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Rigidity of hypersurfaces satisfying an Okumura type inequality, height estimates in warped product spaces and stability in weighted manifolds

por

Eudes Leite de Lima

Campina Grande – PB
Dezembro de 2017

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sob orientação do

Prof. Dr. Henrique Fernandes de Lima

Tese apresentada ao Corpo Docente do Programa Associado de Pós Graduação em Matemática UFPB/UFCG, como requisito parcial para obtenção do título de Doutor em Matemática.

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Dedicatória

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Resumo

Este trabalho está dividido em três partes. Na primeira parte, estudamos hipersuperfícies de curvatura média ou escalar constante imersas em formas espacias Riemannianas ou Lorentzianas satisfazendo uma desigualdade tipo Okumura adequada. Precisamente, obtemos estimativas superiores e inferiores ótimas para a parte sem traço da segunda forma fundamental destas hipersuperfícies. Em particular, resultados de rigidez são provados. Na segunda parte, estamos interessados em hipersuperfícies Weingarten linear generalizadas imersas em produtos warped Riemannianos ou Lorentzianos. Nesta parte, provamos interessantes estimativas de altura bem como teoremas semi-espaço para estas hipersuperfícies. Como aplicação destes resultados, fornecemos informações sobre a topologia no infinito de tais hipersuperfícies. Finalmente, a terceira parte é dedicada ao estudo da estabilidade de hipersurpefícies com f-curvatura média zero imersas em produtos warped semi-Riemannianos weighted. Em particular, damos uma condição suficiente para estas hipersuperfícies serem estáveis com respeito ao seu operador de Jacobi usual.

Palavras-chave: formas espaciais, hipersuperfícies de curvatura média constante, hipersuperfícies de curvatura escalar constante, desigualadade tipo Okumura, espaços produto warped, hipersuperfícies Weingarten linear generalizadas, estimativas de altura, teoremas semi-espaço, produtos warped weighted, hipersuperfícies with f-curvatura média zero, estabilidade.

Abstract

This work is divided into three parts. In the first part, we study constant mean or scalar curvature hypersurfaces immersed into Riemannian or Lorentzian space forms satisfying a suitable Okumura type inequality. Precisely, we obtain sharp upper and lower estimates for the traceless part of the second fundamental form of these hypersurfaces. In particular, rigidity results are proved. In the second part, we are interested in generalized linear Weingarten hypersurfaces immersed into Riemannian or Lorentzian warped products. In this part, we proved interesting height estimates as well as half-space theorems for these hypersurfaces. As application of these results, we provide informations regarding the topology at infinity of such hypersurfaces. Finally, the third part is dedicated to the study of the stability of hypersurfaces with zero f-mean curvature immersed into weighted semi-Riemannian warped products. In particular, we give a sufficient condition for these hypersurfaces be stable with respect to the its standard Jacobi operator.

Keywords: space forms, constant mean curvature hypersurfaces, constant scalar curvature hypersurfaces, Okumura type inequality, warped product spaces, generalized linear Weingarten hypersurfaces, height estimates, half-space theorems, weighted warped products, zero f-mean curvature hypersurfaces, stability.

Contents

Introduction

This thesis is divided into three independent parts as follows:

Part I: Rigidity of hypersurfaces satisfying an Okumura type inequality

The problem of characterizing hypersurfaces with constant mean or scalar curvature immersed into Riemannian or Lorentzian space forms of constant sectional curvature constitutes an important and fruitful topic in the theory of isometric immersions, which has being widely approached by many authors.

Let us first describe about the Riemannian context. Regarding to the constant mean curvature case, in 1966 Klotz and Osserman [83] proved that the totally umbilical spheres and circular cylinders are the only complete surfaces immersed into the Euclidean space $\mathbb{R}^3$ with nonzero constant mean curvature and whose Gaussian curvature does not change of sign. In seventies, Hoffman [76] obtained an extension of that result to the case of surfaces with constant mean curvature in the 3-dimensional sphere $\mathbb{S}^3$ and Tribuzy [114] showed the case of surfaces with constant mean curvature in the 3-dimensional hyperbolic space $\mathbb{H}^3$.

In higher dimensions, the first results in this direction are due to Simons [110], Lawson [87] and Chern et al. [45], which can be grouped in the following way: if the squared norm of the second fundamental form $|A|^2$ of a compact minimal hypersurface Σ^n immersed into the n-dimensional sphere $\mathbb{S}^{n+1}$ satisfies $|A|^2 \leq n$, then either $|A|^2 = 0$ and the hypersurface is totally geodesic, or $|A|^2 = n$ and the hypersurface is a minimal Clifford torus. Afterwards, Alencar and do Carmo [5] studied the case of compact hypersurfaces in the sphere with constant mean curvature. Specifically, they introduced a tensor Φ, nowadays called the total umbilicity tensor, and showed that if the squared norm of Φ is bounded from above by certain constant $\beta(H, n)$, then either the hypersurface is totally umbilical or the equality $|\Phi|^2 = \beta(H, n)$ holds, where $\beta(H, n)$ depends only on the mean curvature H and the dimension n of the hypersurface. In the last case, they characterized all hypersurfaces with this property. This extended the previous results of [110], [87] and [45].

More recently, Alías and García-Martínez [14] used the weak Omori-Yau's maximum principle due to Pigola et al. [103, 104] to study the behavior of the squared norm of the total umbilicity tensor of a complete hypersurface with constant mean curvature immersed into a Riemannian space form deriving a sharp estimate for the supremum of $|\Phi|^2$. In particular, they gave a generalization of the result due to Alencar and do Carmo.

Following the approach developed in [14], Meléndez [93] introduced an interesting and suitable Okumura type inequality on the total umbilicity tensor Φ and was able to prove a nice generalization of the results of [14], in the sense that he characterized new isoparametric hypersurfaces of the Riemannian space forms. More precisely, denoting by $\mathbb{M}_c^{n+1}$ an $(n+1)$-dimensional Riemannian space form of constant sectional curvature $c \in \{0, 1, -1\}$, he proved:

Theorem (Theorem 1.4 of [93]). *Let Σ^n be a complete hypersurface immersed into a Riemannian space form $\mathbb{M}_c^{n+1}$, $n \geq 3$, with constant mean curvature H such that $H^2 + c > 0$. If its total umbilicity tensor Φ satisfies*

$$|\mathrm{tr}(\Phi^3)| \leq \frac{(n-2p)}{\sqrt{np(n-p)}} |\Phi|^3,$$

for some $1 \leq p < \frac{n}{2}$, then

(i) *either* $\sup|\Phi| = 0$ *and Σ^n is a totally umbilical hypersurface,*

(ii) *or*

$$\sup|\Phi| \geq \beta(H, n, p, c) = \frac{\sqrt{n}}{2\sqrt{p(n-p)}} \left(\sqrt{n^2H^2 + 4p(n-p)c} - (n-2p)|H| \right) > 0.$$

Moreover, the equality $\sup|\Phi| = \beta(H, n, p, c)$ *holds and this supremum is attained at some point of Σ^n if and only if*

(a) *$c = 0$ and Σ^n is a circular cylinder $\mathbb{R}^p \times \mathbb{S}^{n-p}(r) \subset \mathbb{R}^{n+1}$, with $r > 0$;*

(b) *$c = 1$ and Σ^n is either a minimal Clifford torus or a constant mean curvature torus $\mathbb{S}^p(\sqrt{1-r^2}) \times \mathbb{S}^{n-p}(r) \subset \mathbb{S}^{n+1}$, with $0 < r^2 < \frac{n-p}{n}$;*

(c) *$c = -1$ and Σ^n is a hyperbolic cylinder $\mathbb{H}^p(-\sqrt{1+r^2}) \times \mathbb{S}^{n-p}(r) \subset \mathbb{H}^{n+1}$, with $r > 0$.*

When we consider the case of hypersurfaces with constant scalar curvature in Riemannian space forms, in lower dimension one has the classical results due to Hilbert [74] and Hartman and Nirenberg [71]. The former says that the sphere is the only surface with constant Gaussian curvature nonzero in the 3-dimensional Euclidean space $\mathbb{R}^3$, and the second asserts that a surface with zero Gaussian curvature in $\mathbb{R}^3$ must be a cylinder or a plane.

As for to the higher dimensional case, in 1977 Cheng and Yau [44] proved the following well known rigidity result concerning compact constant scalar curvature hypersurfaces immersed into a Riemannian space form which, in its original version, states:

Theorem (Theorem 2 of [44]). *Let Σ^n be a compact hypersurface with nonnegative sectional curvature immersed in a manifold with constant sectional curvature c. Suppose that the normalized scalar curvature of Σ^n is constant and greater than or equal to c. Then Σ^n is either totally umbilical, a (Riemannian) product of two totally umbilical constantly curved submanifolds or possibly a flat manifold which is different from the above two types. The last case can happen only if $c = 0$. (If the ambient manifold is the Euclidean space, the last two cases cannot occur because of the compactness of Σ^n.)*

In the noncompact case, they extended the previous theorem when $c = 0$ by characterizing such a hypersurface Σ^n as being a circular cylinder $\mathbb{R}^p \times \mathbb{S}^{n-p}$. More precisely, they proved the following:

Theorem (Theorem 4 of [44]). *Let Σ^n be a complete noncompact hypersurface in the Euclidean space with nonnegative curvature. Suppose that the scalar curvature of Σ^n is constant, then Σ^n is a generalized cylinder $\mathbb{R}^p \times \mathbb{S}^{n-p}$.*

Their approach involves a careful study of a self-adjoint differential operator introduced by them in [44], nowadays called Cheng-Yau's operator. Actually, this operator has become one of the most efficient tools to deal with issue of rigidity concerning constant scalar curvature hypersurfaces in Riemannian space forms. Indeed, there exists a vast literature related to the problem of establishing rigidity results in the same spirit of [44] under various hypothesis on geometry of such hypersurfaces (see, for instance, [15,27,88,115,116] and the references therein). One of the more recentle is due to Alías et al. [15], where the authors obtained a suitable weak maximum principle for the Cheng-Yau's operator of a complete hypersurface with constant scalar curvature immersed into a Riemannian space form and they applied it to estimate the squared norm of the total umbilicity tensor of the hypersurface. In particular, they proved:

Theorem 1 (Theorems 1 and 2 of [15]). *Let Σ^n be a complete hypersurface immersed into a Riemannian space form $\mathbb{M}_c^{n+1}$, $n \geq 3$, with constant normalized scalar curvature satisfying $R \geq 1$, when $c = 1$, and $R > 0$, when $c \in \{0, -1\}$. Then*

(i) either $\sup|\Phi| = 0$ *and Σ^n is a totally umbilical hypersurface,*

(ii) or

$$\sup|\Phi| \geq \alpha(R, n, c) = R\sqrt{\frac{n(n-1)}{(n-2)(nR-(n-2)c)}}.$$

Moreover, if $R > 1$ when $c = 1$, the equality $\sup|\Phi| = \alpha(R, n, c)$ *holds and this supremum is attained at some point of Σ^n if and only if*

(a) $c = 0$ and Σ^n is a circular cylinder $\mathbb{R}^1 \times \mathbb{S}^{n-1}(r) \subset \mathbb{R}^{n+1}$;

(b) $c = 1$ and Σ^n is a Clifford torus $\mathbb{S}^1(\sqrt{1-r^2}) \times \mathbb{S}^{n-1}(r) \subset \mathbb{S}^{n+1}$;

(c) $c = -1$ and Σ^n is a hyperbolic cylinder $\mathbb{H}^1(\sqrt{1-r^2}) \times \mathbb{S}^{n-1}(r) \subset \mathbb{H}^{n+1}$,

with $r^2 = \frac{n-2}{nR}$.

Let us now describe about the Lorentzian context, beginning by the case of constant mean curvature hypersurfaces. Regarding to the Lorentz-Minkowski space $\mathbb{R}_1^{n+1}$ one can truly say that the first remarkable results in this branch were the rigidity theorems of Calabi [28] and Cheng and Yau [43], who showed (the former for $n \leq 4$, and the latter for general n) that the only maximal, complete, noncompact, spacelike hypersurfaces of the Lorentz-Minkowski space $\mathbb{R}_1^{n+1}$ are the spacelike hyperplanes. However, in the case that the mean curvature is a positive constant, Treibergs [113] surprising showed that there are many entire solutions of the corresponding constant mean curvature equation in $\mathbb{R}^{n+1}_1$, which he was able to classify by their projective boundary values at infinity.

In 1977 Goddard [66] conjectured that every complete spacelike hypersurface with constant mean curvature H in the de Sitter space $\mathbb{S}^{n+1}_1$ must be totally umbilical. Although the conjecture turned out to be false in its original statement, it motivated a great deal of work of several authors trying to find a positive answer to the conjecture under appropriate additional hypotheses. The first result in this direction was obtained by Ramanathan [105], who showed that a constant mean curvature complete spacelike surface in $\mathbb{S}^3_1$ satisfying $H^2 < 1$ is totally umbilical. Moreover, by assuming that $H^2 > 1$ he showed that the Goddard's conjecture is false by means of certain non-totally umbilical spacelike surfaces previously studied by Dajczer and Nomizu in [51].

Simultaneous and independently, Akutagawa [2] proved that the Goddard's conjecture is true provided that $H^2 < 1$ in the case $n = 2$, and when $H^2 < \frac{4(n-1)}{n^2}$ if $n > 2$. He also constructed complete spacelike rotation surfaces in $\mathbb{S}^3_1$ with constant mean curvature satisfying $H^2 > 1$ and which are non-totally umbilical. Later on, Montiel [94] proved the conjecture for the compact case and exhibited examples of complete spacelike hypersurfaces in $\mathbb{S}^{n+1}_1$ with constant mean curvature satisfying $H^2 \geq \frac{4(n-1)}{n^2}$ and being non-totally umbilical, the so called hyperbolic cylinders, which are isometric to the Riemannian product $\mathbb{S}^{n-1}(\sqrt{1+r^2}) \times \mathbb{H}^1(r) \subset \mathbb{S}^{n+1}_1$, where $r > 0$, showing that the general conjecture is false.

A few years later, Montiel [95] was able to characterize these hyperbolic cylinders as been the only complete non-totally umbilical spacelike hypersurfaces of $\mathbb{S}^{n+1}_1$ having $H = \frac{2\sqrt{n-1}}{n}$ and such that $\sup|\Phi| = \frac{n-2}{\sqrt{n}}$ is attained at some point, where Φ stands for the total umbilicity tensor of the hypersurface. Afterwards, Brasil et al. [25] generalized the Montiel's result by characterizing the hyperbolic cylinders of $\mathbb{S}^{n+1}_1$ as been the only complete non-totally umbilical spacelike hypersurfaces of constant mean curvature satisfying $\frac{2\sqrt{n-1}}{n} \leq H < 1$ and with

$$\sup|\Phi| = \frac{\sqrt{n}}{2\sqrt{(n-1)}}\left((n-2)|H| - \sqrt{n^2H^2 - 4(n-1)}\right)$$

attained at some point. Among other results, they also characterized all complete spacelike hypersurfaces of constant mean curvature with two distinct principal curvatures as been either rotation hypersurfaces or Riemannian products of the type $\mathbb{S}^{n-p}(\sqrt{1+r^2}) \times \mathbb{H}^p(r)$, that is, of an $(n-p)$-dimensional Euclidean sphere and a p-dimensional hyperbolic space, which are also called hyperbolic cylinders.

In the case of constant scalar curvature hypersurfaces, an interesting result due to Cheng and Ishikawa [40] states that the totally umbilical Euclidean sphere is the only compact spacelike hypersurface in the de Sitter space having constant normalized scalar curvature $R < 1$, generalizing a previuos result proved by Zheng [119] under the additional condition that the sectional curvature of the hypersurface is nonnegative. On the other hand, Li [89] posed the question on whether the only complete spacelike hypersurfaces with constant normalized scalar curvature R satisfying $\frac{n-2}{n} \leq R \leq 1$ in the de Sitter space are totally umbilical ones. In [29] Camargo et al., by extending a technique introduced by Cheng and Yau [44], answered this question positively under the additional assumption that the mean curvature is bounded from above on such hypersurfaces (see also [13, 23, 24, 26, 78, 90, 109] for others results in this context).

This part of the thesis is devoted to generalize and improve some of the above cited notorious results. To do this, our main assumption will be the Okumura type inequality introduced by Melendéz [93] which, as we shall see, gives a natural generalization of the hypothesis contained in aforementioned results. To be more precisely, in Chapter 2 we prove a result like in Theorem 1 with the advantage that we characterize new isoparametric hypersurfaces of the ambient space (see Theorem 2.1.1). Later on, in Chapter 3 we improve Theorem 1.2 and Proposition 1.2 of [25] (see Theorem 3.1.1). Moreover, we give a nice generalization of Theorems 1 and 2 of [82] (see Theorem 3.1.2). Afterwards in Chapter 4 we also improve and generalize Theorem 1.1 of [29] as well as others results contained in [46, 78, 109] (see Theorems 4.1.1 and 4.1.2). To close this part, in Chapter 5 we study hypersurfaces with constant mean curvature immersed into locally symmetric Riemannian manifolds. In particular, we obtain generalizations of various results contained in [14, 93] (see Theorems 5.2.1 and 5.2.2).

Part II: Generalized linear Weingarten hypersurfaces in warped products: height estimates and half-space theorems

The last few decades have seen a steadily growing interest in the study of a priori estimates of the height function of constant mean curvature compact graphs or, more generally, compact hypersurfaces with boundary having some constant higher order mean curvature immersed into semi-Riemannian pruduct spaces of the type $\mathbb{R} \times M^n$ or $-\mathbb{R} \times M^n$, where M^n is an arbitrary Riemannian manifold. This problem has gained special attention, being considered by several authors probably motivated by the fact that these estimates turn out to be a very useful tool in order to investigate existence and uniqueness results of complete hypersurfaces with constant higher order mean curvature as well as to obtain information on the topology at infinity of such hypersurfaces (in the Riemannian setting see, for instance, [3, 11, 42, 61, 72, 77, 85, 86, 107] and, in the Lorentzian setting see, for instance, [47, 52, 92, 96]). Lately, height estimates has becoming very useful even in the case of hypersurfaces immersed into semi-Riemannian warped product spaces (see [64, 65]).

In the context of the Riemannian geometry, in 1969 the first height estimate of compact graphs with positive constant mean curvature in the Euclidean space $\mathbb{R}^3$ and boundary on a plane was obtained by Heinz [72]. More specifically, denoting by H the mean curvature, Heinz proved that the height of such a graph can rise at most $\frac{1}{H}$. In [85], Korevaar et al. obtained a sharp bound of compact embedded hypersurfaces in the hyperbolic space $\mathbb{H}^{n+1}$ with nonzero constant mean curvature and boundary contained into a totally geodesic hyperplane. Next, Rosenberg [107] gave height estimates of compact hypersurfaces with some constant higher order mean curvature and with zero boundary values embedded either in the Euclidean space $\mathbb{R}^{n+1}$ or in the hyperbolic space $\mathbb{H}^{n+1}$, generalizing the previous estimates of [72] and [85].

Later on, Cheng and Rosenberg [42] were able to generalize these estimates for compact graphs with some constant higher order mean curvature in the product manifold $\mathbb{R} \times M^n$, with boundary contained into a slice $\{t_0\} \times M^n$, for some $t_0 \in \mathbb{R}$. When the fiber M^n is compact, as application of their height estimates, they used the Alexandrov's reflection method in order to obtain some informations on the topology at infinity of noncompact properly embedded hypersurface having constant higher order mean curvature, proving that such a hypersurface must have at least two ends or, equivalently, it cannot lie in a half-space.

Afterwards, Alías and Dajczer [11] and García-Martínez et al. [65] gave extensions of the aforementioned results to the case of hypersurfaces immersed into the so called pseudo-hyperbolic spaces, that is, following the terminology introduced by Tashiro [112], Riemannian warped products of the type $\mathbb{R} \times_{e^t} M^n$ or $\mathbb{R} \times_{\cosh t} M^n$. The former [11] focused in constant mean curvature hypersurfaces and the later [65] considered hypersurfaces with some constant higher order mean curvature. Moreover, in [65] was proved topological results for noncompact properly immersed hypersurfaces of constant mean and higher order mean curvature of these pseudo-hyperbolic spaces in the same spirit of [42], by assuming that the fiber M^n is compact.

Towards the Lorentzian context, the first result in this direction is due to López [92], who proved that compact spacelike surfaces with constant mean curvature in the 3-dimensional Lorentz-Minkowski spacetime $\mathbb{R}^3_1$ with boundary on a plane can reach at most a height of $\frac{|H|A}{2\pi}$, where A is the area of the region of the surface above the plane containing its boundary. Later on, Montiel [96] obtained height estimates of compact spacelike graphs in the steady state spacetime and he applied them to prove some existence and uniqueness theorems for complete spacelike hypersurfaces in the de Sitter spacetime with constant mean curvature $H > 1$ and prescribed asymptotic future boundary. Also, de Lima studied height estimates and obtained a sharp estimate of compact spacelike hypersurfaces with some constant higher order mean curvature in the Lorentz-Minkowski spacetime $\mathbb{R}^{n+1}_1$ and with boundary contained into a spacelike hyperplane (see [52]), and after jointly with Colares they were able to generalize these estimates to the case of compact spacelike hypersurfaces of positive constant higher order mean curvature in Lorentzian product spacetime $-\mathbb{R} \times M^n$, whose fiber has nonnegative sectional curvature, and with boundary contained into a slice (see [47]).

More recently, García-Martínez and Impera [64] extended the height estimates proved by Colares and de Lima for spacelike hypersurfaces of constant higher order mean curvature in a Lorentzian warped product $-\mathbb{R} \times_\rho M^n$, so called generalized Robertson-Walker (GRW) spacetimes, with boundary contained into a slice. As application they obtained informations on the topology at infinity of constant higher order mean curvature complete spacelike hypersurfaces immersed into a spatially closed GRW spacetime. Moreover, using a version of the Omori-Yau's maximum principle for trace type differential operators, they also gave some half-space results concerning complete spacelike hypersurfaces of constant higher order mean curvature immersed into the non-spatially closed GRW spacetime.

In this part of the work, the main aim is to extend this investigation to a much more general class of hypersurfaces containing those that naturally appear when dealing with the case of constant higher order mean curvature. More precisely, we consider hypersurfaces which satisfy a natural condition on a linear relation involving the higher order mean curvatures. In particular, when this relation is constant the hypersurface is called a generalized linear Weingarten hypersurface. Then we obtain estimates of the height function of compact generalized linear Weingarten hypersurfaces in semi-Riemannian warped products spaces, without the assumption of any higher order mean curvature to be constant. We point out that our results offer improvements of those obtained in [11, 19, 42, 65, 77] when the ambient space is a Riemannian warped product, and [47, 52, 64] when the ambient space is a Lorentzian warped product.

Furthermore, we are able to study the topology at infinity of complete noncompact generalized linear Weingarten hypersurfaces in semi-Riemannian warped products by proving half-space theorems, generalizing some results obtained in [42, 64, 65, 77].

Part III: On stability of hypersurfaces in weighted semi-Riemannian warped products

Let $(\overline{M}^{n+1}, \langle\ ,\ \rangle)$ be an $(n+1)$-dimensional oriented Riemannian or Lorentzian manifold and let $f : \overline{M}^{n+1} \to \mathbb{R}$ be a smooth function. The weighted manifold $\overline{M}^{n+1}_f$ associated with $\overline{M}^{n+1}$ and f is the triple $(\overline{M}^{n+1}, \langle\ ,\ \rangle, e^{-f} d\overline{M})$, where $d\overline{M}$ denotes the standard volume element of $\overline{M}^{n+1}$ induced by the metric $\langle\ ,\ \rangle$. We will refer to function f as the weight function of the weighted manifold $\overline{M}^{n+1}_f$. In this setting, for a weighted manifold $\overline{M}^{n+1}_f$, an important and natural tensor is the so called Bakry-Émery-Ricci tensor $\overline{\mathrm{Ric}}_f$, which is a generalization of Ricci tensor $\overline{\mathrm{Ric}}$ of $\overline{M}^{n+1}$ and is defined by

$$\overline{\mathrm{Ric}}_f = \overline{\mathrm{Ric}} + \overline{\mathrm{Hess}} f,$$

where $\overline{\mathrm{Hess}} f$ is the Hessian of f on $\overline{M}^{n+1}$. In particular, if f is constant $\overline{\mathrm{Ric}}_f$ is simply the standard Ricci tensor $\overline{\mathrm{Ric}}$ of $\overline{M}^{n+1}$.

Appearing naturally in the study of self-shrinkers, Ricci solitons, harmonic heat flows and many other subjects in differential geometry, weighted manifolds are proved to be important nontrivial generalizations of Riemannian manifolds and, nowadays, there are several geometric investigations concerning them. For a brief overview of results in this scope, we refer the articles of Morgan [98] and Wei and Wylie [117].

Let $\psi : \Sigma^n \to \overline{M}^{n+1}_f$ be an isometrically immersed orientable Riemannian manifold into $\overline{M}^{n+1}_f$. Then Σ^n becomes automatically a weighted Riemannian manifold by weighted structure induced from $\overline{M}^{n+1}_f$. In this case and following Gromov [69], the weighted mean curvature, or simply f-mean curvature, H_f of Σ^n is defined by

$$nH_f = nH + \varepsilon \langle \overline{\nabla} f, N \rangle,$$

where H denotes the standard mean curvature of Σ^n with respect to its orientation, $\varepsilon = 1$ if

$\overline{M}^{n+1}$ is a Riemannian manifold, and $\varepsilon = -1$ if $\overline{M}^{n+1}$ is a Lorentzian manifold. In particular, when f is constant we have $H_f = H$ and we recover the usual definition of mean curvature. When the ambient space is Riemannian and the f-mean curvature H_f vanishes identically on Σ^n we said that Σ^n is a f-minimal hypersurface. In the case in which the ambient space is Lorentzian and the f-mean curvature H_f vanishes identically on Σ^n, it is called a f-maximal hypersurface.

The research on the geometry of hypersurfaces having constant f-mean curvature and, in particular, the investigations on the behavior of hypersurfaces with f-mean curvature vanishes identically immersed into a weighted ambient space, constitutes a recent and fruitful topic into the theory of isometric immersions. It has been already approached by many authors and we may cite, for instance, the works [34, 36, 37, 41, 60, 73, 75, 80, 91, 106, 108].

As in the case of zero mean curvature hypersurfaces, it is well known that the condition of Σ^n has zero f-mean curvature is equivalent to the fact that Σ^n is a critical point of the weighted area functional,

$$\mathrm{vol}_f(\Sigma) = \int_\Sigma e^{-f} d\Sigma,$$

for every variation of Σ^n with compact support and fixed boundary. It is natural to wonder whether these hypersurfaces has the property of to minimize (if the ambient space is Riemannian) or maximize (if the ambient space is Lorentzian) the weighted area functional. Recently many authors has been devoted to the study of this question (see, for instance, [34, 41, 60, 80] and references therein).

In order to answer this question, it is very useful to know the second variation formula of the weighted area functional. Let V be a normal compactly supported variation of Σ^n and take $\varphi \in C_0^\infty(\Sigma)$ such that $V = \varphi N$, where N determines the orientation of Σ^n. If the f-mean curvature H_f of Σ^n vanishes identically, then it is well known that the second variation of the weighted area functional is given by (in the Riemannian case see, for instance, [41], and in the Lorentzian case see, for instance, [57])

$$\frac{d^2}{dt^2}\mathrm{vol}_f(\Sigma)|_{t=0} = -\varepsilon \int_\Sigma \varphi L_f \varphi d\Sigma,$$

where the weighted Jacobi operator L_f is defined by

$$L_f = \Delta_f + \varepsilon\left(|A|^2 + \overline{\mathrm{Ric}}_f(N, N)\right).$$

Here $\Delta_f = \Delta - \langle \nabla f, \nabla \cdot \rangle$ is the f-Laplacian on Σ_f. Then we say that Σ^n is L_f-stable if it minimizes (resp. maximizes) the weighted are functional in the Riemannian case (resp. Lorentzian case), that is, $\frac{d^2}{dt^2}\mathrm{vol}_f(\Sigma)|_{t=0} \geq 0$ (resp. ≤ 0).

This part of the thesis is dedicated to the study of the L_f-stability of zero f-mean curvature hypersurfaces immersed into a weighted semi-Riemannian warped product space. Precisely, our main results give a sufficient condition for these hypersurfaces to be L_f-stable. In particular, we generalize recent results due to Aledo and Rubio [4] in Riemannian warped products.

Part I

Rigidity of hypersurfaces satisfying an Okumura type inequality

Chapter 1

Preliminaries for Part I

In this chapter, for the sake of clarity we shall introduce several useful definitions and notations that will appear throughout Part I of this thesis. For instance, the Riemannian and Lorentzian space forms and some basic equations for hypersurfaces immersed into them as well as two relevant Simons type formulas, the first for the Laplacian acting on the squared norm of the second fundamental form, and the second for the Cheng-Yau's operator acting on the mean curvature of these hypersurfaces. Moreover we shall highlight our Okumura type hypothesis, which must appear in all main results of this part of the thesis.

In this setting, we begin by establishing the notations which will appear in forthcomings Chapters 2, 3 and 4. Let us denote by $\mathbb{M}_c^{n+1}$ the standard model of an $(n+1)$-dimensional Riemannian space form with constant sectional curvature c, with $c \in \{0, 1, -1\}$. Actually, $\mathbb{M}_c^{n+1}$ denotes the Euclidean space $\mathbb{R}^{n+1}$ when $c = 0$, endowed with the standard Riemannian metric

$$\langle\ ,\ \rangle = dx_1^2 + dx_2^2 + \ldots + dx_{n+1}^2, \tag{1.1}$$

the $(n+1)$-dimensional Euclidean sphere $\mathbb{S}^{n+1}$,

$$\mathbb{S}^{n+1} = \{p \in \mathbb{R}^{n+2}\ ;\ \langle p, p\rangle = 1\} \subset \mathbb{R}^{n+2},$$

endowed with the Riemannian metric induced from $\mathbb{R}^{n+2}$ when $c = 1$, and the $(n+1)$-dimensional hyperbolic space $\mathbb{H}^{n+1}$,

$$\mathbb{H}^{n+1} = \{p \in \mathbb{R}_1^{n+2}\ ;\ \langle p, p\rangle_1 = -1,\ p_1 \geq 0\} \subset \mathbb{R}_1^{n+2},$$

furnished with the Riemannian metric induced from $\mathbb{R}_1^{n+2}$ when $c = -1$. Here, $\mathbb{R}_1^{n+2}$ stands for the $(n+2)$-dimensional Euclidean space $\mathbb{R}^{n+2}$ endowed with the Lorentzian metric

$$\langle\ ,\ \rangle_1 = -dx_1^2 + dx_2^2 + \ldots + dx_{n+2}^2. \tag{1.2}$$

In the Lorentzian context, we will denote by $\mathbb{L}_c^{n+1}$ the standard model of an $(n+1)$-dimensional Lorentzian space form with constant sectional curvature c, with $c \in \{0, 1, -1\}$. Then, $\mathbb{L}_c^{n+1}$ denotes the Lorentz-Minkowski space $\mathbb{R}_1^{n+1}$ when $c = 0$, that is, the $(n+1)$-

dimensional Euclidean space $\mathbb{R}^{n+1}$ endowed with the Lorentzian metric (1.2), the $(n+1)$-dimensional de Sitter space $\mathbb{S}_1^{n+1}$,

$$\mathbb{S}_1^{n+1} = \{p \in \mathbb{R}_1^{n+2} \ ; \ \langle p, p\rangle_1 = 1\} \subset \mathbb{R}_1^{n+2},$$

endowed with the Lorentzian metric induced from $\mathbb{R}_1^{n+2}$ when $c = 1$, and the $(n+1)$-dimensional anti-de Sitter space $\mathbb{H}_1^{n+1}$,

$$\mathbb{H}_1^{n+1} = \{p \in \mathbb{R}_2^{n+2} \ ; \ \langle p, p\rangle_2 = -1\} \subset \mathbb{R}_2^{n+2},$$

furnished with the Lorentzian metric induced from $\mathbb{R}_2^{n+2}$ when $c = -1$. Here, $\mathbb{R}_2^{n+2}$ stands for the $(n+2)$-dimensional Euclidean space $\mathbb{R}^{n+2}$ endowed with the semi-Riemannian metric

$$\langle\ ,\ \rangle_2 = -dx_1^2 - dx_2^2 + dx_3^2 + \ldots + dx_{n+2}^2. \tag{1.3}$$

In order to simplify the notation, when $c = \pm 1$ we agree to denote by $\langle\ ,\ \rangle$ without distinction, the Riemannian metric in (1.1) on $\mathbb{R}^{n+2}$, the Lorentzian metric in (1.2) on $\mathbb{R}_1^{n+2}$ and the semi-Riemannian metric in (1.3) on $\mathbb{R}_2^{n+2}$. We also agree to denote by $\langle\ ,\ \rangle$ the corresponding Riemannian metric induced on $\mathbb{M}_c^{n+1} \hookrightarrow \mathbb{R}^{n+2}$ ant the Lorentzian metric induced on $\mathbb{L}_c^{n+1} \hookrightarrow \mathbb{R}^{n+2}$.

Throughout this Part I, we will deal with oriented and connected hypersurfaces Σ^n isometrically immersed into a Riemannian or Lorentzian space form (except in Chapter 5, where the ambient space is a locally symmetric Riemmanian manifold). In the Lorentzian case, we will always assume that Σ^n is a spacelike hypersurfaces by meaning that the induced metric on Σ^n via the immersion is a Riemannian metric. In order to avoid confusion, in this chapter we shall use the term Riemannian hypersurfaces to indicate both the case, hypersurfaces in $\mathbb{M}_c^{n+1}$ and spacelike hypersurfaces in $\mathbb{L}_c^{n+1}$.

With this in mind, let us review some basic facts and terminology about such a Riemannian hypersurface Σ^n. Let ∇ be the Levi-Civita connection of Σ^n. As in [102], the curvature tensor $\mathcal{R}$ of Σ^n is defined as

$$\mathcal{R}(X,Y)Z = \nabla_{[X,Y]}Z - [\nabla_X, \nabla_Y]Z$$

for every tangent vector fields $X, Y, Z \in \mathfrak{X}(\Sigma)$, where $[\ ,\ ]$ denotes the standard Lie bracket. Let $A : \mathfrak{X}(\Sigma) \to \mathfrak{X}(\Sigma)$ be its second fundamental form with respect to a globally defined normal unit vector field N, $AX = \overline{\nabla}_X N$. Here, $\overline{\nabla}$ stands for the Levi-Civita connection of the ambient space. It is well known that the curvature tensor of Σ^n can be described in terms of the second fundamental form A by the Gauss equation as follows

$$\mathcal{R}(X,Y)Z = c\left(\langle X,Z\rangle Y - \langle Y,Z\rangle X\right) + \varepsilon\left(\langle AX,Z\rangle AY - \langle AY,Z\rangle AX\right) \tag{1.4}$$

for every $X, Y, Z \in \mathfrak{X}(\Sigma)$, where $\varepsilon = 1$ when the ambient space is a Riemannian space form and $\varepsilon = -1$ if the ambient space is a Lorentzian space form. Let us also denote by H the mean curvature function of Σ^n,

$$H = \frac{1}{n}\mathrm{tr}(A).$$

In particular, the normalized scalar curvature R of Σ^n is given by

$$n(n-1)R = n(n-1)c + \varepsilon\left(n^2H^2 - |A|^2\right), \tag{1.5}$$

where we used the same criterion of (1.4) for the sign ε. The Codazzi equation is given by

$$(\nabla A)(X,Y) = (\nabla A)(Y,X)$$

for all $X,Y,Z \in \mathfrak{X}(\Sigma)$, where $\nabla A : \mathfrak{X}(\Sigma)\times\mathfrak{X}(\Sigma) \to \mathfrak{X}(\Sigma)$ denotes the covariant differential of A,

$$(\nabla A)(X,Y) = (\nabla_Y A)X = \nabla_Y(AX) - A(\nabla_Y X).$$

For our purposes, it will be appropriate to deal with the so called traceless second fundamental form $\Phi : \mathfrak{X}(\Sigma) \to \mathfrak{X}(\Sigma)$ of the Riemannian hypersurface Σ^n, which is defined by

$$\Phi = A - HI,$$

where I is the identity operator on $\mathfrak{X}(\Sigma)$. From here it is not difficult to verify that Φ is a traceless tensor, that is, $\mathrm{tr}(\Phi) = 0$ and that holds the following relation,

$$|\Phi|^2 = |A|^2 - nH^2. \tag{1.6}$$

Moreover, $|\Phi|$ vanishes identically on Σ^n if and only if Σ^n is a totally umbilical hypersurface. For this reason, Φ is also called the total umbilicity tensor of Σ^n. We also note that, by equation (1.5), the following relation is trivially satisfied:

$$n(n-1)R = n(n-1)(c + \varepsilon H^2) - \varepsilon|\Phi|^2. \tag{1.7}$$

Proceeding with our preliminary, let us denote by $P : \mathfrak{X}(\Sigma) \to \mathfrak{X}(\Sigma)$ the first Newton transformation of Σ^n, which is defined as the tensor $P = nHI - A$. It is easy to see that P is a self-adjoint tensor which commutes with the second fundamental form and satisfies $\mathrm{tr}(P) = n(n-1)H$.

Associated to first Newton transformation P, one has the second order linear differential operator $L : C^\infty(\Sigma) \to C^\infty(\Sigma)$ defined by

$$Lu = \mathrm{tr}(P \circ \mathrm{hess}\, u), \tag{1.8}$$

where $\mathrm{hess}\, u : \mathfrak{X}(\Sigma) \to \mathfrak{X}(\Sigma)$ denotes the self-adjoint linear tensor metrically equivalent to the Hessian of u, which is given by

$$\mathrm{hess}\, u(X) = \nabla_X \nabla u$$

for all $X \in \mathfrak{X}(\Sigma)$. In particular, since the ambient space has constant sectional curvature, it was proved by Rosenberg [107] (see also Caminha [30]) that

$$Lu = \operatorname{div}(P\nabla u),$$

where div denotes the standard divergent operator on Σ^n. This implies that the operator L is elliptic if and only if P is positive definite. The operator L is sometimes called the Cheng-Yau's operator because in the Riemannian setting it was introduced by Cheng and Yau in [44].

The proofs of our main results are based on two Simons type formulas, which has already been used by several authors. To wit, for the Laplacian acting on the squared norm of the second fundamental form $|A|^2$ and for the Cheng-Yau's operator acting on the mean curvature function H. For the sake of completeness, we include here its derivation by following, for instance, [19, 95, 99].

Proposition 1.0.1. *Let Σ^n be a Riemannian hypersurface immersed into an $(n+1)$-dimensional Riemannian or Lorentzian space form. Let us choice ε as in* (1.4)*.*

(i) The formula

$$\frac{1}{2}\Delta|A|^2 = |\nabla A|^2 + n\operatorname{tr}(A \circ \text{hess } H) + \varepsilon nH\operatorname{tr}A^3 - \varepsilon|A|^4 + cn(|A|^2 - nH^2)$$

holds on Σ^n.

(ii) The formula

$$L(nH) = \frac{\varepsilon n(n-1)}{2}\Delta R + |\nabla A|^2 - n^2|\nabla H|^2 + \varepsilon nH\operatorname{tr}A^3 - \varepsilon|A|^4 + nc(|A|^2 - nH^2)$$

holds on Σ^n.

Proof. To prove item (i), let us begin by observing that a standard tensor computation yields

$$\frac{1}{2}\Delta|A|^2 = |\nabla A|^2 + \langle A, \Delta A\rangle, \tag{1.9}$$

where $\Delta A : \mathfrak{X}(\Sigma) \to \mathfrak{X}(\Sigma)$ is the rough Laplacian,

$$\Delta A(X) = \operatorname{tr}(\nabla^2 A(X, \cdot, \cdot)) = \sum_{i=1}^{n} \nabla^2 A(X, E_i, E_i).$$

Here $\{E_1, \ldots, E_n\}$ is a local orthonormal frame on Σ^n. We observe that, in our notation,

$$\nabla^2 A(X, Y, Z) = (\nabla_Z \nabla A)(X, Y) = \nabla_Z(\nabla A(X, Y)) - A(\nabla_Z X, Y) - A(X, \nabla_Z Y).$$

By Codazzi equation, $\nabla^2 A$ is symmetric in its two first variables,

$$\nabla^2 A(X, Y, Z) = \nabla^2 A(Y, X, Z), \quad X, Y, Z \in \mathfrak{X}(\Sigma^n).$$

With respect to the symmetries of $\nabla^2 A$ in the other variables, it is not difficult to see that

$$\nabla^2 A(X,Y,Z) = \nabla^2 A(X,Z,Y) - \mathcal{R}(Z,Y)AX + A(\mathcal{R}(Z,Y)X).$$

Thus, choosing the local orthonormal frame diagonalizing A and using the Gauss equation, it follows from here that

$$\begin{aligned}
\Delta A(X) &= \sum_{i=1}^{n} \big(\nabla^2 A(E_i,E_i,X) - \mathcal{R}(E_i,X)AE_i + A(\mathcal{R}(E_i,X)E_i)\big) \\
&= \mathrm{tr}\,(\nabla_X(\nabla A)) + \varepsilon nHA^2X - \varepsilon|A|^2AX + ncAX - ncHX \\
&= n\nabla_X\nabla H + \varepsilon nHA^2X - \varepsilon|A|^2AX + ncAX - ncHX,
\end{aligned}$$

where we have used the fact that the trace commutes with ∇_X and, by Codazzi equation, that $\mathrm{tr}(\nabla A) = n\nabla H$. Therefore, the previous identity jointly with (1.9) allows us to obtain the desired.

In case (ii), by equation (1.8), the operator L satisfies

$$\begin{aligned}
L(nH) &= nH\Delta(nH) - \mathrm{tr}(A\circ\mathrm{hess}(nH)) \\
&= \frac{1}{2}\Delta(n^2H^2) - n^2|\nabla H|^2 - n\mathrm{tr}(A\circ\mathrm{hess}H).
\end{aligned}$$

Then the Simons' formula in (i) gives

$$\begin{aligned}
L(nH) &= \frac{1}{2}\Delta(n^2H^2) - \frac{1}{2}\Delta|A|^2 - n^2|\nabla H|^2 + |\nabla A|^2 \\
&+ \varepsilon nH\mathrm{tr}A^3 - \varepsilon|A|^4 + cn(|A|^2 - nH^2).
\end{aligned}$$

Finally, it follows from equation (1.5) that

$$L(nH) = \frac{\varepsilon n(n-1)}{2}\Delta R + |\nabla A|^2 - n^2|\nabla H|^2 + \varepsilon nH\mathrm{tr}A^3 - \varepsilon|A|^4 + cn(|A|^2 - nH^2),$$

which concludes the proof of the proposition. □

In order to prove our main results, we also recall the well known Okumura's Lemma due to Okumura in [100], which was completed with the equality case by Alencar and do Carmo in [5].

Lemma 1.0.2. *Let $\kappa_1,\dots,\kappa_n$, $n\geq 3$, be real numbers such that $\sum_i \kappa_i = 0$ and $\sum_i \kappa_i^2 = \beta^2$, where $\beta \geq 0$. Then*

$$-\frac{(n-2)}{\sqrt{n(n-1)}}\beta^3 \leq \sum_i \kappa_i^3 \leq \frac{(n-2)}{\sqrt{n(n-1)}}\beta^3.$$

and the equality holds if and only if either at least $n-1$ of the numbers κ_i are equal.

Next we quote an Okumura type result due to Meléndez [93] which is closely related to the total umbilicity tensor (for more details, see Lemma 2.2 of [93]).

Lemma 1.0.3. *Let $\kappa_1, \ldots, \kappa_n$, $n \geq 3$, be real numbers such that $\sum_i \kappa_i = 0$ and $\sum_i \kappa_i^2 = \beta^2$, where $\beta \geq 0$. Then, the equation*

$$\sum_i \kappa_i^3 = \frac{(n-2p)}{\sqrt{np(n-p)}}\beta^3 \qquad \left(\sum_i \kappa_i^3 = -\frac{(n-2p)}{\sqrt{np(n-p)}}\beta^3\right), \quad 1 \leq p \leq n-1,$$

holds if and only if p of the numbers κ_i are nonnegative (resp. nonpositive) and equal and the rest $n-p$ of the numbers κ_i are nonpositive (resp. nonegative) and equal.

Related to the previous two lemmas, Melendéz [93] introduced an interesting and suitable hypothesis on the total umbilicity tensor of a hypersurface immersed into a Riemannian space form. To be precise and slightly more general, let us consider a Riemanian hypersurface Σ^n immersed into a Riemannian or Lorentzian space form and let Φ be its total umbilicity tensor. We must point out that along this part of the thesis we will always assume in our main results the following Okumura type inequality on Φ:

$$|\mathrm{tr}(\Phi^3)| \leq \frac{(n-2p)}{\sqrt{np(n-p)}}|\Phi|^3, \tag{1.10}$$

for some $1 \leq p < \frac{n}{2}$. It is worth pointing out that since Φ is traceless, by the classical Okumura's Lemma 1.0.2, inequality (1.10) is automatically true when $p = 1$. Furthermore, when $1 < p < \frac{n}{2}$ we claim that to suppose that inequality (1.10) holds is weaker than to assume the geometric condition of the hypersurface has two distinct principal curvatures with multiplicities p and $n-p$. Indeed, in this latter case Φ also has two distinct eigenvalues, said μ and ν, with multiplicity p and $n-p$, respectively. In particular, we get $\mu = -\frac{n-p}{p}\nu$ and $|\Phi|^2 = p\mu^2 + (n-p)\nu^2$, which implies that

$$\mathrm{tr}(\Phi^3) = p\mu^3 + (n-p)\nu^3 = \pm\frac{(2p-n)}{\sqrt{np(n-p)}}|\Phi|^3,$$

proving the claim.

On the other hand, Lemmas 1.0.2 and 1.0.3 say that if the equality in (1.10) holds then the hypersurface must have at most two principal curvatures.

To close this chapter, we quote the following result, which can be found in Lemma 4.1 of [6] or Lemma 2.5 of [31] in the Riemannian setting, and Lemma 2 of [32] in the Lorentzian setting.

Lemma 1.0.4. *Let Σ^n be a Riemannian hypersurface immersed into an $(n+1)$-dimensional Riemannian or Lorentzian space form. Let us choose ε as in* (1.4)*. If the constant normalized scalar curvature satisfies $\varepsilon(R-c) \geq 0$, then*

$$|\nabla A|^2 - n^2|\nabla H|^2 \geq 0.$$

Chapter 2

Constant scalar curvature hypersurfaces in Riemannian space forms

In this chapter, we obtain a sharp lower bound for the supremum of the norm of the traceless second fundamental form of complete hypersurfaces satisfying the Okumura type inequality in (1.10) with constant scalar curvature immersed into a Riemannian space form $\mathbb{M}_c^{n+1}$. The sharpness is proved by showing that the standard products embedding $\mathbb{M}_c^p \times \mathbb{S}^{n-p}(r) \hookrightarrow \mathbb{M}_c^{n+1}$ realize this estimate for a well defined radius r. The results presented in this chapter make part of [54].

2.1 Statement of the main result

In 1977, Cheng and Yau [44] proved the following well known rigidity result concerning compact constant scalar curvature hypersurfaces immersed into a Riemannian space form which, in its original version, states:

Theorem (Theorem 2 of [44])**.** *Let Σ^n be a compact hypersurface with nonnegative sectional curvature immersed in a manifold with constant sectional curvature c. Suppose that the normalized scalar curvature of Σ^n is constant and greater than or equal to c. Then Σ^n is either totally umbilical, a (Riemannian) product of two totally umbilical constantly curved submanifolds or possibly a flat manifold which is different from the above two types. The last case can happen only if $c = 0$. (If the ambient manifold is the Euclidean space, the last two cases cannot occur because of the compactness of Σ^n.)*

In the noncompact case, they extended the previous theorem when $c = 0$ by characterizing such a hypersurface Σ^n as being a circular cylinder $\mathbb{R}^p \times \mathbb{S}^{n-p}$. More precisely, they proved the following:

Theorem (Theorem 4 of [44])**.** *Let Σ^n be a complete noncompact hypersurface in the Euclidean space with nonnegative sectional curvature. Suppose that the scalar curvature of Σ^n is constant, then Σ^n is a generalized cylinder $\mathbb{R}^p \times \mathbb{S}^{n-p}$.*

Their approach involves a careful study of the so called Cheng-Yau's operator L defined in (1.8). Actually, this operator has become one of the most efficient tools to deal with issue of rigidity concerning constant scalar curvature hypersurfaces in Riemannian space forms. Indeed, there exists a vast literature related to the problem of establishing rigidity results in the same spirit of [44] under various hypothesis on the geometry of these hypersurfaces (see, for instance, [15, 27, 88, 115, 116] and the references therein).

In this context, Li [88] extended these results due to Cheng and Yau by assuming a boundedness on the squared norm of the second fundamental form of the hypersurface. More recently, Alías et al. [15] obtained a suitable weak maximum principle for the Cheng-Yau's operator of a complete hypersurface with constant scalar curvature immersed into a Riemannian space form, and they applied it to estimate the squared norm of the traceless part of the second fundamental form of the hypersurface. In particular, they proved:

Theorem (Theorems 1 and 2 of [15])**.** *Let Σ^n be a complete hypersurface immersed into a Riemannian space form $\mathbb{M}_c^{n+1}$, $n \geq 3$, with constant normalized scalar curvature satisfying $R \geq 1$, when $c = 1$, and $R > 0$, when $c \in \{0, -1\}$. Then*

(i) either $\sup|\Phi| = 0$ *and Σ^n is a totally umbilical hypersurface,*

(ii) or

$$\sup|\Phi| \geq \alpha(R, n, c) = R\sqrt{\frac{n(n-1)}{(n-2)(nR-(n-2)c)}}.$$

Moreover, if $R > 1$ when $c = 1$, the equality $\sup|\Phi| = \alpha(R, n, c)$ *holds and this supremum is attained at some point of Σ^n if and only if*

(a) $c = 0$ and Σ^n is a circular cylinder $\mathbb{R}^1 \times \mathbb{S}^{n-1}(r) \subset \mathbb{R}^{n+1}$;

(b) $c = 1$ and Σ^n is a Clifford torus $\mathbb{S}^1(\sqrt{1-r^2}) \times \mathbb{S}^{n-1}(r) \subset \mathbb{S}^{n+1}$;

(c) $c = -1$ and Σ^n is a hyperbolic cylinder $\mathbb{H}^1(\sqrt{1-r^2}) \times \mathbb{S}^{n-1}(r) \subset \mathbb{H}^{n+1}$,

with $r^2 = \frac{n-2}{nR}$.

Here, by assuming the Okumura type inequality in (1.10) on the total umbilicity tensor Φ (see Chapter 1 for definition of Φ), the purpose of this chapter is to prove the following rigidity result, similar to some of the results above cited.

Theorem 2.1.1. *Let Σ^n be a complete hypersurface immersed into a Riemannian space form $\mathbb{M}_c^{n+1}$, with constant normalized scalar curvature satisfying $R \geq 1$, when $c = 1$, and $R > 0$, when $c \in \{0, -1\}$. If its total umbilicity tensor Φ satisfies* (1.10) *for some $1 \leq p < \frac{n}{2}$, then*

(i) either $\sup|\Phi| = 0$ *and Σ^n is a totally umbilical hypersurface,*

(ii) or

$$\sup|\Phi| \geq \alpha(R, n, p, c) > 0,$$

where $\alpha(R,n,p,c)$ is a positive constant depending only on R, n, p and c. Moreover, if $R>1$ when $c=1$, the equality $\sup|\Phi| = \alpha(R,p,n,c)$ *holds and this supremum is attained at some point of Σ^n if and only if*

(a) $c=0$ and Σ^n is a circular cylinder $\mathbb{R}^p \times \mathbb{S}^{n-p}(r) \subset \mathbb{R}^{n+1}$, with $r^2 = \frac{(n-p)(n-p-1)}{n(n-1)R} > 0$;

(b) $c=1$ and Σ^n is a Clifford torus $\mathbb{S}^p(\sqrt{1-r^2}) \times \mathbb{S}^{n-p}(r) \subset \mathbb{S}^{n+1}$, with

$$r^2 = \frac{(n-1)(nR+(n-2p)) - \sqrt{[n(n-1)(R-1)+2p(n-p)]^2 - 4p(n-p)(p(n-p)-(n-1))}}{2n(n-1)R};$$

(c) $c=-1$ and Σ^n is a hyperbolic cylinder $\mathbb{H}^p(\sqrt{1-r^2}) \times \mathbb{S}^{n-p}(r) \subset \mathbb{H}^{n+1}$, with

$$r^2 = \frac{-(n-1)(nR-(n-2p)) + \sqrt{[n(n-1)(R+1)-2p(n-p)]^2 - 4p(n-p)(p(n-p)-(n-1))}}{2n(n-1)R}.$$

Regarding Okumura type condition on the tensor Φ, we saw in Chapter 1 that it is automatically true when $p=1$. In this case, Theorem 2.1.1 has been obtained by Alías et al. (see Theorems 1 and 2 of [15]). In this context, the general case of Theorem 2.1.1 is a nice generalization of the results of [15] in the sense that we characterize new isoparametric hypersurfaces of the ambient space, namely, the product of space forms when $p>1$.

Furthermore, when $1<p<\frac{n}{2}$ we proved in Chapter 1 that the hypothesis in (1.10) is weaker than to assume the geometric condition of the hypersurface has two distinct principal curvatures with multiplicities p and $n-p$. For this reason, Theorem 2.1.1 can be regarded as a sort of improvement of the results contained in [38, 39, 79] concerning complete hypersurfaces having two distinct principal curvatures with multiplicities greater than one.

2.2 Auxiliary results

In order to prove Theorem 2.1.1, we will need some auxiliary results. The first one is concerning the ellipticity of the Cheng-Yau's operator L as well as the validity of a generalized version of the Omori-Yau's maximum principle on a hypersurface $\Sigma^n \hookrightarrow \mathbb{M}^{n+1}_c$ for the operator L, meaning that for any function $u \in C^2(\Sigma)$ with $u^* = \sup u < +\infty$, there exists a sequence of points $\{p_j\} \subset \Sigma^n$ satisfying

$$u(p_j) > u^* - \frac{1}{j}, \quad |\nabla u(p_j)| < \frac{1}{j} \quad \text{and} \quad Lu(p_j) < \frac{1}{j},$$

for every $j \in \mathbb{N}$ (for more details, see Appendix A, Lemma A.0.3). More precisely,

Lemma 2.2.1. *Let Σ^n be a complete hypersurface immersed into a Riemannian space form $\mathbb{M}^{n+1}_c$, with constant normalized scalar curvature satisfying $R>c$ (resp. $R \geq c$). In the case where $R=c$, assume in addition that the mean curvature function H does not change sign on Σ^n. The following holds:*

(i) The operator L is elliptic (resp. semi-elliptic) or, equivalently, P is positive definite (resp. semi-definite), for an appropriate choice of the orientation of Σ^n;

(ii) If $\sup|\Phi|^2 < +\infty$, then the Omori-Yau's maximum principle holds on Σ^n for the operator L.

Proof. To prove item (i), let us reason as in the proof of Lemma 4.2 of [31]. When $R > c$ the Gauss equation and our assumption on the scalar curvature of Σ^n imply that the mean curvature function H does not vanish on Σ^n. In particular, for an appropriate choice of the orientation of Σ^n we can assume that $H > 0$. In the case $R = c$, we choose the orientation of Σ^n such that $H \geq 0$. Denoting by $\lambda_1, \ldots, \lambda_n$ the principal curvatures of Σ^n we have that

$$\mu_i = nH - \lambda_i$$

are the eigenvalues of the tensor P, $i = 1, \ldots, n$. Then, by using equation (1.5), it is not difficult to see that $|\lambda_i| \leq nH$, which gives

$$0 \leq \mu_i \leq 2nH, \quad \forall\, i = 1, \ldots, n,$$

with the strict inequalities in the case $R > c$. Hence, writing μ_- and μ_+ to denote the minimum and the maximum of the eigenvalues of P, respectively, we conclude that

$$\mu_- \geq 0 \quad \text{and} \quad \mu_+ \leq 2nH, \tag{2.1}$$

occurring the strict inequalities in the case $R > c$. This proves item (i).

Let us prove item (ii). To do this, we claim that the sectional curvature K_Σ of Σ^n is bounded from below. Indeed, it follows from Gauss equation (1.4) that if $\{X, Y\}$ is an orthonormal basis for an arbitrary plane tangent to Σ^n, then

$$\begin{aligned} K_\Sigma(X,Y) &= c + \langle AX, X\rangle\langle AY, Y\rangle - \langle AX, Y\rangle^2 \\ &\geq c - |AX||AY| - |AX|^2 \\ &\geq c - 2|A|^2, \end{aligned} \tag{2.2}$$

where the last inequality follows from the fact that

$$|AX|^2 \leq \operatorname{tr}(A^2)|X|^2 = |A|^2$$

for every unitary vector field X tangent to Σ^n. On the other hand, equation (1.5) yields

$$|\Phi|^2 = \frac{n-1}{n}|A|^2 - (n-1)(R-c).$$

As we are assuming that $\sup|\Phi|^2 < +\infty$, we get $\sup|A|^2 < +\infty$. Thus, by (2.2), we prove the claim.

Moreover, taking into account once more equation (1.5), we get

$$H^2 = \frac{1}{n(n-1)}|\Phi|^2 + (R - c). \tag{2.3}$$

Then the mean curvature function H also satisfies $\sup H^2 < +\infty$. In particular, equation (2.1) implies that $\sup \mathrm{tr}(P) < +\infty$. Therefore, Lemma A.0.3 guarantees that the Omori-Yau's maximum principles holds on Σ^n for the operator L. $\square$

Secondly, it will also be essential the following lower boundedness of the operator L acting on the squared norm of the total umbilicity tensor Φ of a complete hypersurface immersed into $\mathbb{M}_c^{n+1}$ having constant normalized scalar curvature.

Proposition 2.2.2. *Let Σ^n be a complete hypersurface immersed into a Riemannian space form $\mathbb{M}_c^{n+1}$, with constant normalized scalar curvature satisfying $R \geq c$. In the case where $R = c$, assume in addition that the mean curvature function H does not change sign on Σ^n. If its total umbilicity tensor Φ satisfies* (1.10) *for some $1 \leq p < \frac{n}{2}$, then*

$$\frac{1}{2}L(|\Phi|^2) \geq \frac{1}{\sqrt{n(n-1)}}|\Phi|^2 Q_{R,n,p,c}(|\Phi|)\sqrt{|\Phi|^2 + n(n-1)(R-c)},$$

where the function $Q_{R,n,p,c}(x)$ is given by

$$Q_{R,n,p,c}(x) = -(n-2)x^2 - (n-2p)\frac{\sqrt{n-1}}{\sqrt{p(n-p)}}x\sqrt{x^2 + n(n-1)(R-c)} + n(n-1)R. \tag{2.4}$$

Proof. As in the proof of Lemma 2.2.1, we can choose the orientation of Σ^n so that $H \geq 0$, occurring the strict inequality in the case $R > c$. Since the scalar curvature of Σ^n is constant, we get from (2.3) that

$$\frac{n}{2(n-1)}L(|\Phi|^2) = \frac{1}{2}L(n^2H^2) = nHL(nH) + n^2\langle P\nabla H, \nabla H\rangle. \tag{2.5}$$

By using Lemma 2.2.1 (i), we have that P is positive semi-definite. In particular, from (2.5) we find

$$\frac{1}{2(n-1)}L(|\Phi|^2) \geq HL(nH), \tag{2.6}$$

which jointly with Lemma 1.0.1 (ii) give

$$\frac{1}{2(n-1)}L(|\Phi|^2) \geq H(|\nabla A|^2 - n^2|\nabla H|^2) + nH^2\mathrm{tr}(A^3) - H|A|^4 + ncH(|A|^2 - nH^2). \tag{2.7}$$

But, since $R \geq c$, by Lemma 1.0.4 we know that

$$|\nabla A|^2 - n^2|\nabla H|^2 \geq 0. \tag{2.8}$$

Then, since we are choosing the orientation such that $H \geq 0$, inequalities (2.7) and (2.8) imply that

$$\frac{1}{2(n-1)}L(|\Phi|^2) \geq nH^2\text{tr}(A^3) - H|A|^4 + ncH(|A|^2 - nH^2). \tag{2.9}$$

On the other hand, the squared norm of the total umbilicity tensor Φ is given by $|\Phi|^2 = |A|^2 - nH^2$ and it is not difficult to see that

$$\text{tr}(A^3) = \text{tr}(\Phi^3) + 3H|\Phi|^2 + nH^3. \tag{2.10}$$

Hence putting (2.10) into (2.9) we find

$$\frac{1}{2(n-1)}L(|\Phi|^2) \geq -H|\Phi|^4 + nH^2\text{tr}(\Phi^3) + nH(H^2 + c)|\Phi|^2. \tag{2.11}$$

Taking into account our assumption on Φ we obtain from (2.11) that

$$\frac{1}{2(n-1)}L(|\Phi|^2) \geq H|\Phi|^2 \left(-|\Phi|^2 - \frac{n(n-2p)}{\sqrt{np(n-p)}}H|\Phi| + n(H^2 + c) \right). \tag{2.12}$$

Since $H \geq 0$ we observe that by equation (1.5) the mean curvature can be written as

$$H = \frac{1}{\sqrt{n(n-1)}}\sqrt{|\Phi|^2 + n(n-1)(R-c)}. \tag{2.13}$$

Therefore, substituting (2.13) into (2.12) we get the desired inequality. $\square$

2.3 Proof of Theorem 2.1.1

We begin by observing that if $\sup|\Phi| = +\infty$, then the claim (ii) of Theorem 2.1.1 trivially holds and there is nothing to prove.

So, let us assume, without loss of generality that $\sup|\Phi| < +\infty$ and, as aforementioned, we choose the orientation of Σ^n such that $H \geq 0$. From Lemma 2.2.1 we deduce that the Omori-Yau's maximum principle holds on Σ^n for the operator L. In particular, by applying this result to the function $|\Phi|^2$, we obtain a sequence $\{p_j\}$ in Σ^n satisfying

$$\lim|\Phi|(p_j) = \sup|\Phi| \quad \text{and} \quad L(|\Phi|^2)(p_j) < \frac{1}{j},$$

which jointly with Proposition 2.2.2 gives

$$\frac{1}{j} > L(|\Phi|^2)(p_j) \geq \frac{2}{\sqrt{n(n-1)}}|\Phi|^2(p_j)Q_{R,n,p,c}(|\Phi|(p_j))\sqrt{|\Phi|^2(p_j) + n(n-1)(R-c)},$$

where the function $Q_{R,n,p,c}(x)$ is given by (2.4). Taking the limit as $j \to +\infty$, we infer

$$(\sup|\Phi|)^2\, Q_{R,n,p,c}(\sup|\Phi|)\sqrt{(\sup|\Phi|)^2 + n(n-1)(R-c)} \leq 0.$$

Since we are assuming that $R \geq 1$, when $c = 1$, and $R > 0$, when $c \in \{0, -1\}$, it follows that either $\sup|\Phi| = 0$, which means that $|\Phi| \equiv 0$ and the hypersurface is totally umbilical, or $\sup|\Phi| > 0$ and then $Q_{R,n,p,c}(\sup|\Phi|) \leq 0$. In the latter case, we see that

$$Q_{R,n,p,c}(0) = n(n-1)R > 0$$

and the function $Q_{R,n,p,c}(x)$ is strictly decreasing for $x \geq 0$. In particular, this implies that there exists an unique positive real number $\alpha(R,n,p,c) > 0$, depending only on R, n, p and c, such that $Q_{R,n,p,c}(\alpha(R,n,p,c)) = 0$. Hence, $Q_{R,n,p,c}(\sup|\Phi|) \leq 0$ means that we must have

$$\sup|\Phi| \geq \alpha(R,n,p,c) > 0.$$

This concludes the proof of the first part of Theorem 2.1.1.

Now, let us assume that the equality $\sup|\Phi| = \alpha(R,n,p,c)$ holds. In particular, $Q_{R,n,p,c}(|\Phi|) \geq 0$ on Σ^n and then Proposition 2.2.2 assures that $|\Phi|^2$ is a L-subharmonic function on Σ^n, that is,

$$L(|\Phi|^2) \geq 0 \quad \text{on} \quad \Sigma^n.$$

Furthermore, since $R > 1$ when $c = 1$, Lemma 2.2.1 (i) asserts that the operator L is elliptic. Hence, if there exists a point on Σ^n such that the supremum $\sup|\Phi|$ is attained, then $|\Phi|^2$ is a L-subharmonic function on Σ^n which attains its supremum and, by the stronger maximum principle, it must be constant, that is, $|\Phi| = \alpha(R,n,p,c)$. Thus, it holds the equality in Proposition 2.2.2, namely,

$$\frac{1}{2}L(|\Phi|^2) = 0 = \frac{1}{\sqrt{n(n-1)}}|\Phi|^2 Q_{R,n,p,c}(|\Phi|)\sqrt{|\Phi|^2 + n(n-1)(R-c)}.$$

It follows from here that all the inequalities along the proof of Proposition 2.2.2 must be, in fact, equalities. In particular, we obtain that equation (2.6) must be an equality, and this jointly with the positiveness of the tensor P imply that the mean curvature function H is constant. Moreover, it also occurs the equality in (2.9), that is,

$$|\nabla A|^2 = n^2|\nabla H|^2 = 0.$$

Then, the principal curvatures of Σ^n must be constant and Σ^n is an isoparametric hypersurface. Besides, expression (2.12) is also equality, which implies by Lemma 1.0.3 that Σ^n has exactly two distinct constant principal curvatures with multiplicities p and $n-p$. Then, by the classical results on isoparametric hypersurfaces of Riemannian space forms (see, for instance, Theorem 4 in [87]) we conclude that Σ^n must be one of the following standard products embeddings:

(a) $\mathbb{R}^p \times \mathbb{S}^{n-p}(r) \subset \mathbb{R}^{n+1}$ or $\mathbb{R}^{n-p} \times \mathbb{S}^p(r) \subset \mathbb{R}^{n+1}$, with $r > 0$, if $c = 0$;

(b) $\mathbb{S}^p(\sqrt{1-r^2}) \times \mathbb{S}^{n-p}(r) \subset \mathbb{S}^{n+1}$, with $0 < r < 1$, if $c = 1$;

(c) $\mathbb{H}^p(\sqrt{1+r^2}) \times \mathbb{S}^{n-p}(r) \subset \mathbb{H}^{n+1}$ or $\mathbb{H}^{n-p}(\sqrt{1+r^2}) \times \mathbb{S}^p(r) \subset \mathbb{H}^{n+1}$, with $r > 0$, if $c = -1$.

Concerning the case of the Euclidean space $\mathbb{R}^{n+1}$ (that is, $c = 0$), the positive constant $\alpha(R, n, p, 0) > 0$ is given explicitly by

$$\alpha(R, n, p, 0) = \sqrt{\frac{p(n-1)R}{n-p-1}}.$$

In this case, on the one hand the product $\mathbb{R}^p \times \mathbb{S}^{n-p}(r) \subset \mathbb{R}^{n+1}$, for a given radius $r > 0$, has constant principal curvatures, for an appropriate choice of the orientation, given by

$$\lambda_1 = \ldots = \lambda_p = 0 \quad \text{and} \quad \lambda_{p+1} = \ldots = \lambda_n = \frac{1}{r},$$

so that its constant mean curvature is $H = \frac{n-p}{nr}$ and

$$|\Phi| = \sqrt{\frac{p(n-p)}{nr^2}}.$$

Hence, by equation (1.7), its constant scalar curvature is given by

$$R = \frac{(n-p)(n-p-1)}{n(n-1)r^2} > 0.$$

Thus, we must have

$$|\Phi| = \sqrt{\frac{p(n-1)R}{n-p-1}} = \alpha(R, n, p, 0)$$

and the equality holds. On the other hand, for a given radius $r > 0$, the product $\mathbb{R}^{n-p} \times \mathbb{S}^p(r) \subset \mathbb{R}^{n+1}$ has, for a suitable choice of the normal vector field, constant principal curvatures

$$\lambda_1 = \ldots = \lambda_{n-p} = 0 \quad \text{and} \quad \lambda_{n-p+1} = \ldots = \lambda_n = \frac{1}{r},$$

constant mean curvature $H = \frac{p}{nr}$ and

$$|\Phi| = \sqrt{\frac{p(n-p)}{nr^2}},$$

so that, by using once more equation (1.7), its constant scalar curvature is

$$R = \frac{p(p-1)}{n(n-1)r^2}.$$

In particular, if $p = 1$ we obtain $R = 0$, which cannot happen because of our assumption on R. If $p > 1$, then we find

$$|\Phi| = \sqrt{\frac{(n-1)(n-p)R}{p-1}} > \alpha(R, n, p, 0),$$

that is, the inequality is strict. This gives the characterization of the equality $\sup|\Phi| = \alpha(R, n, p, 0)$ in the case $c = 0$.

Now, we consider the case of the Euclidean sphere $\mathbb{S}^{n+1}$ (that is, $c = 1$). For a radius $0 < r < 1$, the product $\mathbb{S}^p(\sqrt{1-r^2}) \times \mathbb{S}^{n-p}(r) \subset \mathbb{S}^{n+1}$ has, for an appropriate choice of the of the normal vector field, principal curvatures given by

$$\lambda_1 = \ldots = \lambda_p = \frac{r}{\sqrt{1-r^2}} \quad \text{and} \quad \lambda_{p+1} = \ldots = \lambda_n = -\frac{\sqrt{1-r^2}}{r}.$$

Then, its constant mean curvature is

$$H = \frac{nr^2 - (n-p)}{nr\sqrt{1-r^2}}, \tag{2.14}$$

and the norm of the total umbilicity tensor is

$$|\Phi| = \sqrt{\frac{p(n-p)}{nr^2(1-r^2)}}, \tag{2.15}$$

where, by (2.14),

$$r^2 = \frac{nH^2 + 2(n-p) \pm |H|\sqrt{n^2H^2 + 4p(n-p)}}{2n(H^2+1)}$$

and we choose the sign $+$ when $r^2 > \frac{n-p}{n}$ and the sign $-$ when $r^2 \leq \frac{n-p}{n}$. From here, we can write the norm of Φ in terms of the mean curvature as

$$|\Phi| = \frac{\sqrt{n}}{2\sqrt{p(n-p)}} \left(\sqrt{n^2H^2 + 4p(n-p)} \pm (n-2p)|H| \right),$$

where we used the same criterion for the sign.

Let us also observe that, by equation (2.12), it is not difficult to see that when evaluated in $|\Phi|$ the function $Q_{R,n,p,1}(x)$ is given by

$$Q_{R,n,p,1}(|\Phi|) = -(n-1)P_{H,n,p,1}(|\Phi|),$$

where $P_{H,n,p,1}(x)$ is the polynomial

$$P_{H,n,p,1}(x) = x^2 + \frac{n(n-2p)H}{\sqrt{np(n-p)}}x - n(H^2+1). \tag{2.16}$$

On the one hand, if $r^2 \leq \frac{n-p}{n}$, we obtain $P_{H,n,p,1}(|\Phi|) = 0$, which means that $Q_{R,n,p,1}(|\Phi|) = 0$ and, in this case, we must have $|\Phi| = \alpha(R,n,p,1)$ and the equality holds. On the other hand, if $r^2 > \frac{n-p}{n}$ then $P_{H,n,p,1}(|\Phi|) > 0$, which implies $Q_{R,n,p,1}(|\Phi|) < 0$ and, in this case, $|\Phi| > \alpha(R,n,p,1)$, that is, the inequality is strict.

Next, by using equations (1.7), (2.14) and (2.15), we get that the constant scalar curvature of the product $\mathbb{S}^p(\sqrt{1-r^2}) \times \mathbb{S}^{n-p}(r) \subset \mathbb{S}^{n+1}$ is given by

$$n(n-1)(R-1) = \frac{n(n-1)r^4 - 2(n-1)(n-p)r^2 + (n-p)(n-p-1)}{r^2(1-r^2)}, \tag{2.17}$$

which implies $R > 1$ if and only if

$$r^2 < \frac{(n-1)(n-p) - \sqrt{p(n-1)(n-p)}}{n(n-1)} \quad \text{or} \quad r^2 > \frac{(n-1)(n-p) + \sqrt{p(n-1)(n-p)}}{n(n-1)}.$$

In particular, $|\Phi| = \alpha(R, n, p, 1)$ and $R > 1$ if and only if

$$r^2 < \frac{(n-1)(n-p) - \sqrt{p(n-1)(n-p)}}{n(n-1)}. \tag{2.18}$$

Finally, taking into account once more equation (2.17), we find

$$r^2 = \frac{(n-1)(nR + (n-2p)) \pm \sqrt{[n(n-1)(R-1) + 2p(n-p)]^2 - 4p(n-p)(p(n-p) - (n-1))}}{2n(n-1)R}. \tag{2.19}$$

In particular, if $|\Phi| = \alpha(R, n, p, 1)$ and $R > 1$, we must have

$$r^2 = \frac{(n-1)(nR + (n-2p)) - \sqrt{[n(n-1)(R-1) + 2p(n-p)]^2 - 4p(n-p)(p(n-p) - (n-1))}}{2n(n-1)R},$$

otherwise, denoting by r_+^2 the value of r^2 in (2.19) with sign $+$, we can show that

$$r_+^2 > \frac{(n-1)(n-p) - \sqrt{p(n-1)(n-p)}}{n(n-1)}.$$

But this contradicts equation (2.18). This concludes the characterization of the equality $\sup|\Phi| = \alpha(R, n, p, 1)$ in the case $c = 1$.

In the case of the hyperbolic space $\mathbb{H}^{n+1}$ (that is, $c = -1$), for a given $r > 0$ we have that the standard product embedding $\mathbb{H}^p(\sqrt{1+r^2}) \times \mathbb{S}^{n-p}(r) \subset \mathbb{H}^{n+1}$ has constant principal curvatures, for a suitable choice of the orientation, given by

$$\lambda_1 = \ldots = \lambda_p = \frac{r}{\sqrt{1+r^2}} \quad \text{and} \quad \lambda_{p+1} = \ldots = \lambda_n = \frac{\sqrt{1+r^2}}{r}.$$

Thus, its constant mean curvature H is given by

$$H = \frac{nr^2 + (n-p)}{nr\sqrt{1+r^2}}, \tag{2.20}$$

and we also have

$$|\Phi| = \sqrt{\frac{p(n-p)}{nr^2(1+r^2)}}. \tag{2.21}$$

In this case, $H^2 > 1$ and with a straightforward computation one shows that

$$r^2 = \frac{2(n-p) - nH^2 + |H|\sqrt{n^2H^2 - 4p(n-p)}}{2n(H^2-1)},$$

which implies that, in terms of the mean curvature, $|\Phi|$ is given by

$$|\Phi| = \frac{\sqrt{n}}{2\sqrt{p(n-p)}}\left(\sqrt{n^2H^2 - 4p(n-p)} - (n-2p)|H|\right). \tag{2.22}$$

As in the case of the Euclidean sphere, it is not difficult to see that

$$Q_{R,n,p,-1}(|\Phi|) = -(n-1)P_{H,n,p,-1}(|\Phi|),$$

where

$$P_{H,n,p,-1}(x) = x^2 + \frac{n(n-2p)H}{\sqrt{np(n-p)}}x - n(H^2-1). \tag{2.23}$$

In particular, since $|\Phi|$ given by (2.22) is the unique positive root of polynomial $P_{H,n,p,-1}(x)$, we obtain that $Q_{R,n,p,-1}(|\Phi|) = 0$ and, in this case, we must have $|\Phi| = \alpha(R,n,p,-1)$ and the equality holds. Moreover, the constant scalar curvature of $\mathbb{H}^p(\sqrt{1+r^2}) \times \mathbb{S}^{n-p}(r) \subset \mathbb{H}^{n+1}$, as given by (1.7), (2.20) and (2.21), is

$$R = \frac{(n-1)(n-2p)r^2 + (n-p)(n-p-1)}{n(n-1)r^2(1+r^2)} > 0,$$

which gives

$$r^2 = \frac{-(n-1)(nR-(n-2p)) + \sqrt{\left[n(n-1)(R+1) - 2p(n-p)\right]^2 - 4p(n-p)(p(n-p)-(n-1))}}{2n(n-1)R}.$$

On the other hand, the standard product embedding $\mathbb{H}^{n-p}(\sqrt{1+r^2}) \times \mathbb{S}^p(r) \subset \mathbb{H}^{n+1}$ has constant principal curvatures, for a appropriate choice of the orientation, given by

$$\lambda_1 = \ldots = \lambda_{n-p} = \frac{r}{\sqrt{1+r^2}} \quad \text{and} \quad \lambda_{n-p+1} = \ldots = \lambda_n = \frac{\sqrt{1+r^2}}{r}.$$

Then, its constant mean curvature is

$$H = \frac{nr^2+p}{nr\sqrt{1+r^2}},$$

and, in this case, we have

$$|\Phi| = \sqrt{\frac{p(n-p)}{nr^2(1+r^2)}}. \tag{2.24}$$

Hence, its constant scalar curvature is

$$R = \frac{p(p-1) - (n-1)(n-2p)r^2}{n(n-1)r^2(1+r^2)}.$$

We note that $R > 0$ if and only if

$$r^2 < \frac{p(p-1)}{(n-1)(n-2p)}.$$

In particular, $p = 1$ cannot be fulfilled. Now let us assume that $p > 1$ and $R > 0$. Then, by equation (1.7), we obtain $H^2 > 1$. In this case, we find

$$r^2 = \frac{2p - nH^2 + |H|\sqrt{n^2H^2 - 4p(n-p)}}{2n(H^2-1)},$$

which gives, by (2.24),

$$|\Phi| = \frac{\sqrt{n}}{2\sqrt{p(n-p)}}\left(\sqrt{n^2H^2 - 4p(n-p)} + (n-2p)|H|\right).$$

Therefore, in this case, $P_{H,n,p,-1}(|\Phi|) > 0$ and we must have $|\Phi| > \alpha(R,n,p,-1)$, that is, the inequality is strict. This proves the characterization of the equality $\sup|\Phi| = \alpha(R,n,p,-1)$ in the case $c = -1$ and finishes the proof of Theorem 2.1.1.

Chapter 3

Constant mean curvature spacelike hypersurfaces in Lorentzian space forms

In this chapter, we deal with complete constant mean curvature spacelike hypersurfaces immersed into a Lorentzian space form and satisfying the Okumura type inequality introduced in Chapter 1. In this setting, we obtain lower and upper estimates for the norm of the total umbilicity tensor and we show that these estimates are sharp in the sense that the hyperbolic cylinders realize them. The results of this chapter can be found in [48].

3.1 Statement of the main results

In 1977 Goddard [66] conjectured that every complete spacelike hypersurface with constant mean curvature H in the de Sitter space $\mathbb{S}_1^{n+1}$ must be totally umbilical. Although the conjecture turned out to be false in its original statement, it motivated a great deal of work of several authors trying to find a positive answer to the conjecture under appropriate additional hypotheses. The first result in this direction was obtained by Ramanathan [105], who showed that a complete constant mean curvature spacelike surface in $\mathbb{S}_1^3$ satisfying $H^2 < 1$ is totally umbilical. Moreover, if $H^2 > 1$ he showed that the Goddard's conjecture is false by means of certain non-totally umbilical spacelike surfaces previously studied by Dajczer and Nomizu in [51].

Simultaneous and independently, Akutagawa [2] proved that the Goddard's conjecture is true when $H^2 < 1$ in the case $n = 2$, and when $H^2 < \frac{4(n-1)}{n^2}$ if $n > 2$. He also constructed complete spacelike rotation surfaces in $\mathbb{S}_1^3$ with constant mean curvature satisfying $H^2 > 1$ and which are non-totally umbilical. Later on, Montiel [94] proved the conjecture for the compact case and exhibited examples of complete spacelike hypersurfaces in $\mathbb{S}_1^{n+1}$ with constant mean curvature satisfying $H^2 \geq \frac{4(n-1)}{n^2}$ and being non-totally umbilical, the so called hyperbolic cylinders, which are isometric to the Riemannian product $\mathbb{S}^{n-p}(\sqrt{1+r^2}) \times \mathbb{H}^p(r) \subset \mathbb{S}_1^{n+1}$, where $r > 0$, showing that the general conjecture is false.

A few years later, Montiel [95] was able to characterize the hyperbolic cylinder $\mathbb{S}^{n-1}(\sqrt{1+r^2}) \times \mathbb{H}^1(r) \subset \mathbb{S}_1^{n+1}$ as been the only complete non-totally umbilical spacelike hypersurfaces of $\mathbb{S}_1^{n+1}$ having $H = \frac{2\sqrt{n-1}}{n}$ and such that $\sup|\Phi| = \frac{n-2}{\sqrt{n}}$ is attained at some point (see Proposition 2 of

[95]) where, as before, Φ stands for the total umbilicity tensor of the hypersurface. Afterwards, Brasil et al. [25] generalized the Montiel's result by characterizing hyperbolic cylinders of $\mathbb{S}_1^{n+1}$ as been the only complete non-totally umbilical spacelike hypersurfaces of constant mean curvature satisfying $\frac{2\sqrt{n-1}}{n} \leq H < 1$ and with $\sup|\Phi| = \frac{\sqrt{n}}{2\sqrt{n-1}}\left((n-2)|H| - \sqrt{n^2H^2 - 4(n-1)}\right)$ attained at some point. Among other results, they also characterized all complete spacelike hypersurfaces of constant mean curvature with two distinct principal curvatures as been either rotation hypersurfaces or Riemannian products of the type $\mathbb{S}^{n-p}(\sqrt{1+r^2}) \times \mathbb{H}^p(r)$, that is, of an $(n-p)$-dimensional Euclidean sphere and a p-dimensional hyperbolic space, which are also called hyperbolic cylinders.

By assuming the Okumura type inequality introduced in Chapter 1 on the total umbilicity tensor Φ, the first purpose of this chapter is to prove the following result, improving some of the above cited results.

Theorem 3.1.1. *Let Σ^n be a complete spacelike hypersurface immersed into the de Sitter space $\mathbb{S}_1^{n+1}$, with constant mean curvature H. If its total umbilicity tensor Φ satisfies* (1.10) *for some $1 \leq p < \frac{n}{2}$, then*

(i) either $\sup|\Phi| = 0$ *and Σ^n is a totally umbilical hypersurface,*

(ii) or $H^2 \geq \frac{4p(n-p)}{n^2}$ and

$$\beta(H, n, p, 1) \leq \sup|\Phi| \leq \hat{\beta}(H, n, p, 1),$$

where

$$\hat{\beta}(H, n, p, 1) = \frac{\sqrt{n}}{2\sqrt{p(n-p)}}\left((n-2p)|H| + \sqrt{n^2H^2 - 4p(n-p)}\right)$$

and

$$\beta(H, n, p, 1) = \frac{\sqrt{n}}{2\sqrt{p(n-p)}}\left((n-2p)|H| - \sqrt{n^2H^2 - 4p(n-p)}\right).$$

Moreover, if $\frac{4p(n-p)}{n^2} \leq H^2 < 1$ then $\beta(H, n, p, 1) > 0$ and the equality $\sup|\Phi| = \beta(H, n, p, 1)$ *holds and this supremum is attained at some point of Σ^n if and only if Σ^n is a hyperbolic cylinder $\mathbb{S}^{n-p}(\sqrt{1+r^2}) \times \mathbb{H}^p(r) \subset \mathbb{S}_1^{n+1}$, with*

$$r = \sqrt{\frac{nH^2 - 2p + |H|\sqrt{n^2H^2 - 4p(n-p)}}{2n(1-H^2)}} \geq \frac{\sqrt{p}}{\sqrt{n-2p}}.$$

Regarding Theorem 3.1.1 let us recall that the Okumura type condition in (1.10) holds trivially when $p = 1$ because of Lemma 1.0.2. In this case, Theorem 3.1.1 has been obtained by Brasil et al. [25] (see Theorem 1.2 and Proposition 1.1 there). On the other hand, we also recall that when $1 < p < \frac{n}{2}$, then (1.10) is weaker than to assume that the spacelike hypersurface has two distinct principal curvatures with multiplicities p and $n-p$. For this reason, Theorem 3.1.1 can be regarded as an improvement of Theorem 1.2 and Proposition 1.2 of [25].

Proceeding, we obtain other characterization result concerning complete spacelike hypersurfaces satisfying the Okumura type condition in (1.10) and, now, immersed into any Lorentzian space form $\mathbb{L}_c^{n+1}$ of constant sectional curvature $c \in \{0, 1, -1\}$. Specifically:

Theorem 3.1.2. *Let Σ^n be a complete spacelike hypersurface immersed into a Lorentzian space form $\mathbb{L}_c^{n+1}$, with constant mean curvature H. If its total umbilicity tensor Φ satisfies* (1.10) *for some $1 \leq p < \frac{n}{2}$, then*

(i) either $\sup|\Phi| = 0$ *and Σ^n is a totally umbilical hypersurface,*

(ii) or

$$\sup|\Phi| \leq \hat{\beta}(H, n, p, c),$$

where

$$\hat{\beta}(H, n, p, c) = \frac{\sqrt{n}}{2\sqrt{p(n-p)}}\left((n-2p)|H| + \sqrt{n^2H^2 - 4pc(n-p)}\right).$$

Moreover, the equality $|\Phi| = \hat{\beta}(H, n, p, c)$ holds if and only if

(a) $c = 0$ and Σ^n is a hyperbolic cylinder $\mathbb{R}^{n-p} \times \mathbb{H}^p(r) \subset \mathbb{R}_1^{n+1}$, with $r = \frac{p}{n|H|} > 0$,

(b) $c = 1$ and Σ^n is a hyperbolic cylinder $\mathbb{S}^{n-p}(\sqrt{1+r^2}) \times \mathbb{H}^p(r) \subset \mathbb{S}_1^{n+1}$, with either $r = \frac{p}{\sqrt{n(n-2p)}}$ if $H^2 = 1$, or

$$\frac{p}{\sqrt{n(n-2p)}} < r = \sqrt{\frac{nH^2 - 2p + |H|\sqrt{n^2H^2 - 4p(n-p)}}{2n(1-H^2)}} \leq \frac{\sqrt{p}}{\sqrt{n-2p}}$$

when $H^2 < 1$, or

$$r = \sqrt{\frac{2p - nH^2 + |H|\sqrt{n^2H^2 - 4p(n-p)}}{2n(H^2-1)}} < \frac{p}{\sqrt{n(n-2p)}}$$

when $H^2 > 1$.

(c) $c = -1$ and Σ^n is either a maximal hyperbolic cylinder $\mathbb{H}^{n-p}\left(\frac{\sqrt{n-p}}{\sqrt{p}}\right) \times \mathbb{H}^p\left(\frac{\sqrt{n}}{\sqrt{p}}\right) \subset \mathbb{H}_1^{n+1}$, or a hyperbolic cylinder $\mathbb{H}^{n-p}(\sqrt{1-r^2}) \times \mathbb{H}^p(r) \subset \mathbb{H}_1^{n+1}$, with

$$r = \sqrt{\frac{nH^2 + 2p - |H|\sqrt{n^2H^2 + 4p(n-p)}}{2n(1+H^2)}} \leq \frac{\sqrt{p}}{\sqrt{n}}.$$

By using relation (1.6) to rewrite Theorem 3.1.2 in terms of the second fundamental form of the spacelike hypersurface, we see that in the case $p = 1$ our result becomes equivalent to Theorems 1 and 2 in [82]. In this context, the general case of Theorem 3.1.2 is a nice generalization of the results in [82] in the sense that we characterize new isoparametric hypersurfaces in $\mathbb{L}_c^{n+1}$, namely, the hyperbolic cylinders when $p > 1$.

3.2 Auxiliary results

This section is devoted to quote two key lemmas which will be essential to prove Theorems 3.1.1 and 3.1.2 in the next sections, namely: a sufficient condition to the validity of the

Omori-Yau's maximum principle on Σ^n for the Laplacian and a well known sufficient condition to the boundedness of the squared norm of the second fundamental form $|A|^2$ of spacelike hypersurfaces in Lorentzian space forms.

Lemma 3.2.1. *Let Σ^n be a spacelike hypersurface immersed into a Lorentzian space form $\mathbb{L}_c^{n+1}$, with constant mean curvature H. Then, the Omori-Yau's maximum principle holds on Σ^n for the Laplacian.*

Proof. It follows from Gauss equation (1.4) that, taking a local orthonormal frame $\{E_1, \ldots, E_n\}$ tangent to Σ^n, the Ricci curvature Ric of Σ^n is given by

$$\mathrm{Ric}(X, X) = c(n-1)|X|^2 - nH\langle AX, X\rangle + |AX|^2, \tag{3.1}$$

for all vector field X tangent to Σ^n. Since we can write

$$-nH\langle AX, X\rangle + |AX|^2 = \left|AX - \frac{nH}{2}X\right|^2 - \frac{n^2H^2}{4}|X|^2,$$

we get from (3.1) that

$$\mathrm{Ric}(X, X) \geq \left(c(n-1) - \frac{n^2H^2}{4}\right)|X|^2,$$

for all vector field X tangent to Σ^n. In particular, since H is constant this implies that the Ricci curvature of Σ^n is bounded from below, which assures by Lemma A.0.1 of Appendix A that the Omori-Yau's maximum principle holds on Σ^n for the Laplacian. □

Our second lemma gives a sufficient condition that asserts the boundedness of the squared norm of the second fundamental form $|A|^2$ of spacelike hypersurfaces in Lorentzian space forms, which follows from Theorem 1 in [82].

Lemma 3.2.2. *Let Σ^n be a spacelike hypersurface immersed into a Lorentzian space form $\mathbb{L}_c^{n+1}$, with constant mean curvature H satisfying $H^2 \geq \frac{4(n-1)}{n^2}$ when $c = 1$. Then, the squared norm of the second fundamental form $|A|^2$ is bounded from above.*

3.3 Proof of Theorem 3.1.1

Since H is constant and taking into account equations (1.6) and (2.10), we deduce that the Simons' formula (see Proposition 1.0.1 (i)) can be rewritten in terms of the total umbilicity tensor as follows

$$\frac{1}{2}\Delta|\Phi|^2 = |\nabla\Phi|^2 + |\Phi|^4 - nH\mathrm{tr}(\Phi^3) - n(H^2-1)|\Phi|^2. \tag{3.2}$$

Then our assumption (1.10) on the total umbilicity tensor yields

$$\begin{aligned} \frac{1}{2}\Delta|\Phi|^2 &\geq |\nabla\Phi|^2 + |\Phi|^4 - \frac{n(n-2p)}{\sqrt{np(n-p)}}|H||\Phi|^3 - n(H^2-1)|\Phi|^2 \\ &\geq |\Phi|^2\mathcal{P}_{H,n,p,1}(|\Phi|), \end{aligned} \tag{3.3}$$

where $\mathcal{P}_{H,n,p,1}(x)$ is the polynomial given by

$$\mathcal{P}_{H,n,p,1}(x) = x^2 - \frac{n(n-2p)}{\sqrt{np(n-p)}}|H|x - n(H^2-1). \tag{3.4}$$

On the one hand, if $H^2 < \frac{4(n-1)}{n^2}$, it follows by a well known result due to Akutagawa [2] that Σ^n is totally umbilical. In this case, taking into account the classification of the totally umbilical spacelike hypersurfaces by Montiel [94], we conclude that Σ^n must be isometric to an Euclidean sphere.

On the other hand, if $H^2 \geq \frac{4(n-1)}{n^2}$, it follows from Lemma 3.2.2 that $|\Phi|^2$ is bounded from above, because H is constant. Besides, Lemma 3.2.1 says that the Omori-Yau's maximum principle holds on Σ^n. Therefore, from Lemma A.0.1, there is a sequence $\{p_j\} \subset \Sigma^n$ such that

$$\lim |\Phi|(p_j) = \sup |\Phi| \quad \text{and} \quad \Delta|\Phi|^2(p_j) < \frac{1}{j},$$

which jointly with (3.3) imply

$$\frac{1}{j} > \frac{1}{2}\Delta|\Phi|^2(p_j) \geq |\Phi|^2(p_j)\mathcal{P}_{H,n,p,1}(|\Phi|(p_j)).$$

Making $j \to +\infty$ we find

$$(\sup |\Phi|)^2 \, \mathcal{P}_{H,n,p,1}(\sup |\Phi|) \leq 0.$$

From this, we deduce that either $\sup |\Phi| = 0$ and Σ^n is a totally umbilical hypersurface, or $\sup |\Phi| > 0$ and then $\mathcal{P}_{H,n,p,1}(\sup |\Phi|) \leq 0$. In the latter case, when $\frac{4(n-1)}{n^2} \leq H^2 < \frac{4p(n-p)}{n^2}$ the polynomial $\mathcal{P}_{H,n,p,1}(x) > 0$ for every $x \in \mathbb{R}$. Hence, it must necessarily be $H^2 \geq \frac{4p(n-p)}{n^2}$. In this case, the polynomial $\mathcal{P}_{H,n,p,1}(x)$ has two roots, which in fact become a double root when $H^2 = \frac{4p(n-p)}{n^2}$, given by

$$0 < \hat{\beta}(H,n,p,1) = \frac{\sqrt{n}}{2\sqrt{p(n-p)}}\left((n-2p)|H| + \sqrt{n^2H^2 - 4p(n-p)}\right)$$

and

$$\beta(H,n,p,1) = \frac{\sqrt{n}}{2\sqrt{p(n-p)}}\left((n-2p)|H| - \sqrt{n^2H^2 - 4p(n-p)}\right).$$

Therefore, in this case $\mathcal{P}_{H,n,p,1}(\sup |\Phi|) \leq 0$ means that

$$\beta(H,n,p,1) \leq \sup |\Phi| \leq \hat{\beta}(H,n,p,1).$$

Proceeding, if $\frac{4p(n-p)}{n^2} \leq H^2 < 1$, then it is not difficult to verify that $\beta(H,n,p,1) > 0$. In this case, if the equality $\sup |\Phi| = \beta(H,n,p,1)$ holds, we get that $\mathcal{P}_{H,n,p,1}(|\Phi|) \geq 0$ on Σ^n. This together with (3.3) guarantee that $|\Phi|^2$ is a subharmonic function on Σ^n. Therefore, if there exists a point in Σ^n at which this supremum is attained, by the strongly maximum principle, $|\Phi|$ must be constant, namely, $|\Phi| = \beta(H,n,p,1)$. Thus, (3.3) becomes trivially an equality. In particular, this implies that $|\nabla A| = |\nabla\Phi| = 0$, that is, Σ^n is an isoparametric hypersurface. Besides, (3.3) also assures that the equality in (1.10) holds, which gives, by Lemma 1.0.3, that the hypersurface has exactly two distinct constant principal curvatures, with multiplicities p and $n-p$. Then, by the classical results on isoparametric hypersurfaces of the de Sitter space $\mathbb{S}_1^{n+1}$ (see, for instance, Theorem 5.1 in [1]) we conclude that Σ^n must be one of the two following standard products embeddings:

$$\mathbb{S}^{n-p}(\sqrt{1+r^2})\times\mathbb{H}^p(r)\subset\mathbb{S}_1^{n+1} \quad \text{or} \quad \mathbb{S}^p(\sqrt{1+r^2})\times\mathbb{H}^{n-p}(r)\subset\mathbb{S}_1^{n+1}, \quad \text{with} \quad r>0.$$

In the first case, $\mathbb{S}^{n-p}(\sqrt{1+r^2})\times\mathbb{H}^p(r)$ has principal curvatures, for a suitable choice of the normal vector field, given by

$$\lambda_1=\ldots=\lambda_{n-p}=\frac{r}{\sqrt{1+r^2}} \quad \text{and} \quad \lambda_{n-p+1}=\ldots=\lambda_n=\frac{\sqrt{1+r^2}}{r}$$

and its constant mean curvature is

$$H=\frac{nr^2+p}{nr\sqrt{1+r^2}}. \tag{3.5}$$

In this case, by equation (1.6),

$$|\Phi|=\sqrt{\frac{p(n-p)}{nr^2(1+r^2)}}. \tag{3.6}$$

We observe that $H^2<1$ if and only if $r^2>\frac{p^2}{n(n-2p)}$. We also notice that $H^2=\frac{4p(n-p)}{n^2}$ when $r^2=\frac{p}{n-2p}$, while $\frac{4p(n-p)}{n^2}<H^2<1$ in other case. Further, for each value of $H^2\in\left(\frac{4p(n-p)}{n^2},1\right)$ there exist two values of $r^2>\frac{p^2}{n(n-2p)}$ with the same constant mean curvature H^2, which are given by

$$\frac{p^2}{n(n-2p)}<r^2=\frac{nH^2-2p-H\sqrt{n^2H^2-4p(n-p)}}{2n(1-H^2)}<\frac{p}{n-2p} \tag{3.7}$$

and

$$\frac{p}{n-2p}<r^2=\frac{nH^2-2p+H\sqrt{n^2H^2-4p(n-p)}}{2n(1-H^2)}. \tag{3.8}$$

Then from equation (3.6) we find in the first case, by using (3.7),

$$|\Phi|=\frac{\sqrt{n}}{2\sqrt{p(n-p)}}\left((n-2p)H+\sqrt{n^2H^2-4p(n-p)}\right)=\hat{\beta}(H,n,p,1)>\beta(H,n,p,1), \tag{3.9}$$

that is, the inequality is strict, while in the second case one has from (3.8)

$$|\Phi|=\frac{\sqrt{n}}{2\sqrt{p(n-p)}}\left((n-2p)H-\sqrt{n^2H^2-4p(n-p)}\right)=\beta(H,n,p,1)$$

and the equality holds. Besides, if $r^2=\frac{p}{n-2p}$ it is clear that the equality $\sup|\Phi|=\beta(H,n,p,1)$ also happens. Hence, the equality $\sup|\Phi|=\beta(H,n,p,1)$ holds for $r^2\geq\frac{p}{n-2p}$.

On the other hand, $\mathbb{S}^p(\sqrt{1+r^2}) \times \mathbb{H}^{n-p}(r)$ has constant principal curvatures given by

$$\lambda_1 = \ldots = \lambda_p = \frac{r}{\sqrt{1+r^2}} \quad \text{and} \quad \lambda_{p+1} = \ldots = \lambda_n = \frac{\sqrt{1+r^2}}{r}$$

for a suitable choice of the normal vector field, with constant mean curvature

$$H = \frac{nr^2 + (n-p)}{nr\sqrt{1+r^2}}.$$

Then, $H^2 > 1$ and in this case $\beta(H, n, p, 1) < 0$ always. In particular, $\sup|\Phi| > \beta(H, n, p, 1)$ and the equality never happens. This finishes the proof of Theorem 3.1.1.

3.4 Proof of Theorem 3.1.2

Reasoning as in the proof of Theorem 3.1.1, one has that the Omori-Yau's maximum principle holds on Σ^n because of Lemma 3.2.1 and, on the other hand, it is not difficult to see that the Simons' formula in (3.2), in this case, it becomes

$$\frac{1}{2}\Delta|\Phi|^2 = |\nabla\Phi|^2 + |\Phi|^4 - nH\mathrm{tr}(\Phi^3) - n(H^2 - c)|\Phi|^2.$$

Then the Okumura type condition on Φ yields

$$\begin{aligned} \frac{1}{2}\Delta|\Phi|^2 &\geq |\nabla\Phi|^2 + |\Phi|^4 - \frac{n(n-2p)}{\sqrt{np(n-p)}}|H||\Phi|^3 - n(H^2 - c)|\Phi|^2 \\ &\geq |\Phi|^2\mathcal{P}_{H,n,p,c}(|\Phi|), \end{aligned} \tag{3.10}$$

where $\mathcal{P}_{H,n,p,c}(x)$ is the polynomial given by

$$\mathcal{P}_{H,n,p,c}(x) = x^2 - \frac{n(n-2p)}{\sqrt{np(n-p)}}|H|x - n(H^2 - c). \tag{3.11}$$

When $c = 1$ the first part of Theorem 3.1.2 was proved in Theorem 3.1.1. Let us consider for the moment the case $c \in \{0, -1\}$. Then $H^2 - c \geq 0$ and the polynomial $\mathcal{P}_{H,n,p,c}(x)$ has an unique nonnegative root given by

$$\hat{\beta}(H, n, p, c) = \frac{\sqrt{n}}{2\sqrt{p(n-p)}}\left((n-2p)|H| + \sqrt{n^2H^2 - 4pc(n-p)}\right).$$

By using once more Lemma 3.2.2 we get that $|\Phi|^2$ is bounded from above. Then, by applying Lemma A.0.1 to the function $|\Phi|^2$ we know that there exists a sequence of points $\{q_j\} \subset \Sigma^n$ satisfying

$$\lim|\Phi|(q_j) = \sup|\Phi| \quad \text{and} \quad \Delta|\Phi|^2(q_j) < \frac{1}{j}.$$

In particular, from (3.10) we obtain

$$(\sup|\Phi|)^2\,\mathcal{P}_{H,n,p,c}(\sup|\Phi|)\leq 0,$$

which gives that either $\sup|\Phi|=0$ and Σ^n is a totally umbilical hypersurface, or $\sup|\Phi|>0$ and in this case must be $\mathcal{P}_{H,n,p,c}(\sup|\Phi|)\leq 0$. In the latter case, when $c=0$ it follows from rigidity theorems of Calabi [28] and Cheng and Yau [43] that $H\neq 0$. Hence, for any value of $c\in\{0,-1\}$, we have

$$0<\sup|\Phi|\leq\hat{\beta}(H,n,p,c).$$

Going back to the general case, we assume that the equality $|\Phi|=\hat{\beta}(H,n,p,c)$ holds. If $H=0$, which can occur only when $c=-1$, then by a result due to Ishihara [81] we know that Σ^n is a maximal hyperbolic cylinder $\mathbb{H}^{n-p}\left(\frac{\sqrt{n-p}}{\sqrt{p}}\right)\times\mathbb{H}^p\left(\frac{\sqrt{n}}{\sqrt{p}}\right)\subset\mathbb{H}^{n+1}_1$, which satisfies $|\Phi|=\sqrt{n}=\hat{\beta}(0,n,p,-1)$. If $H\neq 0$, then a similar reasoning to that presented in Theorem 3.1.1 shows that Σ^n is an isoparametric hypersurface having exactly two distinct constant principal curvatures with multiplicities p and $n-p$. Again, we can apply a classical result on isoparametric hypersurfaces of Lorentzian space forms (see Theorem 5.1 in [1]) to conclude that Σ^n must be isometric to one of the following standard products embeddings:

(a) $\mathbb{R}^{n-p}\times\mathbb{H}^p(r)\subset\mathbb{R}^{n+1}_1$ or $\mathbb{R}^p\times\mathbb{H}^{n-p}(r)\subset\mathbb{R}^{n+1}_1$, with $r>0$, if $c=0$;

(b) $\mathbb{S}^{n-p}(\sqrt{1+r^2})\times\mathbb{H}^p(r)\subset\mathbb{S}^{n+1}_1$ or $\mathbb{S}^p(\sqrt{1+r^2})\times\mathbb{H}^{n-p}(r)\subset\mathbb{S}^{n+1}_1$, with $r>0$, if $c=1$;

(c) $\mathbb{H}^{n-p}(\sqrt{1-r^2})\times\mathbb{H}^p(r)\subset\mathbb{H}^{n+1}_1$, with $0<r<1$, if $c=-1$.

Concerning to the Lorentz-Minkowski space $\mathbb{R}^{n+1}_1$ (that is, $c=0$), for a given radius $r>0$, $\mathbb{R}^{n-p}\times\mathbb{H}^p(r)\subset\mathbb{R}^{n+1}_1$ has, for a suitable choice of the normal vector field, constant principal curvatures given by

$$\lambda_1=\ldots=\lambda_{n-p}=0\quad\text{and}\quad\lambda_{n-p+1}=\ldots=\lambda_n=\frac{1}{r},$$

so that its constant mean curvature is $H=\frac{p}{nr}$. In this case, we achieve

$$|\Phi|=\frac{\sqrt{p(n-p)}}{\sqrt{nr^2}}=\frac{\sqrt{n(n-p)}}{\sqrt{p}}|H|=\hat{\beta}(H,n,p,0)$$

and the equality holds. On the other hand, for a given radius $r>0$ and a appropriate choice of the orientation, $\mathbb{R}^p\times\mathbb{H}^{n-p}(r)\subset\mathbb{R}^{n+1}_1$ has constant principal curvatures

$$\lambda_1=\ldots=\lambda_p=0\quad\text{and}\quad\lambda_{p+1}=\ldots=\lambda_n=\frac{1}{r}$$

and constant mean curvature $H=\frac{n-p}{nr}$, which gives

$$|\Phi|=\frac{\sqrt{p(n-p)}}{\sqrt{nr^2}}=\frac{\sqrt{np}}{\sqrt{n-p}}|H|<\hat{\beta}(H,n,p,0),$$

that is, the inequality is strict, giving the characterization of the equality $|\Phi| = \hat{\beta}(H, n, p, 0)$ when $c = 0$.

As for the case of the de Sitter space $\mathbb{S}_1^{n+1}$ (that is, $c = 1$), let us consider first the hyperbolic cylinder $\mathbb{S}^{n-p}(\sqrt{1+r^2}) \times \mathbb{H}^p(r) \subset \mathbb{S}_1^{n+1}$. As in Theorem 3.1.1, we infer that $H^2 \leq 1$ if and only if $r^2 \geq \frac{p^2}{n(n-2p)}$ and equality $H^2 = 1$ holds when $r^2 = \frac{p}{n(n-2p)}$. Thus, by equations (3.7), (3.8) and (3.9) we get that in the case $H^2 \leq 1$,

$$|\Phi| = \frac{\sqrt{n}}{2\sqrt{p(n-p)}} \left((n-2p)|H| + \sqrt{n^2H^2 - 4p(n-p)} \right) = \hat{\beta}(H, n, p, 1),$$

when

$$\frac{p^2}{n(n-2p)} < r^2 = \frac{nH^2 - 2p - |H|\sqrt{n^2H^2 - 4p(n-p)}}{2n(1-H^2)} \leq \frac{p}{n-2p}$$

and the equality holds, while

$$|\Phi| = \frac{\sqrt{n}}{2\sqrt{p(n-p)}} \left((n-2p)|H| - \sqrt{n^2H^2 - 4p(n-p)} \right) = \beta(H, n, p, 1) < \hat{\beta}(H, n, p, 1)$$

when

$$\frac{p}{n-2p} < r^2 = \frac{nH^2 - 2p + |H|\sqrt{n^2H^2 - 4p(n-p)}}{2n(1-H^2)},$$

that is, the inequality is strict. On the other hand, $H^2 > 1$ if and only if $r^2 < \frac{p^2}{n(n-2p)}$ and, by (3.5),

$$r^2 = \frac{2p - nH^2 + |H|\sqrt{n^2H^2 - 4p(n-p)}}{2n(H^2-1)}.$$

So, in this case equation (3.6) yields

$$|\Phi| = \frac{\sqrt{n}}{2\sqrt{p(n-p)}} \left((n-2p)|H| + \sqrt{n^2H^2 - 4p(n-p)} \right) = \hat{\beta}(H, n, p, 1).$$

In particular, the equality $|\Phi| = \hat{\beta}(H, n, p, 1)$ holds too.

In the case of the hyperbolic cylinder $\mathbb{S}^p(\sqrt{1+r^2}) \times \mathbb{H}^{n-p}(r) \subset \mathbb{S}_1^{n+1}$ with $r > 0$, which has constant principal curvatures, for a suitable choice of the orientation,

$$\lambda_1 = \ldots = \lambda_p = \frac{r}{\sqrt{1+r^2}} \quad \text{and} \quad \lambda_{p+1} = \ldots = \lambda_n = \frac{\sqrt{1+r^2}}{r},$$

we deduce that its constant mean curvature is given by

$$H = \frac{nr^2 + (n-p)}{nr\sqrt{1+r^2}}$$

and

$$|\Phi| = \sqrt{\frac{p(n-p)}{nr^2(1+r^2)}}. \tag{3.12}$$

We also observe that r^2 is given by

$$0 < r^2 = \frac{2(n-p) - nH^2 + |H|\sqrt{n^2H^2 + 4p(n-p)}}{2n(H^2-1)}.$$

Then, by (3.12),

$$|\Phi| = \frac{\sqrt{n}}{2\sqrt{p(n-p)}}\left(\sqrt{n^2H^2 + 4p(n-p)} - (n-2p)|H|\right) < \hat{\beta}(H, n, p, -1).$$

This gives the characterization of the equality $|\Phi| = \hat{\beta}(H, n, p, 1)$ when $c = 1$.

Finally, in the case of the anti-de Sitter space $\mathbb{H}_1^{n+1}$ (that is, $c = -1$), for a given $0 < r < 1$ we have that the standard product embedding $\mathbb{H}^{n-p}(\sqrt{1-r^2}) \times \mathbb{H}^p(r) \subset \mathbb{H}_1^{n+1}$ has constant principal curvatures

$$\lambda_1 = \ldots = \lambda_{n-p} = \frac{r}{\sqrt{1-r^2}} \quad \text{and} \quad \lambda_{n-p+1} = \ldots = \lambda_n = -\frac{\sqrt{1-r^2}}{r},$$

for a suitable choice of the normal vector field. Then, its constant mean curvature H is given by

$$H = \frac{nr^2 - p}{nr\sqrt{1-r^2}}, \tag{3.13}$$

which implies

$$|\Phi| = \sqrt{\frac{p(n-p)}{nr^2(1-r^2)}}.$$

So, an easy computation using (3.13) shows that

$$r^2 = \frac{nH^2 + 2p \pm |H|\sqrt{n^2H^2 + 4p(n-p)}}{2n(1+H^2)},$$

where we choose the sign $+$ or $-$ according to $r^2 > \frac{p}{n}$ or $r^2 \leq \frac{p}{n}$, respectively. Hence, we obtain

$$|\Phi| = \frac{\sqrt{n}}{2\sqrt{p(n-p)}}\left(\mp(n-2p)|H| + \sqrt{n^2H^2 - 4pc(n-p)}\right),$$

where we use the same criterion for the sign. Therefore, the equality $|\Phi| = \hat{\beta}(H, n, p, -1)$ holds when $r^2 \leq \frac{p}{n}$, and must be $|\Phi| < \hat{\beta}(H, n, p, -1)$ when $r^2 > \frac{p}{n}$. This proves the characterization of the equality $|\Phi| = \hat{\beta}(H, n, p, -1)$ when $c = -1$ and this concludes the proof of Theorem 3.1.2.

Chapter 4

Constant scalar curvature spacelike hypersurfaces in Lorentzian space forms

We provide sharp lower and upper bounds for the supremum of the norm of the total umbilicity tensor of complete spacelike hypersurfaces with constant scalar curvature immersed into a Lorentzian space form satisfying the Okumura type inequality already studied in Chapters 2 and 3. In this chapter we present the results contained in [50].

4.1 Statement of the main results

The problem of characterizing hypersurfaces with constant scalar curvature of the de Sitter space $\mathbb{S}_1^{n+1}$ constitutes an important and fruitful topic in the theory of isometric immersions, which has being widely approached by many authors. For instance, an interesting result due to Cheng and Ishikawa [40] states that the totally umbilical Euclidean sphere is the only compact spacelike hypersurface in the de Sitter space having constant normalized scalar curvature $R < 1$, generalizing a previuos result proved by Zheng [119] under the additional condition that the sectional curvatures of the hypersurface are nonnegative. On the other hand, Li [89] posed the question on whether the only complete spacelike hypersurfaces with constant normalized scalar curvature R satisfying $\frac{n-2}{n} \leq R \leq 1$ in the de Sitter space are totally umbilical ones. In [29] Camargo et al., by extending a technique introduced by Cheng and Yau [44], gave a positive answer to this question positively under the additional assumption that the mean curvature is bounded from above on such hypersurfaces (see also [13, 23, 24, 26, 78, 90, 109] for others results in this context).

In this chapter, by assuming the Okumura type inequality, already studied in Chapters 2 and 3, on the total umbilicity tensor Φ of complete spacelike hypersurfaces with constant scalar curvature, our first purpose is to prove the following rigidity result, which yields a gap between the totally umbilical hypersurfaces and hyperbolic cylinders of the de Sitter space $\mathbb{S}_1^{n+1}$ and improvement some of the aforementioned results.

Theorem 4.1.1. *Let Σ^n be a complete spacelike hypersurface immersed into the de Sitter space $\mathbb{S}_1^{n+1}$, with constant positive normalized scalar curvature satisfying $R \leq 1$. If its total umbilicity* *tensor Φ satisfies* (1.10) *for some $1 \leq p < \frac{n}{2}$, then*

(i) either $\sup|\Phi| = 0$ *and Σ^n is a totally umbilical hypersurface,*

(ii) or $R \leq C(n,p,1)$ and

$$\sup|\Phi| \geq \gamma(R,n,p,1) > 0,$$

where $C(n,p,1) \leq 1$ and $\gamma(R,n,p,1)$ are positive constants depending only on n and p, and R, n and p, respectively. Moreover, the equality $\sup|\Phi| = \gamma(R,n,p,1)$ *holds and this supremum is attained at some point of Σ^n if and only if*

(a) $p = 1$ and Σ^n is a hyperbolic cylinder $\mathbb{S}^{n-1}(\sqrt{1+r^2}) \times \mathbb{H}^1(r) \subset \mathbb{S}_1^{n+1}$ with $r^2 = \frac{n-2-nR}{nR}$;

(b) $1 < p < \frac{n}{2}$ and Σ^n is a hyperbolic cylinder $\mathbb{S}^{n-p}(\sqrt{1+r^2}) \times \mathbb{H}^p(r) \subset \mathbb{S}_1^{n+1}$, with

$$r^2 \geq \frac{p(p-1) + \sqrt{p(n-p)(n-p-1)(p-1)}}{(n-1)(n-2p)}.$$

Again, it is worth pointing out that the Okumura type condition in (1.10) on the tensor Φ is always true when $p = 1$. Moreover, in this case the constant $C(n,1,1)$ is given explicitly by $C(n,1,1) = \frac{n-2}{n}$, as we shall see. In particular, Theorem 4.1.1 gives a meaningful improvement of Theorem 1.1 in [29]. In addtion, still in relation to the case $p = 1$, our Theorem 4.1.1 is an improvement of a recent result obtained by Alías et al. [13]. In this context, the general case of Theorem 4.1.1 becomes a nice generalization of the results aforementioned in the sense that new isoparametric hypersurfaces of the de Sitter space are characterized, namely, the hyperbolic cylinders $\mathbb{S}^{n-p}(\sqrt{1+r^2}) \times \mathbb{H}^p(r) \subset \mathbb{S}_1^{n+1}$ when $p > 1$.

Furthermore, let us recall once more that when $1 < p < \frac{n}{2}$ the assumption in (1.10) is weaker than to assume the geometric condition of the hypersurface has two distinct principal curvatures with multiplicities p and $n-p$. For this reason, Theorem 4.1.1 can be regarded as an improvement of results contained in [78, 109] concerning complete spacelike hypersurfaces in the de Sitter space having two distinct principal curvatures with multiplicities greater than one.

Proceeding, we obtain a sharp upper bound for the norm of the total umbilicity tensor of complete spacelike hypersurfaces satisfying the Okumura type condition in (1.10), which gives another characterization result provided that this bound is reached. Here, the hypersurfaces are immersed into any Lorentzian space form $\mathbb{L}_c^{n+1}$ of constant sectional curvature $c \in \{0, 1, -1\}$.

Theorem 4.1.2. *Let Σ^n be a complete spacelike hypersurface immersed into a Lorentzian space form $\mathbb{L}_c^{n+1}$, with constant normalized scalar curvature R satisfying $R \leq c$. If its total umbilicity tensor Φ satisfies* (1.10) *for some $1 < p < \frac{n}{2}$ and $|\Phi|$ is bounded from above, then*

(i) either $\sup|\Phi| = 0$ *and Σ^n is a totally umbilical hypersurface,*

(ii) or

$$0 < \sup|\Phi| \leq \hat{\gamma}(R,n,p,c),$$

where $\hat{\gamma}(R,n,p,c)$ is a positive constant depending only on R, n, p and c. Moreover, if $R < -1$ when $c = -1$, the equality $|\Phi| = \hat{\gamma}(R,n,p,c)$ holds if and only if *(a) $c = 0$ and Σ^n is a hyperbolic cylinder $\mathbb{R}^{n-p} \times \mathbb{H}^p(r) \subset \mathbb{R}^{n+1}_1$, with $r = -\frac{p(p-1)}{n(n-1)R} > 0$;*

(b) $c = 1$ and Σ^n is a hyperbolic cylinder $\mathbb{S}^{n-p}(\sqrt{1+r^2}) \times \mathbb{H}^p(r) \subset \mathbb{S}^{n+1}_1$, with

$$r^2 \leq \frac{p(p-1) + \sqrt{p(n-p)(n-p-1)(p-1)}}{(n-1)(n-2p)};$$

(c) $c = -1$ and Σ^n is a hyperbolic cylinder $\mathbb{H}^{n-p}(\sqrt{1-r^2}) \times \mathbb{H}^p(r) \subset \mathbb{H}^{n+1}_1$, with

$$r^2 = \frac{(n-1)(nR + (n-2p)) + \sqrt{[n(n-1)(R+1) - 2p(n-p)]^2 - 4p(n-p)(p(n-p) - (n-1))}}{2n(n-1)R}.$$

We observe that our Theorem 4.1.2 gives a sort of improvent of the main result in [46] when $c = -1$, that is, the ambient space is the anti-de Sitter space $\mathbb{H}^{n+1}_1$.

4.2 Auxiliary results

The main tools to prove Theorems 4.1.1 and 4.1.2 are results concerning the ellipticity of the Cheng-Yau's operator L and the validity of the Omori-Yau's maximum principle on a complete spacelike hypersurface $\Sigma^n \hookrightarrow \mathbb{L}^{n+1}_c$ for the operator L as well as a lower boundedness for the operator L acting on the squared norm of the total umbilicity tensor Φ. The proofs of these results are analogous to the proofs of Lemma 2.2.1 and Proposition 2.2.2 given in Chapter 2 in the Riemannian setting. For the sake of completeness, we include here the derivation of them. So, we begin by proving the following:

Lemma 4.2.1. *Let Σ^n be a complete spacelike hypersurface immersed into a Lorentzian space form $\mathbb{L}^{n+1}_c$, with constant normalized scalar curvature satisfying $R < c$ (resp. $R \leq c$). In the case where $R = c$, assume in addition that the mean curvature function H does not change sign on Σ^n. The following holds:*

(i) The operator L is elliptic (resp. semi-elliptic) or, equivalently, P is positive definite (resp. semi-definite), for an appropriate choice of the orientation of Σ^n;

(ii) If $\sup |\Phi|^2 < +\infty$, then the Omori-Yau's maximum principle holds on Σ^n for the operator L.

Proof. We follow the ideas contained in the proof of Lemma 6 of [32]. One has by equation (1.5) that the mean curvature function H does not change sign on Σ^n. In particular, if $R < c$ then for an appropriate choice of the orientation of Σ^n we can assume that $H > 0$. In the case $R = c$, we choose the orientation of Σ^n so that $H \geq 0$. Let $\lambda_1, \ldots, \lambda_n$ be the principal curvatures of Σ^n. Then the eigenvalues of tensor P are given by

$$\mu_i = nH - \lambda_i, \quad i = 1, \ldots, n.$$

Taking into account equation (1.5) one shows that $|\lambda_i| \leq nH$, which gives

$$0 \leq \mu_i \leq 2nH, \quad \forall\, i = 1, \ldots, n,$$

with the strict inequalities in the case $R < c$. If μ_- and μ_+ stand for the minimum and the maximum of the eigenvalues of P, respectively, we infer that

$$\mu_- \geq 0 \quad \text{and} \quad \mu_+ \leq 2nH, \tag{4.1}$$

occurring the strict inequalities in the case $R < c$. In particular, item (i) is proved.

As for to item (ii), we state that the sectional curvature K_Σ of Σ^n is bounded from below. Indeed, Gauss equation (1.4) yields

$$\begin{aligned} K_\Sigma(X, Y) &= c - \langle AX, X\rangle\langle AY, Y\rangle + \langle AX, Y\rangle^2 \\ &\geq c - |AX||AY| \\ &\geq c - |A|^2, \end{aligned}$$

where $\{X, Y\}$ is an orthonormal basis for an arbitrary plane tangent to Σ^n. Besides, the relation in (1.5) implies

$$|\Phi|^2 = \frac{n-1}{n}|A|^2 - (n-1)(c - R).$$

In particular, must be $\sup |A|^2 < +\infty$ proving the claim.

To conclude, it suffices to observe that equation (1.7) gives

$$H^2 = \frac{1}{n(n-1)}|\Phi|^2 + (c - R), \tag{4.2}$$

so that the mean curvature function H also satisfies $\sup H^2 < +\infty$. In particular, equation (4.1) implies that $\sup \operatorname{tr}(P) < +\infty$. Therefore, Lemma A.0.3 of Appendix A guarantees that the Omori-Yau's maximum principles holds on Σ^n for the operator L. $\square$

Next we prove a lower boundedness for the operator L acting on the squared norm of the total umbilicity tensor Φ.

Proposition 4.2.2. *Let Σ^n be a complete spacelike hypersurface immersed into a Lorentzian space form $\mathbb{L}_c^{n+1}$, with constant normalized scalar curvature satisfying $R \leq c$. In the case where $R = c$, assume in addition that the mean curvature function H does not change sign on Σ^n. If its total umbilicity tensor Φ satisfies* (1.10) *for some $1 \leq p < \frac{n}{2}$, then*

$$\frac{1}{2}L(|\Phi|^2) \geq \frac{1}{\sqrt{n(n-1)}}|\Phi|^2 \mathcal{Q}_{R,n,p,c}(|\Phi|)\sqrt{|\Phi|^2 + n(n-1)(c - R)},$$

where the function $\mathcal{Q}_{R,n,p,c}(x)$ is given by

$$\mathcal{Q}_{R,n,p,c}(x) = (n-2)x^2 - (n-2p)\frac{\sqrt{n-1}}{\sqrt{p(n-p)}}x\sqrt{x^2 + n(n-1)(c-R)} + n(n-1)R. \tag{4.3}$$

Proof. Let us choose the orientation of Σ^n such that $H \geq 0$, occurring the strict inequality in the case $R < c$. Since Σ^n has constant scalar curvature, we may use (4.2) to obtain

$$\frac{n}{2(n-1)}L(|\Phi|^2) = \frac{1}{2}L(n^2H^2) = nHL(nH) + n^2\langle P\nabla H, \nabla H\rangle.$$

Then Lemma 4.2.1 guarantees that the operator P is positive semi-definite. In particular, the previous equality gives

$$\frac{1}{2(n-1)}L(|\Phi|^2) \geq HL(nH). \tag{4.4}$$

It follows from Simons' formula (see Lemma 1.0.1 (ii)) that

$$\frac{1}{2(n-1)}L(|\Phi|^2) \geq H(|\nabla A|^2 - n^2|\nabla H|^2) + H|A|^4 - nH^2\mathrm{tr}(A^3) + ncH(|A|^2 - nH^2). \tag{4.5}$$

Since $R \leq c$, Lemma 1.0.4 implies that

$$|\nabla A|^2 - n^2|\nabla H|^2 \geq 0. \tag{4.6}$$

Taking into account that $H \geq 0$, from inequalities (4.5) and (4.6) we deduce that

$$\frac{1}{2(n-1)}L(|\Phi|^2) \geq H|A|^4 - nH^2\mathrm{tr}(A^3) + ncH(|A|^2 - nH^2). \tag{4.7}$$

We also observe that the following relation

$$\mathrm{tr}(A^3) = \mathrm{tr}(\Phi^3) + 3H|\Phi|^2 + nH^3 \tag{4.8}$$

happens trivially. Then taking into account equation (1.6) and putting (4.8) into (4.7) we find

$$\frac{1}{2(n-1)}L(|\Phi|^2) \geq H|\Phi|^4 - nH^2\mathrm{tr}(\Phi^3) - nH(H^2 - c)|\Phi|^2. \tag{4.9}$$

Hence the Okumura type condition on Φ jointly with (4.9) give

$$\frac{1}{2(n-1)}L(|\Phi|^2) \geq H|\Phi|^2\left(|\Phi|^2 - \frac{n(n-2p)}{\sqrt{np(n-p)}}H|\Phi| - n(H^2 - c)\right). \tag{4.10}$$

Moreover, equation (1.7) allows us to rewrite the mean curvature as

$$H = \frac{1}{\sqrt{n(n-1)}}\sqrt{|\Phi|^2 + n(n-1)(c-R)}. \tag{4.11}$$

Therefore, by substituting (4.11) into (4.10) we get the desired inequality. □

4.3 Proof of Theorem 4.1.1

For the sake of clarity, we prove Theorem 4.1.1 in two steps, namely: proof of the inequalities and characterization of the equality.

4.3.1 First part: proof of the inequalities

If $\sup|\Phi| = +\infty$ then inequality (ii) of Theorem 4.1.1 holds automatically and there is nothing to prove.

Then let us assume that $\sup|\Phi| < +\infty$ and choose the orientation of Σ^n so that $H \geq 0$. It follows that Lemma 4.2.1 applies here and using it, we have that the Omori-Yau's maximum principle holds on Σ^n for the Cheng-Yau's operator L. By applying this to the function $|\Phi|^2$, we obtain a sequence of points $\{p_j\} \subset \Sigma^n$ such that

$$\lim|\Phi|(p_j) = \sup|\Phi| \quad \text{and} \quad L(|\Phi|^2)(p_j) < \frac{1}{j}.$$

From here and by Proposition 4.2.2 we deduce

$$\frac{1}{j} > L(|\Phi|^2)(p_j) \geq \frac{2}{\sqrt{n(n-1)}}|\Phi|^2(p_j)\mathcal{Q}_{R,n,p,1}(|\Phi|(p_j))\sqrt{|\Phi|^2(p_j) + n(n-1)(1-R)},$$

where the function $\mathcal{Q}_{R,n,p,1}(x)$ is given by (4.3). Passing the limit as $j \to +\infty$, we get

$$(\sup|\Phi|)^2\,\mathcal{Q}_{R,n,p,1}(\sup|\Phi|)\sqrt{(\sup|\Phi|)^2 + n(n-1)(1-R)} \leq 0.$$

Taking into account that $R \leq 1$ it follows that either $\sup|\Phi| = 0$, meaning that the hypersurface is totally umbilical, or $\sup|\Phi| > 0$ which means that must be

$$\mathcal{Q}_{R,n,p,1}(\sup|\Phi|) \leq 0.$$

In this latter case, we must have $R < 1$.

Let us consider first the case $p = 1$. Then the function $\mathcal{Q}_{R,n,1,1}(x)$ is strictly decreasing for every $x \in \mathbb{R}$ and $\mathcal{Q}_{R,n,1,1}(x) \to \frac{n(n-1)}{2}(nR - (n-2))$ when $x \to +\infty$. Hence, in this case, $\mathcal{Q}_{R,n,1,1}(\sup|\Phi|) \leq 0$ means that $R < \frac{n-2}{n} =: C(n,1,1)$ necessarily. In this latter case, since $\mathcal{Q}_{R,n,1,1}(0) = n(n-1)R > 0$, there exists an unique positive real number $\gamma(R,n,1,1) > 0$ depends only on R and n, such that $\mathcal{Q}_{R,n,1,1}(\gamma(R,n,1,1)) = 0$. In particular, it must be

$$\sup|\Phi| \geq \gamma(R,n,1,1) > 0.$$

As for to the case $1 < p < \frac{n}{2}$ we set, for each $t \in (-\infty, 1)$, the family of functions

$$\mathcal{Q}_{t,n,p,1}(x) = \widetilde{\mathcal{Q}}_{t,n,p,1}(x) + n(n-1)t \tag{4.12}$$

where the functions $\widetilde{\mathcal{Q}}_{t,n,p,1}(x)$ are given by

$$\widetilde{\mathcal{Q}}_{t,n,p,1}(x) = (n-2)x^2 - (n-2p)\frac{\sqrt{n-1}}{\sqrt{p(n-p)}}x\sqrt{x^2 + n(n-1)(1-t)}.$$

Then one verifies that the functions $\widetilde{\mathcal{Q}}_{t,n,p,1}(x)$ are coercive for every $t \in (-\infty, 1)$ and, in particular, the infimum $\inf \widetilde{\mathcal{Q}}_{t,n,p,1}$ must be attained. Let $\eta(t, n, p, 1) \in \mathbb{R}$ be a real number such that $\widetilde{\mathcal{Q}}_{t,n,p,1}(\eta(t, n, p, 1)) = \inf \widetilde{\mathcal{Q}}_{t,n,p,1}$. We also note that for each $t \in (-\infty, 1)$, the function $\widetilde{\mathcal{Q}}_{t,n,p,1}(x)$ is decreasing on $(-\infty, 0]$ and its derivative $\widetilde{\mathcal{Q}}'_{t,n,p,1}(x) < 0$ for every $x \geq 0$ small enough because $t < 1$. Thus the infimum $\inf \widetilde{\mathcal{Q}}_{t,n,p,1} < 0$ for every $t \in (-\infty, 1)$. Since $\widetilde{\mathcal{Q}}_{t,n,p,1}(x)$ has an unique positive critical point, we obtain that $\eta(t, n, p, 1) > 0$ is unique too. Hence $\widetilde{\mathcal{Q}}_{t,n,p,1}(x)$ is decreasing on $(-\infty, \eta(t, n, p, 1)]$ and increasing on $[\eta(t, n, p, 1), +\infty)$. Besides we have that $\inf \widetilde{\mathcal{Q}}_{t,n,p,1}$ decreases as t decreases, which implies that the set $X = \{t \in (0, 1)\ ;\ \inf \mathcal{Q}_{t,n,p,1}(x) \leq 0\} \neq \emptyset$. So we can define $C(n, p, 1) = \sup X$.

By the discussion above, it follows that in this case $\mathcal{Q}_{R,n,p,1}(\sup |\Phi|) \leq 0$ means that must be $R \leq C(n, p, 1)$. Therefore since $\mathcal{Q}_{R,n,p,1}(0) = n(n-1)R > 0$, we obtain that the function $\mathcal{Q}_{R,n,p,1}(x)$ vanishes only in two positive real numbers, $0 < \gamma(R, n, p, 1) \leq \hat{\gamma}(R, n, p, 1)$, which in fact become equals when $R = C(n, p, 1)$, depending only on R, n and p. In particular, it follows that also must be

$$0 < \gamma(R, n, p, 1) \leq \sup |\Phi| \leq \hat{\gamma}(R, n, p, 1). \tag{4.13}$$

This concludes the proof of the first part of Theorem 4.1.1.

Remark 4.3.1. Let us observe that for every $t \leq C(n, p, 1)$ the function $\mathcal{Q}_{t,n,p,1}(x)$ defined in (4.12) vanishes only in two real numbers, $\gamma(t, n, p, 1) \leq \hat{\gamma}(t, n, p, 1)$, and $\hat{\gamma}(t, n, p, 1) > 0$ always. Moreover, given $t_1 < t_2 < 1$ it is not difficult to see that the relation $\mathcal{Q}_{t_1,n,p,1}(x) < \mathcal{Q}_{t_2,n,p,1}(x)$ holds for every $x \geq 0$. In particular, if $t_1 < t_2 \leq C(n, p, 1)$ then

$$\hat{\gamma}(t_1, n, p, 1) > \hat{\gamma}(t_2, n, p, 1),$$

that is, $\hat{\gamma}(t, n, p, 1)$ decrease as t increase, and if $0 < t_1$ we also have

$$\gamma(t_1, n, p, 1) < \gamma(t_2, n, p, 1),$$

that is, $\gamma(t, n, p, 1)$ increase as t increase.

4.3.2 Second part: characterization of the equality

Let us prove the characterization of the equality in Theorem 4.1.1. To this end, let us assume that $\sup |\Phi| = \gamma(R, n, p, 1)$. In particular, $\mathcal{Q}_{R,n,p,1}(|\Phi|) \geq 0$ on Σ^n and then Proposition 4.2.2

yields

$$L(|\Phi|^2) \geq 0 \quad \text{on} \quad \Sigma^n,$$

that is, $|\Phi|^2$ is a L-subharmonic function on Σ^n. Furthermore, since $R < 1$, Lemma 4.2.1 says that the operator L is elliptic. Hence, if there exists a point on Σ^n such that this supremum is attained, by the stronger maximum principle, $|\Phi|$ must be constant, that is, $|\Phi| = \gamma(R, n, p, 1)$. Thus, the inequality in Proposition 4.2.2 becomes an equality,

$$\frac{1}{2}L(|\Phi|^2) = 0 = \frac{1}{\sqrt{n(n-1)}}|\Phi|^2 \mathcal{Q}_{R,n,p,1}(|\Phi|)\sqrt{|\Phi|^2 + n(n-1)(c-R)}.$$

Hence, all the inequalities along the proof of Proposition 4.2.2 must be, in fact, equalities. In particular, we obtain that equation (4.4) must be an equality, which jointly with the positiveness of the operator P imply that the mean curvature H is constant. Moreover, (4.7) is an equality,

$$|\nabla A|^2 = n^2|\nabla H|^2 = 0.$$

Then, the principal curvatures of Σ^n must be constant and Σ^n is an isoparametric hypersurface. Besides, equation (4.10) is an equality too, which implies by Lemma 1.0.3 that Σ^n has exactly two distinct constant principal curvatures with multiplicities p and $n-p$. Hence, by the classical results on isoparametric hypersurfaces of the de Sitter space $\mathbb{S}_1^{n+1}$ (see, for instance, Theorem 5.1 in [1]), we conclude that Σ^n must be one of the two following standard products embeddings:

$$\mathbb{S}^{n-p}(\sqrt{1+r^2}) \times \mathbb{H}^p(r) \subset \mathbb{S}_1^{n+1} \quad \text{or} \quad \mathbb{S}^p(\sqrt{1+r^2}) \times \mathbb{H}^{n-p}(r) \subset \mathbb{S}_1^{n+1}, \quad \text{with} \quad r > 0.$$

When $p = 1$ was proved by Alías et al. [13] that $\gamma(R, n, 1, 1) = R\sqrt{\frac{n(n-1)}{(n-2)(n-2-nR)}}$ and the equality $\sup|\Phi| = \gamma(R, n, 1, 1)$ holds if and only if Σ^n is isometric to the hyperbolic cylinder $\mathbb{S}^{n-1}(\sqrt{1+r^2}) \times \mathbb{H}^1(r)$, with $r^2 = \frac{n-2-nR}{nR}$. This gives the characterization of the equality $\sup|\Phi| = \gamma(R, n, 1, 1)$ in the case $p = 1$.

So let us assume that $1 < p < \frac{n}{2}$. In the first case, $\mathbb{S}^{n-p}(\sqrt{1+r^2}) \times \mathbb{H}^p(r) \subset \mathbb{S}_1^{n+1}$ has principal curvatures, for a suitable choice of the normal vector field, given by

$$\lambda_1 = \ldots = \lambda_{n-p} = \frac{r}{\sqrt{1+r^2}} \quad \text{and} \quad \lambda_{n-p+1} = \ldots = \lambda_n = \frac{\sqrt{1+r^2}}{r}$$

and its constant mean curvature is

$$H = \frac{nr^2 + p}{nr\sqrt{1+r^2}}.$$

As in the proof of Theorem 3.1.1, we observe that $H^2 \leq 1$ if and only if $r^2 \geq \frac{p^2}{n(n-2p)}$. In this case, the equality $H^2 = 1$ holds when $r^2 = \frac{p^2}{n(n-2p)}$, and $H^2 = \frac{4p(n-p)}{n^2}$ when $r^2 = \frac{p}{n-2p}$, while $\frac{4p(n-p)}{n^2} < H^2 < 1$ in other case. Moreover, for each value of $H^2 \in \left(\frac{4p(n-p)}{n^2}, 1\right)$ there exist two values of $r^2 > \frac{p^2}{n(n-2p)}$ with the same constant mean curvature H^2, which are given by

$$\frac{p^2}{n(n-2p)} < r^2 = \frac{nH^2 - 2p - H\sqrt{n^2H^2 - 4p(n-p)}}{2n(1-H^2)} < \frac{p}{n-2p} \tag{4.14}$$

and

$$\frac{p}{n-2p} < r^2 = \frac{nH^2 - 2p + H\sqrt{n^2H^2 - 4p(n-p)}}{2n(1-H^2)}. \tag{4.15}$$

On the other hand, $H^2 > 1$ if and only if $r^2 < \frac{p^2}{n(n-2p)}$ and, in this case,

$$r^2 = \frac{2p - nH^2 + H\sqrt{n^2H^2 - 4p(n-p)}}{2n(H^2-1)}. \tag{4.16}$$

Next, the norm of the total umbilicity tensor is

$$|\Phi| = \sqrt{\frac{p(n-p)}{nr^2(1+r^2)}}. \tag{4.17}$$

Hence by using equation (1.7), we get that $\mathbb{S}^{n-p}(\sqrt{1+r^2}) \times \mathbb{H}^p(r)$ has constant scalar curvature given by

$$n(n-1)(R-1) = \frac{-n(n-1)r^4 - 2p(n-1)r^2 - p(p-1)}{r^2(1+r^2)},$$

which gives that $R < 1$ always happens. Moreover, we find

$$R(r) = \frac{(n-1)(n-2p)r^2 - p(p-1)}{n(n-1)r^2(1+r^2)},$$

which implies $R > 0$ if and only if $r^2 > \frac{p(p-1)}{(n-1)(n-2p)}$. Besides, setting $t_1 := \frac{(n-2p)^2}{p(n-1)(n-p)} < 1$ it is not difficult to show that the following holds:

(i) $R = t_1$ if and only if either $r^2 = \frac{p^2}{n(n-2p)}$ or $r^2 = \frac{(n-p)(p-1)}{n-2p}$;

(ii) $R < t_1$ if and only if either $r^2 < \frac{p^2}{n(n-2p)}$ or $r^2 > \frac{(n-p)(p-1)}{n-2p}$;

(iii) $R > t_1$ if and only if $\frac{p^2}{n(n-2p)} < r^2 < \frac{(n-p)(p-1)}{n-2p}$.

Moreover, we see that the scalar curvature $R(r)$ attains its maximum for

$$r^2_{\max} = \frac{p(p-1) + \sqrt{p(n-p)(n-p-1)(p-1)}}{(n-1)(n-2p)},$$

so that $R(r)$ is increasing for $0 < r^2 \leq r^2_{\max}$ and decreasing for $r^2_{\max} \leq r^2 < +\infty$. Let us also observe that, by equation (4.10), when evaluated in $|\Phi|$ the function $\mathcal{Q}_{R,n,p,1}(x)$ is given by

$$\mathcal{Q}_{R,n,p,1}(|\Phi|) = (n-1)\mathcal{P}_{H,n,p,1}(|\Phi|),$$

where $\mathcal{P}_{H,n,p,1}(x)$ is the polynomial given in (3.4),

$$\mathcal{P}_{H,n,p,1}(x) = x^2 - \frac{n(n-2p)H}{\sqrt{np(n-p)}}x - n(H^2-1).$$

In case (i), $R = t_1$, if $r^2 = \frac{p^2}{n(n-2p)}$ then $H^2 = 1$ and, by equation (4.17), we infer that

$$|\Phi| = \frac{\sqrt{n}}{\sqrt{p(n-p)}}(n-2p) =: x_1.$$

Hence $\mathcal{P}_{H,n,p,1}(|\Phi|) = 0$, which means that $\mathcal{Q}_{t_1,n,p,1}(|\Phi|) = 0$. Since $\mathcal{Q}'_{t_1,n,p,1}(x_1) > 0$ we obtain that $x_1 = \frac{\sqrt{n}}{\sqrt{p(n-p)}}(n-2p) = \hat{\gamma}(t_1,n,p,1)$ is the biggest positive root of $\mathcal{Q}_{t_1,n,p,1}(x)$. In particular, we have $|\Phi| = \hat{\gamma}(t_1,n,p,1) > \gamma(t_1,n,p,1)$ and the inequality is strict. On the other hand, if $r^2 = \frac{(n-p)(p-1)}{n-2p}$ then

$$r^2 > \frac{p}{n-2p}$$

and must be $H^2 < 1$. In particular, r^2 is also given by (4.15) and, by (4.17), $|\Phi|$ can be write in terms of H as

$$|\Phi| = \frac{\sqrt{n}}{2\sqrt{p(n-p)}}\left((n-2p)H - \sqrt{n^2H^2 - 4p(n-p)}\right).$$

Hence $\mathcal{P}_{H,n,p,1}(|\Phi|) = 0$ and one has $\mathcal{Q}_{t_1,n,p,1}(|\Phi|) = 0$ too. Since in this case $|\Phi| < x_1 = \hat{\gamma}(t_1,n,p,1)$, must be $|\Phi| = \gamma(t_1,n,p,1)$ and the equality holds.

As to case (ii), $R < t_1$, on the one hand if $r^2 < \frac{p^2}{n(n-2p)}$ then $H^2 > 1$ and r^2 is given by (4.16). Thus, equation (4.17) yields

$$|\Phi| = \frac{\sqrt{n}}{2\sqrt{p(n-p)}}\left((n-2p)H + \sqrt{n^2H^2 - 4p(n-p)}\right)$$

and one can check that $\mathcal{P}_{H,n,p,1}(|\Phi|) = 0$. In particular, $\mathcal{Q}_{R,n,p,1}(|\Phi|) = 0$ and again must be either $|\Phi| = \gamma(R,n,p,1)$ or $|\Phi| = \hat{\gamma}(R,n,p,1)$. Since $|\Phi| > x_1 = \hat{\gamma}(t_1,n,p,1)$ and, by Remark 4.3.1,

$$\gamma(R,n,p,1) < \gamma(t_1,n,p,1) < \hat{\gamma}(t_1,n,p,1),$$

we obtain $|\Phi| = \hat{\gamma}(R,n,p,1)$ and the inequality is strict. On the other hand, if $r^2 > \frac{(n-p)(p-1)}{n-2p} > \frac{p}{n-2p}$ then, as in case (i), we get that

$$|\Phi| = \frac{\sqrt{n}}{2\sqrt{p(n-p)}}\left((n-2p)H - \sqrt{n^2H^2 - 4p(n-p)}\right)$$

is a positive root of $\mathcal{P}_{H,n,p,1}(x)$. It follows that either $|\Phi| = \gamma(R,n,p,1)$ or $|\Phi| = \hat{\gamma}(R,n,p,1)$. In this case, since $|\Phi| < x_1 = \hat{\gamma}(t_1,n,p,1)$ and taking into account once more Remark 4.3.1,

$$\gamma(R,n,p,1) < \gamma(t_1,n,p,1) < \hat{\gamma}(t_1,n,p,1) < \hat{\gamma}(R,n,p,1),$$

one has $|\Phi| = \gamma(R,n,p,1)$ and the equality holds.

For case (iii), $R > t_1$, let us observe first that an easy computation gives

$$\frac{p^2}{n(n-2p)} < r_{\max}^2 < \frac{p}{n-2p} < \frac{(n-p)(p-1)}{n-2p}.$$

When $r^2 = \frac{p}{n-2p}$, it follows that $H^2 = \frac{4p(n-p)}{n^2}$, $|\Phi| = \frac{n-2p}{\sqrt{n}} =: x_0$ is the only root of $\mathcal{P}_{H,n,p,1}(x)$ and

$$R = \frac{(n-2p)^2}{n(n-1)} =: t_0.$$

Then, we must have either $|\Phi| = \gamma(t_0, n, p, 1)$ or $|\Phi| = \hat{\gamma}(t_0, n, p, 1)$. But, since in this case $\mathcal{Q}'_{t_0,n,p,1}(x_0) < 0$, we get that x_0 is the smallest positive root of $\mathcal{Q}_{t_0,n,p,1}(x)$. Hence $|\Phi| = \gamma(t_0, n, p, 1)$ and the equality holds.

Let us now consider the case $\frac{p}{n-2p} < r^2 < \frac{(n-p)(p-1)}{n-2p}$. Then $H^2 < 1$, r^2 is given by (4.15) and

$$|\Phi| = \frac{\sqrt{n}}{2\sqrt{p(n-p)}}\left((n-2p)H - \sqrt{n^2H^2 - 4p(n-p)}\right)$$

is a positive root of $\mathcal{P}_{H,n,p,1}(x)$, which means that either $|\Phi| = \gamma(R, n, p, 1)$ or $|\Phi| = \hat{\gamma}(R, n, p, 1)$. Since in this case R decrease as r^2 increase, we have $R < t_0$. In particular,

$$\gamma(R, n, p, 1) < \gamma(t_0, n, p, 1) = x_0 < \hat{\gamma}(R, n, p, 1).$$

Moreover, since in this case $|\Phi| < x_0$, we get $|\Phi| = \gamma(R, n, p, 1)$ and the equality holds.

Let us assume by last that $\frac{p^2}{n(n-2p)} < r^2 < \frac{p}{n-2p}$. Hence, $H^2 < 1$, r^2 is given by (4.14) and

$$|\Phi| = \frac{\sqrt{n}}{2\sqrt{p(n-p)}}\left((n-2p)H + \sqrt{n^2H^2 - 4p(n-p)}\right)$$

is a positive root of $\mathcal{P}_{H,n,p,1}(x)$, which gives again that either $|\Phi| = \gamma(R, n, p, 1)$ or $|\Phi| = \hat{\gamma}(R, n, p, 1)$. Let us suppose for a moment that

$$|\Phi|(r_{\max}) = \gamma(R(r_{\max}), n, p, 1).$$

Then, for $r_{\max}^2 < r^2 < \frac{p}{n-2p}$ we have:

- H decrease in r, which implies that $|\Phi|$ decrease in r;
- R decrease in r, which implies that $\hat{\gamma}(R, n, p, 1)$ increase in r.

In particular,

$$|\Phi|(r) < |\Phi|(r_{\max}) = \gamma(R(r_{\max}), n, p, 1) \leq \hat{\gamma}(R(r_{\max}), n, p, 1) < \hat{\gamma}(R(r), n, p, 1).$$

It follows that $|\Phi| = \gamma(R, n, p, 1)$ for $r_{\max}^2 < r^2 < \frac{p}{n-2p}$. On the other hand, for $\frac{p^2}{n(n-2p)} < r^2 < r_{\max}^2$ we find:

- H decrease in r, which implies that $|\Phi|$ decrease in r;

- R increase in r, which implies that $\gamma(R,n,p,1)$ increase in r.

In particular,
$$|\Phi|(r) > |\Phi|(r_{\max}) = \gamma(R(r_{\max}),n,p,1) > \gamma(R(r),n,p,1).$$
Hence, if $\frac{p^2}{n(n-2p)} < r^2 < r_{\max}^2$ must be $|\Phi| = \hat{\gamma}(R,n,p,1)$.

A similar reasoning shows that if
$$|\Phi|(r_{\max}) = \hat{\gamma}(R(r_{\max}),n,p,1),$$
then $|\Phi| = \gamma(R,n,p,1)$ for $r_{\max}^2 < r^2 < \frac{p}{n-2p}$ and $|\Phi| = \hat{\gamma}(R,n,p,1)$ for $\frac{p^2}{n(n-2p)} < r^2 < r_{\max}^2$. In any case, we get
$$|\Phi| = \gamma(R,n,p,1), \text{ when } r_{\max}^2 < r^2 < \frac{p}{n-2p},$$
and the equality holds, and
$$|\Phi| = \hat{\gamma}(R,n,p,1), \text{ when } \frac{p^2}{n(n-2p)} < r^2 < r_{\max}^2,$$
and the inequality is strict. Finally, by continuity we conclude
$$|\Phi|(r_{\max}) = \gamma(R(r_{\max}),n,p,1) = \hat{\gamma}(R(r_{\max}),n,p,1)$$
and the equality holds too for $r^2 = r_{\max}^2$.

In the second case, the hyperbolic cylinder $\mathbb{S}^p(\sqrt{1+r^2}) \times \mathbb{H}^{n-p}(r) \subset \mathbb{S}_1^{n+1}$ has constant principal curvatures given by
$$\lambda_1 = \ldots = \lambda_p = \frac{r}{\sqrt{1+r^2}} \quad \text{and} \quad \lambda_{p+1} = \ldots = \lambda_n = \frac{\sqrt{1+r^2}}{r},$$
for a suitable choice of the normal vector field, constant mean curvature
$$H = \frac{nr^2 + (n-p)}{nr\sqrt{1+r^2}} \tag{4.18}$$
and the norm of its total umbilicity tensor is
$$|\Phi| = \sqrt{\frac{p(n-p)}{nr^2(1+r^2)}}. \tag{4.19}$$
Hence by using equation (1.7), we get that its constant scalar curvature is given by
$$R = -\frac{(n-1)(n-2p)r^2 + (n-p)(n-p-1)}{n(n-1)r^2(1+r^2)}, \tag{4.20}$$
so that $R < 0$. Therefore, $\mathbb{S}^p(\sqrt{1+r^2}) \times \mathbb{H}^{n-p}(r)$ does not satisfy our hypothesis. This finishes the characterizing of the equality $\sup|\Phi| = \gamma(R,n,p,1)$ when $1 < p < \frac{n}{2}$, proving Theorem 4.1.1.

4.4 Proof of Theorem 4.1.2

First of all, as in the proof of Theorem 4.1.1, on the one hand one has that the Omori-Yau's maximum principle holds on Σ^n for the Cheng-Yau's operator L because of Lemma 4.2.1 and, on the other hand, by applying it to the function $|\Phi|^2$, it yields

$$(\sup|\Phi|)^2\, \mathcal{Q}_{R,n,p,c}(\sup|\Phi|)\sqrt{(\sup|\Phi|)^2+n(n-1)(c-R)}\leq 0,$$

where the function $\mathcal{Q}_{R,n,p,c}(x)$ is given by (4.3). Then either $\sup|\Phi|=0$ and Σ^n is a totally umbilical hypersurface, or $\sup|\Phi|>0$ and in this case must be $\mathcal{Q}_{R,n,p,c}(\sup|\Phi|)\leq 0$. In the latter case, since $p>1$ we have that $\mathcal{Q}_{R,n,p,c}(x)$ is a coercive function. When $c=1$ the first part of Theorem 4.1.2 follows of the proof of Theorem 4.1.1 (see equation (4.13)) and Remark 4.3.1. Indeed, in this case must be $R\leq C(R,n,1)$ and follows from Remark 4.3.1 that $\mathcal{Q}_{R,n,p,1}(x)$ has a positive root $\hat{\gamma}(R,n,p,1)>0$. Then, by equation (4.13), we must have

$$0<\sup|\Phi|\leq\hat{\gamma}(R,n,p,1).$$

So let us consider for the moment the case $c\in\{0,-1\}$. Then, when $c=0$ must be $R<0$, and thus $\mathcal{Q}_{R,n,p,0}(0)=n(n-1)R<0$ and $\mathcal{Q}'_{R,n,p,0}(x)<0$ for every $x\geq 0$ small enough. When $c=-1$ we have $\mathcal{Q}_{R,n,p,-1}(0)=n(n-1)R<0$ and $\mathcal{Q}'_{R,n,p,-1}(x)<0$ for every $x\geq 0$ small enough. Hence, as in the case $c=1$, there exists an unique positive real number $\hat{\gamma}(R,n,p,c)>0$, depending only on R, n, p and c, such that $\mathcal{Q}_{R,n,p,c}(\hat{\gamma}(R,n,p,c))=0$. Hence, $\mathcal{Q}_{R,n,p,c}(\sup|\Phi|)\leq 0$ yields

$$0<\sup|\Phi|\leq\hat{\gamma}(R,n,p,c),$$

proving the desired inequality.

Coming back to the general case, let us assume that the equality $|\Phi|=\hat{\gamma}(R,n,p,c)$ holds. In particular, $\mathcal{Q}_{R,n,p,c}(|\Phi|)=0$ on Σ^n and then Proposition 4.2.2 assures that $|\Phi|^2$ is a L-subharmonic function on Σ^n. Thus, the analogue of Theorem 4.1.1 applies to show that Σ^n is an isoparametric hypersurface having exactly two distinct constant principal curvatures with multiplicities p and $n-p$. Hence a classical result on isoparametric hypersurfaces of Lorentzian space forms (see Theorem 5.1 in [1]) says that Σ^n must be isometric to one of the following standard products embeddings:

(a) $\mathbb{R}^{n-p}\times\mathbb{H}^p(r)\subset\mathbb{R}^{n+1}_1$ or $\mathbb{R}^p\times\mathbb{H}^{n-p}(r)\subset\mathbb{R}^{n+1}_1$, with $r>0$, if $c=0$;

(b) $\mathbb{S}^{n-p}(\sqrt{1+r^2})\times\mathbb{H}^p(r)\subset\mathbb{S}^{n+1}_1$ or $\mathbb{S}^p(\sqrt{1+r^2})\times\mathbb{H}^{n-p}(r)\subset\mathbb{S}^{n+1}_1$, with $r>0$, if $c=1$;

(c) $\mathbb{H}^{n-p}(\sqrt{1-r^2})\times\mathbb{H}^p(r)\subset\mathbb{H}^{n+1}_1$, with $0<r<1$, if $c=-1$.

In the case of the Lorentz-Minkowski space $\mathbb{R}^{n+1}_1$ (that is, $c=0$), the positive constant $\hat{\gamma}(R,n,p,0)>0$ is given explicitly by

$$\hat{\gamma}(R,n,p,0)=\sqrt{-\frac{(n-1)(n-p)R}{p-1}}.$$

In this case, on the one hand the product $\mathbb{R}^{n-p} \times \mathbb{H}^p(r) \subset \mathbb{R}_1^{n+1}$, for a given radius $r > 0$ and an appropriate choose of the orientation, has constant principal curvatures given by

$$\lambda_1 = \ldots = \lambda_{n-p} = 0 \quad \text{and} \quad \lambda_{n-p+1} = \ldots = \lambda_n = \frac{1}{r},$$

so that its constant mean curvature $H = \frac{p}{nr}$ and the norm of Φ is

$$|\Phi| = \sqrt{\frac{p(n-p)}{nr^2}}.$$

Hence, by equation (1.7), its constant scalar curvature is given by

$$R = -\frac{p(p-1)}{n(n-1)r^2} < 0.$$

Thus, we must have

$$|\Phi| = \sqrt{-\frac{p(n-1)(n-p)R}{p-1}} = \hat{\gamma}(R, n, p, 0)$$

and the equality holds. On the other hand, for a given radius $r > 0$, the product $\mathbb{R}^p \times \mathbb{H}^{n-p}(r) \subset \mathbb{R}_1^{n+1}$ has constant principal curvatures, for an suitable choose of the orientation,

$$\lambda_1 = \ldots = \lambda_p = 0 \quad \text{and} \quad \lambda_{p+1} = \ldots = \lambda_n = \frac{1}{r},$$

constant mean curvature $H = \frac{n-p}{nr}$ and

$$|\Phi| = \sqrt{\frac{p(n-p)}{nr^2}}.$$

Then, by using once more equation (1.7), its constant scalar curvature is

$$R = -\frac{(n-p)(n-p-1)}{n(n-1)r^2} < 0,$$

which gives

$$|\Phi| = \sqrt{-\frac{p(n-1)R}{n-p-1}} < \hat{\gamma}(R, n, p, 0),$$

that is, the inequality is strict. This proves the characterization of the equality $|\Phi| = \hat{\gamma}(R, n, p, 0)$ in the case $c = 0$.

As for the case of the de Sitter space $\mathbb{S}_1^{n+1}$ (that is, $c = 1$), let us consider first the hyperbolic cylinder embedding $\mathbb{S}^{n-p}(\sqrt{1+r^2}) \times \mathbb{H}^p(r) \subset \mathbb{S}_1^{n+1}$. As showed in the proof of Theorem 4.1.1 (ii), the equality $|\Phi| = \hat{\gamma}(R, n, p, 1)$ holds if and only if $r^2 \leq r_{\max}^2$.

For the hyperbolic cylinder $\mathbb{S}^p(\sqrt{1+r^2}) \times \mathbb{H}^{n-p}(r) \subset \mathbb{S}_1^{n+1}$, with $r > 0$, it follows from (4.20) that $R < 0$ always. Moreover, by equation (4.18) we obtain that $H^2 > 1$ and

$$r^2 = \frac{2(n-p) - nH^2 + |H|\sqrt{n^2H^2 - 4p(n-p)}}{2n(H^2-1)}.$$

Hence, from (4.19) we can write $|\Phi|$ in terms of the mean curvature H as

$$|\Phi| = \frac{\sqrt{n}}{2\sqrt{p(n-p)}}\left(\sqrt{n^2H^2 - 4p(n-p)} - (n-2p)|H|\right).$$

Then, in this case we see that $\mathcal{P}_{H,n,p,1}(|\Phi|) < 0$, which also implies $\mathcal{Q}_{R,n,p,1}(|\Phi|) < 0$. In particular, $|\Phi| < \hat{\gamma}(R,n,p,1)$ and the inequality is strict. This concludes the characterization of the equality $|\Phi| = \hat{\gamma}(R,n,p,1)$ in the case $c = 1$.

By last, we consider the case of the anti-de Sitter space $\mathbb{H}^{n+1}_1$ (that is, $c = -1$). Then, for a given $0 < r < 1$, we have that the standard product embedding $\mathbb{H}^{n-p}(\sqrt{1-r^2}) \times \mathbb{H}^p(r) \subset \mathbb{H}^{n+1}_1$ has constant principal curvatures

$$\lambda_1 = \ldots = \lambda_{n-p} = \frac{r}{\sqrt{1-r^2}} \quad \text{and} \quad \lambda_{n-p+1} = \ldots = \lambda_n = -\frac{\sqrt{1-r^2}}{r},$$

for an appropriate choose of the normal vector field. Then, its constant mean curvature H is given by

$$H = \frac{nr^2 - p}{nr\sqrt{1-r^2}}, \tag{4.21}$$

which gives

$$|\Phi| = \sqrt{\frac{p(n-p)}{nr^2(1-r^2)}}. \tag{4.22}$$

So, an easy computation using (4.21) shows that

$$r^2 = \frac{nH^2 + 2p \pm |H|\sqrt{n^2H^2 + 4p(n-p)}}{2n(1+H^2)},$$

where we choose the sign $+$ or $-$ according to $r^2 \geq \frac{p}{n}$ or $r^2 \leq \frac{p}{n}$, respectively. Hence, we obtain that $|\Phi|$ is, in terms of H, given by

$$|\Phi| = \frac{\sqrt{n}}{2\sqrt{p(n-p)}}\left(\sqrt{n^2H^2 - 4pc(n-p)} \mp (n-2p)|H|\right),$$

where we use the same criterion for the sign. Let us also observe that, as in the case of the de Sitter space, the following relation holds

$$\mathcal{Q}_{R,n,p,-1}(|\Phi|) = (n-1)\mathcal{P}_{H,n,p,-1}(|\Phi|),$$

where $\mathcal{P}_{H,n,p,-1}(x)$ is the polynomial obtained taking $c = -1$ in (3.11), that is,

$$\mathcal{P}_{H,n,p,-1}(x) = x^2 - \frac{n(n-2p)H}{\sqrt{np(n-p)}}x - n(H^2+1).$$

On the one hand, if $r^2 > \frac{p}{n}$ then $\mathcal{P}_{H,n,p,-1}(|\Phi|) < 0$, which implies $\mathcal{Q}_{R,n,p,-1}(|\Phi|) < 0$ and one has $|\Phi| < \hat{\gamma}(R,n,p,-1)$, that is, the inequality is strict. On the other hand, if $r^2 \leq \frac{p}{n}$, we obtain $\mathcal{P}_{H,n,p,-1}(|\Phi|) = 0$, which means that $\mathcal{Q}_{R,n,p,-1}(|\Phi|) = 0$ and, in this case, we must have $|\Phi| = \hat{\gamma}(R,n,p,-1)$ and the equality holds.

Next, by using equation (4.21) and (4.22), we get that the constant scalar curvature of the product $\mathbb{H}^{n-p}(\sqrt{1-r^2}) \times \mathbb{H}^p(r) \subset \mathbb{H}_1^{n+1}$ is given by

$$n(n-1)(R+1) = \frac{-n(n-1)r^4 + 2p(n-1)r^2 + p(p-1)}{r^2(1-r^2)}, \tag{4.23}$$

which implies $R < -1$ if and only if

$$r^2 < \frac{p(n-1) - \sqrt{p(n-1)(n-p)}}{n(n-1)} \quad \text{or} \quad r^2 > \frac{p(n-1) + \sqrt{p(n-1)(n-p)}}{n(n-1)}.$$

In particular, $|\Phi| = \hat{\gamma}(R,n,p,-1)$ and $R < -1$ if and only if

$$r^2 < \frac{p(n-1) - \sqrt{p(n-1)(n-p)}}{n(n-1)}. \tag{4.24}$$

Finally, taking into account equation (4.23) we find

$$r^2 = \frac{(n-1)(nR+(n-2p)) \pm \sqrt{[n(n-1)(R+1) - 2p(n-p)]^2 - 4p(n-p)(p(n-p)-(n-1))}}{2n(n-1)R}. \tag{4.25}$$

In particular, if $|\Phi| = \hat{\gamma}(R,n,p,1)$ and $R < -1$, we must have

$$r^2 = \frac{(n-1)(nR+(n-2p)) + \sqrt{[n(n-1)(R+1) - 2p(n-p)]^2 - 4p(n-p)(p(n-p)-(n-1))}}{2n(n-1)R},$$

otherwise, denoting by r_-^2 the value of r^2 in (4.25) with sign $-$, we can show that

$$r_-^2 > \frac{p(n-1) - \sqrt{p(n-1)(n-p)}}{n(n-1)},$$

which gives a contradiction because of equation (4.24). This finishes the characterization of the equality $|\Phi| = \hat{\gamma}(R,n,p,-1)$ in the case $c = -1$ and also proves Theorem 4.1.2.

Chapter 5

Further results for hypersurfaces with constant mean curvature in locally symmetric Riemannian manifolds

We obtain a sharp estimate to the scalar curvature of stochastically complete hypersurfaces immersed with constant mean curvature into a locally symmetric Riemannian space obeying standard curvature constraints (which includes, in particular, a Riemannian space with constant sectional curvature). For this, we suppose that these hypersurfaces satisfy the Okumura type inequality introduced in Chapter 1. Our approach is based on the equivalence between stochastic completeness and the validity of the weak version of the Omori-Yau's generalized maximum principle. The results of this chapter can be found in [53].

5.1 Preliminaries: moving frame, a general Simons' formula and locally symmetric spaces

In this section we begin by introducing quickly some notions in Riemmanian geometry using the moving frame formalism which, in this situation, it is more appropriate to deal with an orientable and connected hypersurface Σ^n isometrically immersed into an arbitrary $(n+1)$-dimensional Riemannian manifold $\overline{M}^{n+1}$.

To do that, let us choose a local orthonormal frame $\{e_1, \ldots, e_{n+1}\}$ on $\overline{M}^{n+1}$ with dual coframe $\{\omega_1, \ldots, \omega_{n+1}\}$ such that, at each point of Σ^n, $e_1, \ldots, e_n$ are tangent to Σ^n and e_{n+1} is normal to Σ^n. We will use the following convention for the indices:

$$1 \leq A, B, C, \ldots \leq n+1 \quad \text{and} \quad 1 \leq i, j, k, \ldots \leq n.$$

In this setting, $\overline{R}_{ABCD}, \overline{R}_{AC}$ and $\overline{R}$ denote respectively the Riemannian curvature tensor, the

Ricci tensor and the scalar curvature of the Riemannian manifold $\overline{M}^{n+1}$. So, we have

$$\overline{R}_{AC} = \sum_B \overline{R}_{ABCB} \quad \text{and} \quad \overline{R} = \sum_A \overline{R}_{AA}.$$

Now, restricting all the tensor to Σ^n, we see that $\omega_{n+1} = 0$ on Σ^n. Hence, $0 = d\omega_{n+1} = -\sum_i \omega_{n+1i} \wedge \omega_i$ and as it is well known we get

$$\omega_{n+1i} = \sum_j h_{ij}\omega_j, \quad h_{ij} = h_{ji}.$$

This gives the second fundamental form of Σ^n, $A = \sum_{i,j} h_{ij}\omega_i\omega_j e_{n+1}$ and its squared norm $|A|^2 = \sum_{i,j} h_{ij}^2$. Furthermore, the mean curvature H of Σ^n is defined by $H = \frac{1}{n}\sum_i h_{ii}$.

On the other hand, it follows from Gauss equation that the Ricci curvature and the scalar curvature of Σ^n are given, respectively, by

$$R_{ij} = \sum_k \overline{R}_{ikjk} + nHh_{ij} - \sum_k h_{ik}h_{kj} \quad \text{and} \quad R = \sum_i R_{ii}, \tag{5.1}$$

where R_{ijkl} are the components of the Riemannian curvature tensor of Σ^n. So, by (5.1), we obtain

$$R = \sum_{i,j} \overline{R}_{ijij} + n^2H^2 - |A|^2. \tag{5.2}$$

We also remember that the squared norm of the covariant differential of the second fundamental form A is given by

$$|\nabla A|^2 = \sum_{i,j,k} h_{ijk}^2, \tag{5.3}$$

where h_{ijk} denote the first covariant derivatives of h_{ij}.

Taking a local orthonormal frame $\{e_1, \ldots, e_n\}$ on Σ^n such that $h_{ij} = \lambda_i\delta_{ij}$, the following Simons type formula is well known (see, for instance, equation (2.10) of [67]):

$$\begin{aligned} \frac{1}{2}\Delta|A|^2 &= |\nabla A|^2 + \sum_i \lambda_i (nH)_{ii} + nH\sum_i \lambda_i^3 - |A|^4 \\ &- \sum_{i,j,k} h_{ij}(\overline{R}_{(n+1)ijk;k} + \overline{R}_{(n+1)kik;j}) + \sum_i \overline{R}_{(n+1)i(n+1)i}\left(nH\lambda_i - |A|^2\right) \\ &+ \sum_{i,j}(\lambda_i - \lambda_j)^2\overline{R}_{ijij}. \end{aligned} \tag{5.4}$$

Proceeding as in [12, 67], we will assume that there exist constants c_1 and c_2 such that the sectional curvature $\overline{K}$ of the ambient space $\overline{M}^{n+1}$ satisfies the following two constraints

$$\overline{K}(\eta, v) = \frac{c_1}{n}, \tag{5.5}$$

for vectors $\eta \in T^{\perp}\Sigma$ and $v \in T\Sigma$, and

$$\overline{K}(u, v) \geq c_2, \tag{5.6}$$

for vectors $u, v \in T\Sigma$.

From now on, we will consider $\overline{M}^{n+1}$ being a locally symmetric Riemannian manifold, which means that all the covariant derivative components $\overline{R}_{ABCD;E}$ of its curvature tensor vanish identically.

Remark 5.1.1 (Remark 3.1 of [12])**.** When the ambient manifold $\overline{M}^{n+1}$ has constant sectional curvature $\overline{c}$, then it is locally symmetric and the curvature conditions (5.5) and (5.6) are satisfied for every hypersurface Σ^n immersed into $\overline{M}^{n+1}$, with $c_1/n = c_2 = \overline{c}$. Therefore, in some sense our assumptions are a natural generalization of the case in which the ambient space has constant sectional curvature. Moreover, when the ambient manifold is a Riemannian product of two Riemannian manifolds of constant sectional curvature, say $\overline{M} = M_1(\kappa_1) \times M_2(\kappa_2)$, then $\overline{M}$ is locally symmetric and, if $\kappa_1 = 0$ and $\kappa_2 \geq 0$, then every hypersurface of type $\Sigma = \Sigma_1 \times M_2(\kappa_2)$, where Σ_1 is an orientable and connected hypersurface immersed into $M_1(\kappa_1)$, satisfies the curvature constraints (5.5) and (5.6) with $c_1 = c_2 = 0$ (for more details, see Remark 3.1 of [12]).

Supposing that $\overline{M}^{n+1}$ satisfies condition (5.5) and denoting by $\overline{R}_{AB}$ the components of its Ricci tensor, we have that its scalar curvature $\overline{R}$ is such that

$$\overline{R} = \sum_{A=1}^{n+1} \overline{R}_{AA} = \sum_{i,j=1}^{n} \overline{R}_{ijij} + 2\sum_{i=1}^{n} \overline{R}_{(n+1)i(n+1)i} = \sum_{i,j=1}^{n} \overline{R}_{ijij} + 2c_1. \tag{5.7}$$

Since the scalar curvature of a locally symmetric Riemannian manifold is constant, from (5.7) we see that $\sum_{i,j} \overline{R}_{ijij}$ is a constant naturally attached to $\overline{M}^{n+1}$. So, for sake of simplicity, in the course of this section we will denote the constant $\frac{1}{n(n-1)} \sum_{i,j} \overline{R}_{ijij}$ by $\overline{\mathcal{R}}$ and, assuming that $\overline{M}^{n+1}$ also satisfies condition (5.6), the parameter c will stand for the quantity $2c_2 - \dfrac{c_1}{n}$.

The traceless second fundamental form, in this case and in local coordinates, is given by $\Phi_{ij} = h_{ij} - H\delta_{ij}$, which gives the tensor

$$\Phi = \sum_{i,j} \Phi_{ij}\omega_i \otimes \omega_j.$$

Then we have $|\Phi|^2 = \sum_{i,j} \Phi_{ij}^2$ is the squared norm of Φ and it is not difficult to see that, by (5.2), we get

$$|\Phi|^2 = n(n-1)\left(H^2 + \overline{\mathcal{R}}\right) - R. \tag{5.8}$$

5.2 Main result

This section is devoted to obtain a sharp estimate for the infimum of the scalar curvature of a stochastically complete hypersurface immersed with constant mean curvature into a locally symmetric Riemannian manifold. We recall that the stochastic completeness of the Riemannian manifold Σ^n is equivalent to the validity of the weak Omori-Yau maximum principle on Σ^n (for more details, see Appendix A, Lemma A.0.2). So, we ready to state and prove our main result.

Theorem 5.2.1. *Let Σ^n be a stochastically complete hypersurface immersed into a locally symmetric Riemannian manifold $\overline{M}^{n+1}$ satisfying curvature conditions* (5.5) *and* (5.6)*. Suppose that Σ^n has constant mean curvature H with $H^2+c>0$, where $c=2c_2-c_1/n$. If its total umbilicity tensor Φ satisfies* (1.10) *for some $1\leq p\leq \frac{n}{2}$, then*

(i) either $\inf R = n(n-1)(H^2+\overline{\mathcal{R}})$ *and Σ^n is a totally umbilical hypersurface,*

(ii) or

(a) $\inf R \leq n(n-2)H^2+n(n-1)\overline{\mathcal{R}}-nc$*, if $p=\frac{n}{2}$,*

(b) and if $p<\frac{n}{2}$,

$$\inf R \leq \frac{n(n-2p)}{2p(n-p)}\left(2p(n-p)c_{n,p}+nd_{n,p}H^2+|H|\sqrt{n^2H^2+4p(n-p)c}\right),$$

where the constants $c_{n,p}$ and $d_{n,p}$ are given by

$$c_{n,p}=\frac{\overline{R}-c_1-2nc_2}{n(n-2p)} \quad \text{and} \quad d_{n,p}=\frac{2np-2p^2-n}{n-2p}.$$

Moreover, if the equality holds and this infimum is attained at some point of Σ^n, then Σ^n is an isoparametric hypersurface with (in the case $c>0$, assume in addition that $H\neq 0$) two distinct principal curvatures of multiplicities p and $n-p$.

It follows from (5.8) that $\inf R = n(n-1)(H^2+\overline{\mathcal{R}})-\sup|\Phi|^2$. Hence, Theorem 5.2.1 can be rewritten equivalently in terms of the total umbilicity tensor as follows.

Theorem 5.2.2. *Let Σ^n be a stochastically complete hypersurface immersed into a locally symmetric Riemannian manifold $\overline{M}^{n+1}$ satisfying curvature conditions* (5.5) *and* (5.6)*. Suppose Σ^n has constant mean curvature H such that $H^2+c>0$, where $c=2c_2-c_1/n$. If its total umbilicity tensor Φ satisfies* (1.10) *for some $1\leq p\leq\frac{n}{2}$, then*

(i) either $\sup|\Phi|=0$ *and Σ^n is a totally umbilical hypersurface,*

(ii) or

$$\sup|\Phi|\geq \eta(H,n,p,c)=\frac{\sqrt{n}}{2\sqrt{p(n-p)}}\left(\sqrt{n^2H^2+4p(n-p)c}-(n-2p)|H|\right)>0.$$

Moreover, if the equality holds and this supremum is attained at some point of Σ^n, then Σ^n is an isoparametric hypersurface with (in the case $c > 0$, assume in addition that $H \neq 0$) two distinct principal curvatures of multiplicities p and $n - p$.

Proof. Firstly, taking a local orthonormal frame field $\{e_1, \ldots, e_n\}$ on Σ^n such that

$$h_{ij} = \lambda_i \delta_{ij} \quad \text{and} \quad \Phi_{ij} = \kappa_i \delta_{ij},$$

we can check that

$$\sum_i \kappa_i = 0, \quad \sum_i \kappa_i^2 = |\Phi|^2 \quad \text{and} \quad \sum_i \lambda_i^3 = \sum_i \kappa_i^3 + 3H|\Phi|^2 + nH^3.$$

Now, since $\overline{M}^{n+1}$ is locally symmetric and Σ^n has constant mean curvature, it follows from (5.4) that

$$\begin{aligned} \frac{1}{2}\Delta|\Phi|^2 &= \frac{1}{2}\Delta|A|^2 \\ &= |\nabla A|^2 + nH\sum_i \lambda_i^3 - |A|^4 + \sum_i \overline{R}_{(n+1)i(n+1)i}\left(nH\lambda_i - |A|^2\right) \\ &+ \sum_{i,j}(\lambda_i - \lambda_j)^2 \overline{R}_{ijij}. \end{aligned} \tag{5.9}$$

From curvature conditions (5.5) and (5.6), we get

$$\sum_i \overline{R}_{(n+1)i(n+1)i}(nH\lambda_i - |A|^2) = c_1(nH^2 - |A|^2) = -c_1|\Phi|^2 \tag{5.10}$$

and

$$\begin{aligned} \sum_{i,j} \overline{R}_{ijij}(\lambda_i - \lambda_j)^2 &\geq c_2 \sum_{i,j}(\lambda_i - \lambda_j)^2 \\ &= 2nc_2(|A|^2 - nH^2) = 2nc_2|\Phi|^2. \end{aligned} \tag{5.11}$$

Moreover, it follows from our hypothesis (1.10) that

$$\begin{aligned} nH\sum_i \lambda_i^3 - |A|^4 &= n^2H^4 + 3nH^2|\Phi|^2 + nH\sum_i \kappa_i^3 - \left(|\Phi|^2 + nH^2\right)^2 \\ &\geq n^2H^4 + 3nH^2|\Phi|^2 - n|H|\left|\sum_i \kappa_i^3\right| - |\Phi|^4 - 2nH^2|\Phi|^2 - n^2H^4 \\ &\geq -|\Phi|^4 - \frac{n(n-2p)}{\sqrt{np(n-p)}}|H||\Phi|^3 + nH^2|\Phi|^2. \end{aligned} \tag{5.12}$$

Hence, since $c = 2c_2 - c_1/n$, inserting (5.10), (5.11) and (5.12) into (5.9) we obtain that

$$\begin{aligned} \frac{1}{2}\Delta|\Phi|^2 &\geq |\nabla A|^2 - |\Phi|^4 - \frac{n(n-2p)}{\sqrt{np(n-p)}}|H||\Phi|^3 + n(H^2+c)|\Phi|^2 \\ &\geq -|\Phi|^2 P(H,n,p,c)(|\Phi|), \end{aligned} \tag{5.13}$$

where $P_{H,n,p,c}(x)$ is the polynomial given as in equations (2.16) and (2.23),

$$P_{H,n,p,c}(x) = x^2 + \frac{n(n-2p)}{\sqrt{np(n-p)}}|H|x - n(H^2+c).$$

We observe that, since $H^2 + c > 0$, the polynomial $P_{H,n,p,c}(x)$ has an unique positive root given by

$$\eta(H,n,p,c) = \frac{\sqrt{n}}{2\sqrt{p(n-p)}}\left(\sqrt{n^2H^2 + 4p(n-p)c} - (n-2p)|H|\right).$$

If $\sup|\Phi| = +\infty$, then (ii) holds trivially and there is nothing to prove. If $\sup|\Phi| < +\infty$, then we can apply Lemma A.0.2 to the function $|\Phi|^2$ to assures that there exists a sequence of points $\{p_j\} \subset \Sigma^n$ such that

$$\lim|\Phi|(p_j) = \sup|\Phi| \quad \text{and} \quad \limsup \Delta|\Phi|^2(p_j) \leq 0,$$

which jointly with (5.13) implies

$$(\sup|\Phi|)^2 P_{H,n,p,c}(\sup|\Phi|) \geq 0.$$

It follows from here that either $\sup|\Phi| = 0$, which means that $|\Phi|$ vanishes identically and the hypersurface is totally umbilical, or $\sup|\Phi| > 0$ and then $P_{H,n,p,c}(\sup|\Phi|) \geq 0$. In the latter case, it must be $\sup|\Phi| \geq \eta(H,n,p,c)$, which gives the inequality in (ii).

Moreover, let us now assume that the equality $\sup|\Phi| = \eta(H,n,p,c)$ holds. In this case, $P_{H,n,p,c}(|\Phi|) \leq 0$ on Σ^n, which jointly with (5.13) implies that the function $|\Phi|^2$ is subharmonic on Σ^n. Therefore, if this supremum is attained at some point of Σ^n, it follows from stronger maximum principle that $|\Phi| = \eta(H,n,p,c)$ is constant. Thus, (5.13) becomes trivially an equality,

$$\frac{1}{2}\Delta|\Phi|^2 = 0 = -|\Phi|^2 P_{H,n,p,c}(|\Phi|).$$

From here we obtain that $|\nabla A|^2 = 0$ and, consequently, from (5.3) we conclude that Σ^n is isoparametric hypersurface. Finally, using once more equality (5.13) we also obtain the equality in Lemma 1.0.3, which implies that the hypersurface has exactly two distinct principal curvatures of multiplicities p and $n-p$. This finishes the proof from theorem. □

It is worth pointing out that when $\overline{M}^{n+1}$ is a space form, then the constants $\overline{\mathcal{R}}$ and $c_{n,1}$ in Theorem 5.2.1 agree with its sectional curvature. In this setting, Theorems 5.2.1 and 5.2.2 generalize Theorems 3 and 5 of [14] when $p = 1$, and Theorems 1.2 and 1.4 of [93] when $1 < p \leq \frac{n}{2}$, for the context of hypersurfaces immersed with constant mean curvature into a locally symmetric spaces.

In the particular case where Σ^n is complete, we obtain from Theorem 5.2.1 (or Theorem 5.2.2) the following consequence.

Corollary 5.2.3. *Let Σ^n be a complete hypersurface immersed into a locally symmetric Riemannian manifold $\overline{M}^{n+1}$ satisfying curvature conditions* (5.5) *and* (5.6)*. Suppose that Σ^n has constant mean curvature H with $H^2 + c > 0$, where $c = 2c_2 - c_1/n$. If its total umbilicity tensor Φ satisfies* (1.10) *for some $1 \leq p \leq \frac{n}{2}$, then*

(i) either $\inf R = n(n-1)(H^2 + \overline{\mathcal{R}})$ *and Σ^n is a totally umbilical hypersurface,*

(ii) or

(a) $\inf R \leq n(n-2)H^2 + n(n-1)\overline{\mathcal{R}} - nc$*, if $p = \frac{n}{2}$,*

(b) and if $p < \frac{n}{2}$,

$$\inf R \leq \frac{n(n-2p)}{2p(n-p)} \left(2p(n-p)c_{n,p} + nd_{n,p}H^2 + |H|\sqrt{n^2H^2 + 4p(n-p)c}\right),$$

where the constants $c_{n,p}$ and $d_{n,p}$ are given by

$$c_{n,p} = \frac{\overline{R} - c_1 - 2nc_2}{n(n-2p)} \quad \text{and} \quad d_{n,p} = \frac{2np - 2p^2 - n}{n-2p}.$$

Moreover, if the equality holds and this infimum is attained at some point of Σ^n, then Σ^n is an isoparametric hypersurface with (in the case $c > 0$, assume in addition that $H \neq 0$) two distinct principal curvatures of multiplicities p and $n-p$.

As mentioned before, we can rewritten Corollary 5.2.3 equivalently in terms of the total umbilicity tensor as follows.

Corollary 5.2.4. *Let Σ^n be a complete hypersurface immersed into a locally symmetric Riemannian manifold $\overline{M}^{n+1}$ satisfying curvature conditions* (5.5) *and* (5.6)*. Suppose that Σ^n has constant mean curvature H with $H^2 + c > 0$, where $c = 2c_2 - c_1/n$. If its total umbilicity tensor Φ satisfies* (1.10) *for some $1 \leq p \leq \frac{n}{2}$, then*

(i) either $\sup|\Phi| = 0$ *and Σ^n is a totally umbilical hypersurface,*

(ii) or

$$\sup|\Phi| \geq \eta(H,n,p,c) = \frac{\sqrt{n}}{2\sqrt{p(n-p)}} \left(\sqrt{n^2H^2 + 4p(n-p)c} - (n-2p)|H|\right) > 0.$$

Moreover, if the equality holds and this supremum is attained at some point of Σ^n, then Σ^n is an isoparametric hypersurface with (in the case $c > 0$, assume in addition that $H \neq 0$) two distinct principal curvatures of multiplicities p and $n-p$.

Proof. We note that when $\sup|\Phi| = +\infty$ the result is clearly true. So, we can suppose that $\sup|\Phi| < +\infty$. In this case, since Σ^n has constant mean curvature, we have that $\sup|A|^2 < +\infty$. Hence, from equation (5.1) and our hypothesis on sectional curvature of $\overline{M}^{n+1}$, we get

$$R_{ii} \geq (n-1)c_2 - nH\sup|A| - \sup|A|^2 > -\infty,$$

that is, the Ricci curvature of Σ^n is bounded from below. In particular, Σ^n is stochastically complete and the result follows from Theorem 5.2.2. $\square$

Another consequence of Theorem 5.2.1 is the following result for complete parabolic hypersurfaces in locally symmetric spaces.

Corollary 5.2.5. *Let Σ^n be a complete parabolic hypersurface immersed into a locally symmetric Riemannian manifold $\overline{M}^{n+1}$ satisfying curvature conditions* (5.5) *and* (5.6)*. Suppose that Σ^n has constant mean curvature H with $H^2 + c > 0$, where $c = 2c_2 - c_1/n$. If its total umbilicity tensor Φ satisfies* (1.10) *for some $1 \leq p \leq \frac{n}{2}$, then*

(i) either $\inf R = n(n-1)(H^2 + \overline{\mathcal{R}})$ *and Σ^n is a totally umbilical hypersurface,*

(ii) or

(a) $\inf R \leq n(n-2)H^2 + n(n-1)\overline{\mathcal{R}} - nc$, *if $p = \frac{n}{2}$,*

(b) and if $p < \frac{n}{2}$,

$$\inf R \leq \frac{n(n-2p)}{2p(n-p)}\left(2p(n-p)c_{n,p} + nd_{n,p}H^2 + |H|\sqrt{n^2H^2 + 4p(n-p)c}\right),$$

where the constants $c_{n,p}$ and $d_{n,p}$ are given by

$$c_{n,p} = \frac{\overline{R} - c_1 - 2nc_2}{n(n-2p)} \quad \text{and} \quad d_{n,p} = \frac{2np - 2p^2 - n}{n-2p}.$$

Moreover, if the equality holds, then Σ^n is an isoparametric hypersurface with (in the case $c > 0$, assume in addition that $H \neq 0$) two distinct principal curvatures of multiplicities p and $n-p$.

Equivalently, we can prove the following:

Corollary 5.2.6. *Let Σ^n be a complete parabolic hypersurface immersed into a locally symmetric Riemannian manifold $\overline{M}^{n+1}$ satisfying curvature conditions* (5.5) *and* (5.6)*. Suppose that Σ^n has constant mean curvature H with $H^2 + c > 0$, where $c = 2c_2 - c_1/n$. If its total umbilicity tensor Φ satisfies* (1.10) *for some $1 \leq p \leq \frac{n}{2}$, then*

(i) either $\sup|\Phi| = 0$ *and Σ^n is a totally umbilical hypersurface,*

(ii) or

$$\sup|\Phi| \geq \eta(H,n,p,c) = \frac{\sqrt{n}}{2\sqrt{p(n-p)}}\left(\sqrt{n^2H^2 + 4p(n-p)c} - (n-2p)|H|\right) > 0.$$

Moreover, if the equality holds, then Σ^n is an isoparametric hypersurface with (in the case $c > 0$, assume in addition that $H \neq 0$) two distinct principal curvatures of multiplicities p and $n-p$.

Proof. Firstly, we recall that every parabolic Riemannain manifold is stochastically complete. Then, by the first part of Theorem 5.2.2 we obtain that either $\sup|\Phi| = 0$ and Σ^n is totally umbilical hypersurface, or $\sup|\Phi| \geq \eta(H, n, p, c)$. Moreover, if the equality $\sup|\Phi| = \eta(H, n, p, c)$ holds, then as in the proof of Theorem 5.2.2 we have $P_{H,n,p,c}(|\Phi|) \leq 0$ and $|\Phi|^2$ is a subharmonic function on Σ^n which is bounded from above. Since Σ^n is parabolic, it must be constant $|\Phi| = \eta(H, n, p, c)$. Therefore, at this point we can reason in a similar way to the proof of Theorem 5.2.2. ∎

Part II

Generalized linear Weingarten hypersurfaces in warped products: height estimates and half-space theorems

Chapter 6

Preliminaries for Part II

In this chapter we shall briefly introduce some basic facts and notations that will appear along Part II of this thesis, namely: the semi-Riemannian warped product spaces, some relations between curvatures tensors, the higher order mean curvatures, some formulas involving the linearized operators of the higher order mean curvatures acting in certain support functions defined on hypersurfaces immersed into such warped products, the notion of generalized Weingarten linear hypersurfaces, among others.

To start, throughout this part of the thesis we will always consider M^n a connected n-dimensional Riemannian manifold, $I \subset \mathbb{R}$ an open interval in $\mathbb{R}$ and $\rho : I \to \mathbb{R}$ a positive smooth function on I. Let us denote by $\overline{M}^{n+1} = \varepsilon I \times_\rho M^n$ the product manifold $I \times M^n$ endowed with the semi-Riemannian metric

$$\langle\ ,\ \rangle = \varepsilon \pi_I^*(dt^2) + (\rho \circ \pi_I)^2 \pi_M^*(\langle\ ,\ \rangle_M), \tag{6.1}$$

where $\varepsilon = \pm 1$, π_I and π_M denote the canonical projections from $I \times M^n$ onto each factor, $\langle\ ,\ \rangle_M$ is the Riemannian metric on M^n and I is endowed with the metric εdt^2. The semi-Riemannian manifold $\overline{M}^{n+1}$ is called the semi-Riemannian warped product space with base I, fiber M^n and warping function ρ. In particular, ∂_t is an unitary vector field globally defined on $\overline{M}^{n+1} = \varepsilon I \times_\rho M^n$, which determines on $\overline{M}^{n+1}$ a codimension one foliation by totally umbilical slices $\{t\} \times M$. Besides, we have that $\langle \partial_t, \partial_t \rangle = \varepsilon$.

On the one hand, when $\varepsilon = 1$ equation (6.1) gives a Riemannian metric on $\overline{M}^{n+1}$ and the corresponding Riemannian manifold will be denoted by $\overline{M}^{n+1} = I \times_\rho M^n$ and called a Riemannian warped product.

On the other hand, if $\varepsilon = -1$ then (6.1) is a Lorentzian metric on $\overline{M}^{n+1}$. In this case, the corresponding Lorentzian manifold will be denoted by $\overline{M}^{n+1} = -I \times_\rho M^n$ and called a Lorentzian warped product. When M^n has constant sectional curvature, the warped product $\overline{M}^{n+1} = -I \times_\rho M^n$ has been known in the mathematical literature as a Robertson-Walker (RW) spacetime, an allusion to the fact that, for $n = 3$, it is an exact solution of the Einstein's field equations (see Chapter 12 of [102]). After [20], the warped product $\overline{M}^{n+1} = -I \times_\rho M^n$ has usually been referred to as a generalized Robertson-Walker (GRW) spacetime, and we will stick to this usage along this part.

Given a semi-Riemannian warped product $\overline{M}^{n+1} = \varepsilon I \times_\rho M^n$, let us denote by $\overline{R}$ and R_M the curvatures tensors of the ambient space $\overline{M}^{n+1}$ and of the fiber M^n, respectively. Here, we are following [102] for our definition of the curvature tensor of $\overline{M}^{n+1}$, namely, if $U, V, W \in \mathfrak{X}(\overline{M})$ then

$$\overline{R}(U,V)W = \overline{\nabla}_{[X,Y]}Z - [\overline{\nabla}_X, \overline{\nabla}_Y]Z,$$

where $\overline{\nabla}$ stands for the Levi-Civita connection of $\overline{M}^{n+1}$ and [,] denotes the standard Lie bracket. In what follows we collect some important relations, which are well known and can be found in [102].

Lemma 6.0.1 (Lemma 7.35 and Proposition 7.42 of [102])**.** *Let $\varepsilon I \times_\rho M^n$ be a semi-Riemannian warped product. Let $f : I \to \mathbb{R}$ be a smooth function on I and $U, V, W \in \mathfrak{X}(M)$. We have:*

(a) $\overline{\nabla}(f \circ \pi_I) = \varepsilon f' \partial_t$. *In particular and for the sake of simplicity we will write f to indicate $f \circ \pi_I$ and $\overline{\nabla} f$ to indicate $\overline{\nabla}(f \circ \pi_I)$;*

(b) $\overline{R}(U, \partial_t)\partial_t = \dfrac{\rho''}{\rho} U$;

(c) $\overline{R}(\partial_t, \partial_t)U = \overline{R}(U, V)\partial_t = 0$;

(d) $\overline{R}(\partial_t, U)V = \dfrac{\langle U, V\rangle}{\rho} \varepsilon \rho'' \partial_t$;

(e) $\overline{R}(U,V)W = R_M(U,V)W - \dfrac{\varepsilon(\rho')^2}{\rho^2} (\langle U, W\rangle V - \langle V, W\rangle U)$.

As main consequence of the previous lemma, we shall prove a very useful relationship between the curvatures tensors $\overline{R}$ and R_M as well as the derivatives of the warping function ρ of a semi-Riemannian warped product $\varepsilon I \times_\rho M^n$, which are not easily available in details in the literature. This is the subject of the next lemma whose proof is a technical computation that for the sake of completeness we include it here. So, we have.

Lemma 6.0.2. *Let $\overline{M}^{n+1} = \varepsilon I \times_\rho M^n$ be a semi-Riemannian warped product. For every $U, V, W \in \mathfrak{X}(\overline{M})$, the following relation holds:*

$$\begin{aligned} \overline{R}(U,V)W &= R_M(U^*, V^*)W^* - \varepsilon[(\log\rho)']^2 (\langle U, W\rangle V - \langle V, W\rangle U) \\ &- (\log\rho)''\langle W, \partial_t\rangle (\langle U, \partial_t\rangle V - \langle V, \partial_t\rangle U) \\ &- (\log\rho)'' (\langle U, W\rangle\langle V, \partial_t\rangle - \langle V, W\rangle\langle U, \partial_t\rangle)\, \partial_t, \end{aligned}$$

where we are writing $U^* = U - \varepsilon\langle U, \partial_t\rangle\partial_t$ to denote the orthogonal projection of U onto TM.

Proof. To prove this lemma we will use as main tool Lemma 6.0.1. So, by using the C^∞-linearity of the tensor $\overline{R}$ we get

$$\begin{aligned}
\overline{R}(U,V)W &= \overline{R}(U^*,V)W + \varepsilon\langle U,\partial_t\rangle\overline{R}(\partial_t,V)W \\
&= \overline{R}(U^*,V^*)W + \varepsilon\langle V,\partial_t\rangle\overline{R}(U^*,\partial_t)W \\
&+ \varepsilon\langle U,\partial_t\rangle\overline{R}(\partial_t,V^*)W + \langle U,\partial_t\rangle\langle V,\partial_t\rangle\overline{R}(\partial_t,\partial_t)W \\
&= \overline{R}(U^*,V^*)W^* + \varepsilon\langle W,\partial_t\rangle\overline{R}(U^*,V^*)\partial_t + \varepsilon\langle V,\partial_t\rangle\overline{R}(U^*,\partial_t)W^* \\
&+ \langle V,\partial_t\rangle\langle W,\partial_t\rangle\overline{R}(U^*,\partial_t)\partial_t + \varepsilon\langle U,\partial_t\rangle\overline{R}(\partial_t,V^*)W^* \\
&+ \langle U,\partial_t\rangle\langle W,\partial_t\rangle\overline{R}(\partial_t,V^*)\partial_t + \langle U,\partial_t\rangle\langle V,\partial_t\rangle\overline{R}(\partial_t,\partial_t)W.
\end{aligned}$$

From Lemma 6.0.1 (c) we have $\overline{R}(U^*,V^*)\partial_t = \overline{R}(\partial_t,\partial_t)W = 0$, and the previous equation becomes

$$\begin{aligned}
\overline{R}(U,V)W &= \overline{R}(U^*,V^*)W^* + \varepsilon\langle V,\partial_t\rangle\overline{R}(U^*,\partial_t)W^* \\
&+ \langle V,\partial_t\rangle\langle W,\partial_t\rangle\overline{R}(U^*,\partial_t)\partial_t + \varepsilon\langle U,\partial_t\rangle\overline{R}(\partial_t,V^*)W^* \\
&+ \langle U,\partial_t\rangle\langle W,\partial_t\rangle\overline{R}(\partial_t,V^*)\partial_t,
\end{aligned}$$

which can be rewritten as

$$\begin{aligned}
\overline{R}(U,V)W &= \overline{R}(U^*,V^*)W^* - \varepsilon\langle V,\partial_t\rangle\overline{R}(\partial_t,U^*)W^* \\
&+ \langle V,\partial_t\rangle\langle W,\partial_t\rangle\overline{R}(U^*,\partial_t)\partial_t + \varepsilon\langle U,\partial_t\rangle\overline{R}(\partial_t,V^*)W^* \qquad (6.2)\\
&- \langle U,\partial_t\rangle\langle W,\partial_t\rangle\overline{R}(V^*,\partial_t)\partial_t,
\end{aligned}$$

where we use the symmetries of the curvature tensor $\overline{R}$.

Let us computation each term of the right side of (6.2). To the first, Lemma 6.0.1 (e) implies that

$$\begin{aligned}
\overline{R}(U^*,V^*)W^* &= \overline{R}_M(U^*,V^*)W^* - \frac{\langle\overline{\nabla}\rho,\overline{\nabla}\rho\rangle}{\rho^2}\left(\langle U^*,W^*\rangle V^* - \langle V^*,W^*\rangle U^*\right) \\
&= \overline{R}_M(U^*,V^*)W^* - \frac{\varepsilon(\rho')^2}{\rho^2}\left(\langle U^*,W^*\rangle V^* - \langle V^*,W^*\rangle U^*\right). \qquad (6.3)
\end{aligned}$$

We observe that

$$\langle U^*,W^*\rangle = \langle U,W\rangle - \varepsilon\langle U,\partial_t\rangle\langle W,\partial_t\rangle$$
$$\langle V^*,W^*\rangle = \langle V,W\rangle - \varepsilon\langle V,\partial_t\rangle\langle W,\partial_t\rangle,$$

which gives

$$\langle U^*,W^*\rangle V^* = \langle U,W\rangle V - \varepsilon\langle U,W\rangle\langle V,\partial_t\rangle\partial_t - \varepsilon\langle U,\partial_t\rangle\langle W,\partial_t\rangle V + \langle U,\partial_t\rangle\langle V,\partial_t\rangle\langle W,\partial_t\rangle$$
$$\langle V^*,W^*\rangle U^* = \langle V,W\rangle U - \varepsilon\langle V,W\rangle\langle U,\partial_t\rangle\partial_t - \varepsilon\langle V,\partial_t\rangle\langle W,\partial_t\rangle U + \langle U,\partial_t\rangle\langle V,\partial_t\rangle\langle W,\partial_t\rangle.$$

Hence, we find

$$\begin{aligned}\langle U^*, W^*\rangle V^* - \langle V^*, W^*\rangle U^* &= \langle U, W\rangle V - \langle V, W\rangle U \\ &+ \varepsilon\left(\langle V, W\rangle\langle U, \partial_t\rangle - \langle U, W\rangle\langle V, \partial_t\rangle\right)\partial_t \\ &- \varepsilon\langle W, \partial_t\rangle\left(\langle U, \partial_t\rangle V - \langle V, \partial_t\rangle U\right).\end{aligned}$$

Putting this into (6.3) yields

$$\begin{aligned}\overline{R}(U^*, V^*)W^* &= \overline{R}_M(U^*, V^*)W^* - \frac{\varepsilon(\rho')^2}{\rho^2}\left(\langle U, W\rangle V - \langle V, W\rangle U\right) \\ &- \frac{(\rho')^2}{\rho^2}\left(\langle V, W\rangle\langle U, \partial_t\rangle - \langle U, W\rangle\langle V, \partial_t\rangle\right)\partial_t \\ &+ \frac{(\rho')^2}{\rho^2}\langle W, \partial_t\rangle\left(\langle U, \partial_t\rangle V - \langle V, \partial_t\rangle U\right).\end{aligned} \tag{6.4}$$

Next, by applying Lemma 6.0.1 (d) we see that

$$\begin{aligned}\overline{R}(\partial_t, U^*)W^* &= \frac{\langle U^*, W^*\rangle}{\rho}\overline{\nabla}_{\partial_t}\overline{\nabla}\rho \\ &= \left(\frac{\langle U, W\rangle - \varepsilon\langle U, \partial_t\rangle\langle W, \partial_t\rangle}{\rho}\right)\varepsilon\rho''\partial_t \\ &= \frac{\varepsilon\rho''}{\rho}\langle U, W\rangle\partial_t - \frac{\rho''}{\rho}\langle U, \partial_t\rangle\langle W, \partial_t\rangle\partial_t.\end{aligned} \tag{6.5}$$

The analogous applies to

$$\overline{R}(\partial_t, V^*)W^* = \frac{\varepsilon\rho''}{\rho}\langle V, W\rangle\partial_t - \frac{\rho''}{\rho}\langle V, \partial_t\rangle\langle W, \partial_t\rangle\partial_t. \tag{6.6}$$

We continue by using once more Lemma 6.0.1 (b) to obtain that

$$\overline{R}(U^*, \partial_t)\partial_t = \frac{\rho''}{\rho}U^* = \frac{\rho''}{\rho}U - \varepsilon\frac{\rho''}{\rho}\langle U, \partial_t\rangle\partial_t \tag{6.7}$$

and

$$\overline{R}(V^*, \partial_t)\partial_t = \frac{\rho''}{\rho}V^* = \frac{\rho''}{\rho}V - \varepsilon\frac{\rho''}{\rho}\langle V, \partial_t\rangle\partial_t \tag{6.8}$$

Then, inserting (6.4), (6.5), (6.6), (6.7) and (6.8) into equation (6.2) we get that

$$\begin{aligned}
\overline{R}(U,V)W &= \overline{R}_M(U^*,V^*)W^* - \frac{\varepsilon(\rho')^2}{\rho^2}\left(\langle U,W\rangle V - \langle V,W\rangle U\right)\\
&- \frac{(\rho')^2}{\rho^2}\left(\langle V,W\rangle\langle U,\partial_t\rangle - \langle U,W\rangle\langle V,\partial_t\rangle\right)\partial_t\\
&+ \frac{(\rho')^2}{\rho^2}\langle W,\partial_t\rangle\left(\langle U,\partial_t\rangle V - \langle V,\partial_t\rangle U\right)\\
&- \frac{\rho''}{\rho}\langle U,W\rangle\langle V,\partial_t\rangle\partial_t + \frac{\rho''}{\rho}\langle V,W\rangle\partial_t\\
&+ \frac{\rho''}{\rho}\langle V,\partial_t\rangle\langle W,\partial_t\rangle U - \frac{\rho''}{\rho}\langle U,\partial_t\rangle\langle W,\partial_t\rangle V\\
&= \overline{R}_M(U^*,V^*)W^* - \varepsilon[(\log\rho)']^2\left(\langle U,W\rangle V - \langle V,W\rangle U\right)\\
&- \left(\frac{\rho''}{\rho} - \frac{(\rho')^2}{\rho^2}\right)\langle W,\partial_t\rangle\left(\langle U,\partial_t\rangle V - \langle V,\partial_t\rangle U\right)\\
&- \left(\frac{\rho''}{\rho} - \frac{(\rho')^2}{\rho^2}\right)\left(\langle U,W\rangle\langle V,\partial_t\rangle - \langle V,W\rangle\langle U,\partial_t\rangle\right)\partial_t.
\end{aligned}$$

Finally, we conclude that

$$\begin{aligned}
\overline{R}(U,V)W &= \overline{R}_M(U^*,V^*)W^* - \varepsilon[(\log\rho)']^2\left(\langle U,W\rangle V - \langle V,W\rangle U\right)\\
&- (\log\rho)''\langle W,\partial_t\rangle\left(\langle U,\partial_t\rangle V - \langle V,\partial_t\rangle U\right)\\
&- (\log\rho)''\left(\langle U,W\rangle\langle V,\partial_t\rangle - \langle V,W\rangle\langle U,\partial_t\rangle\right)\partial_t
\end{aligned}$$

as desired. $\square$

Throughout this Part II we will deal with (connected) hypersurfaces $\psi : \Sigma^n \to \overline{M}^{n+1}$ immersed into the semi-Riemannian warped product $\overline{M}^{n+1} = \varepsilon I \times_\rho M^n$. In the case that $\overline{M}^{n+1}$ is a Riemannian manifold, we will assume that Σ^n is a two-sided hypersurface, which means that its normal bundle is trivial, that is, there exists an unitary normal vector field N globally defined on Σ^n. Otherwise, if $\overline{M}^{n+1}$ is a Lorentzian manifold, Σ^n is assumed to be a spacelike hypersurface, meaning that the induced metric on Σ^n via the immersion ψ is a Riemannian metric. In the latter case, since ∂_t is a timelike vector field globally defined on $\overline{M}^{n+1}$, there exists an unique unitary timelike normal vector field (also denoted by) N globally defined on Σ^n which is either in the same time-orientation of ∂_t, so that $\langle N,\partial_t\rangle \leq -1$, or in the opposite time-orientation of ∂_t, that is, $\langle N,\partial_t\rangle \geq 1$. In this case, we will refer to the normal vector field N as been the future-pointing Gauss map of Σ^n when N is in the same time-orientation of ∂_t. Its opposite will be refered as been the past-pointing Gauss map of Σ^n. As usual, in both the cases we also denote by $\langle\ ,\ \rangle$ the metric of Σ^n induced via ψ. Sometimes we will refer to Σ^n as a Riemannian hypersurface to mean that Σ^n is either a two-sided hypersurface or a spacelike hypersurface. We also observe that $\langle N,N\rangle = \varepsilon$.

Let us denote by $A : \mathfrak{X}(\Sigma) \to \mathfrak{X}(\Sigma)$ the second fundamental form (or shape operator) of the Riemannian hypersurface Σ^n in $\overline{M}^{n+1} = \varepsilon I \times_\rho M^n$ with respect to N, which is given by

$AX = -\overline{\nabla}_X N$, where as already mentioned above $\overline{\nabla}$ stands for the Levi-Civita connection of $\overline{M}^{n+1}$. It is well known that the curvature tensor R of the hypersurface Σ^n can be described in terms of the second fundamental form A and of the curvature tensor $\overline{R}$ of the ambient space $\overline{M}^{n+1}$ by the Gauss equation as follows

$$R(X,Y)Z = (\overline{R}(X,Y)Z)^{\top} + \varepsilon\left(\langle AX, Z\rangle AY - \langle AY, Z\rangle AX\right) \tag{6.9}$$

for every tangent vector fields $X, Y, Z \in \mathfrak{X}(\Sigma)$, where $(\ )^{\top}$ denotes the tangential component of a vector field in $\mathfrak{X}(\overline{M})$ along Σ^n.

Associated with the second fundamental form A there are n algebraic invariants, which are the elementary symmetric functions S_r of its principal curvatures $\kappa_1, \ldots, \kappa_n$, given by

$$S_0 = 1 \text{ and } S_r = S_r(\kappa_1, \ldots, \kappa_n) = \sum_{i_1 < \ldots < i_r} \kappa_{i_1} \cdots \kappa_{i_r}, \qquad 1 \leq r \leq n.$$

For each $0 \leq r \leq n$, we define the r-mean curvature H_r of the hypersurface Σ^n by

$$\binom{n}{r} H_r = \varepsilon^r S_r(\kappa_1, \ldots, \kappa_n).$$

In particular, when $r = 1$,

$$H_1 = \varepsilon \frac{1}{n} \sum_i \kappa_i = \varepsilon \frac{1}{n} \mathrm{tr}(A) = H$$

is just the mean curvature of Σ^n, which is the main extrinsic curvature of the hypersurface. When $r = 2$, H_2 defines a geometric quantity which is related to the (intrinsic) scalar curvature S of the hypersurface. For instance, when the ambient space has constant sectional curvature c, it follows from Gauss equation that $S = n(n-1)(c + \varepsilon H_2)$. In general, it also follows from Gauss equation that when r is odd H_r is extrinsic (and its sign depends on the chosen orientation), while when r is even H_r is an intrinsic geometric quantity. Moreover, we also observe that the characteristic polynomial of A can be written in terms of the H_r as

$$\det(xI - A) = \sum_{j=0}^{n} \binom{n}{j} (-\varepsilon)^j H_j x^{n-j}. \tag{6.10}$$

It is usual to refer to the r-mean curvatures as the higher order mean curvatures of the hypersurface.

Regarding the higher order mean curvatures, it is a classical fact that it satisfy a very useful set of inequalities, usually alluded as Newton's inequalities. For future reference, we collect them here. A proof can be found in [70] (see Theorems 51 and 52) jointly with Proposition 3.2 of [42] (see also Proposition 2.3 of [30]).

Lemma 6.0.3. *Let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be an orientable Riemannian hypersurface immersed into a semi-Riemannian warped product $\overline{M}^{n+1} = \varepsilon I \times_\rho M^n$. Suppose that there exists an elliptic point in Σ^n. If H_{r+1} is positive on Σ^n, we have that the same holds for H_k, $k = 1, \cdots, r$. Moreover,* *(a)* $H_kH_{k+2} \leq H_{k+1}^2$ *for every* $k = 1, \cdots, r$*;*

(b) $H_1 \geq H_2^{1/2} \geq \ldots \geq H_r^{1/r}$,

and the equality holds only at umbilical points.

Here, by an elliptic point in a Riemannian hypersurface Σ^n we mean a point $p_0 \in \Sigma^n$ where all principal curvatures $\kappa_i(p_0)$ are positive, when $\overline{M}^{n+1}$ is a Riemannian manifold, and negative, when $\overline{M}^{n+1}$ is a Lorentzian manifold, with respect to an appropriate choice of the orientation N of Σ^n.

For each $0 \leq r \leq n$, one defines the r-th Newton transformation $P_r : \mathfrak{X}(\Sigma) \to \mathfrak{X}(\Sigma)$ of the hypersurface Σ^n by setting $P_0 = I$ (the identity tensor) and, for $1 \leq r \leq n$, via the recurrence relation

$$P_r = \binom{n}{r} H_r I - \varepsilon A P_{r-1}.$$

Equivalently,

$$P_r = \sum_{j=0}^{r} \binom{n}{j} (-\varepsilon)^{r-j} H_j A^{r-j},$$

so that the Cayley-Hamilton's Theorem and (6.10) give $P_n = 0$. We also observe that when r is even, the definition of P_r does not depend on the chosen of the unitary normal vector field N, but when r is odd there is a change of sign in the definition of P_r according to choose of the orientation. Moreover, it is easy to see that each P_r is a self-adjoint operator which commutes with the second fundamental form A, that is, if a local orthonormal frame on Σ^n diagonalizes A, then it also diagonalizes each P_r. More specifically, if $\{E_1, \ldots, E_n\}$ is a local orthonormal frame with $A(E_i) = \kappa_i E_i$, then (see, for instance, Lemma 2.1 of [22])

$$P_r(E_i) = \mu_{i,r} E_i,$$

where

$$\mu_{i,r} = \sum_{i_1 < \cdots < i_r, i_j \neq i} \kappa_{i_1} \cdots \kappa_{i_r}$$

are the eigenvalues of P_r, $i = 1, \ldots, n$. It follows from here that for each $0 \leq r \leq n-1$, we have (see, for instance, Lemma 2.1 of [22])

$$\mathrm{tr}(P_r) = c_r H_r, \quad \text{with} \quad c_r := (n-r)\binom{n}{r} = (r+1)\binom{n}{r+1}. \tag{6.11}$$

Let ∇ be the Levi-Civita connection of the hypersurface Σ^n. Associated to each Newton transformation P_r, one has the second order linear differential operator $L_r : C^\infty(\Sigma) \to C^\infty(\Sigma)$, for $r = 0, 1, \ldots, n-1$, defined by

$$L_r u = \mathrm{tr}(P_r \circ \mathrm{hess}\ u),$$

where hess $u : \mathfrak{X}(\Sigma) \to \mathfrak{X}(\Sigma)$ denotes the self-adjoint linear tensor metrically equivalent to the

Hessian of u, Hess u, which are given by

$$\text{hess } u(X) = \nabla_X \nabla u \quad \text{and} \quad \text{Hess } u(X,Y) = \langle \text{hess } u(X), Y \rangle,$$

respectively, for all $X, Y \in \mathfrak{X}(\Sigma)$. In particular, $L_0 = \Delta$ is the Laplacian of Σ^n, which is always an elliptic operator in the divergence form. More generally, denoting by div the standard divergent operator on Σ^n, we have for a given local orthonormal frame $\{E_1, \ldots, E_n\}$ on Σ^n that

$$\begin{aligned} \text{div}(P_r \nabla u) &= \sum_{i=1}^{n} \langle (\nabla_{E_i} P_r)(\nabla u), E_i \rangle + \sum_{i=1}^{n} \langle P_r(\nabla_{E_i} \nabla u), E_i \rangle \\ &= \langle \text{div} P_r, \nabla u \rangle + L_r u, \end{aligned} \tag{6.12}$$

where the divergence of P_r on Σ^n is given by

$$\text{div} P_r = \text{tr}(\nabla P_r) = \sum_{i=1}^{n} (\nabla_{E_i} P_r)(E_i).$$

In particular, when the ambient space has constant sectional curvature equation (6.12), reduces to $L_r u = \text{div}(P_r \nabla u)$, because $\text{div}(P_r) = 0$ (see [107] for more details). Moreover, we get from equation (6.12) that the operator L_r is elliptic if and only if P_r is positive definite.

For our purpose, it will be useful to have some geometric conditions which guarantee the ellipticity of the operators L_r when $r \geq 1$. For $r = 1$, the next lemma assures the ellipticity of L_1 (see Lemma 3.10 of [59] and Lemma 3.2 of [7]).

Lemma 6.0.4. *Let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be an orientable Riemannian hypersurface immersed into a semi-Riemannian warped product $\overline{M}^{n+1} = \varepsilon I \times_\rho M^n$. If $H_2 > 0$ on Σ, then the operator L_1 is elliptic or, equivalently, P_1 is positive definite (for an appropriate choice of the orientation N).*

When $r \geq 2$, the following lemma give us sufficient conditions to assert the ellipticity of L_r. The proof is given in Proposition 3.2 of [22] (see also Proposition 3.2 of [42]).

Lemma 6.0.5. *Let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be an orientable Riemannian hypersurface immersed into a semi-Riemannian warped product $\overline{M}^{n+1} = \varepsilon I \times_\rho M^n$ with $H_{r+1} > 0$ on Σ^n, for some $2 \leq r \leq n-1$. If there exists an elliptic point in Σ^n, with respect an appropriate choice of the orientation N, then for all $1 \leq k \leq r$ the operator L_k is elliptic or, equivalently, P_k is positive definite (for an appropriate choice of the orientation N, if k is odd).*

Remark 6.0.6. As showed in Proposition 3.2 of [42], in the Riemannian case Theorem 6.0.5 still holds with the weaker assumption that there is a point $p \in \Sigma^n$ such that all the principal curvatures at p are nonnegative.

We continue our preliminaries by considering two particular functions naturally attached to the hypersurface $\psi : \Sigma^n \to \varepsilon I \times_\rho M^n$, namely, the (vertical) height function

$$h := \pi_{\mathbb{R}} \circ \psi : \Sigma^n \to \mathbb{R}$$

and the angle function

$$\Theta := \langle N, \partial_t \rangle : \Sigma^n \to \mathbb{R}.$$

With a simple computation we can show that the gradient of projection π_I on $\varepsilon I \times_\rho M^n$ is given by

$$\overline{\nabla}\pi_I = \partial_t.$$

In particular, the gradient of the height function h on Σ^n is

$$\nabla h = \partial_t^\top = \partial_t - \varepsilon\langle N, \partial_t\rangle N = \partial_t - \varepsilon\Theta N,$$

so that

$$|\nabla h|^2 = \varepsilon(1 - \Theta^2).$$

Besides, the angle function also satisfies $|\Theta| \leq 1$ when the ambient space is Riemannian and $|\Theta| \geq 1$ if the ambient space is Lorentzian.

Next, we recall some formulas concerning height and angle functions, which were obtained by Alías et al. [17] in the Riemannian case, whereas in the Lorentzian case were given by Alías and Colares [8], and will be essential for the proofs of the main results of this part of the thesis (for more details see Proposition 6 and Lemma 27 of [17] and Lemma 4.1 and Corollary 8.5 of [8]).

Proposition 6.0.7. *Let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be an orientable Riemannian hypersurface immersed into a semi-Riemannian warped product $\overline{M}^{n+1} = \varepsilon I \times_\rho M^n$. For every $r = 0, \ldots, n-1$, the following formulas hold:*

(a) The height function h satisfies

$$\text{Hess } h(X,Y) = \varepsilon(\log\rho)'(h)(\langle X,Y\rangle - \varepsilon\langle\nabla h, X\rangle\langle\nabla h, Y\rangle) + \Theta\langle AX, Y\rangle$$

and

$$L_r h = \varepsilon(\log\rho)'(h)\,(c_r H_r - \varepsilon\langle P_r\nabla h, \nabla h\rangle) + \varepsilon c_r \Theta H_{r+1},$$

where $c_r := (n-r)\binom{n}{r} = (r+1)\binom{n}{r+1}$.

(b) Let $\sigma(t)$ be a primitive of $\rho(t)$. Then

$$L_r\sigma(h) = \varepsilon c_r\,(\rho'(h)H_r + \rho(h)\Theta H_{r+1})$$

(c) Set $\tilde{\Theta} := \rho(h)\Theta$*. Then,*

$$\begin{aligned} L_r\tilde{\Theta} &= -\varepsilon\frac{c_r\rho(h)}{r+1}\langle\nabla H_{r+1},\nabla h\rangle - \varepsilon c_r\rho'(h)H_{r+1} \\ &- \varepsilon\frac{c_r\rho(h)\Theta}{r+1}\left(nH_1H_{r+1}-(n-r-1)H_{r+2}\right) \\ &- \varepsilon\frac{\tilde{\Theta}}{\rho^2(h)}\sum_{i=1}^{n}\mu_{i,r}K_M(N^*,E_i^*)|N^*\wedge E_i^*|^2 \\ &- \tilde{\Theta}(\log\rho)''(h)\left(c_r|\nabla h|^2H_r - \langle P_r\nabla h,\nabla h\rangle\right), \end{aligned}$$

where $\{E_1,\dots,E_n\}$ *is an orthonormal frame on* Σ^n *diagonalizing* A*,* K_M *denotes the sectional curvature of the fiber* M^n*,* $\mu_{i,r}$ *stands for the eigenvalues of* P_r*, for every vector field* $X\in\mathfrak{X}(\overline{M})$*,* X^* *is the orthogonal projection on* TM *and*

$$|X^*\wedge Y^*|^2 = |X^*|^2|Y^*|^2 - \langle X^*,Y^*\rangle^2.$$

We conclude this chapter by introducing a wide class of Riemannian hypersurfaces, the so called generalized linear Weingarten hypersurfaces. Let $\psi:\Sigma^n\to\overline{M}^{n+1}$ be a Riemannian hypersurface immersed into a semi-Riemannian warped product $\overline{M}^{n+1}=\varepsilon I\times_\rho M^n$. We say that Σ^n is (r,s)-linear Weingarten, for some $0\le r\le s\le n-1$, if there exist nonnegative real numbers $b_r,\dots,b_s$ (at least one of them nonzero) such that the following linear relation holds on Σ^n:

$$\sum_{k=r}^{s} b_kH_{k+1} = d\in\mathbb{R}. \tag{6.13}$$

Thus, naturally attached to a (r,s)-linear Weingarten Riemannian hypersurface we have the constant d given by (6.13). We note that the (r,r)-linear Weingarten Riemannian hypersurfaces are exactly the Riemannian hypersurfaces having $d=H_{r+1}$ constant. In particular, this class of hypersurfaces is more general than those having some constant higher order mean curvature. On the other hand, if the ambient space has zero sectional curvature and taking into account that in this case $\overline{S}=\varepsilon H_2$, where $\overline{S}$ stands for the normalized scalar curvature of Σ^n, we observe that the $(0,1)$-linear Weingarten Riemannian hypersurfaces are called simply linear Weingarten hypersurfaces. Throughout this Part II, we will always denote by d the constant given by relation (6.13).

Chapter 7

The Riemannian case

In this chapter we give height estimates and study the topology at infinity (in form of half-space theorems) of (r,s)-linear Weingarten two-sided hypersurfaces immersed into a Riemannian warped product of the type $\mathbb{R} \times_\rho M^n$. Here we present results of [55, 56].

7.1 Height estimates

The main intention of this section is to provide height estimates of compact (r,s)-linear Weingarten hypersurfaces immersed into Riemannian warped product spaces of the type $\mathbb{R} \times_\rho M^n$. Moreover, some particular cases are also studied. To wit, when the ambient space is a pseudo-hyperbolic space, that is, $\rho(t) = e^t$ or $\rho(t) = \cosh t$, and the case of standard products $\mathbb{R} \times M^n$, that is, when the warping function is constant.

The starting point is to prove that under suitable assumption on a (not necessarily constant) linear combination involving some of the higher order mean curvatures, any compact two-sided hypersurface immersed into a Riemannian warped product $\mathbb{R} \times_\rho M^n$ with non-empty boundary contained into a slice $\{t_0\} \times M^n$, for some $t_0 \in \mathbb{R}$, must lie entirely in one of the two regions of the ambient space bounded by the slice. We point out that this was proved in [65] (see Proposition 3.3 of [65]) considering only one of the higher order mean curvatures being constant. For that reason and for the sake of completeness we give here a proof of this fact.

In order to do this we recall an interesting tangency principle due to Fontenele and Silva [63]. Let Σ_1 and Σ_2 be a two hypersurfaces in an arbitrary Riemannian manifold $\overline{N}^{n+1}$ that are tangent at a common point p_0, that is, that satisfy $T_{p_0}\Sigma_1 = T_{p_0}\Sigma_2$. Fix a normal vector η_0 at p_0 and locally parametrize both hypersurfaces in a neighborhood U of zero in $T_{p_0}\Sigma_1 = T_{p_0}\Sigma_2$ by means of the exponential map of $\overline{N}^{n+1}$ as follows:

$$\varphi_i(x) = \exp_{p_0}(x + \delta_i(x)\eta_0), \quad i = 1, 2,$$

where $\delta_i : U \to \mathbb{R}$ are well-determined functions satisfying $\delta_i(0) = 0$. Following [63], we say that Σ_1 remains above Σ_2 in a neighborhood of p_0 with respect to η_0 if $\delta_1(x) \geq \delta_2(x)$ in a neighborhood of zero. This is equivalent to requiring that the geodesics of the ambient space

$\overline{N}^{n+1}$ normal to the hypersurface $\exp_{p_0}(U)$ in a neighborhood of p_0 in the orientation determined by η_0 intercept Σ_2 before Σ_1. So, with this preliminaries we can state the following result (for more details, see Theorem 1.1 of [63]).

Lemma 7.1.1. *Let Σ_1 and Σ_2 be hypersurfaces as above such that Σ_1 remains above Σ_2 in a neighborhood of p_0 with respect to η_0. Assume that, for some $0 \leq r \leq n-1$, we have $H_{r+1}^{\Sigma_1}(x) \leq H_{r+1}^{\Sigma_2}(x)$ in a neighborhood of zero in $T_{p_0}\Sigma_1 = T_{p_0}\Sigma_2$ and, if $r \geq 1$, the principal curvature vector $\kappa^{\Sigma_2}(0) = \left(\kappa_1^{\Sigma_2}(0), \ldots, \kappa_n^{\Sigma_2}(0)\right)$ of Σ_2 at zero belongs to connected component in $\mathbb{R}^{n+1}$, Γ_{r+1}, of the set $\{S_{r+1} > 0\}$ containing $(1, \ldots, 1)$. Then, Σ_1 and Σ_2 coincide in a neighborhood of p_0.*

Remark 7.1.2. We note that in the case in which $H_{r+1}^{\Sigma_2}$ is positive, the assumption $\kappa^{\Sigma_2}(0) \in \Gamma_{r+1}$ holds trivially.

So, as aforementioned our first result is the following, which generalizes Proposition 3.3 of [65].

Proposition 7.1.3. *Let $\psi : \Sigma^n \to \mathbb{R} \times_\rho M^n$ be a compact two-sided hypersurface with positive $(s+1)$-mean curvature, for some $0 \leq s \leq n-1$, and boundary $\partial\Sigma$ contained into the slice $\{t_0\} \times M^n$ for some $t_0 \in \mathbb{R}$. The following holds:*

(a) If ρ' does not change sign on $(-\infty, t_0]$ and, for some $0 \leq r \leq s$, we have

$$\sum_{k=r}^{s} b_k H_{k+1} \geq \sum_{k=r}^{s} b_k \sup_{(-\infty,t_0]} [(\log \rho)']^{k+1}$$

for certain nonnegative constants b_k and, when $s \geq 2$, there exists an elliptic point in Σ^n, then $h \geq t_0$;

(b) If $\rho' > 0$ on $[t_0, +\infty)$ and, for some $0 \leq r \leq s$, we have

$$\sum_{k=r}^{s} b_k H_{k+1} \leq \sum_{k=r}^{s} b_k \inf_{[t_0,+\infty)} [(\log \rho)']^{k+1}$$

for certain nonnegative constants b_k and, when $s \geq 2$, there exists an elliptic point in Σ^n, then $h \leq t_0$.

Proof. To prove (a) we begin by stating that there exists $i_0 \in \{r, \ldots, s\}$ satisfying

$$H_{i_0+1}(p) \geq \sup_{(-\infty,t_0]} [(\log \rho)']^{i_0+1}, \ \forall p \in \Sigma^n. \tag{7.1}$$

It is clear that we can suppose $r < s$ and, by Lemma 6.0.3, we have that H_{k+1} is positive for every $k = r, \ldots, s$. Since ρ' does not change sign on $(-\infty, t_0]$, we have two possibilities: either $\rho' \geq 0$ on $(-\infty, t_0]$ or $\rho' \leq 0$ on $(-\infty, t_0]$. In the first case, we will prove the claim by contradiction, that is, let us assume that (7.1) is false. So, we can choose a point $p \in \Sigma^n$ such that $H_{r+1}(p)^{1/(r+1)} < \sup_{(-\infty,t_0]} (\log\rho)'$. On the other hand, from our hypothesis, must there exists $r < i \leq s$ such that $H_{i+1}(p)^{1/(i+1)} > \sup_{(-\infty,t_0]} (\log\rho)'$, which gives $H_{r+1}(p)^{1/(r+1)} < H_{i+1}(p)^{1/(i+1)}$ leading to a contradiction because of Lemma 6.0.3. Consequently $i_0 = r$ meets the desired. In the latter case, the claim follows because at least one of the numbers $r+1,\ldots,s+1$ is odd.

To close the part (a), we argue once more by contradiction. So, let us assume that $h \geq t_0$ is false. Then, there is an interior point $p_1 \in \Sigma^n$ such that

$$\min h = h(p_1) = t_1 < t_0.$$

Setting $\Sigma_1 = \{t_1\} \times M^n$ and $\Sigma_2 = \Sigma^n$, we see that Σ_1 and Σ_2 are tangent at the common point p_1 and that Σ_1 remains above Σ_2 in a neighborhood of p_1 with respect to $-\partial_t$. Moreover, the claim above yields

$$H_{i_0+1}^{\Sigma_2} = H_{i_0+1} \geq \sup_{(-\infty,t_0]} [(\log\rho)']^{i_0+1} \geq (\log\rho)'(t_1)^{i_0+1} = H_{i_0+1}^{\Sigma_1}.$$

Hence, we can apply the tangency principle (see Theorem 7.1.1) to conclude that h is constant equal to t_1 in a neighborhood of p_1. Therefore, the set $\{h = t_1\}$ is open and closed in Σ^n and by using that Σ^n is connected we get a contradiction.

Now we prove part (b). As above, one shows that $H_{s+1}(p) \leq \inf_{[t_0,+\infty)} [(\log\rho)']^{s+1}$ for every $p \in \Sigma^n$. Assuming again by contradiction that $h \leq t_0$ is false, must there exists an interior point $p_2 \in \Sigma^n$ satisfying

$$\max h = h(p_2) = t_2 > t_0.$$

Taking $\Sigma_1 = \Sigma^n$ and $\Sigma_2 = \{t_2\} \times M^n$, we obtain that Σ_1 and Σ_2 are tangent at the common point p_2, Σ_1 remains above Σ_2 in a neighborhood of p_2 with respect to $-\partial_t$ and

$$H_{s+1}^{\Sigma_1} = H_{s+1} \leq \inf_{[t_0,+\infty)} [(\log\rho)']^{s+1} \leq [(\log\rho)'(t_2)]^{s+1} = H_{s+1}^{\Sigma_2}.$$

Since $\rho' > 0$, it follows that p_2 is an elliptic point of Σ_2. From now on, reasoning as in the part (a) we arrive to a contradiction. $\square$

Remark 7.1.4. In the special case $r = s$, it is easy to see that we do not need to assume in Proposition 7.1.3 (a) that ρ' has a sign.

In our next results, we focus our attention on Riemannian warped product spaces of the type $\mathbb{R} \times_\rho M^n$ satisfying the convergence condition

$$K_M \geq \sup\{(\rho')^2 - \rho\rho''\}, \tag{7.2}$$

where K_M stands for the sectional curvature of the fiber M^n. Warped products satisfying this convergence condition has been studied, for instance, in [17, 65]. In the case in which this condition is assumed on the Ricci curvature instead sectional curvature, it is also well known (see, for instance, [9, 11, 97]).

Now we are ready to state and prove our first main result. More precisely, we will establish an estimate for the height function h of compact (r,s)-linear Weingarten two-sided hypersurfaces in Riemannian warped product spaces of the type $\mathbb{R}\times_\rho M^n$. Before, let us recall that we will always denote by d the constant given in equation (6.13).

Theorem 7.1.5. *Let $\mathbb{R}\times_\rho M^n$ be a Riemannian warped product satisfying the convergence condition in (7.2) and with non-decreasing warping function. Let $\psi:\Sigma^n\to\mathbb{R}\times_\rho M^n$ be a compact (r,s)-linear Weingarten two-sided hypersurface with positive $(s+1)$-mean curvature, for some $0\le s\le n-1$, boundary $\partial\Sigma^n$ contained into the slice $\{t_0\}\times M^n$, for some $t_0\in\mathbb{R}$, and $d\ge\sum_{k=r}^s b_k\sup_{(-\infty,t_0]}[(\log\rho)']^{k+1}$. Suppose that $H_1\ge\max|H_{r+1}|^{1/(r+1)}$ and, if $s\ge 2$, there exists an elliptic point in Σ^n. If the angle function Θ does not change sign on Σ^n, then*

$$\Sigma^n\subset\left[t_0,t_0+\frac{\rho(\max h)}{\rho(t_0)\min H_1}\right]\times M^n.$$

Proof. We may assume without loss of generality that Σ^n is not a slice, otherwise there is nothing to prove. From Proposition 7.1.3 we have $h\ge t_0$. In particular, we can choose an interior point p_0 in Σ^n such that the height function reaches its maximum. By Proposition 6.0.7 we get, at p_0, that

$$0\ge\Delta h(p_0)=n(\log\rho)'(h(p_0))+n\Theta(p_0)H_1(p_0)\ge n\Theta(p_0)H_1(p_0).$$

Since, by Lemma 6.0.3, the mean curvature is positive, we obtain that $\Theta(p_0)\le 0$ and Θ is a nonpositive function.

Let us consider on Σ^n the smooth function $\varphi=c\sigma(h)+\tilde{\Theta}$, where $c\in\mathbb{R}$ is a positive constant to be chosen in an appropriate way, $\sigma(t)$ is a primitive of the warping function $\rho(t)$ and $\tilde{\Theta}=\rho\Theta$. Then Proposition 6.0.7 yields

$$\begin{aligned}L_k\varphi &= -\frac{c_k\rho(h)}{k+1}\langle\nabla H_{k+1},\nabla h\rangle+c_k\rho'(h)\,(cH_k-H_{k+1})\\ &- \rho(h)\Theta\binom{n}{k+1}(nH_1H_{k+1}-(n-k-1)H_{k+2}-(k+1)cH_{k+1})\\ &- \frac{\tilde{\Theta}}{\rho^2(h)}\sum_{i=1}^n\mu_{i,k}K_M(N^*,E_i^*)|N^*\wedge E_i^*|^2\\ &- \tilde{\Theta}(\log\rho)''(h)\left(c_k|\nabla h|^2H_k-\langle P_k\nabla h,\nabla h\rangle\right),\end{aligned}\tag{7.3}$$

where $\{E_1,\dots,E_n\}$ is an orthonormal frame on Σ^n diagonalizing A with $P_kE_i=\mu_{i,k}E_i$, for every $i=1,\dots,n$ and $k=r,\dots,s$, X^* denotes the orthogonal projection on TM and the constants c_k are defined in (6.11).

Since H_{s+1} is positive and Σ^n has an elliptic point, from Lemma 6.0.3, we get that

$$H_1H_{k+1}-H_{k+2}\ge H_1H_{k+1}-H_{k+1}^2H_k^{-1}=\frac{H_{k+1}}{H_k}\left(H_1H_k-H_{k+1}\right).$$

By using once more Lemma 6.0.3 it follows that

$$H_1H_{k+1} - H_{k+2} \geq \frac{H_{k+1}}{H_k}\left(H_1H_k - H_k^{(k+1)/k}\right) = H_{k+1}(H_1 - H_k^{1/k}) \geq 0.$$

Then the previous inequality gives

$$\begin{aligned} nH_1H_{k+1} - (n-k-1)H_{k+2} - (k+1)cH_{k+1} &= (k+1)H_{k+1}(H_1 - c) \\ &+ (n-k-1)(H_1H_{k+1} - H_{k+2}) \\ &\geq (k+1)H_{k+1}(H_1 - c) \geq 0, \end{aligned} \tag{7.4}$$

provided that $c := \min H_1$. In particular, for this choose of c, it follows from our hypothesis on H_1 and Lemma 6.0.3 that

$$cH_k - H_{k+1} \geq H_{k+1}^{1/(k+1)}(H_k - H_{k+1}^{k/(k+1)}) \geq 0. \tag{7.5}$$

On the other hand, by our assumptions we can apply Lemma 6.0.5 (or Lemma 6.0.4 if $s = 1$) for to guarantee the ellipticity of the operators L_k for every $k = r, \ldots, s$ or, equivalently, P_k is positive definite. In particular, its eigenvalues $\mu_{i,k}$ are all positive on Σ^n, and from the convergence condition in (7.2) we have

$$\mu_{i,k}K_M(N^*, E_i^*)|N^* \wedge E_i^*|^2 \geq \mu_{i,k}C|N^* \wedge E_i^*|^2, \tag{7.6}$$

for every $i = 1, \ldots, n$ and $k = r, \ldots, s$, where we are writing $C = \sup\{(\rho')^2 - \rho\rho''\}$. With a straightforward computation, one shows that

$$|N^* \wedge E_i^*|^2 = |\nabla h|^2 - \langle E_i, \nabla h\rangle^2,$$

which jointly with (7.6) imply

$$\begin{aligned} \sum_{i=1}^{n} \mu_{i,k}K_M(N^*, E_i^*)|N^* \wedge E_i^*|^2 &\geq C\left(\mathrm{tr}(P_k)|\nabla h|^2 - \sum_{i=1}^{n} \mu_{i,k}\langle E_i, \nabla h\rangle^2\right) \\ &= C\left(\mathrm{tr}(P_k)|\nabla h|^2 - \langle P_k\nabla h, \nabla h\rangle\right), \end{aligned}$$

Then, since $\mathrm{tr}(P_k) = c_kH_k$ and $C/\rho^2(h) + (\log\rho)''(h) \geq 0$, we conclude that

$$\frac{1}{\rho^2(h)}\sum_{i=1}^{n} \mu_{i,k}K_M(N^*, E_i^*)|N^* \wedge E_i^*|^2 + (\log\rho)''(h)\left(c_k|\nabla h|^2H_k - \langle P_k\nabla h, \nabla h\rangle\right) \geq 0, \tag{7.7}$$

where the last inequality follows from the fact that P_k is positive definite. Hence, by using (7.4), (7.5) and (7.7), and taking into account that $\Theta \leq 0$ and the warping function is non-decreasing, we infer from (7.3) that

$$L_k\varphi \geq -\frac{c_k\rho(h)}{k+1}\langle\nabla H_{k+1}, \nabla h\rangle. \tag{7.8}$$

Next, let us introduce the second order linear differential operator $L : C^\infty(\Sigma) \to C^\infty(\Sigma)$ defined by

$$\begin{aligned} L &= \sum_{k=r}^{s} (k+1) c_k^{-1} b_k L_k \\ &= \mathrm{tr}\,(P \circ \mathrm{hess})\,, \end{aligned} \tag{7.9}$$

where the tensor $P : \mathfrak{X}(\Sigma) \to \mathfrak{X}(\Sigma)$ is given by

$$P = \sum_{k=r}^{s} (k+1) c_k^{-1} b_k P_k. \tag{7.10}$$

Since $(k+1)c_k^{-1} b_k > 0$ for every $k = r, \ldots, s$ and each operator L_k is elliptic (equivalently, each P_k is positive definite) we see that the tensor P is positive definite and, consequently, the operator L is elliptic too. So, equation (7.8) and the fact that Σ^n is (r, s)-linear Weingarten imply that

$$L\varphi \geq 0.$$

Since Σ^n is compact, we can apply the weak maximum principle for the elliptic operator L and, taking into account that Θ is a nonpositive function, we find

$$c\sigma(h) - \rho(h) \leq \varphi \leq \max_{\partial\Sigma} \varphi = c\sigma(t_0) + \rho(t_0) \max_{\partial\Sigma} \Theta \leq c\sigma(t_0), \tag{7.11}$$

which implies

$$c(\sigma(h) - \sigma(t_0)) \leq \rho(h) \leq \rho(\max h). \tag{7.12}$$

By using once more that ρ is non-decreasing and σ is increasing, it is not difficult to see that for any $t \geq t_0$ it holds

$$\sigma(t) - \sigma(t_0) \geq \rho(t_0)(t - t_0).$$

Since the height function satisfies $h \geq t_0$, from (7.12) and previous inequality we get

$$c\rho(t_0)(h - t_0) \leq \rho(\max h).$$

Therefore, we conclude that

$$h \leq t_0 + \frac{\rho(\max h)}{\rho(t_0) \min H_1}.$$

This finishes the proof of the theorem. □

It is worth pointing out that when $r = s$, that is, the hypersurface has constant positive $(s+1)$-mean curvature H_{s+1}, our assumption $H_1 \geq \max |H_{s+1}|^{1/(s+1)}$ in Theorem 7.1.5 holds trivially because of Lemma 6.0.3. In particular, we have:

Corollary 7.1.6. *Let $\mathbb{R} \times_\rho M^n$ be a Riemannian warped product satisfying the convergence condition in* (7.2) *and with non-decreasing warping function. Let $\psi : \Sigma^n \to \mathbb{R} \times_\rho M^n$ be a compact* *two-sided hypersurface with constant positive $(s+1)$-mean curvature, for some $0 \le s \le n-1$, boundary $\partial\Sigma^n$ contained into the slice $\{t_0\} \times M^n$, for some $t_0 \in \mathbb{R}$, and $H_{s+1} \ge \sup\limits_{(-\infty,t_0]} [(\log\rho)']^{s+1}$. Suppose that there exists an elliptic point in Σ^n. If the angle function Θ does not change sign on Σ^n, then*

$$\Sigma^n \subset \left[t_0, t_0 + \frac{\rho(\max h)}{\rho(t_0)\min H_1}\right] \times M^n.$$

When the warping function ρ is either exponential or hyperbolic cosine function, following the terminology introduced by Tashiro [112], the corresponding warped product $\mathbb{R} \times_{e^t} M^n$ or $\mathbb{R} \times_{\cosh t} M^n$ has been referred to as a pseudo-hyperbolic space. The Tashiro's terminology is due to the fact that with suitable choices of the fiber M^n we obtain warped products which are isometric to the hyperbolic space. For more details about these spaces and others topics related see, for instance, [10, 11, 65, 97].

In the special case in which the warping function is given by $\rho(t) = e^t$, we are able to improve the estimate of Theorem 7.1.5 so that it does not depend on the height function h of the hypersurface Σ^n. More specifically, we get the following result.

Theorem 7.1.7. *Let $\mathbb{R} \times_{e^t} M^n$ be a pseudo-hyperbolic space whose fiber has nonnegative sectional curvature. Let $\psi : \Sigma^n \to \mathbb{R} \times_{e^t} M^n$ be a compact (r,s)-linear Weingarten two-sided hypersurface with positive $(s+1)$-mean curvature, for some $0 \le s \le n-1$, boundary $\partial\Sigma^n$ contained into the slice $\{t_0\} \times M^n$, for some $t_0 \in \mathbb{R}$, and $d > \sum_{k=r}^{s} b_k$. Suppose that $H_1 \ge \max|H_{r+1}|^{1/(r+1)}$ and, if $s \ge 2$, there exists an elliptic point in Σ^n. If the angle function Θ does not change sign on Σ^n, then*

$$\Sigma^n \subset \left[t_0, t_0 + \log\left(\frac{\min H_1}{\min H_1 - 1}\right)\right] \times M^n.$$

Proof. From Proposition 7.1.3 it is clear that $\min H_1 > 1$ and the height function satisfies $h \ge t_0$. We also note that in this case the function φ in Theorem 7.1.5 is given by $\varphi = e^h(c + \Theta)$, where $c = \min H_1$ and Θ must be a nonpositive function. So, by equation (7.11), it follows that

$$e^h(\min H_1 - 1) \le e^{t_0} \min H_1$$

proving the result. □

In the particular situation of hypersurfaces having constant $(s+1)$-mean curvature H_{s+1}, Theorem 7.1.7 improves the estimate obtained by García-Martínez et al. in Theorem 3.10 of [65]. Indeed, there the authors proved that, in this case, the height function satisfies

$$t_0 \le h \le t_0 + \log\left(\frac{H_{s+1}^{1/(s+1)}}{H_{s+1}^{1/(s+1)} - 1}\right)$$

On the other hand, since the function $f(x) = \frac{x}{x-1}$ is decreasing on $(1, +\infty)$, it is easy to see that the inequality

$$\log\left(\frac{\min H_1}{\min H_1 - 1}\right) \le \log\left(\frac{H_{s+1}^{1/(s+1)}}{H_{s+1}^{1/(s+1)} - 1}\right)$$

holds for every $s = 0, \dots, n-1$, which gives the claim.

Next, we also consider the case when the warping function is given by $\rho(t) = \cosh t$. As in the previous theorem, we can improve the estimate of Theorem 7.1.5 as follows.

Theorem 7.1.8. *Let $\mathbb{R} \times_{\cosh t} M^n$ be a pseudo-hyperbolic space whose fiber has sectional curvature satisfying $K_M \geq -1$. Let $\psi : \Sigma^n \to \mathbb{R} \times_{\cosh t} M^n$ be a compact (r,s)-linear Weingarten two-sided hypersurface with positive $(s+1)$-mean curvature, for some $0 \leq s \leq n-1$, boundary $\partial\Sigma^n$ contained into the slice $\{0\} \times M^n$ and $d > \sum_{k=r}^{s} b_k$. Suppose that $H_1 \geq \max |H_{r+1}|^{1/(r+1)}$ and, if $s \geq 2$, there exists an elliptic point in Σ^n. If the angle function Θ does not change sign on Σ^n, then*

$$\Sigma^n \subset \left[0, \tanh^{-1}\left(\frac{1}{\min H_1}\right)\right] \times M^n.$$

Proof. By applying Proposition 7.1.3 we see that $h \geq 0$. Moreover, in this case, the smooth function φ defined in Theorem 7.1.5 is given by $\varphi = c \sinh h + \Theta \cosh h$, where $c = \min H_1$ and the angle function Θ is taken nonpositive. Since $\sinh t \geq 0$ for every $t \geq 0$, reasoning as in the proof of Theorem 7.1.5 we arrive to equation (7.11), which implies that

$$c \sinh h - \cosh h \leq c \sinh 0 = 0.$$

Therefore the previous inequality yields

$$h \leq \tanh^{-1}\left(\frac{1}{\min H_1}\right)$$

and the theorem follows. □

We observe that in the case in which some higher order mean curvature is constant, Theorem 7.1.8 improves the estimate given in Theorem 3.11 of [65]. Indeed, Theorem 3.11 of [65] says that, in this situation,

$$0 \leq h \leq \tanh^{-1}\left(\frac{1}{H_{s+1}^{1/(s+1)}}\right).$$

But, it is clear that

$$\tanh^{-1}\left(\frac{1}{\min H_1}\right) \leq \tanh^{-1}\left(\frac{1}{H_{s+1}^{1/(s+1)}}\right)$$

is true for every $s = 0, \dots, n-1$, as stated.

We continue by treating the simplest case in which the warping function is constant. In this case, Theorem 7.1.5 becomes the following result, where we do not need to assume the existence of an elliptic point on the hypersurface.

Theorem 7.1.9. *Let $\mathbb{R} \times M^n$ be a product space whose fiber has nonnegative sectional curvature K_M. Let $\psi : \Sigma^n \to \mathbb{R} \times M^n$ be a compact (r,s)-linear Weingarten two-sided hypersurface with $(s+1)$-mean curvature $H_{s+1} \neq 0$ on Σ^n, for some $0 \leq s \leq n-1$, and boundary $\partial\Sigma^n$ contained into the slice $\{t_0\} \times M^n$ for some $t_0 \in \mathbb{R}$. Suppose that the angle function Θ does not change sign on Σ^n. Then,* *(a) Either* $\max h \neq t_0$ *and*

$$\Sigma^n \subset \left[t_0, t_0 + \frac{1}{\min H_1}\right] \times M^n,$$

(b) or $\min h \neq t_0$ *and*

$$\Sigma^n \subset \left[t_0 - \frac{1}{\min H_1}, t_0\right] \times M^n.$$

Proof. First of all it is clear from our hypothesis on the $(s+1)$-mean curvature that either $\max h \neq t_0$ or $\min h \neq t_0$. So, we begin by assuming that $\max h \neq t_0$ and let us choose an interior point p_0 of Σ^n such that the height function reaches its maximum and the orientation so that $\Theta \leq 0$. Then, from Proposition 6.0.7 we get

$$0 \geq \text{Hess } h(p_0)(v,v) = \Theta(p_0)\langle Av, v\rangle(p_0), \quad \forall v \in T_{p_0}\Sigma,$$

that is, at p_0 all the principal curvatures are nonnegative. Since we are assume that $H_{s+1} \neq 0$ on Σ^n, we must have $H_{s+1} > 0$ on Σ^n. In particular, we can apply Lemma 6.0.5 (or Lemma 6.0.4 if $s = 1$; see also Remark 6.0.6) to guarantee the ellipticity of the operator L_k and that H_{k+1} is positive for every $k = 0, \ldots, s$. So, for instance, we have

$$L_s h = c_s \Theta H_{s+1} \leq 0$$

and, consequently, by the weak maximum principle, we obtain that $h \geq t_0$ on Σ^n. Hence, by applying Theorem 7.1.5, part (a) follows.

In the case $\min h \neq t_0$, we choose an interior point q_0 of Σ^n satisfying $\min h = h(q_0)$ and the orientation so that $\Theta \geq 0$. Then,

$$0 \leq \text{Hess } h(q_0)(v,v) = \Theta(q_0)\langle Av, v\rangle(q_0), \quad \forall v \in T_{q_0}\Sigma,$$

that is, at q_0 all the principal curvatures must be nonnegative. So, reasoning as previous case we see that each operator L_k is elliptic and H_{k+1} is positive for every $k = 0, \ldots, s$. Besides must be $h \leq t_0$ on Σ^n.

Moreover, keeping the notation of Theorem 7.1.5, it follows that $\varphi = ch + \Theta$ satisfies, by equations (7.3), (7.4) and our assumption on K_M,

$$L_k \varphi \leq -\frac{c_k}{k+1}\langle \nabla H_{k+1}, \nabla h\rangle,$$

which implies that $L\varphi \leq 0$, where the operator L is given by (7.9). Therefore, by weak maximum principle, we conclude that

$$\varphi \geq \min_{\partial\Sigma} \varphi \geq ct_0,$$

that is,

$$h \geq t_0 - \frac{1}{\min H_1}.$$

This finishes the proof of the theorem. □

Remark 7.1.10. We note that the estimate given in Theorem 7.1.9 is sharp in the sense that it is reached by the hemisphere $\Sigma_+ = \{x \in \mathbb{S}^n \; ; \; x_1 \geq 0\}$ of the standard sphere $\mathbb{S}^n$ in $\mathbb{R}^{n+1}$. Indeed, it follows easily that Σ_+ is a totally umbilical hypersurface (in fact, it is a vertical graph) with $H_1 = 1$, boundary $\{0\} \times \mathbb{S}^{n-1} \subset \{0\} \times \mathbb{R}^n$ and has the maximum height 1.

It is worth pointing out that for hypersurface having constant $(s+1)$-mean curvature H_{s+1}, Theorem 7.1.9 improves the estimate obtained by Cheng and Rosenberg in Theorem 4.1(i) of [42]. Indeed, in [42] the authors showed that

$$t_0 \leq h \leq t_0 + \frac{1}{H_{s+1}^{1/(s+1)}}.$$

Here it is easy to see that the inequality

$$\frac{1}{\min H_1} \leq \frac{1}{H_{s+1}^{1/(s+1)}}$$

holds for every $s = 0, \ldots, n-1$. Moreover, this result is also an extension of Theorem 3.5 in [11] (case $\alpha = 0$) and Proposition 1 in [77] (case $\tau = 0$).

Proceeding, we are able to relax the assumption on sectional curvature K_M of the fiber M^n letting it be bounded from below by a negative constant. For this, we will assume that the mean curvature satisfies a certain condition, which holds automatically when the sectional curvature of the fiber is nonnegative. In what follows, we continue denoting by $c = \min H_1$. So, we get the following result.

Theorem 7.1.11. *Let $\mathbb{R} \times M^n$ be a product space whose fiber has sectional curvature satisfying $K_M \geq -\alpha$, for some positive constant $\alpha \in \mathbb{R}$. Let $\psi : \Sigma^n \to \mathbb{R} \times M^n$ be a compact (r,s)-linear Weingarten two-sided hypersurface with $(s+1)$-mean curvature $H_{s+1} \neq 0$ on Σ^n, for some $0 \leq s \leq n-1$, and boundary $\partial\Sigma^n$ contained into the slice $\{t_0\} \times M^n$ for some $t_0 \in \mathbb{R}$. Suppose that the angle function Θ does not change sign on Σ^n and $c(r+1)\min H_{k+1} > \alpha(s+1)\max H_k$ for every $k = r\ldots, s$. Then,*

(a) Either $\max h \neq t_0$ and

$$\Sigma^n \subset \left[t_0, t_0 + \frac{(r+1)d}{(r+1)dc - (s+1)\alpha\beta}\right] \times M^n,$$

where d is given by (6.13) and $\beta = \sum_{k=r}^{s} b_k \max H_k$.

(b) or $\min h \neq t_0$ and

$$\Sigma^n \subset \left[t_0 - \frac{(r+1)d}{(r+1)dc - (s+1)\alpha\beta}, t_0\right] \times M^n,$$

where d is given by (6.13) and $\beta = \sum_{k=r}^{s} b_k \max H_k$.

Proof. In what follows, we keep the notations established in Theorem 7.1.5. Let us suppose $\max h \neq t_0$ first. Then, as in Theorem 7.1.9, taking the angle function Θ nonpositive, it is easy to see that, for every $k = 0, \dots, s$, the operator L_k is elliptic and the $(k+1)$-mean curvature H_{k+1} is positive, and $h \geq t_0$. Moreover, by equations (7.3) and (7.4) we get that the function φ defined in Theorem 7.1.5 satisfies (note that in this case $\varphi = ch + \Theta$)

$$L_k \varphi \geq -\frac{c_k}{k+1}\langle \nabla H_{k+1}, \nabla h\rangle - \Theta \sum_{i=1}^{n} \mu_{i,k} K_M(N^*, E_i^*)|N^* \wedge E_i^*|^2. \tag{7.13}$$

Since the eigenvalues $\mu_{i,k}$ are all positive on Σ^n and using our assumption on K_M we have

$$\mu_{i,k} K_M(N^* \wedge E_i^*)|N^* \wedge E_i^*|^2 \geq -\mu_{i,k}\alpha|N^* \wedge E_i^*|^2 \geq -\mu_{i,k}\alpha, \tag{7.14}$$

for every $i = 1, \dots, n$ and $k = r, \dots, s$, because $|N^* \wedge E_i^*|^2 = |\nabla h|^2 - \langle E_i, \nabla h\rangle^2 \leq 1$. Then, equation (7.14) yields

$$\sum_{i=1}^{n} \mu_{i,k} K_M(N^* \wedge E_i^*)|N^* \wedge E_i^*|^2 \geq -\alpha \mathrm{tr}(P_k) = -\alpha c_k H_k \geq -\alpha c_k \max H_k.$$

From here and (7.13) we get

$$L\varphi \geq \sum_{k=r}^{s} (k+1)\alpha\Theta b_k \max H_k \geq (s+1)\alpha\beta\Theta, \tag{7.15}$$

where $\beta = \sum_{k=r}^{s} b_k \max H_k$. On the other hand, by using Proposition 6.0.7 we see that

$$Lh = \sum_{k=r}^{s} (k+1) c_k^{-1} b_k L_k h = \sum_{k=r}^{s} (k+1)\Theta b_k H_{k+1} \leq (r+1)d\Theta. \tag{7.16}$$

So, let us consider on Σ^n the smooth function given by

$$\tilde{\varphi} = \varphi - \frac{(s+1)\alpha\beta}{(r+1)d} h = \frac{(r+1)dc - (s+1)\alpha\beta}{(r+1)d} h + \Theta.$$

Then, from equations (7.15) and (7.16) we obtain

$$L\tilde{\varphi} = L\varphi - \frac{(s+1)\alpha\beta}{(r+1)d} Lh \geq (s+1)\alpha\beta\Theta - (s+1)\alpha\beta\Theta = 0.$$

Hence, we can apply once more the weak maximum principle to conclude that

$$\tilde{\varphi} \leq \max_{\partial\Sigma} \tilde{\varphi} \leq \frac{(r+1)dc - (s+1)\alpha\beta}{(r+1)d} t_0,$$

that is,

$$\frac{(r+1)dc - (s+1)\alpha\beta}{(r+1)d}(h - t_0) \leq 1. \tag{7.17}$$

Now, from our assumption on c we get

$$\begin{aligned}(r+1)dc-(s+1)\alpha\beta &= (r+1)c\sum_{k=r}^{s} b_k H_{k+1} - (s+1)\alpha \sum_{k=r}^{s} b_k \max H_k \\ &= \sum_{k=r}^{s} b_k \left((r+1)cH_{k+1} - (s+1)\alpha \max H_k\right) > 0.\end{aligned}$$

Therefore, from equation (7.17), we arrive to

$$h \leq t_0 + \frac{(r+1)d}{(r+1)dc-(s+1)\alpha\beta}$$

as desired.

Finally, the case $\min h \neq t_0$ it follows in analogues way and this finishes the proof of the theorem. □

In the case of hypersurfaces with constant $(s+1)$-mean curvature H_{s+1}, our assumption on c in Theorem 7.1.11 becomes $cH_{s+1} > \alpha \max H_s$. In particular, it is weaker than assumption (7.77) of Theorem 7.19 in [19],

$$H_{s+1}^{(s+2)/(s+1)} > \alpha \max H_s,$$

because, in this case, $c \geq H_{s+1}^{1/(s+1)}$ by Lemma 6.0.3. Moreover, the constant $\frac{(r+1)d}{(r+1)dc-(s+1)\alpha\beta}$ is just given by $\frac{H_{s+1}}{cH_{s+1}-\alpha \max H_s}$. Furthermore

$$\frac{H_{s+1}}{cH_{s+1}-\alpha\max H_s} \leq \frac{H_{s+1}}{H_{s+1}^{(s+2)/(s+1)}-\alpha \max H_s}.$$

In this setting, our estimate improves that given in Theorem 7.19 of [19] for the case in which these hypersurfaces are compacts.

7.2 Half-space theorems and topology at infinity

The aim of this section is to obtain information on the topology at infinity, in the form of half-space theorems, regarding complete two-sided hypersurfaces in Riemannian warped product spaces of the type $\mathbb{R} \times_\rho M^n$. We point out that the results presented here do not assume that the hypersurface has some constant higher order mean curvature. As we shall see, this gives generalizations of results contained in [42, 65, 77].

In the first part of this section, we consider the case in which the fiber M^n is compact. Then as consequence of the height estimates proved in the previous section, we get various half-space theorems. In the next step we analyze hypersurfaces in Riemannian warped products whose fiber is not necessarily compact and, in the same spirit of the first part, now using a generalized version of the Omori-Yau's maximum principle for trace type differential operators, we prove new half-space theorems which seem to be interesting in its own.

Let us begin with the following definition.

Definition 7.2.1. *We say that a two-sided hypersurface in a Riemannian warped product space $\mathbb{R} \times_\rho M^n$ lies in an upper or lower half-space if it is, respectively, contained into a region of $\mathbb{R} \times M^n$ of the form*

$$[a, +\infty) \times M^n \qquad \text{or} \qquad (-\infty, a] \times M^n,$$

for some real number $a \in \mathbb{R}$.

Our first half-space theorem is regarding more general case of a Riemannian warped product arbitrary, which follows as an application of Proposition 7.1.3.

Theorem 7.2.2. *Let $\mathbb{R} \times_\rho M^n$ be a Riemannian warped product whose fiber is compact. Let $\psi : \Sigma^n \to \mathbb{R} \times_\rho M^n$ be a noncompact two-sided properly immersed hypersurface with positive $(s+1)$-mean curvature, for some $0 \leq s \leq n-1$. The following holds:*

(a) If ρ' does not change sign on $\mathbb{R}$ and, for some $0 \leq r \leq s$, we have

$$\sum_{k=r}^{s} b_k H_{k+1} \geq \sum_{k=r}^{s} b_k \sup[(\log \rho)']^{k+1}$$

for certain nonnegative constants b_k and, if $s \geq 2$, there exists an elliptic point in Σ^n, then Σ^n cannot lie in an upper half-space. In particular, Σ^n must have at least one bottom end.

(b) If $\rho' > 0$ on $\mathbb{R}$ and, for some $0 \leq r \leq s$, we have

$$\sum_{k=r}^{s} b_k H_{k+1} \leq \sum_{k=r}^{s} b_k \inf[(\log \rho)']^{k+1}$$

for certain nonnegative constants b_k and, if $s \geq 2$, there exists an elliptic point in Σ^n, then Σ^n cannot lie in a lower half-space. In particular, Σ^n must have at least one top end.

Proof. We prove first part (a). Let us assume by contradiction that Σ^n lies in an upper half-space, that is, $\Sigma^n \subset [a, +\infty) \times M^n$, for some $a \in \mathbb{R}$. For any number $t_0 > a$ let Σ_{t_0} be the hypersurface

$$\Sigma_{t_0} = \{(t, p) \in \Sigma^n \; ; \; t \leq t_0\}.$$

Then Σ_{t_0} is a compact two-sided hypersurface with boundary contained into the slice $\{t_0\} \times M$, because M^n is compact and the immersion is proper. Moreover, the following inequality holds:

$$\sum_{k=r}^{s} b_k H_{k+1} \geq \sum_{k=r}^{s} b_k \sup[(\log \rho)']^{k+1} \geq \sum_{k=r}^{s} b_k \sup_{(-\infty, t_0]} [(\log \rho)']^{k+1}.$$

Hence, by Proposition 7.1.3 we conclude that the height function of Σ_{t_0} satisfies $h \geq t_0$, leading to a contradiction since t_0 is arbitrary.

Let us consider part (b). We reason again by contradiction assuming that Σ^n is contained into a lower half-space of the form $(-\infty, a] \times M^n$, for some $a \in \mathbb{R}$. We set

$$\Sigma_{t_0} = \{(t, p) \in \Sigma^n \; ; \; t \geq t_0\},$$

where $t_0 < a$ is arbitrary. It follows that Σ_{t_0} is a compact two-sided hypersurface with boundary contained into the slice $\{t_0\} \times M$. Besides it is also true that

$$\sum_{k=r}^{s} b_k H_{k+1} \leq \sum_{k=r}^{s} b_k \inf[(\log \rho)']^{k+1} \leq \sum_{k=r}^{s} b_k \inf_{[t_0,+\infty)} [(\log \rho)']^{k+1}.$$

By using once more Proposition 7.1.3 we get $h \leq t_0$ characterizing a contradiction. This proof the theorem. □

When we consider a pseudo-hyperbolic space of the type $\mathbb{R} \times_{e^t} M^n$, using Theorem 7.1.7 we can get a stronger result than Theorem 7.2.2 as follows, which gives a generalization of Theorem 4.3 of [65].

Theorem 7.2.3. *Let $\mathbb{R} \times_{e^t} M^n$ be a pseudo-hyperbolic space whose fiber is compact and has nonnegative sectional curvature. Let $\psi : \Sigma^n \to \mathbb{R} \times_{e^t} M^n$ be a noncompact (r, s)-linear Weingarten two-sided properly immersed hypersurface with positive $(s+1)$-mean curvature, for some $0 \leq s \leq n-1$, and $d > \sum_{k=r}^{s} b_k$. Suppose that $H_1 \geq \sup |H_{r+1}|^{1/(r+1)}$ and, if $s \geq 2$, there exists an elliptic point in Σ^n. If the angle function Θ does not change sign on Σ^n, then Σ^n cannot lie in a half-space. In particular, Σ^n must have at least one bottom and one top end.*

Proof. It is immediate from Theorem 7.2.2 that Σ^n cannot lie in an upper half-space. On the other hand, if $\Sigma^n \subset (-\infty, a] \times M^n$, for some $a \in \mathbb{R}$, then as above let us consider for any $t_0 < a$ the hypersurface

$$\Sigma_{t_0} = \{(t, p) \in \Sigma^n \ ; \ t \geq t_0\}.$$

Thus Σ_{t_0} satisfies the assumptions of Theorem 7.1.7, which implies that

$$a - t_0 \leq \log \left(\frac{\min_{\Sigma_{t_0}} H_1}{\min_{\Sigma_{t_0}} H_1 - 1} \right). \tag{7.18}$$

Moreover, taking into account the assumption on d and H_1, we must have $\inf H_1 > 1$. Hence, equation (7.18) yields

$$a - t_0 \leq \log \left(\frac{\min_{\Sigma_{t_0}} H_1}{\min_{\Sigma_{t_0}} H_1 - 1} \right) \leq \log \left(\frac{\inf H_1}{\inf H_1 - 1} \right).$$

Letting t_0 small enough we reached a contradiction. □

In the case of a pseudo-hyperbolic space of the type $\mathbb{R} \times_{\cosh t} M^n$, reasoning as in Proposition 7.1.3 we obtain the following generalization of Theorem 4.5 of [65].

Theorem 7.2.4. *Let $\mathbb{R} \times_{\cosh t} M^n$ be a pseudo-hyperbolic space whose fiber is compact. Let $\psi : \Sigma^n \to \mathbb{R} \times_{\cosh t} M^n$ be a noncompact two-sided properly immersed hypersurface with positive $(s+1)$-mean curvature, for some $0 \leq s \leq n-1$. Suppose that for some $0 \leq r \leq s$ we have $\sum_{k=r}^{s} b_k H_{k+1} \geq \sum_{k=r}^{s} b_k$, for certain nonnegative constants b_k. If $s \geq 2$, assume that there exists an elliptic point in Σ^n. Then, Σ^n cannot lie in an upper half-space. In particular, Σ^n must have at least one bottom end.*

Proof. We argue once more by contradiction, that is, let us assume that Σ^n is contained into an upper half-space $[a, +\infty) \times M^n$, for some $a \in \mathbb{R}$. As in the proof of Proposition 7.1.3 we can get $H_{r+1} \geq 1$. Given $t_0 > a$, we set the hypersurface

$$\Sigma_{t_0} = \{(t, p) \in \Sigma^n \ ; \ t \leq t_0\}.$$

It is clear that Σ_{t_0} is a compact two-sided hypersurface with boundary $\partial\Sigma_{t_0} \subset \{t_0\} \times M^n$. Furthermore,

$$\sup_{(-\infty, t_0]} (\tanh t)^{r+1} \leq 1 \leq H_{r+1}.$$

Therefore, by Proposition 7.1.3 (see also Remark 7.1.4) we conclude that $h \geq t_0$, which gives a contradiction. $\square$

Next, we prove half-space theorems regarding two-sided hypersurfaces immersed into standard product space $\mathbb{R} \times M^n$. Its prove follows from Theorem 7.1.9.

Theorem 7.2.5. *Let $\mathbb{R} \times M^n$ be a product space whose fiber is compact and has nonnegative sectional curvature. Let $\psi : \Sigma^n \to \mathbb{R} \times M^n$ be a noncompact (r, s)-linear Weingarten two-sided properly immersed hypersurface with bounded away from zero $(s+1)$-mean curvature, for some $0 \leq s \leq n-1$, and such that its angle function Θ does not change sign. Then, Σ^n cannot lie in a half-space. In particular, Σ^n must have at least one bottom and one top end.*

Proof. Let us assume by contradiction that Σ^n lies in an upper half-space, that is, $\Sigma^n \subset [a, +\infty) \times M^n$, for some $a \in \mathbb{R}$. As in the proof of Theorem 7.2.2, we denote by Σ_{t_0} the hypersurface

$$\Sigma_{t_0} = \{(t, p) \in \Sigma^n \ ; \ t \leq t_0\},$$

where $t_0 > a$ is arbitrary. Then, Σ_{t_0} is a compact (r, s)-linear Weingarten two-sided hypersurface with boundary contained into the slice $\{t_0\} \times M$ and $\min h \neq t_0$. Hence, by Theorem 7.1.9 we must have $H_{s+1} > 0$ on Σ_{t_0} and $\Sigma_{t_0} \subset [t_0 - \frac{1}{c(t_0)}, t_0] \times M^n$, where $c(t_0) = \min_{\Sigma_{t_0}} H_1 > 0$, that is,

$$t_0 - a \leq \frac{1}{c(t_0)}.$$

Because H_{s+1} is bounded away from zero we get $\inf H_{s+1} > 0$, which implies $\inf H_1 > 0$. Thus

$$t_0 - a \leq \frac{1}{c(t_0)} \leq \frac{1}{\inf H_1}.$$

Then choosing t_0 large enough we reached a contradiction.

Finally, if Σ^n is contained in a lower half-space, we may apply the some argument above to arrive at a contradiction. $\square$

Similarly, we can reason as in Theorem 7.2.5 to obtain as consequence of Theorem 7.1.11 the following result, where we keep the notation $c = \min H_1$.

Theorem 7.2.6. *Let $\mathbb{R} \times M^n$ be a product space whose fiber is compact with sectional curvature satisfying $K_M \geq -\alpha$, for some positive constant $\alpha \in \mathbb{R}$. Let $\psi : \Sigma^n \to \mathbb{R} \times M^n$ be a noncompact (r, s)-linear Weingarten two-sided properly immersed hypersurface with bounded away from zero $(s+1)$-mean curvature, for some $0 \leq s \leq n-1$, and such that its angle function Θ does not change sign. Suppose that $c(r+1) \min H_{k+1} > \alpha(s+1) \max H_k$ for every $k = r \ldots, s$. Then, Σ^n cannot lie in a half-space. In particular, Σ^n must have at least one bottom and one top end.*

We observe that our results generalize those obtained by Cheng and Rosenberg [42] and Hoffman et al. [77] for the case in which the mean curvature or some higher order mean curvature is constant.

In order to treat the case in which the fiber is not compact, we will make use of a generalized version of the Omori-Yau's maximum principle for trace type differential operators proved in [19] (for more details, see Appendix A, Lemma A.0.3). Let us recall that, given a Riemannian manifold Σ^n and a semi-elliptic operator $\mathcal{L} = \mathrm{tr}(\mathcal{P} \circ \mathrm{hess})$, where $\mathcal{P} : \mathfrak{X}(\Sigma) \to \mathfrak{X}(\Sigma)$ is a positive semi-definite symmetric tensor, we say that the Omori-Yau's maximum principle holds on Σ^n for the operator $\mathcal{L}$ if, for any function $u \in C^2(\Sigma)$ with $u^* = \sup u < +\infty$, there exists a sequence of points $\{p_j\} \subset \Sigma^n$ satisfying

$$u(p_j) > u^* - \frac{1}{j}, \quad |\nabla u(p_j)| < \frac{1}{j} \quad \text{and} \quad \mathcal{L}u(p_j) < \frac{1}{j}$$

for every $j \in \mathbb{N}$. Equivalently, for any smooth function $u \in C^2(\Sigma)$ with $u_* = \inf u > -\infty$ there exists a sequence of points $\{p_j\} \subset \Sigma^n$ satisfying

$$u(p_j) < u_* + \frac{1}{j}, \quad |\nabla u(p_j)| < \frac{1}{j} \quad \text{and} \quad \mathcal{L}u(p_j) > -\frac{1}{j}$$

for every $j \in \mathbb{N}$.

Now we are ready to state and prove our next half-space theorem.

Theorem 7.2.7. *Let $\mathbb{R} \times_\rho M^n$ be a Riemannian warped product satisfying the convergence condition in (7.2) and with non-decreasing warping function. Let $\psi : \Sigma^n \to \mathbb{R} \times_\rho M^n$ be a complete noncompact (r, s)-linear Weingarten two-sided hypersurface with positive $(s+1)$-mean curvature, for some $1 \leq r \leq s \leq n-1$, and $d > \sum_{k=r}^{s} b_k \sup[(\log \rho)']^{k+1}$. Suppose that $\sup \frac{\rho''(h)}{\rho(h)} < +\infty$ and, if $s \geq 2$, there exists an elliptic point in Σ^n. Assume further that $\sup |H_r| < +\infty$ and the second fundamental form satisfies $|A| \leq G(r_o)$, where $G \in C^1([0, +\infty))$ is such that*

$$G(0) > 0, \quad G'(t) \geq 0 \quad \text{and} \quad \frac{1}{G(t)} \notin L^1(+\infty)$$

and r_o is the distance function from a reference point in Σ^n. Then, either $\sup \Theta > 0$ or Σ^n cannot lie in an upper half-space.

Proof. We begin by stating that the sectional curvature K_Σ of Σ satisfies the assumption (A.1) of Lemma A.0.3. Indeed, denoting by $\overline{K}$ the sectional curvature of the ambient space, it follows from Gauss equation (6.9) that if $\{X, Y\}$ is an orthonormal basis for an arbitrary plane tangent to Σ^n, then

$$\begin{aligned} K_\Sigma(X,Y) &= \overline{K}(X,Y) + \langle AX, X\rangle\langle AY, Y\rangle - \langle AX, Y\rangle^2 \\ &\geq \overline{K}(X,Y) - |AX||AY| - |AX|^2 \\ &\geq \overline{K}(X,Y) - 2|A|^2, \end{aligned} \tag{7.19}$$

where the last inequality follows from the fact that

$$|AX|^2 \leq \mathrm{tr}(A^2)|X|^2 = |A|^2$$

for every unitary vector X tangent to Σ^n. On the other hand, Lemma 6.0.2 gives

$$\begin{aligned} \overline{R}(U,V)W &= R_M(U^*, V^*)W^* - [(\log\rho)']^2(\langle U, W\rangle V - \langle V, W\rangle U) \\ &- (\log\rho)''\langle W, \partial_t\rangle(\langle U, \partial_t\rangle V - \langle V, \partial_t\rangle U) \\ &- (\log\rho)''(\langle U, W\rangle\langle V, \partial_t\rangle - \langle V, W\rangle\langle U, \partial_t\rangle)\partial_t, \end{aligned}$$

for every vector U, V, W tangent to $\mathbb{R}\times_\rho M^n$, where U^* denotes the orthogonal projection of U on TM. In particular, for the base $\{X, Y\}$ we get

$$\begin{aligned} \overline{K}(X,Y) &= \langle \overline{R}(X,Y)X, Y\rangle \\ &= \frac{1}{\rho^2(h)}K_M(X^*, Y^*)|X^*\wedge Y^*|^2 - (\log\rho)'(h)^2 \\ &- (\log\rho)''(h)(\langle X, \partial_t\rangle^2 + \langle Y, \partial_t\rangle^2). \end{aligned}$$

By using the convergence condition in (7.2) and taking into account that $|X^*\wedge Y^*|^2 = 1 - \langle X, \partial_t\rangle^2 - \langle Y, \partial_t\rangle^2$ we obtain

$$\overline{K}(X,Y) \geq -(\log\rho)''(h) - (\log\rho)'(h)^2 = -\frac{\rho''(h)}{\rho(h)}.$$

Hence, since the second fundamental form satisfies $|A| \leq G(r_o)$, we infer from equation (7.19) that

$$K_\Sigma \geq -\sup\frac{\rho''(h)}{\rho(h)} - 2G^2(r_o), \tag{7.20}$$

which concludes the claim.

From now on, we assume that the angle function Θ is nonpositive and argue by contradiction, that is, we suppose that Σ^n lies in an upper half-space. Equivalently, the height function of Σ^n satisfies $h_* = \inf h > -\infty$.

Following [17], for each $k = r, \ldots, s$, let $\mathcal{L}_k^+ : C^\infty(\Sigma) \to C^\infty(\Sigma)$ be the second order linear differential operator given by

$$\begin{aligned}\mathcal{L}_k^+ &= \sum_{i=0}^{k}(-1)^i\frac{c_k}{c_i}(\log\rho)'(h)^{k-i}\Theta^i L_i \\ &= \operatorname{tr}(\mathcal{P}_k^+\circ\operatorname{hess}),\end{aligned} \tag{7.21}$$

where $\mathcal{P}_k^+ : \mathfrak{X}(\Sigma)\to\mathfrak{X}(\Sigma)$ is defined by

$$\mathcal{P}_k^+ = \sum_{i=0}^{k}(-1)^i\frac{c_k}{c_i}(\log\rho)'(h)^{k-i}\Theta^i P_i. \tag{7.22}$$

In particular, as showed in Section 6 of [17], we have the following equality

$$\mathcal{L}_k^+\sigma(h) = c_k\rho(h)\left((\log\rho)'(h)^{k+1} + (-1)^k\Theta^{k+1}H_{k+1}\right), \tag{7.23}$$

where $\sigma(t)$ denotes a primitive of the warping function. We also note that, by Lemma 6.0.5 and since the angle function was supposed to be nonpositive, it follows that $\mathcal{P}_k^+$ is a positive semi-definite symmetric tensor for every $k = r,\dots,s$. Besides, since $d > b_k[\sup(\log\rho)']^{k+1}$ for every $k = r,\dots,s$, and equation (6.11) assures that $\operatorname{tr}(P_i) = c_iH_i$, by Lemma 6.0.3 we get

$$\operatorname{tr}(\mathcal{P}_k^+) \le c_k\sum_{i=0}^{k}\left(\frac{d}{b_k}\right)^{(k-i)/(k+1)}H_r^{i/r},$$

which implies that $\sup\operatorname{tr}(\mathcal{P}_k^+) < +\infty$ for every $k = r,\dots,s$, because of the assumption on H_r.

Now we set the following second order linear differential operator $\mathcal{L}^+ : C^\infty(\Sigma)\to C^\infty(\Sigma)$ by

$$\mathcal{L}^+ = \sum_{k=r}^{s} b_kc_k^{-1}\mathcal{L}_k^+ = \operatorname{tr}(\mathcal{P}^+\circ\operatorname{hess}),$$

where $\mathcal{P}^+ : \mathfrak{X}(\Sigma)\to\mathfrak{X}(\Sigma)$ is given by

$$\mathcal{P}^+ = \sum_{k=r}^{s} b_kc_k^{-1}\mathcal{P}_k^+.$$

Then $\mathcal{P}^+$ is a positive semi-definite symmetric tensor with $\sup\operatorname{tr}(\mathcal{P}^+) < \infty$. In particular, $\mathcal{L}^+$ is a semi-elliptic operator. So, we are ready to apply Lemma A.0.3 to guarantee that the Omori-Yau's maximum principle holds on Σ^n for the operator $\mathcal{L}^+$.

Besides it is clear that $\sigma(t)$ satisfies $\sigma(h) \ge \sigma(h_*) > -\infty$. Hence there exists a sequence of points $\{p_j\}\subset\Sigma^n$ with the following properties

$$\lim\sigma(h)(p_j) = \sigma(h_*), \quad |\nabla\sigma(h)(p_j)| = \rho(h(p_j))|\nabla h(p_j)| < \frac{1}{j} \quad\text{and}\quad \mathcal{L}^+\sigma(h)(p_j) > -\frac{1}{j}.$$

In particular, equation (7.23) yields

$$-\frac{1}{j} < \mathcal{L}^{+}\sigma(h)(p_j) = \sum_{k=r}^{s} \rho(h(p_j))(b_k(\log\rho)'(h(p_j))^{k+1} + (-1)^k\Theta(p_j)^{k+1}b_k H_{k+1}(p_j)).$$

Letting $j \to +\infty$ we must have $\Theta(p_j) \to -1$, because $|\nabla h|^2 = 1 - \Theta^2$, which gives

$$d \leq \sum_{k=r}^{s} b_k(\log\rho)'(h_*)^{k+1} \leq \sum_{k=r}^{s} b_k \sup[(\log\rho)']^{k+1},$$

leading to a contradiction. This finishes the proof of the theorem. $\square$

Let us observe that the proof of Theorem 7.2.7 remains true with the stronger assumption that K_Σ is bounded from below by a constant, which implies the validity of the Omori-Yau's maximum principle. For instance, if we assume that $\sup \frac{\rho''(h)}{\rho(h)} < +\infty$, reasoning as in the proof of Theorem 7.2.7, we see that K_Σ is bounded from below since $\sup|A|^2 < +\infty$. On the other hand, the hypothesis on H_r in Theorem 7.2.7, $\sup|H_r| < \infty$, can be replaced by $\sup H_1 < \infty$, because of Lemma 6.0.3. In this case, taking into account the relation

$$|A|^2 = n^2H_1^2 - n(n-1)H_2,$$

it follows that the condition $\sup|A|^2 < +\infty$ is equivalent to $\sup H_1 < +\infty$. This proves the following result:

Corollary 7.2.8. *Let $\mathbb{R} \times_\rho M^n$ be a Riemannian warped product satisfying the convergence condition in* (7.2) *and with non-decreasing warping function. Let $\psi : \Sigma^n \to \mathbb{R} \times_\rho M^n$ be a complete noncompact (r,s)-linear Weingarten two-sided hypersurface with positive $(s+1)$-mean curvature, for some $0 \leq r \leq s \leq n-1$, and $d > \sum_{k=r}^{s} b_k \sup[(\log\rho)']^{k+1}$. Suppose that $\sup|H_1| < +\infty$, $\sup \frac{\rho''(h)}{\rho(h)} < +\infty$ and, if $s \geq 2$, there exists an elliptic point in Σ^n. Then, either $\sup\Theta > 0$ or Σ^n cannot lie in an upper half-space.*

In the case of hypersurfaces having constant mean curvature the assumption that the warping function is non-decreasing can be dropped as follows.

Corollary 7.2.9. *Let $\mathbb{R} \times_\rho M^n$ be a Riemannian warped product satisfying the convergence condition in* (7.2)*. Let $\psi : \Sigma^n \to \mathbb{R} \times_\rho M^n$ be a complete noncompact two-sided hypersurface with constant mean curvature satisfying $H_1 > \sup(\log\rho)'$. Suppose that $\inf H_2 > -\infty$ and $\sup \frac{\rho''(h)}{\rho(h)} < +\infty$. Then Σ^n cannot lie in an upper half-space.*

Proof. Let us reason by contradiction that Σ^n lies in an upper half-space, that is, $\inf h = h_* > -\infty$. As in the proof of Theorem 7.2.7 and by remark above, we might see that the Omori-Yau's maximum principle holds on Σ^n for the Laplacian. Then there is a sequence of points $\{p_j\} \subset \Sigma^n$ satisfying

$$\lim h(p_j) = h_*, \quad |\nabla h(p_j)| < \frac{1}{j} \quad \text{and} \quad \Delta h(p_j) > -\frac{1}{j}.$$

By applying Proposition 6.0.7 we find

$$-\frac{1}{j} < \Delta h(p_j) = (\log\rho)'(h(p_j))\left(n - |\nabla h(p_j)|^2\right) + n\Theta(p_j)H_1$$

Since the angle function is bounded, taking limits here and choosing the orientation so that $H_1 > 0$, we conclude that

$$H_1 \leq \sup(\log\rho)',$$

which gives a contradiction. □

For instance, when the warping function is $\rho(t) = e^t$ we get.

Corollary 7.2.10. *Let $\mathbb{R} \times_{e^t} M^n$ be a pseudo-hyperbolic space whose fiber has nonnegative sectional curvature. Let $\psi : \Sigma^n \to \mathbb{R} \times_{e^t} M^n$ be a complete noncompact two-sided hypersurface with constant mean curvature satisfying $H_1 > 1$. Suppose that* $\inf H_2 > -\infty$*. Then Σ^n cannot lie in an upper half-space.*

More generally, for hypersurfaces having some constant higher order mean curvature we get the following result:

Corollary 7.2.11. *Let $\mathbb{R} \times_\rho M^n$ be a Riemannian warped product satisfying the convergence condition in* (7.2) *and with non-decreasing warping function. Let $\psi : \Sigma^n \to \mathbb{R} \times_\rho M^n$ be a complete noncompact two-sided hypersurface with constant $(s+1)$-mean curvature satisfying $H_{s+1} > \sup[(\log\rho)']^{s+1}$ for some $1 \leq s \leq n-1$. Suppose that* $\sup|H_1| < +\infty$, $\sup\frac{\rho''(h)}{\rho(h)} < +\infty$ *and, if $s \geq 2$, there exists an elliptic point in Σ^n. Then, either* $\sup\Theta > 0$ *or Σ^n cannot lie in an upper half-space.*

In particular, in the pseudo-hyperbolic space $\mathbb{R} \times_{e^t} M^n$ we obtain:

Corollary 7.2.12. *Let $\mathbb{R} \times_{e^t} M^n$ be a pseudo-hyperbolic space whose fiber has nonnegative sectional curvature. Let $\psi : \Sigma^n \to \mathbb{R} \times_{e^t} M^n$ be a complete noncompact two-sided hypersurface with constant $(s+1)$-mean curvature satisfying $H_{s+1} > 1$, for some $1 \leq s \leq n-1$. Suppose that* $\sup|H_1| < +\infty$ *and, if $s \geq 2$, there exists an elliptic point in Σ^n. Then, either* $\sup\Theta > 0$ *or Σ^n cannot lie in an upper half-space.*

We proceed by proving the version of Theorem 7.2.7 in the case in which the warping function is non-increasing.

Theorem 7.2.13. *Let $\mathbb{R} \times_\rho M^n$ be a Riemannian warped product satisfying the convergence condition in* (7.2) *and with non-increasing warping function. Let $\psi : \Sigma^n \to \mathbb{R}\times_\rho M^n$ be a complete noncompact (r,s)-linear Weingarten two-sided hypersurface with positive $(s+1)$-mean curvature, for some $1 \leq r \leq s \leq n-1$, and $d > \sum_{k=r}^{s} b_k \sup[-(\log\rho)']^{k+1}$. Suppose that* $\sup\frac{\rho''(h)}{\rho(h)} < +\infty$ *and, if $s \geq 2$, there exists an elliptic point in Σ^n. Assume further that* $\sup|H_r| < +\infty$ *and the second fundamental form satisfies $|A| \leq G(r_o)$, where $G \in C^1([0,+\infty))$ is such that*

$$G(0) > 0, \quad G'(t) \geq 0 \quad \text{and} \quad \frac{1}{G(t)} \notin L^1(+\infty)$$

and r_o is the distance function from a reference point in Σ^n. Then, either $\inf \Theta < 0$ *or* Σ^n *cannot lie in a lower half-space.*

Proof. Again, we reason by contradiction as follows: we suppose that Θ is a nonnegative function and Σ^n lies in a lower half-space, that is, the height function of Σ^n satisfies $h^* = \sup h < +\infty$.

We consider, for each $k = r, \dots, s$, the second order linear differential operator $\mathcal{L}_k^- : C^\infty(\Sigma) \to C^\infty(\Sigma)$ defined as

$$\begin{aligned} \mathcal{L}_k^- &= \sum_{i=0}^{k} (-1)^{k-i} \frac{c_k}{c_i} (\log \rho)'(h)^{k-i} \Theta^i L_i \\ &= \operatorname{tr}\left(\mathcal{P}_k^- \circ \operatorname{hess}\right), \end{aligned} \tag{7.24}$$

where $\mathcal{P}_k^- : \mathfrak{X}(\Sigma) \to \mathfrak{X}(\Sigma)$ is the positive semi-definite symmetric tensor given by

$$\mathcal{P}_k^- = \sum_{i=0}^{k} (-1)^{k-i} \frac{c_k}{c_i} (\log \rho)'(h)^{k-i} \Theta^i P_i. \tag{7.25}$$

As in Theorem 7.2.7, the sectional curvature K_Σ of Σ^n satisfies condition (7.20) and $\operatorname{tr}(\mathcal{P}_k^-) < +\infty$. Hence, by Lemma A.0.3 the Omori-Yau's maximum principle holds on Σ^n for the semi-elliptic second order linear differential operator $\mathcal{L}^- : C^\infty(\Sigma) \to C^\infty(\Sigma)$ given by

$$\mathcal{L}^- = \sum_{k=r}^{s} b_k c_k^{-1} \mathcal{L}_k^- = \operatorname{tr}(\mathcal{P}^- \circ \operatorname{hess}),$$

where $\mathcal{P}^- : \mathfrak{X}(\Sigma) \to \mathfrak{X}(\Sigma)$, defined as

$$\mathcal{P}^- = \sum_{k=r}^{s} b_k c_k^{-1} \mathcal{P}_k^-,$$

is a positive semi-definite symmetric tensor and satisfies $\operatorname{tr}(\mathcal{P}^-) < +\infty$.

Now let $\sigma(t)$ be a primitive of the warping function, which must satisfies $\sigma(h) \le \sigma(h^*)$. Then there exists a sequence of points $\{q_j\} \subset \Sigma^n$ with the following properties:

$$\lim \sigma(h)(q_j) = \sigma(h^*), \quad |\nabla \sigma(h)(q_j)| < \frac{1}{j} \quad \text{and} \quad \mathcal{L}^- \sigma(h)(q_j) < \frac{1}{j}.$$

Taking into account that $\mathcal{L}_k^- = (-1)^k \mathcal{L}_k^+$, where $\mathcal{L}_k^+$ is defined in the proof of Theorem 7.2.7, jointly with (7.23), we find

$$\frac{1}{j} > \mathcal{L}^- \sigma(h)(q_j) = \sum_{k=r}^{s} \rho(h(q_j)) \left(-b_k[-(\log \rho)'(h(q_j))]^{k+1} + \Theta(q_j)^{k+1} b_k H_{k+1}(q_j)\right).$$

Therefore, making $j \to \infty$ we see that $\Theta(q_j) \to 1$ implying that

$$d \leq \sum_{k=r}^{s} b_k \sup[-(\log \rho)']^{k+1}.$$

This characterizes a contradiction and proves the result. $\square$

As in Theorem 7.2.7, Theorem 7.2.13 remains true if we replace the conditions $|A| \leq G(r_0)$ and $\sup |H_r| < \infty$ by the stronger condition on the mean curvature H_1, namely: $\sup H_1 < \infty$. More precisely,

Corollary 7.2.14. *Let $\mathbb{R} \times_\rho M^n$ be a Riemannian warped product satisfying the convergence condition in* (7.2) *and with non-increasing warping function. Let $\psi : \Sigma^n \to \mathbb{R} \times_\rho M^n$ be a complete noncompact (r,s)-linear Weingarten two-sided hypersurface with positive $(s+1)$-mean curvature, for some $1 \leq r \leq s \leq n-1$, and $d > \sum_{k=r}^{s} b_k \sup[-(\log \rho)']^{k+1}$. Suppose that* $\sup |H_1| < \infty$, $\sup \frac{\rho''(h)}{\rho(h)} < +\infty$ *and, if $s \geq 2$, there exists an elliptic point in Σ^n. Then, either* $\inf \Theta < 0$ *or Σ^n cannot lie in a lower half-space.*

As other immediate consequence we get the following result for hypersurfaces with constant mean curvature, where the assumption that the warping function is non-increasing can be dropped.

Corollary 7.2.15. *Let $\mathbb{R} \times_\rho M^n$ be a Riemannian warped product satisfying the convergence condition in* (7.2)*. Let $\psi : \Sigma^n \to \mathbb{R} \times_\rho M^n$ be a complete noncompact two-sided hypersurface with constant mean curvature satisfying $H_1 > \sup -(\log \rho)'$. Suppose that* $\inf H_2 > -\infty$ *and* $\sup \frac{\rho''(h)}{\rho(h)} < +\infty$. *Then Σ^n cannot lie in a lower half-space.*

In the case of hypersurfaces having some constant higher order mean curvature Theorem 7.2.13 becomes:

Corollary 7.2.16. *Let $\mathbb{R} \times_\rho M^n$ be a Riemannian warped product satisfying the convergence condition in* (7.2) *and with non-increasing warping function. Let $\psi : \Sigma^n \to \mathbb{R} \times_\rho M^n$ be a complete noncompact two-sided hypersurface with constant $(s+1)$-mean curvature satisfying $H_{s+1} > \sup[-(\log \rho)']^{s+1}$, for some $1 \leq r \leq n-1$. Suppose that* $\sup |H_1| < +\infty$, $\sup \frac{\rho''(h)}{\rho(h)} < +\infty$ *and, if $s \geq 2$, there exists an elliptic point in Σ^n. Then, either* $\inf \Theta < 0$ *or Σ^n cannot lie in a lower half-space.*

Finally, we close this section by stating the following result in the case of product spaces $\mathbb{R} \times M^n$, which is a consequence of Theorems 7.2.7 and 7.2.13. It is worth pointing out that in this case our conclusion is stronger than that one of Theorems 7.2.7 and 7.2.13.

Theorem 7.2.17. *Let $\mathbb{R} \times M^n$ be a product space whose fiber has sectional curvature satisfying $K_M \geq -\alpha$, for some positive constant $\alpha \in \mathbb{R}$. Let $\psi : \Sigma^n \to \mathbb{R} \times M^n$ be a complete noncompact (r,s)-linear Weingarten two-sided hypersurface with positive $(s+1)$-mean curvature, for some $1 \leq r \leq s \leq n-1$. Suppose that* $\sup |H_r| < +\infty$ *and, if $s \geq 2$, there exists an elliptic point in Σ^n.*

Assume further that the second fundamental form satisfies $|A| \leq G(r_o)$*, where* $G \in C^1([0, +\infty))$ *is such that*

$$G(0) > 0, \quad G'(t) \geq 0 \quad \text{and} \quad \frac{1}{G(t)} \notin L^1(+\infty)$$

and r_o *is the distance function from a reference point in* Σ^n*. The following holds:*

(a) Either $\sup \Theta > 0$ *or* Σ^n *cannot lie in an upper half-space;*

(b) Either $\inf \Theta < 0$ *or* Σ^n *cannot lie in a lower half-space.*

In particular, when the hypersurface has constant mean curvature or some higher order mean curvature, Theorem 7.2.17 enables us to draw the following conclusion:

Corollary 7.2.18. *Let* $\mathbb{R} \times M^n$ *be a product space whose fiber has sectional curvature satisfying* $K_M \geq -\alpha$*, for some positive constant* $\alpha \in \mathbb{R}$*. Let* $\psi : \Sigma^n \to \mathbb{R} \times M^n$ *be a complete noncompact two-sided hypersurface with positive constant mean curvature and such that* $\inf H_2 > -\infty$*. The following holds.*

(a) Either $\sup \Theta > 0$ *or* Σ^n *cannot lie in an upper half-space;*

(b) Either $\inf \Theta < 0$ *or* Σ^n *cannot lie in a lower half-space.*

In other words we have:

(a') There is no complete noncompact two-sided hypersurface having positive constant mean curvature, $\inf H_2 > -\infty$*, angle function nonpositive and contained into an upper half-space;*

(b') There is no complete noncompact two-sided hypersurface having positive constant mean curvature, $\inf H_2 > -\infty$*, angle function nonnegative and contained into a lower half-space.*

Corollary 7.2.19. *Let* $\mathbb{R} \times M^n$ *be a product space whose fiber has sectional curvature satisfying* $K_M \geq -\alpha$*, for some positive constant* $\alpha \in \mathbb{R}$*. Let* $\psi : \Sigma^n \to \mathbb{R} \times M^n$ *be a complete noncompact two-sided hypersurface with positive constant* $(s+1)$*-mean curvature, for some* $1 \leq s \leq n-1$*. Suppose that* $\sup |H_1| < +\infty$ *and, if* $s \geq 2$*, there exists an elliptic point in* Σ^n*. The following holds.*

(a) Either $\sup \Theta > 0$ *or* Σ^n *cannot lie in an upper half-space;*

(b) Either $\inf \Theta < 0$ *or* Σ^n *cannot lie in a lower half-space.*

In other words we have:

(a') There is no complete noncompact two-sided hypersurface having $H_{s+1} > 0$*, an elliptic point, with* $\sup |H_1| < +\infty$*, angle function nonpositive and contained into an upper half-space;*

(b') There is no complete noncompact two-sided hypersurface having $H_{s+1} > 0$*, an elliptic point, with* $\sup |H_1| < +\infty$*, angle function nonnegative and contained into a lower half-space.*

Finally we collect (a) and (b) in the previous corollaries in order to obtain the following result.

Corollary 7.2.20. *Let* $\mathbb{R} \times M^n$ *be a product space whose fiber has sectional curvature satisfying* $K_M \geq -\alpha$*, for some positive constant* $\alpha \in \mathbb{R}$*. Let* $\psi : \Sigma^n \to \mathbb{R} \times M^n$ *be a complete noncompact two-sided hypersurface with positive constant* $(s+1)$*-mean curvature, for some* $0 \leq s \leq n-1$*. In addition, if* $s = 0$ *assume that* $\inf H_2 > -\infty$*. Suppose further that* $\sup |H_1| < +\infty$ *and, if* $s \geq 2$*, there exists an elliptic point in* Σ^n*. Then, either* Θ *does not vanish identically or* Σ^n *cannot lie in a half-space.*

Chapter 8

The Lorentzian case

This chapter is dedicated to provide height estimates and half-space theorems of generalized linear Weingarten spacelike hypersurfaces in generalized Robertson-Walker (GRW, for short) spacetimes $-\mathbb{R} \times_\rho M^n$, which also enable us to obtain information about the topology at infinity of these hypersurfaces. In this chapter we include the results of [21, 49].

8.1 Height estimates

The goal of this section is to give height estimates of compact generalized linear Weingarten spacelike hypersurfaces immersed into a GRW spacetime $-\mathbb{R} \times_\rho M^n$. To do this, in general way, we follows the techniques used in the Riemannian setting in Section 7.1. However, as we shall see, in this case our estimates are considerably different of those obtained in the Riemannian case. For this and for the sake of completeness, we would like to present the proofs of our results in details.

In this setting, we start by establishing that, under a suitable assumption on a linear combination involving some of the higher order mean curvatures (not necessarily constant), any compact spacelike hypersurface immersed into a GRW spacetime $-\mathbb{R} \times_\rho M^n$ with non-empty boundary contained into a slice must lie entirely in one of the two regions of the spacetime bounded by the slice. We point out that, when the warping function is increasing and the Gauss map N is future-pointing, this was proved in [64] (see Proposition 14 of [64]) considering only one of the higher order mean curvatures. We also observe that in [64] the authors considered the case in which the warping function is decreasing. However, such situation do not have any application in your results. Here we also consider the case in which the warping function is decreasing jointly with the assumption that the Gauss map N is past-pointing and we are able to obtain new estimates for the height function of these hypersurfaces (see Theorem 8.1.5 below). For that reason we give here a proof of it.

Proposition 8.1.1. *Let $\psi : \Sigma^n \to -\mathbb{R} \times_\rho M^n$ be a compact spacelike hypersurface with positive $(s+1)$-mean curvature, for some $0 \le s \le n-1$, and boundary $\partial\Sigma$ contained into the slice $\{t_0\} \times M^n$ for some $t_0 \in \mathbb{R}$. The following holds:*

(a) If $\rho'(h) > 0$ and, for some $0 \leq r \leq s$, we have

$$\sum_{k=r}^{s} b_k H_{k+1} \geq \sum_{k=r}^{s} b_k \sup[(\log \rho)']^{k+1}$$

for certain nonnegative constants b_k and, when $s \geq 2$, there exists an elliptic point in Σ^n with respect the future-pointing Gauss map, then $h \leq t_0$;

(b) If $\rho'(h) < 0$ and, for some $0 \leq r \leq s$, we have

$$\sum_{k=r}^{s} b_k H_{k+1} \geq \sum_{k=r}^{s} b_k \sup[-(\log \rho)']^{k+1}$$

for certain nonnegative constants b_k and, when $s \geq 2$, there exists an elliptic point in Σ^n with respect the past-pointing Gauss map, then $h \geq t_0$.

Proof. Let us prove part (a). As in the proof of Proposition 7.1.3 we can show that

$$H_{r+1}^{1/(r+1)}(p) \geq \sup(\log \rho)', \ \forall p \in \Sigma^n.$$

Indeed, if $r = s$ there is nothing to prove. Otherwise, assuming by contradiction that there is a point $p \in \Sigma^n$ such that $H_{r+1}(p)^{1/(r+1)} < \sup(\log \rho)'$, it follows from our hypothesis that must there exists $r < i \leq s$ with $H_{i+1}(p)^{1/(i+1)} > \sup(\log \rho)'$, which gives $H_{r+1}(p)^{1/(r+1)} < H_{i+1}(p)^{1/(i+1)}$ leading to a contradiction.

From now on, we follow the ideas of Proposition 14 of [64]. Let $\mathcal{L}_r^+ : C^\infty(\Sigma) \to C^\infty(\Sigma)$ be the operator defined in (7.21),

$$\begin{aligned} \mathcal{L}_r^+ &= \sum_{i=0}^{r} (-1)^i \frac{c_r}{c_i} (\log \rho)'(h)^{r-i} \Theta^i L_i \\ &= \operatorname{tr}\left(\mathcal{P}_r^+ \circ \operatorname{hess}\right), \end{aligned}$$

where the tensor $\mathcal{P}_r^+ : \mathfrak{X}(\Sigma) \to \mathfrak{X}(\Sigma)$ is given by equation (7.22),

$$\mathcal{P}_r^+ = \sum_{i=0}^{r} (-1)^i \frac{c_r}{c_i} (\log \rho)'(h)^{r-i} \Theta^i P_i.$$

Taking into account that in this case the angle function satisfies $\Theta \leq -1$ jointly with Lemmas 6.0.4 and 6.0.5, we infer that $\mathcal{L}_r^+$ is an elliptic operator. Besides, equation (3.4) of [16] yields

$$\mathcal{L}_r^+ \sigma(h) = c_r \rho(h) \left(-[(\log \rho)'(h)]^{r+1} + (-\Theta)^{r+1} H_{r+1}\right), \tag{8.1}$$

where $\sigma(t)$ is a primitive of the warping function $\rho(t)$. Hence, by claim proved above we have

$$\mathcal{L}_r^+ \sigma(h) \geq c_r \rho(h) H_{r+1} \left(-1 + (-\Theta)^{r+1}\right) \geq 0.$$

Then, by the weak maximum principle, $\sigma(h)$ must attain its maximum on $\partial\Sigma$, in others words, $\sigma(h) \leq \sigma(t_0)$. Since σ is an increasing function, this implies that $h \leq t_0$, which proves part (a).

Now we prove part (b). To this end, we observe that the analogous applies here to obtain that $H_{r+1}^{1/(r+1)}(p) \geq -\sup(\log\rho)'$, $\forall p \in \Sigma^n$. Next, let us consider the second order linear differential operator defined in (7.24),

$$\begin{aligned} \mathcal{L}_r^- &= \sum_{i=0}^{r}(-1)^{r-i}\frac{c_r}{c_i}(\log\rho)'(h)^{r-i}\Theta^i L_i \\ &= \operatorname{tr}\left(\mathcal{P}_r^- \circ \operatorname{hess}\right), \end{aligned}$$

where the tensor $\mathcal{P}_r^- : \mathfrak{X}(\Sigma) \to \mathfrak{X}(\Sigma)$ is defined as in (7.25),

$$\mathcal{P}_r^- = \sum_{i=0}^{r}(-1)^{r-i}\frac{c_r}{c_i}(\log\rho)'(h)^{r-i}\Theta^i P_i.$$

Since $\rho' < 0$, $\Theta \geq 1$ and the operator L_i is elliptic for every $i = 0, \ldots, r$, then $\mathcal{L}_r^-$ must be elliptic too. Moreover, as already observed, we have that $\mathcal{L}_r^- = (-1)^r\mathcal{L}_r^+$. So, by using once more equation (3.4) of [16] we find

$$\mathcal{L}_r^-\sigma(h) = c_r\rho(h)\left([-(\log\rho)'(h)]^{r+1} - \Theta^{r+1}H_{r+1}\right). \tag{8.2}$$

It follows from here that

$$\mathcal{L}_r^-\sigma(h) \leq c_r\rho(h)H_{r+1}\left(1 - \Theta^{r+1}\right) \leq 0.$$

Finally, using again the weak maximum principle, we conclude that $h \geq t_0$, as desired. $\square$

Remark 8.1.2. Regarding the condition on Σ^n of having an elliptic point when either the warping function is increasing and the Gauss map is future-pointing or the warping function is decreasing and the Gauss map is past-pointing in Proposition 8.1.1, it is a natural condition. For instance, Alías and Colares [8] proved that if the GRW spacetime is spatially closed, that is, the Riemannian fact is compact, then any compact spacelike hypersurface immersed into such a spacetime admit an elliptic point in these conditions on the warping function and the Gauss map N of the hypersurface (see Lemma 5.3 of [8]). In this context, this assumption seems very natural.

Following the terminology introduced in [8], we recall that a GRW spacetime $-I \times_\rho M^n$ satisfies the strong null convergence condition (strong NCC, for short) if the sectional curvature K_M of the fiber M^n satisfies

$$K_M \geq \sup\{\rho\rho'' - (\rho')^2\}.$$

Having this in mind, we are ready to state and prove our next result regarding estimate of the height function of compact (r, s)-linear Weingarten spacelike hypersurfaces in a GRW spacetime.

Theorem 8.1.3. *Let $-\mathbb{R}\times_\rho M^n$ be a GRW spacetime satisfying the strong NCC and with increasing warping function. Let $\psi : \Sigma^n \to -\mathbb{R}\times_\rho M^n$ be a compact (r,s)-linear Weingarten spacelike hypersurface with positive $(s+1)$-mean curvature, boundary $\partial\Sigma^n$ contained into the slice $\{t_0\}\times M^n$, for some $t_0 \in \mathbb{R}$, and $d \geq \sum_{k=r}^{s} b_k \sup[(\log\rho)']^{k+1}$. Suppose that $H_1 \geq \sup|H_{r+1}|^{1/(r+1)}$ and, if $s \geq 2$, there exists an elliptic point in Σ^n with respect the future-pointing Gauss map. Then,*

$$\Sigma^n \subset [t_0-\alpha, t_0]\times M^n,$$

where

$$\alpha = \frac{\frac{\rho(t_0)}{\rho(\min h)}\max_{\partial\Sigma}(-\Theta)-1}{\min H_1}.$$

Proof. We closely follow the proof of Theorem 7.1.5. Let us consider on Σ^n the smooth function $\varphi = c\sigma(h)+\tilde{\Theta}$, where $c\in\mathbb{R}$ is a positive constant to be chosen in an appropriate way, $\sigma(t)$ is a primitive of $\rho(t)$ and $\tilde{\Theta}=\rho\Theta$. By Proposition 6.0.7 we have

$$\begin{aligned} L_k\varphi &= \frac{c_k\rho(h)}{k+1}\langle\nabla H_{k+1},\nabla h\rangle + c_k\rho'(h)\left(H_{k+1}-cH_k\right)\\ &+ \rho(h)\Theta\binom{n}{k+1}\left(nH_1H_{k+1}-(n-k-1)H_{k+2}-(k+1)cH_{k+1}\right)\\ &+ \frac{\tilde{\Theta}}{\rho^2(h)}\sum_{i=1}^{n}\mu_{i,k}K_M(N^*,E_i^*)|N^*\wedge E_i^*|^2\\ &- \tilde{\Theta}(\log\rho)''(h)\left(c_k|\nabla h|^2H_k-\langle P_k\nabla h,\nabla h\rangle\right), \end{aligned} \tag{8.3}$$

where $P_kE_i=\mu_{i,k}E_i$, for every $i=1,\dots,n$ and $k=r,\dots,s$.

Since H_{s+1} is positive and Σ^n has an elliptic point, Lemma 6.0.3 gives $H_1H_{k+1}\geq H_{k+2}$, which implies

$$\begin{aligned} nH_1H_{k+1}-(n-k-1)H_{k+2}-(k+1)cH_{k+1} &= (k+1)H_{k+1}(H_1-c)\\ &+ (n-k-1)(H_1H_{k+1}-H_{k+2})\\ &\geq (k+1)H_{k+1}(H_1-c)\geq 0, \end{aligned} \tag{8.4}$$

provided that $c:=\min H_1$. In particular, with this choose of c, it follows from our hypothesis on H_1 and Lemma 6.0.3 that

$$H_{k+1}-cH_k \leq H_{k+1}^{1/(k+1)}\left(H_{k+1}^{k/(k+1)}-H_k\right)\leq 0. \tag{8.5}$$

On the other hand, by our assumptions we can apply Lemma 6.0.5 (or Lemma 6.0.4 if $s=1$) to obtain the ellipticity of the operator L_k for every $k=r,\dots,s$, in others words, P_k is positive definite. In particular, its eigenvalues $\mu_{i,k}$ are all positive on Σ^n, and from the strong NCC we get

$$\mu_{i,k}K_M(N^*,E_i^*)|N^*\wedge E_i^*|^2 \geq \mu_{i,k}C|N^*\wedge E_i^*|^2, \tag{8.6}$$

for every $i=1,\dots,n$ and $k=r,\dots,s$, where we are writing $C=\sup\{\rho\rho''-(\rho')^2\}$. With a straightforward computation we find

$$|N^* \wedge E_i^*|^2 = |N^*|^2|E_i^*|^2 - \langle N^*, E_i^*\rangle^2 = |\nabla h|^2 - \langle E_i, \nabla h\rangle^2,$$

which jointly with (8.6) imply

$$\begin{aligned}\sum_{i=1}^n \mu_{i,k} K_M(N^*, E_i^*)|N^* \wedge E_i^*|^2 &\geq C\left(\operatorname{tr}(P_k)|\nabla h|^2 - \sum_{i=1}^n \mu_{i,k}\langle E_i, \nabla h\rangle^2\right)\\ &= C\left(\operatorname{tr}(P_k)|\nabla h|^2 - \langle P_k\nabla h, \nabla h\rangle\right).\end{aligned}$$

Then, since $\operatorname{tr}(P_k) = c_k H_k$ and $C/\rho^2(h) - (\log\rho)''(h) \geq 0$, we obtain that

$$\frac{1}{\rho^2(h)}\sum_{i=1}^n \mu_{i,k} K_M(N^*, E_i^*)|N^* \wedge E_i^*|^2 - (\log\rho)''(h)\left(c_k|\nabla h|^2 H_k - \langle P_k\nabla h, \nabla h\rangle\right) \geq 0, \tag{8.7}$$

where the last inequality follows from the fact that P_k is positive definite. Hence putting (8.4), (8.5) and (8.7) into (8.3) and taking into account that the warping function is increasing and $\Theta < 0$, we infer that

$$L_k\varphi \leq \frac{c_k\rho(h)}{k+1}\langle\nabla H_{k+1}, \nabla h\rangle. \tag{8.8}$$

Proceeding, let us consider the operator $L : C^\infty(\Sigma) \to C^\infty(\Sigma)$ defined in equation (7.9),

$$\begin{aligned}L &= \sum_{k=r}^s (k+1)c_k^{-1}b_k L_k\\ &= \operatorname{tr}(P \circ \operatorname{hess}),\end{aligned}$$

where the tensor $P : \mathfrak{X}(\Sigma) \to \mathfrak{X}(\Sigma)$ is given by equation (7.10),

$$P = \sum_{k=r}^s (k+1)c_k^{-1}b_k P_k.$$

As in the proof of Theorem 7.1.5, we have that L is an elliptic operator, because $(k+1)c_k^{-1}b_k > 0$ and each operator L_k is elliptic, for every $k = r, \ldots, s$. It follows from here jointly with equation (8.8) and the fact that Σ^n is (r,s)-linear Weingarten that $L\varphi \leq 0$, that is,

$$L(-\varphi) \geq 0.$$

We observe that, by compactness of Σ^n, the weak maximum principle applies to the elliptic operator L. Then we must have

$$-c\sigma(h) - \rho(h)\Theta \leq \max_{\partial\Sigma}(-\varphi) = -c\sigma(t_0) + \rho(t_0)\max_{\partial\Sigma}(-\Theta),$$

which implies

$$c(\sigma(h) - \sigma(t_0)) \geq \rho(h) - \rho(t_0)\max_{\partial\Sigma}(-\Theta). \tag{8.9}$$

By using once more that ρ and σ are increasing functions, is not difficult to see that, for any $t \leq t_0$, the following holds:

$$\sigma(t_0) - \sigma(t) \geq \rho(t)(t_0 - t).$$

Since Proposition 8.1.1 says that $h \leq t_0$, we can apply equation (8.9) to get

$$c\rho(h)(h - t_0) \geq \rho(h) - \rho(t_0) \max_{\partial\Sigma}(-\Theta).$$

Therefore, we conclude that

$$c(h - t_0) \geq 1 - \frac{\rho(t_0)}{\rho(h)} \max_{\partial\Sigma}(-\Theta),$$

that is,

$$h \geq t_0 - \frac{\frac{\rho(t_0)}{\rho(\min h)} \max_{\partial\Sigma}(-\Theta) - 1}{\min H_1}.$$

This finishes the proof of the theorem. $\square$

As in the Riemannian case, it turns out that for hypersurfaces with constant $(s+1)$-mean curvature H_{s+1} our assumption $H_1 \geq \sup |H_{s+1}|^{1/(s+1)}$ in Theorem 8.1.3 holds trivially because of Lemma 6.0.3. Moreover, we observe that in this case Theorem 8.1.3 improves the estimate obtained by García-Martínez and Impera in Theorem 16 of [64], which states that the hight function satisfies

$$t_0 - \frac{\frac{\rho(t_0)}{\rho(\min h)} \max_{\partial\Sigma}(-\Theta) - 1}{H_{s+1}^{1/(s+1)}} \leq h \leq t_0.$$

Since the inequality

$$\frac{\frac{\rho(t_0)}{\rho(\min h)} \max_{\partial\Sigma}(-\Theta) - 1}{\min H_1} \leq \frac{\frac{\rho(t_0)}{\rho(\min h)} \max_{\partial\Sigma}(-\Theta) - 1}{H_{s+1}^{1/(s+1)}}$$

holds for every $s = 0, \ldots, n-1$, we get the improvement desired.

We also observe that Theorem 8.1.3 does not contemplate the case in which the warping function is constant. However, a similar argument to that given by Colares and de Lima in [47] allows us to obtain the next result, which improves Theorem 3.3 of [47] for the case of standard Lorentzian product spaces of the type $-\mathbb{R} \times M^n$.

Theorem 8.1.4. *Let $-\mathbb{R} \times M^n$ be a Lorentzian product whose fiber has nonnegative sectional curvature. Let $\psi : \Sigma^n \to -\mathbb{R} \times M^n$ be a compact (r, s)-linear Weingarten spacelike hypersurface with positive $(s+1)$-mean curvature and boundary $\partial\Sigma^n$ contained into the slice $\{t_0\} \times M^n$. Suppose that, if $s \geq 2$, there exists an elliptic point in Σ^n with respect the future-pointing Gauss map. Then,*

$$\Sigma^n \subset [t_0 - \alpha, t_0] \times M^n,$$

where

$$\alpha = \frac{\max_{\partial\Sigma}(-\Theta) - 1}{\min H_1}.$$

Proof. The proof follows as in Theorem 7.1.9. We begin by observing that, in this case, we also can apply Lemma 6.0.5 (or Lemma 6.0.4 if $s=1$) for to assures the ellipticity of the operators L_k for every $k=r,\dots,s$. Proposition 6.0.7 yields, for instance, $L_r h = -c_r H_{r+1}\Theta \geq 0$, which gives $h \leq t_0$ on Σ^n.

As before, let $\varphi = ch + \Theta$ be the smooth function on Σ^n where $c = \min H_1$. By using once more Proposition 6.0.7 we have

$$\begin{aligned} L_k\varphi &= \frac{c_k}{k+1}\langle \nabla H_{k+1}, \nabla h\rangle \\ &+ \Theta\binom{n}{k+1}\left(nH_1H_{k+1} - (n-k-1)H_{k+2} - (k+1)cH_{k+1}\right) \\ &+ \Theta\sum_{i=1}^{n}\mu_{i,k}K_M(N^*,E_i^*)|N^*\wedge E_i^*|^2, \end{aligned}$$

where $P_kE_i = \mu_{i,k}E_i$, for every $i=1,\dots,n$ and $k=r,\dots,s$. Then one has

$$L_k\varphi \leq \frac{c_k}{k+1}\langle \nabla H_{k+1}, \nabla h\rangle.$$

Hence, reasoning as in the proof of Theorem 8.1.3 we see that $-\varphi \leq \max_{\partial\Sigma}(-\varphi)$, that is,

$$-ch + 1 \leq -ct_0 + \max_{\partial\Sigma}(-\Theta),$$

which finishes the proof of the theorem. □

As aforementioned, we also consider the case in which the warping function is decreasing and the Gauss map is past-pointing, keeping positive $(s+1)$-mean curvature. This is the subject of the our next theorem.

Theorem 8.1.5. *Let $-\mathbb{R}\times_\rho M^n$ be a GRW spacetime satisfying the strong NCC and with decreasing warping function. Let $\psi : \Sigma^n \to -\mathbb{R}\times_\rho M^n$ be a compact (r,s)-linear Weingarten spacelike hypersurface with positive $(s+1)$-mean curvature, boundary $\partial\Sigma^n$ contained into the slice $\{t_0\}\times M^n$, for some $t_0\in\mathbb{R}$, and $d \geq \sum_{k=r}^{s} b_k \sup[-(\log\rho)']^{k+1}$. Suppose that $H_1 \geq \sup|H_{r+1}|^{1/(r+1)}$ and, if $s\geq 2$, there exists an elliptic point in Σ^n with respect the past-pointing Gauss map. Then,*

$$\Sigma^n \subset [t_0, t_0+\beta]\times M^n,$$

where

$$\beta = \frac{\frac{\rho(t_0)}{\rho(\max h)}\max_{\partial\Sigma}(\Theta) - 1}{\min H_1}.$$

Proof. For the sake of simplicity, here we keep the notation of Theorem 8.1.3. Since $\rho' < 0$ and $\Theta \geq 1$, it follows from (8.4), (8.5), (8.7) and (8.3) that

$$L_k\varphi \geq \frac{c_k\rho(h)}{k+1}\langle \nabla H_{k+1}, \nabla h\rangle,$$

which gives

$$L\varphi \geq 0.$$

By the weak maximum principle we get

$$c\sigma(h) + \rho(h)\Theta \leq \max_{\partial\Sigma} \varphi = c\sigma(t_0) + \rho(t_0) \max_{\partial\Sigma} \Theta,$$

that is,

$$c(\sigma(h) - \sigma(t_0)) \leq \rho(t_0) \max_{\partial\Sigma} \Theta - \rho(h). \tag{8.10}$$

Besides, it is easy to see that for every $t \geq t_0$ the inequality

$$\sigma(t) - \sigma(t_0) \geq \rho(t)(t - t_0) \tag{8.11}$$

holds. Since Proposition 8.1.1 gives $h \geq t_0$, we must have from equations (8.10) and (8.11) that

$$c\rho(h)(h - t_0) \leq \rho(t_0) \max_{\partial\Sigma} \Theta - \rho(h).$$

Therefore,

$$c(h - t_0) \leq \frac{\rho(t_0)}{\rho(h)} \max_{\partial\Sigma} \Theta - 1$$

and the result follows. □

Reasoning as in Theorem 8.1.4 we get the following result:

Theorem 8.1.6. *Let $-\mathbb{R} \times M^n$ be a Lorentzian product whose fiber has nonnegative sectional curvature. Let $\psi : \Sigma^n \to -\mathbb{R} \times M^n$ be a compact (r, s)-linear Weingarten spacelike hypersurface with positive $(s + 1)$-mean curvature and boundary $\partial\Sigma^n$ contained into the slice $\{t_0\} \times M^n$. Suppose that, if $s \geq 2$, there exists an elliptic point in Σ^n with respect the past-pointing Gauss map. Then,*

$$\Sigma^n \subset [t_0, t_0 + \beta] \times M^n,$$

where

$$\beta = \frac{\max_{\partial\Sigma}(\Theta) - 1}{\min H_1}.$$

To conclude this section let us consider as ambient space the Lorentz-Minkowski spacetime $\mathbb{L}_1^{n+1}$. For convenience, we will adopt as model for the Lorentz-Minkowski spacetime the product manifold $-\mathbb{R} \times \mathbb{R}^n$ endowed with the Lorentzian metric

$$\langle\ ,\ \rangle = -\pi_{\mathbb{R}}^*(dt^2) + \pi_{\mathbb{R}^n}^*(dx^2),$$

where $\pi_{\mathbb{R}}^*$ and $\pi_{\mathbb{R}^n}^*$ denote the canonical projections from $\mathbb{R} \times \mathbb{R}^n$ on each factor, $dx^2 = dx_1^2 + \cdots + dx_n^2$ is the canonical Riemannian metric on the n-dimensional Euclidean space $\mathbb{R}^n$ and $-\mathbb{R}$ stands for the line $\mathbb{R}$ furnished with the metric $-dt^2$. We also note that the Gauss map $N \in \mathfrak{X}^\perp(\Sigma)$ of a spacelike hypersurface Σ^n immersed into the Lorentz-Minkowski spacetime can be regarded as a map $N : \Sigma^n \to \mathbb{H}^n$, where $\mathbb{H}^n$ denotes the n-dimensional hyperbolic space, that is,

$$\mathbb{H}^n = \{p \in \mathbb{L}^{n+1} \ ; \ \langle p, p\rangle = -1, \ p_1 \geq 0\}.$$

In this setting, the image $N(\Sigma)$ will be called the *hyperbolic image* of Σ^n. Furthermore, given a geodesic ball $B(a, \varrho) \subset \mathbb{H}^n$ centered at a point $a \in \mathbb{H}^n$ and radius $\varrho > 0$, it is well known that $B(a, \varrho)$ is characterized as

$$B(a, \varrho) = \{p \in \mathbb{H}^n \ ; \ -\cosh\varrho \leq \langle p, a\rangle \leq -1\}.$$

In particular, if the hyperbolic image of Σ is contained into some geodesic ball $B(a, \varrho)$, then

$$1 \leq |\langle N, a\rangle| \leq \cosh\varrho.$$

Hence if Σ^n is compact (necessarily with nonempty boundary; see, for instance, Section 2 of [18]) one has

$$\max_{\partial\Sigma} |\Theta| \leq \cosh\varrho,$$

where ϱ is the radius of a geodesic ball of center $\partial_t := e_1 = (1, 0, \ldots, 0)$. With this preliminaries, we are ready to prove the following result, where the assumption of the hypersurface has an elliptic point is replaced by a condition of boundedness on the hyperbolic image of the hypersurface.

Theorem 8.1.7. *Let $\psi : \Sigma^n \to \mathbb{L}^{n+1}$ be a compact (r, s)-linear Weingarten spacelike hypersurface immersed into the Lorentz-Minkowski space such that H_{s+1} has strict sign on it and whose boundary $\partial\Sigma$ is contained into the hyperplane $\{0\}\times\mathbb{R}^n$. If the hyperbolic image of Σ^n is contained into a geodesic ball of center $e_1 \in \mathbb{H}^n$ and radius $\varrho > 0$, then the height function h of Σ^n satisfies the following estimate*

$$|h| \leq \frac{\cosh\varrho - 1}{\min H_1}. \tag{8.12}$$

Moreover, estimate (8.12) *is sharp in the sense that it is reached by the hyperbolic cap*

$$\Sigma_\lambda = \left\{x \in \mathbb{L}^{n+1}; \langle x, x\rangle = -\lambda^2, \ \lambda \leq x_1 \leq \sqrt{1+\lambda^2}\right\}, \tag{8.13}$$

where λ is the positive constant given by $\lambda = (\cosh\varrho - 1)^{-1/2}$.

Proof. From Lemma 1 of [18], our assumption that the boundary of Σ^n is contained into the hyperplane $\{0\} \times \mathbb{R}^n$ implies that (after an appropriate choice of orientation on Σ^n) there exists an elliptic point in Σ^n. Hence the height estimate in (8.12) follows of Theorems 8.1.4 and 8.1.6.

Finally, it is not difficult to verify that the hyperbolic cap Σ_λ defined in (8.13) is a spacelike hypersurface of the Lorentz-Minkowski spacetime which has constant $(s + 1)$-mean curvature given by

$$H_{s+1} = \frac{1}{\lambda^{rs+1}} > 0,$$

for every $0 \leq s \leq n - 1$ (if we choose the Gauss map N in the same time-orientation of e_1, for the case in which s is even). Moreover, the hyperbolic image of Σ_λ is contained in the geodesic ball of center $e_1 \in \mathbb{H}^{n+1}$ and radius

$$\varrho = \cosh^{-1} \sqrt{1 + \frac{1}{\lambda^2}}.$$

Thus, the height function of Σ_λ is given by

$$h = \frac{\cosh \varrho - 1}{\min_{\Sigma_\lambda} H_1},$$

showing that the estimate in (8.12) is sharp. $\square$

We point out that for a spacelike hypersurface with constant $(s+1)$-mean curvature H_{s+1}, Theorem 8.1.7 improves the estimate obtained by de Lima in Theorem 4.2 of [52]. Indeed, the de Lima's result says that

$$|h| \leq \frac{\cosh \varrho - 1}{H_{s+1}^{1/(s+1)}}.$$

On the other hand, it follows from Lemma 6.0.3 that

$$\frac{\cosh \varrho - 1}{\min H_1} \leq \frac{\cosh \varrho - 1}{H_{s+1}^{1/(s+1)}}$$

for every $s = 0, \ldots, n-1$, that is, (8.12) is a best estimate.

8.2 Half-space theorems and topology at infinity

The purpose of this section is to recover some of the half-space theorems given in Section 7.2 for the case of complete spacelike hypersurfaces immersed into a GRW spacetime $-\mathbb{R} \times M^n$. Following [64], our approach is based on the generalized version of the Omori-Yau's maximum principle for trace type differential operators given by Lemma A.0.3.

It is worth pointing out that our results give an improvement of those obtained by García-Martínez and Impera [64] for hypersurfaces having some constant higher order mean curvature in a GRW spacetime with warping function non-decreasing (see Theorem 8.2.2 below). Moreover, we are able to consider the case in which the warping function is non-increasing (see Theorem 8.2.5 below).

Before, let us recall the following definition, which in the Lorentzian setting was first introduced in [64]. We say that a spacelike hypersurface in a GRW spacetime $-\mathbb{R} \times_\rho M^n$ lies in an upper or lower half-space if it is, respectively, contained into a region of $-\mathbb{R} \times_\rho M^n$ of the form

$$[a, +\infty) \times M^n \qquad \text{or} \qquad (-\infty, a] \times M^n,$$

for some real number $a \in \mathbb{R}$.

We also recall that a GRW spacetime $-\mathbb{R} \times_\rho M^n$ is said spatially closed if its fiber M^n is compact. In this setting, as an application of Proposition 8.1.1 we get the following result, which is a generalization of Theorem 26 in [64] for the case in which the warping function is increasing and the Gauss map is future-pointing. In particular, information on the topology at infinity of these hypersurfaces are given.

Theorem 8.2.1. *Let $-\mathbb{R}\times_\rho M^n$ be a spatially closed GRW spacetime and let $\psi : \Sigma^n \to -\mathbb{R}\times_\rho M^n$ be a properly immersed complete spacelike hypersurface with positive $(s+1)$-mean curvature, for some $0 \le s \le n-1$. The following holds:*

(a) If $\rho'(h) > 0$ and, for some $0 \le r \le s$, we have

$$\sum_{k=r}^{s} b_k H_{k+1} \ge \sum_{k=r}^{s} b_k \sup[(\log\rho)']^{k+1}$$

for certain nonnegative constants b_k and, when $s \ge 2$, there exists an elliptic point in Σ^n with respect the future-pointing Gauss map, then Σ^n cannot lie in a lower half-space. In particular, Σ^n must have at least one top end.

(b) If $\rho'(h) < 0$ and, for some $0 \le r \le s$, we have

$$\sum_{k=r}^{s} b_k H_{k+1} \ge \sum_{k=r}^{s} b_k \sup[-(\log\rho)']^{k+1}$$

for certain nonnegative constants b_k and, when $s \ge 2$, there exists an elliptic point in Σ^n with respect the past-pointing Gauss map, then Σ^n cannot lie in an upper half-space. In particular, Σ^n must have at least one bottom end;

Proof. Since the proof is analogues to the Riemannian case, it is sufficient to prove, for instance, item (b). For this, let us assume by contradiction that Σ^n lies in an upper half-space, that is,

$$\Sigma^n \subset [a, +\infty) \times M^n$$

for some $a \in \mathbb{R}$. For any number $t_0 > a$, we denote by Σ_{t_0} the hypersurface

$$\Sigma_{t_0} = \{(t,p) \in \Sigma^n \ ; \ t \le t_0\}.$$

Then, Σ_{t_0} is a compact spacelike hypersurface with boundary contained into the slice $\{t_0\} \times M$, because M^n is compact and the immersion is proper. Therefore, by Proposition 8.1.1 we get $h \ge t_0$ characterizing a contradiction since t_0 is arbitrary. This finishes the proof of the theorem. □

From now on, the aim is to study the case in which the fiber is not necessarily compact. More precisely, following ideas already presented in the Riemannian setting, given a GRW spacetime $-\mathbb{R}\times M^n$ we are interest in to prove half-space theorems for noncompact generalized Weingarten spacelike hypersurfaces immersed in these ambient spaces. The first one is the following:

Theorem 8.2.2. *Let $-\mathbb{R}\times_\rho M^n$ be a GRW spacetime satisfying the strong NCC and with nondecreasing warping function. Let $\psi:\Sigma^n\to-\mathbb{R}\times_\rho M^n$ be a complete noncompact (r,s)-linear Weingarten spacelike hypersurface with positive $(s+1)$-mean curvature, for some $1\leq r\leq s\leq n-1$, and $d>\sum_{k=r}^s b_k\sup[(\log\rho)']^{k+1}$. Suppose that $\inf\frac{\rho''(h)}{\rho(h)}>-\infty$ and, if $s\geq 2$, there exists an elliptic point in Σ^n with respect the future-pointing Gauss map. Assume further that $\sup|H_r|<+\infty$ and the second fundamental form satisfies $|A|\leq G(r_o)$, where $G\in C^1([0,+\infty))$ is such that*

$$G(0)>0,\quad G'(t)\geq 0\quad\text{and}\quad\frac{1}{G(t)}\notin L^1(+\infty)$$

and r_o is the distance function from a reference point in Σ^n. Then Σ^n cannot lie in a lower half-space.

Proof. As in the proof of Theorem 7.2.7, we states that in this case the assumptions of Lemma A.0.3 of Appendix A also holds. Indeed, we note that by Lemma 6.0.2

$$\begin{aligned}\overline{R}(U,V)W &= R_M(U^*,V^*)W^*+[(\log\rho)']^2(\langle U,W\rangle V-\langle V,W\rangle U)\\ &- (\log\rho)''\langle W,\partial_t\rangle(\langle U,\partial_t\rangle V-\langle V,\partial_t\rangle U)\\ &- (\log\rho)''(\langle U,W\rangle\langle V,\partial_t\rangle-\langle V,W\rangle\langle U,\partial_t\rangle)\partial_t,\end{aligned}$$

for every U,V,W tangent to $-\mathbb{R}\times_\rho M^n$. In particular, for an orthonormal base $\{X,Y\}$ of an arbitrary plane tangent to Σ^n we get

$$\begin{aligned}\overline{K}(X,Y) &= \frac{1}{\rho^2(h)}K_M(X^*,Y^*)|X^*\wedge Y^*|^2+(\log\rho)'(h)^2\\ &- (\log\rho)''(h)(\langle X,\nabla h\rangle^2+\langle Y,\nabla h\rangle^2).\end{aligned}$$

By the strong NCC and the fact that $|X^*\wedge Y^*|^2=1+\langle X,\nabla h\rangle^2+\langle Y,\nabla h\rangle^2$ we find

$$\begin{aligned}\overline{K}(X,Y) &\geq (\log\rho)''(h)+(\log\rho)'(h)^2\\ &= \frac{\rho''(h)}{\rho(h)}.\end{aligned}$$

On the other hand, by using the Gauss equation (6.9) and the previous inequality, we infer that the sectional curvature K_Σ of Σ^n satisfies

$$\begin{aligned}K_\Sigma &= \overline{K}(X,Y)-\langle AX,X\rangle\langle AY,Y\rangle+\langle AX,Y\rangle^2\\ &\geq \frac{\rho''(h)}{\rho(h)}-|AX||AY|\\ &\geq \frac{\rho''(h)}{\rho(h)}-|A|^2.\end{aligned}$$

Hence, the assumption on the warping function and the second fundamental form imply that K_Σ satisfies (A.1) of Lemma A.0.3 of Appendix A, proving the claim.

From now on, we argue by contradiction. Let us suppose that Σ^n lies in a lower half-space.

In others words, the height function of Σ^n satisfies $h^* = \sup h < +\infty$.

Let $\mathfrak{L}_k : C^\infty(\Sigma) \to C^\infty(\Sigma)$ be the second order linear differential operator, for each $k = r, \dots, s$, given by

$$\begin{aligned} \mathfrak{L}_k &= \frac{1}{(-\Theta)^k}\sum_{i=0}^{k}(-1)^i\frac{c_k}{c_i}(\log\rho)'(h)^{k-i}\Theta^i L_i \\ &= \operatorname{tr}(\mathfrak{P}_k \circ \operatorname{hess}), \end{aligned} \tag{8.14}$$

where the tensor $\mathfrak{P}_k : \mathfrak{X}(\Sigma) \to \mathfrak{X}(\Sigma)$ is defined by

$$\mathfrak{P}_k = \frac{1}{(-\Theta)^k}\sum_{i=0}^{k}(-1)^i\frac{c_k}{c_i}(\log\rho)'(h)^{k-i}\Theta^i P_i.$$

We note that

$$\mathfrak{L}_k = \frac{1}{(-\Theta)^k}\mathcal{L}_k^+ \quad \text{and} \quad \mathfrak{P}_k = \frac{1}{(-\Theta)^k}\mathcal{P}_k^+,$$

where $\mathcal{L}_k^+$ and $\mathcal{P}_k^+$ are given by equations (7.21) and (7.22), respectively. Besides, taking in mind our assumption we have that $\mathfrak{L}_k$ is a semi-elliptic operator or, equivalently, $\mathfrak{P}_k$ is a positive semi-definite tensor. Moreover, since $d > b_k \sup[(\log\rho)']^{k+1}$ for every $k = r, \dots, s$, it follows from Lemma 6.0.3 that

$$\begin{aligned} \operatorname{tr}(\mathfrak{P}_k) &\le \frac{c_k}{(-\Theta)^k}\sum_{i=0}^{k}(-\Theta)^i\left(\frac{d}{b_k}\right)^{(k-i)/(k+1)} H_r^{i/r} \\ &\le c_k\sum_{i=0}^{k}\left(\frac{d}{b_k}\right)^{(k-i)/(k+1)} H_r^{i/r}, \end{aligned}$$

which implies that $\sup\operatorname{tr}(\mathfrak{P}_k) < +\infty$.

We set the second order linear differential operator $\mathfrak{L} : C^\infty(\Sigma) \to C^\infty(\Sigma)$ by

$$\mathfrak{L} = \sum_{k=r}^{s} b_k c_k^{-1}\mathfrak{L}_k^+ = \operatorname{tr}(\mathfrak{P} \circ \operatorname{hess}), \tag{8.15}$$

where the tensor $\mathfrak{P} : \mathfrak{X}(\Sigma) \to \mathfrak{X}(\Sigma)$ is given by

$$\mathfrak{P} = \sum_{k=r}^{s} b_k c_k^{-1}\mathfrak{P}_k^+.$$

Then $\mathfrak{P}$ is a positive semi-definite symmetric tensor with $\sup\operatorname{tr}(\mathfrak{P}) < \infty$. In particular, $\mathfrak{L}$ is an semi-elliptic operator. It follows from here jointly with the claim proved above that Lemma A.0.3 applies in this case, that is, the Omori-Yau's maximum principle holds on Σ^n for the operator $\mathfrak{L}$.

Since $\sigma(h) \le \sigma(h^*) < +\infty$, we can to assure the existence of a sequence of points $\{p_j\} \subset \Sigma^n$

with the following properties:

$$\lim \sigma(h)(p_j) = \sigma(h^*), \quad |\nabla\sigma(h)(p_j)| = \rho(h)(p_j)|\nabla h(p_j)| < \frac{1}{j} \quad \text{and} \quad \mathfrak{L}\sigma(h)(p_j) < \frac{1}{j}.$$

Hence equation (8.1) yields

$$\frac{1}{j} \quad > \quad \sum_{k=r}^{s} \frac{\rho(h(p_j))}{(-\Theta(p_j))^k} b_k \left(-[(\log\rho)'(h(p_j))]^{k+1} + (-\Theta(p_j))^{k+1} H_{k+1}(p_j)\right).$$

By relation $|\nabla h|^2 = \Theta^2 - 1$, making $j \to +\infty$ we get $d \leq \sum_{k=r}^{s} b_k \sup[(\log\rho)']^{k+1}$, characterizing a contradiction. □

Let us recall the following remark already mentioned in a similar way after Theorem 7.2.7. From the equality

$$|A|^2 = n^2 H_1^2 - n(n-1)H_2$$

it follows that under the assumption $\inf H_2 > -\infty$ the condition $\sup |A|^2 < +\infty$ is equivalent to $\sup |H_1| < +\infty$. More generally, if there exists an elliptic point for an appropriate choice of the Gauss map and H_{s+1} does not change sign on Σ^n for some $s = 2, \ldots, n-1$, then by Lemma 6.0.3 the condition $\sup |A|^2 < +\infty$ is equivalent to $\sup H_1 < +\infty$. Moreover, if $\inf \frac{\rho''(h)}{\rho(h)} > -\infty$, reasoning as in the proof of Theorem 8.2.2 we get that, under assumption $\sup |H_1| < +\infty$, the sectional curvature of hypersurface is bounded from below. In particular, the following result holds.

Corollary 8.2.3. *Let $-\mathbb{R} \times_\rho M^n$ be a GRW spacetime satisfying the strong NCC and with non-decreasing warping function. Let $\psi : \Sigma^n \to -\mathbb{R} \times_\rho M^n$ be a complete (r,s)-linear Weingarten spacelike hypersurface with positive $(s+1)$-mean curvature, for some $0 \leq r \leq s \leq n-1$, and $d > \sum_{k=r}^{s} b_k \sup[(\log\rho)']^{k+1}$. Suppose that $\sup |H_1| < +\infty$, $\inf \frac{\rho''(h)}{\rho(h)} > -\infty$ and, if $s \geq 2$, there exists an elliptic point in Σ^n with respect the future-pointing Gauss map. Then, Σ^n cannot lie in a lower half-space.*

We note that, if Σ^n is a (s,s)-linear Weingarten hypersurface in Corollary 8.2.3, that is, if Σ^n has constant $(s+1)$-mean curvature, we recover Theorem 35 (i) of [64]:

Corollary 8.2.4 (Theorem 35 (i) of [64])**.** *Let $-\mathbb{R} \times_\rho M^n$ be a GRW spacetime satisfying the strong NCC and with non-decreasing warping function. Let $\psi : \Sigma^n \to -\mathbb{R} \times_\rho M^n$ be a complete spacelike hypersurface with constant $(s+1)$-mean curvature satisfying $H_{s+1}^{1/(s+1)} > \sup(\log\rho)'$ for some $0 \leq s \leq n-1$. Suppose that $\sup |H_1| < +\infty$, $\inf \frac{\rho''(h)}{\rho(h)} > -\infty$ and, if $s \geq 2$, there exists an elliptic point in Σ^n with respect the future-pointing Gauss map. Then, Σ^n cannot lie in a lower half-space.*

We observe that as showed in Theorem 32 of [64], when the hypersurface has constant mean curvature we do not need to assume that ρ' does not change sign in Corollary 8.2.4.

Finally, we close this section by proving the version of the previous theorem for the case in which the warping function is non-increasing.

Theorem 8.2.5. *Let $-\mathbb{R}\times_\rho M^n$ be a GRW spacetime satisfying the strong NCC and with non-increasing warping function. Let $\psi:\Sigma^n\to-\mathbb{R}\times_\rho M^n$ be a complete (r,s)-linear Weingarten spacelike hypersurface with positive $(s+1)$-mean curvature, for some $1\le r\le s\le n-1$, and $d>\sum_{k=r}^s b_k\sup[-(\log\rho)']^{k+1}$. Suppose that* $\inf\frac{\rho''(h)}{\rho(h)}>-\infty$ *and, if $s\ge 2$, there exists an elliptic point in Σ^n with respect the past-pointing Gauss map. Assume further that* $\sup|H_r|<+\infty$ *and the second fundamental form satisfies $|A|\le G(r_o)$, where $G\in C^1([0,+\infty))$ is such that*

$$G(0)>0,\quad G'(t)\ge 0\quad\text{and}\quad \frac{1}{G(t)}\notin L^1(+\infty)$$

and r_o is the distance function from a reference point in Σ^n. Then Σ^n cannot lie in an upper half-space.

Proof. Let us assume by contradiction that Σ^n lies in an upper half-space, that is, the height function of Σ^n satisfies $h_*=\inf h>-\infty$. For each $k=r,\dots,s$, let $\mathfrak{L}_k$ the operator given in (8.14). We observer that it can be rewrite as

$$\mathfrak{L}_k=\frac{1}{\Theta^k}\sum_{i=0}^{k}(-1)^{k-i}\frac{c_k}{c_i}(\log\rho)'(h)^{k-i}\Theta^iL_i=\frac{1}{\Theta^k}\mathcal{L}_k^-,$$

where $\mathcal{L}_k^-$ is defined in (7.24). Hence, in this case, $\mathfrak{L}_k$ is also a semi-elliptic operator. In particular, $\mathfrak{L}$ defined in (8.15) is a semi-elliptic operator too. Furthermore, as in the previous theorem the Omori-Yau's maximum principle holds on Σ^n for the semi-elliptic operator $\mathfrak{L}$.

Since $\sigma(h)\ge\sigma(h_*)$, there exists a sequence of points $\{q_j\}\subset\Sigma^n$ satisfying

$$\lim\sigma(h)(q_j)=\sigma(h_*),\quad |\nabla\sigma(h)(q_j)|<\frac{1}{j}\quad\text{and}\quad \mathfrak{L}\sigma(h)(q_j)>-\frac{1}{j},$$

which jointly with (8.2) implies that

$$-\frac{1}{j}\ <\ \sum_{k=r}^{s}\frac{\rho(h(q_j))}{\Theta(q_j)^k}b_k\left([-(\log\rho)'(h(q_j))]^{k+1}-\Theta(q_j)^{k+1}H_{k+1}(q_j)\right).$$

Therefore, letting $j\to\infty$ we get

$$d\le\sum_{k=r}^{s}b_k\sup[-(\log\rho)']^{k+1}$$

giving a contradiction. This proves the result. □

In particular, we get.

Corollary 8.2.6. *Let $-\mathbb{R}\times_\rho M^n$ be a GRW spacetime satisfying the strong NCC and with non-increasing warping function. Let $\psi:\Sigma^n\to-\mathbb{R}\times_\rho M^n$ be a complete (r,s)-linear Weingarten spacelike hypersurface with positive $(s+1)$-mean curvature, for some $0\le r\le s\le n-1$, and $d>\sum_{k=r}^s b_k\sup[-(\log\rho)']^{k+1}$. Suppose that* $\sup|H_1|<+\infty$, $\inf\frac{\rho''(h)}{\rho(h)}>-\infty$ *and, if $s\ge 2$, there* *exists an elliptic point in Σ^n with respect the past-pointing Gauss map. Then, Σ^n cannot lie in an upper half-space.*

As an immediate consequence we obtain the following result for hypersurfaces with some constant higher order mean curvature.

Corollary 8.2.7. *Let $-\mathbb{R} \times_\rho M^n$ be a GRW spacetime satisfying the strong NCC and with non-increasing warping function. Let $\psi : \Sigma^n \to -\mathbb{R} \times_\rho M^n$ be a complete spacelike hypersurface with constant $(s+1)$-mean curvature satisfying $H_{s+1}^{1/(s+1)} > \sup[-(\log\rho)']$, for some $0 \leq s \leq n-1$. Suppose that $\sup|H_1| < +\infty$, $\inf \frac{\rho''(h)}{\rho(h)} > -\infty$ and, if $s \geq 2$, there exists an elliptic point in Σ^n with respect the past-pointing Gauss map. Then, Σ^n cannot lie in an upper half-space.*

Part III

On stability of hypersurfaces in weighted semi-Riemannian warped products

Chapter 9

Preliminaries for Part III

Let $(\overline{M}^{n+1}, \langle\ ,\ \rangle)$ be an $(n+1)$-dimensional oriented Riemannian or Lorentzian manifold and let $f : \overline{M}^{n+1} \to \mathbb{R}$ be a smooth function. The weighted manifold $\overline{M}_f^{n+1}$ associated with $\overline{M}^{n+1}$ and f is the triple $(\overline{M}^{n+1}, \langle\ ,\ \rangle, e^{-f}d\overline{M})$, where $d\overline{M}$ denotes the standard volume element of $\overline{M}^{n+1}$ induced by the metric $\langle\ ,\ \rangle$. We will refer to function f as the weight function of the weighted manifold $\overline{M}_f^{n+1}$. In this setting, for a weighted manifold $\overline{M}_f^{n+1}$, an important and natural tensor is the so called Bakry-Émery-Ricci tensor $\overline{\mathrm{Ric}}_f$, which is a generalization of Ricci tensor $\overline{\mathrm{Ric}}$ of $\overline{M}^{n+1}$ which is defined by

$$\overline{\mathrm{Ric}}_f = \overline{\mathrm{Ric}} + \overline{\mathrm{Hess}} f,$$

where $\overline{\mathrm{Hess}} f$ is the Hessian of f on $\overline{M}^{n+1}$. In particular, if f is constant $\overline{\mathrm{Ric}}_f$ is simply the standard Ricci tensor $\overline{\mathrm{Ric}}$ of $\overline{M}^{n+1}$.

Appearing naturally in the study of self-shrinkers, Ricci solitons, harmonic heat flows and many other subjects in differential geometry, weighted manifolds are proved to be important nontrivial generalizations of Riemannian manifolds and, nowadays, there are several geometric investigations concerning them. For a brief overview of results in this scope, we refer the articles of Morgan [98] and Wei and Wylie [117].

Let $\psi : \Sigma^n \to \overline{M}_f^{n+1}$ be an isometrically immersed orientable Riemannian manifold into $\overline{M}_f^{n+1}$. Then Σ^n becomes automatically a weighted Riemannian manifold by weighted structure induced from $\overline{M}_f^{n+1}$. In this case and following Gromov [69], the weighted mean curvature, or simply f-mean curvature, H_f of Σ^n is defined by

$$nH_f = nH + \varepsilon\langle\overline{\nabla} f, N\rangle,$$

where H denotes the standard mean curvature of Σ^n with respect to its orientation, $\varepsilon = 1$ if $\overline{M}^{n+1}$ is a Riemannian manifold, and $\varepsilon = -1$ if $\overline{M}^{n+1}$ is a Lorentzian manifold. In particular, when f is constant we have $H_f = H$ and we recover the usual definition of mean curvature. When the ambient space is Riemannian and the f-mean curvature H_f vanishes identically on Σ^n we said that Σ^n is a f-minimal hypersurface. In the case in which the ambient space is Lorentzian and the f-mean curvature H_f vanishes identically on Σ^n, it is called a f-maximal hypersurface. In both the case, its mean curvature H satisfies

$$nH = -\varepsilon\langle\overline{\nabla}f, N\rangle. \tag{9.1}$$

The research on the geometry of hypersurfaces having constant f-mean curvature and, in particular, the investigations on the behavior of hypersurfaces with f-mean curvature vanishes identically immersed into a weighted ambient space, constitutes a recent and fruitful topic into the theory of isometric immersions. It has been already approached by many authors and we may cite, for instance, the works [34, 36, 37, 41, 60, 73, 75, 80, 91, 106, 108].

As in the case of zero mean curvature hypersurfaces, it is well known that the condition of Σ^n has zero f-mean curvature is equivalent to the fact that Σ^n is a critical point of the weighted area functional,

$$\mathrm{vol}_f(\Sigma) = \int_\Sigma e^{-f}d\Sigma,$$

for every variation of Σ^n with compact support and fixed boundary. It is natural to wonder whether these hypersurfaces has the property of to minimize (if the ambient space is Riemannian) or maximize (if the ambient space is Lorentzian) the weighted area functional. Recently many authors has been devoted to the study of this question (see, for instance, [34, 41, 60, 80] and references therein).

In order to answer this question, it is very useful to know the second variation formula of the weighted area functional. To this end, let us recall that the f-divergence operator on Σ^n is defined by

$$\mathrm{div}_f(X) = e^f\mathrm{div}(e^{-f}X), \tag{9.2}$$

where X is a tangent vector field on Σ^n and div denotes the standard divergence operator of Σ^n. From (9.2) we can define the f-Laplacian of Σ^n by

$$\Delta_f u = \mathrm{div}_f(\nabla u) = \Delta u - \langle\nabla f, \nabla u\rangle, \tag{9.3}$$

where u is a smooth function on Σ^n, Δ denotes the Laplacian induced of div and ∇ stands for the Levi-Civita connection of Σ^n induced from Levi-Civita connection $\overline{\nabla}$ from ambient space $\overline{M}^{n+1}_f$.

Now let V be a normal compactly supported variation of Σ^n and take $\varphi \in C_0^\infty(\Sigma)$ such that $V = \varphi N$, where N determines the orientation of Σ^n. If the f-mean curvature H_f of Σ^n vanishes identically, then it is well known that the second variation of the weighted area functional is given by (in the Riemannian case see, for instance, [41], and in the Lorentzian case see, for instance, [57])

$$\frac{d^2}{dt^2}\mathrm{vol}_f(\Sigma)|_{t=0} = -\varepsilon\int_\Sigma \varphi L_f\varphi d\Sigma, \tag{9.4}$$

where the weighted Jacobi operator L_f is defined by

$$L_f = \Delta_f + \varepsilon\left(|A|^2 + \overline{\mathrm{Ric}}_f(N, N)\right).$$

Then we say that Σ^n is L_f-stable if it minimizes (resp. maximizes) the weighted are functional in the Riemannian case (resp. Lorentzian case), that is, $\frac{d^2}{dt^2}\mathrm{vol}_f(\Sigma)|_{t=0} \geq 0$ (resp. ≤ 0).

This part of the thesis is dedicated to the study of the L_f-stability of zero f-mean curvature hypersurfaces immersed into a weighted semi-Riemannian warped product space. In this setting and for the sake of simplicity, we will adopt all notations and definitions already introduced in Chapter 6 regarding warped product spaces.

Chapter 10

The Riemannian case

In this chapter we give sufficient conditions to guarantee L_f-stability of f-minimal hypersurfaces immersed into a weighted Riemannian warped product space, where L_f stands for the weighted Jacobi operator. The results presented herein make part of [58].

10.1 L_f-Stability of f-minimal hypersurfaces in weighted Riemannian warped products

Let $\overline{M}_f^{n+1} = (I \times_\rho M^n)_f$ be a weighted Riemannian warped product and let $\psi : \Sigma^n \to \overline{M}_f^{n+1}$ be a f-minimal two-sided hypersurface. Then equation (9.4) says that the second variation formula of weighted area functional is given by

$$\frac{d^2}{dt^2}\mathrm{vol}_f(\Sigma)|_{t=0} = -\int_\Sigma \varphi L_f \varphi d\Sigma, \tag{10.1}$$

where $V = \varphi N$ is a normal compactly supported variation of Σ^n and the weighted Jacobi operator L_f is defined by

$$L_f = \Delta_f + |A|^2 + \overline{\mathrm{Ric}}_f(N, N). \tag{10.2}$$

In particular, (10.1) depends only on $\varphi \in C_0^\infty(\Sigma)$. In this setting, let us emphasize the following definition introduced in previous chapter:

Definition 10.1.1. *Let Σ^n be a hypersurface as above. We say that Σ^n is L_f-stable if, for any compactly supported smooth function $\varphi \in C_0^\infty(\Sigma)$, it holds that*

$$\frac{d^2}{dt^2}\mathrm{vol}_f(\Sigma)|_{t=0} = -\int_\Sigma \varphi L_f \varphi e^{-f} d\Sigma \geq 0.$$

In order to proof our main theorem of this section, we will need use the following auxiliary result, which gives a sufficient condition for a f-minimal hypersurfaces be L_f-stable.

Lemma 10.1.2. *Let $\psi : \Sigma^n \to \overline{M}_f^{n+1}$ be a f-minimal two-sided hypersurface immersed into a weighted Riemannian warped product $\overline{M}_f^{n+1} = (I \times_\rho M^n)_f$. If there exists a positive smooth function $u \in C^\infty(\Sigma)$ such that $L_f u \leq 0$, then Σ^n is L_f-stable.*

Proof. Assume that there exists such a function u and take $\varphi \in C_0^\infty(\Sigma)$. Then, we can choose $\eta \in C_0^\infty(\Sigma)$ satisfying $\varphi = \eta u$. Hence, from (10.2) we have

$$\begin{aligned}
\int_\Sigma \varphi L_f \varphi e^{-f} d\Sigma &= \int_\Sigma \eta u L_f(\eta u) e^{-f} d\Sigma \\
&= \int_\Sigma \left[\eta^2 u L_f u + \eta u^2 \Delta\eta + 2\eta u \langle \nabla u, \nabla \eta\rangle - \eta u^2 \langle \nabla\eta, \nabla f\rangle\right] e^{-f} d\Sigma \\
&\leq \int_\Sigma \left[\eta u^2 \Delta\eta + 2\eta u \langle \nabla u, \nabla \eta\rangle - \eta u^2 \langle \nabla\eta, \nabla f\rangle\right] e^{-f} d\Sigma \\
&= \int_\Sigma \left[\eta u^2 \Delta\eta + \frac{1}{2}\langle \nabla u^2, \nabla \eta^2\rangle - \eta u^2 \langle \nabla\eta, \nabla f\rangle\right] e^{-f} d\Sigma. \qquad (10.3)
\end{aligned}$$

On the other hand, we can see that

$$\operatorname{div}(u^2 \nabla \eta^2) = \langle \nabla u^2, \nabla \eta^2\rangle + u^2 \Delta \eta^2 = \langle \nabla u^2, \nabla \eta^2\rangle + 2\eta u^2 \Delta\eta + 2u^2 |\nabla \eta|^2.$$

Therefore, from the weighted version of divergence theorem (see Lemma 2.2 of [34]), we get from last equation together with (10.3) that

$$\begin{aligned}
\int_\Sigma \varphi L_f \varphi e^{-f} d\Sigma &\leq \int_\Sigma \left[\frac{1}{2}\operatorname{div}(u^2 \nabla \eta^2) - \eta u^2 \langle \nabla\eta, \nabla f\rangle - u^2 |\nabla\eta|^2\right] e^{-f} d\Sigma \\
&= \int_\Sigma \left[\frac{1}{2}\operatorname{div}_f(u^2 \nabla \eta^2) - u^2 |\nabla\eta|^2\right] e^{-f} d\Sigma \\
&\leq - \int_\Sigma u^2 |\nabla \eta|^2 e^{-f} d\Sigma \leq 0
\end{aligned}$$

This shows that Σ^n is L_f-stable, as desired. □

Remark 10.1.3. It is worth to observe that the converse of Lemma 10.1.2 is also true and can be found in Lemma 2.1 of [60] (see also Proposition 3 of [80]).

Proceeding, it follows from a splitting theorem due to Fang et al. (see Theorem 1.1 of [62]) that if a weighted Riemannian warped product $\overline{M}_f^{n+1} = (I \times_\rho M^n)_f$ with bounded weight function f is such that $\overline{\mathrm{Ric}}_f$ is nonnegative, then f must be constant along I. Motivated by this fact, in our main result we will consider weighted Riemannian warped product $\overline{M}_f^{n+1}$ whose weight function f does not depend on the parameter $t \in I$, that is, $\langle \overline{\nabla} f, \partial_t\rangle = 0$ and, for the sake of simplicity, we will denote such a manifold by $\overline{M}_f^{n+1} = I \times_\rho M_f^n$.

Now we are ready to state main theorem.

Theorem 10.1.4. *Let $\psi : \Sigma^n \to \overline{M}_f^{n+1}$ be a f-minimal two-sided hypersurface immersed into a weighted Riemannian warped product $\overline{M}_f^{n+1} = I \times_\rho M_f^n$. Setting $\tilde{\Theta} = \rho\Theta$ we have*

$$L_f \tilde{\Theta} = -n\frac{\rho''}{\rho}\tilde{\Theta}. \qquad (10.4)$$

Moreover, the following holds:

(a) If the angle function Θ has strict sign and the warping function satisfies $\rho'' \geq 0$ on Σ^n, then Σ^n is L_f-stable;

(b) If Σ^n is compact, the angle function Θ has strict sign and the warping function satisfies $\rho'' \leq 0$ on Σ^n, then Σ^n is L_f-stable if and only if $\rho'' = 0$ on Σ^n;

(c) If Σ^n is compact, Θ does not vanish identically and $\rho'' < 0$ on Σ^n, then Σ^n cannot be L_f-stable.

Proof. To prove the first part, we observe that by applying Proposition 6.0.7 in this case we get (see also, for instance, Proposition 2.1 of [33])

$$\Delta\tilde{\Theta} = -n\rho\partial_t^\top(H) - n\rho' H - nN(\rho') - (|A|^2 + \overline{\mathrm{Ric}}(N,N))\tilde{\Theta}. \tag{10.5}$$

Besides, since $\rho\partial_t$ is a conformal vector field on $\overline{M}_f^{n+1}$, then $\nabla\tilde{\Theta} = -\rho A(\partial_t^\top)$. Hence from (9.1) jointly with our restrictions on the weight function and Hessian's definition, with a straightforward computation, we obtain

$$\begin{aligned} n\partial_t^\top(H) &= -\partial_t^\top\langle\overline{\nabla} f, N\rangle \\ &= -\langle\overline{\nabla}_{\partial_t}\overline{\nabla} f, N\rangle + \Theta\langle\overline{\nabla}_N\overline{\nabla} f, N\rangle + \langle\overline{\nabla} f, A(\partial_t^\top)\rangle \\ &= -\overline{\mathrm{Hess}} f(N,\partial_t) + \Theta\overline{\mathrm{Hess}} f(N,N) - \rho^{-1}\langle\nabla f, \nabla\tilde{\Theta}\rangle. \end{aligned} \tag{10.6}$$

On the other hand, it is not difficult to verify that

$$\overline{\mathrm{Hess}} f(N,\partial_t) = -\rho^{-1}\rho'\langle\overline{\nabla} f, N\rangle = n\rho^{-1}\rho' H. \tag{10.7}$$

Hence, equations (10.5), (10.6) and (10.7) yield

$$\Delta\tilde{\Theta} = \langle\nabla f, \nabla\tilde{\Theta}\rangle - nN(\rho') - (|A|^2 + \overline{\mathrm{Ric}}_f(N,N))\tilde{\Theta}.$$

Thus, from (9.3) we obtain that

$$\begin{aligned} \Delta_f\tilde{\Theta} &= \Delta\tilde{\Theta} - \langle\nabla f, \nabla\tilde{\Theta}\rangle \\ &= -nN(\rho') - (|A|^2 + \overline{\mathrm{Ric}}_f(N,N))\tilde{\Theta}, \end{aligned}$$

which implies that equation (10.4) holds.

To prove (a), we observe that since the angle function has strict sign, for an appropriated choose of N, we can suppose that Θ is a positive function. Therefore, since the warping function satisfies $\rho'' \geq 0$ on Σ^n, it follows from Lemma 10.1.2 that Σ^n is L_f-stable.

Now, let us prove (b). Again, we can suppose that Θ is a positive function. By using that Σ^n is L_f-stable, we infer

$$0 \leq -\int_\Sigma \tilde{\Theta} L_f\tilde{\Theta} e^{-f} d\Sigma = \int_\Sigma n\frac{\rho''}{\rho}\tilde{\Theta}^2 e^{-f} d\Sigma \leq 0,$$

which gives immediately that $\rho'' = 0$ on Σ^n. The converse follows.

Finally, item (c) follows of the fact that in this case

$$-\int_\Sigma \tilde{\Theta} L_f \tilde{\Theta} e^{-f} d\Sigma = \int_\Sigma n \frac{\rho''}{\rho} \tilde{\Theta}^2 e^{-f} d\Sigma < 0,$$

that is, Σ^n is not L_f-stable. $\square$

To close this chapter, it is worth point out that Theorem 10.1.4 gives a generalization of Theorems 3, 13 and 14 due to Aledo and Rubio [4].

Chapter 11

The Lorentzian case

This chapter is dedicated to the study of the L_f-stability of f-maximal spacelike hypersurfaces immersed into a weighted GRW spacetime. In particular, the main result gives a sufficient condition for these hypersurfaces be L_f-stable. In this chapter we present the results of the paper [57].

11.1 L_f-Stability of f-maximal hypersurfaces in weighted Lorentzian warped products

Let $\overline{M}_f^{n+1} = (-I \times_\rho M^n)_f$ be a weighted GRW spacetime and let $\psi : \Sigma^n \to \overline{M}_f^{n+1}$ be a f-maximal spacelike hypersurface. Let $V = \varphi N$ be a normal compactly supported variation of Σ^n. Equation (9.4) yields

$$\frac{d^2}{dt^2}\mathrm{vol}_f(\Sigma)|_{t=0} = \int_\Sigma \varphi L_f \varphi d\Sigma,$$

where the weighted Jacobi operator L_f, in this case, is given by

$$L_f = \Delta_f - \left(|A|^2 + \overline{\mathrm{Ric}}_f(N,N)\right). \tag{11.1}$$

Hence, the second variation of Σ^n depends only on $\varphi \in C_0^\infty(\Sigma)$. As in the previous chapter, this motivates the following definition:

Definition 11.1.1. *Let Σ^n be a hypersurface as above. We say that Σ^n is L_f-stable if, for any compactly supported smooth function $\varphi \in C_0^\infty(\Sigma)$, it holds that*

$$\frac{d^2}{dt^2}\mathrm{vol}_f(\Sigma)|_{t=0} = \int_\Sigma \varphi L_f \varphi e^{-f} d\Sigma \leq 0.$$

Reasoning in analogous way to Lemma 10.1.2 it is not difficult to obtain a version for the Lorentzian case. For the sake of completeness, we include the proof of this fact here.

Lemma 11.1.2. *Let $\psi : \Sigma^n \to \overline{M}_f^{n+1}$ be a f-maximal spacelike hypersurface immersed into a weighted GRW spacetime $\overline{M}_f^{n+1} = -I \times_\rho M_f^n$. If there exists a positive smooth function $u \in C^\infty(\Sigma)$ such that $L_f u \leq 0$, then Σ^n is L_f-stable.*

Proof. Let u be such a function and take $\varphi \in C_0^\infty(\Sigma)$. Then, we can choose $\eta \in C_0^\infty(\Sigma)$ satisfying $\varphi = \eta u$. Hence, from (11.1) we have

$$\begin{aligned}\int_\Sigma \varphi L_f \varphi e^{-f} d\Sigma &= \int_\Sigma \eta u L_f(\eta u) e^{-f} d\Sigma \\ &\leq \int_\Sigma \left[\eta u^2 \Delta\eta + 2\eta u\langle \nabla u, \nabla\eta\rangle - \eta u^2 \langle \nabla\eta, \nabla f\rangle\right] e^{-f} d\Sigma \\ &= \int_\Sigma \left[\eta u^2 \Delta\eta + \frac{1}{2}\langle \nabla u^2, \nabla\eta^2\rangle - \eta u^2 \langle \nabla\eta, \nabla f\rangle\right] e^{-f} d\Sigma. \end{aligned} \tag{11.2}$$

On the other hand, it is not difficult to verify that

$$\mathrm{Div}(u^2 \nabla\eta^2) = \langle \nabla u^2, \nabla\eta^2\rangle + u^2\Delta\eta^2 = \langle \nabla u^2, \nabla\eta^2\rangle + 2\eta u^2 \Delta\eta + 2u^2|\nabla\eta|^2. \tag{11.3}$$

Hence, using once more the weighted version of divergence theorem jointly with (11.2) and (11.3) we achieve

$$\begin{aligned}\int_\Sigma \varphi L_f \varphi e^{-f} d\Sigma &\leq \int_\Sigma \left[\frac{1}{2}\mathrm{Div}(u^2\nabla\eta^2) - \eta u^2\langle\nabla\eta, \nabla f\rangle - u^2|\nabla\eta|^2\right] e^{-f} d\Sigma \\ &= \int_\Sigma \left[\frac{1}{2}\mathrm{Div}_f(u^2\nabla\eta^2) - u^2|\nabla\eta|^2\right] e^{-f} d\Sigma \\ &\leq -\int_\Sigma u^2|\nabla\eta|^2 e^{-f} d\Sigma \leq 0.\end{aligned}$$

Therefore Σ^n is L_f-stable. So the proof is completed. □

Let $\overline{M}^{n+1} = -I \times_\rho M^n$ be a GRW spacetime and $f : \overline{M}^{n+1} \to \mathbb{R}$ a smooth function on $\overline{M}^{n+1}$. It follows from a splitting theorem due to Case (see Theorem 1.2 of [35]) that if weight function f is bounded and $\overline{\mathrm{Ric}}_f(T,T) \geq 0$ for all timelike vector fields $T \in \mathfrak{X}(\Sigma)$, then f must be constant along I. Motivated by this result, here we will consider weighted GRW spacetimes $\overline{M}_f^{n+1}$ whose weight function f does not depend on the parameter $t \in I$, that is, $\langle \overline{\nabla} f, \partial_t\rangle = 0$ and, for sake of simplicity, we will denote them by $\overline{M}_f^{n+1} = -I \times_\rho M_f^n$. In what follows, we taken the orientation N in the same time-orientation of ∂_t, that is, $\Theta = \langle N, \partial_t\rangle \leq -1$.

Theorem 11.1.3. *Let $\psi : \Sigma^n \to \overline{M}_f^{n+1}$ be a f-maximal spacelike hypersurface immersed into a weighted GRW spacetime $\overline{M}_f^{n+1} = -I \times_\rho M_f^n$. Setting $\tilde{\Theta} = \rho\Theta$ we have*

$$L_f\tilde{\Theta} = n\frac{\rho''}{\rho}\tilde{\Theta}. \tag{11.4}$$

Moreover, the following holds:

(a) If $\rho'' \leq 0$ on Σ^n, then Σ^n is L_f-stable.

(b) If Σ^n is compact and $\rho'' \geq 0$ on Σ^n, then Σ^n is L_f-stable if and only if $\rho'' = 0$ on Σ^n.

(c) If Σ^n is compact and $\rho'' > 0$ on Σ^n, then Σ^n cannot be L_f-stable.

Proof. Let us prove equation (11.4) first. By applying Proposition 6.0.7 one has (see also, for instance, Proposition 2.1 of [33])

$$\Delta\tilde{\Theta} = n\rho\partial_t^\top(H) + n\rho' H - nN(p') + (|A|^2 + \overline{\mathrm{Ric}}(N,N))\tilde{\Theta}. \tag{11.5}$$

On the other hand, as in the proof of Theorem 10.1.4 we have

$$n\partial_t^\top(H) \;=\; \overline{\mathrm{Hess}} f(N,\partial_t) + \Theta\overline{\mathrm{Hess}} f(N,N) + \rho^{-1}\langle\nabla f, \nabla\tilde{\Theta}\rangle \tag{11.6}$$

and

$$\overline{\mathrm{Hess}} f(N,\partial_t) = -\rho^{-1}\rho'\langle\overline{\nabla} f, N\rangle = -n\rho^{-1}\rho' H. \tag{11.7}$$

Then equations (11.5), (11.6) and (11.7) imply that

$$\Delta\tilde{\Theta} = \langle\nabla f, \nabla\tilde{\Theta}\rangle - nN(\rho') + (|A|^2 + \overline{\mathrm{Ric}}_f(N,N))\tilde{\Theta}.$$

Thus, from (9.3) we get that

$$\begin{aligned}
\Delta_f\tilde{\Theta} &= \Delta\tilde{\Theta} - \langle\nabla f, \nabla\tilde{\Theta}\rangle \\
&= -nN(\rho') + (|A|^2 + \overline{\mathrm{Ric}}_f(N,N))\tilde{\Theta}.
\end{aligned}$$

Hence equation (11.4) holds.

To prove item (a), it suffices to notice that, by equation (11.4), we have $L_f(-\tilde{\Theta}) \geq 0$ and, since Θ is taken negative, Lemma 11.1.2 assures that Σ^n is L_f-stable.

Now, let us deal of item (b). In this case, we have that $C_0^\infty(\Sigma) = C^\infty(\Sigma)$. So, if Σ^n is L_f-stable, we obtain

$$0 \geq \int_\Sigma \tilde{\Theta} L_f \tilde{\Theta} e^{-f} d\Sigma = \int_\Sigma n\frac{\rho''}{\rho}\tilde{\Theta}^2 e^{-f} d\Sigma \geq 0,$$

that is, $\rho'' = 0$ on Σ^n. The reciprocal statement follows from item (a).

Finally, we prove item (c). To do so, we use the definition of L_f-stability to infer

$$\int_\Sigma \tilde{\Theta} L_f \tilde{\Theta} e^{-f} d\Sigma = \int_\Sigma n\frac{\rho''}{\rho}\tilde{\Theta}^2 e^{-f} d\Sigma > 0.$$

Therefore, Σ^n cannot be L_f-stable, which finishes the proof. □

Appendix

Appendix A

A brief comment about the generalized Omori-Yau's maximum principle

In this appendix, we recall briefly a generalized version of the Omori-Yau's maximum principle for trace type differential operators proved in [19] as well as the well known Omori-Yau's maximum principle for the Laplacian operator. Let Σ^n be a Riemannian manifold and let $\mathcal{L} = \mathrm{tr}(\mathcal{P} \circ \mathrm{hess})$ be a semi-elliptic operator, where $\mathcal{P} : \mathfrak{X}(\Sigma) \to \mathfrak{X}(\Sigma)$ is a positive semi-definite symmetric tensor. Following the terminology introduced by Pigola et al. [104], we say that the Omori-Yau maximum's principle holds on Σ^n for the operator $\mathcal{L}$ if, for any function $u \in C^2(\Sigma)$ with $u^* = \sup u < +\infty$, there exists a sequence of points $\{p_j\} \subset \Sigma^n$ satisfying

$$u(p_j) > u^* - \frac{1}{j}, \quad |\nabla u(p_j)| < \frac{1}{j} \quad \text{and} \quad \mathcal{L}u(p_j) < \frac{1}{j}$$

for every $j \in \mathbb{N}$.

In this sense, the classical result given by Omori and Yau in [101, 118] states that the Omori-Yau's maximum principle holds for the Laplacian on every complete Riemannian manifold with Ricci curvature bounded from below, that is:

Lemma A.0.1. *Let Σ^n be a complete Riemannian manifold whose Ricci curvature is bounded from below and $u \in C^2(\Sigma)$ satisfying $u^* < +\infty$. Then, there exists a sequence of points $\{p_j\} \subset \Sigma^n$ such that*

$$u(p_j) > u^* - \frac{1}{j}, \quad |\nabla u(p_j)| < \frac{1}{j} \quad \text{and} \quad \Delta u(p_j) < \frac{1}{j}.$$

On the other hand, as observed also by Pigola et al. [104], the validity of Omori-Yau's maximum principle on Σ^n does not depend on curvatures bounds as much as one would expect. For instance, the Omori-Yau's maximum principle holds on every Riemannian manifolds which is properly immersed into a Riemannian space form with controlled mean curvature (see [104], Example 1.14). In particular, it holds for every constant mean curvature hypersurface properly immersed into a Riemannian space form.

More generally, and following again the terminology introduced in [104], the weak Omori-Yau's maximum principle is said to hold on a (not necessarily complete) n-dimensional Riemannian manifold Σ^n if, for any smooth function $u \in C^2(\Sigma)$ with $u^* < +\infty$ there exists a sequence of points $\{p_j\} \subset \Sigma^n$ with the properties

$$u(p_j) > u^* - \frac{1}{j} \quad \text{and} \quad \Delta u(p_j) < \frac{1}{j}.$$

As proved by Pigola et al. [103, 104], the fact that the weak Omori-Yau's maximum principle holds on Σ^n is equivalent to the stochastic completeness of the manifold, that is:

Lemma A.0.2. *A Riemannian manifold Σ^n is stochastically complete if and only if for every $u \in C^2(\Sigma)$ satisfying $u^* < +\infty$, there exists a sequence of points $\{p_j\} \subset \Sigma^n$ such that*

$$u(p_j) > u^* - \frac{1}{j} \quad \text{and} \quad \Delta u(p_j) < \frac{1}{j}.$$

In particular, the weak Omori-Yau's maximum principle holds on every parabolic Riemannian manifold (see Corollary 6.4 of [68]).

In the more general setting, we quote a suitable version of the Omori-Yau's maximum principle for trace type differential operators on a complete noncompact Riemannian manifold (see Theorem 6.13 of [19]).

Lemma A.0.3. *Let Σ^n be a complete noncompact Riemannian manifold; let $o \in \Sigma^n$ be a reference point and denote by r_o the Riemannian distance function from o. Assume that the sectional curvature of Σ^n satisfies*

$$K_\Sigma \geq -G^2(r_o), \tag{A.1}$$

with $G \in C^1([0, +\infty))$ is such that

$$G(0) > 0, \quad G'(t) \geq 0 \quad \text{and} \quad \frac{1}{G(t)} \notin L^1(+\infty). \tag{A.2}$$

Let $\mathcal{P}$ be a positive semi-definite symmetric tensor on Σ^n. If $\sup \operatorname{tr}(\mathcal{P}) < +\infty$, then the Omori-Yau's maximum principle holds on Σ^n for the semi-elliptic operator $\mathcal{L} = \operatorname{tr}(\mathcal{P} \circ \operatorname{hess})$.

In particular, Lemma A.0.3 remains true if we replace the condition in (A.1) by the stronger condition of Σ^n having sectional curvature bounded from below by a constant.

Remark A.0.4. As it is well known, examples of functions G satisfying condition (A.2) in Lemma A.0.3 are given by (see, for instance [19, 104])

$$G(t) = t \prod_{j=1}^{N} \log^j(t), \quad t \gg 1,$$

where $\log^j$ stands for the j-th iterated logarithm.

Recently, many authors has been studied new forms of the Omori-Yau's maximum principle in order to extend the investigation to a much more general class of differential operators containing the Laplacian operator. We refer to the interested reader the comprehensive book [19] for a complete background about this topic.

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