeras equações passíveis de análises. Não por menos, o campo da matemática que se dedica a encontrar soluções aproximadas destas equações se desenvolveu tanto no último século.

Certamente, depois de se prever o comportamento de um determinado fenômeno, um passo seguinte é influenciá-lo. É nesse sentido que atua a teoria de controle: atuar e influenciar o comportamento de certas variáveis em tais fenômenos.

Um sistema de controle é uma equação de evolução (EDO ou EDP) que depende de um parâmetro u, que escreveremos da seguinte forma:

$$y' = f(t, y, u),$$

onde $t \in [0, T]$ é o tempo, $y : [0, T] \mapsto Y$ é a função estado, $u : [0, T] \mapsto U$ é o controle e Y e U são espaços de funções adequados. Na equação acima, y' representa a derivada de y em relação ao tempo t.

O problema de controle consiste em encontrar um controle u tal que a função estado se comporta de uma forma desejada. Exemplificaremos alguns, dentre os vários, problemas de controlabilidade presentes na literatura.

Controle Ótimo: Encontrar um controle que minimiza algum funcional custo, por exemplo,

$$J(u) = \|y(T;u) - \bar{y}\|_U^2 + \|u\|_U^2,$$

em que $\bar{y}$ é um alvo desejado e $y(T;u)$ é o estado alcançado pelo sistema no tempo final T.

Controlabilidade Exata: Dado dois tempos $T_0 < T_1$ e y_0, y_1 dois possíveis estados do sistema, encontrar $u : [T_0, T_1] \mapsto U$ tal que

$$\begin{cases} y' = f(y,u) \ \text{ em } \ [T_0, T_1] \\ y(T_0) = y_0, \ \ y(T_1) = y_1. \end{cases}$$

Em outras palavras, partindo de qualquer configuração inicial y_0, podemos conduzir a solução y para o estado y_1 sob a ação do controle u.

Controlabilidade Aproximada: Dados $T_0 < T_1$, dois possíveis estados y_0, y_1 e $\epsilon > 0$, encontrar $u : [T_0, T_1] \mapsto U$ tal que

$$\begin{cases} y' = f(y,u) \ \text{ em } \ [T_0, T_1] \\ y(T_0) = y_0, \ \ \|y(T_1) - y_1\|_U < \epsilon. \end{cases}$$

A controlabilidade aproximada é uma versão mais fraca se comparada a controlabilidade exata. De fato, em vez de pedirmos que a função estado seja exatamente y_1 em T_1, pedimos apenas que o estado esteja arbitrariamente perto de y_1.

Controlabilidade Nula: Dados dois tempos $T_0 < T_1$ e y_0 um estado do sistema, encontrar $u : [T_0, T_1] \mapsto U$ tal que

$$\begin{cases} y' = f(y,u) \ \text{ em } \ [T_0, T_1] \\ y(T_0) = y_0, \ \ y(T_1) = 0. \end{cases}$$

Controlabilidade Exata para as Trajetórias: Dados $T_0 < T_1$, $y_0 \in Y$ e $\bar{y}$ uma trajetória (uma solução com controle $\bar{u} : [T_0, T_1] \mapsto U$), encontrar um controle $u : [T_0, T_1] \mapsto U$ tal que

$$\begin{cases} y' = f(y,u) \ \text{ em } \ [T_0, T_1] \\ y(T_0) = y_0, \ \ y(T_1) = \bar{y}(T_1). \end{cases}$$

Os conceitos de controlabilidade nula e controlabilidade exata para as trajetórias são de especial importância em sistemas não reversíveis e sistemas com efeito regularizante. Nestes casos, a controlabilidade exata não é esperada.

0.2 Controlabilidade nula de equações parabólicas degeneradas

O estudo da controlabilidade de equações diferenciais parciais atraiu o interesse de vários cientistas nas últimas décadas. Diversos resultados foram desenvolvidos sobre problemas semi-lineares, problemas em domínios ilimitados, sistemas de dinâmica dos fluidos entre outros. Nessas direções, alguns trabalhos notáveis são [15, 18, 20, 21, 24]. Por outro lado, no caso particular de equações parabólicas degeneradas, ainda pouco se sabe, veja [7, 19, 27].

Primeiramente, consideremos o modelo mais básico de equação parabólica degenerada estudado na última década:

$$\begin{cases} u_t - (x^\alpha u_x)_x + b_0(x,t)u = g1_{\mathcal{O}} & \text{em} \quad (0,1)\times(0,T), \\ u(1,\cdot)=0 \ \text{ e } \ \begin{cases} u(0,\cdot)=0 & \text{se} \ \ \alpha\in(0,1) \\ (x^\alpha u_x)(0,\cdot)=0 & \text{se} \ \ \alpha\in[1,2) \end{cases} & \text{em} \quad (0,T), \\ u(\cdot,0)=u_0 & \text{em} \quad (0,1), \end{cases} \tag{1}$$

onde $\alpha \in (0,2)$, $\omega \subset (0,1)$ é um aberto e 1_ω sua função característica associada, $T>0$, $b_0 \in L^\infty(0,1)$, $g \in L^2(\omega \times (0,T))$ e $u_0 \in L^2(\Omega)$.

Dizemos que (1) é *nulamente controlável* quando dado $u_0 \in L^2(0,1)$, existe $g \in L^2((0,1)\times(0,T))$ tal que a solução u de (1) satisfaz

$$u(\cdot,T)=0. \tag{2}$$

Na década de 90, no trabalho [20], o método HUM começa a ser popularizado e se consagra como o principal método para provar a controlabilidade nula de equações diferenciais parciais. Tal método consiste em reduzir o problema da controlabilidade nula ao problema de obter uma certa desigualdade para o estado adjunto do sistema original. Tal desigualdade, chamada de *desigualdade de observabilidade*, viria a se tornar na década seguinte, o principal método para provar a controlabilidade nula de problemas parabólicos degenerados. Mais especificamente, em [7], os autores consideraram o sistema adjunto de (1), isto é,

$$\begin{cases} v_t + (x^\alpha v_x)_x + b_0(x,t)v = h & \text{em} \quad (0,1)\times(0,T), \\ v(1,\cdot)=0 \ \text{ e } \begin{cases} v(0,\cdot)=0 & \text{se } \ \alpha\in(0,1) \\ (x^\alpha v_x)(0,\cdot)=0 & \text{se } \ \alpha\in[1,2) \end{cases} & \text{em} \quad (0,T), \\ v(\cdot,T)=v_T & \text{em} \quad (0,1). \end{cases} \tag{3}$$

e provaram a seguinte desigualdade de observabilidade para (3):

Proposition 0.2.1 *Sejam $\alpha \in (0,2)$ e $T>0$ dados e seja ω um subintervalo aberto e não vazio de $(0,1)$. Então existe $C>0$ tal que, para todo $v_T \in L^2(0,1)$, a solução v de (3) satisfaz*

$$\int_0^1 x^\alpha |v_x(0,x)|^2\,dx \le C \iint_{\omega\times(0,T)} |v(x,t)|^2\,dx\,dt. \tag{4}$$

Desde sua popularização na década de 90, a principal ferramenta usada para obter desigualdades de observabilidade tem sido as famigeradas *Desigualdades de Carleman.* Na década seguinte, com as EDPs degeneradas, não foi diferente, apesar exigirem outras ferramentas adicionais. Em [7] os autores provaram a seguinte desigualdade de Carleman:

Proposition 0.2.2 *Assuma $\alpha\in(0,2)$. Existem constantes positivas s_0 e C tais que, para qualquer $s\ge s_0$ e qualquer solução v de (3), vale:*

$$\iint_Q e^{-2s\sigma_0}\left[s\theta x^\alpha |v_x|^2 + s^3\theta^3 x^{2-\alpha}|v|^2\right]dx\,dt \le C\left[\|e^{-s\sigma_0}h\|^2 + s\int_0^T e^{-2s\sigma_0}\theta|v_x|^2 dt\Big|_{x=1}\right]. \tag{5}$$

A inclusão de um termo de primeira ordem espacial na equação de (1) é uma questão delicada que ainda não foi totalmente solucionada. Em [19] os autores estenderam os resultados de [7] para o seguinte problema

$$\begin{cases} u_t - (x^\alpha u_x)_x + (x^q b_1(x,t)u)_x + b_0(x,t)u = g1_{\mathcal{O}} & \text{em} \quad (0,1)\times(0,T), \\ u(1,\cdot)=0 \ \text{ and } \begin{cases} u(0,\cdot)=0 & \text{se } \ \alpha\in(0,1) \\ (x^\alpha u_x)(0,\cdot)=0 & \text{se } \ \alpha\in[1,2) \end{cases} & \text{em} \quad (0,T), \\ u(\cdot,0)=u_0 & \text{em} \quad (0,1), \end{cases} \tag{6}$$

onde $q\ge\alpha/2$ e $b_1\in L^\infty(0,1)$. Neste caso a Proposição 0.2.2 ainda vale para o sistema adjunto:

$$\begin{cases} v_t + (x^\alpha v_x)_x + x^q b_1 v_x + b_0 v = h & \text{em} \quad (0,1)\times(0,T), \\ v(1,\cdot)=0 \ \text{ and } \begin{cases} v(0,\cdot)=0 & \text{se } \ \alpha\in(0,1) \\ (x^\alpha v_x)(0,\cdot)=0 & \text{se } \ \alpha\in[1,2) \end{cases} & \text{em} \quad (0,T), \\ v(\cdot,0)=v_0 & \text{em} \quad (0,1). \end{cases} \tag{7}$$

A controlabilidade nula de (6) sem o peso x^q, isto é, do problema

$$\begin{cases} u_t - (x^\alpha u_x)_x + (b_1(x,t)u)_x + b_0(x,t)u = g1_{\mathcal{O}} & \text{em} \quad (0,1)\times(0,T), \\ u(1,\cdot)=0 \;\; \text{and} \;\; \begin{cases} u(0,\cdot)=0 & \text{se} \;\; \alpha\in(0,1) \\ (x^\alpha u_x)(0,\cdot)=0 & \text{se} \;\; \alpha\in[1,2) \end{cases} & \text{em} \quad (0,T), \\ u(\cdot,0)=u_0 & \text{em} \quad (0,1), \end{cases} \tag{8}$$

até onde sabemos, permanece aberta. Em [27], considerando $\alpha \in (0,1/2)$, os autores fizeram modificações nas já clássicas funções peso introduzidas em [7] e conseguiram provar estimativas de Carleman semelhantes para o problema adjunto:

$$\begin{cases} v_t + (x^\alpha v_x)_x + b_1 v_x + b_0 v = h & \text{em} \quad (0,1)\times(0,T), \\ v(1,\cdot)=0 \;\; \text{and} \;\; \begin{cases} v(0,\cdot)=0 & \text{se} \;\; \alpha\in(0,1) \\ (x^\alpha v_x)(0,\cdot)=0 & \text{se} \;\; \alpha\in[1,2) \end{cases} & \text{em} \quad (0,T), \\ v(\cdot,0)=v_0 & \text{em} \quad (0,1). \end{cases} \tag{9}$$

Com isso, a controlabilidade nula de (8) foi estabelecida para $\alpha \in (0,1/2)$.

O caso $\alpha = 2$ é interessante do ponto de vista das aplicações, pois a equação de (6), com $\alpha = 2$, tem como caso particular a célebre equação de Black Scholes [6], que modela o preço de opções de compra de ativos financeiros. Porém, já se sabe desde [7], que o problema (1) com $\alpha = 2$, isto é, o problema

$$\begin{cases} u_t - (x^2 u_x)_x + b_0(x,t)u = g1_{\mathcal{O}} & \text{em} \quad (0,1)\times(0,T), \\ u(1,\cdot)=0 \quad \text{e} \quad (x^2u_x)(0,\cdot)=0 & \text{em} \quad (0,T), \\ u(\cdot,0)=u_0 & \text{em} \quad (0,1), \end{cases} \tag{10}$$

não é, em geral, nulamente controlável.

Passando a dimensões espaciais superiores, até onde sabemos, o único trabalho publicado é [10], onde os autores obtiveram controlabilidade nula (usando novamente estimativas de Carleman) do seguinte sistema:

$$\begin{cases} u_t - div(A_0\nabla u) + b_0 u = g_0 1_{\omega_0} & \text{em} \quad Q_0, \\ \begin{cases} u=0 & \text{if} \;\; \alpha\in(0,1) \\ \dfrac{\partial u}{\partial \nu}=0 & \text{if} \;\; \alpha\in[1,2) \end{cases} & \text{sobre} \quad \Sigma_0, \\ u(\cdot,0)=u_0 & \text{em} \quad \Omega_0, \end{cases} \tag{11}$$

onde $Q_0 := \Omega_0 \times (0,T)$, $T > 0$, $\Omega_0 \subset \mathbb{R}^2$ é um dominio limitado com fronteira Γ_0 de classe C^4, $\Sigma_0 := \Gamma_0 \times (0,T)$, $\omega_0 \subset \Omega_0$ é aberto, $u_0 \in L^2(\Omega_0)$, $g_0 \in L^2(Q_0)$, $b_0 \in L^\infty(Q_0)$, $\alpha \in (0,2)$, e $A_0 : \overline{\Omega}_0 \longrightarrow M_{2\times 2}(\mathbb{R})$ satisfaz as seguintes condições:

(i) $a_{ij} \in C^3(\Omega_0;\mathbb{R}) \cap C^0(\overline{\Omega}_0;\mathbb{R})$, onde $A_0(x) = (a_{ij}(x))$;

(ii) $A_0(x)$ é simétrica $\forall x \in \overline{\Omega}_0$;

(iii) $A_0(x)$ é positiva definida $\forall x \in \Omega_0$;

(iv) Sejam $r_i(x)$, os autovalores e $\varepsilon_i(x)$ os autovetores unitários correspondentes à $A_0(x)$, $i = 1, 2$. Denotemos por $P_{\Gamma_0}(x)$ a projeção de x até a fronteira Γ_0 e $O(\Gamma_0;\delta) := \{x \in \Omega_0 : d(x,\Gamma_0) < \delta\}$. Existe $\delta > 0$ tal que

   1. $r_1(x) = d(x,\Gamma_0)^\alpha$, $\forall x \in O(\Gamma_0;\delta)$,
   2. $r_2(x) > 0$ $\forall x \in \overline{\Omega_0 \backslash O(\Gamma_0;\delta)}$;
   3. $\varepsilon_1(x) = \nu(P_{\Gamma_0}(x))$ $\forall x \in O(\Gamma_0;\delta)$.

Vale salientar a discrepância entre o sistema (11), cujo operador diferencial degenera em toda a fronteira, e o sistema (1), cujo operador diferencial degenera em apenas uma parte da fronteira. Tal discrepância é fruto da dificuldade de construir pesos adequados para a desigualdade de Carleman.

0.3 Controlabilidade multi-objetivo

Diferentemente dos conceitos de controlabilidade usuais ja descritos acima, a controlabilidade multi-objetivo, como o próprio nome sugere, consiste em buscar um ou mais controles que façam o estado atender mais de um requisito. Neste trabalho, temos como objetivo principal provar resultados de controlabilidade hierárquica. Neste tipo de controlabilidade o objetivo é, além de controlar o estado no instante final T, controlar o estado também ao longo do processo evolutivo, pelo menos em uma parte do domínio.

Para sermos mais claros considere agora o sistema

$$\begin{cases} u_t - \triangle u = f1_\omega + v_1 1_{\omega_1} + v_2 1_{\omega_2} & \text{em} \quad Q \\ u = 0 & \text{sobre} \quad \Sigma \\ u(x,0) = u_0(x) & \text{em} \quad \Omega, \end{cases} \tag{12}$$

onde $\Omega \in \mathbb{R}^n$ é um domínio limitado com fronteira Γ suave, $Q = \Omega \times (0,T)$, $\Sigma = \Gamma \times (0,T)$ e $\omega, \omega_i \subset \Omega$ são abertos. Fixemos abertos $\omega_{i,d} \subset \Omega$ e funções $u_{i,d} \in L^2(\Omega)$ e introduzimos os funcionais

$$J_i(f;v_1,v_2)=\frac{\beta_i}{2}\iint_{\omega_{i,d}\times(0,T)}|u-u_{i,d}|^2\,dx\,dt+\mu_i\iint_{\omega_i\times(0,T)}|v_i|^2\,dx\,dt, \tag{13}$$

onde $\beta_i,\mu_i>0$ são constantes fixadas e u é o estado de (12) associado à terna $(f;v_1,v_2)$.

O problema da controlabilidade nula hierárquica consiste em buscar controles f (o líder) e (v_1,v_2) (os seguidores) que levem o estado u de (12) ao estado nulo em T e, ademais, mantenham o estado u o mais próximo possível dos estados $u_{i,d}$ ao longo de todo o processo evolutivo, pelo menos nas regiões de controle $\omega_{i,d}$. Em termos mais técnicos isto significa que os controles f e (v_1,v_2) devem ser tais que a solução de (12) satisfaz $u(x,T)=0$ em Ω e minimizam (em um certo sentido) os funcionais J_i.

Diferentes formas de minimizar os funcionais J_i conduzem a diferentes tipos de controlabilidade hierárquica.

Definição: Dado $f\in L^2(Q)$, dizemos que um par $(v_1,v_2)\in L^2(\omega_1\times(0,T),\omega_2\times(0,T))$ é um *equilíbrio de Nash* associado à f se

$$\begin{aligned}J_1(g;f_1,f_2)&\le J_2(g;\bar{f}_1,f_2),\quad \forall \bar{f}_1\in L^2(Q),\\ J_2(g;f_1,f_2)&\le J_2(g;f_1,\bar{f}_2),\quad \forall \bar{f}_2\in L^2(Q).\end{aligned} \tag{14}$$

Definição: Dado $f\in L^2(\mathcal{O}\times(0,T))$, dizemos que um par $(v_1,v_2)\in L^2(\omega_1\times(0,T),\omega_2\times(0,T))$ é um *equilíbrio de Pareto* associado à f se, para qualquer $(h_1,h_2)\in L^2(\omega_1\times(0,T),\omega_2\times(0,T))$ tivermos

1. $J_1(f;h_1,h_2)\le J_1(f;v_1,v_2) \;\Rightarrow\; J_2(f;v_1,v_2)\le J_2(f;h_1,h_2),$
2. $J_2(f;h_1,h_2)\le J_2(f;v_1,v_2) \;\Rightarrow\; J_1(f;v_1,v_2)\le J_1(f;h_1,h_2).$ (15)

A principal diferença entre os conceitos de equilíbrio de Nash e Pareto é que o equilíbrio de Pareto é *cooperativo*, enquanto o de Nash não.

Em [4], os autores provaram a existência e unicidade do equilíbrio de Nash, assim como resultados de controlabilidade nula hierárquica.

0.4 Contribuições e organização do trabalho

Este trabalho está organizado da seguinte maneira:

Capítulo 1: Neste capítulo nos dedicamos inteiramente à controlabilidade nula hierárquica de Stackelberg-Nash para o sistema parabólico degenerado unidimensional.

A principal ferramenta para obter resultados dessa classe, utilizando as técnicas desenvolvidas em [4], são "boas" estimativas de Carleman. "Boas" no sentido de que, entre outras coisas, a desigualdade precisa ter derivadas temporal de primeira ordem e derivadas espaciais de segunda ordem no lado esquerdo da desigualdade. Apesar desse não ser o caso da desigualdade na Proposição 0.2.2, não é difícil incluir estes termos na mesma. O real problema da desigualdade na Proposição 1, reside no fato do termo de observação atuar na fronteira. Para obter a desigualdade de observabilidade com observação no interior do domínio, em [7], os autores contornaram esse inconveniente usando a desigualdade de Cacciopolli. Infelizmente essa desigualdade não ajuda para obter a observabilidade necessária que conduz a controlabilidade nula hierárquica. Dessa forma, construímos novos pesos para obter uma estimativa de Carleman com observação atuando no interior do domínio. Esta nova estimativa de Carleman se estende naturalmente ao sistema (7). Os novos pesos também podem ser alterados como em [27] para obter estimativas de Carleman para o sistema (9) no caso $\alpha \in (0, 1/2)$. Com estas novas estimativas de Carleman, somos capazes de aplicar as técnicas de [4] e provar a controlabilidade nula hierarquica de Stackelberg-Nash para os sistemas (1), (6) e (8). Quanto ao sistema (10), a questão é um pouco mais delicada, pois até já se sabe que o sistema sequer é, em geral, nulamente controlável. Não obstante, apresentamos hipóteses geométricas sobre o domínio de controle, sobre as quais é possível obter boas estimativas de Carleman e resultados de controlabilidade nula hierárquica de Stackelberg Nash.

Capítulo 2: Neste capítulo começamos a estender os resultados do Capítulo 1 para duas dimensões espaciais. Motivados por problemas financeiros, Consideramos o seguinte sistema

$$\begin{cases} u_t - div(A\nabla u) + bu = g1_\omega & \text{em} \quad Q, \\ B.C. & \text{sobre} \quad \Sigma, \\ u(\cdot, 0) = u_0 & \text{em} \quad \Omega, \end{cases} \tag{16}$$

onde $\Omega = (0,1)\times(0,1)$, $\Gamma := \partial\Omega$, $T > 0$, $Q = \Omega\times(0,T)$, $\Sigma := \Gamma\times(0,T)$, $\omega \subset \Omega$ é aberto e 1_ω é a função característica, $b \in L^\infty(Q)$, $g \in L^2(Q)$, $u_0 \in L^2(\Omega)$, $A : \overline{\Omega} \mapsto M_{2\times 2}(\mathbb{R})$ é dada por

$$A(x) = diag(x_1^{\alpha_1}, x_2^{\alpha_2}),$$

$$B.C. := \begin{cases} u = 0 \ \text{ sobre } \ \Sigma & \text{se } \ \alpha_1, \ \alpha_2 \in [0,1), \\ u = 0 \ \text{ sobre } \ \Sigma_{3,4} \ \text{ e } \ (A\nabla u)\nu = 0 \ \text{ sobre } \ \Sigma_{1,2} & \text{se } \ \alpha_1, \ \alpha_2 \in [1,2], \\ u = 0 \ \text{ sobre } \ \Sigma_{1,3,4} \ \text{ e } \ (A\nabla u)\nu = 0 \ \text{ sobre } \ \Sigma_2 & \text{se } \ \alpha_1 \in [0,1) \ \text{ e } \ \alpha_2 \in [1,2], \\ u = 0 \ \text{ sobre } \ \Sigma_{2,3,4} \ \text{ e } \ (A\nabla u)\nu = 0 \ \text{ sobre } \ \Sigma_1 & \text{se } \ \alpha_1 \in [1,2] \ \text{ e } \ \alpha_2 \in [0,1), \end{cases}$$

$\alpha = (\alpha_1, \alpha_2) \in [0,2] \times [0,2]$, $\Sigma_{i,j,l} := (\Gamma_i \cup \Gamma_j \cup \Gamma_l) \times (0,T)$, e

$$\Gamma_1 := \{0\} \times [0,1], \ \Gamma_2 := [0,1] \times \{0\}, \ \Gamma_3 := \{1\} \times [0,1], \ \ \Gamma_4 := [0,1] \times \{1\}.$$

Vale notar que, comparando com o sistema (11), o sistema (16) é uma extensão mais fiel para duas dimensões do sistema unidimensional (1). Neste capítulo, provamos a boa colocação do sistema (16) e, sob certas condições geométricas sobre o domínio de controle, estimativas de Carleman para o caso $\alpha_1, \alpha_2 \in (0,2)$. Como consequência obtemos resultados de controlabilidade nula hierárquica de Stackelberg-Nash.

Capítulo 3: No capítulo anterior, provamos estimativas de Carleman para o sistema (16) apenas para o caso $\alpha_1, \alpha_2 \in (0,2)$. A técnica usada para construir os pesos necessários para a desigualdade de Carleman enfrenta dificuldades técnicas nos demais casos de combinações de valores de α_i. Neste capítulo propomos outra técnica, com outros pesos, para contemplar as demais combinações de valores de α_i. Dessa forma, no caso em que $\alpha_1 = 0$ ou $\alpha_2 = 0$, obtemos resultados com hipóteses geométricas menos restritivas, enquanto que nos demais casos precisamos de hipóteses mais restritivas às consideradas no Capítulo 2. Ademais, estendemos os resultados para sistemas em dimensões espaciais superiores.

Capítulo 1

Controlabilidade nula de Stackelberg-Nash para algumas equações parabólicas degeneradas lineares e semilineares

Stackelberg-Nash null controllability for some linear and semilinear degenerate parabolic equations

F. D. ARARUNA, B. S. V. de ARAÚJO, and E. FERNÁNDEZ-CARA

Abstract

This paper deals with the application of Stackelberg-Nash strategies to the null controllability of degenerate parabolic equations. We assume that we can act on the system through a hierarchy of controls. A first control (the leader) is assumed to determine the policy; then, a Nash equilibrium pair (corresponding to a noncooperative multi-objective optimization strategy) is found; this governs the action of other controls (the followers). This way, the state of the system is driven to zero and, consequently, we solve a hierarchical null controllability problem. The main novelty in this paper is that the physical systems are governed by linear or semilinear 1D heat equations with degenerate coefficients.

Keywords: Null controllability, Stackelberg-Nash strategies, degenerate parabolic equations, Carleman inequalities.

Mathematics Subject Classification: 34K35, 49J20, 35K10.

1.1 Introduction

The study of the controllability of partial differential equations and systems has attracted the interest of many authors. The theory has been extended to semilinear problems, equations in unbounded domains, and systems in fluid dynamics, among others; see for instance [15, 18, 20, 21, 24]. In the particular case of degenerate parabolic equation, still not many things are known, see [7, 19, 27].

In this paper, we assume that $\alpha \in [0,2]$ is an exponent.

Let us first consider the following degenerate systems:

$$\begin{cases} u_t - (x^\alpha u_x)_x + b_0(x,t)u = g1_{\mathcal{O}} & \text{in} \quad Q, \\ u(1,\cdot) = 0 \ \text{ and } \begin{cases} u(0,\cdot) = 0 & \text{if } \alpha \in [0,1) \\ (x^\alpha u_x)(0,\cdot) = 0 & \text{if } \alpha \in [1,2] \end{cases} & \text{on} \quad (0,T), \\ u(\cdot,0) = u_0 & \text{in} \quad (0,1), \end{cases} \tag{1.1}$$

$$\begin{cases} u_t - (x^\alpha u_x)_x + (x^q b_1(x,t)u)_x + b_0(x,t)u = g1_{\mathcal{O}} & \text{in} \quad Q, \\ u(1,\cdot) = 0 \ \text{ and } \begin{cases} u(0,\cdot) = 0 & \text{if } \alpha \in [0,1) \\ (x^\alpha u_x)(0,\cdot) = 0 & \text{if } \alpha \in [1,2] \end{cases} & \text{on} \quad (0,T), \\ u(\cdot,0) = u_0 & \text{in} \quad (0,1), \end{cases} \tag{1.2}$$

(with $q \geq \alpha/2$), and

$$\begin{cases} u_t - (x^\alpha u_x)_x + (b_1(x,t)u)_x + b_0(x,t)u = g1_{\mathcal{O}} & \text{in} \quad Q, \\ u(1,\cdot) = 0 \ \text{ and } \begin{cases} u(0,\cdot) = 0 & \text{if } \alpha \in [0,1) \\ (x^\alpha u_x)(0,\cdot) = 0 & \text{if } \alpha \in [1,2] \end{cases} & \text{on} \quad (0,T), \\ u(\cdot,0) = u_0 & \text{in} \quad (0,1), \end{cases} \tag{1.3}$$

where $Q = (0,1)\times(0,T)$, $\mathcal{O} \subset (0,1)$ is a non-empty open subset and $1_{\mathcal{O}}$ is its associated characteristic function, $b_0 \in L^\infty(Q)$, $b_1 \in L^\infty(0,T;H^1(0,1))$, $g \in L^2(Q)$, and $u_0 \in L^2(0,1)$.

When $\alpha \in [0,1)$, we say that the problems (1.1), (1.2) and (1.3) are *weakly degenerate*; contrarily, we say that they are *strongly degenerate* if $\alpha \in [1,2)$. When

$\alpha = 2$, an appropriate change of variables show that (1.1), (1.2) and (1.3) are equivalent to some nondegenerate problems in an unbounded domain. In particular, (1.2) and (1.3) can be viewed as generalizations of the celebrated Black-Scholes equation

$$u_t - \frac{x^2}{2}\sigma u_{xx} - rxu_x + ru = 0, \tag{1.4}$$

which models the behavior of an option $u = u(x,t)$ as a function of the portfolio cotization and time. Here, σ is the volatility and r is the risk free rate [6].

From the control viewpoint, a relevant question is whether or not these systems are null-controllable. In other words, for each $u_0 \in L^2(0,1)$, we try to elucidate if there exist controls $g \in L^2(Q)$ such as the associated states u (the corresponding solutions to (1.1), (1.2) or (1.3)) satisfy

$$u(x,T) = 0 \quad \text{in} \quad (0,1). \tag{1.5}$$

In the nondegenerate case, that is, when the spatial domain is replaced by an interval (a,b), with $a > 0$, the null controllability of these systems is a trivial consequence of the (classical) results in [20] and [25].

The null controllability of systems (1.1), (1.2) and (1.3) have been proved, repectively, in [7] (for $\alpha \in (0,2)$), [12] (for $\alpha \in (0,2)$), and [27] (for $\alpha \in (0,1/2)$). These works are based on classical duality arguments, which reduces a null controllability property to an observability inequality for the solutions of the adjoint system. In these cases, the adjoint system of (1.1), (1.2) and (1.3) are given respectively by

$$\begin{cases} v_t + (x^\alpha v_x)_x + b_0(x,t)v = h & \text{in} \quad Q, \\ v(1,\cdot) = 0 \ \text{ and } \begin{cases} v(0,\cdot) = 0 & \text{if } \ \alpha \in [0,1) \\ (x^\alpha v_x)(0,\cdot) = 0 & \text{if } \ \alpha \in [1,2) \end{cases} & \text{on} \quad (0,T), \\ v(\cdot,0) = v_0 & \text{in} \quad (0,1), \end{cases} \tag{1.6}$$

$$\begin{cases} v_t + (x^\alpha v_x)_x + x^q b_1 v_x + b_0 v = h & \text{in} \quad Q, \\ v(1,\cdot) = 0 \ \text{ and } \begin{cases} v(0,\cdot) = 0 & \text{if } \ \alpha \in [0,1) \\ (x^\alpha v_x)(0,\cdot) = 0 & \text{if } \ \alpha \in [1,2) \end{cases} & \text{on} \quad (0,T), \\ v(\cdot,0) = v_0 & \text{in} \quad (0,1), \end{cases} \tag{1.7}$$

and

$$
\begin{cases}
v_t + (x^\alpha v_x)_x + b_1 v_x + b_0 v = h & \text{in} \quad Q, \\
v(1,\cdot) = 0 \;\; \text{and} \;\; \begin{cases} v(0,\cdot) = 0 & \text{if} \;\; \alpha \in [0,1) \\ (x^\alpha v_x)(0,\cdot) = 0 & \text{if} \;\; \alpha \in [1,2) \end{cases} & \text{on} \quad (0,T), \\
v(\cdot,0) = v_0 & \text{in} \quad (0,1).
\end{cases}
\tag{1.8}
$$

The observability inequalities for (1.6), (1.7), and (1.8) were obtained making use of suitable Carleman estimates for themselves.

The inclusion of a first-order term in the equation in (1.1) is a delicate question that, in general, remains open. If it is a multiple of an appropriate power of x, like in (1.2), it is easy to show that the Carleman estimates proved in [7] again hold for its adjoint system and, consequently, null controllability holds too, see [19]. However, the more general situation presented in (1.3) has been solved only for $\alpha \in [0, 1/2)$, see [27].

Now, taking $\alpha = 2$ in (1.1), we have the following system:

$$
\begin{cases}
u_t - (x^2 u_x)_x + b_0(x,t) u = g 1_{\mathcal{O}} & \text{in} \quad Q, \\
u_x(1,\cdot) = u_x(0,\cdot) = 0 & \text{on} \quad (0,T), \\
u(\cdot,0) = u_0 & \text{in} \quad (0,1).
\end{cases}
\tag{1.9}
$$

In this case, an additional difficulty is found: there is no known Carleman estimate for the solution to the associated adjoint. To overcome this difficulty, we introduce the change of variables

$$
y = log(1/x), \qquad U(y,t) = x^{1/2} u(x,t), \tag{1.10}
$$

which transform (1.9) into

$$
\begin{cases}
U_t - U_{yy} + B(y,t) U = G(y,t) 1_{\vartheta} & \text{in} \quad Q', \\
\left(U_y + \frac{1}{2} U \right)(0,\cdot) = 0 \;\; \text{and} \;\; \lim\limits_{y \to +\infty} \left(U_y + \frac{1}{2} U \right)(y,\cdot) = 0 & \text{on} \quad (0,T), \\
U(\cdot,0) = U_0 & \text{in} \quad (0,+\infty),
\end{cases}
\tag{1.11}
$$

where

$$
Q' = (0,+\infty) \times (0,T), \quad B(y,t) = b_0(x,t) + 1/4, \quad G = e^{-y/2} g,
$$

and

$$
\vartheta = \{ y \in (0,+\infty) : e^{-y} \in \mathcal{O} \}.
$$

Since we are dealing here with a problem in unbounded domain, the null controllability properties depend on the choice of ϑ. Indeed, in [21] the authors present a Carleman estimate for solutions of the adjoint system of a similar one to (1.11), but with Dirichlet boundary conditions. It is well know that the boundary conditions have an important role in the Carleman inequality, hence, to deal with (1.11), a new inequality must be proved for its adjoint system:

$$\begin{cases} w_t + w_{yy} + B(y,t)w = F & \text{in} \quad Q', \\ \left(w_y + \dfrac{1}{2}w\right)(0,\cdot) = 0 \ \text{ and } \ \lim\limits_{y\to\infty}\left(w_y + \dfrac{1}{2}w\right)(y,\cdot) = 0 & \text{on} \quad (0,T), \\ w(\cdot,T) = w_T & \text{in} \quad (0,+\infty). \end{cases} \tag{1.12}$$

As mentioned in [21], there are few situations where we can get a Carleman estimate for systems in unbounded domain. The most simple of this situations is when the non-controllable region is a bounded set. To deal with this case, $(0,+\infty)\backslash\vartheta$ must be a bounded set; and this happens if and only if $\mathcal{O} \ni 0$.

This paper deals with Stackelberg-Nash strategies for the null controllability of degenerate parabolic systems similar to those above. To be more specific, let us fix the non-empty open subsets $\mathcal{O}_i \subset (0,1)$ $(i=1,2)$, and for each triplet $(g,f_1,f_2) \in [L^2(Q)]^3$, let us consider systems (1.1), (1.3) and (1.9) with $g1_{\mathcal{O}}$ replaced by $g1_{\mathcal{O}} + f_1 1_{\mathcal{O}_1} + f_2 1_{\mathcal{O}_2}$, i.e.

$$\begin{cases} u_t - (x^\alpha u_x)_x + b_0(x,t)u = g1_{\mathcal{O}} + f_1 1_{\mathcal{O}_1} + f_2 1_{\mathcal{O}_2} & \text{in} \quad Q, \\ u(1,\cdot) = 0 \ \text{ and } \ \begin{cases} u(0,\cdot) = 0 & \text{if} \quad \alpha \in [0,1) \\ (x^\alpha u_x)(0,\cdot) = 0 & \text{if} \quad \alpha \in [1,2) \end{cases} & \text{on} \quad (0,T), \\ u(\cdot,0) = u_0 & \text{in} \quad (0,1), \end{cases} \tag{1.13}$$

$$\begin{cases} u_t - (x^\alpha u_x)_x + (x^q b_1(x,t)u)_x + b_0(x,t)u = g1_{\mathcal{O}} + f_1 1_{\mathcal{O}_1} + f_2 1_{\mathcal{O}_2} & \text{in} \quad Q, \\ u(1,\cdot) = 0 \ \text{ and } \ \begin{cases} u(0,\cdot) = 0 & \text{if} \quad \alpha \in [0,1) \\ (x^\alpha u_x)(0,\cdot) = 0 & \text{if} \quad \alpha \in [1,2) \end{cases} & \text{on} \quad (0,T), \\ u(\cdot,0) = u_0 & \text{in} \quad (0,1), \end{cases} \tag{1.14}$$

$$\begin{cases} u_t - (x^\alpha u_x)_x + (b_1(x,t)u)_x + b_0(x,t)u = g1_{\mathcal{O}} + f_1 1_{\mathcal{O}_1} + f_2 1_{\mathcal{O}_2} & \text{in} \quad Q, \\ u(1,\cdot) = 0 \ \text{ and } \ \begin{cases} u(0,\cdot) = 0 & \text{if} \quad \alpha \in [0,1) \\ (x^\alpha u_x)(0,\cdot) = 0 & \text{if} \quad \alpha \in [1,2) \end{cases} & \text{on} \quad (0,T), \\ u(\cdot,0) = u_0 & \text{in} \quad (0,1), \end{cases} \tag{1.15}$$

and

$$\begin{cases} u_t - (x^2 u_x)_x + b_0(x,t)u = g1_{\mathcal{O}} + f_1 1_{\mathcal{O}_1} + f_2 1_{\mathcal{O}_2} & \text{in} \quad Q, \\ u_x(1,\cdot) = u_x(0,\cdot) = 0 & \text{on} \quad (0,T), \\ u(\cdot,0) = u_0 & \text{in} \quad (0,1). \end{cases} \tag{1.16}$$

For simplicity, we will assume that only three controls are applied (one leader and two followers), but very similar considerations hold for systems with a higher number of controls.

Now, for $i = 1, 2$, let us introduce the non-empty open sets $\mathcal{O}_{i,d} \subset (0,1)$, the functions $u_{i,d} \in L^2(\mathcal{O}_{i,d} \times (0,T))$, and the functionals $J_i : L^2(\mathcal{O} \times (0,T)) \times \mathcal{U} \mapsto \mathbb{R}$ given by

$$J_i(g; f_1, f_2) := \frac{\beta_i}{2} \iint_{\mathcal{O}_{i,d}\times(0,T)} |u - u_{i,d}|^2 \, dx\, dt + \mu_i \iint_{\mathcal{O}_i\times(0,T)} |f_i|^2 \, dx\, dt, \tag{1.17}$$

where $\mathcal{U} := \mathcal{U}_1 \times \mathcal{U}_2$, with $\mathcal{U}_i := L^2(\mathcal{O} \times (0,T))$, β_i and μ_i are positive constants, and u is the associated state to (1.13), (1.14), (1.15) or (1.16).

For a fixed $g \in L^2(\mathcal{O}\times(0,T))$, we say that a pair $(f_1, f_2) \in \mathcal{U}$ is a *Nash equilibrium* for (J_1, J_2) associated to g when

$$\begin{aligned} J_1(g; f_1, f_2) &\le J_2(g; \bar{f}_1, f_2), \quad \forall \bar{f}_1 \in L^2(Q), \\ J_2(g; f_1, f_2) &\le J_2(g; f_1, \bar{f}_2), \quad \forall \bar{f}_2 \in L^2(Q). \end{aligned} \tag{1.18}$$

Our goal is to prove that, for any $u_0 \in L^2(0,1)$, there exist a control $g \in L^2(\mathcal{O}\times(0,T))$ (called *leader*) and a associated Nash equilibrium $(f_1, f_2) \in \mathcal{U}$ (called *followers*) such that the associated state u of (1.13), (1.14), (1.15) or (1.16) satisfies (1.5).

Results of this type, i.e. Stackelberg-Nash null controllability, were proved for the first time in [4], in the context of nondegenerate parabolic equations. After that, we can mention [23]. Most of the works dealing with Stackelberg-Nash strategy are in the context of the approximate controllability. In this issue, we cite [22].

The rest of this paper is organized as follows. In Section 2 we present the notations, the definitions of some spaces and some preliminary results. We also provide in this section the main tool of this work: Carleman estimates for degenerate parabolic equations with control regions in the interior of the domain. In Section 3, we establish and prove our main results on Stackelberg-Nash null controllability. In Section 4, we make some additional comments, discuss open questions and advance some future work. The paper ends with three appendices containing the proofs of the Carleman estimates.

1.2 Notations and preliminary results

1.2.1 Notations and spaces

The usual norm and inner product in $L^2(0,1)$ and in $L^2(Q)$ will be denoted respectvely by $|\cdot|$ and $(\cdot,\cdot)$, and $\|\cdot\|$ and $((\cdot,\cdot))$. The norms in $L^\infty(0,1)$ and in $L^\infty(Q)$ will be denoted respectively by $|\cdot|_\infty$ and $\|\cdot\|_\infty$.

Let us consider the sets

$$\begin{aligned} H_\alpha \ &:= \ \{ \ u \in L^2(0,1) : u \text{ is absolutely continuous in } [0,1], \\ &\qquad x^{\alpha/2} u_x \in L^2(0,1), \ u(0) = u(1) = 0 \ \}, \end{aligned}$$

for $\alpha \in [0,1)$ and

$$\begin{aligned} H_\alpha \ &:= \ \{ \ u \in L^2(0,1); u \text{ is locally absolutely continuous in } (0,1], \\ &\qquad x^{\alpha/2} u_x \in L^2(0,1), \ u(1) = 0 \ \} \end{aligned}$$

for $\alpha \in [1,2]$.

In the sequel, for any two Banach spaces X and Y, the notation $X \hookrightarrow Y$ indicates that $X \subset Y$ and, moreover, the embedding $X \mapsto Y$ is continuous.

It is easy see that $L^2(0,1) \hookrightarrow H'_\alpha$ for all $\alpha \in [0,2]$, so that $L^2(Q) \hookrightarrow L^2(0,T;H'_\alpha)$. Moreover, it is shown in [1] that the embbeding $H_\alpha \hookrightarrow L^2(0,1)$ is compact. Hence, from the well known Aubin-Lions Compactness Theorem, we see that the Hilbert space

$$W_\alpha := \{ \ y \in L^2(0,T;H_\alpha) : y_t \in L^2(0,T;H'_\alpha) \ \}$$

is compactly embedded in $L^2(Q)$.

1.2.2 Existence and estimates

The following result has been proved in [1, 19, 27]:

Proposition 1.2.1 *For any $g \in L^2(Q)$ and any $u_0 \in L^2(0,1)$, there exists exactly one solution u to each of the systems (1.1), (1.2), (1.3) and (1.9), with*

$$u \in L^2(0,T;H_\alpha) \cap C^0([0,T];L^2(0,1)).$$

Furthermore, there exists a constant $C > 0$ only depending on T, α, b_0 and b_1, such that

$$\sup_{t\in[0,T]} |u(\cdot,t)|^2 + \iint_Q x^\alpha |u_x|^2\, dx\, dt \leq C(\|g\|^2 + |u_0|^2).$$

Hardy inequalities are a standard tool in the analysis of degenerate equations. The following result is proved in [1]:

Proposition 1.2.2 (Hardy's inequality) *Assume that $\alpha < 2$ and $\alpha \neq 1$. Let $z : [0,1] \mapsto \mathbb{R}$ be locally absolutely continuous in $(0,1]$, with*

$$\int_0^1 x^\alpha |z_x|^2\, dx < +\infty.$$

Then, for all $\delta \in (0,1]$, one has

$$\begin{cases} \displaystyle\int_0^\delta x^{\alpha-2}|z|^2\, dx \leq \frac{4}{(1-\alpha)^2}\int_0^\delta x^\alpha |z_x|^2\, dx & \text{if} \quad \alpha < 1 \text{ and } \displaystyle\lim_{x\to 0^+} z(x) = 0 \\ \displaystyle\int_0^1 x^{\alpha-2}|z|^2\, dx \leq \frac{4}{(1-\alpha)^2}\int_0^1 x^\alpha |z_x|^2\, dx & \text{if} \quad \alpha \in (1,2) \text{ and } \displaystyle\lim_{x\to 1^-} z(x) = 0. \end{cases}$$

1.2.3 Carleman estimates for (1.6) and (1.7)

Let us introduce the functions θ, p_0 and σ_0 with

$$\theta(t) := \frac{1}{(t(T-t))^4}, \quad p_0(x) := \frac{1-x^{2-\alpha}}{(2-\alpha)^2} \quad \text{and} \quad \sigma_0(x,t) := \theta(t)p_0(x).$$

The following Carleman estimate is proved in [7]:

Proposition 1.2.3 *There exists positive constants s_0 and C such that, for any $s \geq s_0$ and any solution v to (1.6), one has:*

$$\iint_Q e^{-2s\sigma_0}\left[s\theta x^\alpha |v_x|^2 + s^3\theta^3 x^{2-\alpha}|v|^2\right] dx\, dt \leq C\left[\|e^{-s\sigma_0}h\|^2 + s\int_0^T e^{-2s\sigma_0}\theta|v_x|^2 dt\Big|_{x=1}\right] \tag{1.19}$$

We will need below a Carleman estimate with the previous boundary integral replaced by an interior observation term and time derivatives and second-order spatial derivatives in the left hand side. The goal of this section is to present such an estimate. The main idea is to modify the weight functions and then proceed as in [7]; a similar result was recently proved in [10], but there the equation is slightly different (the coefficient degenerate at both $x = 0$ and $x = 1$).

Let us fix $\alpha \in [0,2)$ and the non-empty open sets $\omega \subset (0,1)$ and $\omega_0 = (a_0, c_0) \subset\subset \omega$. We will use the following notations:

$$\omega_{0T} := \omega_0 \times (0,T), \quad \omega_T := \omega \times (0,T) \quad \text{and} \quad Q_0 := (0,a_0)\times(0,T).$$

Let $\eta \in C^\infty(0,1)$ be a function such that

$$\eta(x) = \frac{x^{2-\alpha}}{2-\alpha} \ \text{ in } \ [0, a_0] \ \text{ and } \ \eta(x) = -\frac{x^{2-\alpha}}{2-\alpha} \ \text{ in } \ [c_0, 1].$$

Now, for $\lambda \geq \lambda_0 > 0$ and $s \in \mathbb{R}$, we introduce

$$p(x) := e^{\lambda(2|\eta|_\infty + \eta(x))}, \ \ \xi(x,t) := p(x)\theta(t), \ \sigma(x,t) := \theta(t)e^{4\lambda|\eta|_\infty} - \xi,$$
$$\gamma_1(\lambda) := |1-\alpha| + \lambda^{-1/4}, \quad \text{and} \quad \gamma_2(s) := |1-\alpha| + s^{-1/2}.$$

The main result in this section is the following:

Theorem 1.2.4 *There exist positive constants C, s_0 and λ_0, depending only on ω, ω_0, $\|b_0\|_\infty$, T and α such that, for any $s \geq s_0$, any $\lambda \geq \lambda_0$ and any solution v to (1.6), one has:*

$$\iint_Q e^{-2s\sigma} \left[s^{-1}\gamma_1(\lambda)\xi^{-1}(|v_t|^2 + |(x^\alpha v_x)_x|^2) + s\lambda x^\alpha \xi |v_x|^2 + s\lambda^2 x^{2\alpha}|\eta'|^2\xi|v_x|^2 \right] dx\, dt$$
$$+ \iint_Q e^{-2s\sigma}|v|^2 \left[s^2\lambda^2\gamma_1(\lambda)\xi^2\gamma_2(s\xi) + s^3\lambda^3 x^{2-\alpha}\xi^3 + s^3\lambda^4 x^{2\alpha}|\eta'|^4\xi^3 \right] dx\, dt$$
$$\leq C \left[\|e^{-s\sigma}h\|^2 + s^3\lambda^3 \iint_{\omega_T} e^{-2s\sigma}\xi^3|v|^2\, dx\, dt \right]. \qquad (1.20)$$

The proof of Theorem 1.2.4 is presented in Appendix A.

It is possible to refine the estimate in Theorem 1.2.4 by replacing the powers of $|\eta'|$ by powers of x at the price of increasing the power of λ in the local term:

Corollary 1.2.5 *There exist positive constants C, s_0 and λ_0, depending only on ω, ω_0, $\|b_0\|_\infty$, T and α such that, for any $s \geq s_0$, any $\lambda \geq \lambda_0$, and any solution v to (1.6), one has:*

$$\iint_Q e^{-2s\sigma} \left[s^{-1}\gamma_1(\lambda)\xi^{-1}(|v_t|^2 + |(x^\alpha v_x)_x|^2) + s\lambda x^\alpha \xi |v_x|^2 + s\lambda^2 x^2\xi|v_x|^2 \right] dx\, dt$$
$$+ \iint_Q e^{-2s\sigma}|v|^2 \left[s^2\lambda^2\gamma_1(\lambda)\xi^2\gamma_2(s\xi) + s^3\lambda^3 x^{2-\alpha}\xi^3 + s^3\lambda^4 x^{4-2\alpha}\xi^3 \right] dx\, dt$$
$$\leq C \left[\|e^{-s\sigma}h\|^2 + s^3\lambda^4 \iint_{\omega_T} e^{-2s\sigma}\xi^3|v|^2\, dx\, dt \right]. \qquad (1.21)$$

If v is a solution to (1.7) and we introduce $\tilde{h} = h - x^q b_1 v_x$, we see that v solves a system of the kind (1.6). Thus, the following result holds.

Corollary 1.2.6 *Let us assume that $q \geq \alpha/2$ and $b_0, b_1 \in L^\infty(Q)$. Then, there exists positive constants C, s_0, and λ_0, depending only on ω, $\|b_0\|_\infty$, $\|b_1\|_\infty$, T, and α such that, for any $s \geq s_0$, any $\lambda \geq \lambda_0$, and any solution v for (1.7), the estimates (1.20) and (1.21) still hold.*

1.2.4 Carleman estimate for (1.8)

Let us assume that $\alpha \in [0,1/2)$. To our knowledge, the arguments in the proofs of the Carleman estimates in Section 2.3 cannot be adapted to the solutions to (1.8). Therefore, we have to establish other estimates with other weight functions.

Let $\tilde\eta \in C^\infty(0,1)$ be a function such that

$$\tilde\eta(x) = \frac{3x^{(4-2\alpha)/3}}{4-2\alpha} \quad \text{in} \quad [0,a_0] \quad \text{and} \quad \tilde\eta(x) = -\frac{3x^{(4-2\alpha)/3}}{4-2\alpha} \quad \text{in} \quad [c_0,1].$$

For simplicity, we will keep the same notation for the functions σ and ξ with the function η replaced by $\tilde\eta$. With this function, we get the following result:

Theorem 1.2.7 *There exist positive constants C, s_0 and λ_0, depending only on ω, ω_0, $\|b_0\|_\infty$, $\|b_1\|_\infty$, T and α such that, for any $s \geq s_0$, any $\lambda \geq \lambda_0$ and any solution v to (1.8) one has*

$$\begin{aligned}\iint_Q e^{-2s\sigma}\left[s^{-1}\gamma_1(\lambda)\xi^{-1}(|v_t|^2+|(x^\alpha v_x)_x|^2) + s\lambda x^{(4\alpha-2)/3}\xi|v_x|^2 + s\lambda^2 x^{2\alpha}|\tilde\eta'|^2\xi|v_x|^2\right] dx\,dt \\ + \iint_Q e^{-2s\sigma}|v|^2\left[s\lambda^2\xi x^{(2\alpha-4)/3} + s^3\lambda^3\xi^3 + s^3\lambda^4 x^{2\alpha}|\tilde\eta'|^4\xi^3\right] dx\,dt \\ \leq C\left[\|e^{-s\sigma}h\|^2 + s^3\lambda^3\iint_{\omega_T} e^{-2s\sigma}\xi^3|v|^2\,dx\,dt\right]. \qquad (1.22)\end{aligned}$$

The proof of Theorem 1.2.7 is given in Appendix B.

An estimate similar to those in Corollary 1.2.5 can also be obtained.

1.2.5 Carleman estimate for (1.12)

In general, there is no estimate of the Carleman kind for the adjoint system of (1.9). This is expected since, in general, this system is not null-controllable.

However, if the control domain is of the form $\omega = (0,c)$ for some $c \in (0,1)$, then in the system (1.11) obtained after the change of variables (1.10), the new control domain ϑ possesses a bounded complementary set in $(0,+\infty)$. In view of the results in [21], null controllability can be expected in this case and, consequently, it is reasonable to try to prove a Carleman estimate for the solutions to (1.12).

Thus, let us assume that $\omega = (0,c)$ and let us introduce $\vartheta_0 := (a_0,+\infty) \subset\subset \vartheta$, $\vartheta_{0T} := \vartheta_0 \times (0,T)$ and $\vartheta_T := \vartheta \times (0,T)$. Let $\hat\eta \in \mathcal{C}^2([0,\infty))$ be a function such that $\hat{\eta}(y) = -y$ in $[0, a_0]$, $\hat{\eta}$, $\hat{\eta}'$ and $\hat{\eta}''$ are bounded in $[0, +\infty)$ and $|\hat{\eta}'| \geq C > 0$ in $[0, +\infty)\backslash\vartheta_0$. Finally, for $\lambda \geq \lambda_0 > 0$, let us set

$$\hat{p}(y) := e^{\lambda(2|\hat{\eta}|_\infty + \hat{\eta}(y))},\ \tilde{p}(y) := e^{\lambda(2|\hat{\eta}|_\infty - \hat{\eta}(y))},\ \hat{\xi}(y,t) := \theta(t)\hat{p}(y),\quad \tilde{\xi}(y,t) := \theta(t)\tilde{p}(y),$$
$$\hat{\sigma}(y,t) := \theta(t)e^{4\lambda|\hat{\eta}|_\infty} - \hat{\xi},\ \tilde{\sigma}(y,t) := \theta(t)e^{4\lambda|\hat{\eta}|_\infty} - \tilde{\xi} \ \text{ and } \ \varrho := e^{-2s\hat{\sigma}} + e^{-2s\tilde{\sigma}}.$$

The Carleman estimates that we can get for (1.12) are given in the following result:

Theorem 1.2.8 *There exist positive constants C, s_0 and λ_0, depending only on ω, B and T such that, for any $s \geq s_0$, any $\lambda \geq \lambda_0$ and any solution w to (1.12), one has:*

$$\iint_{Q'} \varrho \left[s^{-1}\hat{\xi}^{-1}(|w_t|^2 + |w_{yy}|^2) + s\lambda^2\hat{\xi}|w_y|^2 + s^3\lambda^4\hat{\xi}^3|w|^2 \right] dy\, dt$$
$$\leq C \left[\|\varrho^{1/2}F\|^2 + s^3\lambda^4 \iint_{\vartheta_T} \varrho\hat{\xi}^3|w|^2\, dy\, dt \right]. \tag{1.23}$$

The proof of Theorem 1.2.8 is given in Appendix C.

1.3 Stackelberg-Nash null controllability

In this section we will prove the Stackelberg-Nash null controllabilty of (1.13), (1.14), (1.15) and (1.16).

From the linearity of the systems and the convexity of the functionals J_i, it is clear that (f_1, f_2) is a Nash equilibrium for (J_1, J_2) if and only if

$$J_1'(g; f_1, f_2)(f, 0) = 0,\quad \forall f \in \mathcal{U}_1 \quad \text{and} \quad J_2'(g; f_1, f_2)(0, f) = 0,\quad \forall f \in \mathcal{U}_2.$$

Arguing as in [4, 22] the following result holds.

Proposition 1.3.1 *There exist a constant $\mu_{00} > 0$ such that, if $\mu_i \geq \mu_{00}$ $(i = 1, 2)$, for each $g \in L^2(Q))$ there exists a unique associated Nash equilibrium (f_1, f_2) for (J_1, J_2). Furthermore, there exists a constant $C > 0$ such that*

$$\|(f_1, f_2)\| \leq C(1 + \|g\|).$$

In particular, the corresponding state u satisfies

$$\|u\|_{L^\infty(0,T;L^2(0,1))} + \|u\|_{L^2(0,T;H_\alpha)} + \|u_t\|_{L^2(0,T;H^{-1}(0,1))} \leq C(1 + \|g\|).$$

To establish the Stackelberg-Nash null controllability, in the sequel we will impose the following assumptions:

$$\begin{cases} \mathcal{O}_{1,d} = \mathcal{O}_{2,d}; \text{ the common observability set will be denoted } \mathcal{O}_d. \\ \mathcal{O}_d \cap \mathcal{O} \neq \emptyset. \\ \displaystyle\iint_{\mathcal{O}_d \times (0,T)} \theta^2 |u_{i,d}|^2 \, dx \, dt < +\infty \quad \text{for } i = 1, 2. \\ \text{In the case of system (1.3)} \quad \alpha \in [0, 1/2). \\ \text{In the case of system (1.9)} \quad \mathcal{O} = (0, c) \text{ for some } c \in [0, 1). \end{cases} \tag{1.24}$$

The main result in this section is the following.

Theorem 1.3.2 *Assume that (1.24) holds. There exists $\mu_0 \geq \mu_{00}$ such that, if $\mu_1, \mu_2 \geq \mu_0$, for every $u_0 \in L^2(0,1)$ there exist a leader control $g \in L^2(Q)$ and a unique associated Nash equilibrium (f_1, f_2) such that the corresponding state, i.e. the solution to (1.13), (1.14), (1.15) or (1.16), satisfies (1.5).*

1.3.1 Proof of Theorem 1.3.2

The strategy will be the following:

- First, we will characterize the Nash equilibrium associated to g as the solution, together with u, to an appropriate coupled system.
- Then, we will prove that, if μ_1 and μ_2 are large enough, any solution to the corresponding adjoint system satisfies an observability estimate.

As a consequence of a well known duality argument, this will imply the desired result. This strategy is very similar to systems (1.13), (1.14), (1.15) and (1.16). Hence we will only consider the system (1.13).

Arguing as in [4, 22], the following result can be easily proved.

Proposition 1.3.3 *Let $g \in L^2(\mathcal{O} \times (0,T))$ be given. Then (f_1, f_2) is a Nash equilibrium of (1.13) associated to g if and only if*

$$f_i = -\frac{1}{\mu_i} \phi_i|_{\mathcal{O}_i \times (0,T)},$$

where the ϕ_i $(i = 1, 2)$ solve, together with u, the following coupled system:

$$\begin{cases} u_t - (x^\alpha u_x)_x + b_0(x,t)u = 1_{\mathcal{O}} f - \dfrac{1}{\mu_1}\phi_1 1_{\mathcal{O}_1} - \dfrac{1}{\mu_2}\phi_2 1_{\mathcal{O}_2} & \text{in} \quad Q, \\ -(\phi_i)_t - (x^\alpha (\phi_i)_x)_x + b_0(x,t)\phi_i = \alpha_i(u - u_{i,d})1_{\mathcal{O}_{i,d}} & \text{in} \quad Q, \\ u(1,t) = \phi_i(1,t) = 0 & \text{on} \quad (0,T), \\ \begin{cases} u(0,t) = \phi_i(0,t) = 0 & \text{if} \quad \alpha \in [0,1) \\ (x^\alpha u_x)(0,t) = (x^\alpha (\phi_i)_x)(0,t) = 0 & \text{if} \quad \alpha \in [1,2) \end{cases} & \text{on} \quad (0,T), \\ u(x,0) = u_0(x), \quad \phi_i(x,T) = 0 & \text{in} \quad (0,1). \end{cases} \tag{1.25}$$

As we said previously, to prove Theorem 1.3.2 it is sufficient to find an observability estimate for the adjoint system of (**??**), which is given by

$$\begin{cases} -z_t - (x^\alpha z_x)_x + b_0(x,t)z = \beta_1\varphi_1 1_{\mathcal{O}_{1,d}} + \beta_2\varphi_2 1_{\mathcal{O}_{2,d}} & \text{in} \quad Q, \\ (\varphi_i)_t - (x^\alpha (\varphi_i)_x)_x + b_0(x,t)\varphi_i = -\dfrac{1}{\mu_i} z 1_{\mathcal{O}_i} \quad (i = 1,2) & \text{in} \quad Q, \\ z(1,t) = \varphi_i(1,t) = 0 & \text{on} \quad (0,T), \\ \begin{cases} z(0,t) = \varphi_i(0,t) = 0 & \text{if} \quad \alpha \in [0,1) \\ (x^\alpha z_x)(0,t) = (x^\alpha (\varphi_i)_x)(0,t) = 0 & \text{if} \quad \alpha \in [1,2) \end{cases} & \text{on} \quad (0,T), \\ z(x,T) = z_T(x), \quad \varphi_i(x,0) = 0 & \text{in} \quad (0,1). \end{cases} \tag{1.26}$$

For this, we consider the following result.

Theorem 1.3.4 *Let assumptions in (1.24) hold. Then there exist positive constants μ_0 and C, and a weight function $\rho = \rho(t)$ blowing up at $t = T$ such that, if $\mu_i \geq \mu_0$ $(i = 1, 2)$, for any $z^T \in L^2(0,1)$ and any solution (z, φ_i) to (1.26), the following inequality holds:*

$$|z(\cdot,0)|^2 + \sum_{i=1}^{2} \iint_Q \rho^{-2}|\varphi_i|^2 \, dx \, dt \leq C \iint_{\mathcal{O}\times(0,T)} |z|^2 dx \, dt. \tag{1.27}$$

Proof. Let us fix ω' and ω_1 with

$$\omega' = (a', c') \subset\subset \omega_1 = (a_1, c_1) \subset\subset \mathcal{O} \cap \mathcal{O}_d$$

and a function $\psi \in C^\infty(0,1)$ such that

$$0 \leq \psi \leq 1, \ \psi = 1 \ \text{ in } \ \omega' \ \text{ and } \ supp(\psi) \subset\subset \omega_1.$$

We will use the notations $\omega'_T := \omega' \times (0,T)$, $\omega_{1T} := \omega_1 \times (0,T)$ and we will denote by $E(v)$ the left hand side of (1.20).

Let us assume that $(z, \varphi_1, \varphi_2)$ solves (1.26) and let us introduce $h := \beta_1\varphi_1 + \beta_2\varphi_2$. It is clear that Theorem 1.2.4 can be applied to z and also to h. Thus, there exist positive constants λ_0, s_0, and C such that, for $s \geq s_0$ and $\lambda \geq \lambda_0$, one has

$$E(z) \;\leq\; C\left[\iint_Q e^{-2s\sigma}|h|^2 dx\,dt + s^3\lambda^4 \iint_{\omega'_T} e^{-2s\sigma}\xi^3|z|^2\,dx\,dt\right]$$

and

$$E(h) \;\leq\; C\left[\iint_Q e^{-2s\sigma}|z|^2 dx\,dt + s^3\lambda^4 \iint_{\omega'_T} e^{-2s\sigma}\xi^3|h|^2\,dx\,dt\right].$$

Then, for s_0 large enough, we get

$$E(z) + E(h) \leq C\left[s^3\lambda^4 \iint_{\omega'_T} e^{-2s\sigma}\xi^3|z|^2\,dx\,dt + s^3\lambda^4 \iint_{\omega'_T} e^{-2s\sigma}\xi^3|h|^2\,dx\,dt\right]. \tag{1.28}$$

Since $h = -z_t - (x^\alpha z_x)_x + bz$ in ω_1, it follows that

$$s^3\lambda^4 \iint_{\omega'_T} e^{-2s\sigma}\xi^3|h|^2\,dx\,dt \;\leq\; s^3\lambda^4 \iint_{\omega_{1T}} e^{-2s\sigma}\xi^3\psi h\left[-z_t - (x^\alpha z_x)_x + bz\right]\,dx\,dt \tag{1.29}$$

Using integration by parts and Young's inequality, we find

$$-s^3\lambda^4 \iint_{\omega_{1T}} e^{-2s\sigma}\xi^3\psi h z_t\,dx\,dt \;\leq\; Cs^{-1}E(h) + s^8\lambda^8 \iint_{\omega_{1T}} e^{-2s\sigma}\xi^7|z|^2\,dx\,dt, \tag{1.30}$$

$$-s^3\lambda^4 \iint_{\omega_{1T}} e^{-2s\sigma}\xi^3\psi h(x^\alpha z_x)_x\,dx\,dt \;\leq\; Cs^{-1}E(h) + Cs^8\lambda^8 \iint_{\omega_{1T}} e^{-2s\sigma}\xi^7|z|^2\,dx\,dt, \tag{1.31}$$

and

$$s^3\lambda^4 \iint_{\omega_{1T}} e^{-2s\sigma}\xi^3\psi h b_0 z\,dx\,dt \;\leq\; Cs^{-1}E(h) + Cs^4\lambda^4 \iint_{\omega_{1T}} e^{-2s\sigma}\xi^3|z|^2\,dx\,dt. \tag{1.32}$$

Combining the estimates in (1.29), (1.30), (1.31) and (1.32) we see that

$$s^3\lambda^4 \iint_{\omega'_T} e^{-2s\sigma}\xi^3|h|^2\,dx\,dt \;\leq\; Cs^{-1}E(h) + Cs^8\lambda^8 \iint_{\omega_{1T}} e^{-2s\sigma}\xi^7|z|^2\,dx\,dt$$

and, from (1.28),

$$E(z) + E(h) \;\leq\; Cs^8\lambda^8 \iint_{\omega_{1T}} e^{-2s\sigma}\xi^7 z^2\,dx\,dt, \tag{1.33}$$

for large enough s and λ.

From now on, the constant C may depend on s and λ.

Let $\psi_1 \in C^1([0,T])$ be such that

$$0 \le \psi_1 \le 1, \ \psi_1 = 1 \text{ in } [0,T/2] \ \text{ and } \ \psi_1 = 0 \text{ in } [3T/4,T]$$

and let us modify the functions θ, ξ and σ as follows:

$$\bar\theta(t) := \begin{cases} (4/T^2)^4 & \text{if} \quad t \in [0,T/2], \\ \theta(t) & \text{if} \quad t \in [T/2,T], \end{cases} \qquad \bar\xi := p\bar\theta, \ \text{ and } \ \bar\sigma := \bar\theta e^{\lambda|\eta|_\infty} - \bar\xi.$$

Then $Z := \psi_1 z$ is a solution to the following system

$$\begin{cases} Z_t - (x^\alpha Z_x)_x + b_0(x,t)Z = \psi_1(t)h + \psi_1'(t)z & \text{in} \quad Q \\ Z(1,\cdot) = 0 \ \text{ and } \begin{cases} Z(0,\cdot) = 0 & \text{if} \quad \alpha \in [0,1) \\ (x^\alpha Z_x)(0,\cdot) = 0 & \text{if} \quad \alpha \in [1,2) \end{cases} & \text{on} \quad (0,T) \end{cases}$$

and, from the energy estimate in Proposition 1.2.1, the following estimate holds:

$$|z(\cdot,0)|^2 + \int_0^{\frac{T}{2}}\int_0^1 \left[|z|^2 + x^\alpha|z_x|^2\right]\,dx\,dt \le C\left[\int_0^{\frac{3T}{4}}\int_0^1 |h|^2\,dx\,dt + \int_{\frac{T}{2}}^{\frac{3T}{4}}\int_0^1 |z|^2\,dx\,dt\right].$$

Using this inequality and the fact that $e^{-2s\bar\sigma}$ and $\bar\xi$ are bounded from above in $[0,T/2]$, we obtain

$$\begin{aligned} &|z(\cdot,0)|^2 + \int_0^{T/2}\int_0^1 e^{-2s\bar\sigma}\left[\xi^3 x^{2-\alpha}|z|^2 + \bar\xi x^\alpha |z_x|^2\right]\,dx\,dt \\ &\quad \le C\left[\int_0^{3T/4}\int_0^1 |h|^2\,dx\,dt + \int_{T/2}^{3T/4}\int_0^1 |z|^2\,dx\,dt\right]. \end{aligned}$$

Now, the fact that $e^{-2s\sigma}$ and ξ are bounded from below in $[T/2,3T/4]$, we get

$$\begin{aligned} &|z(\cdot,0)|^2 + \int_0^{T/2}\int_0^1 e^{-2s\bar\sigma}\left[\bar\xi^3 x^{2-\alpha}|z|^2 + \bar\xi x^\alpha |z_x|^2\right]\,dx\,dt \\ &\quad \le C\left[\int_0^{3T/4}\int_0^1 |h|^2\,dx\,dt + E(z)\right]. \end{aligned} \tag{1.34}$$

On the other hand, since $\bar\xi = \xi$ in $[T/2,T]$, we have

$$\int_{T/2}^{T}\int_0^1 e^{-2s\bar\sigma}\left[\bar\xi^3 x^{2-\alpha}|z|^2 + \bar\xi x^\alpha |z_x|^2\right]\,dx\,dt \ \le \ E(z). \tag{1.35}$$

From (1.34) and (1.35), a new estimate is found:

$$|z(\cdot,0)|^2 + \iint_Q e^{-2s\bar\sigma}\left[\bar\xi^3 x^{2-\alpha}|z|^2 + \bar\xi x^\alpha |z_x|^2\right]\,dx\,dt \le C\left[\int_0^{\frac{3T}{4}}\int_0^1 |h|^2\,dx\,dt + E(z)\right] \tag{1.36}$$

Using again Proposition 1.2.1 and the inequalities $e^{-2s\bar{\sigma}}$, $\bar{\xi} \geq C > 0$ in $[0, 3T/4]$, we see that

$$\int_0^{3T/4}\int_0^1 |h|^2 dx\,dt \;\leq\; C\left[\frac{\beta_1}{\mu_1}+\frac{\beta_2}{\mu_2}\right]\left[\iint_Q e^{-2s\bar{\sigma}}\left[\bar{\xi}^3 x^{2-\alpha}|z|^2+\bar{\xi}x^\alpha|z_x|^2\right]\,dx\,dt\right] \tag{1.37}$$

and, from (1.36) and (1.37), we conclude that

$$|z(\cdot,0)|^2+\iint_Q e^{-2s\bar{\sigma}}\left[\bar{\xi}^3x^{2-\alpha}|z|^2+\bar{\xi}x^\alpha|z_x|^2\right]\,dx\,dt \;\;\leq\;\; CE(z), \tag{1.38}$$

whenever μ_1 and μ_2 are large enough. By (1.33) and (1.38), it follows that

$$|z(\cdot,0)|^2+E(z)+E(h) \;\;\leq\;\; C\iint_{\omega_T'} e^{-2s\sigma}\xi^7|z|^2\,dx\,dt. \tag{1.39}$$

Let us introduce the functions

$$\sigma_0(t) := \max_{x\in[0,1]}\bar{\sigma}(x,t) = C\bar{\theta}(t) \;\;\text{and}\;\; \rho(t) := e^{s\sigma_0(t)}.$$

Note that ρ is a positive nondecreasing function in $C^1([0,T])$ that blows up at $t=T$. Multiplying the equation satisfied by φ_i by $\rho^{-2}\varphi_i$ and integrating in space, we find that

$$\frac{1}{2}\frac{d}{dt}|\rho^{-1}\varphi_i|^2+\rho^{-2}|x^{\alpha/2}(\varphi_i)_x|^2 \;\;\leq\;\; C\rho^{-2}|z|^2+C|\rho^{-1}\varphi_i|^2$$

and, from Gronwall's Lemma, $\rho^{-1}\varphi_i$ can be bounded as follows:

$$|\rho(t)^{-1}\varphi_i(\cdot,t)|^2 \;\;\leq\;\; C\iint_Q \rho^{-2}|z|^2\,dx\,dt, \quad \forall t\in[0,T]. \tag{1.40}$$

Using (1.39) and the fact that $\rho^{-2} \leq e^{-2s\bar{\sigma}}$, we get the estimates

$$\iint_Q \rho^{-2}|z|^2\,dx\,dt \leq CE(z) \leq C\iint_{\omega_T'} e^{-2s\sigma}\xi^7|z|^2\,dx\,dt.$$

Finally, in view of (1.39) and (1.40), (1.27) holds. ■

As mentioned above, the observability estimate (1.27) implies the null controllability of (1.13). We still have an estimate of the control:

$$\iint_Q |g|^2\,dx\,dt \leq C\left[|u_0|^2+\sum_{i=1}^2\iint_{\mathcal{O}_d\times(0,T)}\rho^2|u_{i,d}|^2\,dx\,dt\right],$$

where C is the constant in (1.27). This ends the proof of Theorem 4.

1.3.2 Similar results for semilinear problems

In this section, we extend Theorem 1.3.2 to semilinear systems of the form

$$\begin{cases} u_t - (x^\alpha u_x)_x + b_0(x,t)u = F_0(u) + g 1_{\mathcal{O}} + f_1 1_{\mathcal{O}_1} + f_2 1_{\mathcal{O}_2} & \text{in} \quad Q, \\ u(1,t) = 0 \ \text{ and } \begin{cases} u(0,t) = 0 & \text{if} \quad \alpha \in [0,1) \\ (x^\alpha u_x)(0,t) = 0 & \text{if} \quad \alpha \in [1,2) \end{cases} & \text{on} \quad (0,T), \\ u(x,0) = u_0(x) & \text{in} \quad (0,1), \end{cases} \tag{1.41}$$

where $F_0 : \mathbb{R} \mapsto \mathbb{R}$ is a globally Lipschitz-continuous function satisfying $F_0(0) = 0$.

Note that an existence-uniqueness result like Proposition 1.2.1 still holds for (1.41).

In this semilinear framework, the functionals J_i in (1.17) are not convex in general. Accordingly, we must consider a weaker concept of Nash equilibrium:

Definition 1.3.5 *Let $g \in L^2(\mathcal{O} \times (0,T))$ be given. The pair $(f_1, f_2) \in \mathcal{U}$ is called a Nash quasi-equilibrium of (1.41) associated to g if*

$$\begin{aligned} J_1'(g; f_1, f_2)(f, 0) &= 0, \quad \forall\, f \in L^2(\omega_1 \times (0,T)), \\ J_2'(g; f_1, f_2)(0, f) &= 0, \quad \forall\, f \in L^2(\omega_2 \times (0,T)). \end{aligned}$$

The main result in this section is the following:

Theorem 1.3.6 *Assume (1.24) holds and μ_1 and μ_2 are large enough. Then, for each $u_0 \in L^2(0,1)$, there exist a leader control $g \in L^2(\omega \times (0,1))$ and an associated Nash quasi-equilibrium (f_1, f_2) such that the corresponding solution to (1.41) satisfies (1.5).*

The proof relies on a standard and well known fixed-point argument. For details, the reader is referred to [4], where the same result is proved for a nondegenerate semilinear parabolic PDE.

Completely similar results can be obtained for semilinear systems of the kind (1.2), (1.3) and (1.9).

1.4 Additional comments and open questions

1.4.1 On the assuption $\mathcal{O}_{1,d} = \mathcal{O}_{2,d}$

The assumption $(1.24)_1$ can be suppressed if we assume that $\mathcal{O}_{i,d} \cap \mathcal{O} \neq \emptyset$ for $i = 1, 2$ and $\mathcal{O}_{1,d} \cap \mathcal{O} \neq \mathcal{O}_{2,d} \cap \mathcal{O}$, see [5] for details. However, when $\mathcal{O}_{1,d} \neq \mathcal{O}_{2,d}$, but $\mathcal{O}_{1,d} \cap \mathcal{O} = \mathcal{O}_{2,d} \cap \mathcal{O}$, the Stackelberg-Nash null controllability is open.

1.4.2 Higher dimensions

Little is known on the control of degenerate parabolic PDEs in higher dimensions. Recently, some results have been established on the global null controllability of a model similar to (1.1) that degenerate at the whole border, see [10]. They rely on suitable Carleman estimates similar to (1.20), whence it is reasonable to think that the results in this work can be extended to this model.

In a forthcoming paper we will present Carleman estimates for some models that degenerates in only a part of the border.

1.4.3 Wave equations

An interesting question is whether Stackelberg-Nash null controllability can be proved for wave equations. This question is open; one of the main difficulties is that, to our best knowledge, the few Carleman estimates with interior control region already in the literature do not have all the necessary terms in the left hand side.

1.5 Appendix A: Proof of Theorem 1

It is suficient to prove Theorem 3 assuming that $b_0 = 0$.

Let v be the solution to (1.6) (where $v_T \in L^2(0,1)$ and $g \in L^2(Q)$). For any $s \geq s_0 > 0$, we set $z = e^{-s\sigma}v$. By a density argument we can assume without loss of generality that v is regular enough. We have:

$$v_t = e^{s\sigma}[s\sigma_t z + z_t], \quad (x^\alpha v_x)_x = e^{s\sigma}[s^2\sigma_x^2 x^\alpha z + 2s\sigma_x x^\alpha z_x + s(\sigma_x x^\alpha)_x z + (x^\alpha z_x)_x]$$

and, consequently,

$$P^+ z + P^- z = e^{-s\sigma} g,$$

where $P^+ z = s\sigma_t z + s^2 x^\alpha \sigma_x^2 z + (x^\alpha z_x)_x$ and $P^- z = z_t + s(x^\alpha \sigma_x)_x z + 2s x^\alpha \sigma_x z_x$. This gives:

$$\|e^{-s\sigma} g\|^2 = \|P^+ z\|^2 + \|P^- z\|^2 + 2((P^+ z, P^- z)). \tag{1.42}$$

We have that $(\!(P^+z, P^-z)\!) = I_1 + ... + I_4$, where

$$
\begin{aligned}
I_1 &= (\!(s\sigma_t z + s^2\sigma_x^2 x^\alpha z + (x^\alpha z_x)_x, z_t)\!)\\
I_2 &= s^2(\!(\sigma_t z, (x^\alpha\sigma_x)_x z + 2\sigma_x x^\alpha z_x)\!)\\
I_3 &= s^3(\!(\sigma_x^2 x^\alpha z, (x^\alpha\sigma_x)_x z + 2\sigma_x x^\alpha z_x)\!)\\
I_4 &= s(\!(\,(x^\alpha z_x)_x, (x^\alpha\sigma_x)_x z + 2\sigma_x x^\alpha z_x)\!).
\end{aligned}
$$

The next step is to compute $I_1,\ I_2,\ I_3$ and I_4. To this purpose, we will use that $z = z_x = 0$ at $t = 0$ and $t = T$ and, also,

$$\iint_Q (x^\alpha z_x)_x z_t \, dx\, dt = 0.$$

After integrating by parts, we deduce easily that

$$
\begin{aligned}
I_1 &= -\frac{s}{2}\iint_Q |z|^2(\sigma_{tt} + 2s\sigma_x\sigma_{xt}x^\alpha)\, dx\, dt\\
I_2 &= -s^2\iint_Q x^\alpha\sigma_x\sigma_{xt}|z|^2 dx\, dt;\\
I_3 &= -s^3\iint_Q x^\alpha\sigma_x(x^\alpha\sigma_x^2)_x|z|^2\, dx\, dt;\\
I_4 &= -s\iint_Q (x^\alpha\sigma_x)_{xx}x^\alpha z z_x dx\, dt - 2s\iint_Q (x^\alpha\sigma_x)_x x^\alpha|z_x|^2\, dx\, dt\\
&\quad + s\alpha\iint_Q \sigma_x x^{2\alpha-1}|z_x|^2\, dx\, dt + s\int_0^T \sigma_x x^{2\alpha}|z_x|^2 dt\Big|_{x=0}^{x=1}.
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
(\!(P^+z, P^-z)\!) = {} & s\int_0^T \sigma_x x^{2\alpha}|z_x|^2\, dt\Big|_{x=0}^{x=1} - s^3\iint_Q x^\alpha\sigma_x(x^\alpha\sigma_x^2)_x|z|^2\, dx\, dt\\
& -2s\iint_Q (x^\alpha\sigma_x)_x x^\alpha|z_x|^2\, dx\, dt - 2s^2\iint_Q \sigma_x\sigma_{xt}x^\alpha|z|^2\, dx\, dt\\
& +\alpha s\iint_Q \sigma_x x^{2\alpha-1}|z_x|^2\, dx\, dt - s\iint_Q (x^\alpha\sigma_x)_{xx}x^\alpha z z_x\, dx\, dt\\
& -\frac{s}{2}\iint_Q \sigma_{tt}|z|^2\, dx\, dt. \qquad (1.43)
\end{aligned}
$$

Using the definitions of the functions σ and ξ, we see that

$$\begin{aligned}
\sigma_x \sigma_{xt} x^\alpha &= \lambda^2 x^\alpha |\eta'|^2 \xi \xi_t \\
x^\alpha \sigma_x (x^\alpha \sigma_x^2)_x &= -\lambda^3 x^\alpha \eta' (x^\alpha |\eta'|^2)_x \xi^3 - 2\lambda^4 x^{2\alpha} |\eta'|^4 \xi^3 \\
(x^\alpha \sigma_x)_x x^\alpha &= -\lambda x^\alpha (x^\alpha \eta')_x \xi - \lambda^2 x^{2\alpha} |\eta'|^2 \xi \\
\sigma_x x^{2\alpha-1} &= -\lambda x^{2\alpha-1} \eta' \xi \\
(x^\alpha \sigma_x)_{xx} x^\alpha &= -\lambda x^\alpha (x^\alpha \eta')_{xx} \xi - 2\lambda^2 x^\alpha \eta' (x^\alpha \eta')_x \xi - \lambda^2 x^{2\alpha} \eta' \eta'' \xi - \lambda^3 x^{2\alpha} (\eta')^3 \xi \\
\sigma_x x^{2\alpha} &= -\lambda x^{2\alpha} \eta' \xi.
\end{aligned}$$

With this information, we will now estimate each term in the right hand side of (1.43). For s_0 and λ_0 large enough, we have:

$$s \int_0^T \sigma_x x^{2\alpha} |z_x|^2 \, dt \Big|_{x=0}^{x=1} \geq 0, \tag{1.44}$$

$$-s^3 \iint_Q x^\alpha \sigma_x (x^\alpha \sigma_x^2)_x |z|^2 \, dx \, dt \geq C \left[s^3 \lambda^4 \iint_Q x^{2\alpha} |\eta'|^4 \xi^3 |z|^2 \, dx \, dt + s^3 \lambda^3 \iint_{Q_0} x^{2-\alpha} \xi^3 |z|^2 \, dx \, dt \right] -C s^3 \lambda^3 \iint_{\omega_{0T}} \xi^3 |z|^2 \, dx \, dt, \tag{1.45}$$

$$-2s \iint_Q x^\alpha (x^\alpha \sigma_x)_x |z_x|^2 \, dx \, dt \geq C s \lambda^2 \iint_Q x^{2\alpha} |\eta'|^2 \xi |z_x|^2 \, dx \, dt + 2s\lambda \iint_{Q_0} x^\alpha \xi |z_x|^2 \, dx \, dt -C s \lambda \iint_{\omega_{0T}} \xi |z_x|^2 \, dx \, dt, \tag{1.46}$$

$$-2s^2 \iint_Q x^\alpha \sigma_x \sigma_{xt} |z|^2 \, dx \, dt \geq -C s^2 \lambda^2 \left[\iint_Q x^{2\alpha} |\eta'|^4 \xi^3 |z|^2 \, dx \, dt + \iint_{Q_0} x^{2-\alpha} \xi^3 |z|^2 \, dx \, dt + \iint_{\omega_{0T}} \xi^3 |z|^2 \, dx \, dt \right], \tag{1.47}$$

and

$$\alpha s \iint_Q x^{2\alpha-1} \sigma_x |z_x|^2 \, dx \, dt \geq -\alpha s \lambda \iint_{Q_0} x^\alpha \xi |z_x|^2 \, dx \, dt - C s \lambda \iint_{\omega_{0T}} \xi |z_x|^2 \, dx \, dt, \tag{1.48}$$

where C only depends on a_0, T and α.

On the other hand,

$$-s \iint_Q x^\alpha (x^\alpha \sigma_x)_{xx} z z_x dx \, dt = s\lambda \iint_Q x^\alpha (x^\alpha \eta')_{xx} \xi z z_x dx \, dt + s\lambda^2 \iint_Q x^{2\alpha} \eta' \eta'' \xi z z_x dx \, dt +2s\lambda^2 \iint_Q x^\alpha \eta' (x^\alpha \eta')_x \xi z z_x dx \, dt + s\lambda^3 \iint_Q x^{2\alpha} (\eta')^3 \xi z z_x dx \, dt \tag{1.49}$$

In order to estimate this last term, we note that

$$s\lambda \iint_Q x^\alpha (x^\alpha \eta')_{xx} \xi z z_x \, dx\, dt \ \geq \ -Cs\lambda \iint_{\omega_{0T}} \left(\xi^3 |z|^2 + \xi |z_x|^2\right) \, dx\, dt, \qquad (1.50)$$

$$\begin{aligned} s\lambda^2 \iint_Q x^{2\alpha} \eta' \eta'' \xi z z_x \, dx\, dt \ &\geq \ -C \iint_Q \left(s^2 \lambda^4 x^{2\alpha} |\eta'|^4 \xi^3 |z|^2 + x^{2\alpha} |\eta'|^2 \xi |z_x|^2\right) \, dx\, dt \\ &\quad -C \iint_{Q_0} \left(s^2 \lambda^3 x^{2-\alpha} \xi^3 |z|^2 + \lambda x^\alpha \xi |z_x|^2\right) \, dx\, dt \\ &\quad -C \iint_{\omega_{0T}} \left(s^2 \lambda^3 \xi^3 |z|^2 + \lambda \xi |z_x|^2\right) \, dx\, dt, \qquad (1.51) \end{aligned}$$

$$\begin{aligned} 2s\lambda^2 \iint_Q x^\alpha \eta' (x^\alpha \eta')_x \xi z z_x \, dx\, dt \ &\geq \ -C \iint_Q \left(s^2 \lambda^4 x^{2\alpha} |\eta'|^4 \xi^3 |z|^2 + x^{2\alpha} |\eta'|^2 \xi |z_x|^2\right) \, dx\, dt \\ &\quad -C \iint_{Q_0} \left(s^2 \lambda^3 x^{2-\alpha} \xi^3 |z|^2 + \lambda x^\alpha \xi |z_x|^2\right) \, dx\, dt \\ &\quad -C \iint_{\omega_{0T}} \left(s^2 \lambda^3 \xi^3 |z|^2 + \lambda \xi |z_x|^2\right) \, dx\, dt, \qquad (1.52) \end{aligned}$$

and

$$s\lambda^3 \iint_Q x^{2\alpha} (\eta')^3 \xi z z_x \, dx\, dt \ \geq \ -C \iint_Q \left(\lambda^2 x^{2\alpha} |\eta'|^2 \xi |z_x|^2 + s^2 \lambda^4 x^{2\alpha} |\eta'|^4 \xi^3 |z|^2\right) \, dx\, dt \qquad (1.53)$$

From (1.49)-(1.53), we conclude that

$$\begin{aligned} -s \iint_Q x^\alpha (x^\alpha \sigma_x)_{xx} z z_x \, dx\, dt \ &\geq \ -C \iint_Q \left(s^2 \lambda^4 x^{2\alpha} |\eta'|^4 \xi^3 |z|^2 + \lambda^2 x^{2\alpha} |\eta'|^2 \xi |z_x|^2\right) \, dx\, dt \\ &\quad -C \iint_{Q_0} \left(s^2 \lambda^3 x^{2-\alpha} \xi^3 |z|^2 + \lambda x^\alpha \xi |z_x|^2\right) \, dx\, dt \\ &\quad -C \iint_{\omega_{0T}} \left(s^2 \lambda^3 \xi^3 |z|^2 + s\lambda \xi |z_x|^2\right) \, dx\, dt. \qquad (1.54) \end{aligned}$$

From (1.43)-(1.48) and (1.54), we conclude that

$$\begin{aligned} ((P^+ z, P^- z)) \ \geq \ & C \left[\iint_Q \left(s^3 \lambda^4 x^{2\alpha} |\eta'|^4 \xi^3 |z|^2 + s\lambda^2 x^{2\alpha} |\eta'|^2 \xi |z_x|^2\right) \, dx\, dt \right. \\ & \left. + \int_{Q_0} \left(s^3 \lambda^3 x^{2-\alpha} \xi^3 |z|^2 + s\lambda x^\alpha \xi |z_x|^2\right) \, dx\, dt \right] \\ & -C \left[\iint_{\omega_{0T}} \left(s^3 \lambda^3 \xi^3 |z|^2 + s\lambda \xi |z_x|^2\right) \, dx\, dt + s \iint_Q \xi^{3/2} |z|^2 \, dx\, dt \right] \end{aligned}$$

and, using (1.42), we find that

$$\begin{aligned} & \|P^+ z\|^2 + \|P^- z\|^2 + \iint_Q \left(s^3 \lambda^4 x^{2\alpha} |\eta'|^4 \xi^3 |z|^2 + s\lambda^2 x^{2\alpha} |\eta'|^2 \xi |z_x|^2\right) \, dx\, dt \\ & \quad + \iint_{Q_0} \left(s^3 \lambda^3 x^{2-\alpha} \xi^3 |z|^2 + s\lambda x^\alpha \xi |z_x|^2\right) \, dx\, dt \\ & \leq C \left[\|e^{-s\sigma} g\|^2 + \iint_{\omega_{0T}} \left(s^3 \lambda^3 \xi^3 |z|^2 + s\lambda \xi |z_x|^2\right) \, dx\, dt + s \iint_Q \xi^{3/2} |z|^2 \, dx\, dt \right]. \qquad (1.55) \end{aligned}$$

Furthermore, we see that

$$s^3\lambda^3 \iint_Q \xi^3 x^{2-\alpha}|z|^2\,dx\,dt \leq Cs^3\lambda^3 \Big[\iint_{Q_0} \xi^3 x^{2-\alpha}|z|^2\,dx\,dt + \iint_Q \xi^3 x^{2\alpha}|\eta'|^4|z|^2\,dx\,dt + \iint_{\omega_{0T}} \xi^3|z|^2\,dx\,dt\Big]$$

and

$$s\lambda \iint_Q \xi x^{\alpha}|z_x|^2\,dx\,dt \leq Cs\lambda \Big[\iint_{Q_0} \xi x^{\alpha}|z_x|^2\,dx\,dt + \iint_Q \xi x^{2\alpha}|\eta'|^2|z_x|^2\,dx\,dt + \iint_{\omega_{0T}} \xi|z_x|^2\,dx\,dt.\Big]$$

Hense, from (1.55) we obtain

$$\|P^+z\|^2 + \|P^-z\|^2 + \iint_Q \left(s^3\lambda^4 x^{2\alpha}|\eta'|^4\xi^3|z|^2 + s\lambda^2 x^{2\alpha}|\eta'|^2\xi|z_x|^2\right)\,dx\,dt + \iint_Q \left(s^3\lambda^3 x^{2-\alpha}\xi^3|z|^2 + s\lambda x^{\alpha}\xi|z_x|^2\right)\,dx\,dt \leq C\left[\|e^{-s\sigma}g\|^2 + \iint_{\omega_{0T}} \left(s^3\lambda^3\xi^3|z|^2 + s\lambda\xi|z_x|^2\right)\,dx\,dt + s\iint_Q \xi^{3/2}|z|^2\,dx\,dt\right]. \tag{1.56}$$

Let us denote by $L(z)$ all the terms in the left hand side of (1.56). For instance, assume that $\alpha \neq 1$. Using Young and Hardy's Inequality we deduce that

$$\begin{aligned} s^2\lambda^2 \iint_Q \xi^2|z|^2\,dx\,dt &\leq s^3\lambda^3 \iint_Q \xi^3 x^{2-\alpha}|z|^2\,dx\,dt + s\lambda \iint_Q x^{\alpha-2}|\xi^{1/2}z|^2\,dx\,dt \\ &\leq s^3\lambda^3 \iint_Q \xi^3 x^{2-\alpha}|z|^2\,dx\,dt + s\lambda \iint_Q x^{\alpha}|(\xi^{1/2}z)_x|^2\,dx\,dt \\ &\leq CL(z). \end{aligned} \tag{1.57}$$

Now assume that $\alpha = 1$. Using Hölder, Young and Hardy's inequality we see that

$$\begin{aligned} s^{3/2}\lambda^{7/4} \iint_Q \xi^{3/2}|z|^2\,dx\,dt &= \iint_{Q_0} (s^3\lambda^4 x^2\xi^3|z|^2)^{1/4}.(s\lambda x^{-2/3}\xi|z|^2)^{3/4}\,dx\,dt \\ &\quad + s^{3/2}\lambda^{7/4}\int_0^T\int_{a_0}^1 \xi^{3/2}|z|^2\,dx\,dt \\ &\leq Cs^3\lambda^4 \iint_Q \xi^3 x^{2\alpha}|\eta'|^4|z|^2\,dx\,dt + s\lambda \iint_Q x^{-2+4/3}(\xi^{1/2}z)^2\,dx\,dt \\ &\leq CL(z). \end{aligned} \tag{1.58}$$

From (1.56), (1.57) and (1.58) we deduce that

$$\|P^+z\|^2 + \|P^-z\|^2 + \iint_Q \left(s^3\lambda^4 x^{2\alpha}|\eta'|^4\xi^3|z|^2 + s\lambda^2 x^{2\alpha}|\eta'|^2\xi|z_x|^2\right)\,dx\,dt + \iint_Q \left(s^3\lambda^3 x^{2-\alpha}\xi^3|z|^2 + s^2\lambda^2\gamma_1(\lambda)\xi^2\gamma_2(s\xi)|z|^2 + s\lambda x^{\alpha}\xi|z_x|^2\right)\,dx\,dt \leq C\left[\|e^{-s\sigma}g\|^2 + \iint_{\omega_{0T}} \left(s^3\lambda^3\xi^3|z|^2 + s\lambda\xi|z_x|^2\right)\,dx\,dt\right]. \tag{1.59}$$

Now, we will work to include the terms with a first-order time derivative and second order spatial derivatives in the left hand side. Using the estimate (1.59) and the definitions of P^-z and P^+z, we see that

$$s^{-1}\gamma_1(\lambda)\iint_Q \xi^{-1}|z_t|^2\,dx\,dt \;\le\; C\left[\|e^{-s\sigma}g\|^2+\iint_{\omega_{0T}}\left(s^3\lambda^3\xi^3|z|^2+s\lambda\xi|z_x|^2\right)\,dx\,dt\right] \tag{1.60}$$

and

$$s^{-1}\gamma_1(\lambda)\iint_Q \xi^{-1}|(x^\alpha z_x)_x|^2\,dx\,dt \le C\left[\|e^{-s\sigma}g\|^2+\iint_{\omega_{0T}}\left(s^3\lambda^3\xi^3|z|^2+s\lambda\xi|z_x|^2\right)\,dx\,dt\right] \tag{1.61}$$

From (1.59), (1.60) and (1.61), it is now clear that

$$\begin{aligned}
&\iint_Q \left(s^{-1}\gamma(\lambda)\xi^{-1}\left[|z_t|^2+|(x^\alpha z_x)_x|^2\right]+s\lambda x^\alpha\xi|z_x|^2+s\lambda^2x^{2\alpha}|\eta'|^2\xi|z_x|^2\right)\,dx\,dt\\
&+\iint_Q\left(s^2\lambda^2\gamma_1(\lambda)\xi^2\gamma_2(s\xi)|z|^2+s^3\lambda^3x^{2-\alpha}\xi^3|z|^2+s^3\lambda^4x^{2\alpha}|\eta'|^4\xi^3|z|^2\right)\,dx\,dt\\
&\qquad\le C\left[\|e^{-s\sigma}g\|^2+\iint_{\omega_{0T}}\left(s^3\lambda^3\xi^3|z|^2+s\lambda\xi|z_x|^2\right)\,dx\,dt\right].
\end{aligned}$$

Coming back to the original variable v and using the estimate

$$\begin{aligned}
\int_{\omega_{0T}} e^{-2s\sigma}\xi|v_x|^2dx\,dt \;&\le\; Cs^2\lambda^2\iint_{\omega_T} e^{-2s\sigma}\xi^3|v|^2dx\,dt\\
&\quad+Cs^{-2}\lambda^{-2}\iint_Q e^{-2s\sigma}\xi^{-1}|(x^\alpha v_x)_x|^2\,dx\,dt
\end{aligned}$$

we find the desired inequality (1.20). □

1.6 Appendix B: Proof of Theorem 2

Again, we will assume that $b_0=0$. Let $g_0=g-b_1v_x$. Arguing as before, we can deduce that

$$\begin{aligned}
((P^+z,P^-z)) \;=\;& s\int_0^T \sigma_x x^{2\alpha}|z_x|^2\,dt\Big|_{x=0}^{x=1} - s^3\iint_Q x^\alpha\sigma_x(x^\alpha\sigma_x^2)_x|z|^2\,dx\,dt\\
&-2s\iint_Q(x^\alpha\sigma_x)_x x^\alpha|z_x|^2\,dx\,dt-2s^2\iint_Q\sigma_x\sigma_{xt}x^\alpha|z|^2\,dx\,dt\\
&+\alpha s\iint_Q\sigma_x x^{2\alpha-1}|z_x|^2\,dx\,dt-\frac{s}{2}\iint_Q\sigma_{tt}|z|^2\,dx\,dt\\
&-s\iint_Q(x^\alpha\sigma_x)_{xx}x^\alpha zz_x\,dx\,dt.
\end{aligned} \tag{1.62}$$

Furthermore,

$$s\int_0^T x^{2\alpha}\sigma_x|z_x|^2\,dt\Big|_{x=0}^{x=1} \;\geq\; 0, \tag{1.63}$$

$$-s^3\iint_Q x^\alpha\sigma_x(x^\alpha\sigma_x^2)_x|z|^2\,dx\,dt \;\geq\; C\left[s^3\lambda^4\iint_Q x^{2\alpha}|\eta'|^4\xi^3|z|^2\,dx\,dt + s^3\lambda^3\iint_{Q_0}\xi^3|z|^2\,dx\,dt\right]$$
$$-Cs^3\lambda^3\iint_{\omega_{0T}}\xi^3|z|^2\,dx\,dt, \tag{1.64}$$

$$-2s\iint_Q x^\alpha(x^\alpha\sigma_x)_x|z_x|^2\,dx\,dt \;\geq\; Cs\lambda^2\iint_Q x^{2\alpha}|\eta'|^2\xi|z_x|^2\,dx\,dt - Cs\lambda\iint_{\omega_{0T}}\xi|z_x|^2\,dx\,dt$$
$$+\frac{2}{3}(1+\alpha)s\lambda\iint_{Q_0} x^{(4\alpha-2)/3}\xi|z_x|^2\,dx\,dt, \tag{1.65}$$

$$-2s^2\iint_Q x^\alpha\sigma_x\sigma_{xt}|z|^2\,dx\,dt \;\geq\; -Cs^2\lambda^2\iint_{Q_0}\xi^3|z|^2\,dx\,dt - Cs^2\lambda^2\iint_{\omega_{0T}}\xi^3|z|^2\,dx\,dt$$
$$-Cs^2\lambda^2\iint_Q x^{2\alpha}|\eta'|^4\xi^3|z|^2\,dx\,dt, \tag{1.66}$$

$$\alpha s\iint_Q x^{2\alpha-1}\sigma_x|z_x|^2\,dx\,dt \;\geq\; -\alpha s\lambda\iint_{Q_0} x^{(4\alpha-2)/3}\xi|z_x|^2\,dx\,dt - Cs\lambda\iint_{\omega_{0T}}\xi|z_x|^2\,dx\,dt \tag{1.67}$$

and

$$-\frac{s}{2}\iint_Q \sigma_{tt}|z|^2\,dx\,dt \;\geq\; -Cs\iint_{Q_0}\xi^3|z|^2\,dx\,dt - Cs\iint_{\omega_{0T}}\xi^3|z|^2\,dx\,dt$$
$$-Cs\iint_Q x^{2\alpha}|\eta'|^4\xi^3|z|^2\,dx\,dt. \tag{1.68}$$

Note that in this case we can control the term on the left hand side of (1.68) much more easily than in the previous case. However, to estimate the last term in the right hand side of (1.62) is more difficult. It is just here where the restriction $\alpha < 1/2$ is required. We have:

$$-s\iint_Q x^\alpha(x^\alpha\sigma_x)_{xx}zz_x\,dx\,dt \;=\; s\lambda\iint_Q x^\alpha(x^\alpha\eta')_{xx}\xi zz_x\,dx\,dt$$
$$+s\lambda^3\iint_Q x^{2\alpha}(\eta')^3\xi zz_x\,dx\,dt + 2s\lambda^2\iint_Q x^\alpha\eta'(x^\alpha\eta')_x\xi zz_x\,dx\,dt$$
$$+s\lambda^2\iint_Q x^{2\alpha}\eta'\eta''\xi zz_x\,dx\,dt. \tag{1.69}$$

In the previous case we had $(x^\alpha\eta')_{xx} = 0$ in $(0,1)\backslash\omega_0$. Here, this is lost and, therefore, we have to work differently.

Using integration by parts, we have that

$$
\begin{aligned}
& s\lambda \iint_Q x^\alpha (x^\alpha \eta')_{xx} \xi z z_x \, dx \, dt \\
& \qquad = -\iint_Q \frac{s\lambda}{2} \xi |z|^2 \left[[x^\alpha (x^\alpha \eta')_{xx}]_x + \lambda x^\alpha \eta' (x^\alpha \eta')_{xx} \right] \, dx \, dt. \qquad (1.70)
\end{aligned}
$$

In view of Proposition 1.2.2, we have

$$
\begin{aligned}
-\frac{s\lambda}{2} \iint_Q & [x^\alpha (x^\alpha \eta')_{xx}]_x \xi |z|^2 \, dx \, dt \\
& \geq -\frac{s\lambda}{54} (1+\alpha)(2-\alpha)(5-4\alpha) \iint_{Q_0} x^{\frac{4\alpha-2}{3}-2} (\xi^{1/2} z)^2 \, dx \, dt \\
& \quad - Cs\lambda \iint_{\omega_{0T}} \xi |z|^2 \, dx \, dt - Cs\lambda \int_0^T \int_{b_0}^1 \xi |z|^2 \, dx \, dt \\
& \geq -\frac{2}{3} s\lambda \frac{(1+\alpha)(2-\alpha)}{5-4\alpha} \iint_{Q_0} x^{(4\alpha-2)/3} (\xi^{1/2} z)_x^2 \, dx \, dt \\
& \quad - Cs\lambda \iint_{\omega_{0T}} \xi^3 |z|^2 \, dx \, dt - Cs\lambda \iint_Q x^{2\alpha} |\eta'|^4 \xi^3 |z|^2 \, dx \, dt. \qquad (1.71)
\end{aligned}
$$

Furthermore,

$$
\begin{aligned}
& -\frac{2}{3} s\lambda \frac{(1+\alpha)(2-\alpha)}{5-4\alpha} \iint_{Q_0} x^{(4\alpha-2)/3} (\xi^{1/2} z)_x^2 \, dx \, dt \\
& = -m(\alpha) s\lambda \iint_{Q_0} x^{(4\alpha-2)/3} \left[\xi |z_x|^2 + \lambda x^{(1-2\alpha)/3} \xi z z_x + \frac{\lambda^2}{4} x^{(2-4\alpha)/3} \xi |z|^2 \right] \, dx \, dt, \qquad (1.72)
\end{aligned}
$$

where $m(\alpha) := \frac{2}{3}(1+\alpha)(2-\alpha)(5-4\alpha)^{-1}$ and

$$
-Cs\lambda^2 \iint_{Q_0} x^{(2\alpha-1)/3} \xi z z_x \, dx \, dt \ \geq\ -C\lambda \iint_{Q_0} \left(\lambda x^{(4\alpha-2)/3} \xi |z_x|^2 + s^2 \lambda^3 \xi^3 |z|^2 \right) \, dx \, dt. \quad (1.73)
$$

Then from (1.71), (1.72) and (1.73), we get

$$
\begin{aligned}
-\frac{s\lambda}{2} \iint_Q [x^\alpha (x^\alpha \eta')_{xx}]_x \xi |z|^2 \, dx \, dt \ \geq\ & -m(\alpha) s\lambda \iint_{Q_0} x^{(4\alpha-2)/3} \xi |z_x|^2 \, dx \, dt \\
& - Cs^2\lambda^3 \iint_{Q_0} \xi^3 |z|^2 \, dx \, dt - Cs\lambda \iint_Q x^{2\alpha} |\eta'|^4 \xi^3 |z|^2 \, dx \, dt \\
& - C\lambda \iint_{Q_0} x^{(4\alpha-2)/3} \xi |z_x|^2 \, dx \, dt - Cs\lambda \iint_{\omega_{0T}} \xi^3 |z|^2 \, dx \, dt. \quad (1.74)
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
-\frac{s\lambda^2}{2} \iint_Q x^\alpha \eta' (x^\alpha \eta')_{xx} \xi |z|^2 \, dx \, dt \ \geq\ & \frac{(1+\alpha)(2-\alpha)}{18} s\lambda^2 \iint_{Q_0} x^{(2\alpha-4)/3} \xi |z|^2 \, dx \, dt \\
& - Cs\lambda^2 \iint_{\omega_{0T}} \xi |z|^2 \, dx \, dt - Cs\lambda^2 \int_0^T \int_{b_0}^1 x^{(2\alpha-4)/3} \xi |z|^2 \, dx \, dt \qquad (1.75)
\end{aligned}
$$

and, from (1.70), (1.74) and (1.75), we see that

$$\begin{aligned} s\lambda\iint_Q x^\alpha(x^\alpha\eta')_{xx}\xi z z_x\,dx\,dt \;&\geq\; Cs\lambda^2\iint_{Q_0} x^{(2\alpha-4)/3}\xi|z|^2\,dx\,dt - Cs^2\lambda^3\iint_{Q_0}\xi^3|z|^2dx\,dt \\ &-m(\alpha)s\lambda\iint_{Q_0} x^{(4\alpha-2)/3}\xi|z_x|^2\,dx\,dt - C\lambda\iint_{Q_0} x^{(4\alpha-2)/3}\xi|z_x|^2\,dx\,dt \\ &-Cs\lambda^2\iint_Q x^{2\alpha}|\eta'|^4\xi^3|z|^2\,dx\,dt - Cs\lambda^2\iint_{\omega_{0T}}\xi^3|z|^2\,dx\,dt. \qquad (1.76) \end{aligned}$$

Note that the new strategy makes appear a new term in the right hand side of (1.76). For the other terms in (1.69), we have that

$$\begin{aligned} 2s\lambda^2\iint_Q x^\alpha\eta'(x^\alpha\eta')_x\xi z z_x\,dx\,dt \;&\geq\; -C\iint_Q\left(s^2\lambda^4x^{2\alpha}|\eta'|^4\xi^3|z|^2 + x^{2\alpha}|\eta'|^2\xi|z_x|^2\right)\,dx\,dt \\ &-C\iint_{Q_0}\left(s^2\lambda^3\xi^3|z|^2+\lambda x^{(4\alpha-2)/3}\xi|z_x|^2\right)\,dx\,dt \\ &-C\iint_{\omega_{0T}}\left(s^2\lambda^3\xi^3|z|^2+\lambda\xi|z_x|^2\right)\,dx\,dt, \qquad (1.77) \end{aligned}$$

$$\begin{aligned} s\lambda^2\iint_Q x^{2\alpha}\eta'\eta''\xi z z_x\,dx\,dt \;&\geq\; -C\iint_Q\left(s^2\lambda^4x^{2\alpha}|\eta'|^4\xi^3|z|^2 + x^{2\alpha}|\eta'|^2\xi|z_x|^2\right)\,dx\,dt \\ &-C\iint_{Q_0}\left(s^2\lambda^3\xi^3|z|^2+\lambda x^{(4\alpha-2)/3}\xi|z_x|^2\right)\,dx\,dt \\ &-C\iint_{\omega_{0T}}\left(s^2\lambda^3\xi^3|z|^2+\lambda\xi|z_x|^2\right)\,dx\,dt, \qquad (1.78) \end{aligned}$$

and

$$s\lambda^3\iint_Q x^{2\alpha}(\eta')^3\xi z z_x\,dx\,dt \;\geq\; -C\iint_Q\left(\lambda^2x^{2\alpha}|\eta'|^2\xi|z_x|^2 + s^2\lambda^4x^{2\alpha}|\eta'|^4\xi^3|z|^2\right)\,dx\,dt \qquad (1.79)$$

From (1.69), (1.76), (1.77), (1.78) and (1.79), we conclude that

$$\begin{aligned} -s\iint_Q x^\alpha(x^\alpha\sigma_x)_{xx}z z_x\,dx\,dt \;&\geq\; -C\iint_Q\left(\lambda^2x^{2\alpha}|\eta'|^2\xi|z_x|^2 + s^2\lambda^4x^{2\alpha}|\eta'|^4\xi^3|z|^2\right)\,dx\,dt \\ &+Cs\lambda^2\iint_{Q_0}x^{(2\alpha-4)/3}\xi|z|^2\,dx\,dt - C\iint_{\omega_{0T}}\left(s^2\lambda^3\xi^3|z|^2+s\lambda\xi|z_x|^2\right)\,dx\,dt \\ &-C\iint_{Q_0}\left(s^2\lambda^3\xi^3|z|^2+\lambda x^{(4\alpha-2)/3}\xi|z_x|^2\right)\,dx\,dt \\ &-m(\alpha)s\lambda\iint_{Q_0}x^{(4\alpha-2)/3}\xi|z_x|^2\,dx\,dt. \qquad (1.80) \end{aligned}$$

It is important to take care of the constants accompanying the term

$$s\lambda\iint_{Q_0}x^{(4\alpha-2)/3}\xi|z_x|^2dx\,dt$$

in the estimates (1.65), (1.67) and (1.80). The sum of these constants is

$$\frac{(2-\alpha)(1-2\alpha)}{5-4\alpha},$$

that is only positive for $\alpha \in [0, 1/2)$.

From (1.62), (1.63), (1.64), (1.65), (1.66), (1.67), (1.80) and (1.68), we deduce that

$$\begin{aligned}
\|P^+z\|^2 + \|P^-z\|^2 + \iint_Q \left(s^3\lambda^4 x^{2\alpha}|\eta'|^4\xi^3|z|^2 + s\lambda^2 x^{2\alpha}|\eta'|^2\xi|z_x|^2\right)\,dx\,dt \\
+ \iint_{Q_0} \left(s^3\lambda^3\xi^3|z|^2 + s\lambda^2 x^{(2\alpha-4)/3}\xi|z|^2 + s\lambda x^{(4\alpha-2)/3}\xi|z_x|^2\right)\,dx\,dt \\
\le C\left[\|e^{-s\sigma}g_0\|^2 + \iint_{\omega_{0T}} \left(s^3\lambda^3\xi^3|z|^2 + s\lambda\xi|z_x|^2\right)\,dx\,dt\right].
\end{aligned}$$

Arguing as in Appendix A, we can replaced the integral in Q_0 in the left hand side by integrals in Q. Moreover, thanks to the additional term in the left hand side of this inequality, we can incorporate terms with time derivatives and second-order spatial derivatives more easily and also return to the variable v:

$$\begin{aligned}
\iint_Q e^{-2s\sigma}\left[s^{-1}\xi^{-1}\left[|v_t|^2 + |(x^\alpha v_x)_x|^2\right] + s\lambda x^{(4\alpha-2)/3}\xi|v_x|^2 + s\lambda^2 x^{2\alpha}|\eta'|^2\xi|v_x|^2\right]\,dx\,dt \\
+ \iint_Q e^{-2s\sigma}|v|^2\left[s\lambda^2 x^{(2\alpha-4)/3}\xi + s^3\lambda^3\xi^3 + s^3\lambda^4 x^{2\alpha}|\eta'|^4\xi^3|v|^2\right]\,dx\,dt \\
\le C\left[\|e^{-s\sigma}(g - b_1 v_x)\|^2 + \iint_{\omega_{0T}} e^{-2s\sigma}\left[s^3\lambda^3\xi^3|z|^2 + s\lambda\xi|z_x|^2\right]\,dx\,dt\right].
\end{aligned} \tag{1.81}$$

Since the power of x in the local term with first-order spatial derivatives is negative, and we can deduce that (1.81) remains true with $(g - b_1 v_x)$ replaced by g. Finally, to eliminate the term with derivatives in the right hand side, it suffices to work as in the case considered in Appendix A. □

1.7 Appendix C: Proof of Theorem 3

Again we will prove Theorem 1.2.8 when $B = 0$. In view of the presence of Robin conditions in (1.12), we will perform another change of variables:

$$v(y,t) = e^{-\eta(y)}w(y,t).$$

Now, (1.12) becomes

$$\begin{cases} v_t + v_{yy} = g_0 & \text{in} \quad Q', \\ \left(v_x - \frac{1}{2}v\right)(0,t) = 0 \ \text{ and } \ \displaystyle\lim_{y\to\infty}\left(v_y - \frac{1}{2}v\right)(y,t) = 0 & \text{on} \quad (0,T), \\ v(x,T) = v_T(x) & \text{in} \quad (0,1), \end{cases} \tag{1.82}$$

where $g_0 := e^{-\eta}F - (\eta'' + |\eta'|^2)v - 2\eta' v_y$.

Let v be a solution to (1.82). For any $s \geq s_0 > 0$, we set $z = e^{-s\sigma}v$ and $\tilde{z} = e^{-s\tilde{\sigma}}v$. We have that z and $\tilde{z}$ satisfy the following initial, final and boundary conditions:

$$z = \tilde{z} = z_y = \tilde{z}_y = 0 \ \text{ at } \ t = 0 \ \text{ and } \ t = T,$$
$$\left(z_y + \left(s\lambda\xi - \frac{1}{2}\right)z\right)(0,t) = 0 \quad \text{and} \quad \lim_{y\to+\infty} z_y(y,t) = \lim_{y\to+\infty} z(y,t) = 0 \ \text{ on } \ [0,T],$$
$$\left(\tilde{z}_y + \left(-s\lambda\tilde{\xi} - \frac{1}{2}\right)\tilde{z}\right)(0,t) = 0 \quad \text{and} \quad \lim_{y\to+\infty} \tilde{z}_y(y,t) = \lim_{y\to+\infty} \tilde{z}(y,t) = 0 \ \text{ on } \ [0,T].$$

Again, we assume that v is sufficintly regular. We have:

$$v_t = e^{s\sigma}[s\sigma_t z + z_t], \qquad v_{yy} = e^{s\sigma}[z_{yy} + s^2\sigma_y^2 z + 2s\sigma_y z_y + s\sigma_{yy}z]$$

and, consequently,

$$M_1 z + M_2 z = g_1,$$

with $M_1 z := I_{11} + I_{12} + I_{13} := -2s\lambda^2|\eta'|^2\xi z - 2s\lambda|\eta'|\xi z_y + z_t$, $M_2 z := I_{21} + I_{22} + I_{23} := s^2\lambda^2|\eta'|^2\xi^2 z + z_{yy} + s\sigma_t z$ and $g_1 := e^{-s\sigma}g_0 + s\lambda\eta''\xi z - s\lambda^2|\eta'|^2\xi z$. This gives

$$\|M_1 z\|^2 + \|M_2 z\|^2 + 2(\!(M_1 z, M_2 z)\!) \quad = \quad \|g_1\|^2. \tag{1.83}$$

Let us estimate $(\!(M_1 z, M_2 z)\!)$:

$$(\!(I_{11}, I_{21})\!) \quad = \quad -2s^3\lambda^4 \iint_{Q'} |\eta'|^4\xi^3|z|^2\,dy\,dt,$$

$$\begin{aligned} (\!(I_{12}, I_{21})\!) \quad = \quad & 3s^3\lambda^3 \iint_{Q'} |\eta'|^2\eta''\xi^3|z|^2\,dy\,dt + 3s^3\lambda^4 \iint_{Q'} |\eta'|^4\xi^3|z|^2\,dy\,dt \\ & + s^3\lambda^3 \int_0^T (\eta')^3\xi^3|z|^2 dt\bigg|_{y=0}, \end{aligned}$$

$$(\!(I_{13}, I_{21})\!) \quad = \quad -s^2\lambda^2 \iint_{Q'} |\eta'|^2\xi^2\theta(2t - T)|z|^2\,dy\,dt,$$

$$((I_{11}, I_{22})) = 4s\lambda^2 \iint_{Q'} \eta'\eta''\xi z z_y \, dy\, dt - s\lambda^3 \iint_{Q'} (3|\eta'|^2\eta'' + \lambda|\eta'|^4)\xi|z|^2 \, dy\, dt$$
$$+2s\lambda^2 \iint_{Q'} |\eta'|^2\xi|z_y|^2 \, dy\, dt - s\lambda^3 \int_0^T (\eta')^3\xi|z|^2 \, dt\Big|_{y=0} + 2s\lambda^2 \int_0^T |\eta'|^2\xi z z_y \, dt\Big|_{y=0},$$

$$((I_{12}, I_{22})) = s\lambda \iint_{Q'} \eta''\xi|z_y|^2 \, dy\, dt + s\lambda^2 \iint_{Q'} |\eta'|^2\xi|z_y|^2 \, dy\, dt + s\,\lambda \int_0^T \eta'\xi|z_y|^2 \, dt\Big|_{y=0},$$

$$((I_{13}, I_{22})) = \int_0^T z z_{ty} \, dt\Big|_{y=0},$$

$$((I_{11}, I_{23})) = -2s^2\lambda^2 \iint_{Q'} |\eta'|^2\xi\sigma_t|z|^2 \, dy\, dt,$$

$$((I_{12}, I_{23})) = s^2\lambda \iint_{Q'} \eta''\xi\sigma_t|z|^2 \, dy\, dt + s^2\lambda^2 \iint_{Q'} |\eta'|^2\xi\sigma_t|z|^2 \, dy\, dt$$
$$+s^2\lambda \iint_{Q'} \eta'\xi\sigma_{yt}|z|^2 \, dy\, dt + s^2\lambda \int_0^T \eta'\xi\sigma_t|z|^2 \, dt\Big|_{y=0},$$

and

$$((I_{13}, I_{23})) = -\frac{s}{2} \iint_{Q'} \sigma_{tt}|z|^2 \, dy\, dt.$$

Adding all theses inequalities, we obtain after some work that

$$((M_1 z, M_2 z)) \geq C \iint_{Q'} \left(s^3\lambda^4\xi^3|z|^2 + s\lambda^2\xi|z_x|^2\right) dy\, dt$$
$$-C \iint_{\omega_{0T}} \left(s^3\lambda^4\xi^3|z|^2 + s\lambda^2\xi|z_y|^2\right) dy\, dt$$
$$+ \int_0^T \left[s^3\lambda^3(\eta')^3\xi^3|z|^2 - s\lambda^3(\eta')^3\xi|z|^2 + s^2\lambda\eta'\xi\sigma_t|z|^2\right] dt\Big|_{y=0}$$
$$+ \int_0^T \left[s\lambda\eta'\xi|z_y|^2 + 2s\lambda^2|\eta'|^2\xi z z_y + z z_{yt}\right] dt\Big|_{y=0}. \quad (1.84)$$

Working similarly with the function $\tilde{z}$, we obtain

$$\tilde{M}_1\tilde{z} + \tilde{M}_2\tilde{z} = \tilde{g}_1$$

with $\tilde{M}_1\tilde{z} := \tilde{I}_{11} + \tilde{I}_{12} + \tilde{I}_{13} := -2s\lambda^2|\eta'|^2\tilde{\xi}\tilde{z} + 2s\lambda\eta'\tilde{\xi}\tilde{z}_y + \tilde{z}_t$, $\tilde{M}_2\tilde{z} := \tilde{I}_{21} + \tilde{I}_{22} + \tilde{I}_{23} := s^2\lambda^2|\eta'|^2\tilde{\xi}^2\tilde{z} + \tilde{z}_{yy} + s\tilde{\sigma}_t\tilde{z}$ and $\tilde{g}_1 := e^{-s\tilde{\sigma}}g_0 - s\lambda\eta''\tilde{\xi}\tilde{z} - s\lambda^2|\eta'|^2\tilde{\xi}\tilde{z}$. This gives:

$$\|\tilde{g}_1\|^2 = \|\tilde{M}_1\tilde{z}\|^2 + \|\tilde{M}_2\tilde{z}\|^2 + 2((\tilde{M}_1\tilde{z}, \tilde{M}_2\tilde{z})) \quad (1.85)$$

and

$$
\begin{aligned}
((\tilde{M}_1\tilde{z}, \tilde{M}_2\tilde{z})) \;\geq\;& C\iint_{Q'} s^3\lambda^4\tilde{\xi}^3|\tilde{z}|^2 + s\lambda^2\tilde{\xi}|\tilde{z}_y|^2\,dy\,dt \\
& -C\iint_{\omega_{0T}} s^3\lambda^4\tilde{\xi}^3|\tilde{z}|^2 + s\lambda^2\tilde{\xi}|\tilde{z}_y|^2\,dy\,dt \\
& + \int_0^T [-s^3\lambda^3(\eta')^3\tilde{\xi}^3\tilde{z}^2 + s\lambda^3(\eta')^3\tilde{\xi}\tilde{z}^2 - s^2\lambda\eta'\tilde{\xi}\tilde{\sigma}_t\tilde{z}^2]\,dt\Big|_{y=0} \\
& + \int_0^T -s\lambda\eta'\tilde{\xi}\tilde{z}_y^2 + 2s\lambda^2|\eta'|^2\tilde{\xi}\tilde{z}\tilde{z}_y + \tilde{z}\tilde{z}_{yt}\,dt\Big|_{y=0}. \qquad (1.86)
\end{aligned}
$$

Note that $z = \tilde{z}$, $\xi = \tilde{\xi}$, $\sigma = \tilde{\sigma}$ and $\sigma_t = \tilde{\sigma}_t$ for $y = 0$. Hence, from (1.84) and (1.86), we find

$$
\begin{aligned}
((M_1z, M_2z)) + ((\tilde{M}_1\tilde{z}, \tilde{M}_2\tilde{z})) \;\geq\;& C\iint_{Q'}\Big[s^3\lambda^4(\xi^3|z|^2 + \tilde{\xi}^3|\tilde{z}|^2) + s\lambda^2(\xi|z_y|^2 + \tilde{\xi}|\tilde{z}_y|^2)\Big]\,dy\,dt \\
& -C\iint_{\omega_{0T}}\Big[s^3\lambda^4(\xi^3|z|^2 + \tilde{\xi}^3|\tilde{z}|^2) + s\lambda^2(\xi|z_y|^2 + \tilde{\xi}|\tilde{z}_y|^2)\Big]\,dy\,dt \\
& +\int_0^T\Big[2s\lambda^2|\eta'|^2(\tilde{\xi}\tilde{z}\tilde{z}_y + \xi zz_y) - s\lambda\eta'(\tilde{\xi}\tilde{z}_y^2 - \xi z_y^2) + \tilde{z}\tilde{z}_{ty} + zz_{ty}\Big]\,dt\Big|_{y=0}. \quad (1.87)
\end{aligned}
$$

On the other hand, in view of the boundary conditions satisfied by z and $\tilde{z}$ we see that

$$
\begin{aligned}
&\int_0^T [zz_{yt} + \tilde{z}\tilde{z}_{yt}]dt\Big|_{y=0} \\
&= \int_0^T \Big[s\lambda\xi_t(|\tilde{z}|^2 - |z|^2) + \frac{1}{4}(|\tilde{z}|^2 + |z|^2)_t + \frac{1}{2}s\lambda\xi(|\tilde{z}|^2 - |z|^2)_t\Big]\,dt\Big|_{y=0} = 0,
\end{aligned}
$$

$$
2s\lambda^2\int_0^T |\eta'|^2\xi[zz_y + \tilde{z}\tilde{z}_y]\,dt\Big|_{y=0} \;=\; 2s\lambda^2\int_0^T |\eta'|^2\xi|z|^2\,dt\Big|_{y=0} \geq 0,
$$

and

$$
-\,s\lambda\int_0^T \eta'\xi[\tilde{z}_y^2 - z_y^2]\,dt\Big|_{y=0} \;=\; 2s^2\lambda^2\int_0^T \xi^2|z|^2\,dt\Big|_{y=0} \geq 0.
$$

As a consequence, from (1.87), (1.85) and (1.83), we deduce that

$$
\begin{aligned}
&\sum_{i=1}^{2}(\|M_iz\|^2 + \|\tilde{M}_i\tilde{z}\|^2) + \iint_{Q'}\Big[s\lambda^2(\xi|z_y|^2 + \tilde{\xi}|\tilde{z}_y|^2) + s^2\lambda^4(\xi^3|z|^2 + \tilde{\xi}^3|\tilde{z}|^2)\Big]\,dy\,dt \\
&\leq C\Big[\|g_1\|^2 + \|\tilde{g}_1\|^2 + \iint_{\omega_{0T}}\Big[s\lambda^2(\xi|z_y|^2 + \tilde{\xi}|\tilde{z}_y|^2) + s^3\lambda^4(\xi^3|z|^2 + \tilde{\xi}^3|\tilde{z}|^2)\Big]\,dy\,dt\Big] \quad (1.88)
\end{aligned}
$$

Using (1.88) and the definitions of g_1, $\tilde{g}_1$, $M_i z$ and $\tilde{M}_i\tilde{z}$, we see that, for s and λ large enough,

$$
\begin{aligned}
s^{-1} &\iint_{Q'} \left[\xi^{-1}(|z_t|^2 + |z_{yy}|^2) + \tilde{\xi}^{-1}(|\tilde{z}_t|^2 + |\tilde{z}_{yy}|^2)\right]\, dx\, dt \\
&+ \iint_{Q} \left[s\lambda^2(\xi|z_y|^2 + \tilde{\xi}|\tilde{z}_y|^2) + s^2\lambda^4(\xi^3|z|^2 + \tilde{\xi}^3|\tilde{z}|^2)\right]\, dy\, dt \\
\leq C &\left[\|\varrho^{1/2} g_0\|^2 + \iint_{\omega_{0T}} \left[s\lambda^2(\xi|z_y|^2 + \tilde{\xi}|\tilde{z}_y|^2) + s^3\lambda^4(\xi^3|z|^2 + \tilde{\xi}^3|\tilde{z}|^2)\right]\, dy\, dt\right].
\end{aligned}
$$

From classical arguments, we can eliminate the terms with derivatives in the right hand side of the inequality and find that

$$
\begin{aligned}
s^{-1} &\iint_{Q'} \left[\xi^{-1}(|z_t|^2 + |z_{yy}|^2) + \tilde{\xi}^{-1}(|\tilde{z}_t|^2 + |\tilde{z}_{yy}|^2)\right]\, dx\, dt \\
&+ \iint_{Q} \left[s\lambda^2(\xi|z_y|^2 + \tilde{\xi}|\tilde{z}_y|^2) + s^2\lambda^4(\xi^3|z|^2 + \tilde{\xi}^3|\tilde{z}|^2)\right]\, dy\, dt \\
&\leq C \left[\|\varrho^{1/2} g_0\|^2 + s^3\lambda^4 \iint_{\omega_{0T}} \left[\xi^3|z|^2 + \tilde{\xi}^3|\tilde{z}|^2\right]\, dy\, dt\right].
\end{aligned}
$$

We then conclude the proof coming back to the original variable w and using the definition of g_0 and the fact that $\xi \leq C\tilde{\xi} \leq C\xi$. □

Capítulo 2

Estimativas de Carleman para algumas EDPs parabólicas degeneradas em dimensão 2 e aplicações

Carleman estimates for some 2-D degenerate parabolic PDEs and applications

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Abstract

This paper deals with a class of two-dimensional degenerate parabolic equations in a squere. The goal is to obtain Carleman estimates to obtain controllability results. In order to be more faithfull with the 1D degenerate problem, we consider that the degeneracy ocour only in a part of the boundary. Then, we present well posedness results and, under some geometrical assumptions, we build suitable weight functions that allows to deduce a global Carleman estimates. As an application, we prove some null controllability and Stackelberg-Nash null controllability results.

Keywords: Null controllability, degenerate parabolic equations, Carleman inequalities, Stackelber-Nash strategies.

Mathematics Subject Classification: 34K35, 49J20, 35K10.

2.1 Introduction

The study of the controllability of partial differential equations and systems has attracted the interest of many authors. The theory has been extended to semilinear problems, equations in unbounded domains of some kinds, and systems in fluid dynamics, among others; see for instance [15, 18, 20, 21, 24].

The study of controllability of degenerate parabolic equations started in the last decade with the works [1, 7, 8, 9, 12, 13, 14]. In this paper, we will analyze the following problem in two spatial dimensions:

$$\begin{cases} u_t - div(A\nabla u) + bu = g1_\omega & \text{in} \quad Q, \\ B.C. & \text{on} \quad \Sigma, \\ u(\cdot,0) = u_0 & \text{in} \quad \Omega, \end{cases} \tag{2.1}$$

where $\Omega = (0,1)\times(0,1)$, $\Gamma := \partial\Omega$, $T>0$, $Q = \Omega\times(0,T)$, $\Sigma := \Gamma\times(0,T)$, $\omega\subset\Omega$ is a non-empty open set, $b\in L^\infty(Q)$, $g\in L^2(Q)$, $u_0\in L^2(\Omega)$, $A:\overline{\Omega}\mapsto M_{2\times 2}(\mathbb{R})$ is given by

$$A(x) = diag(x_1^{\alpha_1}, x_2^{\alpha_2}),$$

and the boundary conditions are given by

$$B.C. := \begin{cases} u = 0 \ \text{ on } \ \Sigma & \text{if} \ \ \alpha_1,\ \alpha_2 \in [0,1), \\ u = 0 \ \text{ on } \ \Sigma_{3,4} \ \text{ and } \ (A\nabla u)\nu = 0 \ \text{ on } \ \Sigma_{1,2} & \text{if} \ \ \alpha_1,\ \alpha_2\in[1,2], \\ u = 0 \ \text{ on } \ \Sigma_{1,3,4} \ \text{ and } \ (A\nabla u)\nu = 0 \ \text{ on } \ \Sigma_2 & \text{if} \ \ \alpha_1\in[0,1), \ \ \alpha_2\in[1,2], \\ u = 0 \ \text{ on } \ \Sigma_{2,3,4} \ \text{ and } \ (A\nabla u)\nu = 0 \ \text{ on } \ \Sigma_1 & \text{if} \ \ \alpha_1\in[1,2], \ \ \alpha_2\in[0,1), \end{cases}$$

with $\alpha = (\alpha_1,\alpha_2)\in[0,2]\times[0,2]$, $\Sigma_{i,j,l} := (\Gamma_i\cup\Gamma_j\cup\Gamma_l)\times(0,T)$, and

$$\Gamma_1 := \{0\}\times[0,1],\ \Gamma_2 := [0,1]\times\{0\},\ \Gamma_3 := \{1\}\times[0,1],\ \Gamma_4 := [0,1]\times\{1\}.$$

In previous papers, the main model is the following:

$$\begin{cases} u_t - (x^\alpha u_x)_x = g(x,t)1_{\mathcal{O}} & \text{in} \quad (0,1)\times(0,T), \\ u(1,\cdot) = 0 \ \text{ and } \begin{cases} u(0,\cdot) = 0 & \text{if} \ \ \alpha\in[0,1) \\ (x^\alpha u_x)(0,\cdot) = 0 & \text{if} \ \ \alpha\in[1,2) \end{cases} & \text{on} \quad (0,T), \\ u(\cdot,0) = u_0 & \text{in} \quad (0,1), \end{cases} \tag{2.2}$$

where $\alpha\in(0,2)$, $T>0$, $u_0\in L^2(0,1)$, $g\in L^2((0,1)\times(0,T))$, $\mathcal{O}\subset(0,1)$ is a non-empty open set, and $1_{\mathcal{O}}$ is the associated characteristic function. The global null controllability of this system is proved using Carleman estimates with appropriate weight functions [1].

Recently, some results on 2D degenerate parabolic equations have appeared in [10, 11]. There, the authors study the well-posedness and the global null controllability of the following system:

$$\begin{cases} u_t - div(A_0 \nabla u) + b_0 u = g_0 1_{\omega_0} & \text{in} \quad Q_0, \\ \begin{cases} u = 0 \quad \text{if} \quad \alpha \in (0,1) \\ \dfrac{\partial u}{\partial \nu} = 0 \quad \text{if} \quad \alpha \in [1,2) \end{cases} & \text{on} \quad \Sigma_0, \\ u(\cdot, 0) = u_0 & \text{in} \quad \Omega_0, \end{cases} \tag{2.3}$$

where $Q_0 := \Omega_0 \times (0,T)$, $T > 0$, $\Omega_0 \subset \mathbb{R}^2$ is a bounded domain with boundary Γ_0 of class C^4, $\Sigma_0 := \Gamma_0 \times (0,T)$, $\omega_0 \subset \Omega_0$ is open, $u_0 \in L^2(\Omega_0)$, $g_0 \in L^2(Q_0)$, $b_0 \in L^\infty(Q_0)$, $\alpha \in (0,2)$, and $A_0 : \overline{\Omega}_0 \mapsto M_{2\times 2}(\mathbb{R})$ satisfies the following conditions:

(i) $A_0(x) = \{a_{ij}(x)\}$, with the $a_{ij} \in C^3(\Omega_0; \mathbb{R}) \cap C^0(\overline{\Omega}_0; \mathbb{R})$;

(ii) $A_0(x)$ is simetric for all $x \in \overline{\Omega}_0$ and positive definite for all $x \in \Omega_0$;

(iii) Let $r_i(x)$ be the eigenvalues and let $\varepsilon_i(x)$ be the associated unit-norm eigenvectors of $A_0(x)$ for $i = 1, 2$. Let us denote by $P_{\Gamma_0}(x)$ the projection of x onto the boundary Γ_0 and $O(\Gamma_0; \delta) := \{x \in \Omega_0 : d(x, \Gamma_0) < \delta\}$. There exists $\delta > 0$ such that the following holds:

   1. $r_1(x) = d(x, \Gamma_0)^\alpha \quad \forall x \in O(\Gamma_0; \delta)$,
   2. $r_2(x) > 0 \quad \forall x \in \overline{\Omega_0 \backslash O(\Gamma_0; \delta)}$;
   3. $\varepsilon_1(x) = \nu(P_{\Gamma_0}(x)) \;\; \forall x \in O(\Gamma_0; \delta)$.

It is convenient to enhance the main differences between systems (2.3) and (2.1). First, contrarily to (2.1), (2.3) degenerates on the whole border. This is crucial, because Carleman estimates need weight functions with a specific behavior near the part of the boundary where the degeneracy occurs. The techniques used in [10] can be extended to problems where the degeneracy appears on a part of the boundary, but this part must be separated from the rest. A typical example is when Ω_0 is an annulus and the coefficients degenerate just one of the components of the boundary. Hence, a new thecnique is required to deal with (2.1). Here, the main idea is to use the control domain to "separate" the part of the boundary where the degeneracy does not occur from a neighborhood of the origin. Note that not only one but the two eigenvalues of the matrix A can degenerate.

In this work, we will prove the well-posedness of (2.1) using semigroup theory in all cases $\alpha_i \in [0,2]$. However, the proof of Carleman estimates requires different techniques and weights in each case; therefore, for this purpose, we will strict to the case $\alpha_1,\ \alpha_2 \in (0,2)$. The other cases will be considered in a forthcoming paper.

It will be seen that, as an application of Carleman estimates, we can prove results on the null controllability and Stackelberg-Nash controllability for linear and semilinear degenerate parabolic systems.

This paper is organized as follows. In Section 2.2, we will present some results on the degenerate operator associated to (2.1) and we will deduce the well posedness. In Section 2.3, we will present Carleman estimates for the solutions to the adjoint of (2.1) when $\alpha_i \in (0,2)$. This will allow to prove results concerning the null controllability of linear and semilinear problems of the kind (2.1). In Section 2.4, we will use these Carleman estimates to prove the Stackelberg-Nash controllability. In Section 2.5, we will present some extensions of the results, open questions and future work. Finally, the paper contains an Appendix where the proof of the Carleman estimate presented in the Section 2.3 is given.

2.2 Preliminar results and well-posedness

2.2.1 Notations, spaces and operators

The usual norm and inner product in $L^2(\Omega)$ and in $L^2(Q)$ will be denoted respectively by $|\cdot|$ and $(\cdot\,,\cdot)$, and $\|\cdot\|$ and $(\!(\cdot\,,\cdot)\!)$. The norms in $L^\infty(\Omega)$ and $L^\infty(Q)$ will be denoted respectively by $|\cdot|_\infty$ and $\|\cdot\|_\infty$.

Now, let us introduce some matrices, spaces, and operators.

- $A^r(x) := diag(x_1^{\alpha_1 r}, x_2^{\alpha_2 r})$, with $r \in \mathbb{R}$,
- $H^1_\alpha(\Omega) := \{u \in L^2(\Omega) : \nabla u A \nabla u \in L^1(\Omega)\}$,
- $H^2_\alpha(\Omega) := \{u \in H^1_\alpha(\Omega) : div(A\nabla u) \in L^2(\Omega)\}$,

- $H^{div}(\Omega) := \{w \in L^2(\Omega)^2 : div(w) \in L^2(\Omega)\}$,
- $L^2_{\alpha^{-1}}(\Omega) := \{w \in L^2(\Omega)^2 : wA^{-1}w \in L^1(\Omega)\}$,
- $H^{div}_\alpha(\Omega) := \{w \in L^2_{\alpha^{-1}}(\Omega) : div(w) \in L^2(\Omega)\}$,
- $\nabla_0 u := A^{1/2}\nabla u, \quad u \in H^1_\alpha(\Omega)$,
- $\Delta_0 u := div(A\nabla u), \quad u \in H^2_\alpha(\Omega)$.

Now, let us consider the following norms:

- $|u|_\alpha := (|u|^2 + |\nabla_0 u|^2)^{1/2}, \quad u \in H^1_\alpha(\Omega)$,
- $|u|_{2,\alpha} := (|u|^2_\alpha + |\triangle_0 u|^2)^{1/2}, \quad u \in H^2_\alpha(\Omega)$,
- $|w|_{div,\alpha} := \left(\int_\Omega wA^{-1}w\,dx + |div(w)|^2\right)^{1/2}, \quad w \in H^{div}_\alpha(\Omega)$.

Note that, for these (natural) norms, $H^1_\alpha(\Omega)$, $H^2_\alpha(\Omega)$ and $H^{div}_\alpha(\Omega)$ are Hilbert spaces and one has the following continuous embbedings:

$$H^1(\Omega) \hookrightarrow H^1_\alpha(\Omega) \hookrightarrow L^2(\Omega), \quad H^2_\alpha(\Omega) \hookrightarrow H^1_\alpha(\Omega), \quad H^{div}_\alpha(\Omega) \hookrightarrow H^{div}(\Omega).$$

Furthermore, $H^1_\alpha(\Omega) \subset H^1_{loc}(\Omega)$ and $H^2_\alpha(\Omega) \subset H^2_{loc}(\Omega)$.

Lemma 2.2.1 *$C^\infty(\overline{\Omega})$ is dense in $H^1_\alpha(\Omega)$.*

Proof. We know that $H^1(\Omega) \hookrightarrow H^1_\alpha(\Omega)$ and that $C^\infty(\overline{\Omega})$ is dense in $H^1(\Omega)$. Hence, it is sufficient to prove that $H^1(\Omega)$ is dense in $H^1_\alpha(\Omega)$. Let us fix $u \in H^1_\alpha(\Omega)$. For $\lambda > 1$ let us introduce the diffeomorphism $f_\lambda : \Omega \to \Omega_\lambda$ where

$$f_\lambda(x) = \frac{x}{\lambda} + x_{0,\lambda} \;\text{ and }\; x_{0,\lambda} = \frac{1}{2\lambda}(\lambda-1, \lambda-1).$$

It is clear that $\Omega_\lambda \subset\subset \Omega$ and $|\Omega_\lambda| \to |\Omega|$, as $\lambda \to 1$.

Now let us introduce

$$u_\lambda(x) := \begin{cases} u(x) & \text{if} \quad x \in \Omega_\lambda \\ u(f_\lambda(x)) & \text{if} \quad x \in \Omega\backslash\Omega_\lambda. \end{cases}$$

Using that $H^1_\alpha(\Omega) \subset H^1_{loc}(\Omega)$ and $\Omega_\lambda \subset\subset \Omega$ we deduce that $u_\lambda \in H^1(\Omega)$. Furthermore,

$$\begin{aligned} |u_\lambda - u|^2_\alpha &= \int_{\Omega\backslash\Omega_\lambda} |u(f_\lambda(x)) - u(x)|^2 + |\nabla_0(u\circ f_\lambda - u)(x)|^2\,dx \\ &\le \int_{f_\lambda(\Omega\backslash\Omega_\lambda)} (\lambda^2|u|^2 + |\nabla_0 u|^2)\,dx + \int_{\Omega\backslash\Omega_\lambda} (|u|^2 + |\nabla_0 u|^2)\,dx \to 0 \;\text{ as }\; \lambda \to 1. \end{aligned}$$

Therefore, $H^1(\Omega)$ is dense in $H^1_\alpha(\Omega)$. ■

Lemma 2.2.1 leads to the following definition

$$H^1_{\alpha,0}(\Omega) := \overline{\mathcal{D}_0}^{H^1_\alpha(\Omega)}, \tag{2.4}$$

where the definition of the space $\mathcal{D}_0$ depends on α:

$$\mathcal{D}_0 := \begin{cases} \{v \in C^\infty(\overline{\Omega}) : supp(v) \subset\subset \Omega\} & \text{if} \quad \alpha_1,\ \alpha_2 \in [0,1), \\ \{v \in C^\infty(\overline{\Omega}) :\ \exists \delta > 0; supp(v) \subset (0, 1-\delta) \times (0, 1-\delta)\} & \text{if} \quad \alpha_1,\ \alpha_2 \in [1,2], \\ \{v \in C^\infty(\overline{\Omega}) :\ \exists \delta > 0; supp(v) \subset (\delta, 1-\delta) \times (0, 1-\delta)\} & \text{if} \quad \alpha_1 \in [0,1),\ \ \alpha_2 \in [1,2], \\ \{v \in C^\infty(\overline{\Omega}) :\ \exists \delta > 0; supp(v) \subset (0, 1-\delta) \times (\delta, 1-\delta)\} & \text{if} \quad \alpha_1 \in [1,2],\ \ \alpha_2 \in [0,1). \end{cases}$$

2.2.2 Trace operators

We know that the trace operator $T : H^1(\Omega) \mapsto L^2(\Gamma)$ is continuous and

$$H^{1/2}(\Gamma) := T(H^1(\Omega))$$

is a Hilbert space for the norm

$$|v|_{H^{1/2}(\Gamma)} := \inf\{|u|_{H^1(\Omega)} : u \in H^1(\Omega),\ \ T(u) = v\}.$$

Moreover, there exists a unique normal trace operator $T_\nu : H^{div}(\Omega) \mapsto H^{-1/2}(\Gamma)$, which is continuous and satisfies

- $T_\nu(w) = (w \cdot \nu)|_\Gamma \ \forall w \in C^\infty(\overline{\Omega})^2$,
- $\displaystyle\int_\Omega (div(w)u + w \cdot \nabla u)\, dx = \langle T_\nu(w), T(u)\rangle \ \ \forall w \in H^{div}(\Omega) \ \ \forall u \in H^1(\Omega)$.

Given $w \in H^{div}(\Omega)$, we can introduce a continuous linear form $\mathcal{T}_w : H^1(\Omega) \mapsto \mathbb{R}$ by putting

$$\mathcal{T}_w(u) := \int_\Omega (div(w)u + w \cdot \nabla u)\, dx \ \ \forall u \in H^1(\Omega).$$

Moreover, we have

$$\mathcal{T}_w(u) = \langle T_\nu(w), T(u)\rangle.$$

Now, for any $\delta > 0$, we introduce

- $\Omega_\delta := \{x \in \Omega : d(x,\Gamma) > \delta\}$ and $\Gamma_\delta := \partial\Omega_\delta$,
- $r_\delta : H^1_\alpha(\Omega) \mapsto H^1(\Omega)$, with $r_\delta(u) := u|_{\Omega_\delta}$,

- $R_\delta : H^{div}_\alpha(\Omega) \mapsto H^{div}(\Omega)$, with $R_\delta(w) = w|_{\Omega_\delta}$,
- $T^\delta := T_\delta \circ r_\delta$, where $T_\delta : H^1(\Omega_\delta) \mapsto H^{1/2}(\Gamma_\delta)$ is the trace operator,
- $T^\delta_\nu := T_{\nu,\delta} \circ R_\delta$, where $T_{\nu,\delta} : H^{div}(\Omega_\delta) \mapsto H^{-1/2}(\Gamma_\delta)$ is the normal trace operator.

Lemma 2.2.2 *If $w \in H^{div}_\alpha(\Omega)$, then the functional $\mathcal{T}_w$ can be continuously extended to the space $H^1_\alpha(\Omega)$. Moreover,*

$$\mathcal{T}_w(u) = \lim_{\delta \to 0} \langle T^\delta_\nu(w), T^\delta(u) \rangle \quad \forall u \in H^1_\alpha(\Omega).$$

Proof. For $u \in H^1_\alpha(\Omega)$ we have

$$\begin{aligned} \left| \int_\Omega (div(w)u + w \cdot \nabla u)dx \right| &= \left| \int_\Omega (div(w)u + [(A^{1/2})^{-1}w][A^{1/2}\nabla u])dx \right| \\ &\leq |div(w)||u| + |(A^{1/2})^{-1}w||A^{1/2}\nabla u| \\ &\leq (|(A^{1/2})^{-1}w| + |div(w)|)(|u| + |A^{1/2}\nabla u|) \leq |w|_{div,\alpha}|u|_\alpha. \end{aligned}$$

This conclude the proof. ■

In view to obtain appropriate results on the normal traces in $H^{div}_\alpha(\Omega)$, we will consider the cases $\alpha_i \in [0,1)$ and $\alpha_i \in [1,2]$ separately.

The case $\alpha_1,\ \alpha_2 \in [0,1)$

The classical results on the normal trace theory remain true for the spaces $H^{div}_\alpha(\Omega)$ and $H^1_\alpha(\Omega)$.

Lemma 2.2.3 *There exists a unique trace operator $T^\alpha : H^1_\alpha(\Omega) \mapsto L^2(\Gamma)$ which extends $T : H^1(\Omega) \mapsto L^2(\Gamma)$. Moreover T^α is continuous and $H^{1/2}_\alpha(\Omega) := T^\alpha(H^1_\alpha(\Omega))$ is a Hilbert space equipped with the norm*

$$|v|_{H^{1/2}_\alpha(\Gamma)} := \inf\{|u|_\alpha : u \in H^1_\alpha(\Omega) \;\; and \;\; T^\alpha(u) = v\}.$$

Proof. We know that $C^\infty(\overline{\Omega})$ is dense in $H^1_\alpha(\Omega)$. Consequently, it is suficient to prove that $T : (C^\infty(\overline{\Omega}), |\cdot|_\alpha) \to L^2(\Gamma)$ is continuous. First let us consider the following open cover $U_0, ..., U_4$ of Ω, where $U_0 := B_{1/4}((1/2,1/2))$, $U_1 := B_{1/2}((0,1/2))$, $U_2 := B_{1/2}((1/2,0))$, $U_3 := B_{1/2}((1,1/2))$ and $U_4 := B_{1/2}((1/2,1))$. Let $\varphi_0, ..., \varphi_4 \in C^\infty(\overline{\Omega})$ be a partition of unity subordinate to the cover $\{U_0, ..., U_4\}$, i.e. $\varphi_0 + ... + \varphi_4 = 1$,

$0 \leq \varphi \leq 1$, *supp* φ is a compact set and *supp* $\varphi \subset U_i$. Given $u \in C^\infty(\overline{\Omega})$, let $u_i = \varphi^{1/2}u$. Using that $\alpha_i \in [0,1)$ we deduce that

$$|u_1(0,x_2)| = \left| \int_0^{3/4} x_1^{\alpha_1/2} \frac{\partial u_1}{\partial x_1} x_1^{-\alpha_1/2}\, dx_1 \right| \leq C \left(\int_0^1 x_1^{\alpha} \left| \frac{\partial u_1}{\partial x_1} \right|^2 dx_1 \right)^{1/2}.$$

Hence

$$\int_{\Gamma_1} |u_1|^2\, ds \leq C \int_\Omega x_1^{\alpha_1} \left| \frac{\partial u_1}{\partial x_1} \right|^2 dx \leq C|u|_\alpha^2.$$

In a similar way we conclude that

$$\int_{\Gamma_i} |u_i|^2\, ds \leq C|u|_\alpha^2, \quad i = 1, ..., 4.$$

Therefore

$$|T(u)|^2 = \int_\Gamma |u|^2\, ds = \sum_{i=0}^4 \int_\Gamma \varphi_i |u|^2\, ds = \sum_{i=0}^4 \int_{\Gamma_i} |u_i|^2\, ds \leq C|u|_\alpha^2.$$

This conclude the proof. ■

The following result is a fundamental tool, not only to deduce the existence of a normal trace operator, but also to prove the Carleman estimates present in the next section.

Lemma 2.2.4 (Hardy inequality I) *Assume that* $\alpha_1, \alpha_2 \in [0,1)$. *There exists a positive constant* $C = C(\alpha_1, \alpha_2)$ *such that*

$$\int_\Omega x_i^{\alpha_i - 2} |u|^2\, dx \leq C \int_\Omega x_i^{\alpha_i} \left| \frac{\partial u}{\partial x_i} \right|^2 dx \quad \forall u \in ker(T^\alpha).$$

Before proving this Hardy Inequality, we consider the following result:

Lemma 2.2.5 *Assume* $\alpha_1, \alpha_2 \in [0,1)$. *There exist* $C = C(\alpha) > 0$ *such that, for any* $u \in C^\infty(\overline{\Omega})$, *one has*

$$\int_\Omega x_1^{\alpha_1 - 2} |u(x) - u(x_1, 0)|^2\, dx \leq C \int_\Omega x_1^{\alpha_1} \left| \frac{\partial u}{\partial x_1} \right|^2 dx$$

and

$$\int_\Omega x_2^{\alpha_2 - 2} |u(x) - u(0, x_2)|^2\, dx \leq C \int_\Omega x_2^{\alpha_2} \left| \frac{\partial u}{\partial x_2} \right|^2 dx.$$

Proof. Let us fix $\max\{\alpha_1,\alpha_2\} < a < b < 1$. We have that

$$\begin{aligned}
\int_\Omega x_2^{\alpha_2-2}|u(x)-u(0,x_2)|^2\,dx &= \int_0^1\left[\int_0^1\left(\int_0^{x_2} t^{b/2}\frac{\partial u}{\partial x_2}(x_1,t)t^{-b/2}\,dt\right)^2 x_2^{\alpha_2-2}\,dx_2\right]dx_1\\
&\le \int_0^1\left[\int_0^1\left(\int_0^{x_2} t^{b}\left|\frac{\partial u}{\partial x_2}(x_1,t)\right|^2 dt\right)\left(\int_0^{x_2} t^{-b}\,dt\right) x_2^{\alpha_2-2}\,dx_2\right]dx_1\\
&= \frac{1}{1-b}\int_0^1\left[\int_0^1\left(\int_0^{x_2} t^{b}\left|\frac{\partial u}{\partial x_2}(x_1,t)\right|^2 dt\right) x_2^{\alpha_2-1-b}\,dx_2\right]dx_1\\
&\le \frac{1}{1-b}\int_0^1\left[\int_0^1 t^{b}\left|\frac{\partial u}{\partial x_2}(x_1,t)\right|^2 \left(\int_t^1 x_2^{\alpha_2-1-b}\,,dx_2\right)dt\right]dx_1
\end{aligned}$$

and

$$\begin{aligned}
\int_1^t x_2^{\alpha_2-1-b}\,dx_2 &= \int_t^1 x_2^{\alpha_2-a}.x_2^{a-1-b}dx_2 \le t^{\alpha_2-a}\int_t^1 x_2^{a-1-b}dx_2\\
&= \frac{t^{\alpha_2-a}}{a-b}(1-t^{a-b}) \le \frac{t^{\alpha_2-b}}{b-a}.
\end{aligned}$$

Thus

$$\int |u(x)-u(0,x_2)|^2 x_2^{\alpha_2-2}\,dx \le C\int_\Omega x_2^{\alpha_2}\left|\frac{\partial u}{\partial x_2}\right|^2 dx.$$

The other inequality can be proved in a similar way. ■

Now, we will return to the Lemma 2.2.4 to establish its proof.

Proof. [*Proof of Lemma 2.2.4*]

Now, let us fix $u \in \ker(T^\alpha)$ and $\delta > 0$. There exists a sequence $u_n \in C^\infty(\overline{\Omega})$ such that $u_n \to u$ in $H^1_\alpha(\Omega)$. Let us introduce two auxiliary sequences given by $u_{1,n}(x) = u_n(x_1,0)$ and $u_{2,n}(x) = u_n(0,x_2)$. From $T^\alpha(u_n) \to 0$ in $L^2(\Gamma)$ we deduce that $u_{i,n} \to 0$ in $L^2(\Omega)$. Using Lemma 2.2.5 we have that

$$\int_{\Omega_\delta} |u_n - u_{i,n}|^2 x_i^{\alpha_i-2}\,dx \le C\int_\Omega \left|\frac{\partial u_n}{\partial x_i}\right|^2 x_i^{\alpha_i}\,dx.$$

Passing to the limit $n \to +\infty$ we obtain that

$$\int_{\Omega_\delta} |u|^2 x_i^{\alpha-2}\,dx \le C\int_\Omega \left|\frac{\partial u}{\partial x_i}\right|^2 x_i^{\alpha}\,dx, \quad \forall \delta > 0.$$

This conclude the proof. ■

From Lemma 2.2.4 the following result holds:

Lemma 2.2.6 *ker*$(T^\alpha) = H^1_{\alpha,0}(\Omega)$ *(recall the definition of this space in (2.4)).*

Proof. It is clear that $H^1_{\alpha,0}(\Omega) \subset \ker(T^\alpha)$. Let us fix $u \in \ker(T^\alpha)$ and introduce $f(x) := x_1 x_2$, $f_n(x) := min\{nf(x), 1\}$, $u_n := f_n u$ and $\Omega_n := \{x \in \Omega : f_n(x) = 1\}$. It is clear that $f_n = 0$ on Γ, $\Omega_n \subset\subset \Omega_{n+1}$ and $|\Omega_n| \to |\Omega|$. Furthermore,

$$\int_\Omega (|u_n|^2 + |\nabla u_n|^2)\, dx \ \leq \ C(n) \left(\int_\Omega |u|^2\, dx + \int_\Omega |\nabla_0 u|^2\, dx \right) < +\infty$$

and $u_n \in H^1(\Omega)$. On the other hand $T(u_n) = T^\alpha(u_n) = f_n|_\Gamma T^\alpha(u) = 0$, then $u_n \in \ker(T) = H^1_0(\Omega)$.

Moreover, we have that

$$\begin{aligned} \int_\Omega (|u-u_n|^2 + |\nabla_0(u-u_n)|^2)\, dx &= \int_{\Omega\setminus\Omega_n} (|u - f_n u|^2 + |\nabla_0(u - f_n u)|^2)\, dx \\ &\leq C \int_{\Omega\setminus\Omega_n} (|u|^2 + |\nabla_0 u|^2)\, dx + n^2 \int_{\Omega\setminus\Omega_n} |\nabla_0 f|^2 |u|^2\, dx. \qquad (2.5) \end{aligned}$$

The first integral in the right hand side of (2.5) tends to 0 as $n \to +\infty$. With respect to the second integral, using Lemma 2.2.4 we have that

$$\begin{aligned} n^2 \int_{\Omega\setminus\Omega_n} |\nabla_0 f|^2 |u|^2\, dx &\leq \int_{\Omega\setminus\Omega_n} (x_1x_2)^{-2}(x_1^{\alpha_1} x_2^2 + x_2^{\alpha_2} x_1^2)|u|^2\, dx \\ &\leq \int_\Omega (x_1^{\alpha_1-2} + x_2^{\alpha_2-2})|u|^2\, dx \to 0. \end{aligned}$$

We conclude that $u_n \to u$ in $H^1_\alpha(\Omega)$ and, consequently, $\ker(T^\alpha) \subset \overline{H^1_0(\Omega)}^{H^1_\alpha(\Omega)} \subset \overline{H^1_{\alpha,0}(\Omega)}^{H^1_\alpha(\Omega)} = H^1_{\alpha_0}(\Omega)$. ■

As a consequence of Lemmas 2.2.4 and 2.2.6, the space $H^1_{\alpha,0}(\Omega)$ can be endowed with the norm

$$|u|_{\alpha,0} := \int_\Omega |\nabla_0 u|^2\, dx,$$

that is equivalent to the norm $|\cdot|_\alpha$ on $H^1_{\alpha,0}(\Omega)$.

Lemma 2.2.7 *There exists a unique normal trace operator $T^\alpha_\nu : H^{div}_\alpha(\Omega) \mapsto H^{-1/2}(\Gamma)$ that is continuous and satisfies*

- $T^\alpha_\nu(w) = (w \cdot \nu)|_\Gamma \ \forall w \in C^\infty(\overline{\Omega})^2$,
- $\mathcal{T}_w(u) = \langle T^\alpha_\nu(w), T^\alpha(u)\rangle \ \ \forall w \in H^{div}_\alpha(\Omega) \ \ \forall u \in H^1_\alpha(\Omega)$.

To prove Lemma 2.2.7, before we need to stablish some preliminaries results.

Lemma 2.2.8 *Assume that $\alpha_i \in [0,1)$. Then $H^1_\alpha(\Omega) \hookrightarrow W^{1,1}(\Omega)$.*

Proof. From Hölder inequality

$$\int_\Omega \left|\frac{\partial u}{\partial x_i}\right| dx \leq \left[\int_\Omega x_i^{-\alpha_i}\, dx. \int_\Omega x_i^{\alpha_i} \left|\frac{\partial u}{\partial x_i}\right|^2 dx\right] < \infty \quad \forall u \in H^1_\alpha(\Omega).$$

This conclude the proof. ■

Lemma 2.2.9 *Assume $\alpha_i \in [0,1)$. Given $u \in H^1_\alpha(\Omega)$, let us introduce*

$$\tilde{u}(x) := \begin{cases} u(x) & \text{if} \quad x \in \Omega \\ 0 & \text{if} \quad x \in \mathbb{R}^2 \backslash \Omega. \end{cases}$$

If $\tilde{u} \in W^{1,1}(\mathbb{R}^2)$, then $u \in H^1_{\alpha,0}(\Omega)$.

Proof. From Lemma 2.2.8, we have that $H^1_{\alpha,0}(\Omega) = H^1_\alpha(\Omega) \cap W^{1,1}_0(\Omega)$. Hence, it is suficient to show that $u \in W^{1,1}_0(\Omega)$.

Given $\lambda > 1$ let us introduce the diffeomorphism $f_\lambda : \Omega \to \Omega_\lambda$ where

$$f_\lambda(x) = \lambda x - x_{1,\lambda} \;\text{ and }\; x_{0,\lambda} := \frac{1}{2\lambda}(\lambda - 1, \lambda - 1).$$

It is clear that $\Omega \subset\subset \Omega_\lambda$ and $|\Omega_\lambda| \to |\Omega|$ as $\lambda \to 1$.

Now let us introduce

$$u_\lambda(x) := \begin{cases} u(x) & \text{if} \quad x \in f_\lambda^{-1}(\Omega) \\ \tilde{u}(f_\lambda(x)) & \text{if} \quad x \notin f_\lambda^{-1}(\Omega). \end{cases}$$

It is clear that $supp(u_\lambda) \subset\subset \Omega$, whence it remains to conclude that $u_\lambda \in W^{1,1}(\Omega)$ and $u_\lambda \to u$ in $W^{1,1}(\Omega)$. In fact, we have that

$$\begin{aligned} |u_\lambda|_{W^{1,1}(\Omega)} &= \int_{f_\lambda^{-1}(\Omega)} (|u| + |\nabla u|)\, dx + \int_{\Omega \backslash f_\lambda^{-1}(\Omega)} (|u(f_\lambda(x))| + |\nabla u(f_\lambda(x))|)\, dx \\ &\leq |u|_\alpha + \int_{f_\lambda(\Omega \backslash f_\lambda^{-1}(\Omega))} (|u|\lambda^2 + |\nabla u|\lambda^3)\, dx < +\infty. \end{aligned}$$

Hence, $u_\lambda \in W^{1,1}(\Omega)$. Moreover, arguing as in the proof of Lemma 2.2.1 we conclude that $u_\lambda \to u$ in $W^{1,1}(\Omega)$. ■

Lemma 2.2.10 *Assume that the $\alpha_i \in [0,1)$. Then $C^\infty(\overline{\Omega})^2$ is dense in $H^{div}_\alpha(\Omega)$.*

Proof. Let us denote by $(\cdot,\cdot)_{\alpha,div}$ the inner product of $H^{div}_\alpha(\Omega)$. Let us fix $w \in (C^\infty(\overline{\Omega})^2)^\perp := \{w \in H^{div}_\alpha(\Omega) : (w,v)_{\alpha,div} = 0 \;\forall v \in C^\infty(\overline{\Omega})^2\}$ and $u = div(w) \in L^2(\Omega)$. We have that

$$\int_\Omega u\mathrm{div}(v)\, dx = -\int_\Omega (A^{-1}w)v\, dx \quad \forall v \in C^\infty(\overline{\Omega})^2.$$

Thus $\nabla u = A^{-1}w$ in the sense of distributions. Consequently, $A\nabla u = w \in L^2_{\alpha^1}(\Omega)$, that is to say, $\nabla u A \nabla u \in L^1(\Omega)$. Then $u \in H^1_\alpha(\Omega)$. Moreover, since $\alpha_i \in [0,1)$, we deduce that $\nabla u = A^{-1}w \in L^1(\Omega)$ and $\widetilde{A^{-1}w} \in L^1(\mathbb{R}^2)^2$. Using that $\nabla \tilde{u} = \widetilde{A^{-1}w}$ in the sense of distributions we conclude that $\tilde{u} \in W^{1,1}(\mathbb{R}^2)$. From Lemma A.3, we see that $u \in H^1_{\alpha,0}(\Omega)$. Consequently there exists a sequence $u_n \in C_0^\infty(\overline{\Omega})$ such that $u_n \to u$ in $H^1_\alpha(\Omega)$.

Using that $H^{div}_\alpha(\Omega) \subset H^{div}(\Omega)$, from standard normal trace theory we get

$$\int_\Omega (v\nabla u_n + div(v)u_n)\,dx = 0 \quad \forall v \in H^{div}_\alpha(\Omega).$$

Consequently, for any fixed $v \in H^{div}_\alpha(\Omega)$, we have that

$$\begin{aligned} |(w,v)_{\alpha,div}| &= \left|\int_\Omega (wA^{-1}v + \mathrm{div}(w)\mathrm{div}(v))\,dx\right| = \left|\int_\Omega (v\nabla u + u\mathrm{div}(v))\,dx\right| \\ &= \left|\int_\Omega (v\nabla(u-u_n) + (u-u_n)\mathrm{div}(v))\,dx\right| \\ &\leq \int_\Omega (|A^{1/2}\nabla(u-u_n)||(A^{1/2})^{-1}v| + |u-u_n||div(v)|)\,dx \\ &\leq |u-u_n|_\alpha |v|_{div,\alpha} \to 0. \end{aligned}$$

Therefore $w = 0$ and thus $(C^\infty(\overline{\Omega})^2)^\perp = \{0\}$. ■

Finally, we are ready to give the proof of Lemma 2.2.7.

Proof. [*Proof of Lemma 2.2.7*]

Let $T_\nu : H^{div}(\Omega) \mapsto H^{-1/2}(\Gamma)$ be the standard normal trace operator. We know that $H^{div}_\alpha(\Omega) \subset H^{div}(\Omega)$. Then, from Lemma A.4 it is sufficient to prove that $T_\nu : (C^\infty(\overline{\Omega})^2, |\cdot|_{div,\alpha}) \mapsto H_\alpha^{-1/2}(\Gamma)$ is continuous. For this purpose, let us fix $w \in C^\infty(\overline{\Omega})^2$.

Given $u \in H^1_\alpha(\Omega)$ there exists a sequence $u_n \in C^\infty(\overline{\Omega})$ such that $u_n \to u$ in $H^1_\alpha(\Omega)$. We have that $\mathcal{T}_w(u_n) = \langle T_\nu(w), T^\alpha(u_n)\rangle$ and from Lemma 2.2.1 we get $\mathcal{T}_w(u) = \langle T_\nu(w), T^\alpha(u)\rangle$. This way, we can deduce that

$$|\langle T_\nu(w), T^\alpha(u)\rangle| \leq |w|_{div,\alpha}.|u|_\alpha \quad \forall u \in H^1_\alpha(\Omega). \tag{2.6}$$

Given $v \in H_\alpha^{1/2}(\Gamma)$, there exists a sequence $u_n \in H^1_\alpha(\Omega)$ such that $T^\alpha(u_n) = v$ and $|u_n|_\alpha \to |v|_{H_\alpha^{1/2}(\Gamma)}$. From (2.6) we get:

$$|\langle T_\nu(w), v\rangle| \leq |w|_{div,\alpha}.|u_n|_\alpha.$$

Consequently $|\langle T_\nu(w), v\rangle| \leq |w|_{div,\alpha}.|v|_{H_\alpha^{1/2}(\Gamma)}$ $\forall v \in H_\alpha^{1/2}(\Gamma)$ and therefore $|T_\nu(w)|_{H_\alpha^{-1/2}(\Gamma)} \leq |w|_{div,\alpha}$. Hence $T_\nu : (C^\infty(\overline{\Omega})^2, |\cdot|_{div,\alpha}) \mapsto L^2(\Gamma)$ is continuous. ■

The case where $\alpha_i \in [1,2]$ for some i

Now, let us assume that $\alpha_i \in [1,2]$ for some $i \in \{1,2\}$. It is no possible to prove the existence of a trace operator, but we have the following:

Lemma 2.2.11 *For any $w \in H^{div}_\alpha(\Omega)$, one has*

$$\int_\Omega (w \cdot \nabla u + div(w)u)\, dx = 0 \quad \forall u \in H^1_{\alpha,0}(\Omega).$$

Before to prove Lemma 2.2.11, we will stablish the following result:

Lemma 2.2.12 *Assume $w = (w,1,w_2) \in H^{div}(\Omega)$ and $u \in \mathcal{D}_0$. The functions $f, g : [0,1/2] \longrightarrow \mathbb{R}$, given by*

$$f(\delta) := \int_{\Gamma_\delta} |T^\delta u|^2\, ds \quad and \quad g(\delta) := \langle T^\delta_\nu(w), T^\delta(u)\rangle$$

are continuous in $\delta = 0$.

Proof. There exists $\delta_0 > 0$ such that $supp(u) \subset (0,1-\delta_0) \times (0,1-\delta_0)$. Thus, $f(\delta) = f_1(\delta) + f_2(\delta)$ $\forall \delta \in [0,\delta_0/2]$ where

$$f_1(\delta) := \int_\delta^{1-\delta} |u(\delta,x_2)|^2\, dx_2 \quad and \quad f_2(\delta) := \int_\delta^{1-\delta} |u(x_1,\delta)|^2\, dx_1.$$

Given $\delta_1, \delta_2 \in [0,\delta_0/2]$, with $\delta_1 < \delta_2$, we have that

$$\begin{aligned}
|f_1(\delta_1) - f_1(\delta_2)| &\le \left| \int_{\delta_2}^{1-\delta_2} (|u(\delta_1,x_2)|^2 - |u(\delta_2,x_2)|^2)\, dx_2 \right| + \int_{\delta_1}^{\delta_2} |u(\delta_1,x_2)|^2\, dx_2 \\
&\quad + \int_{1-\delta_2}^{1-\delta_1} |u(\delta_1,x_2)|^2\, dx_2 \\
&\le |u|_{L^\infty(\Omega)}|\delta_1-\delta_2| + \left| \int_{\delta_2}^{1-\delta_2} \int_{\delta_1}^{\delta_2} \frac{\partial}{\partial x_1}(|u|^2)\, dx_1\, dx_2 \right| \\
&\le (|u|_{L^\infty}(\Omega) + |\nabla u|_{L^\infty(\Omega)})|\delta_1 - \delta_2|.
\end{aligned}$$

Hence, f_1 is continuous. In a similar way we deduce that f_2 is continuous. Therefore f is continuous in $[0,\delta_0/2]$.

Now let us fix $\delta_1, \delta_2 \in [0,\delta_0/2]$ with $\delta_1 < \delta_2$. From standard normal trace theory,

$$|g(\delta_1) - g(\delta_2)| \le \int_{\Omega_{\delta_1} \setminus \Omega_{\delta_2}} (div(w)u + w.\nabla u)\, dx \le |w|_{H^{div}(\Omega_{\delta_1}\setminus\Omega_{\delta_2})}.|u|_{H^1(\Omega_{\delta_1}\setminus\Omega_{\delta_2})}.$$

Thus, since $|\Omega_{\delta_1} \setminus \Omega_{\delta_2}| \to 0$ as $|\delta_1 - \delta_2| \to 0$ the conclusion follows.

■

Now we will to come back to proof of Lemma 2.2.11.

proof of Lemma 2.2.11.

To fix ideas, we will present the proof only in the case $\alpha_1, \alpha_2 \in [1,2]$. The other cases are similar.

Let us fix $w = (w_1, w_2) \in C^\infty(\overline{\Omega})^2$, $u \in \mathcal{D}_{01}$, $\delta_0 > 0$ such that $supp(u) \subset (0, 1-\delta_0) \times (0, 1-\delta_0)$ and $\delta_1, \delta_2 \in (0, \delta_0/2)$ with $\delta_1 < \delta_2$. We have that

$$\begin{aligned}\int_{\delta_1}^{\delta_2} \frac{1}{\delta} |\langle T_\nu^\delta(w), T^\delta(u)\rangle|^2 \, d\delta \;&\leq\; \int_{\delta_1}^{\delta_2} \frac{1}{\delta} \left(\int_\delta^{1-\delta} |w_1(\delta, x_2)|^2 \, dx_2 . \int_\delta^{1-\delta} |u(\delta, x_2)|^2 \, dx_2 \right. \\ &\quad \left. + \int_\delta^{1-\delta} |w_2(x_1, \delta)|^2 \, dx_1 . \int_\delta^{1-\delta} |u(x_1, \delta)|^2 \, dx_1 \right) d\delta \\ &\leq\; |u|^2_{L^\infty(\Omega)} \left(\int_{(\delta_1,\delta_2)\times(0,1)} x_1^{-1} |w_1|^2 \, dx + \int_{(0,1)\times(\delta_1,\delta_2)} x_2^{-1} |w_2|^2 \, dx \right).\end{aligned}$$

We know that $C^\infty(\overline{\Omega})^2$ is dense in $H^{div}(\Omega)$. Hence, for all $w \in H^{div}(\Omega)$ one has

$$\int_{\delta_1}^{\delta_2} \frac{1}{\delta} |\langle T_\nu^\delta(w), T^\delta(u)\rangle|^2 \, d\delta \;\leq\; |u|^2_{L^\infty(\Omega)} \left(\int_{(\delta_1,\delta_2)\times(0,1)} x_1^{-1} |w_1|^2 \, dx + \int_{(0,1)\times(\delta_1,\delta_2)} x_2^{-1} |w_2|^2 \, dx \right).$$

Moreover, if $w \in H^{div}_\alpha(\Omega)$ we deduce that

$$\int_{\delta_1}^{\delta_2} \frac{1}{\delta} |\langle T_\nu^\delta(w), T^\delta(u)\rangle|^2 \, d\delta \;\leq\; |u|^2_{L^\infty(\Omega)} |w|_{div.\alpha} \;\forall\; 0 < \delta_1 < \delta_2 < \delta_0/2.$$

This gives

$$\int_0^{\delta_2} \frac{1}{\delta} |\langle T_\nu^\delta(w), T^\delta(u)\rangle|^2 \, d\delta \;\leq\; |u|^2_{L^\infty(\Omega)} |w|_{div.\alpha} < +\infty.$$

From the continuity of g in $\delta = 0$ we deduce that

$$g(0) = \lim_{\delta \to 0} g(\delta) = 0.$$

From Lemma 2.2.1 we conclude that

$$\mathcal{T}_w(u) = 0 \;\forall w \in H^{div}_\alpha(\Omega) \;\text{ and }\; u \in \mathcal{D}_{01}.$$

Now the result follows from the density of $\mathcal{D}_{01}$ in $H^1_{\alpha,01}(\Omega)$. ■

We end this section with a Hardy-like inequality corresponding to this case.

Lemma 2.2.13 (Hardy inequality II) *Assume that $\alpha_i \neq 1$ for $i = 1, 2$. There exist $C = C(\alpha_1, \alpha_2)$ such that*

$$\int_\Omega x_i^{\alpha_i - 2} |u|^2 \, dx \leq C \int_\Omega x_i^{\alpha_i} \left| \frac{\partial u}{\partial x_i} \right|^2 dx \quad \forall u \in H^1_{\alpha,0}(\Omega).$$

Proof. Again, we will present the proof only for the case $\alpha_1, \alpha_2 \in (1,2]$. The other cases could be deduced using these ideas combined with the present in the case $\alpha_1, \alpha_2 \in [0,1)$. Let us fix $v \in \mathcal{D}_0$, $x_1 \in (0,1)$ and put $\beta = 2 - \alpha_1$. We have that

$$
\begin{aligned}
|v(x_1,x_2)|^2 &= \left| \int_{x_1}^{1} s^{(3-\beta)/4} \frac{\partial v}{\partial s} v(s,x_2) s^{-(3-\beta)/4} \, ds \right|^2 \\
&\leq \int_{x_1}^{1} s^{(3-\beta)/2} \left| \frac{\partial v}{\partial s}(s,x_2) \right|^2 ds . \int_{x_1}^{1} s^{-(3-\beta)/2} \, ds \\
&\leq \int_{x_1}^{1} s^{(3-\beta)/2} \left| \frac{\partial v}{\partial s}(s,x_2) \right|^2 ds . \frac{2}{1-\beta} (x_1^{(\beta-1)/2} - 1) \\
&\leq \int_{x_1}^{1} s^{(3-\beta)/2} \left| \frac{\partial v}{\partial s}(s,x_2) \right|^2 ds . \frac{2}{\alpha - 1} x_1^{(\beta-1)/2} . \qquad (2.7)
\end{aligned}
$$

Multiplying (2.7) by $x_1^{-\beta}$ and integrating x_1 in $(0,1)$, we get:

$$
\begin{aligned}
\int_0^1 x_1^{-\beta} |v(x_1,x_2)|^2 \, dx_1 &\leq \frac{2}{\alpha_1 - 1} \int_0^1 \left(\int_{x_1}^{1} s^{(3-\beta)/2} \left| \frac{\partial v}{\partial s}(s,x_2) \right|^2 ds \right) x_1^{(-1-\beta)/2} \, dx_1 \\
&= \frac{2}{\alpha_1 - 1} \int_0^1 s^{(3-\beta)/2} \left| \frac{\partial v}{\partial s}(s,x_2) \right|^2 \left(\int_0^s x_1^{(-1-\beta)/2} \, dx_1 \right) ds \\
&= \frac{4}{(\alpha_1 - 1)^2} \int_0^1 s^{(3-\beta)/2} \left| \frac{\partial v}{\partial s}(s,x_2) \right|^2 s^{(1-\beta)/2} \, ds \\
&= \frac{4}{(\alpha_1 - 1)^2} \int_0^1 x_1^{\alpha_1} \left| \frac{\partial v}{\partial x_1}(x_1,x_2) \right|^2 dx_1 . \qquad (2.8)
\end{aligned}
$$

Now, integrating (2.8) with respect to x_2 in $(0,1)$ we obtain:

$$
\int_\Omega x_1^{\alpha_1 - 2} |v|^2 \, dx \leq C \int_\Omega x_1^{\alpha_1} \left| \frac{\partial v}{\partial x_1} \right|^2 dx \quad \forall v \in \mathcal{D}_0 .
$$

In a similar way we deduce that

$$
\int_\Omega x_2^{\alpha_2 - 2} |v|^2 \, dx \leq C \int_\Omega x_2^{\alpha_2} \left| \frac{\partial v}{\partial x_2} \right|^2 dx \quad \forall v \in \mathcal{D}_0 .
$$

The result follows by density argument. ■

2.2.3 Well posedness

Let us fix $\alpha = (\alpha_1, \alpha_2)$ in $[0,2] \times [0,2]$.

Lemma 2.2.14 *The operator $-\Delta_0 : D(\Delta_0) \mapsto L^2(\Omega)$, where $D(\Delta_0) := H^2_\alpha(\Omega) \cap H^1_{\alpha,0}(\Omega)$, is m-dissipative and self-adjoint. Moreover, if $\alpha_1 \neq 1 \neq \alpha_2$, then $-\Delta_0$ is strictly dissipative.*

Proof. [*Proof of Lemma 2.2.14*] Let us consider the bilinear form $q : H^1_\alpha(\Omega)^2 \longrightarrow \mathbb{R}$ given by

$$q(u,v) := \int_\Omega \nabla u A \nabla v \, dx.$$

It is clear that q is simmetric and $q(u,u) \geq 0$. Furthermore

$$|q(u,v)| \leq |u|_\alpha . |v|_\alpha .$$

Hence q is continuous. Moreover, from Lemmas 2.2.4 and 2.2.13 we deduce that q is coercive on $H^1_{\alpha,0}(\Omega)$ if $\alpha \neq 1$.

Now supose that $(u,v) \in H^2_\alpha(\Omega) \times H^1_{\alpha,0}(\Omega)$. From Lemmas 2.2.7 and 2.2.11 we have that

$$q(u,v) = -\int_\Omega \triangle_0 u . v \, dx.$$

From the properties of q, the result follows. ■

As consequence of Lemma 2.2.13, $-\Delta_0$ is the infinitesimal generator of a strongly continuous semigroups. Thus, using standard techniques, we can prove the following well-posedness result:

Theorem 2.2.15 *For any $g \in L^2(Q)$ and any $u_0 \in L^2(\Omega)$, there exists a unique solution $u \in C^0([0,T]; L^2(\Omega)) \cap L^2(0,T; H^1_{\alpha,0}(\Omega))$ to (2.1). Furthermore, there exists a positive constant C such that*

$$\sup_{t\in[0,T]} \|u(t)\|^2 + \int_0^T |u(t)|^2_\alpha \, dt \leq C \left(|u_0|^2 + \|g\|^2 \right).$$

2.3 Carleman estimates and null controllability results

In this section we will assume that $\alpha_1, \alpha_2 \in (0,2)$. Let us consider the adjoint of (2.1):

$$\begin{cases} -w_t - \Delta_0 w + bw = f & \text{in} \quad Q, \\ B.C. & \text{on} \quad \Sigma, \\ w(\cdot,0) = w_0 & \text{in} \quad \Omega. \end{cases} \tag{2.9}$$

In order to establish a Carleman estimate for the solutions to (2.9), we will assume the following on the observabillity domain:

$$
\begin{cases} \exists\ \delta_0 > 0, \quad \exists\ a_0, b_0 \in (2\delta_0, 1-2\delta_0) \quad \text{such that} \\ \omega_0 := B_{\delta_0}(P_1) \cup B_{\delta_0}(P_2) \subset \omega, \\ \text{where} \quad B_{\delta_0}(P_i) := \{x \in \Omega : |x - P_i| < \delta_0\}, \quad P_1 = (0, b_0), \quad P_2(a_0, 0). \end{cases} \tag{2.10}
$$

Lemma 2.3.1 *Assume that (2.10) holds. There exist* $C > 0$, $c_1 \in (b_0 - \delta_0, b_0)$, $c_2 \in (a_0 - \delta_0, a_0)$, $d_1 \in (b_0, b_0 + \delta_0)$, $d_2 \in (a_0, a_0 + \delta_0)$, *and* $\eta \in C^\infty(\Omega) \cap C^0(\overline{\Omega})$ *such that, for* $V := \{x \in \Omega : c_1 x_1 + c_2(x_2 - c_1) < 0\}$ *and* $W := \{x \in \Omega : d_1 x_1 + d_2(x_2 - d_1) > 0\}$, *one has:*

(i) $\eta(x) = \sum_{i=1}^{2} \frac{x_i^{2-\alpha_i}}{2-\alpha_i}$ *in* V, $\quad \eta(x) = -\sum_{i=1}^{2} \frac{x_i^{2-\alpha_i}}{2-\alpha_i}$ *in* W,

(ii) $|\nabla \eta|$, $|\nabla \eta A \nabla \eta|$, $|A \nabla \eta| \geq C > 0$ *in* $(\overline{\Omega \backslash \omega_0}) \backslash V$,

(iii) $A\nabla\eta \cdot \nabla(\nabla\eta A \nabla\eta), A\nabla(div(A\nabla\eta)), \sum_{i=1}^{2} A\nabla\left(x_i^\alpha \frac{\partial \eta}{\partial x_i}\right), \sum_{i=1}^{2} x_i^{2\alpha-1} \frac{\partial \eta}{\partial x_i} \in C^0(\overline{\Omega})$.

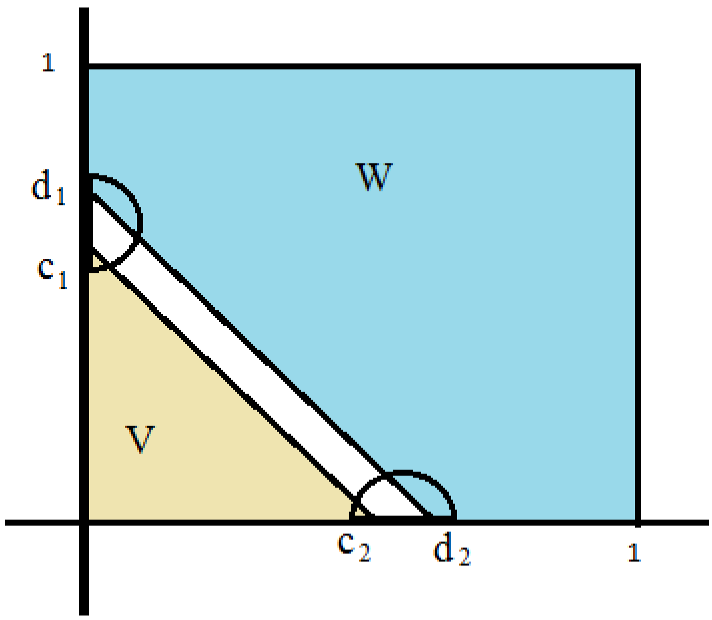

Figura 2.1: illustration of the sets V and W.

Proof. For instance, let us fix $V_1 = V$ with $c_1 = b_0 - \delta_0/2$ and $c_2 = a_0 - \delta_0/2$ and $W_1 = W$ with $d_1 = b_0 + \delta_0/2$ and $d_2 = a_0 + \delta_0/2$.

Let us consider $h \in C^\infty(\overline{\Omega})$ such that

$$
h(x) = \frac{x_1}{2-\alpha_1} + \frac{x_2}{2-\alpha_2} \quad \text{in} \quad V_1 \quad \text{and} \quad h(x) = -\frac{x_1}{2-\alpha_1} - \frac{x_2}{2-\alpha_2} \quad \text{in} \quad W_1.
$$

From Morse theory, there exists a sequence $h_n \in C^\infty(\overline{\Omega})$ of Morse functions such that $h_n \to h$ in $C^\infty(\overline{\Omega})$.

Now let us fix $V_2 = V$ with $c_1 = b_0 - 3\delta_0/4$ and $c_2 = a_0 - 3\delta_0/4$ and $W_2 = W$ with $d_1 = b_0 + 3\delta_0/4$ and $d_2 = a_0 + 3\delta_0/4$. We have that $V_2 \subset\subset V_1$ and $W_2 \subset\subset W_1$. Let be $\psi \in C^\infty(\overline{\Omega})$ such that $0 \le \psi \le 1$ in $\overline{\Omega}$, $\psi = 1$ in $V_2 \cup W_2$ and $\psi = 0$ in $\Omega \backslash (V_1 \cup W_1)$.

Using that $h_n \to h$ in $C^\infty(\overline{\Omega})$ it is possible to prove that the function

$$\eta_0 = h_n + \psi(h - h_n)$$

is a Morse function for all n sufficiently large. Moreover, $\eta_0 \in C^\infty(\overline{\Omega})$,

$$\eta_0(x) = \frac{x_1}{2-\alpha_1} + \frac{x_2}{2-\alpha_2} \quad \text{in} \quad V_2 \quad \text{and} \quad \eta_0(x) = -\frac{x_1}{2-\alpha_1} - \frac{x_2}{2-\alpha_2} \quad \text{in} \quad W_2.$$

Since η_0 is a Morse function, it follows that the set $D = \{x \in \Omega : |\nabla \eta_0| = 0\}$ is finite. Using classical arguments (see [10]) we can move the set D to $\omega_0 \backslash (V_1 \cup W_1)$. Thus, we can assume that

$$|\nabla \eta_0| > C > 0 \quad \text{in} \quad \overline{\Omega} \backslash (\omega_0 \backslash (V_1 \cup W_1)).$$

Now, let us consider $\eta_1 \in C^\infty(\Omega)$ with all the smothness properties presented in item 3 and such that

$$\eta_1(x) = \frac{x_1^{2-\alpha_1}}{2-\alpha_1} + \frac{x_2^{2-\alpha_2}}{2-\alpha_2} \quad \text{in} \quad V_1 \quad \text{and} \quad \eta_1(x) = -\frac{x_1^{2-\alpha_1}}{2-\alpha_1} - \frac{x_2^{2-\alpha_2}}{2-\alpha_2} \quad \text{in} \quad W_1.$$

Again, we will fix $V_3 = V$ with $c_1 = b_0 - 5\delta_0/6$ and $c_2 = a_0 - 5\delta_0/6$ and $W_3 = W$ with $d_1 = b_0 + 5\delta_0/6$ and $d_2 = a_0 + 5\delta_0/6$. We have that $V_3 \subset\subset V_2$ and $W_3 \subset\subset W_2$. Let be $\varphi \in C^\infty(\overline{\Omega})$ such that $0 \le \varphi \le 1$ in $\overline{\Omega}$, $\varphi = 1$ in $V_3 \cup W_3$, $\varphi = 0$ in $\Omega \backslash (V_2 \cup W_2)$, $\frac{\partial \varphi}{\partial x_i} \le 0$ in $V_2 \backslash V_3$ and $\frac{\partial \varphi}{\partial x_i} \ge 0$ in $W_2 \backslash W_3$.

We finally introduce the function

$$\eta = \varphi \eta_1 + m(1-\varphi)\eta_0,$$

where $m > 0$ is such that $x_i^{2-\alpha_i} - m x_i \le 0 \ \ \forall x_i \in [0,1]$.

Sinse $\eta_0, \varphi \in C^\infty(\overline{\Omega})$, $\eta_1 \in C^\infty(\Omega) \cap C^0(\overline{\Omega})$ and η_1 has all the smothness properties of item 3 we deduce that $\eta \in C^\infty(\Omega) \cap C^0(\overline{\Omega})$ and the item 3. Furthermore

$$\eta(x) = \frac{x_1^{2-\alpha}}{2-\alpha_1} + \frac{x_2^{2-\alpha}}{2-\alpha_2} \quad \text{in} \quad V_3 \quad \text{and} \quad \eta_1(x) = -\frac{x_1^{2-\alpha}}{2-\alpha_1} - \frac{x_2^{2-\alpha}}{2-\alpha_2} \quad \text{in} \quad W_3.$$

This proves item 1.

Since $\eta = \eta_1$ in $V_3 \cup W_3$, the item 2 follows in $V_3 \cup W_3$. On the other hand $\eta = \eta_0$ in $\Omega - (V_2 \cup W_2)$ and the item 2 follows in $\Omega \backslash (V_2 \cup W_2)$ too. In $V_2 - V_3$ we have that

$$\frac{\partial \eta}{\partial x_i} = \varphi x_i^{1-\alpha_i} + m\frac{1-\varphi}{2-\alpha_i} + \frac{\partial \varphi}{\partial x_i}\sum_{j=1}^{2}\frac{1}{2-\alpha_j}(x_j^{2-\alpha_j} - m x_j),$$

and easily we deduce that item 2 holds in $V_2 \backslash V_3$. In a similar way we deduce that item 2 holds in $W_2 \backslash W_3$. The proof is now complete. ■

Now, for $\lambda > \lambda_0$ and $s \in \mathbb{R}$, let us introduce the following functions and constants

$$\theta(t) := [t(T-t)]^{-4}, \quad \xi(x,t) := \theta(t)e^{2\lambda(|\eta|_\infty + \eta(x))}, \quad \sigma(x,t) := \theta(t)e^{4\lambda|\eta|_\infty} - \xi(x,t),$$
$$\gamma_1(\lambda) := |\alpha_1 - 1| + |\alpha_2 - 1| + \lambda^{-1/4}, \quad \gamma_2(s) = |\alpha_1 - 1| + |\alpha_2 - 1| + s^{-1/2}.$$

The main result in this section is the following:

Theorem 2.3.2 *Assume that (2.10) holds. There exist positive constants C, s_0, λ_0, such that, for any $\lambda \geq \lambda_0$, $s \geq s_0$ and any solution w to (2.9), one has:*

$$\iint_Q e^{-2s\sigma}\left[s^{-1}\gamma_1(\lambda)\xi^{-1}(|w_t|^2 + |\triangle_0 w|^2) + s\lambda\xi|\nabla_0 w|^2 + s\lambda^2|\nabla w A \nabla \eta|^2\right] dx\, dt$$
$$+ \iint_Q e^{-2s\sigma}|w|^2\left[s^2\lambda^2\gamma_1(\lambda)\xi^2\gamma_2(s\xi) + s^3\lambda^3\xi^3(x_1^{2-\alpha} + x_2^{2-\alpha}) + s^3\lambda^4\xi^3|\nabla_0\eta|^4\right] dx\, dt$$
$$\leq C\left[\|e^{-s\sigma}f\|^2 + s^3\lambda^3\iint_{\omega\times(0,T)} e^{-2s\sigma}\xi^3|w|^2\, dx\, dt\right].$$

2.3.1 Application to null controllability

This section deals with the null controllability of (2.1). The first main result is following:

Theorem 2.3.3 *Let us fix $T > 0$, $\alpha_1, \alpha_2 \in (0,2)$ and an open set $\omega \subset \Omega$. Assume that (2.10) holds. Then, for any $u_0 \in L^2(\Omega)$, there exists a control $g \in L^2(Q)$ such that the solution u of (2.1) satisfies*

$$u(\cdot, T) = 0 \quad in \quad \Omega. \tag{2.11}$$

Moreover, there exists a constant $C = C(T, \alpha, \omega) > 0$ such that

$$\|g\| \leq C|u_0|.$$

As usual in null controllability problems, Theorem 2.3.3 is equivalent to an observabillity property for the adjoint system (2.9):

Proposition 2.3.4 *Let us fix $T > 0$, $\alpha_1, \alpha_2 \in (0,2)$ and an open set $\omega \subset \Omega$. Assume that (2.10) holds. Then there exists a positive constant $C = C(T, \alpha, \omega)$ such that, for any $w_T \in L^2(\Omega)$, the solution to (2.9) satisfies*

$$\int_\Omega |w(x,0)|^2\, dx \le C \iint_{\omega\times(0,T)} |w|^2\, dx\, dt.$$

The way that the Carleman estimate in Theorem 2.3.1 leads to Theorem 2.3.4 is standard; we refer for details to [10].

2.4 Stackelberg-Nash null controllability

In this section we will prove the Stackelberg-Nash null controllabilty of (2.1). In the sequel we will use the spaces $\mathcal{U}_i = L^2(\omega_i \times (0,T))$ and $\mathcal{U} = \mathcal{U}_1 \times \mathcal{U}_2$.

Let us consider the linear system

$$\begin{cases} u_t - \triangle_0 u + bu = g1_\omega + f_1 1_{\omega_1} + f_2 1_{\omega_2} & \text{in} \quad Q, \\ \ B.C. & \text{on} \quad \Sigma, \\ u(\cdot,0) = u_0 & \text{in} \quad \Omega, \end{cases} \tag{2.12}$$

where $\alpha_1,\ \alpha_2 \in (0,2)$, the $\omega_i \subset \Omega$ are non-empty open sets, $(f_1, f_2) \in \mathcal{U}$ and $u_0 \in L^2(\Omega)$.

Let us fix new non-empty open sets $\omega_{i,d} \subset (0,1)$, $u_{i,d} \in L^2(\omega_{i,d} \times (0,T))$ and $\beta_i, \mu_i > 0$ and let us introduce the following functionals:

$$J_i(g; f_1, f_2) := \frac{\beta_i}{2} \iint_{\omega_{i,d}\times(0,T)} |u - u_{i,d}|^2\, dx\, dt + \mu_i \iint_{\omega_i\times(0,T)} |f_i|^2\, dx\, dt. \tag{2.13}$$

Definition 2.4.1 *The pair $(f_1, f_2) \in \mathcal{U}$ is called a Nash equilibrium of (2.12) associated to g if*

$$J_1(g; f_1, f_2) = \min_{f\in\mathcal{U}_1} J_1(g; f, f_2) \quad \textit{and} \quad J_2(g; f_1, f_2) = \min_{f\in\mathcal{U}_2} J_1(g; f_1, f).$$

where these minima are respectively taken in $\mathcal{U}_1$ and $\mathcal{U}_2$.

From the convexity of the functionals J_i, we have that (f_1, f_2) is a Nash equilibrium if, and only if

$$J_1'(g; f_1, f_2)(\bar f_1, 0) = 0 \ \ \forall \bar f_1 \in \mathcal{U}_1 \ \text{ and } \ J_2'(g; f_1, f_2)(0, \bar f_2) = 0 \ \ \forall \bar f_2 \in \mathcal{U}_2.$$

Arguing as in [4, 22] the following is obtained:

Proposition 2.4.2 *There exist positive constants μ_{00} and C such that, if $\mu_1, \mu_2 \geq \mu_{00}$, for every $g \in L^2(\omega \times (0,T))$ there exists a unique associated Nash equilibrium (f_1, f_2) for (2.12), furthermore satisfying*

$$\|(f_1, f_2)\| \leq C(1 + \|g\|).$$

In particular, the corresponding state u satisfies

$$\|u\|_{L^\infty(0,T;L^2(\Omega))} + \|u\|_{L^2(0,T;H^1_{\alpha,0}(\Omega))} + \|u_t\|_{L^2(0,T;H^{-1}(\Omega))} \leq C(1 + \|g\|).$$

In order to establish the Stackelberg-Nash null controllability, that is, the existence of g such that the solution to (2.12) (where (f_1, f_2) is the Nash-equilibrium associated to g) satisfies (2.11), we will impose the following assuptions:

$$\begin{cases} \omega_{1,d} = \omega_{2,d}; \text{ the common observability set will be denoted } \omega_d. \\ \omega_d \cap \omega \neq \emptyset \quad \text{and satisfies (2.10).} \\ \displaystyle\iint_{\omega_d \times (0,T)} \theta^2 |u_{i,d}|^2 \, dx \, dt < +\infty \quad \text{for } i = 1, 2. \end{cases} \tag{2.14}$$

The main result in this section is the following:

Theorem 2.4.3 *Assume that (2.10) and (2.14) hold. There exists $\mu_0 \geq \mu_{00}$ such that, if $\mu_1, \mu_2 \geq \mu_0$, for every $u_0 \in L^2(\Omega)$, there exist a leader control $g \in L^2(\omega \times (\Omega))$ and a unique associated Nash equilibrium (f_1, f_2), such that the solution u of (2.12) satisfies (2.11).*

2.4.1 Proof of Theorem 2.4.3

The proof of Theorem 2.4.3 follows the same steps of the proof of the similar result in [2]. Hence we will present only a sketch and we will consider only the case $\alpha_1, \alpha_2 \in (0, 1)$.

We first note that, arguing as in [4, 22], the following result can be stablished:

Proposition 2.4.4 *Let $g \in L^2(\omega \times (0,T))$ be given. Then (f_1, f_2) is a Nash equilibrium of (2.12) associated to g if and only if*

$$f_i = -\frac{1}{\mu_i} \phi_i|_{\omega_i \times (0,T)},$$

where the ϕ_i, $i = 1, 2$, solve, together with u, the following coupled optimality system:

$$\begin{cases} u_t - \triangle_0 u + bu = 1_\omega g - \dfrac{1}{\mu_1}\phi_1 1_{\omega_1} - \dfrac{1}{\mu_2}\phi_2 1_{\omega_2} & in \quad Q, \\ -(\phi_i)_t - \triangle_0 \phi_i + b\phi_i = \beta_i (u - u_{i,d}) 1_{\omega_{i,d}} & in \quad Q, \\ u = \phi_i = 0 & on \quad \Sigma, \\ u(\cdot, 0) = u_0, \quad \phi_i(\cdot, T) = 0 & in \quad \Omega. \end{cases} \tag{2.15}$$

In view of Proposition 2.4.4, if $\mu_1, \mu_2 \geq \mu_0$, there exists a unique solution to (2.15). If we prove that this system is null-controllable, we will have achieved in fact the proof of Theorem 2.4.3. Therefore, what we need is an observabillity estimate for the adjoint system

$$\begin{cases} -z_t - \triangle_0 z + bz = \beta_1 \varphi_1 1_{\omega_{1,d}} + \beta_2 \varphi_2 1_{\omega_{2,d}} & \text{in} \quad Q, \\ (\varphi_i)_t - \triangle_0 \varphi_i + b\varphi_i = -\dfrac{z}{\mu_i} 1_{\omega_i} & \text{in} \quad Q, \\ z = \varphi_i = 0 & \text{on} \quad \Sigma, \\ z(\cdot, T) = z^T, \quad \varphi_i(\cdot, 0) = 0 & \text{in} \quad \Omega. \end{cases} \tag{2.16}$$

This is established in the following result:

Theorem 2.4.5 *Assume that (2.10) and (2.14) hold. There exist $\mu_0, C > 0$ and a weight function $\rho = \rho(t)$ blowing up at $t = T$ such that, if $\mu_1, \mu_2 \geq \mu_0$, for any $z^T \in L^2(0,1)$, the associated solution $(z, \varphi_1, \varphi_2)$ to (2.16)satisfies:*

$$|z(\cdot, 0)|^2 + \sum_{i=1}^{2} \iint_Q \rho^{-2} |\varphi_i|^2 \, dx \, dt \leq C \iint_{\omega \times (0,T)} e^{-2s\sigma} \xi^9 |z|^2 \, dx \, dt. \tag{2.17}$$

The proof of (2.17) is very similar to the proof of the corresponding result in [2] and, for brevity, we will not be given.

2.5 Further extensions and open questions

In this section we will present some additional comments on the controllability of degenerate parabolic equations.

2.5.1 On systems with gradients

First, consider the problems

$$\begin{cases} u_t - \triangle_0 u + B \cdot (A^{1/2} \nabla u) + bu = g 1_\omega & \text{in} \quad Q, \\ B.C. & \text{on} \quad \Sigma, \\ u(\cdot, 0) = u_0 & \text{in} \quad \Omega, \end{cases} \tag{2.18}$$

and

$$\begin{cases} u_t - \triangle_0 u + B \cdot \nabla u + bu = g 1_\omega & \text{in} \quad Q, \\ B.C. & \text{on} \quad \Sigma, \\ u(\cdot, 0) = u_0 & \text{in} \quad \Omega, \end{cases} \tag{2.19}$$

where $B \in L^\infty(Q)$.

If u is a solution to (2.18), then u solves (2.1) with $g1_\omega$ replaced by $g1_\omega - B(A^{1/2}\nabla u)$. Therefore, the Carleman estimate proved in Theorem 2.3.2 holds for the solutions to the adjoint of (2.18) and the control results in this paper are again true for (2.18).

The null controllability of (2.19) is an open question, even in spatial dimension 1. However, if $\alpha_1, \alpha_2 \in (0, 1/2)$, using an appropriate function η satisfying

$$\eta(x) = 3\sum_{i=1}^{2} \frac{x_i^{(4-2\alpha_i)/3}}{4 - 2\alpha_i} \quad \text{in} \quad V \quad \text{and} \quad \eta(x) = -3\sum_{i=1}^{2} \frac{x_i^{(4-2\alpha_i)/3}}{4 - 2\alpha_i} \quad \text{in} \quad W$$

as defined in [27], we conclude again that all the results in this paper are satisfied.

2.5.2 On other degenerate operators

Using the Carleman estimates stablished in [10], we can argue as before and prove that Theorem 2.4.3 is still true for system (2.3).

The half-degenerate problem

The half-degenerate problem appears when the PDE degenerates only with respect to one variable, that is, $\alpha_1 \in (0, 2]$ and $\alpha_2 = 0$.

This case, apparently more simple, is in fact a little more delicate. The main reason is that the constrution os a function η appropriate for Carleman estimates is more complicate. In order to be more precise, let us remember that, for nondegenerate problems, an important property that the function η must have is

$$\frac{\partial \eta}{\partial \nu} \le 0 \quad \text{on} \quad \Sigma.$$

In the degenerate problems in the previous section, this requirement can be weakened to

$$\frac{\partial \eta}{\partial \nu} \le 0 \quad \text{on} \quad (\Gamma_3 \cup \Gamma_4) \times (0, T).$$

But in this half-degenerate case, we must have

$$\frac{\partial \eta}{\partial \nu} \le 0 \quad \text{on} \quad (\Gamma_2 \cup \Gamma_3 \cup \Gamma_4) \times (0, T)$$

and it is not easy to combine this property and other properties that the function η must have.

This suggests to consider another kind of η and work differently. In a forthcoming paper we will present some Carleman estimates for this case.

The problem with $\alpha_1 = 2$ or $\alpha_2 = 2$

It is well known that, in general, (2.2) is not null controllable, see for exemple [9, 16, 17]. However, as we have shown in [2], there exist situations, depending on the observation domain, where it is possible to prove null controllability. A new technique will be presented in a next work to deal with this case and deduce appropriate Carleman estimates (also valid in higher dimensions).

2.5.3 On spatial dimension 3

Assume that $\Omega := (0,1)^3$ with $a_i, b_i \in (0,1)$, $a_i < b_i$ and let us introduce the sets

$$V := \prod_{i=1}^{3}[0, a_i], \quad \omega_0 := \left(\prod_{i=1}^{3}(0, b_i)\right)\backslash V \quad \text{and} \quad W := \overline{\Omega}\backslash(V \cup \omega_0).$$

Then, there exist functions $\eta \in C^\infty(\overline{\Omega})$ such that

$$\eta(x) = \frac{1}{2-\alpha}(x_1^{2-\alpha} + x_2^{2-\alpha}) \ \text{ in } \ V \quad \text{and} \quad \eta(x) = \frac{-1}{2-\alpha}(x_1^{2-\alpha} + x_2^{2-\alpha}) \ \text{ in } \ W.$$

Consequently, the results in this paper can be adapted to this situation if we assume that ω contains a set like ω_0.

2.5.4 On the assumptions on ω

The main open question left in this work is about the observation domain ω. The key point is Theorem 2.3.1. With a function η satisfying all the properties in Theorem 2.3.1, we can deduce a Carleman estimate and, consequently, the other results follow. In general, with $\omega \subset\subset \Omega$, we do not know how to build a function η satisfying all the properties needed in the proof of Theorem 2.3.1. The best we can do is to build a function η that satisfies properties 1 and 3 and fulfills $|\nabla\eta| \geq C > 0$ in $(\overline{\Omega\backslash\omega})\backslash V$; but we cannot prove for η that $|\nabla\eta A\nabla\eta|, |A\nabla\eta| \geq C > 0$ in $(\overline{\Omega\backslash\omega})\backslash V$.

2.6 Appendix A: Proof of Theorem 2.3.2

Proof. In the sequel, $C > 0$ is a generic constant that depends on T and α and w is a solution of (2.9). From a density argument we can assume that w is sufficiently regular.

For $s > s_0 > 0$ we introduce

$$z = e^{-s\sigma} w.$$

We see that

(i) $z = \dfrac{\partial z}{\partial x_i} = 0$ at $t = 0$ and $t = T$,

(ii) $B.C.$ holds on Σ,

(iii) If $P^-(z) := z_t + s div(zA\nabla\sigma) + s\nabla\sigma A\nabla z$ and $P^+ := div(A\nabla z) + s^2 z\nabla\sigma A\nabla\sigma + s\sigma_t z$ one has

$$P^- z + P^+ z = e^{-s\sigma} f,$$

From item 3, we have that

$$\|P^- z\|^2 + \|P^+ z\|^2 + 2((P^- z, P^+ z)) = \|e^{-s\sigma} f\|^2. \tag{2.20}$$

Let us set $((P^- z, P^+ z)) = I_1 + ... + I_4$ where

$$\begin{aligned} I_1 &:= ((div(A\nabla z) + s^2 z\nabla\sigma A\nabla\sigma + s\sigma_t z, z_t)),\\ I_2 &:= s^2((\sigma_t z, div(zA\nabla\sigma) + \nabla\sigma A\nabla z)),\\ I_3 &:= s^3((z\nabla\sigma A\nabla\sigma, div(zA\nabla\sigma) + \nabla A\nabla z)),\\ I_4 &:= s((div(A\nabla z), div(zA\nabla\sigma) + \nabla\sigma A\nabla\sigma)). \end{aligned}$$

From $B.C.$ we have that

$$\begin{aligned} \iint_Q div(A\nabla z) z_t \, dx\, dt &= -\iint_Q div(A\nabla z_t) z \, dx\, dt = \iint_Q \nabla z A\nabla z_t \, dx\, dt\\ &= \frac{1}{2}\iint_Q \frac{\partial}{\partial t}\left(\nabla z A\nabla z\right) \, dx\, dt = 0. \end{aligned}$$

Hence,

$$I_1 = \frac{-s}{2}\iint_Q (2s\nabla\sigma A\nabla\sigma_t|z|^2 + \sigma_{tt}|z|^2)\, dx\, dt. \tag{2.21}$$

Again, from $B.C.$ we get

$$\begin{aligned} I_2 &= -s^2\iint_Q zA\nabla\sigma\nabla(\sigma_t z)\, dx\, dt + s^2\iint_Q zA\nabla\sigma(\sigma_t\nabla z)\, dx\, dt\\ &= -s^2\iint_Q \nabla\sigma A\nabla\sigma_t|z|^2\, dx\, dt \end{aligned} \tag{2.22}$$

and

$$\begin{aligned} I_3 &= -s^2 \iint_Q zA\nabla\sigma\nabla(z\nabla\sigma A\nabla\sigma)\,dx\,dt + s^3 \iint_Q z(\nabla\sigma A\nabla\sigma)\nabla z A\nabla\sigma\,dx\,dt \\ &= -s^3 \iint_Q A\nabla\sigma\nabla(\nabla\sigma A\nabla\sigma)|z|^2\,dx\,dt. \end{aligned} \tag{2.23}$$

Now let us to compute I_4.

$$\begin{aligned} I_4 &= s\iint_Q div(A\nabla z)[z div(A\nabla\sigma) + 2\nabla z A\nabla\sigma]\,dx\,dt \\ &= -s\iint_Q A\nabla z[\nabla z div(A\nabla\sigma) + z\nabla(div(A\nabla\sigma)) + 2\nabla(\nabla z A\nabla\sigma)]\,dx\,dt \\ &\quad + s\iint_\Sigma (z div(A\nabla\sigma) + 2\nabla z A\nabla\sigma)A\nabla z.z\nu\,ds\,dt. \end{aligned} \tag{2.24}$$

On the other hand we have that

$$\begin{aligned} &-2s\iint_Q A\nabla z\nabla(\nabla z A\nabla\sigma)\,dx\,dt \\ &= -2s\sum_{i=1}^{2}\iint_Q A\nabla z\left[x_i^{\alpha_i}\frac{\partial\sigma}{\partial x_i}\nabla\frac{\partial z}{\partial x_i} + \frac{\partial z}{\partial x_i}\nabla\left(x_i^{\alpha_i}\frac{\partial\sigma}{\partial x_i}\right)\right]\,dx\,dt \end{aligned} \tag{2.25}$$

and

$$\begin{aligned} -2s\sum_{i=1}^{2}\iint_Q A\nabla x_i^{\alpha_i}\frac{\partial\sigma}{\partial x_i}z\nabla\frac{\partial z}{\partial x_i}\,dx\,dt &= -s\sum_{i,j=1}^{2}\iint_Q x_i^{\alpha_i}x_j^{\alpha_j}\frac{\partial\sigma}{\partial x_i}\frac{\partial}{\partial x_i}\left(\left|\frac{\partial z}{\partial x_j}\right|^2\right)\,dx\,dt \\ &= s\sum_{i,j=1}^{2}\left[\iint_Q\left|\frac{\partial z}{\partial x_j}\right|^2\frac{\partial}{\partial x_i}\left(x_i^{\alpha_i}x_j^{\alpha_j}\frac{\partial\sigma}{\partial x_i}\right)\,dx\,dt - \iint_\Sigma x_i^{\alpha_i}x_j^{\alpha_j}\frac{\partial\sigma}{\partial x_i}\left|\frac{\partial z}{\partial x_j}\right|^2\nu_i\,ds\,dt\right] \\ &= s\sum_{i,j=1}^{2}\iint_Q\left[x_j^{\alpha_j}\left|\frac{\partial z}{\partial x_j}\right|^2\frac{\partial}{\partial x_i}\left(x_i^{\alpha_i}\frac{\partial\sigma}{\partial x_i}\right) + x_i^{\alpha_i}\frac{\partial\sigma}{\partial x_i}\frac{\partial}{\partial x_i}(x_j^{\alpha_j})\left|\frac{\partial z}{\partial x_j}\right|^2\right]\,dx\,dt \\ &\quad - s\iint_\Sigma|\nabla z A\nabla z|A\nabla\sigma\nu\,ds\,dt \\ &= s\iint_Q|\nabla z A\nabla z|div(A\nabla\sigma)\,dx\,dt + s\sum_{i=1}^{2}\iint_Q\alpha_i x_i^{2\alpha_i-1}\frac{\partial\sigma}{\partial x_i}\left|\frac{\partial z}{\partial x_i}\right|^2\,dx\,dt \\ &\quad - s\iint_\Sigma|\nabla z A\nabla z|A\nabla\sigma\nu\,ds\,dt. \end{aligned} \tag{2.26}$$

From (2.24)-(2.26), we get:

$$\begin{aligned} I_4 = \iint_Q &\left(-szA\nabla z\nabla(div(A\nabla\sigma)) - 2s\sum_{i=1}^{2}\frac{\partial z}{\partial x_i}A\nabla z\nabla\left(x_i^{\alpha_i}\frac{\partial\sigma}{\partial x_i}\right) + s\sum_{i=1}^{2}\alpha_i x_i^{2\alpha_i-1}\frac{\partial\sigma}{\partial x_i}\left|\frac{\partial z}{\partial x_i}\right|^2\right)\,dx\,dt \\ &+ \iint_\Sigma\left(2s(\nabla z A\nabla\sigma)A\nabla z\nu - |\nabla z A\nabla z|A\nabla\sigma\nu\right)\,ds\,dt. \end{aligned} \tag{2.27}$$

From (2.21)-(2.23), (2.27) and (2.29) we conclude that

$$
\begin{aligned}
((P^-z, P^+z)) &= -s^3 \iint_Q A\nabla\sigma\nabla(\nabla\sigma A\nabla\sigma)|z|^2\, dx\, dt - 2s\sum_{i=1}^{2}\iint_Q \frac{\partial z}{\partial x_i} A\nabla z\nabla\left(x_i^{\alpha}\frac{\partial \sigma}{\partial x_i}\right) dx\, dt \\
& -s^2\iint_Q \nabla\sigma A\nabla\sigma_t|z|^2\, dx\, dt + s\sum_{i=1}^{2}\iint_Q \alpha_i x_i^{2\alpha-1}\frac{\partial\sigma}{\partial x_i}\left|\frac{\partial z}{\partial x_i}\right|^2 dx\, dt \\
& -s\iint_Q zA\nabla z\nabla(div(A\nabla\sigma))\, dx\, dt - \frac{s}{2}\iint_Q \sigma_{tt}|z|^2\, dx\, dt \\
& +\iint_\Sigma (2s(\nabla zA\nabla\sigma)A\nabla z\nu - |\nabla zA\nabla z|A\nabla\sigma\nu)\ ds\, dt. \qquad (2.28)
\end{aligned}
$$

Let us denote by T_1,..., T_7 the seven integrals in the right hand side of (2.28). Now, we will estimate all them. For the integral on the boundary we will use the following result:

Lemma A.1: If $v \in H^2_\alpha(\Omega)\cap H^1_{\alpha,0}(\Omega)$, one has

$$\frac{\partial v}{\partial x_2} = 0 \ \text{ on } \ \Gamma_3 \ \text{ and } \ \frac{\partial v}{\partial x_1} = 0 \ \text{ on } \ \Gamma_4.$$

Futhermore,

$$
\begin{aligned}
\frac{\partial v}{\partial x_2} &= 0 \ \text{ on } \ \Gamma_1 \ \text{ if } \ \alpha_1\in(0,1) \ \text{ and } \ \alpha_2\in[1,2), \\
\frac{\partial v}{\partial x_1} &= 0 \ \text{ on } \ \Gamma_2 \ \text{ if } \ \alpha_2\in(0,1) \ \text{ and } \ \alpha_1\in[1,2).
\end{aligned}
$$

Proof: Let us fix $\varphi \in C_0^\infty(\Gamma_3)$. There exist $\epsilon > 0$ such that $\varphi = 0$ in $\Gamma_3\backslash\Gamma_3^\epsilon$, where $\Gamma_3^\epsilon := \{(1, x_2) \in \Gamma_3 : x_2 \in (\epsilon, 1-\epsilon)\}$. Now let us extend φ to Γ puting $\varphi = 0$ in $\Gamma_1\cup\Gamma_2\cup\Gamma_4$ and let us consider $\Phi \in C^1(\overline{\Omega})$ and $\Psi = (\Psi_1, \Psi_2) \in C^1(\overline{\Omega})^2$ such that $\Psi = \varphi$ on Γ and $\Psi = \nu = (\nu_1, \nu_2) = (1, 0)$ on Γ_3^ϵ. Using that $v = 0$ in Γ_3 we have that

$$
\begin{aligned}
0 &= \sum_{j=1}^{2}\int_{\Gamma_3^\epsilon}\frac{\partial}{\partial x_j}(x_j^{\alpha_j/2}v\Psi_j\Phi).0\, ds = \sum_{j=1}^{2}\int_{\Gamma}\frac{\partial}{\partial x_j}(x_j^{\alpha_j/2}v\Psi_j\Phi)\nu_2\, ds \\
&= \sum_{j=1}^{2}\int_{\Omega}\frac{\partial}{\partial x_2}\frac{\partial}{\partial x_j}(x_j^{\alpha_j/2}v\Psi_j\Phi)\, dx = \sum_{j=1}^{2}\int_{\Gamma}\frac{\partial}{\partial x_2}(x_j^{\alpha_j/2}v\Psi_j\Phi)\nu_j\, ds \\
&= \sum_{j=1}^{2}\int_{\Gamma_3}\frac{\partial}{\partial x_2}(x_j^{\alpha_j/2}v)|\nu_j|^2\varphi\, ds = \int_{\Gamma_3}\frac{\partial}{\partial x_2}(x_1^{\alpha_1/2}v)\varphi\, ds \\
&= \int_{\Gamma_3}\frac{\partial v}{\partial x_2}.\varphi\, ds.
\end{aligned}
$$

Hence, $\dfrac{\partial v}{\partial x_2} = 0$ on Γ_3. In a similar way we conclude that $\dfrac{\partial v}{\partial x_1} = 0$ on Γ_4 and the other identities. ◇

Using Lemma B.1 we deduce that

$$\iint_{\Sigma} (2s(\nabla z A \nabla \sigma) A \nabla z \nu - |\nabla z A \nabla z| A \nabla \sigma \nu)\, ds\, dt \geq 0. \tag{2.29}$$

Now, using the definitions of σ and ξ and the properties of η we deduce that

$$\begin{aligned} T_1 \;\geq\; & C s^3 \lambda^4 \iint_Q \xi^3 |\nabla_0 \eta|^4 |z|^2\, dx\, dt + C s^3 \lambda^3 \int_0^T \int_V \xi^3 (x_1^{2-\alpha_1} + x_2^{2-\alpha_2}) |z|^2\, dx\, dt \\ & - C \int_0^T \int_{\omega_0} \xi^3 |z|^2\, dx\, dt, \end{aligned} \tag{2.30}$$

$$\begin{aligned} T_2 \;\geq\; & 2 s \lambda^2 \iint_Q \xi |\nabla z A \nabla \eta|^2\, dx\, dt + 2 s \lambda \int_0^T \int_V \xi |\nabla_0 z|^2\, dx\, dt \\ & - C s \lambda \int_0^T \int_{\Omega \setminus V} \xi |\nabla_0 z|^2\, dx\, dt, \end{aligned} \tag{2.31}$$

$$\begin{aligned} T_3 \;\geq\; & - C s^2 \lambda^2 \Big[\iint_Q \xi^3 |\nabla_0 \eta|^4 |z|^2\, dx\, dt + \int_0^T \int_V \xi^3 (x_1^{2-\alpha_1} + x_2^{2-\alpha_2}) |z|^2\, dx\, dt \\ & + \int_0^T \int_{\omega_0} \xi^3 |z|^2\, dx\, dt \Big] \end{aligned} \tag{2.32}$$

and

$$\begin{aligned} T_4 \;\geq\; & - s \lambda \sum_{i=1}^{2} \int_0^T \int_V \xi \left[\alpha_1 x_1^{\alpha_1} \frac{\partial z}{\partial x_1} + \alpha_2 x_2^{\alpha_2} \frac{\partial z}{\partial x_2} \right] dx\, dt \\ & - C s \lambda \int_0^T \int_{\Omega \setminus V} \xi |\nabla_0 z|^2\, dx\, dt. \end{aligned} \tag{2.33}$$

Using the definitions of σ and ξ we get

$$\begin{aligned} T_5 \;=\; & s \lambda^3 \iint_Q \xi z |\nabla_0 \eta|^4 \nabla A \nabla \eta\, dx\, dt + s \lambda \iint_Q \xi z A \nabla z \nabla (|\nabla_0 \eta|^2)\, dx\, dt \\ & + s \lambda^2 \iint_Q \xi z\, div(A \nabla \eta) \nabla z A \nabla \eta\, dx\, dt + s \lambda \iint_Q \xi z A \nabla z \nabla (div(A \nabla \eta))\, dx\, dt \end{aligned} \tag{2.34}$$

Now, using Young's inequality we have that

$$s \lambda^3 \iint_Q \xi z |\nabla_0 \eta|^4 \nabla A \nabla \eta\, dx\, dt \geq - s^2 \lambda^4 \iint_Q \xi^3 |\nabla_0 \eta|^4 |z|^2\, dx\, dt - \lambda^2 \iint_Q \xi |\nabla A \nabla \eta|^2\, dx\, dt \tag{2.35}$$

$$\begin{aligned} s \lambda \iint_Q \xi z A \nabla z \nabla (|\nabla_0 \eta|^2)\, dx\, dt \geq & - C \Big[s^2 \lambda^3 \iint_Q \xi^3 |\nabla_0 \eta|^4 |z|^2\, dx\, dt + \lambda \int_0^T \int_{\Omega \setminus V} \xi |\nabla_0 z|^2\, dx\, dt \\ & + \int_0^T \int_V (s^2 \lambda^3 \xi^3 (x_1^{2-\alpha_1} + x_2^{2-\alpha_2}) |z|^2 + \lambda \xi |\nabla_0 z|^2)\, dx\, dt + s^2 \lambda^3 \int_0^T \int_{\omega_0} \xi^3 |z|^2\, dx\, dt, \Big] \end{aligned} \tag{2.36}$$

$$s\lambda^2 \iint_Q \xi z div(A\nabla\eta)\nabla z A\nabla\eta \,dx\,dt \geq -C\left[s^2\lambda^3 \iint_Q \xi^3|\nabla_0\eta|^4|z|^2\,dx\,dt + \lambda\int_0^T\!\!\int_{\Omega\setminus V}\xi|\nabla_0 z|^2\,dx\,dt\right.$$
$$\left. + \int_0^T\!\!\int_V (s^2\lambda^3\xi^3(x_1^{2-\alpha_1}+x_2^{2-\alpha_2})|z|^2 + \lambda\xi|\nabla_0 z|^2)\,dx\,dt + s^2\lambda^3\int_0^T\!\!\int_{\omega_0}\xi^3|z|^2\,dx\,dt,\right] \quad (2.37)$$

and

$$s\lambda \iint_Q \xi z A\nabla z\nabla(div(A\nabla\eta))\,dx\,dt \geq -C\left[s^2\lambda^3\iint_Q \xi^3|\nabla_0\eta|^4|z|^2\,dx\,dt\right.$$
$$\left. + s^2\lambda^3\int_0^T\!\!\int_{\omega_0}\xi^3|z|^2\,dx\,dt + \lambda\int_0^T\!\!\int_{\Omega\setminus V}\xi|\nabla_0 z|^2\,dx\,dt\right]. \quad (2.38)$$

From (2.34)-(2.38) we get:

$$T_5 \geq -C\left[s^2\lambda^4\iint_Q \xi^3|\nabla_0\eta|^4|z|^2\,dx\,dt \int_0^T\!\!\int_V (s^2\lambda^3\xi^3(x_1^{2-\alpha_1}+x_2^{2-\alpha_2})|z|^2 + \lambda\xi|\nabla_0 z|^2\,)dx\,dt\right.$$
$$\left. s^2\lambda^3\int_0^T\!\!\int_{\omega_0}\xi^3|z|^2\,dx\,dt + \lambda\int_0^T\!\!\int_{\Omega\setminus V}\xi|\nabla_0 z|^2\,dx\,dt\right]. \quad (2.39)$$

Finally, we have

$$T_6 \geq -Cs\iint_Q \xi^{3/2}|z|^2\,dx\,dt \quad \text{and} \quad T_7 \geq 0. \quad (2.40)$$

From (2.28)-(2.33), (2.39) and (2.40), we deduce that

$$((P^-z, P^+z)) \geq C\left[\iint_Q (s^3\lambda^4\xi^3|\nabla_0\eta|^4|z|^2 + s\lambda^2\xi|\nabla z A\nabla\eta|^2)\,dx\,dt\right.$$
$$\left. + \int_0^T\!\!\int_V (s^3\lambda^3\xi^3(x_1^{2-\alpha}+x_2^{2-\alpha})|z|^2 + s\lambda\xi|\nabla_0 z|^2)\,dx\,dt\right]$$
$$-C\left[s^3\lambda^3\int_0^T\!\!\int_{\omega_0}\xi^3|z|^2\,dx\,dt s\lambda\int_0^T\!\!\int_{\Omega\setminus V}\xi|\nabla_0 z|^2\,dx\,dt + s\iint_Q \xi^{3/2}|z|^2\,dx\,dt\right]. \quad (2.41)$$

Combining (2.20) and (2.41), we conclude that

$$\|P^-z\|^2 + \|P^+z\|^2 + \iint_Q (s^3\lambda^4\xi^3|\nabla_0\eta|^4|z|^2 + s\lambda^2\xi|\nabla z A\nabla\eta|^2)\,dx\,dt$$
$$\int_0^T\!\!\int_V (s^3\lambda^3\xi^3(x_1^{2-\alpha_1}+x_2^{2-\alpha_2})|z|^2 + s\lambda\xi|\nabla_0 z|^2)\,dx\,dt \leq C\left[\|e^{-s\sigma}f\|^2 + s\iint_Q \xi^{3/2}|z|^2\,dx\,dt\right.$$
$$\left. + s\lambda\int_0^T\!\!\int_{\Omega\setminus V}\xi|\nabla_0 z|^2\,dx\,dt + s^3\lambda^3\int_0^T\!\!\int_{\omega_0}\xi^3|z|^2\,dx\,dt\right]. \quad (2.42)$$

Let us denote by $L(z)$ the sum of all the terms in the left hand side of (2.42) and by $R(z)$ the sum of all the terms in the right.

For instance assume that some $\alpha_i \neq 1$. Using Hardy inequality, we deduce that

$$\begin{aligned}
s^2\lambda^2 \iint_Q \xi^2|z|^2\,dx\,dt &= \iint_Q (s^{3/2}\lambda^{3/2}\xi^{3/2}x_i^{(2-\alpha_i)/2}|z|)(s^{1/2}\lambda^{1/2}\xi^{1/2}x_i^{(\alpha_i-2)/2}|z|)\,dx\,dt \\
&\leq s^3\lambda^3 \iint_Q \xi^3 x_i^{2-\alpha_i}|z|^2\,dx\,dt + s\lambda \iint_Q x_i^{\alpha_i-2}|\xi^{1/2}z|^2\,dx\,dt \\
&\leq s^3\lambda^3 \int_0^T\int_V \xi^3 x_i^{2-\alpha_i}|z|^2\,dx\,dt + s^3\lambda^3 \int_0^T\int_{\omega_0} \xi^3|z|^2\,dx\,dt \\
&\quad + Cs^3\lambda^3 \iint_Q \xi^3|\nabla_0\eta|^4|z|^2\,dx\,dt + s\lambda \iint_Q x_i^{\alpha_i}\left|\frac{\partial}{\partial x_i}(\xi^{1/2}z)\right|^2\,dx\,dt \\
&\leq CR(z). \qquad (2.43)
\end{aligned}$$

Now, let us assume that $\alpha_1 = \alpha_2 = 1$. Using Hölder and Hardy inequalities, we see that

$$\begin{aligned}
s^{3/2}\lambda^{7/4}\int_0^T\int_V \xi^{3/2}|z|^2\,dx\,dt &= \frac{1}{2}\sum_{i=1}^{2}\int_0^T\int_V (s^3\lambda^4 x_i^2\xi^3|z|^2)^{1/4}(s\lambda x_i^{-2/3}\xi|z|^2)^{3/4}\,dx\,dt \\
&\leq s^3\lambda^4 \iint_Q \xi^3|\nabla_0\eta|^4|z|^2\,dx\,dt + s\lambda\sum_{i=1}^{2}\iint_Q x_i^{-2+4/3}(\xi^{1/2}z)^2\,dx\,dt \\
&\leq CR(z). \qquad (2.44)
\end{aligned}$$

From (2.43) and (2.44), we get:

$$s^2\lambda^2\gamma_1(\lambda)\int_0^T\int_V \xi^2\gamma_2(s\xi)|z|^2\,dx\,dt \leq CR(z).$$

Furthermore,

$$\begin{aligned}
s^2\lambda^2\gamma_1(\lambda)\iint_Q \xi^2\gamma_2(s\xi)|z|^2\,dx\,dt &\leq s^2\lambda^2\gamma_1(\lambda)\int_0^T\int_V \xi^2\gamma_2(s\xi)|z|^2\,dx\,dt \\
&+ s^2\lambda^2\int_0^T\int_{\omega_0}\xi^3|z|^2\,dx\,dt + s^2\lambda^2\iint_Q \xi^3|\nabla_0\eta|^4|z|^2\,dx\,dt.
\end{aligned}$$

From (2.42), we deduce that

$$L(z) + s^2\lambda^2\gamma_1(\lambda)\iint_Q \xi^2\gamma_2(s\xi)|z|^2\,dx\,dt \leq CR(z). \qquad (2.45)$$

On the other hand, we have:

$$\begin{aligned}
&\iint_Q (s^3\lambda^3\xi^3(x_1^{2-\alpha_1}+x_2^{2-\alpha_2})|z|^2 + s\lambda\xi|\nabla_0 z|^2)\,dx\,dt \\
&\quad\leq \int_0^T\int_V (s^3\lambda^3\xi^3(x_1^{2-\alpha_1}+x_2^{2-\alpha_2})|z|^2 + s\lambda\xi|\nabla_0 z|^2)\,dx\,dt + s^3\lambda^3\int_0^T\int_{\omega_0}\xi^3|z|^2\,dx\,dt \\
&\qquad + Cs^3\lambda^3\iint_Q \xi^3|\nabla_0\eta|^4|z|^2\,dx\,dt + s\lambda\int_0^T\int_{\Omega\setminus V}\xi|\nabla_0 z|^2\,dx\,dt.
\end{aligned}$$

Therefore, from (2.45) we conclude that

$$\|P^-z\|^2 + \|P^+z\|^2 + \iint_Q (s\lambda\xi|\nabla_0 z|^2 + s\lambda^2\xi|\nabla z A\nabla\eta|^2 + s^2\lambda^2\gamma_1(\lambda)\xi^2\gamma_2(s\xi)|z|^2)\,dx\,dt$$
$$+ \iint_Q s^3\xi^3|z|^2(\lambda^3(x_1^{2-\alpha_1} + x_2^{2-\alpha_2}) + \lambda^4|\nabla_0\eta|^4)\,dx\,dt$$
$$\leq C\left[\|e^{-s\sigma}f\|^2 + s^3\lambda^3\int_0^T\int_{\omega_0}\xi^3|z|^2\,dx\,dt + s\lambda\int_0^T\int_{\Omega\backslash V}\xi|\nabla_0 z|^2\,dx\,dt\right]. \qquad (2.46)$$

Now, using the definitions of P^-z and P^+z, we see that

$$s^{-1}\gamma_1(\lambda)\iint_Q \xi^{-1}|z_t|^2\,dx\,dt \leq s^{-1}\gamma(\lambda)\|P^-z\|^2 + s\gamma_1(\lambda)\iint_Q \xi^{-1}|z|^2|div(A\nabla\sigma)|^2\,dx\,dt$$
$$+Cs\gamma_1(\lambda)\iint_Q \xi^{-1}|\nabla z A\nabla\sigma|^2\,dx\,dt$$
$$\leq s^{-1}\gamma(\lambda)\|P^-z\|^2 + Cs\lambda^4\gamma_1(\lambda)\iint_Q \xi^3|\nabla_0\eta|^4|z|^2\,dx\,dt$$
$$+Cs\lambda^2\gamma_1(\lambda)\iint_Q \xi|z|^2\,dx\,dt + Cs\lambda^2\gamma_1(\lambda)\iint_Q \xi|\nabla z A\nabla\eta|^2\,dx\,dt$$

and

$$s^{-1}\gamma_1(\lambda)\iint_Q \xi^{-1}|\triangle_0 z|^2\,dx\,dt \leq s^{-1}\gamma(\lambda)\|P^+z\|^2 + s^3\gamma_1(\lambda)\iint_Q \xi^{-1}|z|^2|\nabla_0\sigma|^2\,dx\,dt$$
$$+Cs\gamma_1(\lambda)\iint_Q \xi^{-1}|\sigma_t|^2|z|^2\,dx\,dt$$
$$\leq s^{-1}\gamma(\lambda)\|P^+z\|^2 + Cs^3\lambda^4\gamma_1(\lambda)\iint_Q \xi^3|\nabla_0\eta|^4|z|^2\,dx\,dt$$
$$+Cs\gamma_1(\lambda)\iint_Q \xi^{3/2}|z|^2\,dx\,dt.$$

From (2.46), we get:

$$\iint_Q [s^{-1}\gamma_1(\lambda)\xi^{-1}(|z_t|^2 + |\triangle_0 z|^2) + s\lambda\xi|\nabla_0 z|^2 + s\lambda^2\xi|\nabla z A\nabla\eta|^2]\,dx\,dt$$
$$\iint_Q |z|^2[s^2\lambda^2\gamma_1(\lambda)\xi^2\gamma_2(s\xi) + s^3\lambda^3\xi^3(x_1^{2-\alpha_1} + x_2^{2-\alpha_2}) + s^3\lambda^4\xi^3|\nabla_0\eta|^4]\,dx\,dt$$
$$\leq C\left[\|e^{-s\sigma}f\|^2 + s^3\lambda^3\int_0^T\int_{\omega_0}\xi^3|z|^2\,dx\,dt + s\lambda\int_0^T\int_{\Omega\backslash V}\xi|\nabla_0 z|^2\,dx\,dt\right]. \qquad (2.47)$$

Let us consider a set $V_1 \subset\subset V$ such that item 2 of Theorem 2.3.1 is still true in $(\overline{\Omega\backslash\omega_0})\backslash V_1$ and let us take $\psi \in C^\infty(\overline{\Omega})$ such that $0 \leq \psi \leq 1$ in Ω, $\psi = 0$ in V_1 and

$\psi = 1$ in $\Omega\backslash V$. Using integration by parts, we have that

$$\begin{aligned}
s\lambda \int_0^T\int_{\Omega\backslash V} \xi|\nabla_0 z|^2\,dx\,dt &\leq s\lambda \iint_Q \nabla z.(\xi\psi A\nabla z)\,dx\,dt = -s\lambda \iint_Q z.div(\xi\psi A\nabla z)\,dx\,dt \\
&\leq s\lambda \iint_Q \xi\nabla\psi A\nabla z z\,dx\,dt + s\lambda^2 \iint_Q \xi z\psi\nabla z A\nabla\eta\,dx\,dt \\
&\quad + s\lambda \iint_Q \xi z\psi\triangle_0 z\,dx\,dt \\
&\leq Cs^{3/2}\lambda \iint_Q \xi^{3/2}|z|^2\,dx\,dt + Cs^{1/2}\lambda \iint_Q \xi|\nabla_0 z|^2\,dx\,dt \\
&\quad + Cs^2\lambda^2 \iint_Q \xi^3(x_1^{2-\alpha_1} + x_2^{2-\alpha_2})|z|^2\,dx\,dt + \lambda^2 \iint_Q \xi|\nabla z A\nabla\eta|^2\,dx\,dt \\
&\quad + Cs^3\lambda^{5/2} \iint_Q \xi^3(x_1^{2-\alpha_1} + x_2^{2-\alpha_2})|z|^2\,dx\,dt + s^{-1}\lambda^{-1/2} \iint_Q \xi^{-1}|\triangle_0 z|^2\,dx\,dt.
\end{aligned}$$

Hense, from (2.47), for s_0 and λ_0 large enough we deduce that

$$\begin{aligned}
&\iint_Q [s^{-1}\gamma_1(\lambda)\xi^{-1}(|z_t|^2 + |\triangle_0 z|^2) + s\lambda\xi|\nabla_0 z|^2 + s\lambda^2\xi|\nabla z A\nabla\eta|^2]\,dx\,dt \\
&\iint_Q |z|^2[s^2\lambda^2\gamma_1(\lambda)\xi^2\gamma_2(s\xi) + s^3\lambda^3\xi^3(x_1^{2-\alpha_1} + x_2^{2-\alpha_2}) + s^3\lambda^4\xi^3|\nabla_0\eta|^4]\,dx\,dt \\
&\qquad \leq C\left[\|e^{-s\sigma}f\|^2 + s^3\lambda^3 \int_0^T\int_{\omega_0} \xi^3|z|^2\,dx\,dt\right].
\end{aligned}$$

Using classical arguments we can come back to the original variable w and conclude the result. ■

Capítulo 3

Estimativas de Carleman para alguns operadores parabólicos degenerados em dimensões superiores

Carleman estimates for some degenerate parabolic operators in higher dimensions

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Abstract

This paper complements the results on Carleman estimates for degenerate parabolic equations present in [3]. We use geometrical assumptions on the observation domain, that depends on the tipe of degenerate operator, to build suitable weight fuctions that allows to deduce Carleman estimates.

Keywords: Null controllability, degenerate parabolic equations, Carleman inequalities.

Mathematics Subject Classification: 34K35, 49J20, 35K10.

3.1 Introduction

The study of the controllability of parabolic equations and systems has attracted the interest of a lot of authors. The theory has been extended to semilinear problems, equations in unbounded domains and Stokes and Navier-Stokes equations; see for instance [15, 18, 20, 21, 24].

On the other hand, it can be said that the study of controllability of degenerate parabolic equations started in the last decade with the works [1, 7, 8, 9, 12, 13, 14].

In this paper we will give continuity to the results present in [3] to the following system in two spatial dimensions:

$$\begin{cases} u_t - div(A\nabla u) + b(x,t)u = g(x,t)1_\omega & \text{in} \quad Q, \\ B.C. & \text{on} \quad \Sigma, \\ u(x,0) = u_0(x) & \text{in} \quad \Omega, \end{cases} \tag{3.1}$$

where $\Omega = (0,1)\times(0,1)$, $\Gamma := \partial\Omega$, $T > 0$, $Q = \Omega\times(0,T)$, $\Sigma := \Gamma\times(0,T)$, $\omega \subset \Omega$ is open, 1_ω is the characteristic function, $b \in L^\infty(Q)$, $g \in L^2(Q_i)$, $u_0 \in L^2(\Omega)$, $A : \overline{\Omega} \mapsto M_{2\times 2}(\mathbb{R})$ is given by

$$A(x) = diag(x_1^{\alpha_1}, x_2^{\alpha_2}),$$

$$B.C. := \begin{cases} u = 0 \ \text{ on } \ \Sigma & \text{if} \ \ \alpha_1, \alpha_2 \in [0,1), \\ u = 0 \ \text{ on } \ \Sigma_{3,4} \ \text{ and } \ (A\nabla u)\nu = 0 \ \text{ on } \ \Sigma_{1,2} & \text{if} \ \ \alpha_1, \alpha_2 \in [1,+\infty), \\ u = 0 \ \text{ on } \ \Sigma_{1,3,4} \ \text{ and } \ (A\nabla u)\nu = 0 \ \text{ on } \ \Sigma_2 & \text{if} \quad \alpha_1 \in [0,1), \ \ \alpha_2 \in [1,+\infty), \\ u = 0 \ \text{ on } \ \Sigma_{2,3,4} \ \text{ and } \ (A\nabla u)\nu = 0 \ \text{ on } \ \Sigma_1 & \text{if} \quad \alpha_1 \in [1,+\infty), \ \ \alpha_2 \in [0,1), \end{cases}$$

$\alpha = (\alpha_1, \alpha_2) \in [0,+\infty) \times [0,+\infty)$, $\Sigma_{i,j,l} := (\Gamma_i \cup \Gamma_j \cup \Gamma_l) \times (0,T)$, and

$$\Gamma_1 := \{0\} \times [0,1], \ \Gamma_2 := [0,1] \times \{0\}, \ \Gamma_3 := \{1\} \times [0,1] \ \Gamma_4 := [0,1] \times \{1\}.$$

In [3] we has been present the well posedness for (3.1). Furthermore, there we has been proved Carleman estimates for the adjoint system

$$\begin{cases} w_t + \triangle_0 w + b(x,t)w = f & \text{in} \quad Q \\ B.C. & \text{on} \quad \Sigma \\ w(x,0) = w_0(x) & \text{in} \quad \Omega \end{cases} \tag{3.2}$$

in the case $\alpha_1, \alpha_2 \in (0,2)$. Infortunately the thecnique used is not adequate to deal with the case of $\alpha_i = 0$ or $\alpha_i \geq 2$ for some $i \in \{1,2\}$. The goal of this work is to present modifications of the thecnique to deduce Carleman estimates for solution of (3.2) in these cases and extend the results to higher spatial dimensions. As we will see, different combinations of the values of α_i requires different weight functions to deduce an Carleman estimate. As usual, we will present some applications of the results in null controllability problems.

This paper is organized as follows: In Section 3.2 we introduced notations and some Hilbert's spaces. Furthermore we recall the results on the well posedness of (3.1). In Section 3.3 we presented the suitable weight function and the associated Carleman estimate for solutions of (3.2). In Section 3.4 we presented some applications on null controllability problems. In Section 3.5 we commented some extensions of the results to higher dimensions. In Section 3.6 we extend the previous results to a problem with a more general degenerate parabolic operator. In Section 3.7 we make other comments on extensions and open questions. The work end's with two appendices contained the proof of the Carleman's estimate presented in the sections 3.3 and 3.6.

3.2 Notations, spaces and preliminaries results

The usual norm and inner product in $L^2(\Omega)$ will be denoted by $|\cdot|$ and $(\cdot,\cdot)$; on the other hand, $\|\cdot\|$ (resp. $|\cdot|_\infty$ and $\|\cdot\|_\infty$) and $((\cdot,\cdot))$ will stand for the norm and the inner product in $L^2(Q)$ (resp. $L^\infty(\Omega)$ and $L^\infty(Q)$). The results present in this section has been proven in [3].

Let us introduce some matrices, spaces and operators:

- $A^r(x) := diag(x_1^{\alpha_1 r}, x_2^{\alpha_2 r})$, with $x \in \overline{\Omega}$, $r \in \mathbb{R}$,
- $H^1_\alpha(\Omega) := \{u \in L^2(\Omega) : \nabla u A \nabla u \in L^1(\Omega)\}$,
- $H^2_\alpha(\Omega) := \{u \in H^1_\alpha(\Omega) : div(A\nabla u) \in L^2(\Omega)\}$,
- $\nabla_0 u := A^{1/2}\nabla u$, $u \in H^1_\alpha(\Omega)$,
- $\Delta_0 u := div(A\nabla u)$, $u \in H^2_\alpha(\Omega)$.

Now let us consider the following norms

- $|u|_\alpha := (|u|^2 + |\nabla_0 u|^2)^{1/2}$, $u \in H^1_\alpha(\Omega)$,
- $|u|_{2,\alpha} := (|u|^2_\alpha + |\triangle_0 u|^2)^{1/2}$, $u \in H^2_\alpha(\Omega)$,

Note that, for those (natural) norms, $H^1_\alpha(\Omega)$, $H^2_\alpha(\Omega)$ and $H^{div}_\alpha(\Omega)$ are Hilbert spaces and one has the following continuous embbedings:

$$H^1(\Omega) \hookrightarrow H^1_\alpha(\Omega) \hookrightarrow L^2(\Omega), \quad H^2_\alpha(\Omega) \hookrightarrow H^1_\alpha(\Omega) \quad \text{and} \quad H^{div}_\alpha(\Omega) \hookrightarrow H^{div}(\Omega).$$

Furthermore, $H^1_\alpha(\Omega) \subset H^1_{loc}(\Omega)$ and $H^2_\alpha(\Omega) \subset H^2_{loc}(\Omega)$.

Lemma 3.2.1 *$H^1_\alpha(\Omega)$ and $H^2_\alpha(\Omega)$ are Hilbert spaces. Furthermore, $C^\infty(\overline{\Omega})$ is dense in $H^1_\alpha(\Omega)$.*

The Lemma 3.2.1 lead us to the following definition

$$H^1_{\alpha,0}(\Omega) := \overline{\mathcal{D}_0}^{H^1_\alpha(\Omega)},$$

where the definition of the space $\mathcal{D}_0$ depends on α:

$$\mathcal{D}_0 := \begin{cases} \{v \in C^\infty(\overline{\Omega}) : supp(v) \subset\subset \Omega\} & \text{if } \alpha_1, \alpha_2 \in [0,1) \\ \{v \in C^\infty(\overline{\Omega}) : \ \exists \delta > 0; supp(v) \subset (0, 1-\delta) \times (0, 1-\delta)\} & \text{if } \alpha_1, \alpha_2 \in [1, +\infty) \\ \{v \in C^\infty(\overline{\Omega}) : \ \exists \delta > 0; supp(v) \subset (\delta, 1-\delta) \times (0, 1-\delta)\} & \text{if } \alpha_1 \in [0,1), \ \alpha_2 \in [1, +\infty) \\ \{v \in C^\infty(\overline{\Omega}) : \ \exists \delta > 0; supp(v) \subset (0, 1-\delta) \times (\delta, 1-\delta)\} & \text{if } \alpha_1 \in [1, +\infty), \ \alpha_2 \in [0,1). \end{cases}$$

Lemma 3.2.2 *The operator $-\Delta_0 : D(\Delta_0) \mapsto L^2(\Omega)$, where $D(\Delta_0) := H^2_\alpha(\Omega) \cap H^1_{\alpha,0}(\Omega)$, is m-dissipative and self-adjoint. Moreover, if $\alpha \neq 1$, then $-\Delta_0$ is strictly dissipative.*

As consequence of Lemma 3.2.2, $-\triangle_0$ is the infinitesimal generator of a strongly continuous semigroup. Thus, using standard thecniques, we can prove the following well-posedness result.

Theorem 3.2.3 *For any $g \in L^2(Q)$ and any $u_0 \in L^2(\Omega)$, there exists a unique solution $u \in C^0([0,T]; L^2(\Omega)) \cap L^2(0,T; H^1_{\alpha,0}(\Omega))$. Furthermore, there exist a constant $C > 0$ such that*

$$\sup_{t\in[0,T]} \|u(t)\|^2 + \int_0^T |u(t)|^2_\alpha \, dt \leq C\left(|u_0|^2 + \|g\|^2\right).$$

3.3 Carleman estimates and null controllability results

In this section we will present the suitable weight functions that allows to deduce the Carleman estimates to solutions of (3.2). Each combination of the values of α_i requires different geometrical assumptions and different weight funtions.

In this section we will assume that there exists $a_0, b_0, \delta_0 \in (0,1)$ such that

$$\omega_0 := \begin{cases} (0,\delta)\times(0,\delta)\subset\omega & \text{if} \quad \alpha_1,\alpha_2\in(0,+\infty) \\ (0,\delta)\times(a_0,b_0)\subset\omega & \text{if} \quad \alpha_1\in(0,+\infty) \ \text{ and } \ \alpha_2=0 \\ (a_0,b_0)\times(0,\delta)\subset\omega & \text{if} \quad \alpha_1=0 \ \text{ and } \ \alpha_2\in(0,+\infty). \end{cases} \tag{3.3}$$

For comparison, in [3], where we has considered the case $\alpha_1, \alpha_2 \in (0,2)$, the geometrial assumption is: There exists numbers $a_i, b_i, \delta \in (0,1)$ with $a_i < b_i$, such that

$$\omega_0 := (a_1,b_1)\times(0,\delta)\cup(0,\delta)\times(a_2,b_2)\subset\omega. \tag{3.4}$$

We note that if some $\alpha_i = 0$, then (3.4) implies (3.3), but not conversly. On the other hand, if $\alpha_1, \alpha_2 \in (0,2)$, then (3.3) implies (3.4), but not conversly.

Now, we will introduce the weight function $\eta \in C^\infty(\overline{\Omega})$ given by

$$\eta(x) := \begin{cases} -(x_1^2+x_2^2)/2 & \text{if} \quad \alpha_1,\alpha_2\in(0,+\infty) \\ \eta_0(x_2)-x_1^2/2 & \text{if} \quad \alpha_1\in(0,+\infty) \ \text{ and } \ \alpha_2=0 \\ \eta_0(x_1)-x_2^2/2 & \text{if} \quad \alpha_1=0 \ \text{ and } \ \alpha_2\in(0,+\infty), \end{cases} \tag{3.5}$$

where $\eta_0 \in C^\infty([0,1])$ is such that $\eta_0(x) = x$ in $[0, a_0]$ and $\eta_0(x) = -x$ in $[b_0, 1]$. Furthermore, for $\lambda > \lambda_0$ and $s \in \mathbb{R}$, let us introduce

$$\theta(t) := [t(T-t)]^{-4}, \quad \xi(x,t) := \theta(t)e^{2\lambda(|\eta|_\infty+\eta(x))}$$
$$\text{and} \quad \sigma(x,t) := \theta(t)e^{4\lambda|\eta|_\infty} - \xi(x,t).$$

Note that, in any case we have that

$$|\nabla_0\eta|^2 > C > 0 \quad \text{in} \quad \overline{\Omega\backslash\omega_0}. \tag{3.6}$$

The property (3.6) is fundamental to deduce an Carleman estimate. Furthermore, there are other properties that the function η must have and one of them is:

$$\iint_\Sigma \xi\left[|\nabla_0 z|^2\nabla\eta A\nu - 2(\nabla zA\nabla\eta)A\nabla z.\nu\right]\,ds\,dt \geq 0 \quad \forall z\in L^2(0,T;H^2_\alpha(\Omega)\cap H^1_{\alpha,0}(\Omega)) \tag{3.7}$$

The property (3.7) can be easily deduced using the following result:

Lemma 3.3.1 *Assume that $v \in H^2_\alpha(\Omega) \cap H^1_{\alpha,0}(\Omega)$. If $v = 0$ on Γ_1 (respec Γ_3), then*

$$\frac{\partial v}{\partial x_2} = 0 \quad on \quad \Gamma_1 \quad (respec \quad \Gamma_3).$$

Moreover, If $v = 0$ on Γ_2 (respec Γ_4), then

$$\frac{\partial v}{\partial x_1} = 0 \quad on \quad \Gamma_2 \quad (respec \quad \Gamma_4).$$

The proof of Lemma 3.3.1 is given in [3].

The main result in this section is the following:

Theorem 3.3.2 *Assume (3.3). There exists constants C, s_0, λ_0 such that for any $\lambda > \lambda_0$, $s > s_0$ and w solution of (3.2) one has*

$$\iint_Q e^{-2s\sigma}\left[s^{-1}\xi^{-1}\left(|w_t|^2 + |div(A\nabla w)|^2\right) + s\lambda^2\xi\left(|\nabla_0 w|^2 + |\nabla w A\nabla\eta|^2\right)\right] dx\,dt$$
$$+s^3\lambda^4\iint_Q e^{-2s\sigma}\xi^3|w|^2\,dx\,dt \leq C\left(\|e^{-s\sigma}f\|^2 + s^3\lambda^4\iint_{\omega_T} e^{-2s\sigma}\xi^3|w|^2\,dx\,dt\right).$$

The proof of Theorem 3.3.2 is given in Appendix A.

3.4 Application to null controllability

In this section we will discuss the null controllability of (3.1). This is stablished in the following result:

Theorem 3.4.1 *Let us fix $T > 0$, $\alpha_1, \alpha_2 \in [0, +\infty)$ and a open set $\omega \subset \Omega$. Assume that (3.3) holds. Then, for any $u_0 \in L^2(\Omega)$, there exists a control g such that the solution u to (3.1) satisfies*

$$u(., T) = 0 \quad in \quad \Omega. \tag{3.8}$$

Moreover, there exist a constant $C = C(T, \alpha, \omega) > 0$ such that

$$\|g\| \leq C|u_0|.$$

As is classical in null controllability problems, the Theorem 3.4.1 is equivalent to an observabillity property for the adjoint system (3.2). This is the goal of the following result:

Theorem 3.4.2 *Let us fix $T > 0$, $\alpha_1, \alpha_2 \in [0, +\infty)$ and a open set $\omega \subset \Omega$. Assume that (3.3) holds. Then there exist a constant $C = C(T, \alpha, \omega) > 0$ soch that, for any $w_T \in L^2(\Omega)$, the solution w of (3.2) satisfies*

$$\int_\Omega |w(x,0)|^2\,dx \leq C\int_0^T\int_\omega |w|^2\,dx\,dt.$$

The way that leads the Carleman estimate present in Theorem 3.3.2 to Theorem 3.4.2 is standard and we refer to [10] for details.

3.5 Extension to higher dimensions

In this section we will discuss the problem in spatial dimension higher than 2. In order to not deal with a large set of notations, we will consider only the spatial dimension 3.

Hense, let us consider the system (3.1) with

$$\Omega := \prod_{i=1}^{3}(0,1), \quad \Gamma_1 := \{0\} \times [0,1] \times [0,1], \quad \Gamma_2 := [0,1] \times \{0\} \times [0,1],$$
$$\Gamma_3 := [0,1] \times [0,1] \times \{0\}, \quad \Gamma_4 := \{1\} \times [0,1] \times [0,1], \quad \Gamma_5 := [0,1] \times \{1\} \times [0,1],$$
$$\Gamma_6 := [0,1] \times [0,1] \times \{1\}, \quad A(x) := diag(x_1^{\alpha_1}, x_2^{\alpha_2}, x_3^{\alpha_3}),$$

where $\alpha_i \in [0,+\infty)$ and analogous boundary conditions depending on the degeneracy.

Now let us assume the following geometrical condition: There exists numbers $a_i, b_i, \delta \in (0,1)$, $i = 1,2,3$, such that

$$\omega_0 := \begin{cases} (0,\delta)\times(0,\delta)\times(0,\delta) \subset \omega & \text{if} \quad \alpha_1,\alpha_2,\alpha_3 \in (0,+\infty) \\ (0,\delta)\times(0,\delta)\times(a_3,b_3) \subset \omega & \text{if} \quad \alpha_1,\alpha_2 \in (0,+\infty), \quad \alpha_3 = 0 \\ (0,\delta)\times(a_2,b_2)\times(0,\delta) \subset \omega & \text{if} \quad \alpha_1,\alpha_3 \in (0,+\infty), \quad \alpha_2 = 0 \\ (a_1,b_1)\times(0,\delta)\times(0,\delta) \subset \omega & \text{if} \quad \alpha_2,\alpha_3 \in (0,+\infty), \quad \alpha_1 = 0 \\ (0,\delta)\times(a_2,b_2)\times(a_3,b_3) \subset \omega & \text{if} \quad \alpha_1 \in (0,+\infty), \quad \alpha_2 = \alpha_3 = 0 \\ (a_1,b_1)\times(0,\delta)\times(a_3,b_3) \subset \omega & \text{if} \quad \alpha_2 \in (0,+\infty), \quad \alpha_1 = \alpha_3 = 0 \\ (a_1,b_1)\times(a_2,b_2)\times(0,\delta) \subset \omega & \text{if} \quad \alpha_3 \in (0,+\infty), \quad \alpha_1 = \alpha_2 = 0 \end{cases} \tag{3.9}$$

and let us introduce the weight function $\eta \in C^\infty(\overline{\Omega})$ given by

$$\eta(x) := \begin{cases} -(x_1^2+x_2^2+x_3^2)/2 & \text{if} \quad \alpha_1,\alpha_2,\alpha_3 \in (0,+\infty) \\ -(x_1^2+x_2^2)/2 + \eta_3(x_3) & \text{if} \quad \alpha_1,\alpha_2 \in (0,+\infty), \quad \alpha_3 = 0 \\ -(x_1^2+x_3^2)/2 + \eta_2(x_2) & \text{if} \quad \alpha_1,\alpha_3 \in (0,+\infty), \quad \alpha_2 = 0 \\ -(x_2^2+x_3^2)/2 + \eta_1(x_1) & \text{if} \quad \alpha_2,\alpha_3 \in (0,+\infty), \quad \alpha_1 = 0 \\ -x_1^2/2 + \eta_2(x_2) + \eta_3(x_3) & \text{if} \quad \alpha_1 \in (0,+\infty), \quad \alpha_2 = \alpha_3 = 0 \\ -x_2^2/2 + \eta_1(x_1) + \eta_3(x_3) & \text{if} \quad \alpha_2 \in (0,+\infty), \quad \alpha_1 = \alpha_3 = 0 \\ -x_3^2/2 + \eta_1(x_1) + \eta_2(x_2) & \text{if} \quad \alpha_3 \in (0,+\infty), \quad \alpha_1 = \alpha_2 = 0 \end{cases} \tag{3.10}$$

where $\eta_i \in C^\infty([0,1])$ is such that $\eta_i(x) = x$ in $[0,a_i]$ and $\eta_i(x) = -x$ in $[b_i,1]$.

Note that, in any case, one has that (3.6) holds. Now, to get (3.7) we must extend the Lemma 3.3.1:

Lemma 3.5.1 *Assume that* $v \in H^2_\alpha(\Omega) \cap H^1_{\alpha,0}(\Omega)$. *If* $i \in \{1,2,3\}$ *and* $v = 0$ *on* Γ_i *(respec* Γ_{i+3}*), then*

$$\frac{\partial v}{\partial x_j} = 0 \quad on \quad \Gamma_i \quad (respec \quad \Gamma_{i+3}), \quad for \quad j \in \{1,2,3\} \quad and \quad j \neq i.$$

The proof of Lemma 3.5.1 is very similar to the Lemma 3.3.1 and will not be given.

Now, with properties (3.6) and (3.7), we can repeat the same calculations present in Appendix A and deduce that Theorem 3.3.2 still true assuming (3.9).

3.6 Extension to more general degenerate operators

In this section we will consider the problem with a more general degenerate operator:

$$\begin{cases} u_t - div(\tilde{A}\nabla u) + b(x,t)u = g(x,t)1_\omega & \text{in} \quad Q, \\ \widetilde{B.C.} & \text{on} \quad \Sigma, \\ u(x,0) = u_0(x) & \text{in} \quad \Omega, \end{cases} \tag{3.11}$$

where $\widetilde{B.C}$ is similar to $B.C.$ with A replaced by $\tilde{A} := A + B$ and $B : \overline{\Omega} \mapsto M_{2\times 2}(\mathbb{R})$ is such that

1. $b_{ij} \in C^1(\overline{\Omega})$, where $B(x) = (b_{ij}(x))$;
2. $B(x)$ is simetric $\forall x \in \overline{\Omega}$;
3. $B(x)\zeta.\zeta \geq 0 \ \ \forall x \in \overline{\Omega}$ and $\zeta \in \mathbb{R}^2$;
4. $1 + b_{11} + x_2 b_{12} \geq 0$ on Γ_3 and $1 + b_{22} + x_1 b_{12} \geq 0$ on Γ_4;
5. $1 + b_{11} - x_2 b_{12} \geq 0$ on Γ_1 if $\alpha_1 = 0$ and $1 + b_{22} + x_1 b_{12} \geq 0$ on Γ_2 if $\alpha_2 = 0$;
6. there exists $\epsilon > 0$ and $p_{ij} \in C^1(\overline{\Omega})$ such that
$$b_{ij}(x) = x_i^{\alpha_i/2} x_j^{\alpha_j/2} p_{ij}(x) \quad \text{and} \quad p_{12}(x) \geq 0 \ \ \forall x \in \overline{\Omega} \backslash (\epsilon, 1] \times (\epsilon, 1]. \tag{3.12}$$

From item 3 of (3.12) we have that

$$|\nabla \eta \tilde{A} \nabla \eta| \geq |\nabla_0 \eta| \geq C > 0 \quad \text{in} \quad \overline{\Omega \backslash \omega_0}. \tag{3.13}$$

Now we need an estimate similar to (3.7). This is the goal of the following result

Lemma 3.6.1 *Assume that (3.12) holds. For all* $z \in L^2(0,T; H^2_\alpha(\Omega) \cap H^1_{\alpha,0}(\Omega))$ *one has*

$$\iint_\Sigma \xi \left[|\nabla z \tilde{A} \nabla z| \nabla \eta \tilde{A} \nu - 2(\nabla z \tilde{A} \nabla \eta) \tilde{A} \nabla z.\nu \right] ds\, dt \geq 0.$$

Proof. For instace assume that $\alpha_1, \alpha_2 \in (0, +\infty)$. Using Lemma 3.3.1 it is easy to see that $\nabla z \tilde{A} \nabla z = \nabla z \tilde{A} \nu = 0$ on $[\Gamma_1 \cup \Gamma_2] \times (0,T)$. On the other hand we have that

$$|\nabla z \tilde{A} \nabla z| \nabla \eta \tilde{A} \nu - 2(\nabla z \tilde{A} \nabla \eta) \tilde{A} \nabla z.\nu = \left| \frac{\partial z}{\partial x_1} \right|^2 (1 + b_{11})(1 + b_{11} + x_2 b_{12}) \quad \text{on} \quad \Gamma_3 \tag{3.14}$$

and

$$|\nabla z \tilde{A} \nabla z| \nabla \eta \tilde{A} \nu - 2(\nabla z \tilde{A} \nabla \eta) \tilde{A} \nabla z.\nu = \left| \frac{\partial z}{\partial x_2} \right|^2 (1 + b_{22})(1 + b_{22} + x_1 b_{12}) \quad \text{on} \quad \Gamma_4 \tag{3.15}$$

Hense the result follows from item 4 of (3.12).

Now, assume that $\alpha_1 \in (0, +\infty)$ and $\alpha_2 = 0$. Analogously to the previous case, we have that (3.14) and (3.15) holds and $\nabla z \tilde{A} \nabla z = \nabla z \tilde{A} \nu = 0$ on $\Gamma_1 \times (0,T)$. Furthermore, one has

$$|\nabla z \tilde{A} \nabla z| \nabla \eta \tilde{A} \nu - 2(\nabla z \tilde{A} \nabla \eta) \tilde{A} \nabla z.\nu = \left| \frac{\partial z}{\partial x_2} \right|^2 (1 + b_{22})(1 + b_{22} - x_1 b_{12}) \quad \text{on} \quad \Gamma_2.$$

Hense, again the result follows from item 4 of (3.12). In a similar way we conclude the result if $\alpha_1 = 0$ and $\alpha_2 \in (0, +\infty)$. ■

The adjoint system assossiated to (3.11) is

$$\begin{cases} w_t + div(\tilde{A} \nabla w) + b(x,t) w = f & \text{in} \quad Q \\ B.C. & \text{on} \quad \Sigma \\ w(x,0) = w_0(x) & \text{in} \quad \Omega \end{cases} \tag{3.16}$$

The main result in this section is

Theorem 3.6.2 *Assume (3.3) and (3.12). There exists constants* C, s_0, λ_0 *such that for any* $\lambda > \lambda_0$, $s > s_0$ *and* w *solution of (3.16) one has*

$$\iint_Q e^{-2s\sigma} \left[s^{-1} \xi^{-1} \left(|w_t|^2 + |div(\tilde{A} \nabla w)|^2 \right) + s \lambda^2 \xi \left(|\nabla w \tilde{A} \nabla w|^2 + |\nabla w \tilde{A} \nabla \eta|^2 \right) \right] dx\, dt$$
$$+ s^3 \lambda^4 \iint_Q e^{-2s\sigma} \xi^3 |w|^2 \, dx\, dt \leq C \left(\|e^{-s\sigma} f\|^2 + s^3 \lambda^4 \iint_{\omega_T} e^{-2s\sigma} \xi^3 |w|^2 \, dx\, dt \right).$$

The proof of Theorem 3.6.2 is very similar to the previous theorem, but for completeness, we give a sketch in Appendix B.

3.7 Open questions on others extensions

The study of the null controllability of degenerate parabolic equations in higher spatial dimensions started few years ago and still there many opens questions. In this section we discuss some opens questions and comment other extensions of the results proved.

3.7.1 On systems with gradient

Let us consider the following systems

$$\begin{cases} u_t - \Delta_0 u + B.(A^{1/2}\nabla u) + bu = g1_\omega & \text{in} \quad Q \\ B.C. & \text{on} \quad \Sigma \\ u(x,0) = u_0(x) & \text{in} \quad \Omega \end{cases} \tag{3.17}$$

and

$$\begin{cases} u_t - \Delta_0 u + B.\nabla u + bu = g1_\omega & \text{in} \quad Q \\ B.C. & \text{on} \quad \Sigma \\ u(x,0) = u_0(x) & \text{in} \quad \Omega \end{cases} \tag{3.18}$$

where $B \in L^\infty(Q)^2$.

As we comment in [2], all the results in this paper still true for (3.17) and, for (3.18) if $\alpha_1, \alpha_2 \in (0, 1/2)$. To our best knowledge, an Carleman estimate for (3.18) with $\alpha_i \in [1/2, +\infty)$ remains as a open question, even in spatial dimension 1.

3.7.2 On semilinear problems

The results of this work can be extend, in a standard way, to semilinear problems of the kind:

$$\begin{cases} u_t - div(A\nabla u) + F_0(u) = g(x,t)1_\omega & \text{in} \quad Q, \\ B.C. & \text{on} \quad \Sigma, \\ u(x,0) = u_0(x) & \text{in} \quad \Omega, \end{cases} \tag{3.19}$$

where $F_0 : \mathbb{R} \mapsto \mathbb{R}$ is globally lipschitz continous.

Now, let us consider the following semilinear problem

$$\begin{cases} u_t - div(A\nabla u) + F_1(\nabla u, u) = g(x,t)1_\omega & \text{in} \quad Q, \\ B.C. & \text{on} \quad \Sigma, \\ u(x,0) = u_0(x) & \text{in} \quad \Omega, \end{cases} \tag{3.20}$$

where $F_1 : \mathbb{R}^2 \times \mathbb{R} \mapsto \mathbb{R}$ is globally lipschitz continous. This problem remains unsolved even in spatial dimension 1. The reason is that the most natural way to solve him is by the use of Carleman estimates for solutions of the adjoint of (3.18). Hense, from the discussion of the previous section, the results of this paper only can be extended to (3.20) if one of the following assumptions are satisfied:

1. $|F_1(V,t) - F_1(U,r)| \leq C[|A(V-U)| + |t-r|], \quad \forall U, V \in \mathbb{R}^2 \text{ and } t, r \in \mathbb{R};$
2. $\alpha_1, \alpha_2 \in (0, 1/2)$.

3.7.3 On others degenerate operators

With respect to other degenerate parabolic operators, there is a whole range of inexplored problems. In [10], the authors deal with a spatial domain Ω with boundary Γ of class C^4 and a matrix A whose only the first eigenvalue degenerates and this degeneration ocours on the whole border. Hense, questions as matrix with more eigenvalues degenerating are interesting. Problems where the degeneration ocours only in a part of the boundary is very interesting too. This work deal with this situation, but the question still open in other kind of domains.

3.7.4 On assumptions on the observations domains

It is well know that (3.1), in space dimension 1, is, in general, not null controllable if $\alpha = 2$, see [2, 7]. This "in general", should be understood in the sense that the null controllability does not hold for an arbitrary observation domain ω. However, as we has been proved in [2], if ω has some suitable geometrical proprieties, the null controllability's holds. When $\alpha \in (0, 2)$ the null controllability of (3.1) holds without restrictions for ω. In spatial dimension 2, we can't deduce Carleman estimates for solutions of (3.2) without impose some geometrical restrictions on ω, even if $\alpha_i \in (0, 2)$. Hense, an interesting question is study other kind of geometrical properties on ω that leads to Carleman estimates to solutions of (3.2).

3.8 Appendix A: Proof of Theorem 3.3.2

It is sufficient consider the case $b = 0$. In the sequel $C > 0$ is a constant that depends on T and ω and w is a solution of (3.2). From a density argument we can assume that w is sufficiently regular.

For $s > s_0 > 0$ we introduce

$$z = e^{-s\sigma} w.$$

We see that

$$z = \frac{\partial z}{\partial x_i} = 0 \text{ at } t = 0 \text{ and } t = T, \tag{3.21}$$

$$B.C. \text{ holds on } \Sigma, \tag{3.22}$$

$$P^- z + P^+ z = g, \tag{3.23}$$

where

$$g := e^{-s\sigma} f - s\lambda^2 \xi |\nabla_0 \eta|^2 z + s\lambda\xi div(A\nabla\eta) z,$$

$$P^-(z) := I_{11} + I_{12} + I_{13} := -2s\lambda^2 \xi |\nabla_0 \eta|^2 z - 2s\lambda\xi \nabla z A \nabla\eta + z_t$$

and

$$P^+(z) := I_{21} + I_{22} + I_{23} := s^2\lambda^2\xi^2 |\nabla_0\eta|^2 z + div(A\nabla z) + s\sigma_t z.$$

From (3.23) we have that

$$\|P^- z\|^2 + \|P^+ z\|^2 + 2((P^- z, P^+ z)) = \|g\|^2. \tag{3.24}$$

Now, let us estimate $((P^- z, P^+ z))$. Using (3.21) and (3.22) we see that

$$((I_{11}, I_{21})) = -2s^3\lambda^4 \iint_Q \xi^3 |\nabla_0\eta|^4 |z|^2 \, dy \, dt,$$

$$\begin{aligned} ((I_{12}, I_{21})) &= -s^3\lambda^3 \iint_Q \xi^3 |\nabla_0\eta|^2 \nabla(|z|^2) A\nabla\eta \, dx \, dt = s^3\lambda^3 \iint_Q |z|^2 div(\xi^3 |\nabla_0\eta|^2 A\nabla\eta) \, dx \, dt \\ &= 3s^3\lambda^4 \iint_Q \xi^3 |\nabla_0\eta|^4 |z|^2 \, dx \, dt + s^3\lambda^3 \iint_Q \xi^3 div(|\nabla_0\eta|^2 A\nabla\eta) |z|^2 \, dx \, dt \end{aligned}$$

and

$$((I_{13}, I_{21})) = -s^2\lambda^2 \iint_Q \xi\xi_t |\nabla_0\eta|^2 |z|^2 \, dy \, dt.$$

Hense, from (3.6), for s_0 and λ_0 sufficiently large one has

$$
\begin{aligned}
((P^-z, I_{21})) &= s^3\lambda^4 \iint_Q \xi^3|\nabla_0\eta|^4|z|^2\,dx\,dt + s^3\lambda^3 \iint_Q \xi^3 div(|\nabla_0\eta|^2 A\nabla\eta)|z|^2\,dx\,dt \\
&\quad -s^2\lambda^2 \iint_Q \xi\xi_t|\nabla_0\eta|^2|z|^2\,dy\,dt \\
&\geq Cs^3\lambda^4 \iint_Q \xi^3|z|^2\,dx\,dt - Cs^3\lambda^4 \int_0^T\int_{\omega_0} \xi^3|z|^2\,dx\,dt. \qquad (3.25)
\end{aligned}
$$

Furthermore,

$$
((I_{11}, I_{23})) = -2s^2\lambda^2 \iint_Q \xi\sigma_t|\nabla_0\eta|^2|z|^2\,dy\,dt,
$$

$$
\begin{aligned}
((I_{12}, I_{23})) &= s^2\lambda \iint_Q |z|^2 div(\xi\sigma_t A\nabla\eta)\,dx\,dt \\
&= s^2\lambda^2 \iint_Q \xi\sigma_t|\nabla_0\eta|^2|z|^2\,dx\,dt - s^2\lambda^2 \iint_Q \xi\xi_t|\nabla_0\eta|^2|z|^2\,dx\,dt \\
&\quad +s^2\lambda \iint_Q \xi div(A\nabla\eta)|z|^2\,dx\,dt
\end{aligned}
$$

and

$$
((I_{13}, I_{23})) = -\frac{s}{2} \iint_Q \sigma_{tt}|z|^2\,dx\,dt.
$$

Hense, from (3.25) we see that

$$
((P^-z, I_{12} + I_{32})) \geq Cs^3\lambda^4 \iint_Q \xi^3|z|^2\,dx\,dt - Cs^3\lambda^4 \int_0^T\int_{\omega_0} \xi^3|z|^2\,dx\,dt. \qquad (3.26)
$$

On the other hand,

$$
\begin{aligned}
((I_{11}, I_{22})) &= 2s\lambda^2 \iint_Q A\nabla z.\nabla(\xi|\nabla_0\eta|^2 z)\,dx\,dt \\
&= 2s\lambda^2 \iint_Q \xi|\nabla_0\eta|^2|\nabla_0 z|^2\,dx\,dt + 2s\lambda^3 \iint_Q \xi|\nabla_0\eta|^2 z\nabla z A\nabla\eta\,dx\,dt \\
&\quad +2s\lambda^2 \iint_Q \xi z\nabla z A\nabla(|\nabla_0\eta|^2)\,dx\,dt \\
&= -s\lambda^3 \iint_Q \xi|z|^2\nabla\eta A\nabla(\nabla_0\eta|^2)\,dx\,dt - s\lambda^2 \iint_Q \xi div(A\nabla(|\nabla_0\eta|^2))|z|^2\,dx\,dt \\
&\quad +2s\lambda^2 \iint_Q \xi|\nabla_0\eta|^2|\nabla_0 z|^2\,dx\,dt + 2s\lambda^3 \iint_Q \xi|\nabla_0\eta|^2 z\nabla z A\nabla\eta\,dx\,dt, \qquad (3.27)
\end{aligned}
$$

$$
\begin{aligned}
((I_{13}, I_{22})) &= -\iint_Q z div(A\nabla z_t)\,dx\,dt = \iint_Q \nabla z_t A\nabla z\,dx\,dt \\
&= \frac{1}{2} \iint_Q \frac{\partial}{\partial t}(|\nabla_0 z|^2)\,dx\,dt = 0, \qquad (3.28)
\end{aligned}
$$

and

$$
\begin{aligned}
((I_{12}, I_{22})) &= 2s\lambda \iint_Q A\nabla z.\nabla(\xi \nabla z A \nabla \eta)\, dx\, dt - 2s\lambda \iint_\Sigma \xi(\nabla z A \nabla \eta) A \nabla z.\nu\, ds\, dt \\
&= 2s\lambda^2 \iint_Q \xi |\nabla z A \nabla \eta|^2\, dx\, dt + 2s\lambda \iint_Q \xi \nabla z A \nabla(\nabla z A \nabla \eta)\, dx\, dt \\
&\quad -2s\lambda \iint_\Sigma \xi(\nabla z A \nabla \eta) A \nabla z.\nu\, ds\, dt. \qquad (3.29)
\end{aligned}
$$

Now we will to compute the second integral on the right hand side of (3.29). First, note that in any case we have that

$$
\frac{\partial}{\partial x_i}\left(x_j^{\alpha_j} \frac{\partial \eta}{\partial x_j} \right) = 0 \quad \text{for} \quad i \neq j.
$$

Hense

$$
\begin{aligned}
2s\lambda \iint_Q \xi \nabla z A \nabla(\nabla z A \nabla \eta)\, dx\, dt &= -2s\lambda \sum_{i,j=1}^{2} \iint_Q \xi \frac{\partial z}{\partial x_i} x_i^{\alpha_i} \frac{\partial}{\partial x_i}\left(\frac{\partial z}{\partial x_j} x_j^{\alpha_j} \frac{\partial \eta}{\partial x_j} \right) dx\, dt \\
&= s\lambda \sum_{i,j=1}^{2} \iint_Q \xi x_i^{\alpha_i} x_j^{\alpha_j} \frac{\partial \eta}{\partial x_j} \frac{\partial}{\partial x_j}\left(\left| \frac{\partial z}{\partial x_i} \right|^2 \right) dx\, dt + 2s\lambda \sum_{i=1}^{2} \iint_Q \xi \left| \frac{\partial z}{\partial x_i} \right|^2 x_i^{\alpha_i} \frac{\partial}{\partial x_i}\left(x_i^{\alpha_i} \frac{\partial \eta}{\partial x_i} \right) dx\, dt \\
&= -s\lambda \sum_{i,j=1}^{2} \iint_Q \left| \frac{\partial z}{\partial x_i} \right|^2 \xi \left[\lambda x_i^{\alpha_i} x_j^{\alpha_j} \left| \frac{\partial \eta}{\partial x_j} \right|^2 + \frac{\partial}{\partial x_j}(x_i^{\alpha_i})\, x_j^{\alpha_j} \frac{\partial \eta}{\partial x_j} + x_i^{\alpha_i} \frac{\partial}{\partial x_j}\left(x_j^{\alpha_j} \frac{\partial \eta}{\partial x_j} \right) \right] dx\, dt \\
&\quad +2s\lambda \sum_{i=1}^{2} \iint_Q \xi \left| \frac{\partial z}{\partial x_i} \right|^2 x_i^{\alpha_i} \frac{\partial}{\partial x_i}\left(x_i^{\alpha_i} \frac{\partial \eta}{\partial x_i} \right) dx\, dt + s\lambda \sum_{i,j=1}^{2} \iint_\Sigma \xi x_i^{\alpha_i} \left| \frac{\partial z}{\partial x_i} \right|^2 \frac{\partial \eta}{\partial x_j} x_j^{\alpha_j} \nu_j\, ds\, dt \\
&= -s\lambda^2 \iint_Q \xi |\nabla_0 \eta|^2 |\nabla_0 z|^2\, dx\, dt + s\lambda \sum_{i=1}^{2} \iint_Q \xi \left| \frac{\partial z}{\partial x_i} \right|^2 x_i^{\alpha_i} \left(\alpha_i x_i^{\alpha_i - 1} \frac{\partial \eta}{\partial x_i} \right) dx\, dt \\
&\quad +2s\lambda \sum_{i=1}^{2} \iint_Q \xi x_i^{\alpha_i} \left| \frac{\partial z}{\partial x_i} \right|^2 x_i^{\alpha_i} \frac{\partial^2 \eta}{\partial x_i^2}\, dx\, dt + s\lambda \iint_\Sigma \xi |\nabla_0 z|^2 \nabla \eta A \nu\, ds\, dt. \qquad (3.30)
\end{aligned}
$$

Using (3.7), from (3.27), (3.28), (3.29) and (3.30) we get

$$
\begin{aligned}
((P^- z, I_{22})) &\geq Cs\lambda^2 \iint_Q \xi \left[|\nabla_0 \eta|^2 |\nabla_0 z|^2 + |\nabla z A \nabla \eta|^2 \right] dx\, dt \\
&\quad -Cs\lambda \iint_Q \xi |\nabla_0 z|^2\, dx\, dt - Cs\lambda^3 \iint_Q \xi^3 |z|^2\, dx\, dt \qquad (3.31)
\end{aligned}
$$

Using (3.6), from (3.26) and (3.31) we see that

$$
\begin{aligned}
((P^- z, P^+ z)) &\geq C \iint_Q \left[s\lambda^2 \xi (|\nabla_0 z|^2 + |\nabla z A \nabla \eta|^2) + s^3 \lambda^4 \xi^3 |z|^2 \right] dx\, dt \\
&\quad -C \int_0^T \int_{\omega_0} \left[s\lambda^2 \xi |\nabla_0 z|^2 + s^3 \lambda^4 \xi^3 |z|^2 \right] dx\, dt \qquad (3.32)
\end{aligned}
$$

From (3.24) we deduce that

$$\|P^-z\|^2 + \|P^+z\| + s\lambda^2 \iint_Q \xi \left[|\nabla_0 z|^2 + |\nabla z A \nabla \eta|^2\right] dx\,dt + s^3\lambda^4 \iint_Q \xi^3 |z|^2\, dx\,dt$$
$$\leq C\left[\|g\|^2 + \iint_{\omega_{0T}} \left[s\lambda^2\xi|\nabla_0 z|^2 + s^3\lambda^4\xi^3|z|^2\right] dx\,dt\right]. \qquad (3.33)$$

From the definitions of P^-z and P^+z we see that

$$s^{-1}\iint_Q \xi^{-1}|z_t|^2\,dx\,dt \leq Cs^{-1}\|P^-z\|^2 + Cs\lambda^4\iint_Q \xi^3|z|^2\,dx\,dt$$
$$+Cs\lambda^2\iint_Q \xi|\nabla z A\nabla\eta|^2\,dx\,dt,$$

and

$$s^{-1}\iint_Q \xi^{-1}|div(A\nabla z)|^2\,dx\,dt \leq s^{-1}\|P^+z\|^2 + Cs^3\lambda^4\iint_Q \xi^3|z|^2\,dx\,dt.$$

Hense, from (3.33) we deduce that

$$s^{-1}\iint_Q \xi^{-1}\left(|z_t|^2 + |div(A\nabla z)|^2\right) dx\,dt + s\lambda^2\iint_Q \xi\left[|\nabla_0 z|^2 + |\nabla z A\nabla\eta|^2\right] dx\,dt$$
$$+s^3\lambda^4\iint_Q \xi^3|z|^2\,dx\,dt \leq C\left[\|g\|^2 + \iint_{\omega_{0T}}\left[s\lambda^2\xi|\nabla_0 z|^2 + s^3\lambda^4\xi^3|z|^2\right] dx\,dt\right]. \qquad (3.34)$$

Now let us consider a open set $\omega_0 \subset\subset \omega_1 \subset\subset \omega$ and a function $\psi \in C^\infty(\overline{\Omega})$ such that $0 \leq \psi \leq 1$, $\psi = 1$ in ω_0 and $\psi = 0$ in $\Omega\backslash\omega_1$. For any $\epsilon > 0$ we have that

$$\begin{aligned}
s\lambda^2\iint_{\omega_{0T}}\xi|\nabla_0 z|^2\,dx\,dt &\leq s\lambda^2\iint_{\omega_{1T}}\xi\psi\nabla z A\nabla z\,dx\,dt \\
&= -s\lambda^3\iint_{\omega_{1T}}\xi\psi z\nabla z A\nabla\eta\,dx\,dt - s\lambda^2\iint_{\omega_{1T}}\xi z\nabla z A\nabla\psi\,dx\,dt \\
&\quad -s\lambda^2\iint_{\omega_{1T}}\xi\psi z div(A\nabla z)\,dx\,dt \\
&\leq Cs^2\lambda^4\iint_Q \xi^3|z|^2\,dx\,dt + \lambda^2\iint_Q \xi|\nabla z A\nabla\eta|^2\,dx\,dt \\
&\quad \iint_Q \xi|\nabla_0 z|^2\,dx\,dt + C\epsilon^{-1}s^3\lambda^4\iint_{\omega_T}\xi^3|z|^2\,dx\,dt \\
&\quad \epsilon^{-1}s^{-1}\iint_Q \xi^{-1}|div(A\nabla z)|^2\,dx\,dt.
\end{aligned}$$

Hense, for ϵ sufficiently small and for s_0 and λ_0 sufficiently large, from (3.34) we deduce that

$$s^{-1}\iint_Q \xi^{-1}\left(|z_t|^2 + |div(A\nabla z)|^2\right) dx\,dt + s\lambda^2\iint_Q \xi\left[|\nabla_0 z|^2 + |\nabla z A\nabla\eta|^2\right] dx\,dt$$
$$+s^3\lambda^4\iint_Q \xi^3|z|^2\,dx\,dt \leq C\left[\|g\|^2 + s^3\lambda^4\iint_{\omega_T}\xi^3|z|^2\,dx\,dt\right]. \qquad (3.35)$$

Using classical arguments we can coming back to the original variable w and deduce that (3.35) still true with z repalced by w and g replaced by f. This conclude the proof.

3.9 Appendix B: Scketch of the proof of the Theorem 3.6.2

The main difference from this case to the previous one is the computation of the term in the left hand side of (3.30) and the estimate (3.31). Hense we will discuss only this part of the proof. Firstly, a straightfull calculations shows that

$$\begin{aligned} 2s\lambda \iint_Q \xi \nabla z \tilde{A} \nabla \left(\nabla z \tilde{A} \nabla \eta \right) \, dx\, dt &= -s\lambda^2 \iint_Q \xi |\nabla z \tilde{A} \nabla z|.|\nabla \eta \tilde{A} \nabla \eta| \, dx\, dt + \sum_{i=1}^{8} T_i \\ &+ s\lambda \iint_\Sigma \xi |\nabla z \tilde{A} \nabla z| (\nabla \eta \tilde{A} \nu) \, ds\, dt, \end{aligned} \tag{3.36}$$

where $T_1, ..., T_8$ are given by

$$\begin{aligned} T_1 &:= 2s\lambda \sum_{i=1}^{2} \iint_Q \xi \left| \frac{\partial z}{\partial x_i} \right|^2 x_i^{\alpha_i} \frac{\partial}{\partial x_i} \left(x_i^{\alpha_i} \frac{\partial \eta}{\partial x_i} \right) dx\, dt, \\ T_2 &:= -s\lambda \sum_{i,j=1}^{2} \iint_Q \xi \left| \frac{\partial z}{\partial x_i} \right|^2 \frac{\partial}{\partial x_j} \left(x_i^{\alpha_i} x_j^{\alpha_j} \frac{\partial \eta}{\partial x_j} \right) dx\, dt \\ T_3 &:= 2s\lambda \sum_{i,j=1}^{2} \iint_Q \xi \frac{\partial z}{\partial x_i} \frac{\partial z}{\partial x_j} b_{ij} \frac{\partial}{\partial x_j} \left(x_j^{\alpha_j} \frac{\partial \eta}{\partial x_j} \right) dx\, dt, \\ T_4 &:= -s\lambda \sum_{i,j,k=1}^{2} \iint_Q \xi \frac{\partial z}{\partial x_i} \frac{\partial z}{\partial x_j} \frac{\partial}{\partial x_k} \left(b_{ij} x_k^{\alpha_k} \frac{\partial \eta}{\partial x_k} \right) dx\, dt \\ T_5 &:= 2s\lambda \sum_{i,j,k=1}^{2} \iint_Q \xi \frac{\partial z}{\partial x_i} \frac{\partial z}{\partial x_j} x_i^{\alpha_i} \frac{\partial}{\partial x_i} \left(b_{jk} \frac{\partial \eta}{\partial x_k} \right) dx\, dt, \\ T_6 &:= -s\lambda \sum_{i,j,k=1}^{2} \iint_Q \xi \left| \frac{\partial z}{\partial x_i} \right|^2 \frac{\partial}{\partial x_j} \left(x_i^{\alpha_i} b_{jk} \frac{\partial \eta}{\partial x_k} \right) dx\, dt, \\ T_7 &:= 2s\lambda \sum_{i,j,k,l=1}^{2} \iint_Q \xi \frac{\partial z}{\partial x_i} \frac{\partial z}{\partial x_k} b_{ij} \frac{\partial}{\partial x_j} \left(b_{kl} \frac{\partial \eta}{\partial x_l} \right) dx\, dt \end{aligned}$$

and

$$T_8 := -s\lambda \sum_{i,j,k,l=1}^{2} \iint_Q \xi \frac{\partial z}{\partial x_i} \frac{\partial z}{\partial x_j} \frac{\partial}{\partial x_k} \left(b_{ij} b_{kl} \frac{\partial \eta}{\partial x_l} \right) dx\, dt.$$

Now, it is easy to see that

$$|T_1+T_2| \;\leq\; Cs\lambda \iint_Q \xi|\nabla z\tilde{A}\nabla z|\,dx\,dt. \tag{3.37}$$

To estimate the other terms we will to use the item 6 of (3.21). We have that

$$\begin{aligned}
|T_3| \;&\leq\; Cs\lambda\sum_{i,j=1}^{2}\iint_Q \xi\left|\frac{\partial z}{\partial x_i}\frac{\partial z}{\partial x_j}b_{ij}\frac{\partial}{\partial x_j}\left(x_j^{\alpha_j}\frac{\partial \eta}{\partial x_j}\right)\right|\,dx\,dt\\
&\leq\; Cs\lambda\sum_{i,j=1}^{2}\left[\int_0^T\!\!\int_{\Omega\setminus(\epsilon,1)\times(\epsilon,1)}\xi\left|\frac{\partial z}{\partial x_i}\frac{\partial z}{\partial x_j}b_{ij}\frac{\partial}{\partial x_j}\left(x_j^{\alpha_j}\frac{\partial \eta}{\partial x_j}\right)\right|\,dx\,dt\right.\\
&\qquad\left.\int_0^T\!\!\int_{(\epsilon,1)\times(\epsilon,1)}\xi\left|\frac{\partial z}{\partial x_i}\frac{\partial z}{\partial x_j}b_{ij}\frac{\partial}{\partial x_j}\left(x_j^{\alpha_j}\frac{\partial \eta}{\partial x_j}\right)\right|\,dx\,dt\right]\\
&\leq\; Cs\lambda\sum_{i,j=1}^{2}\left[\int_0^T\!\!\int_{\Omega\setminus(\epsilon,1)\times(\epsilon,1)}\xi\left|\frac{\partial z}{\partial x_i}\right|x_i^{\alpha_i/2}\left|\frac{\partial z}{\partial x_j}\right|x_j^{\alpha_j/2}\,dx\,dt\right.\\
&\qquad\left.\int_0^T\!\!\int_{(\epsilon,1)\times(\epsilon,1)}\xi\xi\left|\frac{\partial z}{\partial x_i}\right|x_i^{\alpha_i/2}\left|\frac{\partial z}{\partial x_j}\right|x_j^{\alpha_j/2}\,dx\,dt\right]\\
&\leq\; Cs\lambda\iint_Q \xi|\nabla z\tilde{A}\nabla z|\,dx\,dt.
\end{aligned} \tag{3.38}$$

In a similar way we deduce that

$$|T_4+T_5+T_7| \;\leq\; Cs\lambda\iint_Q \xi|\nabla z\tilde{A}\nabla z|\,dx\,dt. \tag{3.39}$$

On the other hand we have that

$$\begin{aligned}
T_6 \;&=\; -s\lambda\iint_Q \xi|\nabla zA\nabla z|\left[\sum_{i,j=1}^{2}\frac{\partial}{\partial x_i}\left(b_{ij}\frac{\partial \eta}{\partial x_j}\right)\right]\,dx\,dt\\
&\qquad -s\lambda\sum_{i,j=1}^{2}\iint_Q \xi\left|\frac{\partial z}{\partial x_i}\right|^2\frac{\partial}{\partial x_i}(x_i^{\alpha_i})b_{ij}\frac{\partial \eta}{\partial x_j}\,dx\,dt\\
&\geq\; -Cs\lambda\iint_Q \xi|\nabla z\tilde{A}\nabla z|\,dx\,dt - s\lambda\sum_{i,j=1}^{2}\left[\int_0^T\!\!\int_{(\epsilon,1)\times(\epsilon,1)}\xi|\nabla z\tilde{A}\nabla z|\,dx\,dt\right.\\
&\qquad\left.+\int_0^T\!\!\int_{\Omega\setminus(\epsilon,1)\times(\epsilon,1)}\xi\left|\frac{\partial z}{\partial x_i}\right|^2\frac{\partial}{\partial x_i}(x_i^{\alpha_i})x_i^{\alpha_i/2}x_j^{\alpha_j/2}p_{ij}\frac{\partial \eta}{\partial x_j}\,dx\,dt\right]\\
&\geq\; -s\lambda\sum_{i,j=1}^{2}\left[\int_0^T\!\!\int_{\Omega\setminus(\epsilon,1)\times(\epsilon,1)}\xi\left|\frac{\partial z}{\partial x_i}\right|^2\frac{\partial}{\partial x_i}(x_i^{\alpha_i})x_i^{\alpha_i/2}x_j^{\alpha_j/2}p_{ij}\frac{\partial \eta}{\partial x_j}\,dx\,dt\right]\\
&\qquad -Cs\lambda\iint_Q \xi|\nabla z\tilde{A}\nabla z|\,dx\,dt
\end{aligned} \tag{3.40}$$

From item 6 of (3.21) and the definition of the function η we see that the first term on the right hand side of (3.40) is non negative. Hense

$$T_6 \geq -Cs\lambda\iint_Q \xi|\nabla z\tilde{A}\nabla z|\,dx\,dt. \tag{3.41}$$

In a similar way we deduce that

$$T_8 \geq -Cs\lambda \iint_Q \xi |\nabla z \tilde{A} \nabla z| \, dx \, dt. \tag{3.42}$$

Arguing as in the previous proof, from (3.36), (3.37), (3.38), (3.39), (3.41) and (3.42) we deduce an estimate similar to (3.31). The rest of the proof is very similar to the previous one and it's left to the reader.

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