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Concentration-compactness principle and applications to nonlocal elliptic problems

por

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sob orientação do

Prof. Dr. João Marcos Bezerra do Ó

Tese apresentada ao Corpo Docente do Programa Associado de Pós-Graduação em Matemática - UFPB/UFCG, como requisito parcial para obtenção do título de Doutor em Matemática.

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Resumo

O objetivo principal deste trabalho é analisar princípios de concentração de compacidade para espaços de Sobolev fracionários baseados na concentração de compacidade de P.-L. Lions e no perfil de decomposição para convergência fraca em espaços de Hilbert devido a K. Tintarev e K.-H Fieseler. Como aplicação, abordamos questões sobre a compacidade do funcional energia associado aos seguintes problems elípticos não locais,

$$\begin{cases} (-\Delta)^s u = f(x,u) \quad \text{em } \mathbb{R}^N, \\ (-\Delta)^s u + a(x)u = f(x,u) \quad \text{em } \mathbb{R}^N, \\ \begin{cases} (-\Delta)^s u + V(x)u + \lambda K(x)\phi u = f(x,u) + g(x,u) & \text{em} \quad \mathbb{R}^3, \\ (-\Delta)^\alpha \phi = K(x)u^2 & \text{em} \quad \mathbb{R}^3, \end{cases} \end{cases}$$

onde $0 < s < 1$, $0 < \alpha < 1$, $2\alpha + 4s \geqslant 3$, $\lambda > 0$ e $K(x) \geqslant 0$ pertence a um espaço de Lebesgue adequado. Obtemos resultados de existência para uma vasta classe de potenciais $a(x)$ possivelmente singulares, não necessariamente limitados por baixo por uma constante positiva e para não linearidades oscilatórias em ambos os crescimentos subcríticos e críticos que podem não satisfazer a condição de Ambrosetti-Rabinowitz.

Palavras-chave: Concentração de compacidade; Laplaciano fracionário; expoente crítico de Sobolev; métodos variacionais.

Abstract

The main goal of this work is to analyze concentration-compactness principles for fractional Sobolev spaces based on the concentration–compactness principle of P.-L. Lions and in the profile decomposition for weak convergence in Hilbert spaces due to K. Tintarev and K.-H Fieseler. As application, we address questions on compactness of the associated energy functional to the following nonlocal elliptic problems,

$$\begin{cases} (-\Delta)^s u = f(x,u) \quad \text{in } \mathbb{R}^N, \\ (-\Delta)^s u + a(x)u = f(x,u) \quad \text{in } \mathbb{R}^N, \\ \begin{cases} (-\Delta)^s u + V(x)u + \lambda K(x)\phi u = f(x,u) + g(x,u) & \text{in} \quad \mathbb{R}^3, \\ (-\Delta)^\alpha \phi = K(x)u^2 & \text{in} \quad \mathbb{R}^3, \end{cases} \end{cases}$$

where $0 < s < 1$, $0 < \alpha < 1$, $2\alpha + 4s \geqslant 3$, $\lambda > 0$ and $K(x) \geqslant 0$ belongs to a suitable Lebesgue space. We obtain existence results for a wide class of possible singular potentials $a(x)$, not necessarily bounded away from zero and for oscillatory nonlinearities in both subcritical and critical growth range that may not satisfy the Ambrosetti-Rabinowitz condition.

Keywords: Concentration-compactness; fractional Laplacian; critical Sobolev exponent, variational methods.

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"I wish it need not have happened in my time," said Frodo. 'So do I', said Gandalf, "and so do all who live to see such times. But that is not for them to decide. All we have to decide is what to do with the time that is given us."

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Introduction

The main goal of the present work is to analyze concentration-compactness principles for fractional Sobolev spaces. As an application, we address questions on compactness of the associated energy functional to the following nonlocal equation,

$$(-\Delta)^s u = h(x,u) + \mathcal{L}(u) \quad \text{in } \mathbb{R}^N, \tag{$\mathcal{P}_s$}$$

where $0 < s < 1$, $(-\Delta)^s$ is the fractional Laplacian, $h(x,t)$ is a given function and $\mathcal{L}(u)$ is a nonlocal integral operator.

During the past years there has been a considerable amount of research involving nonlocal nonlinear stationary Schrödinger problems. This equation arises in the study of the fractional Schrödinger equation when looking for standing waves. Indeed, when u is a solution of Eq. ($\mathcal{P}_s$), it can be seen as stationary states (corresponding to solitary waves) in nonlinear equations of Schrödinger type

$$i\phi_t - (-\Delta)^s \phi + h(x,\phi) = 0 \quad \text{in } \mathbb{R}^N.$$

Fractional Schrödinger equations are also of interest in quantum mechanics (see e.g. the appendix in [31] for details and physical motivations). Moreover, we refer to [4], [5] and [18], where equations involving the operator $(-\Delta)^s$ arise from several areas of science such as biology, chemistry or finance.

Roughly speaking, the approach to obtain solutions for Eq. ($\mathcal{P}_s$) using variational methods and critical point theory relies in associating Eq. ($\mathcal{P}_s$) with a functional I (usually called *energy functional*), defined in a appropriated infinite dimensional Banach space of functions H. One defines as *weak solutions* the critical points of I, and with aid of additional results it is expected that these weak solutions satisfy Eq. ($\mathcal{P}_s$) in each point of $\mathbb{R}^N$. For an introduction to variational methods and critical point theory we suggest [6, 26, 32, 72, 101].

In this context, in order to find critical points for I, one can use *minimax theorems*, such as the *mountain pass theorem* with the so called *compactness conditions.* It is considered that H is continuously embedded in a Banach space L (typically $L = L^p(\mathbb{R}^N)$) and with the help of suitable other assumptions on $h(x,t)$ and $\mathcal{L}(u)$, one can find a bounded sequence (u_k) in H satisfying $I(u_k) \to c > 0$ and $I'(u_k) \to 0$. The next step is to prove that (u_k) satisfies the *Palais-Smale compactness condition* at the level c, that is, (u_k) converges in H, up to subsequence. However, if I is invariant under the action of a non-compact group (such as translations or dilations) with respect to the embedding $H \hookrightarrow L$, it is expected that the Palais-Smale condition does not hold for all $c > c_0$, for some non-negative c_0. Also, when a constrained minimization problem is considered (whose minimizers give critical points for I) a similar difficult appears, more precisely, the problem admits bounded minimizing sequences that do not converge, even in a subsequence. One shall notice that convergence of these involved functional sequences are not difficult to be obtained whenever the embedding $H \hookrightarrow L$ is compact.

Problems like Eq. $(\mathcal{P}_s)$, where there is *a priori* difficulty of dealing with the fact that the aforementioned sequences do not posses adherent points in the strong topology or more generally that the convergence $I(u_k) \to I(u)$ is not guaranteed, are called with *lack of compactness.* Fortunately, the same non-compact group of invariances that generates lack of compactness can be employed to restore it, precisely, to obtain the convergence $I(u_k) \to I(u)$. This approach to get compactness through the study of convergence of sequence under the action of invariant non-compact groups is called in the literature *concentration-compactness principle*, and was first introduced in the 1980's by P.-L. Lions in a series of works [65–68], for problems like Eq. $(\mathcal{P}_s)$ where $s = 1$, $H = H^1(\mathbb{R}^N)$ or $H = \mathcal{D}^{1,2}(\mathbb{R}^N)$, and $L = L^p(\mathbb{R}^N)$ or $L = L^{2^*_s}(\mathbb{R}^N)$ are the standard Sobolev spaces, and $2^* = 2N/(N-2)$ is the critical Sobolev exponent.

A lot of research about concentration-compactness has been made since those works of P.-L. Lions (see, e.g., [53, 99] and the references therein). Some of them describe the concentration-compactness phenomena by means of *profile decomposition of weak convergence* for bounded sequences in the considered space of functions and they can be seen as extensions of the celebrated Banach-Alaoglu-Bourbaki Theorem. This kind of profile decomposition has been widely investigated in various settings, for instance we may cite the ones in [55, 57, 71, 87, 89]. It describes how the convergence of a bounded sequence fails under a continuous embedding of the considered space.

In this thesis we develop a concentration-compactness principle via profile decomposition of weak convergence for the fractional Sobolev spaces $H^s(\mathbb{R}^N)$ and $\mathcal{D}^{s,2}(\mathbb{R}^N)$, for $0 < s < N/2$ and $0 < s < 1$, respectively, considering their corresponding embedding in $L^p(\mathbb{R}^N)$, $2 < p < 2_s^*$ and $L^{2_s^*}(\mathbb{R}^N)$, where $2_s^* = 2N/(N-2s)$ is the fractional critical Sobolev exponent, following the abstract version of profile decomposition in Hilbert spaces due to K. Schindler and K. Tintarev [99] and the recent advances due to G. Palatucci and A. Pisante [71]. As an application, under our settings, we prove that Palais-Smale compactness condition holds at the mountain pass level. We also use the specific description of our concentration-compactness principle to improve some well known existence results for Eq. $(\mathcal{P}_s)$, and with this, we expect that our results can lead to a new way to study existence of solutions for nonlocal problems like $(\mathcal{P}_s)$.

It is well known that Eq. $(\mathcal{P}_s)$ admits a variational setting in fractional Sobolev spaces, and the solutions are constructed with a variational method by a minimax procedure on the associated energy functional. However, we note that the usual variational techniques cannot be applied straightly because of a lack of compactness, which roughly speaking, originates from the invariance of $\mathbb{R}^N$ with respect to translation and dilation and, analytically, appears because of the non-compactness of the Sobolev embedding. For instance, it is not possible to apply the minimax type arguments used by P. Felmer et al. [51] and R. Servadei and E. Valdinoci [80] and [81] because their approach rely strongly on the sub-criticality of the nonlinear terms or the boundedness of the domain.

To be more specific about our results, in the following lines, we describe each chapter of this thesis.

In Chapter 0, we give the basic concepts and results that are used through the text, turning ou exposition self contained.

In Chapter 1, it is proved our profile decomposition of weak convergence for the fractional Sobolev spaces $\mathcal{D}^{s,2}(\mathbb{R}^N)$ and $H^s(\mathbb{R}^N)$, Theorems 1.1.1 and 1.1.2. These results are proved in [41, 43]. In [55] and [71, Theorem 4] the authors introduced the subject in the fractional framework, and also, the problem of cocompactness in the sense of [29] was extensively discussed. It seems for us that these new abstract results are more appropriated to study the existence of non-trivial solutions for nonlocal elliptic equations ($\mathcal{P}_s$) than the profile decomposition developed in [71]. It is not clear how one can apply [71, Theorem 4] to obtain such a result for nonlinearities with asymptotically self-similar oscillations about the fractional critical growth (see Sect. 1.5 for precise definitions).

It is also worth to mention that Theorem 1.1.1 can be used to prove the fractional version of Lions concentration-compactness principle proved in [71, Theorem 5]. Indeed, Theorem 1.1.1 improves [71, Theorem 5] for the case $\Omega = \mathbb{R}^N$, since the sums of Dirac masses that appears in this result comes from the profiles given in (1.1.4). We also call attention to the fact that Theorem 1.1.2 is an alternative to the well known fractional Lions Lemma of compactness (see [51, Lemma 2.2]), as can been seen in Sect. 3.6. Finally, we point out some differences from our Theorem 1.1.1 and some results on profile decompositions contained in [55, 71]. The decomposition in Theorem 1.1.1 is based in a discrete group of operators, that is, the dilations in the following form

$$\delta_j u(x) = \gamma^{\frac{N-2s}{2}j} u(\gamma^j x),\ \gamma > 1,\ j \in \mathbb{Z}. \tag{0.0.1}$$

From (0.0.1), we can decompose (in a similar way as in [99, Theorem 5.1]) the collection of the "dislocated profiles" $w^{(n)}$ in three: dilation by "enlargement" ($\mathbb{N}_-$), dilation by "reducement" ($\mathbb{N}_+$), and no dilation (pure translation $\mathbb{N}_0$). This allow us to study scalar field equations involving nonlinearities with critical growth more general than the pure critical power (see Sect. 1.5), the so called *asymptotic self-similar functions* (assumption (f_5) in Sect. 2.1). On the other hand in [55,71] was considered continuous dilations of the form

$$\delta_\lambda u(x) = \lambda^{\frac{N-2s}{2}} u(\lambda x),\ \lambda > 0,$$

and their decomposition holds for all $0 < s < N/2$. We should mention that Theorem 1.1.1 holds for $0 < s \leqslant 1$ and at this point arise a natural question which is to prove this result for $1 < s < N/2$. In [71, Proposition 1] it was proved that $D_{\mathbb{R}^N}$–weak convergence is equivalent to strong convergence in $L^{2_s^*}(\mathbb{R}^N)$, for $0 < s < N/2$ (see Sect. 0.5), where for $\gamma > 1$ given,

$$D_{\mathbb{R}^N} := \left\{ d_{y,j} : \mathcal{D}^{s,2}(\mathbb{R}^N) \to \mathcal{D}^{s,2}(\mathbb{R}^N) : d_{y,j}u(x) = \gamma^{\frac{N-2s}{2}j} u(\gamma^j(x-y)),\ y \in \mathbb{R}^N, j \in \mathbb{R} \right\}.$$

From this we can conclude that the answer to that question is analogously to prove that $D_{\mathbb{R}^N}$–weak convergence is equivalent to the $D_{\mathbb{Z}^N}$–weak convergence in $\mathcal{D}^{s,2}(\mathbb{R}^N)$, for $1 < s < N/2$, where $D_{\mathbb{Z}^N} := \left\{d_{y,j} : y \in \mathbb{Z}^N,\ j \in \mathbb{Z}\right\}$. In the affirmative case, Theorem 1.1.1 can be seen as a corollary of the decomposition given in Theorem [71, Theorem 4, Theorem 8], with minor changes (provided also in Sect. 1.2). Nevertheless, for the case that $0 < s < 1$, we present a proof of this fact (given in Proposition 1.2.3), which can also be seen as an alternative proof of Theorem 1.1.1.

In Chapter 2, which relates to the study made in [41], we discuss the existence of non-trivial weak solutions for the equation

$$(-\Delta)^s u = f(x,u) \quad \text{in } \mathbb{R}^N, \tag{$\mathcal{E}_s$}$$

where $f(x,t)$ is assumed to have critical growth. It corresponds to the case where we take $\mathcal{L}(u) \equiv 0$ and $h(x,t) \equiv f(x,t)$ in Eq. $(\mathcal{P}_s)$, with $f_t(x,0) = 0$.

A lot of work has been devoted to the existence of solutions for nonlinear scalar field equations like Eq. $(\mathcal{E}_s)$, both for local case ($s = 1$) and nonlocal case $0 < s < 1$, since the celebrated works of H. Berestycki and P.-L. Lions [11,12]. In these two papers, the authors discuss the existence of radial solutions of the semi-linear elliptic equation

$$-\Delta u = g(u), \quad u \in H^1(\mathbb{R}^N)(N \geqslant 3), \tag{0.0.2}$$

where $g : \mathbb{R} \to \mathbb{R}$ is a continuous odd function with subcritical growth. Under some appropriate conditions on $g(t)$, they used minimizing arguments to prove (in part I) the existence of a positive radial ground state for (0.0.2), that is, solution having the property of the least action among all possible solutions. In [98], K. Tintarev has treated the non-autonomous problem

$$-\Delta u = g(x,u), \quad u \in \mathcal{D}^{1,2}(\mathbb{R}^N)(N \geqslant 3),$$

when the nonlinearity $g(x,t)$ is allowed to have critical growth with asymptotically self-similar oscillations about the critical power $|t|^{2^*-2}t$. Recently, using some minimax arguments, X. Chang and Z-Q. Wang [24] proved the existence of a positive ground state for fractional scalar field equations of the form $(\mathcal{E}_s)$ when $f(x,t) \equiv f(t)$ has subcritical growth and satisfies the Berestycki–Lions type assumptions. In [110], J.

Zhang et al., established the existence of ground state solutions to the fractional scalar field equation $(\mathcal{E}_s)$, when $f(x,t) \equiv f(t)$ has critical growth.

Motivated by the results cited above, another important purpose of this chapter is to prove the existence of a ground state solution for the nonlinear scalar field equation $(\mathcal{E}_s)$ in the "zero mass case" with nonlinearities in the critical growth range. The idea for proving such kind of result for Eq. $(\mathcal{E}_s)$ in the autonomous case is based in a constrained minimization argument similar to [11]. We obtain the result by using the invariance of the problem with respect to action of the translation and dilation group in $\mathbb{R}^N$, thanks to our concentration-compactness principle and a specific Pohozaev identity. Our argument allow us to avoid the typical assumption that $t^{-1}f(x,t)$ is an increasing function, which is usually required in the approach of constrained minimization over a Nehari manifold. Moreover, to prove the existence for the autonomous case $f(x,t) = f(t)$, we do not require the well known Ambrosetti-Rabinowitz condition.

The proof of that Pohozaev type identity is essentially based in the use of the so called s-harmonic extension introduced by L. Caffarelli and L. Silvestre [19] and remarks contained in [47] and [59]. To the best of our knowledge, this is the first work that shows a Pohozaev type identity for the homogeneous Sobolev space $\mathcal{D}^{s,2}(\mathbb{R}^N)$ and for $f(t)$ in the critical growth range. Our method is very convenient in the sense that with our arguments we can always derive a Pohozaev type identity in the fractional framework without relying in global regularization of the solutions. In the present literature, there are only Pohozev type identities for solutions in the inhomogeneous fractional Sobolev space $H^s(\mathbb{R}^N)$, $0 < s < \min\{1, N/2\}$, and for $f(t)$ with subcritical growth (cf. [24]). Moreover, the argument for the proof relies in obtaining the behavior of solutions in the whole space $\mathbb{R}^N$ (cf. [54]).

Our main results may be seen as the nonlocal counterpart of some theorems of K. Tintarev et al. [97–99]. In comparison with the local case [98], we also mention some additional difficulties: the Pohozaev type identities for the fractional framework available in the literature (cf. [24,54,75]) do not match with our settings; an additional hypothesis (assumption (f_4)) must be considered in order to achieve the concentration-compactness for the non-autonomous case. In fact, the asymptotic additivity (f_4) takes the role to describe precisely the behavior of weak convergence under our settings

(Proposition 2.4.1). At this point a natural question arises: Is hypothesis (f_4) necessary to describe the limit of the profile decomposition terms (see Theorem 1.1.1)? Indeed, we believe that without condition (f_4) it is possible to find examples for which this description fails.

Additionally, in Chapter 2, we introduce a new class of nonlinearities of the critical growth type for the fractional framework, that include the power $|t|^{2_s^*}$ as an example. We believe that this new notion of criticality together with our concentration-compactness, can lead to a new way to approach elliptic problems involving nonlinearities with critical growth and the fractional Laplacian, for instance, replacing the well known nonlinearity $f(x,t) = K(x)|t|^{2_s^*-2}t$, which is often considered to studied existence of solutions for Eq. $(\mathcal{P}_s)$ with aid of [71, Theorem 5], for a general *self-similar* function under our settings,

$$f(x,t) \equiv f(t) = \exp\{b_0(\sin(\ln|t|) + 2)\}(b_0\cos(\ln|t|) + 2_s^*)|t|^{2_s^*-2}t, \quad b_0 > 0,\ f(0) \equiv 0,$$

see also Example 2.2.8 in Chapter 2. For the local case a class of self-similar function was introduced in [78, 97–99].

Moreover, as it is well known, one of the main difficulties in leading with nonlinearities with critical growth condition is proving that the minimax level of the functional associated to Eq. $(\mathcal{E}_s)$ avoids levels of non-compactness, which usually requires additional description of the nonlinearity growth. We avoid this by considering that $f(x,t)$ has appropriated limits consistent with our concentration-compactness and comparing the minimax level of functional associated to Eq. $(\mathcal{E}_s)$ with the limit ones.

In Chapter 3, which relates to the study made in [43], we consider the following nonlocal Schrödinger equation

$$(-\Delta)^s u + a(x)u = f(x,u) \quad \text{in } \mathbb{R}^N, \tag{$\mathcal{H}_s$}$$

where $f(x,t)$ is assumed to have either subcritical or critical growth. It corresponds to the case where $\mathcal{L}(x,u) \equiv 0$ and $h(x,t) \equiv f(x,t) - a(x)t$ in Eq. $(\mathcal{P}_s)$, with $f_t(x,0) = 0$.

First, we would like to mention the progress involving potentials $a(x)$ bounded away from zero and nonlinearities with *subcritical growth.* In [79] S. Secchi investigated the existence of ground state solutions for fractional Schrödinger equations by using a minimization argument on the Nehari manifold. He proved existence results under suitable assumptions on the behavior of the potential $a(x)$ and superlinear growth conditions on the nonlinearity. See also [52], where B. Feng proved the existence of ground state solutions of $(\mathcal{H}_s)$, for the particular case $f(x,t) = |t|^{p-2}t$, where $2 < p < 2(N+2s)/N$, $N \geqslant 2$, by using the P.-L. Lions concentration-compactness principle (see [66]). R. Lehrer et al. [61] studied the existence of solutions through projection over an appropriated Pohozaev manifold, assuming that $f(x,t) = a(x)f_0(t)$, where $f_0(t)$ is asymptotically linear, that is, $\lim_{t\to\infty} f_0(t)t^{-1} = 1$ and $\lim_{|x|\to\infty} a(x) = a_\infty > 0$. For the local case ($s = 1$), R. de Marchi [33] studied existence of non-trivial solutions for $(\mathcal{H}_s)$ assuming that $a(x)$ and $f(x,t)$ are asymptotically 1−periodic in each x_i, $i = 1, \ldots, N$, combining variational methods and the concentration-compactness principle, and also proved existence of ground state solutions when $a(x)$ and $f(x,t)$ are 1−periodic in each x_i, $i = 1, \ldots, N$, without assuming that $t \mapsto f(x,t)t^{-1}$ is an increasing function. By using similar approach, H. Zhang et al [106], studied existence of ground state and infinitely many geometrically distinct solutions for Eq. $(\mathcal{H}_s)$, based on the method of Nehari manifold and Lusternik-Schnirelmann category theory. Moreover, for recent works on nonlinear Schrödinger equations where the classical Ambrosetti-Rabinowitz condition is not required we cite [33, 61, 106].

Problems involving potentials bounded away from zero and *critical Sobolev exponent*, precisely, when $f(x,t) = g(x,t) + |t|^{2_s^*-2}t$, where $g(x,t)$ have subcritical growth, we may refer to [62, 82, 83] and the references given there. In these works, it was crucial the presence of perturbation $g(x,t)$ of the critical power $|t|^{2_s^*-2}t$. Moreover, it was assumed the following condition on the potential

$$0 < \inf_{x\in\mathbb{R}^N} a(x) < \liminf_{|x|\to\infty} a(x)$$

which was introduced by P.L. Rabinowitz in [73] to study the local case of Eq. $(\mathcal{P}_s)$ (see also for the critical case [69]). We cite [27, 35, 86] for works on local Schrödinger equations with nonlinearities of the pure critical power type (without subcritical perturbation term) and inverse square type potentials. For the fractional case we cite [39], where it was studied the existence qualitative properties of positive solutions.

Motivated by the above works, we obtained existence of non-trival solutions for Eq. $(\mathcal{H}_s)$ in several cases, which were not considered by the aforementioned papers. Our potential $a(x)$ may change sign, can have singular points of blow up and even vanish at the infinity, and the nonlinearity can be considered with critical or subcritical oscillatory growth. In the subcritical case we assume a condition on the potential $a(x)$ which ensures the continuous embedding of the associated space of functions similar to [85]. Nevertheless differently from [85], we do not impose assumption on $a(x)$ to guarantee the compactness of the Sobolev embedding. To compensate, we ask that the limit of $a(x)$, as $|x|$ goes to infinity, exists and is positive, or alternatively, that $a(x)$ is $1-$periodic in x_i, $i = 1, \ldots, N$. Moreover, by considering similar assumptions made in [34], the potential does not need to be bounded from below by a constant. We also take account the case where the nonlinearity has oscillatory behavior and does not satisfies the typical assumption of Ambrosetti-Rabinowitz. Similar to the aforementioned papers, the nonlinearity $f(x,t)$ is supposed to has a periodic asymptote $f_{\mathcal{P}}(x,t)$, which allow us to "transfer" the usual assumptions to it. Also we mention that we complement and improve some results of [33], since we consider the nonlocal equation $(\mathcal{H}_s)$ and a case where we do not need the monotonicity of $t \mapsto f_{\mathcal{P}}(x,t)t^{-1}$.

In the critical case, inspired in some ideas contained in [27], we treated in this chapter a class of potentials somehow different, since we consider a general class that include as a particular case the inverse fractional square potential $a(x) = -\lambda|x|^{-2s}$, where $0 < \lambda < \Lambda_{N,s}$ and $\Lambda_{N,s}$ is the sharp constant of the Hardy-Sobolev inequality

$$\Lambda_{N,s} \int_{\mathbb{R}^N} |x|^{-2s} u^2 \, \mathrm{d}x \leqslant \int_{\mathbb{R}^N} |\xi|^{2s} |\mathscr{F}u|^2 \, \mathrm{d}\xi, \quad \forall\, u \in C_0^\infty(\mathbb{R}^N). \tag{0.0.3}$$

Moreover, the sharp constant is precisely given by

$$\Lambda_{N,s} := 2^{2s} \frac{\Gamma^2\left(\frac{N+2s}{4}\right)}{\Gamma^2\left(\frac{N-2s}{4}\right)}, \quad 0 < s < 1, \quad N > 2s, \tag{0.0.4}$$

where Γ is the well known Gamma function. Further details about (0.0.3) can be found in [56] and [103]. In that case, the nonlinearities are suppose to be "self-similar functions", in the sense introduced in Sect. 1.5.

In this chapter, we also proved a more suitable and general version of the Pohozaev identity studied in Chapter 2 (Proposition 3.5.1), which is used to study existence of ground state solution for the case where the potential $a(x)$ has singularities. As a consequence of this Pohozaev type identity, we also proved some non-existence results for Eq. $(\mathcal{H}_s)$. Moreover, using this kind of identity and our concentration-compactness principle, we could avoid the use of monotonicity $t \mapsto f_{\mathcal{P}}(x,t)t^{-1}$ and prove some existence results by comparing the minimax level of the associated energy functional of Eq. ($\mathcal{H}_s$) with the one of the associated limit problem.

It is worth to mention that in this chapter we prove the existence of ground states in three cases: First when ($\mathcal{P}_s$) is invariant under the action of translations in $\mathbb{Z}^N$ (subcritical growth), second when ($\mathcal{P}_s$) is invariant under dilations $\gamma^{(N-2s)j/2}u(\gamma\cdot)$ (critical growth), and third when the monotonicity of $t \mapsto f(x,t)t^{-1}$ is considered.

In Chapter 4, which relates to the study made in [44], we are concerned with existence and non-existence of solutions for the following nonlinear fractional Schrödinger-Poisson System

$$\begin{cases} (-\Delta)^s u + a(x)u + \lambda K(x)\phi u = f(x,u) + g(x,u) & \text{in} \quad \mathbb{R}^3, \\ (-\Delta)^\alpha \phi = K(x)u^2 & \text{in} \quad \mathbb{R}^3, \end{cases} \tag{$\mathcal{SP}$}$$

where $0 < s < 1$, $0 < \alpha < 1$, $2\alpha + 4s \geqslant 3$, $\lambda > 0$. Under suitable conditions over $K(x), a(x)$, $f(x,t)$ and $g(x,t)$ it can be proved that System ($\mathcal{SP}$) is equivalent to the following nonlinear Schrödinger equation with a non-local term,

$$(-\Delta)^s u + a(x)u + \lambda K(x)\phi_\alpha[u]u = f(x,u) + g(x,u). \tag{$\mathcal{S}_{NL}$}$$

It corresponds to the case where $\mathcal{L}(u) \equiv \lambda K(x)\phi_\alpha[u]u$, $h(x,t) \equiv f(x,t) + g(x,t) - a(x)t$ in Eq. ($\mathcal{P}_s$), also $f(x,t)$ and $g(x,t)$ are assumed to have subcritical growth and critical growth respectively, and $f_t(x,0) = g_t(x,0) = 0$. In particular, when $K(x) \equiv 0$, the system ($\mathcal{SP}$) turns in to the fractional Schrödinger equation ($\mathcal{H}_s$).

When $\alpha = s = 1$, the System ($\mathcal{SP}$) reduces to the classical Schrödinger-Poisson System

$$\begin{cases} -\Delta u + a(x)u + \lambda K(x)\phi u = f(x,u) + g(x,u) & \text{in} \quad \mathbb{R}^3, \\ -\Delta \phi = K(x)u^2 & \text{in} \quad \mathbb{R}^3, \end{cases} \tag{0.0.5}$$

which describes systems of identically charged particles interacting each other in the case where magnetic effects can be neglected (see [10]). System (0.0.5) was extensively studied in the past years by many authors, mainly concerning existence and multiplicity of solutions by using variational methods. Here we would like to cite some related results, for instance the ones in [1, 3, 22, 25, 76, 90, 104, 105, 107, 109, 111].

First, we would like to mention the progress concerning (0.0.5) involving potentials $a(x)$ bounded away from zero and nonlinearities $f(x,t)$ with subcritical growth ($g(x,t)\equiv 0$). In [76], D. Ruiz proved (for the local case) existence and non-existence results by considering that $f(x,t)=|t|^{p-2}t$, $2<p<6$, $K(x)\equiv V(x)\equiv 1$ and analyzing the relation between parameters p and λ. He also proved non-existence of non-trivial solutions of (0.0.5) if $2<p<3$ and $\lambda\geqslant 1/4$. In [90] J. Sun and S. Ma obtained existence of ground state solution if $a(x)$ is a continuous and 1−periodic in x_i, $i=1,\dots,N$. In this work it was assumed that $f(x,t)$ has 4-superlinear growth, that is

$$\lim_{|t|\to\infty}\frac{F(x,t)}{t^4}=\infty,\quad \text{where } F(x,t)=\int_0^t f(x,\tau)\,\mathrm{d}\tau.$$

Some results concerning sign–changing potentials have appeared in [105] and the references given there, where the authors proved existence and multiplicity of solutions for (0.0.5) by using a linking type theorem when $f(x,t)$ is either 4-superlinear or sublinear at the infinity. They also considered the case where $a(x)$ satisfies the following conditions: there exists $M>0$ such that $|\{x\in\mathbb{R}^3 : a(x)\leqslant\infty\}|<\infty$, and $\Omega=\mathrm{int}V^{-1}(0)$ is nonempty, $\overline{\Omega}=V^{-1}(0)$ and has smooth boundary.

For the progress involving nonlinearities with critical growth, we start by citing the work of J. Zhang [109]. In this paper, it was proved existence of non-trivial radial solutions when it is taken into account autonomous $g(x,t)\equiv g(t)$ with critical growth at the infinity, and in particular, it is possible to consider nonlinearities perturbations of the form $f(x,t)+g(x,t)=|t|^{p-2}t+|t|^4t$, for $2<p<6$. By using the method of Nehari manifold and concentration compactness principle of P.-L. Lions [66], in [107] H. Zhang et al. considered the case that $a(x)$ is asymptotically periodic and bounded away from zero. They proved existence of ground state with 4-superlinear nonlinearity asymptotically periodic $f(x,t)$ and with critical perturbation $g(x,t)=Q(x)|t|^{2^*-2}t$, where $Q(x)\in L^\infty(\mathbb{R}^3)$ is bounded away from zero and $2^*=6$ is the critical Sobolev exponent.

Regarding System ($\mathcal{SP}$), to the best of our knowledge, there are few papers in the literature which considered it. Here we cite [93, 94, 108, 110]. In [110] it was considered nonlinearities satisfying the almost optimal condition introduced in the work of Berestycki–Lions [11] to study (0.0.5) when $K(x)\equiv 0$. The authors in [110] have proved existence of non-trivial solutions with critical nonlinearities at the infinity, more precisely, they assumed that $\lim_{t\to\infty} g(t)/t^{2_s^*-1}>0$, where $2_s^*=6/(3-2s)$ is the fractional critical Sobolev exponent. Their approach allow the consider nonlinearities in the form $f(x,t) + g(x,t) = |t|^{p-2}t + |t|^{2_s^*-2}t$, $2 < p < 2_s^*$, but it is required that $a(x) \equiv a_0 > 0$. It is also worth to mention that recently K. Teng [93, 94] studied existence of ground states for (0.0.5) under general assumptions for the potential $a(x)$, for $K(x) \equiv 1$, $f(x,t) \equiv |t|^{p-1}t$ for $2 < p < 2_s^* - 1$, with $3/4 < s < 1$, and $g(x,t) \equiv |t|^{2_s^*-1}t$.

We point out that unlike the local case $s = \alpha = 1$, the critical exponent 2_s^* is close to 2, as s approaches to 0. This increases the particular difficult that appears in the Eq. $(\mathcal{S}_{NL})$, since even with the Ambrosetti-Rabinowitz assumption, it is not know in general, if the Palais-Smale sequences associated with the functional of Eq. $(\mathcal{S}_{NL})$ are bounded, for instance, when $f(x,t) \equiv |t|^{p-2}t$, $3 < p < 4$, and $g(x,t) \equiv 0$. To overcome this difficulty, one can attempt to use the abstract result due to L. Jeanjean [58] to construct a bounded sequence at the Mountain Pass level. Nevertheless, the lack of compactness associated with the boundedness of the domain or criticality of the nonlinearity still has to be compensated.

Motivated by the above works, mainly in the formulation made in [1,76], our goal is to obtain existence of non-trivial solutions for Eq. $(\mathcal{S}_{NL})$ under general assumptions following the same ideas of Chapters 2 and 3.

We deal with the case where $a(x)$ is not necessarily bounded away from zero and the nonlinearity $g(x,t)$ is supposed to be a general self-similar function. Our approach relies in assuming that $K(x)$, $a(x)$, $f(x,t)$ and $g(x,t)$ have periodic asymptotes $K_{\mathcal{P}}(x)$, $a_{\mathcal{P}}(x)$, $f_{\mathcal{P}}(x,t)$ and $g_\infty(t)$, respectively. We study the limit problem

$$\begin{cases} (-\Delta)^s u + a_{\mathcal{P}}(x)u + \lambda K_{\mathcal{P}}(x)\phi u = f_{\mathcal{P}}(x,u) + g_\infty(u) & \text{in } \mathbb{R}^3, \\ (-\Delta)^\alpha \phi = K_{\mathcal{P}}(x)u^2 & \text{in } \mathbb{R}^3. \end{cases} \tag{0.0.6}$$

and then compare with the minimax level associated with the energy functional of the respectively problems. In order to prove one of the existence results (the case where $f(x,t)$ and $g(x,t)$ are nonidentical to zero), we used some ideas of [42]. More precisely, we estimate the minimax level of the functional associated with System (0.0.6) to avoid levels of non-compactness for the functional associated with the standard System $(\mathcal{SP})$. This approach allows to "transfer" the usual assumptions made in the nonlinearity to it periodic asymptote, in particular, avoiding the monotonicity of $t \mapsto t^{-1}(f(x,t) + g(x,t))$.

We also studied the autonomous case of $(\mathcal{S}_{NL})$, precisely, when $f(x,t) \equiv f(t)$ and $g(x,t) \equiv g(t)$, does not depends on x. In this case it is not necessary that the nonlinearity has $f(t)$ has 4-superlinear growth. Under general assumptions we proved that Palais-Smale sequences at the mountain pass level are indeed bounded, avoiding the use of L. Jeanjean Theorem [58, Theorem 1.1] to construct one.

Additionally, this chapter provides some non-existence results. Following the ideas developed in the previous chapters, we establish an improved version of a Pohozaev type identity given in [94] for System $(\mathcal{S}_{NL})$. As a consequence, we prove a general version of the non-existence result establish in [93, Theorem 1.6]. Another important issue of this chapter is the study of the existence of ground state solutions for Eq. $(\mathcal{S}_{NL})$. We prove existence of ground states following the basic ideas of Chapter 3, by considering that Eq. $(\mathcal{S}_{NL})$ is invariant under action of translations in $\mathbb{Z}^3$.

Notation and terminology

- $C, \ C_0, \ C_1, \ C_2, \ldots$ denotes positive constants (possibly different) that are independent of the given parameters of the context;

- We consider $\mathbb{R}^{N+1}_+ = \{z = (x, y) \in \mathbb{R}^{N+1} : y > 0\}$;

- Given $R, \delta > 0$ we set

$$\begin{cases} B_{R,\delta} = \{z = (x, y) \in \mathbb{R}^{N+1}_+ : |z|^2 < R^2, \ y > \delta\}, \\ F^1_{R,\delta} = \{z = (x, y) \in \mathbb{R}^{N+1}_+ : |z|^2 < R^2, \ y = \delta\}, \\ F^2_{R,\delta} = \{z = (x, y) \in \mathbb{R}^{N+1}_+ : |x|^2 + y^2 = R^2, \ y > \delta\}; \end{cases}$$

- Given $R > 0$ we set

$$\begin{cases} B_R = \{z = (x, y) \in \mathbb{R}^{N+1} : |z|^2 < R^2\}, \\ B^+_R = B_R \cap \mathbb{R}^{N+1}_+ \text{ and} \\ B^N_R = \{z = (x, y) \in \mathbb{R}^{N+1}_+ : |z|^2 < R^2, \ y = 0\}; \end{cases}$$

- $|A|$ denotes the Lebesgue measure of a set A in $\mathbb{R}^N$;

- $\mathcal{X}_A$ denotes the characteristic function of the set A;

- We use the notation $\Phi(u) = \int_{\mathbb{R}^N} F(x, u)\,\mathrm{d}x$ and $\Phi_\kappa(u) = \int_{\mathbb{R}^N} F_\kappa(u)\,\mathrm{d}x$ for $\kappa = 0, +, -, \infty$ (see for instance Sect. 3.1);

- Given $u : \mathbb{R}^N \to \mathbb{R}$ we consider $u_-(x) = \min\{u(x), 0\}$ and $u_+(x) = \max\{u(x), 0\}$;

- $\mathrm{supp}(u)$ denotes the support of the function u;

- $C(\Omega)$ denotes the space of continuous real functions in $\Omega \subset \mathbb{R}^N$;

- $C_0(\Omega)$ denotes the subspace of $C(\Omega)$ consisting of functions u whose support $\mathrm{supp}(u)$ is compact in Ω;

- Let $k \geqslant 1$ be a integer and Ω an open subset of $\mathbb{R}^N$. $C^k(\Omega)$, denotes the space of k-times continuously differentiable real functions defined over Ω and $C^\infty(\Omega) = \bigcap_{k=1}^\infty C^k(\Omega)$;

- $C_0^k(\Omega) = C^k(\Omega) \cap C_0(\Omega)$ and $C_0^\infty(\Omega) = C^\infty(\Omega) \cap C_0(\Omega)$;

- Let $0 < \alpha < 1$, we denote

$$C^{0,\alpha}(\overline{\Omega}) = \left\{ u \in C(\overline{\Omega}) : \sup_{x,y \in \Omega} \frac{|u(x) - u(y)|}{|x-y|^\alpha} < \infty \right\}$$

as the standard Hölder space. $C^{k,\alpha}(\overline{\Omega})$ are the functions in $C^k(\Omega)$ whose all derivatives up order k belongs to $C^{0,\alpha}(\overline{\Omega})$;

- Given $u : \mathbb{R}^N \to \mathbb{R}$ we use the notation

$$d_k^{(n)} u(x) = \gamma^{\frac{N-2s}{2} j_k^{(n)}} u\big(\gamma^{j_k^{(n)}}(x - y_k^{(n)})\big),$$

to indicate the action of dilations and translations given by the profile decomposition of Theorem 1.1.1;

- $\| \cdot \|_p$ denotes the standard norm of the space $L^p(\mathbb{R}^N)$, for $1 \leqslant p < \infty$;

- $\| \cdot \|_\infty$ denotes the standard norm of the space $L^\infty(\mathbb{R}^N)$;

- We denote (see Sect. 0.2)

$$\begin{aligned} [u,v]_s &= \int_{\mathbb{R}^N} (-\Delta)^{s/2} u (-\Delta)^{s/2} v \, \mathrm{d}x, \quad u, v \in \mathcal{D}^{s,2}(\mathbb{R}^N), \\ (u,v) &= \int_{\mathbb{R}^N} (-\Delta)^{s/2} u (-\Delta)^{s/2} v + uv \, \mathrm{d}x = [u,v]_s + (u,v)_2, \quad u, v \in H^s(\mathbb{R}^N). \end{aligned}$$

- We denote $a_k = o(b_k)$, when $a_k / b_k \to 0$, as $k \to \infty$.

Chapter 0

Preliminaries

For the reader convenience, we dedicate this chapter to review some basic concepts and results that are used through the text. Here, for the sake of discussion, we restrict ourself to state without proofs the results that we find most suitable for this work, considering that the reader is familiarized with basic concepts of Functional Analysis and Measure Theory. Thus making our exposition self-contained. For the interested reader we refer the classical books [7, 45, 88, 92] and the "Hitchhiker" to the fractional Laplacian [36], which inspired the development of this chapter.

0.1 Fourier Analysis

In this section we develop some of the theory for the Fourier transform, which is a essential concept needed to study nonlinear Schrödinger equations involving the fractional Laplacian. The Fourier transform is also powerful tool used to study linear partial differential equations, turning them into either algebraic equations or else differential equations involving fewer variables. In this section all functions are complex-valued, and $-$ denotes the complex conjugate.

Definition 0.1.1 (Fourier transform)**.** The *Fourier transform* of $u \in L^1(\mathbb{R}^N)$, is defined by

$$\hat{u}(x) = \mathscr{F}u(x) = \frac{1}{(2\pi)^{N/2}} \int_{\mathbb{R}^N} u(\xi) e^{-i\xi \cdot x} \, \mathrm{d}\xi, \; x \in \mathbb{R}^N,$$

and its *inverse Fourier transform* by

$$\check{u}(x) = \mathscr{F}^{-1}u(x) = \frac{1}{(2\pi)^{N/2}} \int_{\mathbb{R}^N} u(\xi) e^{i\xi \cdot x} \, \mathrm{d}\xi, \; x \in \mathbb{R}^N.$$

Since $|e^{\pm ix\cdot y}| = 1$ these integral are finite for all $x \in \mathbb{R}^N$. The Fourier transform and its inverse can be extended to functions in $L^2(\mathbb{R}^N)$ through the next well known result.

Theorem 0.1.2 (Plancherel's Theorem)**.** *Assume* $u \in L^1(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$. *Then* $\hat{u}, \check{u} \in L^2(\mathbb{R}^N)$ *and*

$$\int_{\mathbb{R}^N} |\hat{u}|^2 \,\mathrm{d}x = \int_{\mathbb{R}^N} |\check{u}|^2 \,\mathrm{d}x = \int_{\mathbb{R}^N} |u|^2 \,\mathrm{d}x.$$

This means that through Plancherel's Theorem the restriction of the Fourier transform $\mathscr{F}\big|_{L^1(\mathbb{R}^N)\cap L^2(\mathbb{R}^N)}$ can be uniquely extended to a unitary isomorphism in $L^2(\mathbb{R}^N)$, with inverse $\mathscr{F}^{-1}$ (inverse Fourier transform). As an consequence of Plancherel's Theorem we have the following well known formula,

$$\int_{\mathbb{R}^N} u\bar{v} \,\mathrm{d}x = \int_{\mathbb{R}^N} \mathscr{F}u\overline{\mathscr{F}v} \,\mathrm{d}x, \quad \forall\ u, v \in L^2(\mathbb{R}^N).$$

We now pass to define a suitable space of functions that is used in some density arguments in the proof of some of our results.

Definition 0.1.3 (Schwartz space)**.** For any non-negative integer m and any multi-index α we define

$$\|u\|_{(m,\alpha)} = \sup_{x\in\mathbb{R}^N} (1 + |x|)^m |\partial^\alpha u(x)|$$

and the Schwartz space $\mathscr{S}(\mathbb{R}^N)$ as

$$\mathscr{S}(\mathbb{R}^N) = \left\{u \in C^\infty(\mathbb{R}^N) : \|u\|_{(m,\alpha)} < \infty,\ \forall\ m, \alpha\right\}.$$

Thus, the Schwartz space $\mathscr{S}(\mathbb{R}^N)$ is defined consisting of rapidly decaying C^∞ functions in $\mathbb{R}^N$ which, together with all their derivatives, vanish at infinity faster than any power of $|x|$.

Remark 0.1.4. If u belongs to $\mathscr{S}(\mathbb{R}^N)$ then it belongs to $L^p(\mathbb{R}^N)$, for any $1 \leqslant p \leqslant \infty$.

The space $\mathscr{S}(\mathbb{R}^N)$ is related to the Fourier transform due to the fact that $\mathscr{F}$ is an isomorphism from $\mathscr{S}(\mathbb{R}^N)$ onto itself, with inverse $\mathscr{F}^{-1}$ (inverse Fourier transform).

The fractional Laplacian

We are now in condition to make a brief discussion about the operator $(-\Delta)^s$.

Definition 0.1.5. Let u any real valued function defined in $\mathbb{R}^N$ and $s > 0$. The fractional Laplacian $(-\Delta)^s u$ is defined by the relation

$$(-\Delta)^s u(x) = \mathscr{F}^{-1}\left(|\xi|^{2s}\,\mathscr{F}u\right)(x),\ x \in \mathbb{R}^N.$$

As an example that makes $(-\Delta)^s u$ well defined as a real number, we can take any $u \in \mathscr{S}(\mathbb{R}^N)$. That is, $\mathscr{S}(\mathbb{R}^N)$ is the suitable space of functions that makes the fractional Laplacian well defined. For the special case that $0 < s < 1$ the fractional Laplacian of $u \in \mathscr{S}(\mathbb{R}^N)$, can be computed by the following singular integral,

$$(-\Delta)^s u(x) = C(N,s) \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^N \setminus B_\varepsilon(0)} \frac{u(x)-u(y)}{|x-y|^{N+2s}} \, \mathrm{d}y, \quad x \in \mathbb{R}^N, \text{ for } 0 < s < 1,$$

and a suitable positive normalizing constant

$$C(N,s) = \left(\int_{\mathbb{R}^N} \frac{1-\cos \varsigma_1}{|\varsigma|^{N+2s}} \, \mathrm{d}\varsigma \right)^{-1}. \tag{0.1.1}$$

It is worth to define as well, the *Riesz Potential* of a function $u : \mathbb{R}^N \to \mathbb{R}$,

$$\mathcal{I}_\alpha[u](x) := c_\alpha \int_{\mathbb{R}^3} u(y)|x-y|^{2\alpha - N} \, \mathrm{d}y,$$

where

$$c_\alpha = \frac{\Gamma(\frac{3}{2} - 2\alpha)}{\pi^{\frac{3}{2}} 2^{2\alpha} \Gamma(\alpha)}.$$

In a sense, the Riesz potential defines an inverse (or solution operator) for a power of the Laplace operator on Euclidean space and this concept is often used in Chapter 4. More precisely,

$$(-\Delta)^\alpha (\mathcal{I}_\alpha[\varphi]) = \varphi, \quad \forall \, \varphi \in \mathscr{S}(\mathbb{R}^3).$$

The next result describes the conditions needed to consider the Riesz Potential as an operator on Lebesgue spaces.

Proposition 0.1.6. *Let $0 < 2\alpha < N$ and $1 \leqslant p < q < \infty$ such that $1/q = 1/p - 2\alpha/N$. Then for $u \in L^p(\mathbb{R}^N)$, the Riesz potential converges for almost every x and, moreover, if $p \neq 1$, there exists a positive constant C such that*

$$\|\mathcal{I}_\alpha u\|_q \leqslant C \|u\|_p.$$

0.2 Fractional Sobolev Spaces

This section is devoted to the definition (as well to describe some properties) of the function spaces that are used in this text.

Definition 0.2.1 (Homogeneous fractional Sobolev space)**.** Let $0 < s < N/2$. The Homogeneous fractional Sobolev space $\mathcal{D}^{s,2}(\mathbb{R}^N)$ is defined as the completion of the space $C_0^\infty(\mathbb{R}^N)$ with respect to the norm

$$[u]_s^2 := \int_{\mathbb{R}^N} |\xi|^{2s} |\mathscr{F} u|^2 \, \mathrm{d}\xi.$$

Thus, by the well known inequality

$$\int_{\mathbb{R}^N} |u|^{2_s^*}\,\mathrm{d}x \leqslant \mathcal{K}_* \left(\int_{\mathbb{R}^N} |\xi|^{2s}|\mathscr{F}u|^2\,\mathrm{d}\xi\right)^{2_s^*/2}, \quad \forall\ u \in C_0^\infty(\mathbb{R}^N), \quad 0 < s < N/2,$$

where

$$\mathcal{K}_* = \left[2^{-2s}\frac{\Gamma(\frac{N-2s}{2})}{\Gamma(\frac{N+2s}{2})}\left(\frac{\Gamma(N)}{\Gamma(N/2)}\right)^{2s/N}\right]^{2_s^*/2},$$

the space $\mathcal{D}^{s,2}(\mathbb{R}^N)$ is well defined with continuous embedding

$$\mathcal{D}^{s,2}(\mathbb{R}^N) \hookrightarrow L^{2_s^*}(\mathbb{R}^N), \quad \text{for} \quad 0 < s < N/2. \tag{0.2.1}$$

By Placherel Theorem, for $0 < s < N/2$, we have

$$[u]_s^2 = \int_{\mathbb{R}^N} |(-\Delta)^{s/2}u|^2\,\mathrm{d}x, \quad \forall\ u \in C_0^\infty(\mathbb{R}^N).$$

Consequently we can consider $\mathcal{D}^{s,2}(\mathbb{R}^N)$ as a separable Hilbert space when equipped with the inner product

$$[u,v]_s = \int_{\mathbb{R}^N} (-\Delta)^{s/2}u(-\Delta)^{s/2}v\,\mathrm{d}x, \quad \forall\ u,v \in \mathcal{D}^{s,2}(\mathbb{R}^N),$$

as well the characterization

$$\mathcal{D}^{s,2}(\mathbb{R}^N) = \left\{u \in L^{2_s^*}(\mathbb{R}^N) :\ (-\Delta)^{s/2}u \in L^2(\mathbb{R}^N)\right\}.$$

It is of our interest as well to consider the closed subspace of $\mathcal{D}^{s,2}(\mathbb{R}^N)$ consisting of radial functions, that is,

$$\mathcal{D}^{s,2}_{\mathrm{rad}}(\mathbb{R}^N) := \left\{u \in \mathcal{D}^{s,2}(\mathbb{R}^N) : u(x) = u(y),\ \text{provided that } |x| = |y|\right\}.$$

In opposition to the integer case, it is not true in general that $(-\Delta)^{s/2}u$ has compact support wherever $u \in C_0^\infty(\mathbb{R}^N)$. To overcome this particular difficulty when dealing with fractional Sobolev spaces and use a suitable approximation by smooth functions argument, we consider another space of functions, which we describe next.

Definition 0.2.2. We define $\mathscr{S}_0(\mathbb{R}^N)$ as the subspace of $\mathscr{S}(\mathbb{R}^N)$ consisting in all function u such that $\mathscr{F}u \in C_0^\infty(\mathbb{R}^N\backslash\{0\})$.

Remark 0.2.3. For $0 < s < N/2$, the space $\mathscr{S}_0(\mathbb{R}^N)$ is dense in $\mathcal{D}^{s,2}(\mathbb{R}^N)$. Consequently, the space $\mathscr{S}(\mathbb{R}^N)$ is also dense in $\mathcal{D}^{s,2}(\mathbb{R}^N)$.

We now pass to introduce our second main space of functions dealt in this text.

Definition 0.2.4 (Inhomogeneous fractional Sobolev space)**.** Let $0 < s \leqslant N/2$. The inhomogeneous fractional Sobolev space is defined as

$$H^s(\mathbb{R}^N) = \left\{u \in L^2(\mathbb{R}^N) : \; |\xi|^s \mathscr{F} u \in L^2(\mathbb{R}^N)\right\},$$

with norm

$$\|u\|^2 := \int_{\mathbb{R}^N} |\xi|^{2s} |\mathscr{F} u|^2 + u^2 \, \mathrm{d}\xi.$$

Notice that $H^s(\mathbb{R}^N)$ is defined in a similar way as the integer Sobolev space $H^1(\mathbb{R}^N)$. More precisely, it is required that $|\xi|^s \mathscr{F} u$ is well defined and belongs to $L^2(\mathbb{R}^N)$, replacing the weak gradient in the definition of $H^1(\mathbb{R}^N)$.

By Plancherel's Theorem, we have that

$$H^s(\mathbb{R}^N) = \left\{u \in L^2(\mathbb{R}^N) : (-\Delta)^{s/2} u \in L^2(\mathbb{R}^N)\right\}.$$

Moreover, $H^s(\mathbb{R}^N)$ is a separable Hilbert equipped with the norm

$$\|u\|^2 = \int_{\mathbb{R}^N} |(-\Delta)^{s/2} u|^2 \, \mathrm{d}x + u^2 \, \mathrm{d}x,$$

which is induced by the inner product

$$(u, v) := \int_{\mathbb{R}^N} (-\Delta)^{s/2} u (-\Delta)^{s/2} v + uv \, \mathrm{d}x = [u, v]_s + (u, v)_2.$$

Although the nonlocal aspect of the previous concepts, some local properties of $H^s(\mathbb{R}^N)$ can be obtained by considering the next definition.

Definition 0.2.5. For $\Omega \subset \mathbb{R}^N$ open set and $0 < s < 1$, the inhomogeneous fractional Sobolev space is defined as

$$\hat{H}^s(\Omega) = \left\{u \in L^2(\Omega) : \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dxdy < \infty\right\},$$

with the norm

$$\|u\|^2_{\hat{H}^s(\Omega)} := \int_\Omega u^2 \, \mathrm{d}x + \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} \, \mathrm{d}x\mathrm{d}y.$$

Concerning the density of smooth functions in the above fractional Sobolev spaces we have the following result.

Proposition 0.2.6. *For $0 < s \leqslant N/2$ the space $C_0^\infty(\mathbb{R}^N)$ is dense in $H^s(\mathbb{R}^N)$, and for $0 < s < 1$ it is dense in $\hat{H}^s(\mathbb{R}^N)$.*

For the case that $0 < s < 1$, we have

$$[u]_s^2 = \frac{C(N,s)}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,\mathrm{d}x\mathrm{d}y, \quad \forall\ u \in C_0^\infty(\mathbb{R}^N),$$

where the positive constant $C(N,s)$ is given in (0.1.1). Thus, when $\Omega = \mathbb{R}^N$, we see that $H^s(\mathbb{R}^N) = \hat{H}^s(\mathbb{R}^N)$ and the norms $\|\cdot\|$ and $\|\cdot\|_{\hat{H}^s(\mathbb{R}^N)}$ are equivalents.

It turns out that the definition of $H^s(\mathbb{R}^N)$ given in Definition 0.2.4 is more appropriated for the general case $s \geqslant 0$, than Definition 0.2.5, because for $s \geqslant 1$, the integral in (0.2.5) is finite if and only if u is constant (see [13, Proposition 2]). Also we have the continuous embedding

$$H^s(\mathbb{R}^N) \hookrightarrow \begin{cases} L^p(\mathbb{R}^N), & 2 \leqslant p \leqslant 2_s^*, \quad \text{for} \quad 0 < s < N/2, \\ L^p(\mathbb{R}^N), & 2 \leqslant p < \infty, \quad \text{for} \quad s = N/2, \end{cases}$$

and the following compact embedding, for Ω open set of class $C^{0,1}$ with bounded boundary,

$$\hat{H}^s(\Omega) \hookrightarrow L^p(\Omega), \quad 1 \leqslant p < 2_s^*, \quad \text{for } 0 < s < \min\{1, N/2\}. \tag{0.2.2}$$

Since the restriction of functions u in $\mathcal{D}^{s,2}(\mathbb{R}^N)$ to Ω, belongs to $\hat{H}^s(\Omega)$, we have as well the following compact embedding,

$$\mathcal{D}^{s,2}(\mathbb{R}^N) \hookrightarrow L^p_{\mathrm{loc}}(\mathbb{R}^N), \quad 1 \leqslant p < 2_s^*, \quad \text{for} \quad 0 < s < \min\{1, N/2\}. \tag{0.2.3}$$

Thus, every bounded sequence in $H^s(\mathbb{R}^N)$ has subsequence that converges strong in $L^p(\Omega)$, for any compact set Ω of $\mathbb{R}^N$.

We also consider the closed subspace of $H^s(\mathbb{R}^N)$ consisting of radial functions, that is,

$$H^s_{\mathrm{rad}}(\mathbb{R}^N) = \left\{u \in H^s(\mathbb{R}^N) : u(x) = u(y), \text{ provided that } |x| = |y|\right\},$$

which has the well known compact embedding (see [64]),

$$H^s_{\mathrm{rad}}(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N), \quad 2 < p < 2_s^*, \quad \text{for} \quad 0 < s < N/2.$$

We finish this section emphasizing that the Plancherel Theorem also gives the next identity, which is used several times throughout this text

$$\int_{\mathbb{R}^N}(-\Delta)^{s/2}u(-\Delta)^{s/2}v\,\mathrm{d}x = \int_{\mathbb{R}^N}(-\Delta)^s uv\,\mathrm{d}x, \quad \forall\ u \in H^{2s}(\mathbb{R}^N),\ v \in H^s(\mathbb{R}^N). \tag{0.2.4}$$

0.3 The s-harmonic extension

We now introduce the harmonic extension following [59, Sect. 2] and for that we begin defining a class of weighted Sobolev spaces suitable to work with this harmonic extension. First, observe that, for any $0 < s < 1$, the function $z = (x,y) \mapsto |y|^{1-2s}$ belongs to the Muckenhoupt class $\mathcal{A}_2$ of weights in $\mathbb{R}^{N+1}$, that is

$$\left(\frac{1}{|B|}\int_B |y|^{1-2s}\,\mathrm{d}x\mathrm{d}y\right)\left(\frac{1}{|B|}\int_B |y|^{2s-1}\,\mathrm{d}x\mathrm{d}y\right) \leqslant C, \quad \text{for all ball } B \text{ in } \mathbb{R}^{N+1}.$$

More details can be found in [46]. Let Q be a open set in $\mathbb{R}^{N+1}$, we consider $L^2(Q,|y|^{1-2s})$ as the Banach space of the Lebesgue measurable functions v defined in Q such that

$$\|v\|_{L^2(Q,|y|^{1-2s})} = \left(\int_Q |y|^{1-2s}v^2\,\mathrm{d}x\mathrm{d}y\right)^{1/2} < \infty.$$

We also consider the space $H^1(Q,|y|^{1-2s})$ of the functions w in $L^2(Q,|y|^{1-2s})$ such that its weak derivatives w_{z_i} exists and belongs to $L^2(Q,|y|^{1-2s})$ for $i = 1,\dots,N+1$. It is easy to see that $H^1(Q,|y|^{1-2s})$ is a Hilbert space with inner product

$$(v_1,v_2)_{H^1(Q,|y|^{1-2s})} = \int_Q |y|^{1-2s}\langle\nabla v_1,\nabla v_2\rangle + |y|^{1-2s}v_1v_2\,\mathrm{d}x\mathrm{d}y,$$

and the induced norm

$$\|v\|_{H^1(Q,|y|^{1-2s})} = \left(\int_Q |y|^{1-2s}|\nabla v|^2 + |y|^{1-2s}v^2\,\mathrm{d}x\mathrm{d}y\right)^{1/2}.$$

We call attention to the fact that the space of smooth functions $C^\infty(Q)\cap H^1(Q,|y|^{1-2s})$ is dense in the weighted Sobolev space $H^1(Q,|y|^{1-2s})$ (see [100] for further details).

Regarding the space $H^1(Q,y^{1-2s})$ with $Q = \Omega\times(0,R)$, where $\Omega\subset\mathbb{R}^N$ is a domain with Lipschitz boundary, it is well known the existence of a well-defined trace operator

$$t_r : H^1(Q,y^{1-2s}) \to H^s(\Omega)$$

with

$$\|t_r(v)\|_{H^s(\Omega)} \leqslant C\|v\|_{H^1(Q,y^{1-2s})}, \quad \forall\ v \in H^1(Q,y^{1-2s}),$$

where $C > 0$, depends only on N, s and Ω (see also [70]). Moreover, by the continuous embedding $H^s(\Omega)\hookrightarrow L^{2_s^*}(\Omega)$, we have

$$\|t_r(v)\|_{L^{2_s^*}(\Omega)} \leqslant C\|v\|_{H^1(Q,y^{1-2s})}, \quad \forall\ v \in H^1(Q,y^{1-2s}). \tag{0.3.1}$$

Let

$$P_s(x,y) = \beta(N,s)\frac{y^{2s}}{(|x|^2+y^2)^{\frac{N+2s}{2}}},$$

where $\beta(N,s)$ is such that

$$\int_{\mathbb{R}^N} P_s(x,1)\,\mathrm{d}x = 1,$$

and $0 < s < 1$. For $u \in \mathcal{D}^{s,2}(\mathbb{R}^N)$ let us set the $s-$harmonic extension of u,

$$w(x,y) = E_s(u)(x,y) := \int_{\mathbb{R}^N} P_s(x-\xi,y)u(\xi)\,\mathrm{d}\xi, \quad (x,y) \in \mathbb{R}^{N+1}_+.$$

Then, for any compact subset K of $\overline{\mathbb{R}^{N+1}_+}$, we have $w \in L^2(K, y^{1-2s})$, $\nabla w \in L^2(\mathbb{R}^{N+1}_+, y^{1-2s})$ and $w \in C^\infty(\mathbb{R}^{N+1}_+)$. Moreover, w satisfies

$$\begin{cases} \operatorname{div}(y^{1-2s}\nabla w) = 0, & \text{in } \mathbb{R}^{N+1}_+, \\ -\lim\limits_{y\to 0^+} y^{1-2s}w_y(x,y) = \kappa_s(-\Delta)^s u(x) & \text{in } \mathbb{R}^N, \\ \|\nabla w\|^2_{L^2(\mathbb{R}^{N+1}_+, y^{1-2s})} = \kappa_s[u]^2_s, \end{cases} \tag{0.3.2}$$

where we understand (0.3.2) in the distribution sense, where $\kappa_s = 2^{1-2s}\Gamma(1-s)/\Gamma(s)$, and Γ is the gamma function. Precisely,

$$\int_{B_R^+} y^{1-2s}\langle\nabla w, \nabla\varphi\rangle\,\mathrm{d}x\mathrm{d}y = \kappa_s\int_{B_R^N}(-\Delta)^{s/2}u(-\Delta)^{s/2}(t_r\varphi)\,\mathrm{d}x, \quad \forall\ \varphi \in C_0^\infty(B_R^+ \cup B_R^N),$$

where for $R > 0$. More generally, given $h : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ we say that a function $v \in H^1(B_R^+, y^{1-2s})$ is a weak solution of the problem

$$\begin{cases} \operatorname{div}(y^{1-2s}\nabla v) = 0 & \text{in } B_R^+, \\ -\lim\limits_{y\to 0^+} y^{1-2s}v_y(x,y) = \kappa_s h(x, t_r(v)(x)) & \text{in } B_R^N, \end{cases} \tag{0.3.3}$$

if, for all $\varphi \in C_0^\infty(B_R^+ \cup B_R^N)$, we have

$$\int_{B_R^+} y^{1-2s}\langle\nabla v, \nabla\varphi\rangle\,\mathrm{d}x\mathrm{d}y = \kappa_s\int_{B_R^N} h(t_r(v))t_r(\varphi)\,\mathrm{d}x, \tag{0.3.4}$$

and the above integrals are finite.

0.4 Regularity results

Following the approach made in [59] we now describe how the s-harmonic extension can be used to obtain regularity for solutions of elliptic problems involving the fractional Laplacian. For this purpouse we state some results of [59]. Next, we consider $Q_R = B_R^N \times (0, R)$ and $C^\alpha(\Omega)$ to denote $C^{[\alpha],\alpha-[\alpha]}(\Omega)$, where $[\alpha]$ is the integer part of the number $\alpha > 0$. We always assume that $0 < s < 1$.

Proposition 0.4.1. *(i) [59, Proposition 2.6] Let $A(x),\ B(x) \in L^p(B_R)$, for some $p > N/2s$. There exists $\alpha \in (0,1)$ depending only on $N,\ s,\ p,\ \|a(x)\|_{L^p(B_R)}$ such that any weak solution v of (0.3.3), with $h(x,t) = A(x)t + B(x)$, is in $C^\alpha(\overline{Q}_{R/2})$.*

(ii) [59, Theorem 2.14] Let $v \in H^1(Q_R, y^{1-2s})$ be a weak solution of (0.3.3) and $h(x,t) \equiv h(x) \in C^\alpha(B_R^N)$ for some $0 < \alpha \notin \mathbb{N}$. If $2s + \alpha$ is not an integer, then $t_r(v)$ is in $C^{2s+\alpha}(B_{R/4}^N)$.

(iii) [59, Proposition 2.13] Let $A(x),\ B(x) \in C^k(B_R^N)$ and $v \in H^1(Q_R, y^{1-2s})$ be a weak solution of (0.3.3) in Q_R, with $h(x,t) = A(x)t + B(x)$, where k is a positive integer. Then $\nabla_x v \in H^1(Q_R, y^{1-2s}) \cap C^\alpha(\overline{Q}_{2/3R})$, for some $\alpha \in (0,1)$, where $\nabla_x v = (v_{x_1}, \dots, v_{x_N})$.

(iv) [59, Lemma 2.18] or [17, Lemma 4.5] Let $h(x,t) \equiv h(x) \in C^\alpha(B_R)$ for some $\alpha \in (0,1)$ and $v \in L^\infty(Q_R) \cap H^1(Q_R, y^{1-2s})$ be a weak solution of (0.3.3). Then there exists $\beta \in (0,1)$ depending only on $N,\ s,\ \alpha$ such that $y^{1-2s}v_y(x,y) \in C^\beta(\overline{Q}_{R/2})$.

We can resume the previous result in the next one, turning our discussion more clearer.

Proposition 0.4.2. *Let $v \in H^1(B_R^+, y^{1-2s})$ be a weak solution of (0.3.3). Suppose that $h(t) \in C^1(\mathbb{R})$ satisfies*

$$\exists\ \mathcal{C}_1,\ \mathcal{C}_2 > 0,\ 2 < p < 2_s^* : |h(t)| \leqslant \mathcal{C}_1 |t|^{p-1} + \mathcal{C}_2(|t| + |t|^{2_s^*-1}),\ \forall t \in \mathbb{R}.$$

If $t_r(v) \in L^p_{\text{loc}}(\mathbb{R}^N)$, for some $p_0 > 2_s^*$, then for any $R > 0$ there exists $0 < y_0,\ r < R$ with $B_r^N \times [0, y_0] \subset B_R^+$, and $\alpha \in (0,1)$, such that

$$v,\ \nabla_x v,\ y^{1-2s}v_y \in C^{0,\alpha}(B_r^N \times [0, y_0]). \tag{0.4.1}$$

Proof. (i) In fact, since

$$\frac{h(t_r v)}{1 + |t_r v|} \in L^q_{\text{loc}}(\mathbb{R}^N), \quad \forall\ N/2s < q \leqslant p_0/(2_s^* - 2),$$

and

$$h(t_r v) = \frac{h(t_r v)}{1 + |t_r v|}\,\text{sgn}(t_r v) t_r v + \frac{h(t_r v)}{1 + |t_r v|},$$

we can use Proposition 0.4.1–(i) to get that v belongs to $C^\alpha(\overline{Q}_{R/2})$, for some $\alpha \in (0,1)$.
(ii) Since $h(t) \in C^1(\mathbb{R})$, thanks to Proposition 0.4.1–(ii) we can apply a bootstrap

argument to obtain that $t_r(v) \in C^{\alpha_1}(B_{R/4k})$, $\alpha_1 \in (1,2)$, for some positive integer k.
(iii) To get that $\nabla_x v \in H^1(Q_R, y^{1-2s}) \cap C^{\alpha_2}(\overline{Q}_{R/6k})$, for some $\alpha_2 \in (0,1)$, we apply Proposition 0.4.1–(iii) with $A(x) = 0$ and $B(x) = h(v) \in C^1(\mathbb{R})$.
(iv) Finally, the fact that $y^{1-2s}v_y(x,y) \in C^{\alpha_3}(\overline{Q}_{R/2})$, $\alpha_3 \in (0,1)$, follows by using Proposition 0.4.1–(iv) in item (i) of this proof. ■

Remark 0.4.3. Let $v \in H^1(Q_R, y^{1-2s})$ be a weak solution of (0.3.3). If v possess the regularity described in (0.4.1), then v satisfies the conditions in (0.3.3) for each point of $B_R^+ \cup B_R^N$ (classical sense). Moreover, denoting $\mathcal{N}_v(x,y) = y^{1-2s}v(x,y)$, we have that

$$\mathcal{N}_v(x,0) = \kappa_s h(v(x,0)), \quad \forall\, x \in B_R^N. \tag{0.4.2}$$

Indeed, the fact that v satisfies the first equation in (0.3.3) for each point in B_R^+ follows by standard elliptic interior regularity arguments using the difference quotient technique (see [20]). To prove that condition (0.4.2) holds, we take $\varphi \in C_0^\infty(B_R^+ \cup B_R^N)$ and use integration by parts formula to get

$$0 = \int_{B_{R,\delta}} \operatorname{div}(y^{1-2s}\nabla v)\,\mathrm{d}x\mathrm{d}y = \int_{B_{R,\delta}} y^{1-2s}\langle \nabla v, \nabla\varphi\rangle\,\mathrm{d}x\mathrm{d}y - \int_{F_{R,\delta}^1} y^{1-2s}v_y\varphi\,\mathrm{d}x,$$

where it is used the fact that $\varphi = 0$ over $F_{R,\delta}^2$ and that $\eta = (0,\ldots,0,-1)$ is the normal vector of $F_{R,\delta}^1$. Now notice that

$$\begin{aligned}\int_{F_{R,\delta}^1} y^{1-2s}v_y\varphi\,\mathrm{d}x &= \int_{B^N_{\sqrt{R^2-\delta^2}}} \delta^{1-2s}v_y(x,\delta)\varphi(x,\delta)\,\mathrm{d}x\\ &= \int_{B_R^N} \delta^{1-2s}v_y(x,\delta)\mathcal{X}_{B^N_{\sqrt{R^2-\delta^2}}}(x)\varphi(x,\delta)\,\mathrm{d}x,\end{aligned}$$

Thus, by Dominated convergence theorem, we obtain that

$$\lim_{\delta\to 0}\int_{F_{R,\delta}^1} y^{1-2s}v_y\varphi\,\mathrm{d}x = \int_{B_R^N} \mathcal{N}_v(x,0)\varphi(x,0)\,\mathrm{d}x.$$

Consequently, from definition (0.3.4), we have

$$\begin{aligned}\kappa_s\int_{B_R^N} h(v(x,0))\varphi(x,0)\,\mathrm{d}x &= \kappa_s\int_{B_R^N} h(t_r(v))t_r(\varphi)\,\mathrm{d}x\\ &= \int_{B_R^+} y^{1-2s}\langle\nabla v,\nabla\varphi\rangle\,\mathrm{d}x\mathrm{d}y = \int_{B_R^N}\mathcal{N}_v(x,0)\varphi(x,0)\,\mathrm{d}x.\end{aligned}$$

Since, $\varphi \in C_0^\infty(B_R^+ \cup B_R^N)$ is arbitrary, condition (0.4.2) follows.

0.5 D-weak convergence and dislocation spaces

As already mentioned, to achieve the decomposition described in the Introduction, we follow the abstract approach of D-weak convergence and dislocation spaces developed in [99]. In this section we state the basic concepts of this abstract approach.

Definition 0.5.1. [99, Definition 3.1] Let D be a set of bounded linear operators on a Hilbert space H, such that for every $g \in D$, $\inf_{u\in H,\|u\|=1} \|gu\| > 0$. We will say that the sequence $(u_k) \subset H$ converges to u D-weakly in H, which we will denote as

$$u_k \overset{D}{\rightharpoonup} u, \text{ in } H,$$

if for any sequence $(g_k) \subset D$,

$$(g_k^* g_k)^{-1} g_k^* (u_k - u) \rightharpoonup 0 \text{ in } H.$$

Let H be a Hilbert space and (g_k) a sequence of bounded linear operators in H. It is commonly used in [99] the notation $g_k \rightharpoonup 0$ to indicate that $g_k u \rightharpoonup 0$ in H for all $u \in H$.

Definition 0.5.2. [99, Definition 3.2] Let H be a separable infinite-dimensional Hilbert space. A set D of bounded linear operators on H is a set of dislocations if

$$0 < \delta := \inf_{g\in D,\|u\|=1} \|gu\|^2 \leqslant \sup_{g\in D,\|u\|=1} \|gu\|^2 < \infty,$$
$$(u_k) \subset H,\ (g_k) \subset D,\ u_k \rightharpoonup 0 \text{ in } H \Rightarrow g_k^* g_k u_k \rightharpoonup 0 \text{ in } H,$$

and, whenever $(u_k) \subset H$ and $(g_k), (h_k) \subset D$,

$$h_k^* g_k \not\rightharpoonup 0,\ (g_k^* g_k)^{-1} g_k^* u_k \rightharpoonup 0 \text{ in } H \Rightarrow (h_k^* h_k)^{-1} h_k^* u_k \rightharpoonup 0 \text{ in } H.$$

The pair (H, D) is called a dislocation space.

The next result give a sufficient condition to establish if a pair (H, D) is a dislocation space. An linear bounded operator $g : H \to H$ is said to be unitary when $g^* = g^{-1}$.

Proposition 0.5.3. *[99, Proposition 3.1] Let H be a separable infinite-dimensional Hilbert space and D be a group (under the operator multiplication) of unitary operators $g : H \to H$. If*

$$g_k \not\rightharpoonup 0 \text{ in } H,\ g_k \in D \Rightarrow g_k u \text{ has a convergent subsequence, for all } u \in H,$$

then (H, D) is dislocation space.

The next result provides a profile decomposition for bounded sequences in separable Hilbert spaces. It is the main tool of our approach to obtain the decomposition described in the Introduction, and it can be seen as a generalization of the celebrated Banach-Alaoglu-Bourbaki Theorem for Hilbert spaces. In fact, as it can be seen, it gives further properties about the weak convergence in terms of D-weak convergence.

Theorem 0.5.4. *[99, Theorem 3.1] Let (H, D) be a dislocation space. If $(u_k) \subset H$ is a bounded sequence, then there exists a set $\mathbb{N}_0 \subset \mathbb{N}$, and sequences $(w^{(n)})_{n\in\mathbb{N}_0} \subset H, (g_k^{(n)})_{k\in\mathbb{N}} \subset D$, $g_k^{(1)} = id$, with $n \in \mathbb{N}_0$, such that for a subsequence of (u_k),*

$$\left(g_k^{(n)*} g_k^{(n)}\right)^{-1} g_k^{(n)*} u_k \rightharpoonup w^{(n)} \text{ in } H, \tag{0.5.1}$$

$$g_k^{(n)*} g_k^{(m)} \rightharpoonup 0 \text{ for } n \neq m. \tag{0.5.2}$$

$$\sum_{n\in\mathbb{N}_0} \|w^{(n)}\|^2 \leqslant \delta^{-1} \limsup_k \|u_k\|^2. \tag{0.5.3}$$

$$u_k - \sum_{n\in\mathbb{N}_0} g_k^{(n)} w^{(n)} \overset{D}{\rightharpoonup} 0 \text{ in } H, \tag{0.5.4}$$

where the series $\sum_{n\in\mathbb{N}_0} g_k^{(n)} w^{(n)}$ converges uniformly in k.

Remark 0.5.5. As mentioned in [99, proof of Theorem 3.1], estimate (0.5.3) holds, provided conditions (0.5.1) and (0.5.2) are satisfied.

It is also convenient for our objectives to review the notion of cocompact embedding.

Definition 0.5.6. [29, Definition 1.2] Let H and L be Banach spaces such that H is continuously embedded into L. Let D be a group of continuous isomorphism on H. We say that the embedding of H into L is cocompact relative to D if every D-weakly convergent sequence in H converges in L.

It is proved in [29] that the embedding $H^s(\mathbb{R}) \hookrightarrow L^p(\mathbb{R}^N)$, $0 < s < N/2$, $2 < p < 2_s^*$, is cocompact with respect to the group of translations. We prove in Chapter 1 that the embedding $\mathcal{D}^{s,2}(\mathbb{R}^N) \hookrightarrow L^{2_s^*}(\mathbb{R}^N)$, $0 < s < \min\{1, N/2\}$, is cocompact with respect to the group of dilations. See Propositions 1.2.3 and 1.3.2 for the precise statements.

0.6 Some variational results

In this section we review some basic variational concepts and results that are used to prove the main results of this thesis. In what follows we always assume that I is a C^1 functional defined over a real Banach space E.

Definition 0.6.1. We say that I has the mountain pass geometry when

(i) $I(0) = 0$;

(ii) There exists $r,\ b > 0$ such that $I(u) \geqslant b$, whenever $\|u\| = r$;

(iii) There is $e \in E$ with $\|e\| > r$ and $I(e) < 0$;

We define the mountain pass (or minimax) level of I as

$$c = \inf_{\gamma \in \Gamma} \sup_{t \in [0,1]} I(\gamma(t)),$$

where

$$\Gamma = \{\gamma \in C([0,1], E) : \gamma(0) = 0,\ \|\gamma(1)\| > r,\ I(\gamma(1)) < 0\}.$$

The next theorems are the main tool used in this text to obtain existence of non-trivial weak solutions for Eq. $(\mathcal{P}_s)$. They ensure the existence of a Palais-Smale sequence at the level c.

Theorem 0.6.2 (Mountain Pass Theorem, see [2,16])**.** *Suppose that I has the mountain pass geometry. Then there exists $(u_k) \subset E$ such that $I(u_k) \to c$ and $I'(u_k) \to 0$ in E^*.*

Theorem 0.6.3 (see [21, 77])**.** *Assume that I has the mountain pass geometry. Then there exists $(u_k) \subset E$ such that $I'(u_k) \to 0$ and $(1 + \|u_k\|)\|I'(u_k)\|_* \to 0$, where $\|\cdot\|_*$ denote the usual norm of the dual E^*.*

Theorem 0.6.4. *[63, Theorem 2.3] If I has the mountain pass geometry and there exists $\gamma_0 \in \Gamma$ such that*

$$c = \max_{t \in [0,1]} I(\gamma_0(t)),$$

then I possess a non-trivial critical point $u \in \gamma_0([0,1])$ such that $I(u) = c$.

Chapter 1

Profile decomposition for weak convergence in fractional Sobolev spaces

In this chapter we develop our concentration-compactness principle, a refinement of the celebrated Banach-Alaoglu-Bourbarki Theorem for the fractional Sobolev spaces $H^s(\mathbb{R}^N)$ and $\mathcal{D}^{s,2}(\mathbb{R}^N)$.

As mentioned in [29], the abstract Hilbert space version given in Theorem 0.5.4, states that, chosen a suitable group of linear operators D acting in a separable Hilbert space H, every bounded sequence in H has a subsequence that D-weakly converges with the following distinct structure: Each term in the subsequence is the sum of a principal term and a remainder term (see assertion (0.5.4) in Theorem 0.5.4). In particular, taking $g = Id$, in the definition of D-weak convergence in (0.5.4), we have

$$u_k - w^{(1)} - \sum_{n \in \mathbb{N}_0 \backslash \{1\}} g_k^{(n)} w^{(n)} \rightharpoonup 0 \text{ in } H.$$

The corrected sequence form a sequence which converges weakly, and each principal term is a (possibly infinite) sum of "dislocated profiles" $w^{(n)}$. Thus, the statement of Banach-Alaoglu-Bourbarki Theorem (for Hilbert spaces) can be seen when $w^{(n)} = 0$, for all $n \in \mathbb{N}_0 \backslash \{1\}$. At the end of this chapter we discuss the class of nonlinearities in the critical growth range dealt in this thesis.

1.1 Statement of the results

We start by profile decomposition for weak convergence in the homogeneous fractional Sobolev space $\mathcal{D}^{s,2}(\mathbb{R}^N)$, which is used to obtain existence of solution for Eq. $(\mathcal{P}_s)$ when the nonlinearity has critical growth.

Theorem 1.1.1. *Let* $(u_k) \subset \mathcal{D}^{s,2}(\mathbb{R}^N)$ *be a bounded sequence,* $0 < s < \min\{1, N/2\}$ *and* $\gamma > 1$. *Then there exist* $\mathbb{N}_* \subset \mathbb{N}$, *disjoints sets (if non-empty)* $\mathbb{N}_0, \mathbb{N}_-, \mathbb{N}_+ \subset \mathbb{N}$, *with* $\mathbb{N}_* = \mathbb{N}_0 \cup \mathbb{N}_+ \cup \mathbb{N}_-$ *and sequences* $(w^{(n)})_{n\in\mathbb{N}_*} \subset \mathcal{D}^{s,2}(\mathbb{R}^N)$, $(y_k^{(n)})_{k\in\mathbb{N}} \subset \mathbb{Z}^N$, $(j_k^{(n)})_{k\in\mathbb{N}} \subset \mathbb{Z}$, $n \in \mathbb{N}_*$, *such that, up to subsequence of* (u_k),

$$\gamma^{-\frac{N-2s}{2}j_k^{(n)}} u_k\big(\gamma^{-j_k^{(n)}} \cdot + y_k^{(n)}\big) \rightharpoonup w^{(n)}, \text{ as } k \to \infty, \text{ in } \mathcal{D}^{s,2}(\mathbb{R}^N), \tag{1.1.1}$$

$$\big|j_k^{(n)} - j_k^{(m)}\big| + \big|\gamma^{j_k^{(n)}}(y_k^{(n)} - y_k^{(m)})\big| \to \infty, \text{ as } k \to \infty, \text{ for } m \neq n, \tag{1.1.2}$$

$$\sum_{n\in\mathbb{N}_*} [w^{(n)}]_s^2 \leqslant \limsup_{k\to\infty} [u_k]_s^2, \tag{1.1.3}$$

$$u_k - \sum_{n\in\mathbb{N}_*} \gamma^{\frac{N-2s}{2}j_k^{(n)}} w^{(n)}\big(\gamma^{j_k^{(n)}}(\cdot - y_k^{(n)})\big) \to 0, \text{ as } k \to \infty, \text{ in } L^{2_s^*}(\mathbb{R}^N), \tag{1.1.4}$$

and the series in (1.1.4) *converges uniformly in* k. *Furthermore,* $1 \in \mathbb{N}_0$, $y_k^{(1)} = 0$; $j_k^{(n)} = 0$ *whenever* $n \in \mathbb{N}_0$; $j_k^{(n)} \to -\infty$ *whenever* $n \in \mathbb{N}_-$; *and* $j_k^{(n)} \to +\infty$ *whenever* $n \in \mathbb{N}_+$.

As it could be viewed, Theorem 1.1.1 describes how the convergence of bounded sequences in $\mathcal{D}^{s,2}(\mathbb{R}^N)$ fails to converge in $L^{2_s^*}(\mathbb{R}^N)$. This "error" of convergence is generated, roughly speaking, by the invariance of action of the group of translation and dilation in $\mathcal{D}^{s,2}(\mathbb{R}^N)$. Observe that the behavior for the correction term in (1.1.4) is precisely described in the assertions (1.1.1)–(1.1.3).

The following version of Theorem 0.5.4 for the fractional Sobolev space $H^s(\mathbb{R}^N)$ is used to study about existence of solutions for Eq. $(\mathcal{P}_s)$ when $h(x,t)$ admits subcritical growth. Next we set $2_s^* = \infty$, when $s = N/2$.

Theorem 1.1.2. *Let* $(u_k) \subset H^s(\mathbb{R}^N)$ *be a bounded sequence with* $0 < s \leqslant N/2s$. *Then there exist* $\mathbb{N}_0 \subset \mathbb{N}$, *and sequences* $(w^{(n)})_{n\in\mathbb{N}_0}$ *in* $H^s(\mathbb{R}^N)$, $(y_k^{(n)})_{k\in\mathbb{N}}$ *in* $\mathbb{Z}^N$, $n \in \mathbb{N}_0$, *such that, for a subsequence of* (u_k),

$$u_k(\cdot + y_k^{(n)}) \rightharpoonup w^{(n)}, \text{ as } k \to \infty, \text{ in } H^s(\mathbb{R}^N), \tag{1.1.5}$$

$$|y_k^{(n)} - y_k^{(m)}| \to \infty, \text{ as } k \to \infty, \text{ for } m \neq n, \tag{1.1.6}$$

$$\sum_{n\in\mathbb{N}_0} \|w^{(n)}\|^2 \leqslant \limsup_{k\to\infty} \|u_k\|^2, \tag{1.1.7}$$

$$u_k - \sum_{n\in\mathbb{N}_0} w^{(n)}(\cdot + y_k^{(n)}) \to 0, \text{ as } k \to \infty, \text{ in } L^p(\mathbb{R}^N), \tag{1.1.8}$$

for any $p \in (2, 2_s^*)$*. Moreover, the series in* (1.1.8) *converges uniformly in* k*.*

Remark 1.1.3. The profile decompositions in Theorems 1.1.1 and Theorem 1.1.2 are unique up to a permutation of index, and up to constant operator. See [99, Proposition 3.4].

As it can be seen, Theorem 1.1.2 describes how bounded sequences in $H^s(\mathbb{R}^N)$ fail to converge in $L^p(\mathbb{R}^N),\ 2 < p < 2_s^*$. This "error" of convergence is produced by the invariance of action of translations in $H^s(\mathbb{R}^N)$.

1.2 Proof of Theorem 1.1.1

To prove it, roughly speaking, we take D as the group of dilations and translations (the precise description of D is given below) in $\mathcal{D}^{s,2}(\mathbb{R}^N)$, $0 < s < N/2$, and describe the behavior of those operators under the weak convergence. We consider

$$T_{\mathbb{R}^N} := \left\{g_y : \mathcal{D}^{s,2}(\mathbb{R}^N) \to \mathcal{D}^{s,2}(\mathbb{R}^N) : g_y u(x) = u(x-y),\ y \in \mathbb{R}^N\right\},$$

and for $\gamma > 1$,

$$\delta_{\mathbb{R}} := \left\{\delta_j : \mathcal{D}^{s,2}(\mathbb{R}^N) \to \mathcal{D}^{s,2}(\mathbb{R}^N) : \delta_j u(x) = \gamma^{\frac{N-2s}{2}j} u(\gamma^j x),\ j \in \mathbb{R}\right\}, \tag{1.2.1}$$

the groups of operators on $\mathcal{D}^{s,2}(\mathbb{R}^N)$ induced by translations and dilations on $\mathbb{R}^N$, respectively. One can easily check that $T_{\mathbb{R}^N}$ and $\delta_{\mathbb{R}}$ are indeed groups of unitary operators in $\mathcal{D}^{s,2}(\mathbb{R}^N)$, by using the following identities

$$\begin{cases} (-\Delta)^{s/2}\left(u(\cdot - y)\right) = \left((-\Delta)^{s/2}u\right)(\cdot - y), \\ (-\Delta)^{s/2}\left(u(\tau\cdot)\right) = \tau^s\left((-\Delta)^{s/2}u\right)(\tau\cdot), \end{cases} \tag{1.2.2}$$

for $u \in \mathcal{D}^{s,2}(\mathbb{R}^N),\ y \in \mathbb{R}^N$ and $\tau > 0$. Now, we define the group

$$D_{\mathbb{R}^N} = \left\{d_{y,j} : \mathcal{D}^{s,2}(\mathbb{R}^N) \to \mathcal{D}^{s,2}(\mathbb{R}^N) : d_{y,j}u(x) = \gamma^{\frac{N-2s}{2}j} u(\gamma^j(x-y)),\ y \in \mathbb{R}^N, j \in \mathbb{R}\right\},$$

which consists by the composition of the elements of $T_{\mathbb{R}^N}$ with $\delta_{\mathbb{R}}$, i.e., $d_{y,j} = \delta_j \circ g_{\gamma^j y} = g_y \circ \delta j$. By checking that $d_{y,j} \circ d_{z,l} = d_{y+\gamma^{-j}z, j+l}$ and $(d_{y,j})^{-1} = d_{-\gamma^j y, -j}$, it is easy to see that $D_{\mathbb{R}^N}$ is a group of unitary operators in $\mathcal{D}^{s,2}(\mathbb{R}^N)$.

With the preceding notation we first derive the next basic result.

Lemma 1.2.1. *Let* $(y_k, j_k) \subset \mathbb{R}^N \times \mathbb{R}$*, such that* $(y_k, j_k) \to (y, j)$*. Then*

$$d_{y_k, j_k} u \to d_{y,j} u,\ \forall u \in \mathcal{D}^{s,2}(\mathbb{R}^N).$$

Proof. By the density of $\mathscr{S}_0(\mathbb{R}^N)$ in $\mathcal{D}^{s,2}(\mathbb{R}^N)$, we just need to prove in the case where $u \in \mathscr{S}_0(\mathbb{R}^N)$. Note that

$$[d_{y_k,j_k}u - d_{y,j}u]_s^2 = 2[u]_s^2 - 2\gamma^{\frac{N}{2}(j_k+j)} \int_{\mathbb{R}^N} (-\Delta)^{s/2}u(\gamma^{j_k}(x-y_k))(-\Delta)^{s/2}u(\gamma^j(x-y))\,\mathrm{d}x.$$

Moreover, identity (0.2.4) implies

$$\int_{\mathbb{R}^N} (-\Delta)^{s/2}u(\gamma^{j_k}(x-y_k))(-\Delta)^{s/2}u(\gamma^j(x-y))dx \\ = \int_{\mathbb{R}^N} u(\gamma^{j_k}(x-y_k))(-\Delta)^s u(\gamma^j(x-y))\,\mathrm{d}x.$$

Since $(-\Delta)^s u(\gamma^j(\cdot - y)) \in L^1(\mathbb{R}^N)$ and $|u(\gamma^{j_k}(x-y_k))| \leqslant \|u\|_\infty$ almost everywhere in $\mathbb{R}^N$, the assertion follows by the Dominated Convergence Theorem. ■

We shall describe how the elements of $D_{\mathbb{R}^N}$ acts in $\mathcal{D}^{s,2}(\mathbb{R}^N)$. This is done in the next result, which is a slightly different version of [71, Lemma 3].

Lemma 1.2.2. *Let $u \in \mathcal{D}^{s,2}(\mathbb{R}^N)\backslash\{0\}$. The sequence $(d_{y_k,j_k}u)$, with $(y_k, j_k) \subset \mathbb{R}^N \times \mathbb{R}$, converges weakly to zero if and only if $|j_k| + |y_k| \to \infty$.*

Proof. Suppose first that $d_{y_k,j_k}u \rightharpoonup 0$ in $\mathcal{D}^{s,2}(\mathbb{R}^N)$ and assume, by contradiction, that $|j_k| + |y_k| \nrightarrow \infty$. Then, up to subsequences, we may assume that $y_k \to y \in \mathbb{R}^N$ and $j_k \to j \in \mathbb{R}$, as $k \to \infty$. By Lemma 1.2.1,

$$0 = \lim_{k\to\infty} [d_{y_k,j_k}u, d_{y,j}u]_s = [d_{y,j}u]_s^2 = [u]_s^2,$$

a contradiction with the fact that $u \neq 0$.
Conversely, assume that $|j_k| + |y_k| \to \infty$. By density of $\mathscr{S}_0(\mathbb{R}^N)$ in $\mathcal{D}^{s,2}(\mathbb{R}^N)$ it suffices to prove that

$$[d_{y_k,j_k}u, v]_s \to 0, \quad \forall u, v \in \mathscr{S}_0(\mathbb{R}^N).$$

If we prove that every subsequence of $(d_{y_k,j_k}u)$ has a subsequence that weakly converges to zero, the assertion follows. To do this, we divide the proof in two cases:

(i) There exists a subsequence of (j_k), such that $j_k \to +\infty$ or $-\infty$;

(ii) There exists a convergent subsequence of (j_k), such that $j_k \to j_0$ and $|y_k| \to \infty$.

Before we start analyzing each case, we observe that by using identity (0.2.4), one has

$$[d_{y_k,j_k}u, v]_s = \int_{\mathbb{R}^N} (-\Delta)^s v(x) d_{y_k,j_k}u(x)\,\mathrm{d}x. \tag{1.2.3}$$

Therefore it is sufficient to study the desired convergence in the right-hand side of (1.2.3).

Case (i). Assume first that $j_k \to +\infty$. By changing the variables under the integral we have that

$$\begin{aligned}|[d_{y_k,j_k}u,v]_s| &= \gamma^{\frac{N-2s}{2}j_k}\left|\int_{\mathbb{R}^N}(-\Delta)^s v(x)u(\gamma^{j_k}(x-y_k))\,\mathrm{d}x\right| \\ &\leqslant \gamma^{-\frac{N+2s}{2}j_k}\|(-\Delta)^s v\|_\infty\|u\|_1 \to 0, \text{ as } k\to\infty.\end{aligned}$$

The same conclusion holds when $j_k \to -\infty$. Indeed, since $D_{\mathbb{R}^N,\mathbb{R}}$ is a group

$$[d_{y_k,j_k}u,v]_s = [u,(d_{y_k,j_k})^{-1}v]_s = [u, d_{-\gamma^{j_k}y_k,-j_k}v]_s.$$

Hence, by interchanging u and v we get the desired conclusion.
Case (ii). Since $j_k \to j_0$, we have that $d_{y_k,j_k}u(x) \to 0$ almost every where x in $\mathbb{R}^N$. Also,

$$|d_{y_k,j_k}u(x)(-\Delta)^s v(x)| \leqslant C\|u\|_\infty|(-\Delta)^s v(x)|, \quad \text{a.e. in } \mathbb{R}^N.$$

Thus, by the Dominated Convergence Theorem,

$$[d_{y_k,j_k}u,v]_s \to 0, \text{ as } k\to\infty. \qquad \blacksquare$$

Finally we take $D = D_{\mathbb{Z}^N} := \{d_{y,j} \in D_{\mathbb{R}^N} : y \in \mathbb{Z}^N,\ j \in \mathbb{Z}\}$, as the aforementioned group of unity operators in $\mathcal{D}^{s,2}(\mathbb{R}^N)$. As already mentioned, the main reason for this (instead of $D_{\mathbb{R}^N}$) in one of the statements in Theorem 1.1.1: it gives further properties for the weak decomposition (cf. Theorem 0.5.4 or [71, Theorem 8]). Considering the following cocompactness result we are able to prove Theorem 1.1.1 (a similar result can be found in [71, Proposition 1]).

Proposition 1.2.3. *Let (u_k) be a bounded sequence in $\mathcal{D}^{s,2}(\mathbb{R}^N)$ and $0 < s < \min\{1, N/2\}$. Then $u_k \stackrel{D}{\rightharpoonup} 0$ if and only if $u_k \to 0$ in $L^{2_s^*}(\mathbb{R}^N)$.*

Proof. Our proof follows the same ideas of [99, Lemma 5.3]. Since $C_0^\infty(\mathbb{R}^N)$ is a dense subset of $\mathcal{D}^{s,2}(\mathbb{R}^N)$, by the continuous embedding of $\mathcal{D}^{s,2}(\mathbb{R}^N)$ in $L^{2_s^*}(\mathbb{R}^N)$, we can assume without loss of generality that the sequence (u_k) belongs to $C_0^\infty(\mathbb{R}^N)$. Let us suppose first that $u_k \stackrel{D}{\rightharpoonup} 0$. Consider $\xi \in C_0^\infty(\mathbb{R}, [0,\infty))$ such that

$$\xi(t) = \begin{cases} t, & \text{if } \dfrac{1}{4}\gamma^{\frac{N-2s}{2}} \leqslant t \leqslant \dfrac{3}{4}\gamma^{\frac{N-2s}{2}}, \\ 0, & \text{if } t \leqslant 1 \text{ or } t \geqslant \gamma^{\frac{N-2s}{2}}, \end{cases} \quad \text{and} \quad |\xi'(t)| \leqslant C,\ \forall t \in \mathbb{R},$$

where we can assume without loss of generality that $\gamma > 4$, because we can replace it by $\gamma^{n_0} > 4$, for integer n_0 large enough, if necessary. Notice that there exists a positive constant C such that

$$\begin{cases} |\xi(t)|^{2_s^*} \leqslant Ct^2, \\ |\xi(t)|^2 \leqslant C|t|^{2_s^*}, \end{cases} \quad \forall t \in \mathbb{R}. \tag{1.2.4}$$

Given any sequence (j_k) in $\mathbb{Z}$, denote

$$v_k = \gamma^{\frac{N-2s}{2} j_k} u_k(\gamma^{j_k} \cdot).$$

Let $Q_z = (0,1)^N + z$, with $z \in \mathbb{Z}^N$. By the Sobolev embedding (0.2.3), for any $z \in \mathbb{Z}^N$, we get that

$$\int_{Q_z} |\xi(|v_k|)|^{2_s^*} \, \mathrm{d}x \leqslant C \|\xi(|v_k|)\|_{H^s(Q_z)}^2 \left(\int_{Q_z} v_k^2 \, \mathrm{d}x \right)^{1-2/2_s^*}. \tag{1.2.5}$$

Moreover, embedding (0.2.1) and relations (1.2.4) implies that,

$$\begin{aligned} \sum_{z \in \mathbb{Z}} \|\xi(|v_k|)\|_{H^s(Q_z)}^2 &= \sum_{z \in \mathbb{Z}} \int_{Q_z} |\xi(|v_k|)|^2 \, \mathrm{d}x + \int_{Q_z} \int_{Q_z} \frac{|\xi(|v_k|)(x) - \xi(|v_k|)(y)|^2}{|x-y|^{N+2s}} \, \mathrm{d}x \mathrm{d}y \\ &\leqslant \int_{\mathbb{R}^N} |\xi(|v_k|)|^2 \, \mathrm{d}x + \max_{t \geqslant 0} \xi'(t) \sum_{z \in \mathbb{Z}} \int_{Q_z} \int_{Q_z} \frac{|v_k(x) - v_k(y)|^2}{|x-y|^{N+2s}} \, \mathrm{d}x \mathrm{d}y \\ &\leqslant C[v_k]_s^2. \end{aligned}$$

Thus, we can take the sum over $z \in \mathbb{Z}^N$ in (1.2.5) to obtain

$$\int_{\mathbb{R}^N} |\xi(|v_k|)|^{2_s^*} \, \mathrm{d}x \leqslant C \sup_{z \in \mathbb{Z}^N} \left(\int_{Q_z} v_k^2 \, \mathrm{d}x \right)^{1-2/2_s^*}. \tag{1.2.6}$$

For each k, let $z_k \in \mathbb{Z}^N$ such that

$$\sup_{z \in \mathbb{Z}^N} \left(\int_{Q_z} v_k^2 \, \mathrm{d}x \right)^{1-2/2_s^*} \leqslant 2 \left(\int_{Q_{z_k}} v_k^2 \, \mathrm{d}x \right)^{1-2/2_s^*}. \tag{1.2.7}$$

Since $u_k \overset{D}{\rightharpoonup} 0$, we have that $v_k(\cdot - z_k) \rightharpoonup 0$ in $\mathcal{D}^{s,2}(\mathbb{R}^N)$, which allow us to apply embedding (0.2.3) and obtain that

$$\int_{Q_{z_k}} v_k^2 \, \mathrm{d}x = \int_{(0,1)^N} v_k^2(\cdot - z_k) \, \mathrm{d}x \to 0, \text{ as } k \to \infty. \tag{1.2.8}$$

Replacing (1.2.7) and (1.2.8) in (1.2.6) we conclude that

$$\lim_{k \to \infty} \int_{\mathbb{R}^N} |\xi(|v_k|)|^{2_s^*} \, \mathrm{d}x = 0. \tag{1.2.9}$$

Now let

$$\xi_j(t) = \gamma^{-\frac{N-2s}{2} j} \xi(\gamma^{\frac{N-2s}{2} j} t), \quad j \in \mathbb{Z}.$$

From convergence (1.2.9), we get

$$\lim_{k \to \infty} \int_{\mathbb{R}^N} |\xi_{j_k}(|u_k|)|^{2_s^*} \, \mathrm{d}x = \lim_{k \to \infty} \int_{\mathbb{R}^N} |\xi(|v_k|)|^{2_s^*} \, \mathrm{d}x = 0, \quad \forall \ (j_k) \subset \mathbb{Z}. \tag{1.2.10}$$

Now the embedding $\mathcal{D}^{s,2}(\mathbb{R}^N) \hookrightarrow L^{2_s^*}(\mathbb{R}^N)$ enable us to get the following estimate,

$$\int_{\mathbb{R}^N} |\xi_j(|u_k|)|^{2_s^*} \, \mathrm{d}x \leqslant C[\xi_j(|u_k|)]_s^2 \left(\int_{\mathbb{R}^N} |\xi_j(|u_k|)|^{2_s^*} \, \mathrm{d}x \right)^{1-2/2_s^*}. \tag{1.2.11}$$

For $j \in \mathbb{Z}$, let

$$\begin{cases} D_{j,k} = \left\{ x \in \mathbb{R}^N : \gamma^{-\frac{N-2s}{2}j} \leqslant |u_k(x)| < \gamma^{-\frac{N-2s}{2}(j-1)} \right\}; \\ E_{j,k} = (D_{j,k} \times \mathbb{R}^N) \cup (\mathbb{R}^N \times D_{j,k}); \\ L_{j,k} = \left\{ x \in \mathbb{R}^N : \frac{1}{4}\gamma^{-\frac{N-2s}{2}j} \leqslant |u_k(x)| \leqslant \frac{3}{4}\gamma^{-\frac{N-2s}{2}(j-1)} \right\}, \end{cases}$$

Since u_k is smooth and has compact support, there exists j_0 in $\mathbb{Z}$ and l in $\mathbb{N}$ such that

$$\operatorname{supp}(u_k) \subset \bigcup_{j=0}^{l} L_{j+j_0,k} \subset \bigcup_{j=0}^{l} D_{j+j_0,k},$$

We also have that the sets

$$S_{j,k} = \bigcup_{m=0}^{j} E_{j+j_0,k} \cap E_{m+j_0,k}, \quad j = 1, \ldots, l,$$

are disjunct as well $E_{j_0,k}$ and $E_{j+j_0,k} \backslash S_{j,k}$, for $j = 1, \ldots, l$. Thus we may write

$$\begin{aligned} \sum_{j=0}^{l} \iint\limits_{E_{j+j_0,k}} \frac{|u_k(x) - u_k(y)|^2}{|x-y|^{N+2s}} \mathrm{d}x\mathrm{d}y &= \sum_{j=1}^{l} \iint\limits_{S_{j,k}} \frac{|u_k(x) - u_k(y)|^2}{|x-y|^{N+2s}} \mathrm{d}x\mathrm{d}y \\ &+ \iint\limits_{E_{j_0,k}} \frac{|u_k(x) - u_k(y)|^2}{|x-y|^{N+2s}} \mathrm{d}x\mathrm{d}y + \sum_{j=1}^{l} \iint\limits_{E_{j+j_0,k} \backslash S_{j,k}} \frac{|u_k(x) - u_k(y)|^2}{|x-y|^{N+2s}} \mathrm{d}x\mathrm{d}y, \\ &= \iint\limits_{A_{l,k}} \frac{|u_k(x) - u_k(y)|^2}{|x-y|^{N+2s}} \mathrm{d}x\mathrm{d}y + \iint\limits_{B_{l,k}} \frac{|u_k(x) - u_k(y)|^2}{|x-y|^{N+2s}} \mathrm{d}x\mathrm{d}y \end{aligned}$$

where

$$A_{l,k} = E_{j_0,k} \cup \bigcup_{j=1}^{l} E_{j+j_0,k} \backslash S_{j,k} \quad \text{and} \quad B_{l,k} = \bigcup_{j=1}^{l} S_{j,k},$$

to get that the estimate

$$\begin{aligned} \sum_{j=0}^{l} [\xi_j(|u_k|)]_s^2 &= \frac{C(N,s)}{2} \sum_{j=0}^{l} \iint\limits_{E_{j,k}} \frac{|\xi_j(|u_k|)(x) - \xi_j(|u_k|)(y)|^2}{|x-y|^{N+2s}} \mathrm{d}x\mathrm{d}y \\ &\leqslant \frac{C(N,s)}{2} \max_{t \geqslant 0} \xi'(t) \sum_{j=0}^{l} \iint\limits_{E_{j,k}} \frac{|u_k(x) - u_k(y)|^2}{|x-y|^{N+2s}} \mathrm{d}x\mathrm{d}y \leqslant 2 \max_{t \geqslant 0} \xi'(t) [u_k]_s^2. \end{aligned}$$

Moreover,

$$\begin{aligned}\int_{\mathbb{R}^N} |u_k|^{2_s^*}\,\mathrm{d}x &\leqslant \sum_{j=0}^{l} \int_{L_{j,k}} |u_k|^{2_s^*}\,\mathrm{d}x \\ &\leqslant \sum_{j=0}^{l} \int_{L_{j,k}} |u_k|^{2_s^*}\,\mathrm{d}x + \int_{D_{j,k}\setminus L_{j,k}} |\xi_j(|u_k|)|^{2_s^*}\,\mathrm{d}x = \sum_{j=0}^{l} \int_{\mathbb{R}^N} |\xi_j(|u_k|)|^{2_s^*}\,\mathrm{d}x.\end{aligned}$$

In view of that, we take the sum over $j = 0, \ldots, l$ in (1.2.11) to conclude that

$$\int_{\mathbb{R}^N} |u_k|^{2_s^*}\,\mathrm{d}x \leqslant C \sup_{j\in\mathbb{Z}} \left(\int_{\mathbb{R}^N} |\xi_j(|u_k|)|^{2_s^*}\,\mathrm{d}x \right)^{1-2/2_s^*}.$$

Similarly as before, we choose (j_k) such that

$$\sup_{j\in\mathbb{Z}} \left(\int_{\mathbb{R}^N} |\xi_j(|u_k|)|^{2_s^*}\,\mathrm{d}x \right)^{1-2/2_s^*} \leqslant 2 \left(\int_{\mathbb{R}^N} |\xi_{j_k}(|u_k|)|^{2_s^*}\,\mathrm{d}x \right)^{1-2/2_s^*},$$

which, from (1.2.10) implies that $\|u_k\|_{2_s^*} \to 0$.

Now assume that $u_k \to 0$ in $L^{2_s^*}(\mathbb{R}^N)$. Let us argue by contradiction and suppose that there exists (y_k) in $\mathbb{Z}^N$ and (j_k) in $\mathbb{Z}$ such that $d_{y_k,j_k}u_k \rightharpoonup u \neq 0$ in $\mathcal{D}^{s,2}(\mathbb{R}^N)$. The invariance of d_{y_k,j_k} with respect to the $L^{2_s^*}$ norm leads to

$$\|u\|_{2_s^*} \leqslant \liminf_{k\to\infty} \|d_{y_k,j_k}u_k\|_{2_s^*} = \lim_{k\to\infty} \|u_k\|_{2_s^*} = 0,$$

which is a contradiction with the fact that $u \neq 0$. ■

Proof of Theorem 1.1.1 completed. By Theorem 0.5.4, we first need to prove that $(\mathcal{D}^{s,2}(\mathbb{R}^N), D_{\mathbb{Z}^N,\mathbb{Z}})$ is a dislocation space. To do so, we use Proposition 0.5.3. Let $(d_{y_k,j_k}) \subset D_{\mathbb{Z}^N,\mathbb{Z}}$, such that $d_{y_k,j_k} \not\rightharpoonup 0$ in $\mathcal{D}^{s,2}(\mathbb{R}^N)$. Hence by Lemma 1.2.2, $y_k \to y$ and $j_k \to j$, up to a subsequence, and by Lemma 1.2.1, $d_{y_k,j_k}u \to d_{y,j}u$, for all $u \in \mathcal{D}^{s,2}(\mathbb{R}^N)$. Therefore Theorem 0.5.4 holds with $H = \mathcal{D}^{s,2}(\mathbb{R}^N)$ and $D = D_{\mathbb{Z}^N,\mathbb{Z}}$. It follows immediately assertions (1.1.1) and (1.1.3). The assertion (1.1.2) is guaranteed by Lemma 1.2.2, and (1.1.4) follows from Proposition 1.2.3. Finally, for each $n \in \mathbb{N}_*$, if $(j_k^{(n)})$ is unbounded we can replace it by a subsequence convergent to $+\infty$ or $-\infty$, by checking either $\limsup_k j_k^{(n)} = +\infty$ or $\limsup_k j_k^{(n)} = -\infty$. If $(j_k^{(n)})$ is bounded, we can replace it by a constant subsequence, say $j^{(n)}$. Moreover, by taking $v_k^{(n)} = \gamma^{-\frac{N-2s}{2}j^{(n)}}u_k(\gamma^{-j^{(n)}}\cdot + y_k^{(n)})$, the convergence (1.1.1) implies

$$u_k(\cdot + y_k^{(n)}) = \delta_{-j^{(n)}}v_k^{(n)} \rightharpoonup \delta_{-j^{(n)}}w^{(n)} \text{ in } \mathcal{D}^{s,2}(\mathbb{R}^N),$$

The proof now follows by setting $j^{(n)} = 0$ and renaming $\delta_{-j^{(n)}}w^{(n)}$ as $w^{(n)}$. In fact, let us denote

$$\overline{\mathbb{N}} = \left\{ n \in \mathbb{N}_* : (j_k^{(n)}) \text{ is bounded} \right\},$$

and set

$$\begin{cases} \overline{w}^{(n)} = \delta_{-j^{(n)}} w^{(n)}, & \overline{y}_k^{(n)} = y_k^{(n)}, \quad \overline{j}_k^{(n)} = 0, \qquad \text{for } n \in \overline{\mathbb{N}}, \\ \overline{w}^{(n)} = w^{(n)}, & \overline{y}_k^{(n)} = y_k^{(n)}, \quad \overline{j}_k^{(n)} = j_k^{(n)}, \quad \text{for } n \in \overline{\mathbb{N}} \backslash \mathbb{N}_\sharp. \end{cases}$$

It is clear that $(\overline{w}^{(n)})$ satisfies conditions (1.1.1)–(1.1.3). To conclude that $(\overline{w}^{(n)})$ also fulfills condition (1.1.4), we take into account the following estimate

$$\begin{aligned} \left\| u_k - \sum_{n \in \mathbb{N}_*} \overline{d}_k^{(n)} \overline{w}^{(n)} \right\|_{2_s^*} \leqslant & \left\| u_k - \sum_{n \in \mathbb{N}_*} d_k^{(n)} w^{(n)} \right\|_{2_s^*} \\ & + \left\| \sum_{n \in \mathbb{N}_*} d_k^{(n)} w^{(n)} - \sum_{n \in \mathbb{N}_*} \overline{d}_k^{(n)} \overline{w}^{(n)} \right\|_{2_s^*}, \end{aligned} \tag{1.2.12}$$

where it is used the notation

$$\overline{d}_k^{(n)} u = \gamma^{\frac{N-2s}{2} \overline{j}_k^{(n)}} u\big(\gamma^{\overline{j}_k^{(n)}} (\cdot - \overline{y}_k^{(n)})\big), \quad u \in \mathcal{D}^{s,2}(\mathbb{R}^N).$$

The first term in the right-hand side of inequality (1.2.12) goes to zero due to (1.1.4). To prove that the second one goes to zero, we start by noticing that, up to subsequence in $n \in \mathbb{N}_*$, the series

$$\sum_{n \in \mathbb{N}_*} \overline{d}_k^{(n)} \overline{w}^{(n)},$$

is uniformly convergent in k, which can be proved by a standard diagonal argument extracting successive subsequences in $n \in \mathbb{N}_*$. This, together with the uniform convergence of (1.1.4), allows us reduce to the case that $\mathbb{N}_*$ is finite. Since

$$\begin{aligned} \left\| \sum_{n \in \mathbb{N}_*} d_k^{(n)} w^{(n)} - \sum_{n \in \mathbb{N}_*} \overline{d}_k^{(n)} \overline{w}^{(n)} \right\|_{2_s^*} & \leqslant \sum_{n \in \overline{\mathbb{N}}} \left\| g_{y_k^{(n)}} \big(\delta_{j_k^{(n)}} w^{(n)} - \delta_{-j^{(n)}} w^{(n)} \big) \right\|_{2_s^*} \\ & = \sum_{n \in \overline{\mathbb{N}}} \left\| \delta_{j_k^{(n)}} w^{(n)} - \delta_{-j^{(n)}} w^{(n)} \right\|_{2_s^*}. \end{aligned}$$

we have that the convergence to zero for the second term in (1.2.12) follows by using Lemma 1.2.1 and the Brezis-Lieb Lemma. ■

1.3 Proof of Theorem 1.1.2

In this section we shall prove the mentioned profile decomposition for bounded sequences in $H^s(\mathbb{R}^N)$, $0 < s \leqslant N/2$. To achieve that we start by considering

$$D = D_{\mathbb{Z}^N} := \left\{ g_y : H^s(\mathbb{R}^N) \to H^s(\mathbb{R}^N) : g_y u(x) = u(x-y),\ y \in \mathbb{Z}^N \right\},$$

which turns to be a unitary group of operators in $H^s(\mathbb{R}^N)$. Once again, the idea is to obtain Theorem 1.1.2 by means of Theorem 0.5.4. For that, we need first to determine how elements of $H^s(\mathbb{R}^N)$ become asymptotically orthogonal in $H^s(\mathbb{R}^N)$ with respect to any fixed other function under a sequence of dislocations.

Lemma 1.3.1. *Let be (y_k) a sequence in $\mathbb{R}^N$ and $0 \neq u \in H^s(\mathbb{R}^N)$. The sequence $(u(\cdot - y_k))$ converges weakly to zero in $H^s(\mathbb{R}^N)$ if, and only if $|y_k| \to \infty$.*

Proof. Suppose first that $u(\cdot - y_k) \rightharpoonup 0$ in $H^s(\mathbb{R}^N)$, and by contradiction, that $y_k \to y$ on a subsequence. By density argument we may assume that $u \in C_0^\infty(\mathbb{R}^N)$, also using Lemma 1.2.1 we obtain that $u(\cdot - y_k) \to u(\cdot - y)$ in $\mathcal{D}^{s,2}(\mathbb{R}^N)$. Thus

$$\begin{aligned} 0 &= \lim_{k\to\infty} (u(\cdot - y_k), u(\cdot - y)) \\ &= \lim_{k\to\infty} \left[\int_{\mathbb{R}^N} (-\Delta)^{s/2} u(\cdot - y_k)(-\Delta)^{s/2} u(\cdot - y) + u(\cdot - y_k) u(\cdot - y)\,\mathrm{d}x \right] = [u]_s^2, \quad (1.3.1) \end{aligned}$$

where the convergence of the second term in (1.3.1) follows by the Dominated Convergence Theorem. This leads to a contradiction with the assumption that $u \neq 0$. Conversely, assume that $|y_k| \to \infty$. Again, by density argument we may assume $u \in C_0^\infty(\mathbb{R}^N)$, and use Lemma 1.2.2 to obtain that $u(\cdot - y_k) \rightharpoonup 0$ in $\mathcal{D}^{s,2}(\mathbb{R}^N)$. Thus

$$\lim_{k\to\infty} \left[\int_{\mathbb{R}^N} (-\Delta)^{s/2} u(\cdot - y_k)(-\Delta)^{s/2} v + u(\cdot - y_k) v\,\mathrm{d}x \right] = 0, \quad \forall\, v \in C_0^\infty(\mathbb{R}^N),$$

where we have used in the second term that $\operatorname{supp} u(\cdot - y_k) \cap \operatorname{supp} v = \varnothing$, for k large enough. ∎

Next, we complement the discussion made in [29] by establishing a equivalence between the convergence in $L^p(\mathbb{R}^N)$ and $D_{\mathbb{Z}^N}$-convergence. The proof of Theorem 1.1.2 follows next by the same argument found in [99, Corollary 3.3].

Proposition 1.3.2. *Let (u_k) be a bounded sequence in $H^s(\mathbb{R}^N)$. Then $u_k \overset{D_{\mathbb{Z}^N}}{\rightharpoonup} 0$ in $H^s(\mathbb{R}^N)$, if and only if $u_k \to 0$ in $L^p(\mathbb{R}^N)$, for all $2 < p < 2_s^*$.*

Proof. The first part is proved in [29, Theorem 2.4]. Thus, let us suppose that $u_k \to 0$ in $L^p(\mathbb{R}^N)$, $2 < p < 2_s^*$. Take a arbitrary sequence (g_{y_k}) in $D_{\mathbb{Z}^N}$ and let $\varphi \in C_0^\infty(\mathbb{R}^N)$. Using identity (0.2.4) we have

$$\left| \int_{\mathbb{R}^N} (-\Delta)^{s/2} (g_{y_k}^* u_k)(-\Delta)^{s/2} \varphi\,\mathrm{d}x \right| \leqslant \|u_k\|_p \left(\int_{\mathbb{R}^N} |(-\Delta)^s \varphi(\cdot - y_k)|^{\frac{p}{p-1}}\,\mathrm{d}x \right)^{\frac{p-1}{p}}.$$

Thus, using Hölder inequality again in the L^2 term of the inner product of $H^s(\mathbb{R}^N)$, we conclude that $g_{y_k}^* u_k \rightharpoonup 0$ in $H^s(\mathbb{R}^N)$. ∎

Proof of Theorem 1.1.2 completed. We prove by applying Theorem 0.5.4. In fact, let (g_{y_k}) in $D_{\mathbb{Z}^N}$ such that $g_{y_k} \not\rightharpoonup 0$ in $H^s(\mathbb{R}^N)$. By Lemma 1.3.1, $y_k \to y$, up to subsequence, and by [41, Lemma 5.2] $g_{y_k} \to g_y$. Thus, by Proposition 0.5.3, $(H^s(\mathbb{R}^N), D_{\mathbb{Z}^N})$ is a dislocation space. Assertions (1.1.6) and (1.1.8) follows by Lemmas 1.3.1 and Proposition 1.3.2 respectively. ∎

1.4 Additional Properties

We reserve this section to give some additional description about the profiles $w^{(n)}$ in Theorems 1.1.1 and 1.1.2. We start by proving that one can consider $\mathcal{D}^{s,2}_{\text{rad}}(\mathbb{R}^N)$ to obtain more compactness.

Proposition 1.4.1. *Let (u_k) in $\mathcal{D}^{s,2}_{\text{rad}}(\mathbb{R}^N)$ be a bounded sequence and $(w^{(n)})$ and $(y_k^{(n)})$ be the collection of profiles given in Theorem 1.1.1. Then $(y_k^{(n)})_k = 0$ for all $n \in \mathbb{N}_0 \backslash \{1\}$, $w^{(n)} \in \mathcal{D}^{s,2}_{\text{rad}}(\mathbb{R}^N)$ and $\mathbb{N}_0 = \{1\}$.*

Proof. The proof of this fact follows similar arguments as in [99, Proposition 5.1]. The idea is to find a new profile that satisfies the desired conditions and use the uniqueness of the profiles (see Remark 1.1.3). Indeed, let $(y_k^{(n)})$, $(j_k^{(n)})$ the sequences provided by Theorem 1.1.1 and define the set

$$\mathbb{N}_\sharp = \left\{ n \in \mathbb{N}_* \backslash \{1\} : |\gamma^{j_k^{(n)}} y_k^{(n)}| \text{ is bounded} \right\}.$$

Passing a subsequence and using a diagonal argument if necessary, we may assume that each sequence $(\gamma^{j_k^{(n)}} y_k^{(n)})$, $n \in \mathbb{N}_\sharp$, is convergent and we denote

$$a^{(n)} = \lim_{k \to \infty} \gamma^{j_k^{(n)}} y_k^{(n)}, \quad n \in \mathbb{N}_\sharp.$$

Suppose that $n \in \mathbb{N}_\sharp$ and notice that

$$\gamma^{-\frac{N-2s}{2} j_k^{(n)}} u_k(\gamma^{-j_k^{(n)}} \cdot) - \gamma^{-\frac{N-2s}{2} j_k^{(n)}} u_k(\gamma^{-j_k^{(n)}}(\cdot - a^{(n)}) + y_k^{(n)}) \rightharpoonup 0 \text{ in } \mathcal{D}^{s,2}(\mathbb{R}^N), \text{ as } k \to \infty.$$

Since, the map $u \mapsto u(\cdot - a^{(n)})$ is linear and continuous in $\mathcal{D}^{s,2}(\mathbb{R}^N)$, we get

$$\gamma^{-\frac{N-2s}{2} j_k^{(n)}} u_k(\gamma^{-j_k^{(n)}}(\cdot - a^{(n)}) + y_k^{(n)}) \rightharpoonup w^{(n)}(\cdot - a^{(n)}) \text{ in } \mathcal{D}^{s,2}(\mathbb{R}^N), \text{ as } k \to \infty,$$

Therefore

$$\gamma^{-\frac{N-2s}{2} j_k^{(n)}} u_k(\gamma^{-j_k^{(n)}} \cdot) \rightharpoonup w^{(n)}(\cdot - a^{(n)}) \text{ in } \mathcal{D}^{s,2}(\mathbb{R}^N), \text{ as } k \to \infty.$$

We now proceed in a similar way as made in the proof of Theorem 1.1.1. Set

$$\begin{cases} \overline{w}^{(n)} = w^{(n)}(\cdot - a^{(n)}), & \overline{y}_k^{(n)} = 0, & \overline{j}_k^{(n)} = j_k^{(n)}, & \text{for } n \in \mathbb{N}_\sharp, \\ \overline{w}^{(n)} = w^{(n)}, & \overline{y}_k^{(n)} = y_k^{(n)}, & \overline{j}_k^{(n)} = j_k^{(n)}, & \text{for } n \in \mathbb{N}_* \backslash \mathbb{N}_\sharp. \end{cases}$$

It is easy to see that $(\overline{w}^{(n)})$ satisfies conditions (1.1.1)–(1.1.3). To see that $(\overline{w}^{(n)})$ also satisfies (1.1.4), we consider the following estimate

$$\left\| u_k - \sum_{n \in \mathbb{N}_*} \overline{d}_k^{(n)} \overline{w}^{(n)} \right\|_{2_s^*} \leqslant \left\| u_k - \sum_{n \in \mathbb{N}_*} d_k^{(n)} w^{(n)} \right\|_{2_s^*} + \left\| \sum_{n \in \mathbb{N}_*} d_k^{(n)} w^{(n)} - \sum_{n \in \mathbb{N}_*} \overline{d}_k^{(n)} \overline{w}^{(n)} \right\|_{2_s^*}, \tag{1.4.1}$$

where we used the notation

$$\overline{d}_k^{(n)} u = \gamma^{\frac{N-2s}{2}\overline{j}_k^{(n)}} u\big(\gamma^{\overline{j}_k^{(n)}}(\cdot - \overline{y}_k^{(n)}\big), \quad u \in \mathcal{D}^{s,2}(\mathbb{R}^N).$$

The first term in the right-hand side of inequality (1.4.1) goes to zero due to (1.1.4). To prove that the second one goes to zero, we start by noticing that, up to subsequence in $n \in \mathbb{N}_*$, the series

$$\sum_{n\in\mathbb{N}_*} \overline{d}_k^{(n)} \overline{w}^{(n)},$$

is uniformly convergent in k, which can be proved by a standard diagonal argument extracting successive subsequences in $n \in \mathbb{N}_*$. This, together with the uniform convergence of (1.1.4), allows us reduce to the case that $\mathbb{N}_*$ is finite. Since

$$\begin{aligned}\left\| \sum_{n\in\mathbb{N}_*} d_k^{(n)} w^{(n)} - \sum_{n\in\mathbb{N}_*} \overline{d}_k^{(n)} \overline{w}^{(n)} \right\|_{2_s^*} &\leqslant \sum_{n\in\mathbb{N}_\sharp} \left\| \delta_{j_k^{(n)}} (w^{(n)} - g_{a^{(n)} - \gamma^{j_k^{(n)}} y_k^{(n)}} w^{(n)}) \right\|_{2_s^*} \\ &= \sum_{n\in\mathbb{N}_\sharp} \left\| w^{(n)} - g_{a^{(n)} - \gamma^{j_k^{(n)}} y_k^{(n)}} w^{(n)} \right\|_{2_s^*}.\end{aligned}$$

we have that the convergence to zero for the second term in (1.4.1) follows by using Lemma 1.2.1 and the Brezis-Lieb Lemma.

Now let η be an element of $\mathcal{O}(N)$, the group of distance-preserving linear isomorphisms of $\mathbb{R}^N$. For $n \in \mathbb{N}_\sharp$, we have that

$$\begin{aligned}\gamma^{-\frac{N-2s}{2}\overline{j}_k^{(n)}} u_k(\gamma^{-\overline{j}_k^{(n)}} \eta(x)) &= \gamma^{-\frac{N-2s}{2}\overline{j}_k^{(n)}} (u_k \circ \eta)(\gamma^{-\overline{j}_k^{(n)}} x) \\ &= \gamma^{-\frac{N-2s}{2}\overline{j}_k^{(n)}} u_k(\gamma^{-\overline{j}_k^{(n)}} x). \end{aligned} \tag{1.4.2}$$

In view of the fact that the operator $T_\eta(u) = u \circ \eta$ is continuous in $\mathcal{D}^{s,2}(\mathbb{R}^N)$, we can pass the weak limit in (1.4.2) to conclude that

$$\overline{w}^{(n)} \circ \eta = \overline{w}^{(n)}, \quad \forall\ n \in \mathbb{N}_\sharp,\ \eta \in O(N).$$

That is, $\overline{w}^{(n)}$ belongs to $\mathcal{D}^{s,2}_{\mathrm{rad}}(\mathbb{R}^N)$ provided that $n \in \mathbb{N}_\sharp$. To conclude the proof we now show that $\overline{w}^{(n)} = 0$ for all $n \in \mathbb{N}_*\backslash\mathbb{N}_\sharp$. Let us argue by contradiction and assume the existence of $\overline{w}^{(n_0)} \neq 0$ for some $n_0 \in \mathbb{N}_*\backslash\mathbb{N}_\sharp$. Once again, using the continuity of $T_{\eta^{-1}}$ we obtain

$$\begin{aligned}&\gamma^{-\frac{N-2s}{2}\overline{j}_k^{(n)}} u_k(\gamma^{-\overline{j}_k^{(n)}} \cdot + \eta \overline{y}_k^{(n)}) = \\ &\qquad T_{\eta^{-1}}(\gamma^{-\frac{N-2s}{2}\overline{j}_k^{(n)}} u_k(\gamma^{-\overline{j}_k^{(n)}} \cdot + \overline{y}_k)) \rightharpoonup w \circ \eta^{-1} \text{ in } \mathcal{D}^{s,2}(\mathbb{R}^N), \text{ as } k \to \infty.\end{aligned}$$

Let $O_M = \{\eta_i \in \mathcal{O}(N)\backslash\{1\} : i = 1, \dots, M\}$ be an arbitrary distinct collection in $\mathcal{O}(N)$. Since $\gamma^{\overline{j}_k^{(n)}} |\overline{y}_k^{(n)}| \to \infty$, we have that

$$\gamma^{\overline{j}_k^{(n)}} |\eta_i \overline{y}_k^{(n)} - \eta_j \overline{y}_k^{(n)}| \to \infty, \quad \forall\ i \neq j.$$

Consequently, from Remark 0.5.5 we get the following estimate,

$$\begin{aligned}\limsup_{k\to\infty}[u_k]_s^2 &\geqslant \sum_{i=1}^{M}[w^{(n)}\circ\eta_i^{-1}]_s^2\\ &= \sum_{i=1}^{M}[w^{(n)}]_s^2 = M[w^{(n)}]_s^2.\end{aligned}$$

Since M is arbitrary we have a contradiction with the fact that (u_k) is bounded in $\mathcal{D}^{s,2}(\mathbb{R}^N)$. ■

We now can prove the well known compact embedding of $H^s_{\mathrm{rad}}(\mathbb{R}^N)$ into $L^p(\mathbb{R}^N)$, for $2 < p < 2^*_s$, by means of Theorem 1.1.2.

Corollary 1.4.2. *Let (u_k) be a bounded sequence in $H^s_{\mathrm{rad}}(\mathbb{R}^N)$ and the profiles $(y_k^{(n)})$ and $(w^{(n)})$ given by Theorem 1.1.2. Then $(y_k^{(n)})_k = 0$ for all $n \in \mathbb{N}_0\backslash\{1\}$, $w^{(1)} \in H^s_{\mathrm{rad}}(\mathbb{R}^N)$ and $\mathbb{N}_0 = \{1\}$.*

Proof. Consider $j_k^{(n)} \equiv 0$ in the proof of Proposition 1.4.1 and replace $[\,\cdot\,]_s$ by $\|\cdot\|$. ■

From the abstract result [99, Corollary 3.2], which is a direct consequence of Theorem 0.5.4, we also have the next additional property, which might be seen as a Brezis-Lieb type result for the corrected sequences in the convergences (1.1.4) and (1.1.8).

Proposition 1.4.3. *Assume that the same assumptions of Theorem 1.1.1 hold. Set*

$$r_k = u_k - \sum_{n\in\mathbb{N}_*}\gamma^{\frac{N-2s}{2}j_k^{(n)}}w^{(n)}\big(\gamma^{j_k^{(n)}}(\cdot - y_k^{(n)}).\big)$$

Then,

$$[u_k]_s^2 - \sum_{\mathbb{N}_*}[w^{(n)}]_s^2 - [r_k]_s^2 \to 0 \text{ in } \mathcal{D}^{s,2}(\mathbb{R}^N)$$

Similarly, suppose that the conditions of Theorem 1.1.2 are satisfied. Set

$$\bar{r}_k = u_k - \sum_{n\in\mathbb{N}_0} w^{(n)}(\cdot + y_k^{(n)}).$$

Then,

$$\|u_k\|^2 - \sum_{n\in\mathbb{N}_0}\|w^{(n)}\|^2 - \|\bar{r}_k\|^2 \to 0 \text{ in } H^s(\mathbb{R}^N)$$

We end this chapter with the next result, which establish a way to prove Theorem 1.1.2 by using Theorem 1.1.1. It is a key result to develop our results in Chapter 4, when we are dealing with nonlinearities $h(x,t)$ in Eq. $(\mathcal{P}_s)$ that possess critical growth.

Proposition 1.4.4. *Let* (u_k) *be a bounded sequence in* $H^s(\mathbb{R}^N)$, $0 < s < 1$, *and* $(w^{(n)})_{\mathbb{N}_*}$ *provided by Theorem 1.1.1. Then* $w^{(n)} = 0$ *for all* $n \in \mathbb{N}_-$. *Moreover, for* $p \in (2, 2_s^*)$,

$$u_k - \sum_{n\in\mathbb{N}_0} w^{(n)}(\cdot - y_k^{(n)}) \to 0 \quad in \quad L^p(\mathbb{R}^N), \tag{1.4.3}$$

the series in (1.4.3) *converges absolutely in* $H^s(\mathbb{R}^N)$ *uniformly in* k, *and* $w^{(n)}$ *are the weak limits of* $(u_k(\cdot + y_k^{(n)}))$ *in* $H^s(\mathbb{R}^3)$.

Proof. This is proved by using similar arguments as in [99, Lemma 5.4], together with Proposition 1.3.2. In fact, the last assertions follows from the fact that the translated sequence $(u_k(\cdot - y_k^{(n)}))$ is still bounded in $H^s(\mathbb{R}^N)$. By Fatou Lemma, we have

$$\|w^{(n)}\|_2^2 \leqslant \liminf_{k\to\infty} \int_{\mathbb{R}^N} |\gamma^{\frac{N-2s}{2} j_k^{(n)}} w^{(n)}|^2 \,\mathrm{d}x = \lim_{k\to\infty} \gamma^{-2sj_k^{(n)}} \int_{\mathbb{R}^N} u_k^2 \,\mathrm{d}x = 0,$$

from this, we get that $\mathbb{N}_- = \varnothing$. Moreover, by Remark 0.5.5 we obtain estimate (1.1.7), which ensures the uniform convergence in k for the series in (1.4.3). It remains to prove the convergence (1.4.3). Let φ in $C_0^\infty(\mathbb{R}^N)$ and (y_k) an arbitrary sequence in $\mathbb{Z}^N$. In view of Proposition 1.3.2, to obtain (1.4.3) it suffices to prove that

$$\left(u_k - \sum_{n\in\mathbb{N}_0} w^{(n)}(\cdot - y_k^{(n)}), \varphi(\cdot - y_k)\right) \to 0, \text{ as } k \to \infty.$$

Indeed,

$$\begin{aligned}\left(u_k - \sum_{n\in\mathbb{N}_*} d_k^{(n)} w^{(n)}, \varphi(\cdot - y_k)\right)& \\ = \left(u_k - \sum_{n\in\mathbb{N}_0} w^{(n)}(\cdot + y_k^{(n)}), \varphi(\cdot - y_k)\right) &- \left(\sum_{n\in\mathbb{N}_+} d_k^{(n)} w^{(n)}, \varphi(\cdot - y_k)\right).\end{aligned} \tag{1.4.4}$$

The term in the left-hand side of equation (1.4.4) goes to zero as $k \to \infty$ due to convergence (1.1.4) together with the fact that

$$\begin{aligned}\left|\left(u_k - \sum_{n\in\mathbb{N}_*} d_k^{(n)} w^{(n)}, \varphi\right)_2\right| &\leqslant \left\|u_k - \sum_{n\in\mathbb{N}_*} d_k^{(n)} w^{(n)}\right\|_{L^2(\operatorname{supp}\varphi)} \|\varphi\|_{L^2(\operatorname{supp}\varphi)} \\ &\leqslant C(\varphi) \left\|u_k - \sum_{n\in\mathbb{N}_*} d_k^{(n)} w^{(n)}\right\|_{L^{2_s^*}(\operatorname{supp}\varphi)} \to 0,\end{aligned}$$

where $C(\varphi)$ is a positive constant that only depends in φ. In order to prove that the second term in the right-hand side of (1.4.4) goes to zero as $k \to \infty$, we observe that the uniform convergence of the series in (1.1.4) enable us to reduce to the case that $\mathbb{N}_*$ is finite. Thus, the desired convergence follows by Lemmas 1.2.1 and 1.2.2, with aid of the compact embedding (0.2.3). ■

1.5 Self-similar functions

We now pass to study a class of non-linearity consistent with our profile decomposition. As it can be seen in the following examples, this class of nonlinearity can been seen as asymptotically oscillatory about the critical power $|t|^{2_s^*}$.

Definition 1.5.1. We say that $F(t) \in C(\mathbb{R})$ is fractional self-similar if there exist $\gamma > 1$ and $0 < s < \min\{1, N/2\}$ such that

$$F(t) = \gamma^{-Nj} F(\gamma^{\frac{N-2s}{2}j} t), \ \forall j \in \mathbb{Z}, \ t \in \mathbb{R}.$$

In this case we use to say that F is fractional self-similar with factor γ and fraction s.

Example 1.5.2. Typical examples of self-similar functions are

(i) $F(t) = |t|^{2_s^*}$, which is self-similar for every factor γ and fraction $0 < s < \min\{1, N/2\}$;

(ii) $H(t) = \cos(\ln|t|)|t|^{2_s^*}$, $H(0) := 0$, which is self-similar with factor $e^{4\pi/(N-2s)}$ and every fraction $0 < s < \min\{1, N/2\}$.

Remark 1.5.3. The function $F(t) \in C^1(\mathbb{R})$ is self-similar if, and only if

$$F'(t) = \gamma^{-\frac{N+2s}{2}j} F'\left(\gamma^{\frac{N-2s}{2}j} t\right), \ \forall j \in \mathbb{Z}, \text{ and } t \in \mathbb{R}.$$

In the next result we derive the basic properties of self-similar functions.

Lemma 1.5.4. *Assume that $F(t)$ is self-similar.*

(i) For each $u \in L^{2_s^*}(\mathbb{R}^N)$ and $j \in \mathbb{Z}$, we have

$$\int_{\mathbb{R}^N} F\left(\gamma^{\frac{N-2s}{2}j} u(\gamma^j \cdot)\right) \mathrm{d}x = \int_{\mathbb{R}^N} F(u)\, \mathrm{d}x; \tag{1.5.1}$$

(ii) There exists $C > 0$ such that

$$|F(t)| \leqslant C|t|^{2_s^*}, \ \forall t \in \mathbb{R}. \tag{1.5.2}$$

Moreover, if $F \in C^2(\mathbb{R})$, then there exists $C > 0$, such that

$$|F(t)| + |F'(t)t| + |F''(t)t^2| \leqslant C|t|^{2_s^*}, \ \forall t \in \mathbb{R}; \tag{1.5.3}$$

(iii) If $F(t)$ is locally Lipschitz then for each real numbers $a_1, \ldots, a_M$, there exist $C = C(M) > 0$ such that

$$\left| F\left(\sum_{n=1}^{M} a_n\right) - \sum_{n=1}^{M} F(a_n) \right| \leqslant C(M) \sum_{m \neq n \in \{1,\ldots,M\}} |a_n|^{2_s^*-1} |a_m|.$$

Proof. (i) The identity (1.5.1) follows immediately by using the change of variables theorem in the integral on the left side of the equation.

(ii) Fix the interval $L = [\gamma^{-\frac{N-2s}{2}}, \gamma^{\frac{N-2s}{2}}]$. By continuity, there exists $C = C(L)$ such that $|F(t)| \leqslant Ct^{2^*_s}$, for all $t \in L$. Now, let $0 < t < \gamma^{-\frac{N-2s}{2}}$ or $t > \gamma^{\frac{N-2s}{2}}$, then (in any case) there exists $j \in \mathbb{Z}$ such that $\gamma^{\frac{N-2s}{2}j}t \in L$, and consequently,

$$\gamma^{Nj}|F(t)| = |F(\gamma^{\frac{N-2s}{2}j}t)| \leqslant \gamma^{Nj}Ct^{2^*_s}.$$

The case where $t < 0$ is analogous. The proof of (1.5.3) follow a similar argument.

(iii) The proof is by induction in M. So we just need to prove that there exits $C > 0$ such that

$$|F(a_1 + a_2) - F(a_1) - F(a_2)| \leqslant C\left(|a_1||a_2|^{2^*_s-1} + |a_1|^{2^*_s-1}|a_2|\right). \tag{1.5.4}$$

To do so, we first fix the interval $I = [-\gamma^{\frac{N-2s}{2}k}, \gamma^{\frac{N-2s}{2}k}]$, where $k \in \mathbb{Z}$ is taken such that $\gamma^{\frac{N-2s}{2}(k-1)} > 2$, to use the Lipschitz assumption. The proof follows by considering several cases.

Case 1: Suppose that $|a_1| \leqslant 1 \leqslant |a_2|$ and $a_1 + a_2 \in I$. Thus there exists $C = C(I)$ such that

$$|F(a_1 + a_2) - F(a_1) - F(a_2)| \leqslant C(|a_1| + |F(a_1)|).$$

By condition (1.5.2) we can estimate

$$|a_1| + |F(a_1)| \leqslant C(|a_1||a_2|^{2^*_s-1} + |a_1|^{2^*_s-1}|a_2|).$$

Case 2: Assume that $|a_1|, |a_2| \geqslant 1$ and $a_1 + a_2 \in I$. Then, there exists $j_1 \in \mathbb{Z}$, $j_1 \leqslant 0$, such that $|b_1| \leqslant 1$, where $b_1 := \gamma^{\frac{N-2s}{2}j_1}a_1$. It is easy to see that $b_1 + a_2 \in I$, hence by the first case, we have the following estimate

$$\begin{aligned}|F(b_1 + a_2) - F(b_1) - F(a_2)| &\leqslant \gamma^{\frac{N-2s}{2}j_1}C(|a_1|^{2^*_s-1}|a_2| + |a_1||a_2|^{2^*_s-1}) \\ &\leqslant C(|a_1|^{2^*_s-1}|a_2| + |a_1||a_2|^{2^*_s-1}),\end{aligned}$$

Therefore we can estimate as follows

$$\begin{aligned}&|F(a_1 + a_2) - F(a_1) - F(a_2)| \leqslant \\ &\qquad |F(b_1 + a_2) - F(b_1) - F(a_2)| + |F(a_1 + a_2) - F(b_1 + a_2) + F(b_1) - F(a_1)|,\end{aligned}$$

with

$$|F(a_1 + a_2) - F(a_1) - F(b_1 + a_2) + F(b_1)| \leqslant 2C|a_2| \leqslant C|a_1|^{2^*_s}|a_2|.$$

Case 3: Suppose that $|a_1|, |a_2| \leqslant 1$. Since

$$\mathbb{R} = \bigcup_{j \in \mathbb{Z}} I_j^- \cup I_j^+,$$

where $I_j^- = [-\gamma^{\frac{N-2s}{2}j}, -\gamma^{\frac{N-2s}{2}(j-1)}]$ and $I_j^+ = [\gamma^{\frac{N-2s}{2}(j-1)}, \gamma^{\frac{N-2s}{2}j}]$ there exists $j_0 \in \mathbb{Z}$ such that

$$\gamma^{\frac{N-2s}{2}j_0}(a_1 + a_2) \in \left[-\gamma^{\frac{N-2s}{2}k}, -\gamma^{\frac{N-2s}{2}(k-1)}\right] \cup \left[\gamma^{\frac{N-2s}{2}(k-1)}, \gamma^{\frac{N-2s}{2}k}\right]$$

Let $b_1 = \gamma^{\frac{N-2s}{2}j_0}a_1$ and $b_2 = \gamma^{\frac{N-2s}{2}j_0}a_2$, with the necessity $|b_1| \geqslant 1$ or $|b_2| \geqslant 1$, because $\gamma^{\frac{N-2s}{2}(k-1)} > 2$. Consequently we can use the first or the second case to get that

$$\begin{aligned}\gamma^{Nj_0}|F(a_1 + a_2) - F(a_1) - F(a_2)| &= |F(b_1 + b_2) - F(b_1) - F(b_2)| \\ &\leqslant \gamma^{Nj_0}C(|a_1|^{2_s^*-1}|a_2| + |a_1||a_2|^{2_s^*-1}).\end{aligned}$$

The general case follows by a similar argument as above, thus we conclude that (1.5.4) holds. ■

Chapter 2

Concentration-compactness principle for nonlocal scalar field equations with critical growth

In this chapter, we study the existence of non-trivial weak and ground state solutions for the nonlinear scalar field equation

$$(-\Delta)^s u = f(x,u) \quad \text{in } \mathbb{R}^N, \tag{$\mathcal{E}_s$}$$

in the "zero mass case" with nonlinearities in the critical growth range.

Outline. The chapter is organized as follows. In Sect. 2.1, we list our assumptions on the nonlinearity $f(x,t)$ to give in Sect. 2.2 some applications of Theorem 1.1.1 to study the existence of mountain-pass solutions of Eq. ($\mathcal{E}_s$), for the autonomous and non-autonomous case. In Sect. 2.3 we prove that weak solutions of Eq. ($\mathcal{E}_s$) satisfy a Pohozaev type identity and have the regularity described in Proposition 0.4.2. In Sect. 2.4, using the properties obtained in the Sect. 1.5, we describe the limit of the profile decomposition of the Palais-Smale sequence at the mountain pass level of the energy functional related to Eq. ($\mathcal{E}_s$). In Sect. 2.5, we prove the results given in Subsect. 2.2 and describe some properties regarding the minimax levels associated with the functional energy of Eq. ($\mathcal{E}_s$), for the autonomous case. In Sect. 2.6, we prove our result about the existence of non-trivial weak solution of Eq. ($\mathcal{E}_s$) in the non-autonomous case, and for the sake of discussion, we establish a sufficient condition that ensures one of our hypothesis, precisely, the assumption (f_7).

2.1 Hypothesis

In order to describe our results on the energy functional of $(\mathcal{E}_s)$ in a more precisely way, we will make the following assumptions:

(f_1) $f(x,t)$ is a Carathéodory function and there exists $C>0$ such that

$$|f(x,t)| \leqslant C|t|^{2_s^*-1} \text{ almost everywhere (a.e.) } x \in \mathbb{R}^N, \ \forall\, t \in \mathbb{R}.$$

(f_2) There exists $\mu > 2$ such that,

$$\mu F(x,t) := \mu \int_0^t f(x,\tau)\,\mathrm{d}\tau \leqslant f(x,t)t, \quad \text{a.e. } x \in \mathbb{R}^N, \ \forall\, t \in \mathbb{R}.$$

(f_3) There exists $R>0$, $t_0>0$, $x_0 \in \mathbb{R}^N$ such that

$$|B_R| \inf_{B_R(x_0)} F(x,t_0) + |B_{R+1}\backslash B_R| \inf_{(x,t)\in(B_{R+1}(x_0)\backslash B_R(x_0))\times[0,t_0]} F(x,t) > 0,$$

In the study of the autonomous case $f(x,t) = f(t)$ we consider a weak version of (f_3).

(f_3') There exists $t_0 > 0$ such that $F(t_0) > 0$.

(f_4) For each real numbers $a_1, \ldots, a_M$, there exist $C = C(M) > 0$ such that

$$\left| F\left(x, \sum_{n=1}^{M} a_n\right) - \sum_{n=1}^{M} F(x,a_n) \right| \leqslant C(M) \sum_{m\neq n\in\{1,\ldots,M\}} |a_n|^{2_s^*-1}|a_m| \ \text{ a.e. } x \in \mathbb{R}^N.$$

(f_5) The following limits exist and are uniformly convergent in x and in compact sets for t,

$$\begin{aligned} f_0(t) &:= \lim_{|x|\to\infty} f(x,t),\\ f_+(t) &:= \lim_{j\in\mathbb{Z}, j\to+\infty} \gamma^{-\frac{N+2s}{2}j} f\left(\gamma^{-j}x, \gamma^{\frac{N-2s}{2}j}t\right),\\ f_-(t) &:= \lim_{j\in\mathbb{Z}, j\to-\infty} \gamma^{-\frac{N+2s}{2}j} f\left(\gamma^{-j}x, \gamma^{\frac{N-2s}{2}j}t\right), \end{aligned}$$

for some $\gamma > 1$ and $0 < s < \min\{1, N/2\}$. Moreover, the functions F_κ, $\kappa = 0, +, -$ satisfies condition (f_3'), where $F_\kappa(t) = \int_0^t f_\kappa(\tau)d\tau$.

(f_6) $f_0(t)$, $f_+(t)$, $f_-(t)$ are continuously differentiable.

We consider associated with Eq. ($\mathcal{E}_s$), the functional $I : \mathcal{D}^{s,2}(\mathbb{R}^N) \to \mathbb{R}$ given by

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^N} |(-\Delta)^{s/2}u|^2 \,\mathrm{d}x - \int_{\mathbb{R}^N} F(x,u)\,\mathrm{d}x.$$

Assuming that $f(x,t)$ satisfies (f_1) and using the same arguments of [72], $I \in C^1(\mathcal{D}^{s,2}(\mathbb{R}^N))$ and

$$I'(u)\cdot v = \int_{\mathbb{R}^N} (-\Delta)^{s/2}u\ (-\Delta)^{s/2}v\,\mathrm{d}x - \int_{\mathbb{R}^N} f(x,u)v\,\mathrm{d}x, \quad u,v \in \mathcal{D}^{s,2}(\mathbb{R}^N).$$

Thus critical points of I correspond to weak solutions of Eq. ($\mathcal{E}_s$) and conversely.

Regarding the minimax level, we consider

$$\Gamma_I = \left\{\gamma \in C([0,\infty), \mathcal{D}^{s,2}(\mathbb{R}^N)) : \gamma(0) = 0,\ \lim_{t\to\infty} I(\gamma(t)) = -\infty\right\},$$

and

$$c(I) = \inf_{\gamma\in\Gamma_I}\sup_{t\geqslant 0} I(\gamma(t)). \tag{2.1.1}$$

For the nonlinearities $f_0(t), f_+(t), f_-(t)$, we consider the associated energy functionals given by

$$I_\kappa(u) = \frac{1}{2}\int_{\mathbb{R}^N} \left|(-\Delta)^{s/2}u\right|^2 \mathrm{d}x - \int_{\mathbb{R}^N} F_\kappa(u)\,\mathrm{d}x, \quad F_\kappa(t) := \int_0^t f_\kappa(\tau)\,\mathrm{d}\tau$$

and the respectively minimax levels

$$c_\kappa = \inf_{\gamma\in\Gamma_\kappa}\sup_{t\geqslant 0} I_\kappa(\gamma(t))$$

where

$$\Gamma_\kappa = \left\{\gamma \in C([0,\infty), \mathcal{D}^{s,2}(\mathbb{R}^N)) : \gamma(0) = 0,\ \lim_{t\to\infty} I_\kappa(\gamma(t)) = -\infty\right\},$$

for $\kappa = 0,+,-$. Next, we assume a condition that compares the mountain pass levels defined above, precisely,

(f_7) $c(I) < c(I_\kappa)$, for each $\kappa = 0,+,-$.

We also consider the additional assumption for $\kappa = 0,+,-$,

(f_7') The following inequalities holds,

$$F_\kappa(t) \leqslant F(x,t),\ \text{ a.e. } x \in \mathbb{R}^N,\ \forall t \in \mathbb{R}. \tag{2.1.2}$$

Moreover, there exists $\delta > 0$ such that the inequality (2.1.2) is strict for all $t \in (-\delta,\delta)$ and almost everywhere $x \in \mathbb{R}^N$.

We are going to prove in Proposition 2.6.1 that (f_7') implies (f_7). To obtain our main result, we first study the autonomous case. For that we assume that $f(t)$ is self-similar:

(f_8) There exists $\gamma > 1$, $0 < s < \min\{1, N/2\}$ such that

$$F(t) = \gamma^{-Nj} F\left(\gamma^{\frac{N-2s}{2}j} t\right), \ \forall j \in \mathbb{Z}, \ \forall t \in \mathbb{R}$$

This allow us to derive some basic results concerning the behavior upon the functional I as we pass the limit over corrected sequences given in Theorem 1.1.1.

2.2 Statement of main results

Next, we state the main result abouts the autonomous case $f(x,t) = f(t)$.

Theorem 2.2.1. *Suppose that $f(t)$ is locally Lipschitz and satisfies (f_1), (f_3'), (f_8). Consider*

$$\mathcal{S}_l = \sup_{[u]_s^2 = l} \int_{\mathbb{R}^N} F(u) \, \mathrm{d}x. \tag{2.2.1}$$

Then, for any maximizing sequence (u_k) of (2.2.1) there exists $(j_k) \subset \mathbb{Z}$ and $(y_k) \subset \mathbb{Z}^N$ such that $(\gamma^{-\frac{N-2s}{2}j_k} u_k(\gamma^{-j_k} \cdot + y_k))$ contains a convergent subsequence in $\mathcal{D}^{s,2}(\mathbb{R}^N)$. In particular, the supremum in (2.2.1) is attained. Moreover, the same conclusion holds for

$$\mathcal{S}_{l,+} = \sup_{[u]_s^2 = l} \int_{\mathbb{R}^N} F_+(u) \, \mathrm{d}x \quad \textit{and} \quad \mathcal{S}_{l,-} = \sup_{[u]_s^2 = l} \int_{\mathbb{R}^N} F_-(u) \, \mathrm{d}x,$$

provided that $f(t)$ is locally Lipschitz and satisfies (f_1), (f_4), (f_5) with $\mathcal{S}_1 > \max\{\mathcal{S}_{1,+}, \mathcal{S}_{1,-}\}$.

Our next result proves that maximizers of (2.2.1) are indeed non-trivial solutions of Eq. $(\mathcal{E}_s)$. Moreover, the mountain pass level (2.1.1) is attained. The main tool to achieve these facts is a Pohozaev type identity proved in Section 2.3, which holds under the condition $0 < s < 1$ and taking into account the smoothness of the nonlinearity.

Theorem 2.2.2. *Assume that $f(t) \in C^1(\mathbb{R})$ satisfies (f_1) and (f_3').*

(i) If v is a nonzero critical point of I, then $c(I) \leqslant I(v)$;

(ii) If w is a maximizer of $\mathcal{S}_{l_0}$ for $l_0 := (2_s^* \mathcal{S}_1)^{-\frac{N-2s}{2s}}$, then w is a critical point of I. Moreover

$$0 < \max_{t \geqslant 0} I(w(\cdot/t)) = I(w) = c(I).$$

From Theorem 2.2.2, we conclude that to obtain weak solutions for the autonomous case, only assumptions (f_1), (f_3') and $((f_8))$ are needed. Moreover, we are able to prove that the minimax level is attained without the Ambrosetti-Rabinowitz condition (f_2).

Another way to approach Eq. $(\mathcal{E}_s)$ is by the means of constrained minimization. In fact, due Theorem 1.1.1 we can argue as in [99], and thanks to the Pohozaev identity, reasoning as in [11], we can derive existence of a ground state solution (or least energy) for Eq. $(\mathcal{E}_s)$, that is, a solution u of $(\mathcal{E}_s)$ such that $I(u) \leqslant I(v)$, for any other solution v.

Theorem 2.2.3. *Suppose that $f(t) \in C^1(\mathbb{R}^N)$ satisfies (f_3') and (f_8). Let*

$$\mathcal{G} = \left\{ u \in \mathcal{D}^{s,2}(\mathbb{R}^N) : \int_{\mathbb{R}^N} F(u)\,\mathrm{d}x = 1 \right\}$$

and consider

$$\mathcal{I} = \inf_{u \in \mathcal{G}} \int_{\mathbb{R}^N} \left|(-\Delta)^{s/2} u(x)\right|^2 \mathrm{d}x. \tag{2.2.2}$$

Then, for any minimizing sequence (u_k) of (2.2.2) there exists $(j_k) \subset \mathbb{Z}$ and $(y_k) \subset \mathbb{Z}^N$ such that $(\gamma^{-\frac{N-2s}{2} j_k} u_k(\gamma^{-j_k} \cdot + y_k))$ contains a convergent subsequence in $\mathcal{D}^{s,2}(\mathbb{R}^N)$. In particular, there exists a minimizer w for (2.2.2). Furthermore, $u = w(\cdot/\beta)$ is a ground state solution for Eq. $(\mathcal{E}_s)$ for some $\beta > 0$.

In the following result we prove that Palais-Smale condition at the mountain pass level holds for the general non-autonomous case.

Theorem 2.2.4. *If $f(x,t)$ satisfies (f_1)–(f_6) and (2.1.2), then Eq. $(\mathcal{E}_s)$ has a non-trivial weak solution u in $\mathcal{D}^{s,2}(\mathbb{R}^N)$ at the mountain pass level, that is, $I(u) = c(I)$. Moreover, if we assume that (f_7) holds true instead of (2.1.2), then any sequence (u_k) in $\mathcal{D}^{s,2}(\mathbb{R}^N)$ such that $I(u_k) \to c(I)$ and $I'(u_k) \to 0$ has a convergent subsequence.*

Remarks on the hypothesis and in the main results

Remark 2.2.5. Next we give several helpful comments concerning our assumptions.

(i) On assumption (f_1), we recall that $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ satisfies the Carathéodory conditions, if for each fixed $t \in \mathbb{R}$, $f(\cdot, t)$ is measurable, and for a.e. $x \in \mathbb{R}^N$, $f(x, \cdot)$ is continous in $\mathbb{R}$.

(ii) Condition (f_1) includes in particular nonlinearites with critical growth.

(iii) Assumption (f_2) is a weak version of the well-known Ambrosetti-Rabinowitz condition in the sense that we do not require that $F(x,t)$ is positive. (see [2, 73]).

(iv) In order to prove that the functional associated with Eq. $(\mathcal{E}_s)$ has the mountain pass geometry we consider (f_3). Furthermore, since we deal with constrained minimization, an autonomous version of (f_3') is needed (see [11]).

(v) The asymptotic additivity given in (f_4) ensure the convergence of the functional I under the weak profile decomposition for bounded sequences in $\mathcal{D}^{s,2}(\mathbb{R}^N)$ described in Theorem 1.1.1.

(vi) The smoothness condition $f(t) \in C^1(\mathbb{R})$ is the natural hypothesis used in the literature to prove that weak solutions of Eq. $(\mathcal{E}_s)$ satisfies a Pohozaev type identity.

(vii) Once the limits in (f_5) exist, to obtain compactness of Palais-Smale sequences at the minimax levels we need to require the additional conditions over the minimax levels c_0, c_+, c_- given in assumption (f_7). In fact, we do not believe that it is possible, in general, to achieve the compactness described in Theorem 2.2.4 without these conditions. We mention that this kind of approach was introduced by P.-L. Lions in [65–68].

(viii) Observe that the approach to obtain concentration-compactness for the autonomous case $f(x,t) = f(t)$ needs to be different since in this case, $f(t)$ does not satisfies (f_7).

(ix) We also consider the case when (f_7) do not hold. Precisely, when it is allowed $c(I) = c(I_\kappa)$, for some $\kappa = 0, +, -$. In this case, the concentration-compactness argument at the mountain pass level cannot be used. We apply [63, Theorem 2.3] to overcome this difficulty and prove existence of solution at the mountain pass level.

(x) If $f(x,t)$ satisfies (f_5) then

$$
\begin{aligned}
F_0(t) &= \lim_{|x|\to\infty} F(x,t). \\
F_+(t) &= \lim_{j\in\mathbb{Z}, j\to+\infty} \gamma^{-Nj} F\left(\gamma^{-j}x, \gamma^{\frac{N-2s}{2}j}t\right), \\
F_-(t) &= \lim_{j\in\mathbb{Z}, j\to-\infty} \gamma^{-Nj} F\left(\gamma^{-j}x, \gamma^{\frac{N-2s}{2}j}t\right),
\end{aligned}
$$

uniformly convergent in x and in compact sets for t. Furthermore, $F_+(t)$ and $F_-(t)$ are self-similar. Thus, functions $f(x,t)$ that satisfies (f_5) can be seen as being asymptotically self-similar at $\pm\infty$.

(xi) Our main results hold, replacing $\mathcal{D}^{s,2}(\mathbb{R}^N)$ by $\mathcal{D}^{s,2}_{\text{rad}}(\mathbb{R}^N)$, and assuming that $f(x,t) = f(|x|,t)$ is radial in x instead of the existence of the asymptote $f_\infty(t)$ or $f_0(t)$. This fact can be easily verified considering Proposition 1.4.1.

(xii) In Lemma 2.5.3 we proved that $\Gamma_\kappa \neq \varnothing$ is equivalent to: $\exists\, t_\kappa$ such that $F_\kappa(t_\kappa) > 0$. Consequently $((f_3'))$ is the most general assumption to ensure that $c(I)$ given in (2.1.1) is well defined (possible valuing $\pm\infty$).

Remark 2.2.6. We have that $\mathcal{G} \neq \varnothing$ and $\mathcal{S}_l > 0$, provided (f_1) and (f_3') hold. In fact, this follows as in [40, Lemma 2.6 and Remark 2.8]. Let $v_R \in C_0^\infty(\mathbb{R}),\ R > 0$, such that $0 \leqslant v_R(t) \leqslant t_0$ and

$$v_R(t) = \begin{cases} t_0, & \text{if } |t| \leqslant R, \\ 0, & \text{if } |t| > R + 1. \end{cases}$$

For all $x \in \mathbb{R}^N$, taking $\varphi_R(x) := v_R(|x|)$, we have $\varphi_R \in \mathcal{D}^{s,2}(\mathbb{R}^N)$. Moreover,

$$\begin{aligned} \int_{\mathbb{R}^N} F(\varphi_R)\,\mathrm{d}x &= \int_{B_R(x_0)} F(t_0)\,\mathrm{d}x + \int_{B_{R+1}(x_0)\backslash B_R(x_0)} F(\varphi_R)\,\mathrm{d}x \\ &\geqslant F(t_0)|B_R| - |B_{R+1}\backslash B_R| \left(\max_{t\in[0,t_0]} |F(t)| \right). \end{aligned}$$

Thus there exist two positive constants C_1 and C_2 such that

$$\int_{\mathbb{R}^N} F(\varphi_R)\,\mathrm{d}x \geqslant C_1 R^N - C_2 R^{N-1} > 0,$$

provided that R is taken large enough. Taking a suitable $\sigma > 0$, we may conclude that $\Phi(\varphi_R(\cdot/\sigma)) = 1$.

Remark 2.2.7. Using the s-harmonic extension, it can be proved the existence of non-negative weak solutions of $(\mathcal{P}_s)$ if $f(x,t) \geqslant 0$ for all $t \geqslant 0$ and almost everywhere x in $\mathbb{R}^N$. For that one can consider the truncation

$$\bar{f}(x,t) = \begin{cases} f(x,t), & \text{if} \quad t \geqslant 0 \\ 0, & \text{if} \quad t < 0. \end{cases}$$

Thus for u a weak solution of $(\mathcal{P}_s)$, with $f(x,t)$ replaced by $\bar{f}(x,t)$, we have that u is also a weak solution for $(\mathcal{P}_s)$ and is non-negative. To see that, let $\xi \in C_0^\infty(\mathbb{R} : [0,1])$ such that

$$\xi(t) = \begin{cases} 1, & \text{if} \quad t \in [-1,1] \\ 0, & \text{if} \quad |t| \geqslant 2 \end{cases} \qquad \text{and} \qquad |\xi'(t)| \leqslant C,\ \forall\, t \in \mathbb{R}.$$

For each $n \in \mathbb{N}$, define $\xi_n : \mathbb{R}^{N+1} \to \mathbb{R}$ by $\xi_n(z) = \xi(|z|^2/n^2)$. Then $\xi_n \in C_0^\infty(\mathbb{R}^{N+1})$ and verifies

$$|\nabla \xi_n(z)| \leqslant C \qquad \text{and} \qquad |z||\nabla \xi_n(z)| \leqslant C,\ \forall\, z \in \mathbb{R}^{N+1}.$$

By a density argument, we can take $\varphi = \xi_n w_-$ in (0.3.4), where $w_-(z) = \min\{w(z), 0\}$. Since $w_-(z) = E_s(u_-)$, we have that

$$\int_{\mathbb{R}^{N+1}_+} y^{1-2s}\xi_n|\nabla w_-|^2 + y^{1-2s}\xi_n \langle \nabla w_+, \nabla w_-\rangle + y^{1-2s}\langle \nabla w_+ + \nabla w_-, w_-\nabla\xi_n\rangle \,\mathrm{d}x\mathrm{d}y$$
$$= \kappa_s \int_{\mathbb{R}^N} \bar{f}(x,u)\xi_n u_- \,\mathrm{d}x,$$

and we may apply the Dominated Convergence Theorem and (0.3.2) to get

$$\|u_-\|^2 = \int_{\mathbb{R}^N} \bar{f}(x,u)u_- \,\mathrm{d}x = 0,$$

which implies that $u_- = 0$. On the other hand, if u has sufficient regularity one can show that u is positive, by applying the maximum principle for the fractional Laplacian as described in [84] (see also [38, Chapter 5]).

Example 2.2.8. Typical examples (see Section 1.5 and the proof of Lemma 1.5.4) of a functions satisfying (f_4)–(f_7) are given by

(i) $f(x,t) = b(x)|t|^{2_s^*-2}t$, where $b(x) \in C(\mathbb{R}^N)$, $b(0) > 0$ and

$$b(x) > b(0) = \inf_{x\in\mathbb{R}^N} b(x) = \lim_{|x|\to\infty} b(x). \tag{2.2.3}$$

(ii) $f(x,t) = \exp\{b(x)(\sin(\ln|t|) + 2)\}(b(x)\cos(\ln|t|) + 2_s^*)|t|^{2_s^*-2}t$, with $f(x,0) \equiv 0$; where $b(x) \in C(\mathbb{R}^N)$ satisfies (2.2.3), $b(0) = 0$ and moreover

$$\sup_{x\in\mathbb{R}^N} b(x) < 2_s^* - \sigma, \quad \text{for some} \quad \sigma \in (2, 2_s^*).$$

The primitive is given by $F(x,t) = \exp\{b(x)(\sin(\ln|t|) + 2)\}|t|^{2_s^*}$.

Remark 2.2.9. The function $f(t) = (2_s^*\cos(\ln|t|) - \sin(\ln|t|))|t|^{2_s^*-2}t$, $f(0) := 0$, satisfies the assumptions of Theorems 2.2.1–2.2.3.

2.3 Local regularity and Pohozaev Identity

We are in the position to prove that weak solutions of autonomous form of Eq. $(\mathcal{E}_s)$ are $C^1(\mathbb{R}^N)$ and satisfies the Pohozaev identity

$$\int_{\mathbb{R}^N} \left|(-\Delta)^{s/2}u\right|^2 \mathrm{d}x = \frac{2N}{N-2s}\int_{\mathbb{R}^N} F(u)\,\mathrm{d}x, \tag{2.3.1}$$

under suitable assumptions on $f(t)$ (see Proposition 2.3.2 for the precise statement). We refer to [24], where the identity was studied for solutions in $H^s(\mathbb{R}^N)$ and when

$f(t)$ satisfy a fractional version of the H. Berestycki and P.-L. Lions assumptions. The main idea for that, it is to use the so called Caffarelli-Silvestre extension (see [19] for more details) which transform the autonomous non-local Eq. $(\mathcal{E}_s)$ in a local one and use recent regularity results to develop the resultant expression in a such way to apply the argument of [11, Section 2]. Our approach is in some way different from the usual one. Although we continue using Caffarelli-Silvestre extension (also know as harmonic extension), by the results of [47, 59], we can derive a local regularity for weak solutions in $\mathcal{D}^{s,2}(\mathbb{R}^N)$ in a more suitable way to get the desired identity by applying a truncation argument. For bounded domains we refer to [74].

In order to apply Proposition 0.4.1 we need to prove a Brezis-Kato type result (see [14]) for solutions of Eq. $(\mathcal{E}_s)$. Although a similar result can be found in [47, Lemma 3.5], the absence of singularity in Eq. $(\mathcal{E}_s)$ allows us to obtain a simpler proof. To achieve that, we strongly rely in the following lemmas, which enable us to proceed as in [14] (cf. [8, Proposition 5.1] or [102, Theorem 1.2]).

Before we start to develop our regularity results, we find worth to mention the fact that $w = E_s(u)$ is a weak solution of (0.3.3) with $g(t) = f(t)$ if, and only if, u is a weak solution of Eq. $(\mathcal{E}_s)$.

Lemma A. *[46, Theorem 1.3] For any $R > 0$, there exists $\sigma > 1$ and $C_R > 0$ depending on R, such that*

$$\left(\int_{B_R} |y|^{1-2s}|v|^{2\sigma}\,\mathrm{d}x\mathrm{d}y\right)^{1/\sigma} \leqslant C_R \int_{B_R} |y|^{1-2s}|\nabla v|^2\,\mathrm{d}x\mathrm{d}y,\ \forall v \in C_0^\infty(B_R).$$

Lemma B. *[47, Lemma 2.6] Let $\xi \in C(\mathbb{R}^{N+1})$ such that $\xi(z) = 0$ for all $|z| \geqslant R$. There exist $C > 0$ such that*

$$\left(\int_{B_R^N} |v\xi|^{2_s^*}\,\mathrm{d}x\mathrm{d}y\right)^{2/2_s^*} \leqslant C \int_{B_R^+} y^{1-2s}|\nabla(v\xi)|^2\,\mathrm{d}x\mathrm{d}y,\ \forall v \in H^1(B_R^+, y^{1-2s}).$$

Proposition 2.3.1. *Assume that condition (f_1) holds. Let $u \in \mathcal{D}^{s,2}(\mathbb{R}^N)$ be a weak solution of Eq. $(\mathcal{E}_s)$ for the autonomous case, then $u \in L^p_{\mathrm{loc}}(\mathbb{R}^N)$, for all $p \geqslant 1$.*

Proof. Let $w = E_s(u)$ and $\xi \in C_0^\infty(\mathbb{R}^{N+1} : [0,1])$ such that

$$\xi(z) = \begin{cases} 1, & \text{if } |z| < R/2 \\ 0, & \text{if } |z| \geqslant R \end{cases} \qquad \text{and} \qquad |\nabla\xi(z)| \leqslant C \quad \forall z \in \mathbb{R}^{N+1},$$

for some $C > 0$. Since the map $t \mapsto t\min\{|t|^\beta, L\}$, β, $L > 0$, is Lipschitz in $\mathbb{R}$, considering $w_{\beta,L} := \min\{|w|^\beta, L\}$ we have $ww_{\beta,L} \in H^1(B_R^+, y^{1-2s})$, consequently using inequality (0.3.1) in a density argument one can see that $ww_{\beta,L}^2\xi^2$ can be taken as a test function in definition (0.3.4). The main idea is to get the estimate

$$\int_{B_R^+} y^{1-2s}|\nabla(ww_{\beta,L}\xi)|^2 \,\mathrm{d}x\mathrm{d}y \leqslant C, \tag{2.3.2}$$

for a suitable β and $C > 0$ which does not depend on L. The next step is to use Fatou Lemma and Lemma B to obtain

$$\int_{B_R^N} |u|^{(\beta+1)2_s^*} \,\mathrm{d}x \leqslant C.$$

This leads to a iteration procedure in β which implies in $u \in L^p(B_R^N)$ for all $p > 1$. To do so, we start taking

$$a(x) := \frac{|f(u)|}{1+|u|} \in L_{\mathrm{loc}}^{N/2s}(\mathbb{R}^N),$$

which implies

$$\int_{B_R^+} y^{1-2s}\left\langle \nabla w, \nabla(ww_{\beta,L}^2\xi^2)\right\rangle \mathrm{d}x\mathrm{d}y \leqslant 2\kappa_s \int_{B_R^N} a(x)(1+u^2)u_{\beta,L}^2\xi^2 \,\mathrm{d}x, \tag{2.3.3}$$

where we used that $(1+t)t \leqslant 2(1+t^2),\ t > 0$ and $t_r(ww_{\beta,L}^2\xi^2) = uu_{\beta,L}^2\xi(\cdot,0)^2$. We now compute the left side of the inequality (2.3.3) and use the following identity

$$w\left\langle \nabla w, \nabla(|w|^{2\beta})\right\rangle = \frac{\beta}{2}|w|^{2(\beta-1)}|\nabla(w^2)|^2,$$

to conclude

$$\begin{aligned}\int_{B_R^+} & y^{1-2s}\min\{|w|^{2\beta},L^2\}|\nabla w|^2\xi^2 \,\mathrm{d}x\mathrm{d}y \\ &+ \frac{\beta}{2}\int_{\{|w|^{2\beta}\leqslant L^2\}\cap B_R^+} y^{1-2s}|w|^{2(\beta-1)}|\nabla(w^2)|^2\xi^2 \,\mathrm{d}x\mathrm{d}y \\ &\leqslant 2\kappa_s\int_{B_R^N} a(x)(1+u^2)u_{\beta,L}^2\xi^2 \,\mathrm{d}x \\ &\quad - 2\int_{B_R^+} y^{1-2s}w\min\{|w|^{2\beta},L^2\}\xi\left\langle \nabla w,\nabla\xi\right\rangle \mathrm{d}x\mathrm{d}y. \end{aligned}\tag{2.3.4}$$

Using the Cauchy inequality (with $\varepsilon = 1/4$) we have

$$\begin{aligned} -2\int_{B_R^+} & y^{1-2s}w\min\{|w|^{2\beta},L^2\}\xi\left\langle \nabla w,\nabla\xi\right\rangle \mathrm{d}x\mathrm{d}y \\ &\leqslant \frac{1}{2}\int_{B_R^+} y^{1-2s}\min\{|w|^{2\beta},L^2\}|\nabla w|^2\xi^2 \,\mathrm{d}x\mathrm{d}y \\ &\quad + C\int_{B_R^+} y^{1-2s}w^2\min\{|w|^{2\beta},L^2\}|\nabla\xi|^2 \,\mathrm{d}x\mathrm{d}y, \end{aligned}\tag{2.3.5}$$

where $C > 0$ is independent of L. From replacing (2.3.5) in (2.3.4), we obtain

$$\begin{aligned}\frac{1}{2}\int_{B_R^+} & y^{1-2s}\min\{|w|^{2\beta},L^2\}|\nabla w|^2\xi^2\,\mathrm{d}x\mathrm{d}y \\ & +\frac{\beta}{2}\int_{\{|w|^{2\beta}\leqslant L^2\}\cap B_R^+} y^{1-2s}|w|^{2(\beta-1)}|\nabla(w^2)|^2\xi^2\,\mathrm{d}x\mathrm{d}y \\ & \leqslant C\int_{B_R^+} y^{1-2s}w^2\min\{|w|^{2\beta},L^2\}|\nabla\xi|^2\,\mathrm{d}x\mathrm{d}y \\ & \qquad + 2\kappa_s\int_{B_R^N} a(x)(1+u^2)u_{\beta,L}^2\xi^2\,\mathrm{d}x. \quad (2.3.6)\end{aligned}$$

Now using

$$\beta^2|w|^{2(\beta-1)}\left|\nabla(w^2)\right|^2 = 4w^2\left|\nabla(|w|^\beta)\right|^2,$$

together with inequality (2.3.6), we can finally estimate (2.3.2),

$$\begin{aligned}\int_{B_R^+} y^{1-2s}|\nabla(ww_{\beta,L}\xi)|^2\,\mathrm{d}x\mathrm{d}y & \leqslant C\int_{B_R^+} y^{1-2s}w^2\min\{|w|^{2\beta},L^2\}|\nabla\xi|^2\,\mathrm{d}x\mathrm{d}y \\ & \qquad + 2\kappa_s\int_{B_R^N} a(x)(1+u^2)u_{\beta,L}^2\xi^2\,\mathrm{d}x. \quad (2.3.7)\end{aligned}$$

It remains to estimate the last two terms in (2.3.7). Assuming $|u|^{\beta+1}\in L^2(B_R^N)$, we get

$$\begin{aligned}\int_{B_R^N} a(x)u^2u_{\beta,L}^2\xi^2\,\mathrm{d}x & \leqslant L_0\int_{B_R^N}|u|^{2(\beta+1)}\xi^2\,\mathrm{d}x + \int_{\{a(x)\geqslant L_0\}} a(x)u^2u_{\beta,L}^2\xi^2\,\mathrm{d}x \\ & \leqslant C_1L_0 + \tilde{C}_1\varepsilon(L_0)\left(\int_{B_R^+} y^{1-2s}|\nabla(ww_{\beta,L}\xi)|^2\,\mathrm{d}x\mathrm{d}y\right)^{2/2_s^*},\end{aligned}$$

where

$$\varepsilon(L_0) := \left(\int_{\{a(x)\geqslant L_0\}} a^{N/2s}(x)\,\mathrm{d}x\right)^{2s/N} \to 0, \text{ as } L_0\to\infty.$$

By the same calculation and using $\min\{|t|^\beta,L\}\leqslant |t|\min\{|t|^\beta,L\}+1, L>1$, we obtain

$$\begin{aligned}\int_{B_R^N} a(x)u_{\beta,L}^2\xi^2\,\mathrm{d}x & \leqslant C_2L_0 \\ & + \tilde{C}_2\varepsilon(L_0)\left[\left(\int_{B_R^+} y^{1-2s}|\nabla(ww_{\beta,L}\xi)|^2\,\mathrm{d}x\mathrm{d}y\right)^{2/2_s^*} + \left(\int_{B_R^N}|\xi|^{2_s^*}\,\mathrm{d}x\right)^{2/2_s^*}\right],\end{aligned}$$

Thus, we can take L_0 large enough such that

$$\int_{B_R^+} y^{1-2s}|\nabla(ww_{\beta,L}\xi)|^2\,\mathrm{d}x\mathrm{d}y \leqslant C_3\int_{B_R^+} y^{1-2s}w^2\min\{|w|^{2\beta},L^2\}|\nabla\xi|^2\,\mathrm{d}x\mathrm{d}y.$$

Finally, assume that $\beta+1\leqslant\sigma$, where σ is given in Lemma A. Using the operator extension by reflection $\mathcal{R}: H^1(B_R^+,y^{1-2s})\to H^1(B_R,|y|^{1-2s})$ given by

$$\mathcal{R}(w)(x,y) = \begin{cases} w(x,y), & \text{if } y>0, \\ w(x,-y), & \text{if } y\leqslant 0,\end{cases}$$

(see for instance [19, Section 4]), we may apply Lemma A for an appropriated sequence of functions in $C_0^\infty(\mathbb{R}^{N+1})$, converging to $\mathcal{R}(w)$ in $H^1(B_R, |y|^{1-2s})$ to get

$$\int_{B_R^+} y^{1-2s} w^2 \min\{|w|^{2\beta}, L^2\} |\nabla \xi|^2 \,\mathrm{d}x\mathrm{d}y \leqslant C_4 \int_{B_R} |y|^{1-2s} |\nabla(\mathcal{R}(w))|^2 \,\mathrm{d}x\mathrm{d}y \leqslant C_5.$$

We take $\beta = \beta_1 = \min\{2_s^*/2, \sigma\} - 1$ and $\beta_{i+1} = \min\{2_s^*/2, \sigma\}(2_s^*/2)^i - 1$, $i = 0, 1, \ldots$, to obtain that $u \in L^{\beta_{i+1}}_{\mathrm{loc}}(\mathbb{R}^N)$. ∎

Summing up all the previous results we can finally conclude the validity of identity (2.3.1) and the desired local regularity.

Proposition 2.3.2. *If $f(t) \in C^1(\mathbb{R})$ and satisfies (f_1), then every weak solution of Eq. $(\mathcal{E}_s)$ for the autonomous case belongs to $C^1(\mathbb{R}^N)$. Moreover, the Pohozaev identity* (2.3.1) *holds true.*

Proof. Let $u \in \mathcal{D}^{s,2}(\mathbb{R}^N)$ be a weak solution of Eq. $(\mathcal{E}_s)$ for the autonomous case with $f(t)$ satisfying (f_1). Consider $w = E_s(u)$. Then by Propositions 0.4.1, w possess the regularity (0.4.1). In particular, $\nabla u = \nabla w(x, 0) \in C(B_r^N)$ for any $r > 0$. Let $\xi \in C_0^\infty(\mathbb{R} : [0, 1])$ such that

$$\xi(t) = \begin{cases} 1, & \text{if} \quad t \in [-1, 1] \\ 0, & \text{if} \quad |t| \geqslant 2 \end{cases} \qquad \text{and} \qquad |\xi'(t)| \leqslant C \quad \forall t \in \mathbb{R},$$

for some $C > 0$. For each $n \in \mathbb{N}$, define $\xi_n : \mathbb{R}^{N+1} \to \mathbb{R}$ by $\xi_n(z) = \xi(|z|^2/n^2)$. Then $\xi_n \in C_0^\infty(\mathbb{R}^{N+1})$ and verifies

$$|\nabla \xi_n(z)| \leqslant C \qquad \text{and} \qquad |z||\nabla \xi_n(z)| \leqslant C \quad \forall z \in \mathbb{R}^{N+1}, \tag{2.3.8}$$

for some $C > 0$. Now observe that,

$$\begin{aligned} &\operatorname{div}(y^{1-2s}\nabla w)\langle z, \nabla w\rangle \xi_n = \\ &\qquad \operatorname{div}\left[y^{1-2s}\xi_n\left(\langle z, \nabla w\rangle \nabla w - \frac{|\nabla w|^2}{2} z\right)\right] + \frac{N-2s}{2} y^{1-2s}|\nabla w|^2 \xi_n \\ &\qquad\qquad + y^{1-2s}\frac{|\nabla w|^2}{2}\langle z, \nabla \xi_n\rangle - y^{1-2s}\langle \nabla w, z\rangle\langle \nabla w, \nabla \xi_n\rangle. \end{aligned} \tag{2.3.9}$$

Note that $\partial B_{\sqrt{2}n,\delta} = F^1_{\sqrt{2}n,\delta} \cup F^2_{\sqrt{2}n,\delta}$. Let $\eta(z) = (0, \ldots, -1)$ be the unit outward normal vector of $B_{\sqrt{2}n,\delta}$ on $F^1_{\sqrt{2}n,\delta}$. Since $\xi_n = 0$ on $F^2_{\sqrt{2}n,\delta}$, by condition (0.3.2), identity (2.3.9) and Divergence Theorem we get

$$\begin{aligned}0 &= \int_{B_{\sqrt{2}n,\delta}} \operatorname{div}(y^{1-2s}\nabla w)\left\langle z,\nabla w\right\rangle \xi_n \,\mathrm{d}x\mathrm{d}y\\ &= \int_{F^1_{\sqrt{2}n,\delta}} y^{1-2s}\xi_n\left[\left\langle z,\nabla w\right\rangle\left\langle \nabla w,\eta\right\rangle - \frac{|\nabla w|^2}{2}\left\langle z,\eta\right\rangle\right]\mathrm{d}x\mathrm{d}y + \theta_{n,\delta}\\ &= \int_{F^1_{\sqrt{2}n,\delta}} \xi_n\left\langle x,\nabla_x w\right\rangle(-y^{1-2s}w_y)\,\mathrm{d}x\\ &\quad - \int_{F^1_{\sqrt{2}n,\delta}} y^{1-2s}\xi_n w_y^2 y\,\mathrm{d}x + \int_{F^1_{\sqrt{2}n,\delta}} y^{1-2s}\xi_n\frac{|\nabla w|^2}{2}y\,\mathrm{d}x + \theta_{n,\delta}\\ &= I^1_{n,\delta} + I^2_{n,\delta} + I^3_{n,\delta} + \theta_{n,\delta},\end{aligned}$$

where

$$\begin{aligned}\theta_{n,\delta} = &\int_{B_{\sqrt{2}n,\delta}} \frac{N-2s}{2}y^{1-2s}|\nabla w|^2\xi_n\,\mathrm{d}x\mathrm{d}y\\ &+ \int_{B_{\sqrt{2}n,\delta}} y^{1-2s}\frac{|\nabla w|^2}{2}\left\langle z,\nabla\xi_n\right\rangle - y^{1-2s}\left\langle \nabla w,z\right\rangle\left\langle \nabla w,\nabla\xi_n\right\rangle\mathrm{d}x\mathrm{d}y.\end{aligned}$$

Using the same arguments as in [47, proof of Theorem 3.7] we deduce that there exists a sequence $\delta_k \to 0$ such that

$$I^2_{n,\delta_k} + I^3_{n,\delta_k} \to 0, \text{ as } k\to\infty.$$

Some computations leads to

$$\xi_n(x,0)\left\langle x,\nabla u\right\rangle f(u) = \operatorname{div}(\xi_n(x,0)F(u)x) - F(u)\left\langle \nabla\xi_n(x,0),x\right\rangle - \xi_n(x,0)F(u)N.$$

By condition (0.3.2), the Divergence Theorem and Remark 0.4.3 we have

$$\begin{aligned}\lim_{k\to\infty} I^1_{n,\delta_k} &= \kappa_s\int_{B^N_{\sqrt{2}n}} \xi_n(x,0)\left\langle x,\nabla u\right\rangle f(u)\,\mathrm{d}x\\ &= \kappa_s\int_{B^N_{\sqrt{2}n}} \operatorname{div}(\xi_n(x,0)F(u)x)) - F(u)\left\langle \nabla\xi_n(x,0),x\right\rangle - \xi_n(x,0)F(u)N\,\mathrm{d}x\\ &= -N\kappa_s\int_{B^N_{\sqrt{2}n}} \xi_n(x,0)F(u)\,\mathrm{d}x - \kappa_s\int_{B^N_{\sqrt{2}n}} F(u)\left\langle \nabla\xi_n(x,0),x\right\rangle\mathrm{d}x.\end{aligned}$$

Summing up, we get that

$$\begin{aligned}0 &= \lim_{k\to\infty}\left[I^1_{n,\delta_k} + I^2_{n,\delta_k} + I^3_{n,\delta_k} + \theta_{n,\delta_k}\right]\\ &= -\kappa_s\int_{B^N_{\sqrt{2}n}} N\xi_n F(u) + F(u)\left\langle \nabla\xi_n,x\right\rangle\mathrm{d}x + \int_{B^+_{\sqrt{2}n}} \frac{N-2s}{2}y^{1-2s}|\nabla w|^2\xi_n\,\mathrm{d}x\mathrm{d}y\\ &\quad + \int_{B^+_{\sqrt{2}n}} y^{1-2s}\frac{|\nabla w|^2}{2}\left\langle z,\nabla\xi_n\right\rangle\mathrm{d}x\mathrm{d}y - \int_{B^+_{\sqrt{2}n}} y^{1-2s}\left\langle \nabla w,z\right\rangle\left\langle \nabla w,\nabla\xi_n\right\rangle\mathrm{d}x\mathrm{d}y.\end{aligned}$$

Consequently taking $n \to \infty$ and using conditions (2.3.8), we conclude

$$\frac{N-2s}{2}\int_{\mathbb{R}^{N+1}_+} y^{1-2s}|\nabla w|^2 \, \mathrm{d}x\mathrm{d}y = N\kappa_s \int_{\mathbb{R}^N} F(u)\, \mathrm{d}x,$$

which together with condition (0.3.2) implies (2.3.1), and the proof is complete. ■

2.4 Behavior of weak decomposition convergence under nonlinearities

Concerning the assumptions (f_4), (f_5), and (f_8), we have the following results, which provides a way to link the weak convergence decomposition (as also the latter lines of Theorem 1.1.1) and the limit over the energy functional I for bounded sequences in $\mathcal{D}^{s,2}(\mathbb{R}^N)$. They are mainly used to prove the existence results stated in Sect. 2.2. Also, the next result can be seen as a generalization of the well know Brezis-Lieb Lemma [15] (see Corollary 2.4.3).

Proposition 2.4.1. *Let* $0 < s < \min\{1, N/2\}$ *and assume that* $f(x,t)$ *satisfies* (f_1), (f_4) *and* (f_5). *Let* (u_k) *in* $\mathcal{D}^{s,2}(\mathbb{R}^N)$ *be a bounded sequence and* $(w^{(n)})_{n\in\mathbb{N}_*}$ *in* $\mathcal{D}^{s,2}(\mathbb{R}^N)$, $n \in \mathbb{N}_*$, *provided by Theorem 1.1.1. Then*

$$\lim_{k\to\infty}\int_{\mathbb{R}^N} F(x,u_k)\,\mathrm{d}x = \int_{\mathbb{R}^N} F(x,w^{(1)})\,\mathrm{d}x + \sum_{n\in\mathbb{N}_0, n>1}\int_{\mathbb{R}^N} F_0(w^{(n)})\,\mathrm{d}x + \sum_{n\in\mathbb{N}_+}\int_{\mathbb{R}^N} F_+(w^{(n)})\,\mathrm{d}x + \sum_{n\in\mathbb{N}_-}\int_{\mathbb{R}^N} F_-(w^{(n)})\,\mathrm{d}x. \quad (2.4.1)$$

Proof. By condition (f_1), the functional

$$\Phi(u) = \int_{\mathbb{R}^N} F(x,u)\,\mathrm{d}x, \quad u \in \mathcal{D}^{s,2}(\mathbb{R}^N),$$

is uniformly continuous in bounded sets of $L^{2^*_s}(\mathbb{R}^N)$, which implies (by assertions (1.1.3) and (1.1.4) of Theorem 1.1.1) that

$$\lim_{k\to\infty}\left[\Phi(u_k) - \Phi\left(\sum_{n\in\mathbb{N}_*} d_k^{(n)} w^{(n)}\right)\right] = 0.$$

To prove (2.4.1) we observe that the uniform convergence of the series in (1.1.4) allows us to reduce to the case where $\mathbb{N}_* = \{1,\ldots,M\}$. Thus,

$$\lim_{k\to\infty}\left[\sum_{n\in\mathbb{N}_0}\Phi\left(w^{(n)}(\cdot - y_k^{(n)})\right) - \Phi(w^{(1)}) - \sum_{n\in\mathbb{N}_0, n>1}\Phi_0(w^{(n)})\right] = 0,$$

$$\lim_{k\to\infty}\left[\sum_{n\in\mathbb{N}_\pm}\Phi(d_k^{(n)}w^{(n)}) - \sum_{n\in\mathbb{N}_\pm}\Phi_\pm(w^{(n)})\right] = 0,$$

follows immediately from the assumption (f_5), by change of variables and the use of Dominated Convergence Theorem. Therefore it is sufficient to prove that

$$\lim_{k\to\infty}\left[\Phi\left(\sum_{n\in\mathbb{N}_*} d_k^{(n)} w^{(n)}\right) - \sum_{n\in\mathbb{N}_*} \Phi(d_k^{(n)} w^{(n)})\right] = 0. \tag{2.4.2}$$

Indeed, by (f_4) we have for all $m \neq n$,

$$\left|\Phi\left(\sum_{n\in\mathbb{N}_*} d_k^{(n)} w^{(n)}\right) - \sum_{n\in\mathbb{N}_*} \Phi(d_k^{(n)} w^{(n)})\right| \leqslant \sum_{m\neq n\in\mathbb{N}_*} \int_{\mathbb{R}^N} |d_k^{(n)}|^{2_s^*-1} |d_k^{(m)}| \,\mathrm{d}x.$$

But by a change of variable we can see that

$$\int_{\mathbb{R}^N} |d_k^{(n)}|^{2_s^*-1} |d_k^{(m)}| \,\mathrm{d}x = \int_{\mathbb{R}^N} |w^{(n)}|^{2_s^*-1} g_k(|w^{(m)}|) \,\mathrm{d}x,$$

where

$$g_k(|w^{(m)}|) = \gamma^{\frac{N-2s}{2}(j_k^{(m)}-j_k^{(n)})} w^{(m)} \left(\gamma^{j_k^{(m)}-j_k^{(n)}}(\cdot - \gamma^{j_k^{(n)}}(y_k^{(m)} - y_k^{(n)}))\right) \rightharpoonup 0 \text{ in } \mathcal{D}^{s,2}(\mathbb{R}^N),$$

due to assertion (1.1.2) of Theorem 1.1.1 and Lemma 1.2.2. Since

$$\alpha(v) = \int_{\mathbb{R}^N} |w^{(n)}|^{2_s^*-1} v \,\mathrm{d}x$$

is a continuous linear functional in $\mathcal{D}^{s,2}(\mathbb{R}^N)$ we conclude (2.4.2). ∎

Corollary 2.4.2. *Let (u_k) be a bounded sequence in $\mathcal{D}^{s,2}(\mathbb{R}^N)$, $0 < s < \min\{1, N/2\}$, and $(w^{(n)})_{n\in\mathbb{N}_*}$ in $\mathcal{D}^{s,2}(\mathbb{R}^N)$, $n \in \mathbb{N}_*$, provided by Theorem 1.1.1. If $F(x,t) = F(t)$ satisfies (f_8) and is locally Lipschitz then, up to subsequence,*

$$\lim_{k\to\infty} \int_{\mathbb{R}^N} F(u_k) \,\mathrm{d}x = \sum_{n\in\mathbb{N}_*} \int_{\mathbb{R}^N} F(w^{(n)}) \,\mathrm{d}x. \tag{2.4.3}$$

Proof. In this case $F(t)$ satisfies (f_4) and (1.5.2), also $F(t) = F_+(t) = F_-(t) = F_0(t)$. ∎

Corollary 2.4.3. *Let $u_k \rightharpoonup u$ in $\mathcal{D}^{s,2}(\mathbb{R}^N)$ and $F(t)$ be as in Corollary 2.4.2 then, up to subsequence,*

$$\lim_{k\to\infty} \int_{\mathbb{R}^N} F(u_k) - F(u-u_k) - F(u) \,\mathrm{d}x = 0.$$

Proof. Let $v_k = u_k - u = u_k - w^{(1)}$. The profiles of Theorem 1.1.1 for (v_k) are given by $\tilde{w}^{(1)} = 0$, $\tilde{w}^{(n)} = w^{(n)}$, $n \in \mathbb{N}_*\backslash\{1\}$. Thus by Corollary 2.4.2 we have

$$\lim_{k\to\infty} \int_{\mathbb{R}^N} F(u_k - u) \,\mathrm{d}x = \sum_{n\in\mathbb{N}_*, n>1} \int_{\mathbb{R}^N} F(w^{(n)}) \,\mathrm{d}x. \tag{2.4.4}$$

Taking the difference between (2.4.3) and (2.4.4) we get the desired result. ∎

2.5 The autonomous case

The aim of this section is to prove Theorems 2.2.1, 2.2.2 and 2.2.3.

Remark 2.5.1. By embedding (0.2.1), we have $\mathcal{S}_l < \infty$. Also $\mathcal{S}_l$ is attained for some l if and only if it is attained for all l. Indeed, this can be checked by considering the rescaling $v = u_1(l^{-1/(N-2s)}\cdot)$ and $u = v_l(l^{1/(N-2s)}\cdot)$, where $[u_1]_s^2 = 1$ and $[v_l]_s^2 = l$ respectively. In particular,

$$l^{\frac{N}{N-2s}}\mathcal{S}_1 = \mathcal{S}_l. \tag{2.5.1}$$

2.5.1 Proof of Theorem 2.2.1

Proof. Suppose that $F(t)$ is self-similar and satisfies (f_3'). Let $(u_k) \subset \mathcal{D}^{s,2}(\mathbb{R}^N)$ be a maximizing sequence for (2.2.1) with $l = 1$, that is, $\|u_k\|^2 = 1$ and $\Phi(u_k) \to \mathcal{S}_1$. Let be $(w^{(n)})_{n\in\mathbb{N}_*}$, $(y_k^{(n)})_{k\in\mathbb{N}}$ and $(j_k^{(n)})_{k\in\mathbb{N}}$, the sequences provided by Theorem 1.1.1. By Corollary 2.4.2,

$$\mathcal{S}_1 = \lim_{k\to\infty}\Phi(u_k) = \sum_{n\in\mathbb{N}_*}\Phi(w^{(n)}), \tag{2.5.2}$$

and at the same time by assertion (1.1.3) of Theorem 1.1.1

$$\sum_{n\in\mathbb{N}_*}[w^{(n)}]_s^2 \leqslant \limsup_{k\to\infty}[u_k]_s^2 \leqslant 1. \tag{2.5.3}$$

The identity (2.5.2) also implies that there exists $n \in \mathbb{N}_*$ with $w^{(n)} \neq 0$. We may write $v^{(n)} = w^{(n)}(\tau_n\cdot)$ where $\tau_n = [w^{(n)}]_s^{2/(N-2s)}$. Consequently $[v^{(n)}]_s^2 = 1$, $\Phi(v^{(n)}) \leqslant \mathcal{S}_1$ and

$$\mathcal{S}_1 = \sum_{n\in\mathbb{N}_*}\tau_n^N\Phi(v^{(n)}) \leqslant \mathcal{S}_1\sum_{n\in\mathbb{N}_*}\tau_n^N.$$

Moreover,

$$1 \leqslant \sum_{n\in\mathbb{N}_*}\tau_n^N. \tag{2.5.4}$$

From (2.5.3) we have

$$\sum_{n\in\mathbb{N}_*}\tau_n^{N-2s} \leqslant 1. \tag{2.5.5}$$

Relations (2.5.4) and (2.5.5) can hold simultaneous provided that there is a $n_0 \in \mathbb{N}_*$ such that $\tau_{n_0} = 1$, while $\tau_n = 0$, whenever $n \neq n_0$. Therefore, by assertion (1.1.4) of Theorem 1.1.1 we obtain

$$u_k - \gamma^{\frac{N-2s}{2}j_k^{(n_0)}}w^{(n_0)}(\gamma^{j_k^{(n_0)}}(\cdot - y_k^{(n_0)})) \to 0 \text{ in } L^{2_s^*}(\mathbb{R}^N).$$

Since $F(t)$ is self-similar, the sequence

$$v_k = \gamma^{-\frac{N-2s}{2}j_k^{(n_0)}}u_k(\gamma^{-j_k^{(n_0)}}\cdot + y_k^{(n_0)}),$$

is a maximizing sequence for (2.2.1) and $v_k \to w^{(n_0)}$ in $L^{2^*_s}(\mathbb{R}^N)$. Furthermore, the continuity of Φ in $L^{2^*_s}(\mathbb{R}^N)$ implies $\Phi(w^{n_0}) = \mathcal{S}_1$, and since $[w^{(n_0)}]_s^2 = 1$, $w^{(n_0)}$ is a maximizer.

Consider now the case where $\mathcal{S}_1 > \max\{\mathcal{S}_{1,+}, \mathcal{S}_{1,-}\}$. Let again (u_k) be a maximizing sequence for (2.2.1) with $l = 1$. Since $f(t)$ verifies (f_5) we can apply Proposition 2.4.1 to get

$$\mathcal{S}_1 = \lim_{k\to\infty} \Phi(u_k) = \sum_{n\in\mathbb{N}_0} \Phi(w^{(n)}) + \sum_{n\in\mathbb{N}_{-\infty}} \Phi_-(w^{(n)}) + \sum_{n\in\mathbb{N}_{+\infty}} \Phi_+(w^{(n)}),$$

where $(w^{(n)})$, $(y_k^{(n)})$, $(j_k^{(n)})$, are given by Theorem 1.1.1. Considering again $v^{(n)} = w^{(n)}(\tau_n \cdot)$, with $\tau_n = [w^{(n)}]_s^{2/(N-2s)}$, we get

$$1 \leqslant \sum_{n\in\mathbb{N}_0} \tau_n^N + \frac{\mathcal{S}_{1,-}}{\mathcal{S}_1} \sum_{n\in\mathbb{N}_{-\infty}} \tau_n^N + \frac{\mathcal{S}_{1,+}}{\mathcal{S}_1} \sum_{n\in\mathbb{N}_{+\infty}} \tau_n^N. \tag{2.5.6}$$

Since $\mathcal{S}_{1,+}/\mathcal{S}_1 < 1$ and $\mathcal{S}_{1,-}/\mathcal{S}_1 < 1$ by assertion (1.1.3) of Theorem 1.1.1, inequalities (2.5.3) and (2.5.6) can hold simultaneously if and only if there is a $n_0 \in \mathbb{N}_0$ such that $\tau_{n_0} = 1$, while $\tau_n = 0$, whenever $n \neq n_0$. Therefore, by assertion (1.1.4) of Theorem 1.1.1,

$$u_k - w^{(n_0)}(\cdot - y_k^{(n_0)}) \to 0 \text{ in } L^{2^*_s}(\mathbb{R}^N),$$

and using a similar argument as in the previous case, we conclude that $w^{(n_0)}$ is a maximizer. ■

Remark 2.5.2. One always has $\mathcal{S}_1 \geqslant \max\{\mathcal{S}_{1,+}, \mathcal{S}_{1,-}\}$. Indeed, as discussed above, it suffices to prove this in the case that $l = 1$, so let $u \in \mathcal{D}^{s,2}(\mathbb{R}^N)$ with $[u]_s = 1$ and $v_j := \delta_j u$, where δ_j is given in (1.2.1), and $j \in \mathbb{Z}$. Then $[v_j]_s = 1$ implies that $\Phi(v_j) \leqslant \mathcal{S}_1$, and by condition (f_5) we conclude $\Phi(v_j) \to \Phi_+(u)$ as $j \to +\infty$. The case for the inequality $\mathcal{S}_1 \geqslant \mathcal{S}_{1,-}$ follows by using the same argument. Moreover, the inequality $\mathcal{S}_1 > \max\{\mathcal{S}_{1,+}, \mathcal{S}_{1,-}\}$ holds whenever $F(t) \geqslant F_+(t)$ and $F(t) \geqslant F_-(t)$ with the strict inequality in a neighborhood of zero. In fact, since $F_+(t)$ and $F_-(t)$ are self-similar, we may consider w_+ and w_- the maximizers of $\mathcal{S}_{l,+}$ and $\mathcal{S}_{l,-}$, respectively, to obtain, by Theorem 2.2.2, Proposition 2.3.2 and Remark 2.5.1, that $\mathcal{S}_{l,+} < \Phi(w_+) \leqslant \mathcal{S}_l$ and $\mathcal{S}_{l,-} < \Phi(w_-) \leqslant \mathcal{S}_l$.

2.5.2 Characterization of the minimax level

We pass now to the study of the minimax level of the energy functional associated with Eq. $(\mathcal{E}_s)$, proving some useful results. This is made by considering the class of paths $\zeta : [0, +\infty) \to \mathcal{D}^{s,2}(\mathbb{R}^N)$ defined by $\zeta_u(t)(x) = u(x/t)$ for any $u \in \mathcal{D}^{s,2}(\mathbb{R}^N)$, because of it homogeneous property with respect to the norm in $\mathcal{D}^{s,2}(\mathbb{R}^N)$.

Lemma 2.5.3. *Suppose that $F(t)$ satisfies the growing condition* (1.5.2)*. If $u \in \mathcal{D}^{s,2}(\mathbb{R}^N)$ is such that $\Phi(u) > 0$, then the path ζ_u belongs to Γ_I. Thus $\Gamma_I \neq \varnothing$ if and only if (f_3') holds.*

Proof. Let $t_n, t_0 > 0$, $n \in \mathbb{N}$, be such that $t_n \to t_0$ and $u \in \mathscr{S}_0(\mathbb{R}^N)$. Since

$$[\zeta_u(t)]_s^2 = t^{N-2s}[u]_s^2, \ \forall t > 0, \tag{2.5.7}$$

using (1.2.2) we have

$$[\zeta_u(t_n) - \zeta_u(t_0)]_s^2 = \\ t_n^{N-2s}[u]_s^2 - 2t_n^{-s}t_0^{-s}\int_{\mathbb{R}^N}(-\Delta)^{s/2}u(x/t_n)(-\Delta)^{s/2}u(x/t_0)\,\mathrm{d}x + t_0^{N-2s}[u]_s^2. \tag{2.5.8}$$

Also, up to a set of Lebesgue measure zero, by identity (0.2.4) we obtain

$$\begin{cases} |(-\Delta)^s u(x/t_0)u(x/t_n)| \leqslant \|u\|_\infty \left|(-\Delta)^s u(x/t_0)\right|, \\ \lim\limits_{n\to\infty}\left[(-\Delta)^s u(x/t_0)u(x/t_n)\right] = (-\Delta)^s u(x/t_0)u(x/t_0), \end{cases} \quad \forall\, x \in \mathbb{R}^N.$$

Thus by the Dominated Convergence Theorem the left-hand side of the identity (2.5.8) goes to zero as $n \to \infty$. By identity (2.5.7) we conclude $\zeta_u \in C([0,\infty), \mathcal{D}^{s,2}(\mathbb{R}^N))$. The general case follows by a density argument.
Now suppose that (f_3') holds. Then there exists $u \in C_0^\infty(\mathbb{R}^N)$ such that $\Phi(u) > 0$ and consequently $\zeta_u \in \Gamma_I$, since

$$I(\zeta_u(t)) = \frac{1}{2}t^{N-2s}[u]_s^2 - t^N\Phi(u) \to -\infty \text{ as } t \to \infty.$$

Conversely, assume that $\Gamma_I \neq \varnothing$. If (f_3') does not hold, then we would have that $I(u) \geqslant 0$, for all $u \in \mathcal{D}^{s,2}(\mathbb{R}^N)$. Hence $\Gamma_I = \varnothing$, which is impossible. ■

Remark 2.5.4. Let $u \in \mathcal{D}^{s,2}(\mathbb{R}^N)$ be such that $\Phi(u) > 0$. Then

$$\max_{t\geqslant 0} I(\zeta_u(t)) = \frac{1}{2}\left(\frac{[u]_s^2}{2_s^*\Phi(u)}\right)^{\frac{N-2s}{2s}}[u]_s^2 - \left(\frac{[u]_s^2}{2_s^*\Phi(u)}\right)^{N/2s}\Phi(u). \tag{2.5.9}$$

Lemma 2.5.5. *Assume that conditions (f_3') and* (1.5.2) *holds. Consider*

$$\tilde{c}(I) := \inf_{\zeta\in\tilde{\Gamma}_I}\sup_{t\geqslant 0} I(\zeta(t)).$$

where

$$\tilde{\Gamma}_I := \{\zeta \in \Gamma_I : \zeta = \zeta_u \ \textit{ for some } u \in \mathcal{D}^{s,2}(\mathbb{R}^N) \textit{ with } \Phi(u) > 0\}$$

Then $c(I) = \tilde{c}(I)$.

Proof. Since $\tilde{\Gamma} \subset \Gamma$ we have $c(I) \leqslant \tilde{c}(I)$. Suppose the contrary, that $c(I) < \tilde{c}(I)$. Then, there exists $\zeta \in \Gamma_I$ such that $c(I) \leqslant \sup_{t \geqslant 0} I(\zeta(t)) < \tilde{c}(I)$. Observe now that, by embedding (0.2.1) and (f_1), the continuous function

$$h(t) = \frac{1}{2}[\zeta(t)]_s^2 - \frac{2_s^*}{2}\Phi(\zeta(t)),\ t > 0,$$

changes sign. Hence, there exists $t_0 > 0$ such that $g(t_0) = 0$ and $\zeta(t_0) \neq 0$, which implies that $[\zeta(t_0)]_s^2 = 2_s^*\Phi(\zeta(t_0))$. Now taking $u = \zeta(t_0)$ in (2.5.9) we get

$$\sup_{t \geqslant 0} I(\zeta_u(t)) = \frac{1}{2}[\zeta(t_0)]_s^2 - \Phi(\zeta(t_0)) \leqslant \sup_{t \geqslant 0} I(\zeta(t)),$$

which leads to a contradiction with the definition of $\tilde{c}(I)$. ■

Remark 2.5.6. In order to prove our nonlocal counterpart of [98, Proposition 2.4], we have to reduce the class of admissible paths. This is made by noticing that

$$\sup_{t \geqslant 0} I(\zeta_v(t)) = \sup_{t \geqslant 0} I(\zeta_{v_\sigma}(t)),$$

for any rescaling $v_\sigma(x) = v(x/\sigma)$, $\sigma > 0$, and taking account the set

$$\tilde{\Gamma}_I^1 := \left\{\zeta \in \Gamma_I : \zeta = \zeta_u \ \text{ for some } u \in \mathcal{D}^{s,2}(\mathbb{R}^N) \text{ with } \Phi(u) > 0 \text{ and } [u]_s \geqslant 1\right\},$$

and the associated minimax level

$$\tilde{c}_1(I) := \inf_{\zeta \in \tilde{\Gamma}_I^1} \sup_{t \geqslant 0} I(\zeta(t)),$$

to obtain that $\tilde{c}(I) = \tilde{c}_1(I)$.

2.5.3 Proof of Theorem 2.2.2

Proof. (i) Let $v \in \mathcal{D}^{s,2}(\mathbb{R}^N)$ be a non-trivial critical point of I. By Proposition 2.3.2 we have $\zeta_v \in \Gamma_I$ and $t = 1$ is a maximum point for the function $t \mapsto I(\zeta_v(t)) = (1/2)t^{N-2s}[v]_s^2 - t^N\Phi(v)$. Hence $c(I) \leqslant \max_{t \geqslant 0} I(\zeta_v(t)) = I(v)$.
(ii) Since w is a maximizer for (2.2.1) we have

$$\int_{\mathbb{R}^N} f(w)v\,\mathrm{d}x = 2\lambda \int_{\mathbb{R}^N} (-\Delta)^{s/2}w(-\Delta)^{s/2}v\,\mathrm{d}x, \quad \forall v \in \mathcal{D}^{s,2}(\mathbb{R}^N),$$

where λ is a Lagrange multiplier. We claim that $\lambda \neq 0$. Indeed, on the contrary, we get $f(w) = 0$ a.e in $\mathbb{R}^N$, which leads to a contradiction with $\Phi(w) > 0$. Thus, we can apply Proposition 2.3.2 to get

$$2\lambda[w]_s^2 = 2_s^* \int_{\mathbb{R}^N} F(w)\,\mathrm{d}x,$$

which together with relation (2.5.1) implies $2\lambda l_0 = 2_s^* S_1 l_0^{N/(N-2s)}$, and the explicit value of l_0 gives $\lambda = 1/2$. In particular,

$$I(w) = \left(\frac{1}{2} - \frac{1}{2_s^*}\right)[w]_s^2 > 0.$$

Let us prove now the last statement of (ii). By the part (i), it is sufficient to prove that $I(w) \leqslant c(I)$. Let $u \in \mathcal{D}^{s,2}(\mathbb{R}^N)$ with $\Phi(u) > 0$, and denote $\tilde{u} = u(\alpha\cdot)$ where $\alpha = [u]_s^{2/(N-2s)}$. Then $[\tilde{u}]_s^2 = 1$ and consequently

$$\Phi(\zeta_u(t)) = \Phi(\zeta_{\tilde{u}}(t\alpha)) \leqslant [\zeta_u(t)]_s^{\frac{2N}{N-2s}}\mathcal{S}_1, \quad \forall t \geqslant 0,$$

from which we can deduce, by Lemma 2.5.5 and Remark 2.5.6, that

$$\begin{aligned} c(I) &= \inf_{\substack{\Phi(u)>0,\\ [u]_s\geqslant 1}} \sup_{t\geqslant 0}\left[\frac{1}{2}[\zeta_u(t)]_s^2 - \Phi(\zeta_u(t))\right] \\ &\geqslant \inf_{\substack{\Phi(u)>0,\\ [u]_s\geqslant 1}} \sup_{t\geqslant 0}\left[\frac{1}{2}[\zeta_u(t)]_s^2 - [\zeta_u(t)]_s^{\frac{2N}{N-2s}}\mathcal{S}_1\right]. \end{aligned}$$

Moreover, we have

$$\begin{aligned} &\sup_{t\geqslant 0}\frac{1}{2}[\zeta_u(t)]_s^2 - [\zeta_u(t)]_s^{\frac{2N}{N-2s}}\mathcal{S}_1 \\ &\quad= \left[\frac{1}{2}(2_s^*\mathcal{S}_1)^{-\frac{N-2s}{2s}} - \mathcal{S}_1(2_s^*\mathcal{S}_1)^{-\frac{N}{2s}}\right][u]_s^{2_s^*(1-s)}, \ \forall u \in D^{s,2}(\mathbb{R}^N) \text{ with } \Phi(u) > 0. \end{aligned}$$

Consequently,

$$\inf_{\substack{\Phi(u)>0,\\ \|u\|\geqslant 1}} \sup_{t\geqslant 0}\left[\frac{1}{2}[\zeta_u(t)]_s^2 - [\zeta_u(t)]_s^{\frac{2N}{N-2s}}\mathcal{S}_1\right] = \frac{1}{2}(2_s^*\mathcal{S}_1)^{-\frac{N-2s}{2s}} - \mathcal{S}_1(2_s^*\mathcal{S}_1)^{-\frac{N}{2s}}.$$

On the other hand, by the explicit value of l_0 and relation (2.5.1) we have that

$$I(w) = \frac{1}{2}(2_s^*\mathcal{S}_1)^{-\frac{N-2s}{2s}} - \mathcal{S}_1(2_s^*\mathcal{S}_1)^{-\frac{N}{2s}}.$$

Thus $c(I) = I(w)$ and by the proof of the statement (i), the path $\zeta_w \in \Gamma_I$ is minimal. ■

2.5.4 Proof of Theorem 2.2.3

Proof. We start by noting that the embedding (0.2.1) together with condition (f_1) implies $\mathcal{I} > 0$. Let (u_k) be a minimizing sequence, that is, $\Phi(u_k) = 1$ and $[u_k]_s^2 \to \mathcal{I}$. Since this sequence is bounded, we may apply Theorem 1.1.1 to obtain the weak profile described in (1.1.1)–(1.1.4). By the Corollary 2.4.2, we have

$$1 = \sum_{n\in\mathbb{N}_*}\int_{\mathbb{R}^N} F(w^{(n)})\,\mathrm{d}x,$$

which implies that there exists $n \in \mathbb{N}_*$ with $0 < \Phi(w^{(n)}) \leqslant 1$. If $\Phi(w^{(n)}) = 1$, considering d_k as the element of $D_{\mathbb{Z}^N,\mathbb{R}}$ given by assertion (1.1.1), we have by the weak lower semi-continuity of the norm that

$$\mathcal{I} \leqslant [w^{(n)}]_s^2 \leqslant \liminf_{k\to\infty}[d_k^* u_k]_s^2 = \mathcal{I} \quad \text{and} \quad [d_k^* u_k]_s^2 = [u_k]_s^2 \to [w^{(n)}]_s^2,$$

which proves the first part of Theorem 2.2.3. Hence, let us assume that $\Phi(w^{(n)}) < 1$. Set $v_k = d_k^* u_k - w$, where $w = w^{(n)}$. By Corollary 2.4.3 we have

$$\lim_{k\to\infty}\left[1 - \int_{\mathbb{R}^N} F(v_k)\,\mathrm{d}x\right] = \int_{\mathbb{R}^N} F(w)\,\mathrm{d}x \tag{2.5.10}$$

Denote $\delta = \Phi(w)$ and set $\hat{w} = w(\delta^{1/N}\cdot)$. Thus $\Phi(\hat{w}) = 1$ and consequently

$$[w]_s^2 = \delta^{\frac{N-2s}{N}}[\hat{w}]_s^2 \geqslant \delta^{\frac{N-2s}{N}}\mathcal{I}. \tag{2.5.11}$$

Now consider

$$\hat{v}_k = v_k(|1-\delta|^{1/N}\beta_k^{1/N}\cdot), \text{ where } \beta_k = \Phi(b_k) \text{ and } b_k = v_k(|1-\delta|^{1/N}\cdot).$$

Since $\beta_k = |1-\delta|^{-1}\Phi(v_k)$, by convergence (2.5.10) we have $\beta_k \to 1$, and we conclude $\Phi(\hat{v}_k) = 1$ for large k. This leads to

$$[v_k]_s^2 = |1-\delta|^{\frac{N-2s}{N}}\beta_k^{\frac{N-2s}{N}}[\hat{v}_k]_s^2 \geqslant |1-\delta|^{\frac{N-2s}{N}}\beta_k^{\frac{N-2s}{N}}\mathcal{I}, \tag{2.5.12}$$

for large k. In the other hand, since $[u_k]_s^2 = [d_k^* u_k]_s^2$, by relations (2.5.11) and (2.5.12) we may infer

$$\begin{aligned}
[u_k]_s^2 &= [v_k]_s^2 + 2[v_k, w]_s + [w]_s^2 \\
&\geqslant \left[\delta^{\frac{N-2s}{N}} + |1-\delta|^{\frac{N-2s}{N}}\beta_k^{\frac{N-2s}{N}}\right]\mathcal{I} + 2[v_k, w]_s,
\end{aligned}$$

and passing the limit we finally conclude

$$1 \geqslant \delta^{1-\frac{2s}{N}} + |1-\delta|^{1-\frac{2s}{N}},$$

which leads to a contradiction since $0 < \delta < 1$. Thus w is the minimizer in (2.2.2) and consequently we have

$$\int_{\mathbb{R}^N} (-\Delta)^{s/2} w (-\Delta)^{s/2} v\,\mathrm{d}x = \lambda \int_{\mathbb{R}^N} f(w) v\,\mathrm{d}x, \quad \forall\, v \in \mathcal{D}^{s,2}(\mathbb{R}^N),$$

where $\lambda \in \mathbb{R}$ is a Lagrange multiplier. Taking $v = w$ in the above identity we have $\lambda \neq 0$, which allows us to apply Proposition 2.3.2 to get $\lambda = \mathcal{I}/2_s^*$, which by an easy computation using identities (1.2.2) leads us to conclude that $u = w(\cdot/\beta)$ is a non-trivial weak solution of Eq. $(\mathcal{E}_s)$, where $\beta = \lambda^{1/2s} = (\mathcal{I}/2_s^*)^{1/2s}$.

Let us prove now that $u = w(\cdot/\beta)$ is a ground state solution of Eq. $(\mathcal{E}_s)$. We start by applying Proposition 2.3.2 again to obtain

$$I(u) = \left(\frac{1}{2} - \frac{1}{2_s^*}\right)[u]_s^2 = \frac{s}{N}(2_s^*)^{-\frac{N-2s}{2s}}[w]_s^{N/s}. \tag{2.5.13}$$

Now let $v \in \mathcal{D}^{s,2}(\mathbb{R}^N)$ be any non-trivial weak solution of Eq. $(\mathcal{E}_s)$. For any $\sigma > 0$ denote $v_\sigma = v(\cdot/\sigma)$. Choose σ such that $\Phi(v_\sigma) = 1$, that is, $\sigma = (\Phi(v))^{-1/N}$. Replacing this value of $\Phi(v)$ in the identity $[v]_s^2 = 2_s^*\Phi(v)$, we get $\sigma = (2_s^*)^{1/N}[v]_s^{-2/N}$. Consequently, we obtain

$$[v_\sigma]_s^2 = (2_s^*)^{\frac{N-2s}{N}}([v]_s^2)^{2s/N},$$

which implies

$$I(v) = \frac{s}{N}[v]_s^2 = \frac{s}{N}(2_s^*)^{-\frac{N-2s}{2s}}[v_\sigma]_s^{N/s}. \tag{2.5.14}$$

Comparing identities (2.5.13) and (2.5.14), we conclude that $I(u) \leqslant I(v)$, i.e, u is a ground state solution for Eq. $(\mathcal{E}_s)$. ∎

2.6 The non-autonomous case

For the sake of discussion, we are going to compare the minimax level of the asymptotic functional I_κ, with the minimax of the Lagrangian associated with Eq. $(\mathcal{E}_s)$, for $\kappa = 0, +, -$.

Proposition 2.6.1. *Suppose that $f(x,t)$ satisfies conditions (f_1)–(f_6). If $F_0(t)$ is self-similar or $(F_0)_\kappa(t) \leqslant F_\kappa(t)$, for all t, $\kappa = +, -$, then $c(I) \leqslant c(I_\kappa)$, for $\kappa = 0, +, -$. Moreover, under these assumptions, (f_7') implies (f_7).*

Proof. Let be $\mathcal{S}_l^\kappa$, the associated constrained maximum similar to (2.2.1) relative to the primitive F_κ, precisely,

$$\mathcal{S}_l^\kappa = \sup_{[u]_s^2 = l} \int_{\mathbb{R}^N} F_\kappa(u)\,\mathrm{d}x \quad \text{for} \quad \kappa = 0, +, -.$$

For each $\kappa = +, -$, the primitive of the nonlinearity F_κ is auto-similar, thus using Theorems 2.2.1 and 2.2.2, we conclude that there exists w_κ maximizer of $\mathcal{S}_{l_0}^\kappa$ such that

$$c(I_\kappa) = I_\kappa(w_\kappa) = \max_{t \geqslant 0} I_\kappa(\zeta_{w_\kappa}(t)) > 0.$$

For each $\kappa = +, -$, let us consider the sequence

$$w_n^\kappa := \gamma^{\frac{N-2s}{2}j_n^\kappa} w_\kappa\left(\gamma^{j_n^\kappa}\cdot\right),$$

where the sequence $(j_n^\kappa) \subset \mathbb{Z}$ is chosen in such a way that $j_n^+ \to +\infty$ and $j_n^- \to -\infty$. Since for each $\kappa = +, -$,

$$\left|I(\zeta_{w_n^\kappa}(t)) - I_\kappa(\zeta_{w_\kappa}(t))\right| \leqslant t^N \int_{\mathbb{R}^N} \left|\gamma^{-Nj_n^\kappa} F\left(\gamma^{-j_n^\kappa}tx, \gamma^{\frac{N-2s}{2}j_n^\kappa} w_\kappa\right) - F_\kappa(w_\kappa)\right| \mathrm{d}x,$$

the uniformity assumption on the limits in (f_5), guarantees (by a density argument) that

$$\lim_{n\to\infty} I(\zeta_{w_n^\kappa}(t)) = I_\kappa(\zeta_{w_\kappa}(t)), \quad \text{uniformly in compact sets of } \mathbb{R}. \tag{2.6.1}$$

We also have that the path $\zeta_{w_n^\kappa}$, $\kappa = +, -$, belongs to Γ_I, for n large enough. In fact, by the uniformly convergence in x of (f_5) and Proposition 2.3.2, there exists $n_0 > 0$ such that

$$\int_{\mathbb{R}^N} \gamma^{-Nj_n^\kappa} F\left(\gamma^{-j_n^\kappa} tx, \gamma^{\frac{N-2s}{2} j_n^\kappa} w_\kappa\right) \mathrm{d}x > \frac{1}{2}\int_{\mathbb{R}^N} F_\kappa(w_\kappa)\,\mathrm{d}x, \ \forall\, n > n_0 \text{ and } t > 0.$$

Thus, for each n there exist $t_n > 0$ such that

$$I(\zeta_{w_n^\kappa}(t_n)) = \max_{t\geqslant 0} I(\zeta_{w_n^\kappa}(t)) > 0.$$

We claim that the sequence (t_m) is bounded. On the contrary, up to subsequence, we get the following contradiction

$$0 < I(\zeta_{w_n^\kappa}(t_m)) = \frac{1}{2} t_m^{N-2s}[w]_s^2 - t_m^N \int_{\mathbb{R}^N} \gamma^{-Nj_n} F\left(\gamma^{-j_n} t_m x, \gamma^{\frac{N-2s}{2} j_n} w_\kappa\right) \mathrm{d}x \to -\infty,$$

as $m \to \infty$. Therefore, up to subsequence, $t_m \to t_0$, and we have

$$\lim_{m\to\infty} \max_{t\geqslant 0} I(\zeta_{w_n^\kappa}(t_n)) = I_\kappa(\zeta_{w_\kappa}(t_0)),$$

because of (2.6.1). Thus we may conclude

$$c(I) \leqslant \lim_{n\to\infty} \max_{t\geqslant 0} I(\zeta_{w_n^\kappa}(t)) \leqslant \max_{t\geqslant 0} I_\kappa(\zeta_{w_\kappa}(t)) = c(I_\kappa).$$

If there exists maximizer w_0 for $\mathcal{S}_{l_0}^0$, then an similar argument as above leads to $c(I) \leqslant c(I_0)$. In fact, for each n, define the path

$$\lambda_n(t) = w_0\left(\frac{\cdot - y_n}{t}\right), \quad t \geqslant 0,$$

where (y_n) is taken in a such way that $|y_n| \to \infty$. As before, we consider the estimate

$$|I(\lambda_n(t)) - I_0(w_0(\cdot/t))| \leqslant t^N \int_{\mathbb{R}^N} |F(tx + y_n, w_0) - F_0(w_0)|\,\mathrm{d}x,$$

to obtain that

$$\lim_{n\to\infty} I(\lambda_n(t)) = I_0(w_0(\cdot/t)), \quad \text{uniformly in compact sets of } \mathbb{R}.$$

We also have that the path λ_n belongs to Γ_I, for n large enough. Indeed, assuming the contrary, we would obtain n_0 and a sequence $l_n \to \infty$ such that $I(\lambda_{n_0}(l_n)) > 0$, for all n. On the other hand, we would have that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} F(l_n x + y_{n_0}, w_0)\,\mathrm{d}x = \int_{\mathbb{R}^N} F_0(w_0)\,\mathrm{d}x,$$

which, by taking n large enough, leads to the contradiction $I(\lambda_{n_0}(l_n)) < 0$. Let $t_n > 0$ such that

$$I(\lambda_n(t_n)) = \max_{t \geqslant 0} I(\lambda_n(t)) > 0.$$

Once again we get that the sequence (t_n) is bounded. On the contrary, there is a subsequence (t_{k_n}) that implies in the following contradiction

$$0 < I(\lambda_n(t_{k_n})) = \frac{1}{2} t_{k_n}^{N-2s} [w_0]_s^2 - t_{k_n}^N \int_{\mathbb{R}^N} F(t_{k_n} x + y_n, w_0) \, \mathrm{d}x \to -\infty, \text{ as } n \to \infty.$$

Thus, up to subsequence, $t_n \to t_0$ and we obtain that

$$\lim_{n \to \infty} \max_{t \geqslant 0} I(\lambda_n(t)) = I_0(w_0(\cdot/t_0)).$$

As a consequence we conclude that

$$c(I) \leqslant \lim_{n \to \infty} \max_{t \geqslant 0} I(\lambda_n(t_n)) \leqslant \max_{t \geqslant 0} I_0(w_0(\cdot/t)) = c(I_0),$$

where we used Proposition 2.3.2 to induce that $t = 1$ is the unique critical point of $I_0(w_0(\cdot/t))$. Thus, let us assume that $\mathcal{S}_{l_0}^0$ is not attained. By Remarks 2.5.1 and 2.5.2; and Theorem 2.2.1, if $\mathcal{S}_{l_0}^0$ is not attained then $\mathcal{S}_{l_0}^0 = \mathcal{S}_{l_0}^+$ or $\mathcal{S}_{l_0}^0 = \mathcal{S}_{l_0}^-$. Thus, using the definition of $\mathcal{S}_{l_0}^0$ we get

$$c(I_\kappa) \leqslant I_0(u), \ \kappa = +, -, \ \forall \ u \in \mathcal{D}^{s,2}(\mathbb{R}^N) \text{ with } [u]_s^2 = l_0.$$

Let $u \in \mathcal{D}^{s,2}(\mathbb{R}^N)$, $u \neq 0$, and denote $\alpha = [u]_s^2$, then considering the rescaling $u_{l_0} = u(t_0 \cdot)$, where $t_0 = (\alpha/l_0)^{-1/(N-2s)}$, we have $[u_{l_0}]_s^2 = l_0$ and consequently

$$\begin{aligned} c(I_\kappa) \leqslant I(u_{l_0}) &= \frac{1}{2} t_0^{N-2s} [u]_s^2 - t_0^N \Phi_0(u) \\ &\leqslant \max_{t \geqslant 0} I_0(\zeta_u(t)), \text{ for } \kappa = +, -. \end{aligned}$$

By Lemma 2.5.5 we conclude $c(I_\kappa) \leqslant c(I_0)$, $\kappa = +, -$.
Now suppose that (f_7') holds. As seen above, ζ_{w_κ} belongs to Γ_I, thus

$$c(I) \leqslant \max_{t \geqslant 0} I(\zeta_{w_\kappa}(t)) < \max_{t \geqslant 0} I_\kappa(\zeta_{w_\kappa}(t)) = c(I_\kappa), \ \kappa = +, -.$$

We claim that $S_{l_0}^0$ is attained, from which we conclude the desired inequality in (f_7). Assume the contrary, by arguing as before, we have $\mathcal{S}_{l_0}^0 = \mathcal{S}_{l_0}^+$ or $\mathcal{S}_{l_0}^0 = \mathcal{S}_{l_0}^-$. Taking $|x| \to \infty$ in (f_7') we get that $F_0(t) \geqslant F_\kappa(t)$, $\kappa = +, -$, for all $t \in \mathbb{R}$. Consequently, in any case,

$$\begin{aligned} \int_{\mathbb{R}^N} F_0(w_\kappa) \, \mathrm{d}x &\leqslant \sup_{[u]_s^2 = l_0} \int_{\mathbb{R}^N} F_0(u) \, \mathrm{d}x \\ &= \int_{\mathbb{R}^N} F_\kappa(w_\kappa) \, \mathrm{d}x \leqslant \int_{\mathbb{R}^N} F_0(w_\kappa) \, \mathrm{d}x, \ \kappa = +, -, \end{aligned} \tag{2.6.2}$$

a contradiction, because relation (2.6.2) implies that $S_{l_0}^0$ is attained. ■

Summarizing all the discussion until now we can finally prove Theorem 2.2.4.

2.6.1 Proof of Theorem 2.2.4

In order to treat the case without compactness condition (f_7), that is not considered in the local counterpart [98], where the case $c(I_\kappa) = c(I)$, $\kappa = 0, +, -$, may occur, we need Theorem 0.6.4, which states that the existence of a critical point of I is guaranteed whenever the minimax level (2.1.1) is attained.

Remark 2.6.2. We define

$$c_1(I) = \inf_{\gamma\in\Gamma_I} \sup_{t\in[0,1]} I(\gamma(t)),$$

where

$$\Gamma_I^1 = \left\{\gamma \in C([0,1], \mathcal{D}^{s,2}(\mathbb{R}^N)) : \gamma(0) = 0,\ [\gamma(1)]_s > r,\ I(\gamma(1)) < 0\right\},$$

as the usual minimax level. We have that $c_1(I) = c(I)$.

Proof of Theorem 2.2.4 completed. For the reader convenience, we divide the proof in several steps.

(i) We start observing that the assumptions (f_2) and (f_3) implies that the functional I has the mountain pass geometry. In particular, $\Gamma_I \neq \emptyset$ and $0 < c(I) < \infty$. In fact, set $v = \varphi_R(x) := v_R(|x - x_0|)$, where v_R as defined as in Remark 2.2.6. Then $\varphi_R \in \mathcal{D}^{s,2}(\mathbb{R}^N)$ and we have

$$\begin{aligned}\int_{\mathbb{R}^N} F(x,v)\,\mathrm{d}x &= \int_{B_R(x_0)} F(x,t_0)\,\mathrm{d}x + \int_{B_{R+1}(x_0)\setminus B_R(x_0)} F(x,v)\,\mathrm{d}x \\ &\geqslant |B_R| \inf_{B_R(x_0)} F(x,t_0) + |B_{R+1}\setminus B_R| \inf_{(x,t)\in(B_{R+1}(x_0)\setminus B_R(x_0))\times[0,t_0]} F(x,t) > 0\end{aligned}$$

Since (f_2) is equivalent to $d/dt(F(x,t)t^{-\mu}) \geqslant 0$, $t > 0$, we have for $t > 1$ that

$$\int_{\mathbb{R}^N} F(x,tv)\,\mathrm{d}x \geqslant t^\mu \int_{\mathbb{R}^N} F(x,v)\,\mathrm{d}x.$$

Hence

$$I(tv) = \frac{t^2}{2}[v]_s^2 - \int_{\mathbb{R}^N} F(x,tv)\,\mathrm{d}x \leqslant \frac{t^2}{2}[v]_s^2 - t^\mu \int_{\mathbb{R}^N} F(x,v)\,\mathrm{d}x \to -\infty, \text{ as } t \to \infty.$$

In the other hand, by the growth condition (f_1) and the embedding 0.2.1,

$$I(u) \geqslant [u]_s^2\left(\frac{1}{2} - C[u]_s^{2_s^*-2}\right), \quad u \in \mathcal{D}^{s,2}(\mathbb{R}^N),$$

Thus, choosing $[u]_s$ sufficiently small, we have $I(u) > 0$. The same can be concluded for the functionals I_κ, since F_κ satisfies (f_2) and (f_3).
Let (u_k) in $\mathcal{D}^{s,2}(\mathbb{R}^N)$ be such that $I(u_k) \to c(I)$ and $I'(u_k) \to 0$, which the existence can be guaranteed by the Mountain Pass Theorem (see Theorem 0.6.2).

(ii) By assumption (f_2), this sequence is bounded in $\mathcal{D}^{s,2}(\mathbb{R}^N)$, since for large k, we have

$$\begin{aligned}
c(I) + 1 + [u_k]_s &\geqslant I(u_k) - \frac{1}{\mu} I'(u_k) \cdot u_k \\
&= \left(\frac{1}{2} - \frac{1}{\mu}\right)[u_k]_s^2 - \int_{\mathbb{R}^N} F(x, u_k) - \frac{1}{\mu} f(x, u_k) u_k \,\mathrm{d}x \\
&\geqslant \left(\frac{1}{2} - \frac{1}{\mu}\right)[u_k]_s^2.
\end{aligned}$$

Let $(w^{(n)})$, $(y_k^{(n)})$ and $(j_k^{(n)})$ be the sequences provided by Theorem 1.1.1. If $w^{(n)} = 0$ for all $n \geqslant 2$, then by assertions (1.1.1) and (1.1.4) of Theorem 1.1.1,

$$u_k \to w^{(1)} \text{ in } L^{2_s^*}(\mathbb{R}^N) \text{ and } u_k \rightharpoonup w^{(1)} \text{ in } \mathcal{D}^{s,2}(\mathbb{R}^N).$$

Therefore we conclude that $w^{(1)}$ is a critical point of I such that, up to subsequence, $u_k \to w^{(1)}$ in $\mathcal{D}^{s,2}(\mathbb{R}^N)$.

(iii) Let us assume first that condition (f_7) holds true. We argue by contradiction and assume that there exists $n_0 \geqslant 2$, such that $w^{(n_0)} \neq 0$. By the estimate (1.1.3) and Proposition 2.4.1 we have, up to subsequence, that

$$\begin{aligned}
c(I) &= \lim_{k\to\infty} \left[\frac{1}{2}\|u_k\|^2 - \int_{\mathbb{R}^N} F(x, u_k)\,\mathrm{d}x\right] \\
&\geqslant I(w^{(1)}) + \sum_{n\in\mathbb{N}_0, n>1} I_0(w^{(n)}) + \sum_{n\in\mathbb{N}_+} I_+(w^{(n)}) + \sum_{n\in\mathbb{N}_-} I_-(w^{(n)}).
\end{aligned} \tag{2.6.3}$$

Let $\varphi \in C_0^\infty(\mathbb{R}^N)$ and $n \geqslant 1$. Since

$$\left|\gamma^{-\frac{N+2s}{2} j_k^{(n)}} f\left(\gamma^{-j_k^{(n)}} x + y_k^{(n)}, \gamma^{\frac{N-2s}{2} j_k^{(n)}} t\right)\right| \leqslant C|t|^{2_s^*-1}, \ \forall\, x \in \mathbb{R}^N \text{ and } t \in \mathbb{R},$$

by the embedding (0.2.3), we can take the limit

$$\begin{aligned}
I'(u_k) \cdot &\left(\gamma^{\frac{N-2s}{2} j_k^{(n)}} \varphi(\gamma^{j_k^{(n)}}(\cdot - y_k^{(n)}))\right) \\
&= [\gamma^{-\frac{N-2s}{2} j_k^{(n)}} u_k(\gamma^{-j_k^{(n)}} \cdot + y_k^{(n)}), \varphi]_s \\
&\quad - \int_{\mathbb{R}^N} \gamma^{-\frac{N+2s}{2} j_k^{(n)}} f\left(\gamma^{-j_k^{(n)}} x + y_k^{(n)}, \gamma^{\frac{N-2s}{2} j_k^{(n)}} v_k^{(n)}\right) \varphi \,\mathrm{d}x,
\end{aligned}$$

where

$$v_k^{(n)} := \gamma^{-\frac{N-2s}{2} j_k^{(n)}} u_k(\gamma^{-j_k^{(n)}} \cdot + y_k^{(n)}),$$

to conclude that $w^{(1)}$ is a critical point of I and $w^{(n)}$ is a critical point of I_0, I_+ or I_-, provided that $n \in \mathbb{N}_0, \mathbb{N}_+$ or $\mathbb{N}_-$, respectively. Consequently, using assumption (f_2)

$$I_\kappa(w^{(n)}) = \frac{1}{2}\int_{\mathbb{R}^N} f_\kappa(w^{(n)}) w^{(n)} \,\mathrm{d}x - \int_{\mathbb{R}^N} F_\kappa(w^{(n)})\,\mathrm{d}x \geqslant 0, \ \forall\, n \geqslant 2,$$

and $I(w^{(1)}) \geqslant 0$. On the other hand, the assumption $c(I) < c(I_\kappa)$ and the estimate (2.6.3) implies $I_\kappa(w^{(n_0)}) < c(I_\kappa)$, which leads to a contradiction with Theorem 2.2.2 (or Proposition 2.3.2).

(iv) Suppose now that relation (2.1.2) holds instead of (f_7). Condition (2.1.2) implies that the path $\zeta_{w^{(n_0)}}$ belongs to Γ_I and $c(I) \leqslant I_\kappa(w^{(n_0)})$, where κ is the corresponding index for which n_0 belongs. In view of the above discussion and estimate (2.6.3), we conclude that

$$u_k \to w^{(1)} \text{ in a subsequence} \quad \text{or} \quad c(I) = \max_{t \geqslant 0} I(w^{(n_0)}(\cdot/t)).$$

If the minimax level $c(I)$ is attained then we can apply Theorem 0.6.4 to obtain the existence of critical point $u \in \zeta_{w^{(n_0)}}([0,\infty))$ such that $I(u) = c(I)$. ■

Chapter 3

Concentration-compactness at the mountain pass level for nonlocal Schrödinger equations

In this chapter, we study the existence of non-trivial weak and ground state solutions for the following class of fractional Schrödinger equation

$$(-\Delta)^s u + a(x)u = f(x,u) \quad \text{in } \mathbb{R}^N. \tag{$\mathcal{H}_s$}$$

We obtain existence results for a wide class of possible singular potentials $a(x)$, not necessarily bounded away from zero and for oscillatory nonlinearities in both subcritical and critical growth range that may not satisfy the Ambrosetti-Rabinowitz condition.
Outline. The chapter is organized as follows. In Sect. 3.1, we describe the assumptions on the potential $a(x)$ and nonlinearity $f(x,t)$ in Eq. $(\mathcal{H}_s)$ that are used to state our results in Sect. 3.2. In Sect. 3.3, we provide a suitable variational settings to prove our main results, more precisely, we prove that the energy functional associated with $(\mathcal{H}_s)$ possess the mountain pass geometry and Palais-Smale sequences at the mountain pass level are bounded. In Sect. 3.4 we describe the limit under the profile decomposition of the Palais-Smale sequence at the mountain pass level of the energy functional related to $(\mathcal{H}_s)$. In Sect. 3.5 we prove that weak solutions of Eq. $(\mathcal{H}_s)$ in the autonomous case $f(x,t) = f(t)$ satisfy a Pohozaev type identity. Sections 3.6, 3.7, 3.8 and 3.9 are dedicated to the proof of our main results concerning existence of mountain pass solutions for Eq. $(\mathcal{H}_s)$.

3.1 Hypothesis

In order to describe our results on the energy functional of $(\mathcal{H}_s)$ in a more precise way, next we state the main assumptions on the potential $a(x)$ and the nonlinearity $f(x,t)$ respectively. We always assume that $N > 2s$ and $0 < s < 1$.

Subcritical case

- Assumptions on $a(x) = V(x) - b(x)$.

(V_1) $V(x) \in L^{\sigma}_{\text{loc}}(\mathbb{R}^N)$, for some $\sigma > 2N/(N+2s)$ and $V(x)$ is 1−periodic in x_i, $i = 1, \dots, N$.

(V_2) The following infimum

$$\mathcal{C}_V = \inf_{u \in C_0^\infty(\mathbb{R}^N), \|u\|_2 = 1} \int_{\mathbb{R}^N} |(-\Delta)^{s/2} u|^2 + V(x) u^2 \,\mathrm{d}x$$

is positive and $V(x) \geqslant -\mathcal{B}$ a.e. $x \in \mathbb{R}^N$, for some $\mathcal{B} > 0$.

(V_3) $0 \leqslant b(x) \in L^{\beta}(\mathbb{R}^N)$, for some $\beta > N/2s$, and $\|b(x)\|_\beta < \mathcal{C}_V^{(\beta)}$, where

$$\mathcal{C}_V^{(\beta)} = \inf_{u \in H^s_V(\mathbb{R}^N), \|u\|_{2\beta'} = 1} \int_{\mathbb{R}^N} |(-\Delta)^{s/2} u|^2 + V(x) u^2 \,\mathrm{d}x, \quad \beta' = \beta/(\beta - 1).$$

(V_4) $V(x) \in L^{\sigma}_{\text{loc}}(\mathbb{R}^N)$, for some $\sigma > N/2s$ and there exists the limit $0 < V_\infty := \lim_{|x| \to \infty} V(x)$.

- Assumptions on $f(x,t)$.

(f_1) $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function. Moreover, for every $\varepsilon > 0$ there exists $p_\varepsilon \in (2, 2_s^*)$ and $C_\varepsilon > 0$ such that

$$|f(x,t)| \leqslant \varepsilon(|t| + |t|^{2_s^* - 1}) + C_\varepsilon |t|^{p_\varepsilon - 1}, \quad \text{a.e. } x \in \mathbb{R}^N \text{ and } \forall\, t \in \mathbb{R}.$$

(f_2) There exists $\mu > 2$ such that,

$$\mu F(x,t) := \mu \int_0^t f(x,\tau)\,\mathrm{d}\tau \leqslant f(x,t)t, \quad \text{a.e. } x \in \mathbb{R}^N \text{ and } \forall\, t \in \mathbb{R}.$$

(f_3) There exists $R > 0$, $t_0 > 0$, $x_0 \in \mathbb{R}^N$ such that

$$|B_R| \inf_{B_R(x_0)} F(x, t_0) + |B_{R+1} \backslash B_R| \inf_{(x,t) \in (B_{R+1}(x_0) \backslash B_R(x_0)) \times [0, t_0]} F(x,t) > 0,$$

In the autonomous case, where $f(x,t) = f(t)$, we consider the following variant of (f_3).

(f_3') There exists $t_0 > 0$ such that $F(t_0) > 0$.

(f_4) The following limits are uniform in x,

$$\lim_{t\to 0} \frac{f(x,t)}{t} = 0 \quad \text{and} \quad \lim_{|t|\to\infty} \frac{F(x,t)}{t^2} = \infty.$$

Moreover, for any compact set K in $\mathbb{R}$, there is a positive constant $C = C(K)$ such that

$$|f(x,t)| \leqslant C, \quad \text{a.e. } x \in \mathbb{R}^N \text{ and } \forall\, t \in K.$$

(f_5) Let $\mathcal{F}(x,t) := \frac{1}{2} f(x,t)t - F(x,t)$. For any $0 < a < b$, we have that

$$\inf_{x\in\mathbb{R}^N} \inf_{a\leqslant|t|\leqslant b} \mathcal{F}(x,t) > 0.$$

(f_6) There exists $p_0 > \max\{1, N/2s\}$ and $a_0, R_0 > 0$ such that

$$|f(x,t)|^{p_0} \leqslant a_0|t|^{p_0}\mathcal{F}(x,t), \quad \text{a.e. } x \in \mathbb{R}^N, \text{ and } \forall\, |t| > R_0.$$

(f_7) There exists a 1−periodic function $f_{\mathcal{P}}(x,t)$ in x_i $i = 1, \ldots, N$ such that

$$\lim_{|x|\to\infty} |f(x,t) - f_{\mathcal{P}}(x,t)| = 0,$$

uniformly in compact sets of $\mathbb{R}$. In addition, we assume that $f_{\mathcal{P}}(x,t)$ satisfies (f_1) and either (f_2)–(f_3) or (f_4).

(f_8) For a.e. $x \in \mathbb{R}^N$ the function

$$t \mapsto \frac{f_{\mathcal{P}}(x,t)}{|t|}, \quad \text{is strict increasing in } \mathbb{R}.$$

For the next condition we are assuming that $f_{\mathcal{P}}(x,t)$ in (f_7) is independent of t and we denote $f_\infty(t) = f_{\mathcal{P}}(t)$.

(f_9) $f_\infty(t)$ belongs to $C^1(\mathbb{R})$ and there exists $t_0 > 0$ such that

$$F_\infty(t_0) - \frac{V_\infty}{2} t_0^2 > 0, \quad \text{where } F_\infty(t) = \int_0^t f_\infty(\tau)d\tau.$$

We look for solutions in the space $H^s_V(\mathbb{R}^N)$ which is defined as the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|u\|_V^2 := \int_{\mathbb{R}^N} |(-\Delta)^{s/2}u|^2 + V(x)u^2 \, \mathrm{d}x.$$

If we assume $V(x) \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$ and (V_2), then $H^s_V(\mathbb{R}^N)$ is well defined, also $\|\cdot\|_V$ is induced by the inner product

$$(u,v)_V := \int_{\mathbb{R}^N} (-\Delta)^{s/2}u(-\Delta)^{s/2}v + V(x)uv \, \mathrm{d}x,$$

and in view of Proposition 3.3.1, we have that $H^s_V(\mathbb{R}^N)$ is a Hilbert space.

Writing, $a(x) = V(x) - b(x)$, we consider associated with the problem $(\mathcal{H}_s)$, the functional $I : H^s_V(\mathbb{R}^N) \to \mathbb{R}$ given by

$$I(u) = \frac{1}{2}\|u\|_V^2 - \frac{1}{2}\int_{\mathbb{R}^N} b(x)u^2 \, \mathrm{d}x - \int_{\mathbb{R}^N} F(x,u) \, \mathrm{d}x.$$

If in addition we assume (V_3) and (f_1) then $I \in C^1(H^s_V(\mathbb{R}^N))$ and

$$I'(u)\cdot v = \int_{\mathbb{R}^N} (-\Delta)^{s/2}u(-\Delta)^{s/2}v + (V(x)-b(x))uv \, \mathrm{d}x - \int_{\mathbb{R}^N} f(x,u)v \, \mathrm{d}x, \; u,v \in H^{s,2}_V(\mathbb{R}^N).$$

Thus critical points of I correspond to weak solutions of $(\mathcal{H}_s)$ and conversely. We define the minimax level as

$$c(I) = \inf_{\gamma \in \Gamma_I} \sup_{t \geqslant 0} I(\gamma(t)), \tag{3.1.1}$$

where

$$\Gamma_I = \left\{ \gamma \in C([0,\infty), H^s_V(\mathbb{R}^N)) : \gamma(0) = 0, \; \lim_{t\to\infty} I(\gamma(t)) = -\infty \right\}. \tag{3.1.2}$$

We also consider the following C^1 functionals associated with the limits given in (V_4), (f_7) and (f_9),

$$I_{\mathcal{P}}(u) := \frac{1}{2}\|u\|_V^2 - \int_{\mathbb{R}^N} F_{\mathcal{P}}(x,u) \, \mathrm{d}x, \; u \in H^s_V(\mathbb{R}^N),$$
$$I_\infty(u) := \frac{1}{2}\|u\|_{V_\infty}^2 - \int_{\mathbb{R}^N} F_\infty(u) \, \mathrm{d}x, \quad u \in H^s_V(\mathbb{R}^N),$$

where $F_{\mathcal{P}}(x,t) = \int_0^t f(x,\tau)d\tau$. Similarly, as in (3.1.1) and (3.1.2), we can define $c(I_{\mathcal{P}})$, $c(I_\infty), \Gamma_{I_{\mathcal{P}}}$ and Γ_{I_∞}. Next we finally state the assumption relative to the minimax level of the considered functionals, that guarantees compactness of the Palais-Smale sequences at the mountain pass level.

(f_{10}) $c(I) < c(I_{\mathcal{P}})$;

(f'_{10}) $c(I) < c(I_\infty)$;

Critical case

- Assumptions on $a(x)$. Here we assume $b(x) \equiv 0$, that is, $a(x) \equiv V(x)$.

(V_1^*) $V(x) \in L^1_{\text{loc}}(\mathbb{R}^N) \cap C(\mathbb{R}^N \backslash \mathcal{O})$, where $\mathcal{O}$ is a finite set, and $V(x) \leqslant 0$ a.e. $x \in \mathbb{R}^N$. Moreover

$$0 < \mathcal{C}_V^* := \inf_{u \in C_0^\infty(\mathbb{R}^N)\backslash\{0\}} \frac{\int_{\mathbb{R}^N} |(-\Delta)^{s/2} u|^2 + V(x) u^2 \,\mathrm{d}x}{\int_{\mathbb{R}^N} |V(x)| u^2 \,\mathrm{d}x} < \infty.$$

(V_2^*) There exists $a_* \in \mathbb{R}^N$ such that the following limits exist and are uniformly convergent in compact sets

$$V_+(x) = \lim_{\lambda \to \infty} \lambda^{-2s} V(\lambda^{-1}(x + a_*)),$$
$$V_-(x) = \lim_{\lambda \to 0} \lambda^{-2s} V(\lambda^{-1}(x + a_*)).$$

Moreover $\lim_{|x| \to \infty} V(x) = 0$, and $V_\pm(x)$ satisfies (V_1^*), provided that $V_\pm(x) \not\equiv 0$.

(V_3^*) For any given sequence (λ_k) of positive numbers such that either $|\lambda_k| \to \infty$ or $|\lambda_k| \to 0$; and sequence (y_k) in $\mathbb{R}^N$, such that $|\lambda_k y_k| \to \infty$ we have,

$$\lim_{k \to \infty} \lambda_k^{-2s} V(\lambda_k^{-1} x + y_k) = 0, \text{ uniformly in compact sets.}$$

- Assumptions on $f(x, t)$.

(f_1^*) $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ satisfies the Carathéodory conditions. Moreover, there exists $C > 0$ such that

$$|f(x, t)| \leqslant C|t|^{2_s^* - 1}, \quad \text{a.e. } x \in \mathbb{R}^N \text{ and } \forall\, t \in \mathbb{R}.$$

(f_2^*) For each real numbers $a_1, \ldots, a_M$, there exist $C = C(M) > 0$ such that

$$\left| F\left(x, \sum_{n=1}^M a_n\right) - \sum_{n=1}^M F(x, a_n) \right| \leqslant C(M) \sum_{m \neq n \in \{1, \ldots, M\}} |a_n|^{2_s^* - 1} |a_m| \text{ a.e. } x \in \mathbb{R}^N.$$

(f_3^*) The following limits exist and are uniformly convergent in x and in compact sets for t,

$$f_0(t) := \lim_{|x| \to \infty} f(x, t),$$
$$f_+(t) := \lim_{j \in \mathbb{Z}, j \to +\infty} \gamma^{-\frac{N+2s}{2} j} f\left(\gamma^{-j} x, \gamma^{\frac{N-2s}{2} j} t\right),$$
$$f_-(t) := \lim_{j \in \mathbb{Z}, j \to -\infty} \gamma^{-\frac{N+2s}{2} j} f\left(\gamma^{-j} x, \gamma^{\frac{N-2s}{2} j} t\right),$$

for some $\gamma > 1$ and $0 < s < \min\{1, N/2\}$.

(f_4^*) For each $\kappa = 0, +, -$, the function

$$t \mapsto \frac{f_\kappa(t)}{|t|} \quad \text{is strict increasing in } \mathbb{R}.$$

Observe that the assumption (V_1^*) guarantees that $\|\cdot\|_V$ defines a norm in $\mathcal{D}^{s,2}(\mathbb{R}^N)$ which is equivalent to the standard one (see Proposition 3.3.1). Thus in the critical case we consider associated with problem $(\mathcal{H}_s)$ the energy functional $I_* : \mathcal{D}^{s,2}(\mathbb{R}^N) \to \mathbb{R}$ given by

$$I_*(u) = \frac{1}{2}\|u\|_V^2 - \int_{\mathbb{R}^N} F(x,u)\,\mathrm{d}x, \quad u \in \mathcal{D}^{s,2}(\mathbb{R}^N),$$

which is well defined and is C^1 provided (f_1^*) holds. We can define $c(I_*)$ and Γ_{I_*} similarly as in (3.1.1) and (3.1.2), by just replacing $H_V^s(\mathbb{R}^N)$ by $\mathcal{D}^{s,2}(\mathbb{R}^N)$.

We consider the next assumption in order to compare the minimax levels of the resulting limiting energy functionals.

$(\mathscr{H}^*)$ The following inequalities holds,

$$V(x) \leqslant V_\pm(x), \quad \text{for a.e. } x \in \mathbb{R}^N, \tag{3.1.3}$$

$$\text{For each } \kappa = 0, +, -,\ F_\kappa(t) \leqslant F(x,t), \quad \text{a.e. } x \in \mathbb{R}^N \text{ and } \forall\, t \in \mathbb{R} \tag{3.1.4}$$

Moreover, at least one of the next conditions hold,

(i) The inequality (3.1.3) strict in a non-zero measure domain.

(ii) There exists $\delta > 0$ such that the inequality (3.1.4) is strict for all $t \in (-\delta, \delta)$ and a.e. $x \in \mathbb{R}^N$.

Also, to consider the autonomous case $f(x,t) \equiv f(t)$, we assume that the nonlinearity is self-similar,

(f_5^*) There exists $\gamma > 1$ and $0 < s < N/2$ such that

$$F(t) = \gamma^{-Nj} F\left(\gamma^{\frac{N-2s}{2}j} t\right), \quad \forall\, j \in \mathbb{Z} \text{ and } t \in \mathbb{R}.$$

3.2 Statement of the main results

We first state our results concerning existence of ground states solutions for Eq. $(\mathcal{H}_s)$ in both subcritical and critical growth range of the nonlinearity. We say that u is a ground state solution for $(\mathcal{H}_s)$, when $I(u) \leqslant I(v)$ for any other weak solution v in the same considered space of functions.

Theorem 3.2.1.

(i) Suppose that $f(x,t)$ and $a(x) \equiv V(x)$ are $1-$periodic in x_i, $i = 1, \dots, N$ and satisfy (f_1)–(f_3) or (f_3)–(f_6) and (V_1)–(V_2) respectively. Then the equation $(\mathcal{H}_s)$ has a ground state solution.

(ii) Suppose that $f(t) \in C^1(\mathbb{R}^N)$ satisfies (f_3') and (f_5^*) for some $\gamma > 1$. Let

$$\mathcal{G} = \left\{ u \in \mathcal{D}^{s,2}(\mathbb{R}^N) : \int_{\mathbb{R}^N} F(u)\, \mathrm{d}x = 1 \right\},$$

and consider

$$\mathcal{I}_\lambda = \inf_{u \in \mathcal{G}} \left\{ \int_{\mathbb{R}^N} |(-\Delta)^{s/2} u|^2 - \lambda |x|^{-2s} u^2 \, \mathrm{d}x \right\}, \tag{3.2.1}$$

where $0 < \lambda < \Lambda_{N,s}$ is given by (0.0.3). Then, there is a radial minimizer w for (3.2.1). Furthermore, there exists $\alpha > 0$ such that $u = w(\cdot/\alpha)$ is a ground state solution for $(\mathcal{H}_s)$, with $a(x) = -\lambda|x|^{-2s}$.

Theorem 3.2.1 takes into account the invariance of I under the action of translations and dilations in $H^s(\mathbb{R}^N)$ and $\mathcal{D}^{s,2}(\mathbb{R}^N)$, to obtain concentration-compactness of Palais-Smale and minimizing sequences in each case respectively. These properties are enough to ensure existence of ground state solutions. Moreover, our results improve and complement [33] for the fractional framework since here we consider a potential $a(x)$ and nonlinearity $F(x,t)$ which can change sign. Also in Thereom 3.2.1–(ii) we do not require the classical Ambrosetti-Rabinowitz condition (f_2). Our argument to prove Thereom 3.2.1–(ii) involves a Pohozev type identity and as usual for this we required C^1 regularity.

Theorem 3.2.2. *Let*

$$\bar{c}(I) := \inf_{u \in H^s_V(\mathbb{R}^N) \setminus \{0\}} \sup_{t \geqslant 0} I(tu) \quad \textit{and} \quad c_{\mathcal{N}}(I) := \inf_{u \in \mathcal{N}} I(u),$$

where $\mathcal{N} = \left\{ u \in H^s_V(\mathbb{R}^N) \setminus \{0\} : I(u) \cdot u = 0 \right\}$. Suppose that for a.e. $x \in \mathbb{R}^N$ the function

$$t \mapsto \frac{f(x,t)}{|t|} \quad \textit{is strict increasing in } \mathbb{R}. \tag{3.2.2}$$

If $V(x) \in L^1_{\text{loc}}(\mathbb{R}^N)$, $a(x) = V(x) - b(x)$ satisfies (V_2)–(V_3) and $f(x,t)$ fulfills (f_1)–(f_2), then

$$c(I) = \bar{c}(I) = c_{\mathcal{N}}(I).$$

In particular, any non-trivial weak solution u in $H^s_V(\mathbb{R}^N)$ at the mountain pass level is a ground state solution.

Theorem 3.2.2 improves some results in [79] since we deal with the case where $a(x)$ may changes sign and is not necessarily bounded from below, also with nonlinearity having the behavior at 0 described by (f_1'). Moreover, Theorem 3.2.2 proves the existence of ground state by replacing the aforementioned invariance by (3.2.2). In fact, our results below give some conditions that guarantee existence of nontrivial weak solutions in $H^s_V(\mathbb{R}^N)$ at the mountain pass level.

Our next results are on the existence of weak solutions of $(\mathcal{P}_s)$ at the mountain-pass level by using the concentration-compactness principle.

Theorem 3.2.3. *Assume that $f(x,t)$ satisfies (f_1)–(f_3) or (f_3)–(f_6); and additionally (f_7). Suppose also that $a(x)$ and $f(x,t)$ satisfy either one of the following conditions,*

(i) $b(x) \equiv 0$, (V_1)–(V_2), (f_8) *and* (f_{10}); *or*

(ii) $V(x) \geqslant 0$, $b(x)$ *has compact support,* (V_2)–(V_4), (f_9) *and* (f_{10}'); *or*

(iii) Replace conditions (f_{10}) and (f_{10}') in the above items by

$$I(u) \leqslant I_{\mathcal{P}}(u) \quad \text{and} \quad I(u) \leqslant I_\infty(u), \quad \forall\, u \in H^s_V(\mathbb{R}^N), \tag{3.2.3}$$

respectively for each considered case.

Then Eq. $(\mathcal{H}_s)$ possess a non-trivial weak solution u in $H^s_V(\mathbb{R}^N)$ at the mountain pass level, that is, $I(u) = c(I)$. Moreover, under the assumptions of items (i) and (ii), any sequence (u_k) in $H^s_V(\mathbb{R}^N)$ such that $I(u_k) \to c(I)$ and $I'(u_k) \to 0$ has a convergent subsequence.

Theorems 3.2.1–(i) and 3.2.3 extend and complement the existence results of [33, 79, 98] in the fractional framework. In Theorem 3.2.3 the potential $a(x) = V(x) - b(x)$ is not necessarily bounded from below and in Theorem 3.2.3–(ii) we do not ask (f_8) as it was made in these works.

Theorem 3.2.4. *Assume that $f(x,t)$ and $a(x) \equiv V(x)$ satisfy (f_1^*)–(f_4^*), (3.1.3), (3.1.4), (f_2)–(f_3) and (V_1^*)–(V_3^*) respectively. Then Eq. $(\mathcal{H}_s)$ has a non-trivial weak solution in $\mathcal{D}^{s,2}(\mathbb{R}^N)$ at the mountain pass level. If we assume additionally condition $(\mathscr{H}^*)$, then any sequence (u_k) in $\mathcal{D}^{s,2}(\mathbb{R}^N)$ such that $I_*(u_k) \to c(I_*)$ and $I_*'(u_k) \to 0$ has a convergent subsequence.*

Theorems 3.2.1–(ii) and 3.2.4 complement the study made in [39]. Theorem 3.2.4 can be seen as a nonlocal generalization of [27, Theorem 5.2], since we take account that the critical nonlinearity is not autonomous. It also can be seen as complement for many results in the literature about existence of non-trivial weak solution for Schrödinger equation with critical nonlinearity and singular potential (cf. [49, 50, 86, 95] and the references given there).

Remarks on the hypothesis and in the main results

Remark 3.2.5. Next we give several helpful comments concerning our assumptions.

(i) Assumption (f_1) can be seen as a subcritical version of (f_5^*) in the sense that it is oscillating about a subcritical power $|t|^{p-2}t$, $2 < p < 2_s^*$. In fact, it is easy to see that (f_1) holds provided $f(x,t)$ satisfies conditions (f_1') and (f_1'') given below.

(f_1') The following limit is uniform in x,

$$\lim_{t\to 0} \frac{f(x,t)}{|t| + |t|^{2_s^*-1}} = 0;$$

(f_1'') There exists a positive constant C and a function $\varrho(t) \in C(\mathbb{R}\backslash\{0\}) \cap L^\infty(\mathbb{R})$ with $2 < \inf_{t\in\mathbb{R}} \varrho(t) \leqslant \sup_{t\in\mathbb{R}} \varrho(t) < 2_s^*$, such that

$$|f(x,t)| \leqslant C(1 + |t|^{\varrho(t)-1}), \quad \text{a.e. } x \in \mathbb{R}^N \text{ and } \forall\, t \in \mathbb{R};$$

For example of nonlinearity satisfying (f_1') and (f_1'') consider

$$f(x,t) = k(x)\left[\varrho'(t)(\ln|t|t) + \varrho(t)\right]|t|^{\varrho(t)-2}t,\ \ f(x,0) \equiv 0,$$

where

$$\varrho(t) = \frac{2_s^* - 2}{16}\sin\left(\ln(|\ln|t||)\right) + \frac{52_s^* + 6}{8} \quad \text{and} \quad 0 \leqslant k(x) \in C(\mathbb{R}) \cap L^\infty(\mathbb{R}^N).$$

The primitive is given by $F(x,t) = k(x)|t|^{\varrho(t)}$. A version of (f_1) for the local case appeared in [96].

(ii) Using similar arguments as in [33, Lemma 2.1], we have that (f_4) and (f_6) imply (f_1) in a more restrict setting, more precisely, there is $p \in (2, 2_s^*)$ such that for any $\varepsilon > 0$ there is $C_\varepsilon > 0$ with

$$|f(x,t)| \leqslant \varepsilon|t| + C_\varepsilon|t|^{p-1}, \qquad \text{a.e. } x \in \mathbb{R}^N \text{ and } \forall\, t \in \mathbb{R}.$$

That is, in the case that $f(x,t)$ fulfills (f_4) and (f_6) we have that $p_\varepsilon = p$, for all $\varepsilon > 0$ in condition (f_1).

(iii) Conditions (f_4)–(f_6) are an alternative for the Ambrosetti-Rabinowitz condition (f_2), and was first introduced in [37] for the local case. By similar arguments as the ones made in [37], condition (f_6) holds once we take account (f_4), (f_5) and that there exists $p \in (2, 2_s^*)$ and $c_1, c_2, r_1 > 0$ such that

$$|f(x,t)| \leqslant c_1|t|^{p-1} \quad \text{and} \quad F(x,t) \leqslant \left(\frac{1}{2} - \frac{1}{c_2|t|^\nu}\right) f(x,t)t, \quad \text{for } |t| \geqslant r_1.$$

where $1 < \nu < 2$ if $N = 1$, and $1 < \nu < N + p - pN/2s$ if $N \geqslant 2$.

(iv) In view of the boundedness of Palais-Smale sequences we point out that we separate our studies for the subcritical case in two distinct situations: $f(x,t)$ satisfies (f_1)–(f_3) or (f_3)–(f_6). The first one is associated to the case where $f(x,t)$ has oscillatory behavior around the subcritical power and the second one refers to the case where $f(x,t)$ does not satisfies Ambrosetti-Rabinowitz condition.

(v) In [33], considering a local Schrödinger equation with asymptotically periodic terms, in order to prove the mountain pass geometry it was assumed that $F(x,t) > 0$ for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}$ and (f_4). This setting allow the author do not use the classical Ambrosetti-Rabinowitz condition (f_3). Here, in this work, we have an improvement even to the local case because we assume (f_3) instead of assuming that $F(x,t) > 0$ for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}$.

(vi) Assumption (f_5) it is used to prove the boundedness of Palais-Smale sequences of the mountain pass level for the functional of Eq. $(\mathcal{H}_s)$. In [33] to prove similar result the author assumed the following more restrictive condition

$$\mathscr{F}(x,t) = \frac{1}{2} f(x,t)t - F(x,t) \geqslant b(t)t^2, \quad \forall\, (x,t) \in \mathbb{R}^N \times \mathbb{R},$$

for some $b(t) \in C(\mathbb{R}\backslash\{0\}, \mathbb{R}^+)$.

(vii) In our approach to study existence of weak solutions of Eq. $(\mathcal{H}_s)$ we use assumption (f_7), unlike the aforementioned papers, where the authors impose the more tight condition

$$|f(x,t) - f_{\mathcal{P}}(x,t)| \leqslant h(x)|t|^{q-1} \quad \text{a.e. } x \text{ in } \mathbb{R}^N \text{ and } \forall\, t \in \mathbb{R},$$

where $h(x)$ belongs to the class of functions in $C(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ such that for every $\varepsilon > 0$ the set $\{x \in \mathbb{R}^N : |h(x)| \geqslant \varepsilon\}$ has finite Lebesgue measure.

(viii) The smoothness condition assumed in (f_9) is the natural hypothesis used in the literature to prove that weak solutions of Eq. $(\mathcal{H}_s)$ satisfies a Pohozaev type identity.

(ix) We prove in Proposition 3.3.1 that $H^s_V(\mathbb{R}^3)$ is well defined and it is continuous embedded in $H^s(\mathbb{R}^3)$. As a consequence of this we can conclude that the infimum $\mathcal{C}_V^{(\beta)}$ defined in (V_3) is strictly positive.

(x) Once the limits in (V_4), (f_7), (f_9) or (f_3^*) exist, to obtain compactness of Palais-Smale sequences at the minimax levels we need to require the additional conditions over the minimax levels given in assumptions (f_{10}), (f'_{10}), $(\mathscr{H}^*)$. In fact, we do not believe that it is possible, in general, to achieve the compactness described in Theorems 3.2.3 and 3.2.4 without these conditions. We mention that this kind of approach was introduced by P.-L. Lions in [65–68].

(xi) Similarly as made in Chapter 2, we also consider the case when (f_{10}), (f'_{10}), $(\mathscr{H}^*)$ do not hold. Precisely, when it is allowed $c(I) = c(I_{\mathcal{P}})$ or $c(I) = c(I_\infty)$. In this case, the concentration-compactness argument at the mountain pass level cannot be used. We apply Theorem 0.6.4 to overcome this difficulty and prove existence of solution at the mountain pass level.

(xii) For problem $(\mathcal{H}_s)$ involving critical growth we require conditions (V_1^*)–(V_3^*) on the potential and (f_3^*), (f_4^*) and $(\mathscr{H}^*)$ on the nonlinear term $f(x,t)$. These assumptions are suitable for our argument, differently from (f_{10})–(f'_{10}), because the potential that appears in the associated limiting equation depends on the profile decomposition of Theorem 1.1.1 for a given Palais-Smale sequence at the mountain pass level (for more details see estimate (3.9.1)).

(xiii) In our results, one can assume that $f(x,t) = f(|x|,t)$ and $a(x) = a(|x|)$ are radial in x instead of the existence of the asymptote $f_\infty(t)$ or $f_0(t)$. This fact can be easily verified by using Proposition 1.4.1.

Remark 3.2.6. Under the assumptions (V_4) and (f_7) we describe next conditions which guarantee that (f_{10}) and (f'_{10}) hold.

$(\mathscr{H})$ The following inequalities hold,

$$F_{\mathcal{P}}(x,t) \leqslant F(x,t), \quad \text{a.e. } x \in \mathbb{R}^N \text{ and } t \in \mathbb{R}, \tag{3.2.4}$$

$$V(x) \leqslant V_\infty, \quad \text{a.e. } x \in \mathbb{R}^N. \tag{3.2.5}$$

In addition, we assume that either (3.2.4) holds strictly in some open interval contained the origin or (3.2.5) holds in a set of positive measure.

In Proposition 3.8.1, under suitable conditions, we obtained the following estimates for the minimax levels: $c(I) \leqslant c(I_{\mathcal{P}})$ and $c(I) \leqslant c(I_\infty)$. Moreover, we proved that under condition $(\mathscr{H})$ we have that (f_{10}) and (f'_{10}) hold. We observe that on the corresponding assumption of Theorem 3.2.3, it is easy to see that inequalities (3.2.4) and (3.2.5) imply that (3.2.3) is satisfied.

Remark 3.2.7. Using the same argument of Remark 2.2.7 it can be proved the existence of non-negative weak solutions of $(\mathcal{P}_s)$ if $f(x,t) \geqslant 0$ for all $t \geqslant 0$ and almost everywhere x in $\mathbb{R}^N$. In fact, consider the truncation

$$\bar{f}(x,t) = \begin{cases} f(x,t), & \text{if} \quad t \geqslant 0, \\ 0, & \text{if} \quad t < 0. \end{cases}$$

Assume that $a(x) \in L^1_{\text{loc}}(\mathbb{R}^N)$ and that conditions (f_1) and (V_2) hold true with $b(x) \equiv 0$. Thus for u a weak solution of $(\mathcal{P}_s)$, with $f(x,t)$ replaced by $\bar{f}(x,t)$, we have that u is also a weak non-negative solution for $(\mathcal{P}_s)$. To see that, let $\xi \in C_0^\infty(\mathbb{R} : [0,1])$ such that

$$\xi(t) = \begin{cases} 1, & \text{if} \quad t \in [-1,1] \\ 0, & \text{if} \quad |t| \geqslant 2 \end{cases} \qquad \text{and} \qquad |\xi'(t)| \leqslant C \quad \forall\, t \in \mathbb{R},$$

For each $n \in \mathbb{N}$, define $\xi_n : \mathbb{R}^{N+1} \to \mathbb{R}$ by $\xi_n(z) = \xi(|z|^2/n^2)$. Then $\xi_n \in C_0^\infty(\mathbb{R}^{N+1})$ and verifies

$$|\nabla \xi_n(z)| \leqslant C \qquad \text{and} \qquad |z||\nabla \xi_n(z)| \leqslant C \quad \forall\, z \in \mathbb{R}^{N+1}.$$

By a density argument, we can take $\varphi = \xi_n w_-$ in (0.3.4), where $w_-(z) = \min\{w(z), 0\}$. Since $w_-(z) = E_s(u_-)$, we have that

$$\int_{\mathbb{R}^{N+1}_+} y^{1-2s}\xi_n|\nabla w_-|^2 + y^{1-2s}\xi_n \langle \nabla w_+, \nabla w_- \rangle + y^{1-2s} \langle \nabla w_+ + \nabla w_-, w_- \nabla \xi_n \rangle \,\mathrm{d}x\mathrm{d}y$$
$$= \kappa_s \int_{\mathbb{R}^N} (\bar{f}(x,u) - a(x)u)\xi_n u_- \,\mathrm{d}x,$$

and we may apply the Dominated Convergence Theorem and (0.3.2) to get

$$\|u_-\|_V^2 = \int_{\mathbb{R}^N} \bar{f}(x,u)u_- \,\mathrm{d}x = 0,$$

which implies that $u_- = 0$. Once again, if u has sufficient regularity one can show u is positive, by applying the maximum principle for the fractional Laplacian as described in [84]. In order to regularize the solutions, we follow the same arguments of [79, Section 6], but as already mentioned in this paper, we need sufficient regularity in the potential $a(x) \equiv V(x)$, which is beyond our scope (see also [38, Chapter 5]).

Example 3.2.8. Our approach include the following classes of potentials:

(i) For a potential satisfying assumption (V_2) and that is not bounded away from zero, consider $0 \leqslant a(x) \equiv V_0(x) \in L^p_{\text{loc}}(\mathbb{R}^N) \cap (C(\mathbb{R}^N \backslash \mathcal{O})$, where $p \geqslant 1$ and $\mathcal{O}$ is a countable set, and suppose that $Z = \{x \in \mathbb{R}^N : V(x) = 0\} \neq \varnothing$ is a countable discrete set.

(ii) Let $V_0(x)$ the potential given above. For a potential the changes sign and satisfies (V_2), consider $a(x) \equiv V_0(x) - \varepsilon$, where $0 < \varepsilon < \mathcal{C}_{V_0}/2$.

(iii) To study potential of the form $a(x) = V(x) - b(x)$, setting

$$V(x) = 2 - \frac{1}{1+|x|^2} \quad \text{and} \quad V_\infty = 2,$$

and

$$b(x) = \begin{cases} \mathcal{C}_b|x|^{-\delta}, & \text{if } |x| \leqslant 1, \\ 0, & \text{if } |x| > 1, \end{cases}$$

we can verify that $a(x) = V(x) - b(x)$ satisfies conditions (V_2)–(V_4). Here $\mathcal{C}_b$ is a positive normalization constant, $0 < \delta < N/\beta$ and $\beta > N/2s$.

(iv) For potential $a(x) \equiv V(x)$ satisfying assumptions (V_1^*)–(V_3^*) we can consider

$$V(x) = -\frac{1}{L}\sum_{j=1}^{L} \frac{\lambda_j}{|x-x^j|^{2s}}, \quad \text{with} \quad 0 < \lambda_j < \frac{\Gamma_{N,s}}{2}, \quad j = 1, \ldots, L,$$

which is well defined in view of (0.0.3).

Example 3.2.9. Note that the hypotheses of Theorems 3.2.1–3.2.4 are for example satisfied by nonlinearities of the following forms:

(i) Let $\varrho(t)$ be as in Remark 3.2.5–(i) and consider $k(x) = |x|^2/(1+|x|^2)$. One can see that

$$f(x,t) = k(x)\left[\varrho'(t)(\ln|t|t) + \varrho(t)\right]|t|^{\varrho(t)-2}t, \quad f(x,0) \equiv 0,$$

satisfies assumptions (f_1)–(f_3), (f_9) and (f'_{10}).

(ii) For a nonlinearity satisfying conditions (f_3)–(f_8) and (f_{10}) we can define

$$f(x,t) = \begin{cases} h(x,t), & \text{for } t \geqslant 0, \\ -h(x,-t), & \text{for } t < 0, \end{cases}$$

where

$$h(x,t) = k(x)t\ln(1+t) + k_1(x)\left[(1+\cos(t))t^2 + 2(t+\sin(t))t\right],$$

for $t \geqslant 0$, $s > N/6$; $k(x) = |x|^2/(1+|x|^2)$ and $0 \leqslant k_1(x) \in C(\mathbb{R}^N)$ is such that $\lim_{|x|\to\infty} k_1(x) = 0$.

(iii) Let $0 \leqslant c(x)$ be a continuous 1−periodic in x_i, $i = 1, \ldots, N$, and consider $f(x,t) = c(x)\left[ph_\varepsilon(t) + h'_\varepsilon(t)t\right]|t|^{p-1}$, $2 < p < 2_s^*$, where $h_\varepsilon(t) \in C^\infty(\mathbb{R})$ is a non-decreasing cutoff function satisfying

$$\begin{cases} |h'_\varepsilon(t)| \leqslant C/t, \ |h_\varepsilon(t)| \leqslant C, \quad \forall t \in \mathbb{R}, \\ h_\varepsilon(t) = -\varepsilon, \ \text{for } t \leqslant 1/4, \quad h_\varepsilon(t) = \varepsilon, \ \text{for } t \geqslant 1/4, \ \text{with } \varepsilon \text{ small enough.} \end{cases}$$

We empathize the fact that $F(x,t)$ changes sign.

(iv) Suppose that the function $k_0(x)$ is continuous and

$$2_s^* - \mu > \sup_{x\in\mathbb{R}^N} k_0(x) \geqslant k_0(x) > k_0(0) = \inf_{x\in\mathbb{R}^N} k_0(x) = \lim_{|x|\to\infty} k_0(x) = 0.$$

The nonlinearity given below satisfies the hypothesis of Theorem 3.2.4,

$$f(x,t) = \exp\{k_0(x)(\sin(\ln|t|)+2)\}\,[k_0(x)\cos(\ln|t|)+2_s^*]\,|t|^{2_s^*-2}t,\ \ f(x,0)\equiv 0.$$

3.3 Variational settings

This section is devoted to develop the basic background needed in order to apply our variational arguments. We start by establishing the space of functions where the solutions lies.

Proposition 3.3.1. *Suppose that $V(x) \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$ and satisfies (V_2), then $H^s_V(\mathbb{R}^N)$ is a Hilbert space continuously embedded in $H^s(\mathbb{R}^N)$. If $V(x)$ satisfies (V_1^*), then the norm $\|\cdot\|_V$ is equivalent to the standard norm of $\mathcal{D}^{s,2}(\mathbb{R}^N)$.*

Proof. Let us prove first that there exists a positive constant C such that

$$C[\varphi]_s^2 \leqslant \|\varphi\|_V^2, \quad \forall\, \varphi \in C_0^\infty(\mathbb{R}^N). \tag{3.3.1}$$

In fact, on the contrary, there would exist a sequence (φ_n) in $C_0^\infty(\mathbb{R}^N)$, such that

$$[\varphi_n]_s^2 > n\|\varphi_n\|_V^2, \quad \forall\, n \in \mathbb{N}.$$

Taking $v_n = \varphi_n/[\varphi_n]_s$, we have

$$\frac{1}{n} > \|v_n\|_V^2 \quad \text{and} \quad \mathcal{C}_V\|v_n\|_2^2 \leqslant \|v_n\|_V^2, \quad \forall\, n \in \mathbb{N},$$

and consequently $\lim_{n\to\infty}\|v_n\|_V^2 = \lim_{n\to\infty}\|v_n\|_2^2 = 0$. This leads to a contradiction with the fact that

$$1 - \mathcal{B}\|v_n\|_2^2 \leqslant \|v_n\|_V^2, \quad \forall\, n \in \mathbb{N}.$$

Now consider (φ_n) any sequence in $C_0^\infty(\mathbb{R}^N)$. Using inequality (3.3.1) we have

$$C[\varphi_m - \varphi_n]_s^2 \leqslant \|\varphi_m - \varphi_n\|_V^2, \quad \text{for any } m \neq n.$$

Consequently,

$$\|\varphi_m - \varphi_n\|^2 \leqslant \min\{1, C\}^{-1}\left(1 + \frac{1}{\mathcal{C}_V}\right)\|\varphi_m - \varphi_n\|_V^2, \quad \text{for any } m \neq n.$$

Thus $H^s_V(\mathbb{R}^N)$ is well defined. Moreover, Fatou Lemma and embedding (0.2.1) implies

$$H^s_V(\mathbb{R}^N) \subset \left\{u \in H^s(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x)u^2\,\mathrm{d}x < \infty\right\},$$

with the continuous embedding $H^s_V(\mathbb{R}^N) \hookrightarrow H^s(\mathbb{R}^N)$.
Assuming condition (V_1^*), we have

$$[u]_s^2 + \int_{\mathbb{R}^N} V(x)u^2 \,\mathrm{d}x \geqslant \mathcal{C}_V^* \int_{\mathbb{R}^N} |V(x)|u^2 \,\mathrm{d}x, \quad \forall\, u \in C_0^\infty(\mathbb{R}^N),$$

from this we derive

$$\begin{aligned} \mathcal{C}_V^*[u]_s^2 &\leqslant (\mathcal{C}_V^* + 1)[u]_s^2 + \int_{\mathbb{R}^N} (V(x) - \mathcal{C}_V^*|V(x)|)u^2 \,\mathrm{d}x \\ &\leqslant (\mathcal{C}_V^* + 1)\|u\|_V^2, \quad \forall\, u \in C_0^\infty(\mathbb{R}^N). \end{aligned}$$

Since $V(x) \leqslant 0$ a.e. in $\mathbb{R}^N$, we conclude that the norms $[\,\cdot\,]_s$ and $\|\cdot\|_V$ are equivalent in $\mathcal{D}^{s,2}(\mathbb{R}^N)$. ■

Remark 3.3.2. (i) If $V(x)$ fulfills (V_2) and (V_4), then $H^s_V(\mathbb{R}^N) = H^s(\mathbb{R}^N)$. Moreover, the norms $\|\cdot\|$ and $\|\cdot\|_V$ are equivalent. Consequently, the path $\lambda_u(t) := u(\cdot/t)$, $t \geqslant 0$ belongs to $C([0,\infty), H^s_V(\mathbb{R}^N))$ and $u(\cdot - y) \in H^s_V(\mathbb{R}^N)$ for all $u \in H^s_V(\mathbb{R}^N)$ and $y \in \mathbb{R}^N$. Indeed, there is a ball B_{R_1} with center at the origin such that

$$\begin{aligned} \int_{\mathbb{R}^N} V(x)u^2 \,\mathrm{d}x &= \int_{B_{R_1}} V(x)u^2 \,\mathrm{d}x + \int_{\mathbb{R}^N\setminus B_{R_1}} V(x)u^2 \,\mathrm{d}x \\ &\leqslant \left(\int_{B_{R_1}} |V(x)|^\sigma \,\mathrm{d}x\right)^{1/\sigma} \left(\int_{B_{R_1}} |u|^{2\sigma/(\sigma-1)} \,\mathrm{d}x\right)^{(\sigma-1)/\sigma} \\ &\quad + (V_\infty + 1) \int_{\mathbb{R}^N\setminus B_{R_1}} u^2 \,\mathrm{d}x, \quad \forall\, u \in H^s_V(\mathbb{R}^N), \end{aligned}$$

where $2 \leqslant 2\sigma/(\sigma - 1) \leqslant 2_s^*$. So we can apply embedding (0.2.3) to conclude the desired result. To obtain that the path λ_u belongs to $C([0,\infty), H^s_V(\mathbb{R}^N))$ we use Lemma 2.5.3.

(ii) If we assume (V_1)–(V_2), then we can replace $H^s(\mathbb{R}^N)$ by $H^s_V(\mathbb{R}^N)$ in Theorem 1.1.2 and the respectively norms in the assertions (1.1.5)–(1.1.8). In fact, condition (V_1) implies that $D_{\mathbb{Z}^N}$ is a group of unitary operators in $H^s_V(\mathbb{R}^N)$.

Now we prove that our functional I_λ has the Mountain Pass Geometry.

Lemma 3.3.3. *Suppose that $f(x,t)$ satisfies (f_1) and either (f_2)–(f_3) or (f_4). If $a(x) = V(x) - b(x) \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$ fulfills (V_2) and (V_3), then the functional I possess the mountain pass geometry. Precisely,*

(i) $I(0) = 0$;

(ii) There exists $r,\ b > 0$ such that $I(u) \geqslant b$, whenever $\|u\|_V = r$;

(iii) There is $e \in H^s_V(\mathbb{R}^N)$ with $\|e\|_V > r$ and $I(e) < 0$;

In particular $0 < c(I) < \infty$.

Proof. Let $\xi_R \in C_0^\infty(\mathbb{R}),\ R > 0$, such that $0 \leqslant \xi_R(t) \leqslant t_0$ and

$$\xi_R(t) = \begin{cases} t_0, & \text{if } |t| \leqslant R, \\ 0, & \text{if } |t| > R+1. \end{cases}$$

Setting $v(x) := \xi_R(|x - x_0|)$, we have $v \in H_V^s(\mathbb{R}^N)$ and by assumption (f_3) we have

$$\begin{aligned}\int_{\mathbb{R}^N} F(x,v)\,\mathrm{d}x &= \int_{B_R(x_0)} F(x,t_0)\,\mathrm{d}x + \int_{B_{R+1}(x_0)\backslash B_R(x_0)} F(x,v)\,\mathrm{d}x \\ &\geqslant |B_R| \inf_{B_R(x_0)} F(x,t_0) + |B_{R+1}\backslash B_R| \inf_{(x,t)\in(B_{R+1}(x_0)\backslash B_R(x_0))\times[0,t_0]} F(x,t) > 0.\end{aligned}$$

First assume that (f_2) holds. Since $b(x) \in L^\beta(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} b(x)u^2\,\mathrm{d}x \leqslant \left(\int_{\mathbb{R}^N} |b(x)|^\beta\,\mathrm{d}x\right)^{1/\beta} \left(\int_{\mathbb{R}^N} |u|^{2\beta/(\beta-1)}\,\mathrm{d}x\right)^{(\beta-1)/\beta}, \quad \forall\, u \in H_V^s(\mathbb{R}^N),$$

with $2 < 2\beta/(\beta-1) < 2_s^*$, by conditions (f_1) and (V_3), for any ε we get that

$$I(u) \geqslant \left[\frac{1}{2}\left(1 - \frac{\|b(x)\|_\beta}{\mathcal{C}_V^{(\beta)}} - 2\varepsilon\mathcal{C}_2\right) - \varepsilon\mathcal{C}_{2_s^*}\|u\|_V^{2_s^*-2} - C_\varepsilon\mathcal{C}_{p_\varepsilon}\|u\|_V^{p_\varepsilon-2}\right]\|u\|_V^2, \tag{3.3.2}$$

for all $u \in H_V^s(\mathbb{R}^N)$, where $\mathcal{C}_2$, $\mathcal{C}_{2_s^*}$ and $\mathcal{C}_{p_\varepsilon}$ are positive constants provided by the embedding described in Proposition 3.3.1. This allow us to consider ε in a such way that the first term in the right-hand side of (3.3.2) is positive, once $\|u\|_V$ is taken small enough. Hence there exists $r > 0$ such that $I(u) > 0$ provided that $\|u\|_V = r$. Since condition (f_2) is equivalent to $d/dt(F(x,t)t^{-\mu}) \geqslant 0$, for $t > 0$, we have

$$\int_{\mathbb{R}^N} F(x,tv)\,\mathrm{d}x \geqslant t^\mu \int_{\mathbb{R}^N} F(x,v)\,\mathrm{d}x, \quad \text{whenever } t > 1.$$

Hence, as $t \to \infty$,

$$\begin{aligned}I(tv) = \frac{t^2}{2}\|v\|_V^2 - \int_{\mathbb{R}^N} b(x)u^2\,\mathrm{d}x - \int_{\mathbb{R}^N} F(x,tv)\,\mathrm{d}x& \\ \leqslant \frac{t^2}{2}\|v\|_V^2 - t^\mu \int_{\mathbb{R}^N} F(x,v)\,\mathrm{d}x \to -\infty, \text{as } t \to \infty.&\end{aligned}$$

Now suppose that assumption (f_4) holds. By Remark 3.2.5–(ii) we can argue as above to conclude the existence of $r > 0$ such that $I(u) > 0$ wherever $\|u\|_V < r$. For any given $R > 0$, there exists $t_R > 0$ such that

$$F(x,t) > Rt^2, \quad \forall\,|t| > t_R,\ \forall\, x \in \mathbb{R}^N.$$

Let be $A(R,t) := \{x \in \mathbb{R}^N : t|v(x)| > t_R\}$, for $t > 0$. We have that

$$\begin{aligned}\int_{\mathbb{R}^N} F(x,tv)\,\mathrm{d}x &= \int_{K_t} F(x,tv)\,\mathrm{d}x + \int_{A(R,t)} F(x,tv)\,\mathrm{d}x \\ &\geqslant \int_{K_t} F(x,tv)\,\mathrm{d}x + Rt^2 \int_{A(R,t)} v^2\,\mathrm{d}x,\end{aligned} \tag{3.3.3}$$

where $K_t = (\mathbb{R}^N \backslash A(R,t)) \cap \operatorname{supp}(v)$. Using Remark 3.2.5–(ii), for each $t > 0$, we get that

$$|F(x,tv)| \leqslant C, \quad \text{for a.e. } x \in K_t,$$

where C is a positive constant that does not depend in x and t. Consequently, for any $x \in \operatorname{supp}(v)$,

$$F(x,tv)\mathcal{X}_{K_t}(x) \to 0, \quad \text{as } t \to \infty,$$

where we have used that, for any $x \in \operatorname{supp}(v)$,

$$\mathcal{X}_{\mathbb{R}^N \backslash A(R,t)}(x) \to \mathcal{X}_{\mathbb{R}^N \backslash \operatorname{supp}(v)}(x) = 0, \quad \text{as } t \to \infty,$$

Thus Dominated Convergence Theorem implies that the first integral in the right-hand side of inequality (3.3.3) goes to zero as t goes to infinity. By the same reason, we also have

$$\lim_{t\to\infty} \int_{A(R,t)} v^2 \, \mathrm{d}x = \lim_{t\to\infty} \int_{\mathbb{R}^N} v^2 \mathcal{X}_{A(R,t)} \, \mathrm{d}x = \int_{\mathbb{R}^N} v^2 \mathcal{X}_{\{v\neq 0\}} \, \mathrm{d}x = \int_{\mathbb{R}^N} v^2 \, \mathrm{d}x$$

In particular, there exists a positive number $t_{0,R}$ such that

$$\frac{1}{2}\int_{\mathbb{R}^N} v^2 \, \mathrm{d}x < \int_{A(R,t)} v^2 \, \mathrm{d}x, \quad \forall\, t > t_{0,R}. \tag{3.3.4}$$

Replacing (3.3.4) in (3.3.3) we obtain that

$$\begin{aligned} I(tv) &= \frac{t^2}{2}\|v\|_V^2 - \frac{t^2}{2}\int_{\mathbb{R}^N} b(x)v^2 \, \mathrm{d}x - \int_{\mathbb{R}^N} F(x,tv) \, \mathrm{d}x \\ &\leqslant \frac{1}{2}\left(\|v\|_V^2 - R\|v\|_2^2\right)t^2 - \int_{K_t} F(x,tv) \, \mathrm{d}x < 0, \quad \text{for } t > t_{0,R}, \end{aligned}$$

provided that R is sufficiently large enough. ■

Remark 3.3.4. (i) In view of Lemma 3.3.3, we define the set

$$\Gamma_I^1 = \left\{\gamma \in C([0,1], H_V^s(\mathbb{R}^N)) : \gamma(0) = 0,\ \|\gamma(1)\|_V > r,\ I(\gamma(1)) < 0\right\},$$

and

$$c_1(I) = \inf_{\gamma\in\Gamma_I} \sup_{t\in[0,1]} I(\gamma(t)),$$

the usual minimax level. Thus have $c_1(I) = c(I)$.

(ii) When $f(x,t) \equiv f(t)$, the mountain pass geometry can be obtained by replacing condition (f_3) by (f_3'). In fact, let ξ_R as in the proof of Lemma 3.3.3 and define $\eta_R(x) = \xi_R(|x|)$. Then, arguing as in Remark 2.2.6, we have

$$\int_{\mathbb{R}^N} F(\eta_R) \, \mathrm{d}x > 0,$$

provided that R is sufficiently enough. The mountain pass geometry now follows as in the proof of Lemma 3.3.3.

(iii) Assume that $f(x,t)$ satisfies (f_1) and either (f_2)–(f_3) or (f_4); and additionally (f_7). Suppose also that $a(x)$ and $f(x,t)$ fulfills (V_2)–(V_4) and (f_9), respectively. Then the limiting functional I_∞ has the mountain pass geometry. In fact, (f_9) together with Lemma 2.5.3 implies that $\lambda_u(t) := u(\cdot/t)$, $t \geqslant 0$, is an admissible path for Γ_{I_∞}, where $u \in H^s(\mathbb{R}^N)$ is such that

$$\int_{\mathbb{R}^N} F_\infty(u) - \frac{V_\infty}{2} u^2 \,\mathrm{d}x > 0. \tag{3.3.5}$$

Using the same argument as in Remark 3.3.4–(ii) we can see that there exists $\varphi_0 \in C_0^\infty(\mathbb{R}^N)$ satisfying (3.3.5) and

$$I_\infty(\lambda_{\varphi_0}(t)) = \frac{1}{2} t^{N-2s}[\varphi_0]_s^2 - t^N \left[\int_{\mathbb{R}^N} F_\infty(\varphi_0) - \frac{V_\infty}{2}\varphi_0^2\right] \to -\infty, \text{ as } \quad t \to \infty.$$

Moreover, $I(u) > 0$ wherever $\|u\|_V = r$, for $r > 0$ sufficiently small enough (see proof of Lemma 3.3.3).

(iv) In addition to the assumptions of Lemma 3.3.3, assume that $F(x,t) > 0$ for a.e. $x \in \mathbb{R}^N$ and $t \neq 0$. Then, for any $u \in H_V^s(\mathbb{R}^N)\backslash\{0\}$, the path defined by $\zeta(t) = tu$ belongs to Γ_I. In fact, we make the following modification in the proof of Lemma 3.3.3, replacing v by u and taking into account the same notation. We have that

$$\begin{cases} \displaystyle\int_{\mathbb{R}^N} F(x,tu)\,\mathrm{d}x \geqslant Rt^2 \int_{A(R,t)} u^2 \,\mathrm{d}x, \\ \displaystyle\lim_{t\to\infty} \int_{A(R,t)} u^2\,\mathrm{d}x = \lim_{t\to\infty}\int_{\mathbb{R}^N} u^2 \mathcal{X}_{A(R,t)}\,\mathrm{d}x = \int_{\mathbb{R}^N} u^2 \mathcal{X}_{\{u\neq 0\}}\,\mathrm{d}x = \int_{\mathbb{R}^N} u^2\,\mathrm{d}x, \end{cases}$$

which enable us to proceed as in (3.3.4) and get that

$$\varphi(t) := I(tu) \leqslant \frac{1}{2}\left(\|u\|_V^2 - R\|u\|_2^2\right) t^2 \to -\infty, \text{ as } t \to \infty,$$

provided that R is large enough. Moreover, suppose that condition (3.2.2) holds. Taking into account that

$$\varphi'(t) = t\left[\|u\|_V^2 - \int_{\mathbb{R}^N} \frac{f(x,tu)}{t} u\,\mathrm{d}x\right], \quad t > 0,$$

we infer that $\zeta(t)$ has a unique critical point.

As a consequence of the previous result, we can guarantee the existence of bounded Palais-Smale sequence at the mountain pass level $c(I)$.

Proposition 3.3.5. *Assume that $a(x) \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$ satisfies (V_2)–(V_3) and $f(x,t)$ satisfies either*

(i) (f_1)–(f_3); or

(ii) (f_3)–(f_6)*;*

Then there exists a bounded sequence (u_k) *such that* $I(u_k) \to c(I)$ *and* $I'(u_k) \to 0$ *in the dual of* $H^s_V(\mathbb{R}^N)$*.*

Proof. (i) By Lemma 3.3.3, we may apply the standard Mountain Pass Theorem (see [2, 16]) in order to find a sequence (u_k) in $H^s_V(\mathbb{R}^N)$ such that $I(u_k) \to c(I)$ and $I'(u_k) \to 0$. For large k, we have

$$\begin{aligned} c(I) + 1 + \|u_k\|_V &\geqslant I(u_k) - \frac{1}{\mu} I'(u_k) \cdot u_k \\ &\geqslant \left(\frac{1}{2} - \frac{1}{\mu}\right)\left(1 - \frac{\|b(x)\|_\beta}{\mathcal{C}_V^{(\beta)}}\right)\|u_k\|_V^2 - \int_{\mathbb{R}^N} F(x, u_k) - \frac{1}{\mu} f(x, u_k) u_k \,\mathrm{d}x \\ &\geqslant \left(\frac{1}{2} - \frac{1}{\mu}\right)\left(1 - \frac{\|b(x)\|_\beta}{\mathcal{C}_V^{(\beta)}}\right)\|u_k\|_V^2, \end{aligned}$$

which implies that (u_k) is bounded in $H^s_V(\mathbb{R}^N)$.

(ii) The proof of this case is based in the arguments made in [33, Lemma 2.5], which are similar to the ones used in [37]. By Lemma 3.3.3, we can apply a variant of the Mountain Pass Theorem (see [21,77]), to obtain the existence of a Cerami sequence (u_k) for I at the level $c(I)$, more precisely,

$$I(u_k) \to c(I) \text{ and } (1 + \|u_k\|_V)\|I'(u_k)\|_* \to 0,$$

where $\|\cdot\|_*$ denote the usual norm of the dual of $H^s_V(\mathbb{R}^N)$. We claim that (u_k) is bounded in $H^s_V(\mathbb{R}^N)$. Assume by contradiction that, up to subsequence, $\|u_k\|_V \to \infty$. Define the sequence

$$v_k = \frac{u_k}{\|u_k\|_V}.$$

We have that

$$\lim_{k\to\infty}\left[1 - \int_{\mathbb{R}^N} \frac{f(x, u_k)}{\|u_k\|_V} v_k \,\mathrm{d}x - \frac{1}{\|u_k\|_V^2}\int_{\mathbb{R}^N} b(x) u_k^2 \,\mathrm{d}x\right] = \lim_{k\to\infty}\left[\frac{1}{\|u_k\|_V^2} I'(u_k) \cdot u_k\right] = 0.$$

The idea is to use indirect arguments and prove that

$$\lim_{k\to\infty}\int_{\mathbb{R}^N} \frac{f(x, u_k)}{\|u_k\|_V} v_k \,\mathrm{d}x = 0,$$

which, by assumption (V_3), leads to the following contradiction,

$$1 = \lim_{k\to\infty}\frac{1}{\|u_k\|_V^2}\int_{\mathbb{R}^N} b(x) u_k^2 \,\mathrm{d}x < \frac{1}{2}. \tag{3.3.6}$$

For $0 \leqslant a < b \leqslant \infty$, defining

$$\Omega_k(a, b) = \left\{x \in \mathbb{R}^N : a \leqslant |u_k(x)| \leqslant b\right\},$$

we are going to prove that for any given $0 < \varepsilon < 1$, there exists k_ε and real numbers $a_\varepsilon, b_\varepsilon$ such that

$$\int_{\mathbb{R}^N} \frac{f(x,u_k)}{\|u_k\|_V} v_k \,\mathrm{d}x = \int_{\Omega_k(0,a_\varepsilon)} \frac{f(x,u_k)}{\|u_k\|_V} v_k \,\mathrm{d}x + \int_{\Omega_k(a_\varepsilon,b_\varepsilon)} \frac{f(x,u_k)}{\|u_k\|_V} v_k \,\mathrm{d}x + \int_{\Omega_k(b_\varepsilon,\infty)} \frac{f(x,u_k)}{\|u_k\|_V} v_k \,\mathrm{d}x < \varepsilon, \quad \forall\, k > k_\varepsilon. \tag{3.3.7}$$

In order to do that, we first make some estimates involving $\mathcal{F}(x,t)$. Define

$$g(r) = \inf \left\{ \mathcal{F}(x,t) : x \in \mathbb{R}^N,\ |t| > r \right\},$$

which is positive and goes to infinity as $r \to \infty$. Indeed, thanks to assumptions (f_5) and (f_6), we have

$$a_0 \mathcal{F}(x,t) \geqslant \left| \frac{f(x,t)}{t} \right|^{p_0} > \left| 2\frac{F(x,t)}{t^2} \right|^{p_0}, \quad \text{for } |t| > R_0.$$

Consequently, by condition (f_4), we obtain that $\mathcal{F}(x,t) \to \infty$, as $|t| \to \infty$, uniformly in x. Due to assumption (f_5), we also can define the positive number

$$m_a^b = \inf \left\{ \frac{\mathcal{F}(x,t)}{t^2} : x \in \mathbb{R}^N,\ a \leqslant |t| \leqslant b \right\}.$$

Using these notations, we see that there exists k_0 such that

$$\begin{aligned} c(I) + 1 &\geqslant I(u_k) - \frac{1}{2} I'(u_k) \cdot u_k \\ &= \int_{\Omega_k(0,a)} \mathcal{F}(x,u_k) \,\mathrm{d}x + \int_{\Omega_k(a,b)} \mathcal{F}(x,u_k) \,\mathrm{d}x + \int_{\Omega_k(b,\infty)} \mathcal{F}(x,u_k) \,\mathrm{d}x \\ &\geqslant \int_{\Omega_k(0,a)} \mathcal{F}(x,u_k) \,\mathrm{d}x + m_a^b \int_{\Omega_k(a,b)} u_k^2 \,\mathrm{d}x + g(b) |\Omega_k(b,\infty)|, \quad \forall\, k > k_0. \end{aligned} \tag{3.3.8}$$

Inequality (3.3.8) implies

$$\lim_{b \to \infty} |\Omega_k(b,\infty)| = 0, \quad \text{uniformly in } k > k_0.$$

Moreover, fixed $2 < q \leqslant 2_s^*$, we have

$$\int_{\Omega_k(a,b)} |v_k|^q \,\mathrm{d}x \leqslant \left(\int_{\Omega_k(a,b)} |v_k|^{2_s^*} \right)^{q/2_s^*} |\Omega_k(a,b)|^{(2_s^* - q)/2_s^*},$$

in particular,

$$\lim_{b \to \infty} \int_{\Omega_k(a,b)} |v_k|^q \,\mathrm{d}x = 0, \quad \text{uniformly in } k > k_0. \tag{3.3.9}$$

On the hand, it follows that

$$\begin{aligned} \int_{\Omega_k(a,b)} v_k^2 \,\mathrm{d}x &= \frac{1}{\|u_k\|_V^2} \int_{\Omega_k(a,b)} u_k^2 \,\mathrm{d}x \\ &\leqslant \left(\frac{1}{\|u_k\|_V^2} \right) \left(\frac{1}{(c(I)+1) m_a^b} \right) \to 0, \text{ as } k \to \infty. \end{aligned} \tag{3.3.10}$$

We now pass to prove the estimate (3.3.7). By condition (f_4), there exists $a_\varepsilon > 0$ such that

$$|f(x,t)| < \varepsilon|t|, \quad \forall\, x \in \mathbb{R}^N, \text{ provided that } |t| < a_\varepsilon.$$

Thus

$$\int_{\Omega_k(0,a_\varepsilon)} \frac{f(x,u_k)}{\|u_k\|_V} v_k \,\mathrm{d}x \leqslant \int_{\Omega_k(0,a_\varepsilon)\cap\{|u_k|>0\}} \frac{f(x,u_k)}{|u_k|} v_k^2 \,\mathrm{d}x < \varepsilon/3, \quad \forall\, k > k_\varepsilon^{(1)},$$

where $k_\varepsilon^{(1)} > k_0$ is obtained by convergence (3.3.10). Taking $2q_0 := 2p_0/(p_0-1)$ and using assumption (f_6) we have that

$$\begin{aligned}\int_{\Omega_k(b_\varepsilon,\infty)} \frac{f(x,u_k)}{\|u_k\|_V} v_k \,\mathrm{d}x &\leqslant \int_{\Omega_k(b_\varepsilon,\infty)} \frac{f(x,u_k)}{|u_k|} v_k^2 \,\mathrm{d}x \\ &\leqslant (a_0(c(I)+1))^{1/p_0} \left(\int_{\Omega_k(b_\varepsilon,\infty)} |v_k|^{2q_0} \,\mathrm{d}x \right)^{1/q_0} < \varepsilon/3, \quad \forall\, k > k_\varepsilon^{(2)},\end{aligned}$$

where b_ε and $k_\varepsilon^{(2)} > k_0$ are taken from convergence (3.3.9). Finally, using condition (f_4) we get that

$$|f(x,u_k)| \leqslant C_\varepsilon |u_k|, \ \forall\, x \in \Omega_k(a_\varepsilon,b_\varepsilon),$$

and some positive constant C_ε that does not depends on k and x. Thus,

$$\int_{\Omega_k(a_\varepsilon,b_\varepsilon)} \frac{f(x,u_k)}{\|u_k\|_V} v_k \,\mathrm{d}x \leqslant \int_{\Omega_k(a_\varepsilon,b_\varepsilon)} \frac{f(x,u_k)}{|u_k|} v_k^2 \,\mathrm{d}x \leqslant C_\varepsilon \int_{\Omega_k(a_\varepsilon,b_\varepsilon)} v_k^2 \,\mathrm{d}x < \varepsilon/3, \ \forall\, k > k_\varepsilon^{(3)},$$

where $k_\varepsilon^{(3)} > k_0$ is obtained from (3.3.10). The contradiction from (3.3.6) and (3.3.7) follows by taking $k_\varepsilon \geqslant \{k_\varepsilon^{(1)}, k_\varepsilon^{(2)} k_\varepsilon^{(3)}\}$. ■

3.4 Behavior of weak decomposition convergence under nonlinearities

We now pass to describe the limit of the profile decomposition (Theorems 1.1.1 and 1.1.2) for bounded sequences under the considered nonlinearities.

Proposition 3.4.1. *Suppose that $f(x,t)$ satisfies (f_1), $a(x) \equiv V(x) \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$ and (V_2). Let (u_k) be a bounded sequence in $H^s_V(\mathbb{R}^N)$ such that $u_k \to u$ in $L^p(\mathbb{R}^N)$, for some $p \in (2, 2_s^*)$, then*

$$\lim_{k\to\infty} \int_{\mathbb{R}^N} f(x,u_k)u_k \,\mathrm{d}x = \int_{\mathbb{R}^N} f(x,u)u \,\mathrm{d}x,$$

up to subsequence. Moreover, if (v_k) is a bounded sequence in $H^s_V(\mathbb{R}^N)$ with $u_k - v_k \to 0$ in $L^p(\mathbb{R}^N)$, for some $2 < p < 2_s^*$, then, up to subsequence,

$$\lim_{k\to\infty} \int_{\mathbb{R}^N} F(x,u_k) - F(x,v_k) \,\mathrm{d}x = 0. \tag{3.4.1}$$

Proof. First observe that $u_k \to u$ in $L^q(\mathbb{R}^N)$ for all $q \in (2, 2_s^*)$. In fact, this follows by a interpolation inequality, if $q < p$ then

$$\|u_k - u\|_q \leqslant \|u_k - u\|_2^{\theta} \|u_k - u\|_p^{1-\theta}$$

where $1/q = \theta/2 + (1-\theta)/p$, and if $q > p$ then

$$\|u_k - u\|_q \leqslant \|u_k - u\|_p^{\theta} \|u_k - u\|_{2_s^*}^{1-\theta}$$

for $1/q = \theta/p + (1-\theta)/2_s^*$. On the other hand, by embedding (0.2.3) and Proposition 3.3.1, up to subsequence $u \in H_V^s(\mathbb{R}^N)$ with,

$$u_k(x) \to u(x) \text{ as } k \to \infty, \text{ a.e. } x \in \mathbb{R}^N \text{ and } |u_k(x)|, |u(x)| \leqslant h_\varepsilon(x) \text{ a.e. } x \in \mathbb{R}^N, \ k \in \mathbb{N},$$

for some $h_\varepsilon \in L^{p_\varepsilon}(\mathbb{R}^N)$. Now consider that

$$\int_{\mathbb{R}^N} |f(x, u_k)u_k - f(x, u)u| \, \mathrm{d}x \leqslant \int_{\mathbb{R}^N} |f(x, u_k)(u_k - u)| \, \mathrm{d}x + \int_{\mathbb{R}^N} |(f(x, u_k) - f(x, u))u| \, \mathrm{d}x.$$

The first integral can be estimated by Hölder inequality as follows

$$\int_{\mathbb{R}^N} |f(x, u_k)(u_k - u)| \, \mathrm{d}x \leqslant \varepsilon \left(\|u_k\|_2 \|u_k - u\|_2 + \|u_k\|_{2_s^*}^{2_s^*-1} \|u_k - u\|_{2_s^*} \right) + C_\varepsilon \|u_k\|_{p_\varepsilon}^{p_\varepsilon - 1} \|u_k - u\|_{p_\varepsilon}.$$

For the second one, consider

$$E_k^\varepsilon := \left\{ x \in \mathbb{R}^N : \varepsilon(|u_k(x)| + |u_k(x)|^{2_s^*-1}) \leqslant C_\varepsilon |u_k(x)|^{p_\varepsilon - 1} \right\}$$

and

$$E^\varepsilon := \left\{ x \in \mathbb{R}^N : \varepsilon(|u(x)| + |u(x)|^{2_s^*-1}) \leqslant C_\varepsilon |u(x)|^{p_\varepsilon - 1} \right\}.$$

Thus

$$\int_{E_k^\varepsilon} |(f(x, u_k) - f(x, u))u| \, \mathrm{d}x = \int_{\mathbb{R}^N} |(f(x, u_k) - f(x, u))u| \mathcal{X}_{H_k^\varepsilon} \, \mathrm{d}x.$$

Since $\mathcal{X}_{E_k^\varepsilon}(x) \to \mathcal{X}_{E^\varepsilon}(x)$ in each point of $\mathbb{R}^N$ and

$$|(f(x, u_k) - f(x, u))u \mathcal{X}_{H_k^\varepsilon}| \leqslant 2C_\varepsilon h_\varepsilon^{p_\varepsilon} \in L^1(\mathbb{R}^N),$$

we may apply the Dominated Convergence Theorem to conclude

$$\lim_{k \to \infty} \int_{E_k^\varepsilon} |(f(x, u_k) - f(x, u))u| \, \mathrm{d}x = 0.$$

On the other way,

$$\limsup_{k \to \infty} \int_{\mathbb{R}^N \setminus E_k^\varepsilon} |(f(x, u_k) - f(x, u))u| \, \mathrm{d}x \leqslant C\varepsilon.$$

where C is a positive constant that does not depend in ε and k. Since ε is arbitrary, (3.4.1) holds.
Now, let us prove (3.4.1). Choose $(\bar{u}_k), (\bar{v}_k)$ in $C_0^\infty(\mathbb{R}^{\mathbb{N}})$ such that

$$\lim_{k\to\infty} \|\bar{u_k} - u_k\|_V = \lim_{k\to\infty} \|\bar{v_k} - v_k\|_V = 0.$$

Thus it suffices to prove that

$$\lim_{k\to\infty} \int_{\mathbb{R}^N} [F(x, \bar{u_k}) - F(x, \bar{v_k})] \,\mathrm{d}x = 0. \tag{3.4.2}$$

Consider $E := (C_0(\mathbb{R}^N), \|\cdot\|_{p_\varepsilon})$ and the functional $\beta : E \to \mathbb{R}$, given by $\beta(u) = \int_{\mathbb{R}^N} F(x,u)\,\mathrm{d}x$ with Gateaux derivative

$$\beta'_G(u)\cdot v = \int_{\mathbb{R}^N} f(x,u)v\,\mathrm{d}x.$$

Thus, we may apply the Mean Value Theorem to get

$$|\beta(u) - \beta(v)| \leqslant \sup_{w\in E, w\in[u,v]} \|\beta'_G(w)\|_* \, \|u-v\|_{p_\varepsilon}, \ \forall\ u, v \in E, \tag{3.4.3}$$

where $[u,v] = \{tu + (1-t)v : t \in [0,1]\}$. Since $(u_k), (v_k), (\bar{u}_k)$ and $(\bar{v}_k)$ lies in a bounded set B in $H^s_V(\mathbb{R}^N)$, we also have, by the continuous embedding $H^s_V(\mathbb{R}^N) \hookrightarrow L^{p_\varepsilon}(\mathbb{R}^N)$, that $B \cap E$ is bounded in E. Consequently β'_G is bounded in $B \cap E$, which allows us to take $u = \bar{u}_k$ and $v = \bar{v}_k$ in (3.4.3) to conclude the convergence (3.4.2). ■

Our next result can be see as the nonlocal counterpart of [99, Lemma 5.1]. Moreover, it might also be seen as an generalization of the well known Brezis-Lieb Lemma [15].

Proposition 3.4.2. *Assume that $f(x,t)$ satisfies (f_1) and (f_7). Let (u_k) in $H^s(\mathbb{R}^N)$ be a bounded sequence and $(w^{(n)})_{n\in\mathbb{N}_0}$ in $H^s(\mathbb{R}^N)$, given by the Theorem 1.1.2. Then*

$$\lim_{k\to\infty} \int_{\mathbb{R}^N} F(x,u_k)\,\mathrm{d}x = \int_{\mathbb{R}^N} F(x,w^{(1)})\,\mathrm{d}x + \sum_{n\in\mathbb{N}_0, n>1} \int_{\mathbb{R}^N} F_{\mathcal{P}}(x,w^{(n)})\,\mathrm{d}x.$$

Proof. By the Proposition 3.4.1 the functional

$$\Phi(u) := \int_{\mathbb{R}^N} F(x,u)\,\mathrm{d}x, \quad u \in H^s(\mathbb{R}^N),$$

is uniformly continuous in bounded sets of $L^p(\mathbb{R}^N)$, for any $2 < p < 2^*_s$, consequently, by assertions (1.1.7) and (1.1.8) of Theorem 1.1.2, we have that

$$\lim_{k\to\infty}\left[\Phi(u_k) - \Phi\left(\sum_{n\in\mathbb{N}_0} w^{(n)}(\cdot - y_k^{(n)})\right)\right] = 0.$$

The uniform convergence in (1.1.8) of Theorem 1.1.2 allows us to reduce to the case where $\mathbb{N}_0 = \{1, \dots, M\}$. Thus taking

$$\Phi_{\mathcal{P}}(u) := \int_{\mathbb{R}^N} F_{\mathcal{P}}(x, u)\,\mathrm{d}x, \quad u \in H^s(\mathbb{R}^N),$$

it follows from (f_7) and Dominated Convergence Theorem that

$$\lim_{k\to\infty} \left[\sum_{n\in\mathbb{N}_0} \Phi\left(w^{(n)}(\cdot - y_k^{(n)})\right) - \Phi(w^{(1)}) - \sum_{n\in\mathbb{N}_0, n>1} \Phi_{\mathcal{P}}(w^{(n)}) \right] = 0.$$

It remains to prove that

$$\lim_{k\to\infty} \left[\Phi\left(\sum_{n\in\mathbb{N}_0} w^{(n)}(\cdot - y_k^{(n)}) \right) - \sum_{n\in\mathbb{N}_0} \Phi\left(w^{(n)}(\cdot - y_k^{(n)})\right) \right] = 0. \tag{3.4.4}$$

Since Φ is locally Lipschitz in bounded sets of $H^s(\mathbb{R}^N)$, using a density argument, we can assume without loss of generality that $w^{(n)} \in C_0^\infty(\mathbb{R}^N)$, for $n = 1, \dots, M$. Consequently, from 1.1.6,

$$\operatorname{supp}(w^{(n)}(\cdot - y_k^{(n)})) \cap \operatorname{supp}(w^{(m)}(\cdot - y_k^{(m)})) = \varnothing, \text{ for } m \neq n \text{ and } k \text{ large enough},$$

which implies that, for k large enough,

$$\begin{aligned} \int_{\mathbb{R}^N} F\left(x, \sum_{n\in\mathbb{N}_0} w^{(n)}(\cdot - y_k^{(n)})\right) \mathrm{d}x &= \int_{\bigcup_{n=1}^M \operatorname{supp}(w^{(n)}(\cdot - y_k^{(n)}))} F\left(x, \sum_{m=1}^M w^{(m)}(\cdot - y_k^{(m)})\right) \mathrm{d}x \\ &= \sum_{n=1}^M \int_{\operatorname{supp}(w^{(n)})} F(x + y_k^{(n)}, w^{(n)})\,\mathrm{d}x, \end{aligned}$$

from this, (3.4.4) follows immediately. ∎

Corollary 3.4.3. *Let (u_k) in $H^s(\mathbb{R}^N)$ be a bounded sequence and $(w^{(n)})_{n\in\mathbb{N}_0}$ in $H^s(\mathbb{R}^N)$, given by Theorem 1.1.2. If $f(x,t)$ is $1-$periodic in x_i, $i = 1, \dots, N$ and satisfies (f_1),*

$$\lim_{k\to\infty} \int_{\mathbb{R}^N} F(x, u_k)\,\mathrm{d}x = \sum_{n\in\mathbb{N}_0} \int_{\mathbb{R}^N} F(x, w^{(n)})\,\mathrm{d}x. \tag{3.4.5}$$

Corollary 3.4.4. *Let $u_k \rightharpoonup u$ in $H^s(\mathbb{R}^N)$ and $F(x,t)$ as in Corollary 3.4.3 then, up to subsequence,*

$$\lim_{k\to\infty} \int_{\mathbb{R}^N} F(u_k) - F(u - u_k) - F(u)\,\mathrm{d}x = 0.$$

Proof. Since $w^{(1)} = u$, following the proof of Proposition 3.4.2, we obtain

$$\lim_{k\to\infty} \int_{\mathbb{R}^N} F(u_k - u)\,\mathrm{d}x = \sum_{n\in\mathbb{N}_*, n>1} \int_{\mathbb{R}^N} F(w^{(n)})\,\mathrm{d}x. \tag{3.4.6}$$

Taking the difference between (3.4.5) and (3.4.6) we get the desired convergence. ∎

We also need the following result, that can be understood as an generalization of Fatou Lemma, or alternatively, that the functional $u \mapsto \int_{\mathbb{R}^N} V(x)u^2 \, \mathrm{d}x$ is sequentially weakly lower semicontinuous with respect to the profile decomposition of Theorem 1.1.2. Moreover, it is a complement to Proposition 3.4.2.

Proposition 3.4.5. *Suppose that $a(x) \equiv V(x) \geqslant 0$ and that (V_2) holds true. Let (u_k) be a bounded sequence in $H^s(\mathbb{R}^N)$ and $(w^{(n)})_{n \in \mathbb{N}_0}$ given in Theorem 1.1.2.*

(i) If (V_1) holds, we have

$$\liminf_{k \to \infty} \int_{\mathbb{R}^N} V(x)u_k^2 \, \mathrm{d}x \geqslant \sum_{n \in \mathbb{N}_0} \int_{\mathbb{R}^N} V(x)|w^{(n)}|^2 \, \mathrm{d}x.$$

(ii) Under (V_4) we obtain,

$$\liminf_{k \to \infty} \int_{\mathbb{R}^N} V(x)u_k^2 \, \mathrm{d}x \geqslant \int_{\mathbb{R}^N} V(x)|w^{(1)}|^2 \, \mathrm{d}x + \sum_{n \in \mathbb{N}_0, n > 1} \int_{\mathbb{R}^N} V_\infty |w^{(n)}|^2 \, \mathrm{d}x.$$

Proof. We prove only the second inequality, the first one follows by a similar argument. It suffices to prove that

$$\begin{aligned} \int_{\mathbb{R}^N} V(x)u_k^2 \, \mathrm{d}x = \int_{\mathbb{R}^N} \left| |V(x)|^{1/2}(u_k - w^{(1)}) - |V_\infty|^{1/2} \sum_{n=2}^{m} w^{(n)}(\cdot - y_k^{(n)}) \right|^2 \mathrm{d}x, \\ + \int_{\mathbb{R}^N} V(x)|w^{(1)}|^2 \, \mathrm{d}x + \sum_{n=2}^{m} \int_{\mathbb{R}^N} V_\infty |w^{(n)}|^2 \, \mathrm{d}x + o(1), \quad \forall\, m. \end{aligned} \tag{3.4.7}$$

where with the notation $a_k = o(b_k)$ we mean that $a_k/b_k \to 0$. To this end, we proceed as in the proof of the iterated Brezis-Lieb Lemma [29, Proposition 6.7],thus the proof of (3.4.7) is made by induction. We start by checking that (3.4.7) holds for $m = 2$. In fact, by Proposition 3.3.1 it is clear that, up to subsequence, the classical Brezis-Lieb Lemma [15] and assertion (1.1.6) implies that

$$\int_{\mathbb{R}^N} V(x)u_k^2 \, \mathrm{d}x = \int_{\mathbb{R}^N} V(x)|w^{(1)}|^2 \, \mathrm{d}x + \int_{\mathbb{R}^N} V(x)|u_k - w^{(1)}|^2 \, \mathrm{d}x + o(1), \tag{3.4.8}$$

consequently and by the same reason,

$$\begin{aligned} &\int_{\mathbb{R}^N} V(x)|u_k - w^{(1)}|^2 \, \mathrm{d}x = \\ &\quad \int_{\mathbb{R}^N} V(x + y_k^{(2)})|u_k(\cdot + y_k^{(2)}) - w^{(1)}(\cdot + y_k^{(2)})|^2 \, \mathrm{d}x \\ &\quad + \int_{\mathbb{R}^N} \left| |V(x + y_k^{(2)})|^{1/2} \left(u_k(\cdot + y_k^{(2)}) - w^{(1)}(\cdot + y_k^{(2)}) \right) - |V_\infty w^{(2)}|^{1/2} \right|^2 \mathrm{d}x \\ &\quad + \int_{\mathbb{R}^N} V_\infty |w^{(2)}|^2 \, \mathrm{d}x + o(1). \end{aligned} \tag{3.4.9}$$

Replacing identity (3.4.9) in (3.4.8) we obtain (3.4.7) for $m = 2$. We shall now prove that (3.4.7) holds for $m+1$ provided that it is true for m. Indeed, arguing as above,

$$\int_{\mathbb{R}^N} \left| |V(x)|^{1/2}(u_k - w^{(1)}) - V_\infty^{1/2} \sum_{n=2}^{m} w^{(n)}(\cdot - y_k^{(n)}) \right|^2 \mathrm{d}x - \int_{\mathbb{R}^N} V_\infty |w^{(m+1)}|^2 \,\mathrm{d}x$$
$$= \int_{\mathbb{R}^N} \left| |V(x)|^{1/2} \left(u_k - w^{(1)}\right) - V_\infty^{1/2} \sum_{n=2}^{m+1} w^{(n)}(\cdot - y_k^{(n)}) \right|^2 \mathrm{d}x + o(1). \quad (3.4.10)$$

Applying the induction hypothesis in (3.4.10) we obtain (3.4.7). ■

3.5 Pohozaev Identity

We finish the section by proving the aforementioned Pohozaev type identity. The proof follows the same arguments used in Sect. 2.3 with some appropriated modifications. It complements some well known results in the present literature, namely: [23, Theorem 2.3], [24, Proposition 4.1] and [75, Theorem 1.1].

Proposition 3.5.1. *Suppose that $f(x,t) \equiv f(t) \in C^1(\mathbb{R})$ and $a(x) \in C^1(\mathbb{R}^N \backslash \mathcal{O})$, where $\mathcal{O}$ is a finite set. Let $u \in \mathcal{D}^{s,2}(\mathbb{R}^N)$ be a weak solution of $(\mathcal{H}_s)$ such that $f(u)/(1+|u|)$ belongs to $L^{N/2s}_{\mathrm{loc}}(\mathbb{R}^N)$. If $F(u)$, $f(u)u$, $a(x)u^2$ and $\langle \nabla a(x), x\rangle u^2$ belongs to $L^1(\mathbb{R}^N)$, then $u \in C^1(\mathbb{R}^N \backslash \mathcal{O})$ and*

$$\frac{N-2s}{2} \int_{\mathbb{R}^N} |(-\Delta)^{s/2} u|^2 \,\mathrm{d}x + \frac{N}{2} \int_{\mathbb{R}^N} a(x) u^2 \,\mathrm{d}x + \frac{1}{2} \int_{\mathbb{R}^N} \langle \nabla a(x), x\rangle u^2 \,\mathrm{d}x = N \int_{\mathbb{R}^N} F(u) \,\mathrm{d}x.$$

Proof. Firstly we prove the local regularity of u. To do that, we consider $x_0 \in \mathbb{R}^N \backslash \mathcal{O}$, and observe that $\overline{u} = u(\cdot + x_0)$ is a weak solution of

$$(-\Delta)^s \overline{u} + \overline{a}(x)\overline{u} = f(\overline{u}) \text{ in } \mathbb{R}^N,$$

where $\overline{a}(x) = a(x + x_0)$. Taking r small enough, the ball B_r^N does not contains any point of discontinuity of $\overline{a}(x)$ and so

$$\frac{|g(\overline{u})|}{1+|\overline{u}|} \in L^{N/2s}(B_r^N), \quad \text{where} \quad g(\overline{u}) := f(\overline{u}) - \overline{a}(x)\overline{u}.$$

This enable us to proceed as in Proposition 2.3.1, to conclude that $u \in L^p(B_r^N)$, for all $p \geqslant 1$. Moreover, since

$$g(\overline{u}) = f(\overline{u}) - \overline{a}(x)\overline{u} = \left[\frac{f(\overline{u})}{1+|\overline{u}|} \operatorname{sgn}(\overline{u}) - \overline{a}(x)\right] \overline{u} + \frac{f(\overline{u})}{1+|\overline{u}|},$$

we may apply Proposition 0.4.2 (the regularity results of [59]) to conclude that there exists $0 < y_0,\ r_0 < r$ with $B_r^N \times [0, y_0] \subset B_r^+$, and $\alpha \in (0,1)$, such that

$$\overline{w},\ \nabla_x \overline{w},\ y^{1-2s}\overline{w}_y \in C^{0,\alpha}(B_{r_0}^N \times [0, y_0]),$$

where $\overline{w}$ is the s-harmonic extension of $\overline{u}$ and $\nabla_x \overline{w} = (\overline{w}_{x_1}, \ldots, \overline{w}_{x_N})$. In particular, since x_0 is arbitrary,

$$w,\ \nabla_x w,\ y^{1-2s} w_y \in C(B_r^N \backslash \mathcal{O} \times [0, y_0]),\ \forall\, r, y_0 > 0. \tag{3.5.1}$$

Consider now $\xi \in C_0^\infty(\mathbb{R} : [0,1])$ such that

$$\xi(t) = \begin{cases} 1, & \text{if} \quad |t| \leqslant 1 \\ 0, & \text{if} \quad |t| \geqslant 2 \end{cases} \qquad \text{and} \qquad |\xi'(t)| \leqslant C \quad \forall\, t \in \mathbb{R},$$

for some $C > 0$. Let $\mathcal{O} = \{x^{(1)}, \ldots, x^{(l)}\}$, and $z^{(i)} = (x^{(i)}, 0),\ i = 1, \ldots, l$. For each $n = 1, \ldots,$ define $\xi_n : \mathbb{R}^{N+1} \to \mathbb{R}$ by

$$\xi_n(z) = \begin{cases} \xi(|z|^2/n^2), & \text{if } |z - z^{(i)}|^2 > 2/n^2, \\ 1 - \xi(n^2 |z - z^{(i)}|^2), & \text{if } |z - z^{(i)}|^2 \leqslant 2/n^2. \end{cases}$$

Then, for n large enough, $\xi_n \in C_0^\infty(\mathbb{R}^N)$ and verifies

$$|z||\nabla \xi_n(z)| \leqslant C \quad \forall\, z \in \mathbb{R}^{N+1}, \tag{3.5.2}$$

for some $C > 0$. Now observe that

$$\begin{aligned} &\operatorname{div}(y^{1-2s} \nabla w) \langle z, \nabla w \rangle \xi_n \\ &\quad = \operatorname{div}\left[y^{1-2s} \xi_n \left(\langle z, \nabla w \rangle \nabla w - \frac{|\nabla w|^2}{2} z \right) \right] + \frac{N-2s}{2} y^{1-2s} |\nabla w|^2 \xi_n \\ &\qquad + y^{1-2s} \frac{|\nabla w|^2}{2} \langle z, \nabla \xi_n \rangle - y^{1-2s} \langle \nabla w, z \rangle \langle \nabla w, \nabla \xi_n \rangle. \end{aligned} \tag{3.5.3}$$

Note that $\partial B_{\sqrt{2}n,\delta} = F^1_{\sqrt{2}n,\delta} \cup F^2_{\sqrt{2}n,\delta}$. Let $\eta(z) = (0, \ldots, -1)$ be the unit outward normal vector of $B_{\sqrt{2}n,\delta}$ on $F^1_{\sqrt{2}n,\delta}$. Since $\xi_n = 0$ on $F^2_{\sqrt{2}n,\delta}$, by condition (0.3.2), identity (3.5.3) and the Divergence Theorem we get

$$\begin{aligned} 0 &= \int_{B_{\sqrt{2}n,\delta}} \operatorname{div}(y^{1-2s} \nabla w) \langle z, \nabla w \rangle \xi_n \, \mathrm{d}x\mathrm{d}y \\ &= \int_{F^1_{\sqrt{2}n,\delta}} y^{1-2s} \xi_n \left[\langle z, \nabla w \rangle \langle \nabla w, \eta \rangle - \frac{|\nabla w|^2}{2} \langle z, \eta \rangle \right] \mathrm{d}x\mathrm{d}y + \theta_{n,\delta} \\ &= \int_{F^1_{\sqrt{2}n,\delta}} \xi_n \langle x, \nabla_x w \rangle (-y^{1-2s} w_y) \, \mathrm{d}x \\ &\qquad - \int_{F^1_{\sqrt{2}n,\delta}} y^{1-2s} \xi_n w_y^2 y \, \mathrm{d}x + \int_{F^1_{\sqrt{2}n,\delta}} y^{1-2s} \xi_n \frac{|\nabla w|^2}{2} y \, \mathrm{d}x + \theta_{n,\delta} \\ &= I^1_{n,\delta} + I^2_{n,\delta} + I^3_{n,\delta} + \theta_{n,\delta}, \end{aligned}$$

where

$$\begin{aligned} \theta_{n,\delta} = &\int_{B_{\sqrt{2}n,\delta}} \frac{N-2s}{2} y^{1-2s} |\nabla w|^2 \xi_n \, \mathrm{d}x\mathrm{d}y \\ &+ \int_{B_{\sqrt{2}n,\delta}} y^{1-2s} \frac{|\nabla w|^2}{2} \langle z, \nabla \xi_n \rangle - y^{1-2s} \langle \nabla w, z \rangle \langle \nabla w, \nabla \xi_n \rangle \, \mathrm{d}x\mathrm{d}y. \end{aligned}$$

We known that there exists a sequence $\delta_k \to 0$ such that

$$I^2_{n,\delta_k} + I^3_{n,\delta_k} \to 0, \text{ as } k \to \infty.$$

Some computations leads to

$$\begin{aligned}
&\xi_n(x,0)\langle x, \nabla u\rangle (f(u) - a(x)u) \\
&\quad = \operatorname{div}\left[\xi_n(x,0)\left(F(u) - \frac{1}{2}a(x)u^2\right)x\right] - \langle \nabla \xi_n(x,0), x\rangle F(u) \\
&\qquad - N\xi_n(x,0)F(u) + \frac{1}{2}\langle \nabla \xi_n(x,0), x\rangle a(x)u^2 \\
&\qquad + \frac{1}{2}\xi_n(x,0)\langle \nabla a(x), x\rangle u^2 + \frac{N}{2}\xi_n(x,0)a(x)u^2.
\end{aligned}$$

Thus, by Remark 0.4.3, condition (3.5.1) and the Divergence Theorem we have

$$\begin{aligned}
\lim_{k\to\infty} I^1_{n,\delta_k} &= \kappa_s \int_{B^N_{\sqrt{2}n}} \xi_n(x,0)\langle x, \nabla u\rangle (f(u) - a(x)u)\,\mathrm{d}x \\
&= -\kappa_s \int_{B^N_{\sqrt{2}n}} \langle \nabla \xi_n(x,0), x\rangle F(u) + N\xi_n(x,0)F(u)\,\mathrm{d}x \\
&\quad + \frac{\kappa_s}{2}\int_{B^N_{\sqrt{2}n}} \langle \nabla \xi_n(x,0), x\rangle a(x)u^2\,\mathrm{d}x \\
&\quad + \frac{\kappa_s}{2}\int_{B^N_{\sqrt{2}n}} \xi_n(x,0)\langle \nabla a(x), x\rangle u^2 + \frac{N}{2}\xi_n(x,0)a(x)u^2\,\mathrm{d}x.
\end{aligned}$$

Summing up, we get

$$\begin{aligned}
0 &= \lim_{k\to\infty}\left[I^1_{n,\delta_k} + I^2_{n,\delta_k} + I^3_{n,\delta_k} + \theta_{n,\delta_k}\right] \\
&= -\kappa_s \int_{B^N_{\sqrt{2}n}} \langle \nabla \xi_n, x\rangle F(u) + N\xi_n F(u)\,\mathrm{d}x \\
&\quad + \kappa_s \int_{B^N_{\sqrt{2}n}} \frac{1}{2}\langle \nabla \xi_n, x\rangle a(x)u^2 - \frac{1}{2}\xi_n \langle \nabla a(x), x\rangle u^2 - \frac{N}{2}\xi_n a(x)u^2\,\mathrm{d}x \\
&\quad + \int_{B_{\sqrt{2}n}} \frac{N-2s}{2} y^{1-2s}|\nabla w|^2 \xi_n\,\mathrm{d}x\mathrm{d}y \\
&\quad + \int_{B_{\sqrt{2}n}} y^{1-2s}\frac{|\nabla w|^2}{2}\langle z, \nabla \xi_n\rangle - y^{1-2s}\langle \nabla w, z\rangle\langle \nabla w, \nabla \xi_n\rangle\,\mathrm{d}x\mathrm{d}y.
\end{aligned}$$

Consequently using condition (3.5.2) to pass the limit $n \to \infty$, we conclude

$$\begin{aligned}
\frac{N-2s}{2}\int_{\mathbb{R}^N} |(-\Delta)^{s/2}u|\,\mathrm{d}x &= \frac{N-2s}{2\kappa_s}\int_{\mathbb{R}^N} y^{1-2s}|\nabla w|^2\,\mathrm{d}x\mathrm{d}y \\
&= N\int_{\mathbb{R}^N} F(u)\,\mathrm{d}x - \frac{N}{2}\int_{\mathbb{R}^N} a(x)u^2 - \frac{1}{2}\langle \nabla a(x), x\rangle u^2\,\mathrm{d}x,
\end{aligned}$$

where in the first equality we used condition (0.3.2). ■

Remark 3.5.2. In previous proof we have applied [59, Theorem 2.15] and for that is was crucial that $a(x)$ is a C^1–function in $\mathbb{R}^N \backslash \mathcal{O}$.

Corollary 3.5.3. *Assume that $f(x,t) \equiv f(t) \in C^1(\mathbb{R})$ and that fulfills (f_1). Moreover, that $a(x) \equiv a_0 > 0$. If $u \in H^s(\mathbb{R}^N)$ is a weak solution for $(\mathcal{H}_s)$, then*

$$\int_{\mathbb{R}^N} F(u) - \frac{a_0}{2}u^2 \, \mathrm{d}x = \frac{N-2s}{2N} \int_{\mathbb{R}^{\mathbb{N}}} |(-\Delta)^{s/2}u|^2 \, \mathrm{d}x$$

Corollary 3.5.4. *Suppose that $f(x,t) \equiv f(t) \in C^1(\mathbb{R})$ and that fulfills (f_1^*). If $u \in \mathcal{D}^{s,2}(\mathbb{R}^N)$ is a weak solution for $(\mathcal{H}_s)$, then*

$$\int_{\mathbb{R}^N} |(-\Delta)^{s/2}u|^2 - \lambda|x|^{-2s}u^2 \, \mathrm{d}x = \frac{2N}{N-2s} \int_{\mathbb{R}^{\mathbb{N}}} F(u) \, \mathrm{d}x,$$

where $0 < \lambda < \Lambda_{N,s}$ is given by (0.0.4).

As a direct consequence of Proposition 3.5.1, we have the following non-existence results, complementing the discussions made in [48, 74].

Corollary 3.5.5 (Non-existence results)**.** *Assume that $f(x,t) \equiv f(t) \in C^1(\mathbb{R}^N)$ and either one of the following conditions are satisfied,*

(i) $a(x) \in C^1(\mathbb{R}^N \backslash \mathcal{O})$, where $\mathcal{O}$ is a finite set, $2sa(x) + \langle \nabla a(x), x \rangle > 0$ for all x in a non-zero measure domain and $2_s^* F(t) \leqslant f(t)t$, for all $t \in \mathbb{R}$; or

(ii) $a(x) \in C^1(\mathbb{R}^N \backslash \mathcal{O})$, where $\mathcal{O}$ is a finite set, $a(x) > 0$, $\langle \nabla a(x), x \rangle < 0$ for all x in a non-zero measure domain and there exists $0 < \delta \leqslant 2$, such that $\delta F(t) \geqslant f(t)t$, for all $t \in \mathbb{R}$; or

(iii) $a(x) \equiv a_0 > 0$ and there exists $0 \leqslant \delta \leqslant 2s/(N-2s)$, in a such way that $2_s^* F(t) \leqslant f(t)t + \delta a_0 t^2$, for all $t \in \mathbb{R}$;

(iv) $a(x) \equiv 0$ and there exists $0 < p < 2_s^*$ such that $pF(t) \geqslant f(t)t$ for all $t \in \mathbb{R}$.

If $u \in H^s(\mathbb{R}^N)$ is a weak solution of Eq. $(\mathcal{H}_s)$, such that $F(u)$, $f(u)u$, $a(x)u^2$, $\langle \nabla a(x), x \rangle u^2$ belongs to $L^1(\mathbb{R}^N)$ and $f(u)/(1+|u|)$ belongs to $L^{N/2s}_{\mathrm{loc}}(\mathbb{R}^N)$, then $u \equiv 0$.

Proof. (i) Applying Proposition 3.5.1, we get

$$\int_{\mathbb{R}^N} |(-\Delta)^{s/2}u|^2 \, \mathrm{d}x + \frac{N}{N-2s} \int_{\mathbb{R}^N} a(x)u^2 \, \mathrm{d}x + \frac{1}{N-2s} \int_{\mathbb{R}^N} \langle \nabla a(x), x \rangle u^2 \, \mathrm{d}x \leqslant \int_{\mathbb{R}^N} f(u)u \, \mathrm{d}x,$$

furthermore using that $I'(u) \cdot u = 0$, we obtain

$$\int_{\mathbb{R}^N} (2sa(x) + \langle \nabla a(x), x \rangle) u^2 \, \mathrm{d}x \leqslant 0,$$

which leads to $u \equiv 0$.

(ii) Using again Proposition 3.5.1 we obtain that

$$\frac{N-2s}{2N}\delta \int_{\mathbb{R}^N} |(-\Delta)^{s/2}u|^2 \,\mathrm{d}x \\ + \frac{\delta}{2}\int_{\mathbb{R}^N} a(x)u^2 \,\mathrm{d}x + \frac{\delta}{2N}\int_{\mathbb{R}^N} \langle \nabla a(x), x\rangle u^2 \,\mathrm{d}x \geqslant \int_{\mathbb{R}^N} f(u)u \,\mathrm{d}x,$$

and we can derive that $u \equiv 0$, because

$$\left(1-\frac{N-2s}{2N}\delta\right)\int_{\mathbb{R}^N} |(-\Delta)^{s/2}u|^2 \,\mathrm{d}x \\ + \left(1-\frac{\delta}{2}\right)\int_{\mathbb{R}^N} a(x)u^2 \,\mathrm{d}x - \frac{\delta}{2N}\int_{\mathbb{R}^N} \langle \nabla a(x), x\rangle u^2 \,\mathrm{d}x \leqslant 0.$$

(iii) Once more we can use Proposition 3.5.1 to get

$$\int_{\mathbb{R}^N} |(-\Delta)^{s/2}u|^2 \,\mathrm{d}x + \frac{N}{N-2s}a_0 \int_{\mathbb{R}^N} u^2 \,\mathrm{d}x \geqslant \int_{\mathbb{R}^N} f(u)u \,\mathrm{d}x,$$

which implies

$$\left[\frac{N-(1+\delta)(N-2s)}{N-2s}\right] a_0 \int_{\mathbb{R}^N} u^2 \,\mathrm{d}x \leqslant 0.$$

In particular $u \equiv 0$.

(iv) Proposition 3.5.1 implies that

$$\int_{\mathbb{R}^N} |(-\Delta)^{s/2}u|^2 \,\mathrm{d}x = 2_s^* \int_{\mathbb{R}^N} F(u) \,\mathrm{d}x \geqslant \frac{2_s^*}{p}\int_{\mathbb{R}^N} f(u)u \,\mathrm{d}x = \frac{2_s^*}{p}\int_{\mathbb{R}^N} |(-\Delta)^{s/2}u|^2 \,\mathrm{d}x,$$

which yields $u \equiv 0$. ■

3.6 Proof of Theorem 3.2.1

Proof. (i) Here we use the profile decomposition given by Theorem 1.1.2. This makes our argument easier then the one of [33, Theorem 2.1].

By Proposition 3.3.5 we know of the existence of a bounded sequence (u_k) such that $I(u_k) \to c(I)$ and $I'(u_k) \to 0$. Since it is bounded, it has a profile decomposition provided by Theorem 1.1.2. If we have $w^{(n)} = 0$ for all $n \in \mathbb{N}_0$, then by assertion (1.1.8), $u_k \to 0$ in $L^p(\mathbb{R}^N)$, for any $2 < p < 2_s^*$ and by convergence (1.1.5) $u_k \rightharpoonup 0$ in $H_V^s(\mathbb{R}^N)$, up to subsequence. Consequently, by Proposition 3.4.1, we have

$$\begin{cases} o(1) + c(I) = I(u_k) = \dfrac{1}{2}\|u_k\|_V^2 - \displaystyle\int_{\mathbb{R}^N} F(x,u_k)\,\mathrm{d}x = \dfrac{1}{2}\|u_k\|_V^2 + o(1), \\ o(1) = I'(u_k)\cdot u_k = \|u_k\|_V^2 - \displaystyle\int_{\mathbb{R}^N} f(x,u_k)u_k\,\mathrm{d}x = \|u_k\|_V^2 + o(1), \end{cases} \tag{3.6.1}$$

a contradiction, since $c(I) > 0$. Thus, there must be at least one nonzero $w^{(n)}$. Moreover, we have that each $w^{(n)}$ is a critical point of I. In fact, it is well known that, up to subsequence, we can take $h^{(n)}$ in $L^{\sigma'}(\text{supp}(\varphi))$, $n \in \mathbb{N}_0$, such that

$$|u_k(x + y_k^{(n)})| \leqslant h^{(n)}(x), \quad \text{a.e. } x \in \text{supp}(\varphi), \tag{3.6.2}$$

where $\sigma' = \sigma/(\sigma - 1)$ and $\varphi \in C_0^\infty(\mathbb{R}^N)$, which can be done thanks to Proposition 3.3.1. Thus, for a.e. $x \in \mathbb{R}^N$ we have

$$\begin{cases} |V(x + y_k^{(n)})u_k(x + y_k^{(n)})\varphi(x)| \leqslant h^{(n)}(x)|V(x)\varphi(x)| \in L^1(\text{supp}(\varphi)) \\ V(x + y_k^{(n)})u_k(x + y_k^{(n)})\varphi(x) = V(x)u_k(x + y_k^{(n)})\varphi(x) \to V(x)w^{(n)}(x)\varphi(x), \end{cases}$$

which, by the Dominated Convergence Theorem leads to

$$\begin{aligned} \lim_{k\to\infty}(u_k, \varphi(\cdot - y_k^{(n)}))_V &= \lim_{k\to\infty}\left[[u_k(\cdot + y_k^{(n)}), \varphi]_s + \int_{\mathbb{R}^N} V(x + y_k^{(n)})u_k(\cdot + y_k^{(n)})\varphi(x)\,\mathrm{d}x\right] \\ &= [w^{(n)}, \varphi]_s + \int_{\mathbb{R}^N} V(x)w^{(n)}\varphi\,\mathrm{d}x. \end{aligned}$$

By the same reason and (f_1), up to subsequence we have,

$$\lim_{k\to\infty}\int_{\mathbb{R}^N} f(x + y_k^{(n)}, u_k(\cdot + y_k^{(n)}))\varphi\,\mathrm{d}x = \int_{\mathbb{R}^N} f(x, w^{(n)})\varphi\,\mathrm{d}x.$$

Consequently we may pass the limit in

$$I'(u_k)\cdot\varphi(\cdot - y_k^{(n)}) = (u_k, \varphi(\cdot - y_k^{(n)}))_V - \int_{\mathbb{R}^N} f(x + y_k^{(n)}, u_k(\cdot + y_k^{(n)}))\varphi\,\mathrm{d}x,$$

to conclude that $I'(w^{(n)}) = 0$, for all $n \in \mathbb{N}_0$. In particular, we get that

$$\mathcal{G}_{\mathcal{S}} = \inf\left\{I(u) : u \in H_V^s(\mathbb{R}^N)\backslash\{0\},\ I'(u) = 0\right\} \geqslant 0.$$

We are going to prove that is $\mathcal{G}_{\mathcal{S}}$ is attained and is positive. Let (u_k) be a minimizing sequence of $\mathcal{G}_{\mathcal{S}}$, that is $I(u_k) \to \mathcal{G}_{\mathcal{S}}$ and $I'(u_k) = 0$. Arguing as in Proposition 3.3.5 we obtain that (u_k) is bounded. Suppose by contradiction and assume that $w^{(n)} = 0$ for all $n \in \mathbb{N}_0$. In this case we actually have that $\mathcal{G}_{\mathcal{S}} > 0$, because on the contrary, if $\mathcal{G}_{\mathcal{S}} = 0$, then using (3.6.1) we would conclude that $\|u_k\|_V = o(1)$, and at the same time,

$$\|u_k\|_V^2 = \int_{\mathbb{R}^N} f(u_k)u_k\,\mathrm{d}x \leqslant \varepsilon(C_2\|u_k\|_V^2 + C_*\|u_k\|_V^{2_s^*}) + C_\varepsilon\|u_k\|_V^{p_\varepsilon},$$

where $\mathcal{C}_2$, $\mathcal{C}_{2_s^*}$ and $\mathcal{C}_{p_\varepsilon}$ are positive constant obtained by applying the embedding described in Proposition 3.3.1. In particular,

$$(1 - \varepsilon\mathcal{C}_2) \leqslant \varepsilon\mathcal{C}_{2_s^*}\|u_k\|_V^{2_s^*-2} + \mathcal{C}_{p_\varepsilon}\|u_k\|_V^{p_\varepsilon-2},\ \forall\, k \in \mathbb{N},$$

which, by taking ε small enough, would lead to a contradiction with the fact that $\|u_k\|_V = o(1)$. In view of that, in any case, we can argue as above to conclude that there must be a nonzero $w^{(n_0)}$ that is a critical point of I. We know from (1.1.5) that $u_k(x+y_k^{(n_0)}) \to w^{(n_0)}(x)$ a.e. in $\mathbb{R}^N$, up to subsequence, which allows us to apply Fatou Lemma to get

$$\begin{aligned}
\mathcal{G}_{\mathcal{S}} = \lim_{k\to\infty} I(u_k) &= \lim_{k\to\infty} \int_{\mathbb{R}^N} \mathcal{F}(x, u_k(\cdot + y_k^{(n_0)}))\,\mathrm{d}x \\
&= \liminf_{k\to\infty} \int_{\mathbb{R}^N} \mathcal{F}(x, u_k(\cdot + y_k^{(n_0)}))\,\mathrm{d}x \\
&\geqslant \int_{\mathbb{R}^N} \mathcal{F}(x, w^{(n_0)})\,\mathrm{d}x = I(w^{(n_0)}),
\end{aligned}$$

where we used (f_2) or (f_5) to ensure that $\mathcal{F}(x, u_k(\cdot+y_k^{(n_0)})) = \mathcal{F}(x,u_k) \geqslant 0$ a.e. in $\mathbb{R}^N$. Thus, once again using (f_2) or (f_5), we can see that $\mathcal{G}_{\mathcal{S}} = I(w^{(n_0)}) > 0$.

(ii) From Proposition 3.3.1, the norm

$$|||u|||_\lambda^2 = \int_{\mathbb{R}^N} |(-\Delta)^{s/2}u|^2 - \lambda|x|^{-2s}u^2\,\mathrm{d}x, \quad u \in \mathcal{D}^{s,2}(\mathbb{R}^N), \quad 0 < \lambda < \Lambda_{N,s},$$

is equivalent with respect to the norm $[\,\cdot\,]_s$ in $\mathcal{D}^{s,2}(\mathbb{R}^N)$. Let (u_k) be a minimizing sequence for $\mathcal{I}_\lambda$, and for each k, let u_k^* be the Schwarz Symmetrization of u_k (see [60] for more details). Applying the fractional Polya-Szegö inequality (see [9, Theorem 3]), for each k, we have that

$$\begin{cases} \displaystyle\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_k^*(x)-u_k^*(y)|^2}{|x-y|^{N+2s}}\,\mathrm{d}x\mathrm{d}y \leqslant \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_k(x)-u_k(y)|^2}{|x-y|^{N+2s}}\,\mathrm{d}x\mathrm{d}y, \\ \displaystyle\int_{\mathbb{R}^N} F(u_k^*)\,\mathrm{d}x = \int_{\mathbb{R}^N} F(u_k)\,\mathrm{d}x. \end{cases}$$

Thus $(u_k^*) \subset \mathcal{D}^{s,2}_{\mathrm{rad}}(\mathbb{R}^N)$ and is also a minimizing sequence for (3.2.1). Now observe that $|||\cdot|||_\lambda$ is invariant with respect to the action of dilations given in Theorem 1.1.1, more precisely,

$$|||u|||_\lambda^2 = \left|\left|\left|\gamma^{\frac{N-2s}{2}} u(\gamma^j\cdot)\right|\right|\right|_\lambda^2, \ \forall\,\gamma > 1,\ u \in \mathcal{D}^{s,2}(\mathbb{R}^N) \text{ and } j \in \mathbb{Z},$$

and satisfies the homogeneity property,

$$|||u(\cdot/\delta)|||_\lambda^2 = \delta^{N-2s}|||u|||_\lambda^2,\ \forall\,u \in \mathcal{D}^{s,2}(\mathbb{R}^N),\ \delta > 0.$$

In view of Proposition 1.4.1 and Corollary 3.5.4, we may proceed, using exactly the same arguments, as in the proof of Theorem 2.2.3, replacing $[\,\cdot\,]_s$ by $|||\cdot|||_\lambda$. ■

Remark 3.6.1. (i) In the context of the proof of Theorem 3.2.1–(i), if we assume in addition that $f(x,t)$ satisfies (3.2.2), then $\mathcal{G}_{\mathcal{S}} = c(I) = I(w^{n_0})$ and $w^{(n_0)}$ is non-negative. Indeed the truncation given in Remark 3.2.7 satisfies the assumptions of Theorem 3.2.1–(i), and we can apply the same argument there, to conclude that the ground state $w^{(n_0)}$ is non-negative. Furthermore, Remark 3.3.4–(iv) guarantees that the path $\zeta(t) = tw^{(n_0)}$, $t \geqslant 0$, belongs to Γ_I and $c(I) \leqslant I(w^{(n_0)})$.

On the other hand, considering (u_k) given in the beginning of the proof of Theorem 3.2.1, by Corollary 3.4.3, Remark 3.3.2–(ii) and estimate (1.1.7), up to subsequence, we have

$$c(I) = \lim_{k\to\infty}\left[\frac{1}{2}\|u_k\|_V^2 - \int_{\mathbb{R}^N} F(x,u_k)\,\mathrm{d}x\right] \geqslant \sum_{n\in\mathbb{N}_0} I(w^{(n)}).$$

Consequently, using (f_2) or (f_5) to guarantee that each $I(w^{(n)})$ is non-negative, we conclude that $c(I) = \mathcal{G}_{\mathcal{S}}$.

(ii) If we consider the infimum (3.2.1) defined over $\mathcal{D}^{s,2}_{\mathrm{rad}}(\mathbb{R}^N)$, by Proposition 1.4.1 we can obtain concentration-compactness of the minimizing sequences as described in Theorem 2.2.3. More precisely, for any minimizing sequence (u_k) of (3.2.1), there exists a sequence (j_k) in $\mathbb{Z}$ such that the sequence $(\gamma^{-\frac{N-2s}{2}j_k}u_k(\gamma^{-j_k}\cdot))$ contains a convergent subsequence in $\mathcal{D}^{s,2}_{\mathrm{rad}}(\mathbb{R}^N)$, whose the limit is a minimizer of (3.2.1) in $\mathcal{D}^{s,2}_{\mathrm{rad}}(\mathbb{R}^N)$.

(iii) In the context of the proof of Theorem 3.2.1–(ii), assume that $F(t) \geqslant 0$ for all $t \geqslant 0$. Since $|||\,|u_k|\,|||_\lambda \leqslant |||u_k|||_\lambda$, without loss of generality we can assume that each u_k is non-negative. In this case, the obtained minimizer for (3.2.1) is non-negative.

3.7 Proof of Theorem 3.2.2

Proof. As mentioned, we prove Theorem 3.2.2 by using the Nehari manifold method (see [91]). For convenience of the reader we divide the proof in several steps.

(i) For each $u \in H^s_V\backslash\{0\}$ there exists a unique $\tau(u) > 0$ such that $\tau(u)u \in \mathcal{N}$ and $\max_{t\geqslant 0} I(tu) = I(\tau(u)u)$. In particular $\mathcal{N} \neq \varnothing$.

We proceed in a similar way as in the Remark 3.3.4–(iv), to see that the function $h_u(t) = I(tu)$, $t > 0$, has a maximum point t_u. Moreover, $h'(t_u) = 0$, if and only if $t_u u$ belongs to $\mathcal{N}$ and

$$\|u\|_V^2 - \int_{\mathbb{R}^N} b(x)u^2\,\mathrm{d}x = \frac{1}{t_u}\int_{\mathbb{R}^N} f(x,t_u u)u\,\mathrm{d}x. \tag{3.7.1}$$

By condition (3.2.2) the right-hand side of the above identity occurs at most one point. Thus there is a unique maximum point $\tau(u) = t_u$ for the function $h_u(t)$.

(ii) The function $\tau : H^s_V\backslash\{0\} \to (0,\infty)$ is continuous. Thus the map $\eta : H^s_V\backslash\{0\} \to \mathcal{N}$, defined by $\eta(u) = \tau(u)u$ is continuous and $\eta\big|_{\mathcal{S}}$ is a homeomorphism of the unit sphere $\mathcal{S}$ of $H^s_V(\mathbb{R}^N)$ in $\mathcal{N}$.

Assume that $u_n \to u$ in $H_V^s\backslash\{0\}$. It is well known that the positivity of the primitive $F(x,t)$ together with condition (f_2) implies

$$F(x,t) \geqslant C_1|t|^\mu - C_2t^2, \quad \text{for a.e. } x \in \mathbb{R}^N \text{ and } \forall\, t \in \mathbb{R}.$$

Thus, from identity 3.7.1 we obtain that

$$\|u_n\|_V^2 - \int_{\mathbb{R}^N} b(x)u_n^2 \,\mathrm{d}x \geqslant C_1|\tau(u_n)|^{\mu-2}\int_{\mathbb{R}^N} |u_n|^\mu \,\mathrm{d}x - C_2\|u_n\|_V^2, \quad \forall\, n \in \mathbb{N}.$$

That is, $(u_n) \subset L^\mu(\mathbb{R}^N)$ with

$$\|u_n\|_V^2 \geqslant C|\tau(u_n)|^{\mu-2}\int_{\mathbb{R}^N} |u_n|^\mu \,\mathrm{d}x, \quad \forall\, n \in \mathbb{N}.$$

Moreover, since $u \neq 0$, the sequence (u_n) is bounded below in the norm $\|\cdot\|_\mu$ by a positive constant. Thus $(\tau(u_n))$ is a bounded sequence. We now pass to prove that any given subsequence for $(\tau(u_n))$ has a convergent subsequence with the same limit $\tau(u)$, from this we obtain the convergence $\tau(u_n) \to \tau(u)$. It is clear that for a subsequence $\tau(u_n) \to t_0$. We actually have that t_0 is positive. In fact, using conditions (f_1) and (V_3) in identity (3.7.1) we get the following estimate,

$$\|u_n\|_V^2 - \int_{\mathbb{R}^N} b(x)u_n^2 \,\mathrm{d}x \leqslant \varepsilon C\left(\|u_n\|_V^2 + \tau(u_n)^{2_s^*-2}\|u_n\|_V^{2_s^*}\right) + C_\varepsilon \tau(u_n)^{p_\varepsilon-2}\|u_n\|_V^{p_\varepsilon},$$

for all $n \in \mathbb{N}$. From which, we obtain

$$\left(1 - \varepsilon\mathcal{C}_2 - \frac{\|b(x)\|_\beta}{\mathcal{C}_V^{(\beta)}}\right)\|u_n\|_V^2 \leqslant \varepsilon\mathcal{C}_{2_s^*}\tau(u_n)^{2_s^*-2}\|u_n\|_V^{2_s^*} + C_\varepsilon\mathcal{C}_{p_\varepsilon}\tau(u_n)^{p_\varepsilon-2}\|u_n\|_V^{p_\varepsilon}, \quad (3.7.2)$$

for all $n \in \mathbb{N}$, which implies $t_0 > 0$, by taking ε small enough. Thus we may apply the Dominated Convergence Theorem in (3.7.1) to conclude that $t_0 = \tau(u)$ and the continuity of the function τ. Using (3.7.1) to compute $\tau(u/\|u\|_V)$ we obtain that

$$\|u\|_V^2 - \int_{\mathbb{R}^N} b(x)u^2 \,\mathrm{d}x = \frac{1}{\frac{\tau(u/\|u\|_V)}{\|u\|_V}}\int_{\mathbb{R}^N} f\left(x, \frac{\tau(u/\|u\|_V)}{\|u\|_V}\right)u \,\mathrm{d}x,$$

which by uniqueness gives $\tau(u/\|u\|_V) = \tau(u)u$. Consequently the inverse of η is the retraction map given by $\varrho : \mathcal{N} \to \mathcal{S}$, $\varrho(u) = u/\|u\|_V$.

(iii) $\mathcal{N}$ is away from the origin, that is, there exists $R_{\mathcal{N}} > 0$ such that $\|u\|_V > R_{\mathcal{N}}$, whenever $u \in \mathcal{N}$.

Indeed, estimate (3.7.2) implies that

$$1 - \varepsilon\mathcal{C}_2 - \frac{\|b(x)\|_\beta}{\mathcal{C}_V^{(\beta)}} \leqslant \varepsilon\mathcal{C}_{2_s^*}\|u\|_V^{2_s^*-2} + C_\varepsilon\mathcal{C}_{p_\varepsilon}\|u\|_V^{p_\varepsilon-2}, \quad \forall\, u \in \mathcal{N}.$$

Taking ε small enough we see that $\|u\| \geqslant R_{\mathcal{N}}$, for all $u \in \mathcal{N}$.

(iv) For all $\zeta \in \Gamma_I$ we have that $\zeta([0,\infty)) \cap \mathcal{N} \neq \varnothing$.

Let us suppose that this assertion is false, that is, there exists $\zeta_0 \in \Gamma_I$ which does not intercepts $\mathcal{N}$ in any point. Let $t_0 > 0$ such that $I(\zeta_0(t_0)) < 0$ and $\zeta_0(t) \neq 0$, for all $(0, t_0]$. We prove now that $\tau(\zeta(t)) > 1$ for all $t \in (0, t_0]$. In fact, by continuity, there is a positive number δ such that $\|\zeta_0(t)\| < R_{\mathcal{N}}$, for all $t \in [0, \delta]$. At the same time, we have that $\|\tau(\zeta_0(t))\zeta_0(t)\|_V > R_{\mathcal{N}}$, which implies $\tau(\zeta_0(t)) > 1$, for all $t \in (0, \delta]$. The continuity of $\tau(t)$ and the fact that $\zeta_0(t) \notin \mathcal{N}$, for all t, allow us to choose $\delta = t_0$. On the other hand, by conditions (f_2) and (3.2.2), we have that

$$\begin{aligned} h_{\zeta(t_0)}(t) &\geqslant \frac{t^2}{2}\left[\|\zeta_0(t_0)\|_V^2 - \int_{\mathbb{R}^N} b(x)|\zeta_0(t_0)|^2\,\mathrm{d}x - \frac{2}{\mu}\int_{\mathbb{R}^N} \frac{f(x,t\zeta_0(t_0))}{t\zeta_0(t_0)}|\zeta_0(t_0)|^2\,\mathrm{d}x\right] \\ &> \frac{t^2}{2}\left[\int_{\mathbb{R}^N} \frac{f(x,\tau(\zeta_0(t_0))\zeta_0(t_0))}{\tau(\zeta_0(t_0))\zeta_0(t_0)}|\zeta_0(t_0)|^2 - \frac{f(x,t\zeta_0(t_0))}{t\zeta_0(t_0)}|\zeta_0(t_0)|^2\,\mathrm{d}x\right] \\ &> 0, \quad \forall\, t \in (0, \tau(\zeta(t_0))]. \end{aligned}$$

In particular, $0 < h_{\zeta(t_0)}(1) = I(\zeta_0(t_0))$, which is a contradiction with the choice of $\zeta_0(t_0)$.

(v) $c_{\mathcal{N}}(I) = \bar{c}(I)$.

In fact, since $\eta\big|_{\mathcal{S}}$ is a homeomorphism, we have

$$\bar{c}(I) = \inf_{u \in H_V^s\setminus\{0\}} I(\tau(u)u) = \inf_{u\in\mathcal{S}} I(\tau(u)u) = c_{\mathcal{N}}(I).$$

(vi) $\bar{c}(I) = c(I)$.

Given $u \in H_V^s\setminus\{0\}$, define the path $\zeta(t) = tt_0u$, where $t_0 > 0$ is chosen in such way that $I(t_0u) < 0$. Then, by Remark 3.3.4–(iv), it is easy to see that $\zeta \in \Gamma_I$ and

$$\max_{t\geqslant 0} I(tu) = \max_{t\geqslant 0} I(\zeta(t)) \geqslant c(I).$$

Consequently $c(I) \leqslant \bar{c}(I)$. On the other hand, given $\zeta \in \Gamma_I$, we know about the existence of t_0 such that $\zeta(t_0)$ belongs to $\mathcal{N}$. Thus,

$$\max_{t\geqslant 0} I(\zeta(t)) \geqslant I(\zeta(t_0)) \geqslant c_{\mathcal{N}}(I) = \bar{c}(I).$$

Since $\zeta \in \Gamma_I$ is arbitrary, we conclude $c(I) \geqslant \bar{c}(I)$. ∎

Remark 3.7.1. In this remark we illustrate how one can apply Theorem 3.2.2. Assume that $a(x) = a(|x|)$ and $f(x,t) = f(|x|,t)$ are radial. Let E be the space defined as the completion of $C_{0,\mathrm{rad}}^\infty(\mathbb{R}^N)$ with respect to the norm $\|\cdot\|_V$. In view of Proposition 3.3.1 it is clear that E is a closed subspace of $H_{\mathrm{rad}}^s(\mathbb{R}^N)$, thus we consider $I_0 = I\big|_{H_{\mathrm{rad}}^s(\mathbb{R}^N)}$ as the energy functional associated with $(\mathcal{H}_s)$ under the considerate settings. By Remark 3.3.4–(iv), it is also clear that Proposition 3.3.5 holds true in this case, that is, there is a sequence (u_k) in E such that $I_0(u_k) \to c(I_0)$ and $I_0'(u_k) \to 0$. Moreover by Corollary 1.4.2 the sequence (u_k) has a convergent subsequence $u_k \to u$ in E. Thus, applying Proposition 3.4.1, we see that u is a radial ground state solution for $(\mathcal{H}_s)$ in E. Moreover, as a consequence of the Principle of Symmetric Criticality, we have that u is a critical point of I.

Remark 3.7.2. In view of Remark 3.2.7, if $b(x) \equiv 0$, then the radial ground state solution u obtained above can be considered as being non-negative.

3.8 Proof of Theorem 3.2.3

Before the proof of Theorem 3.2.3, for the sake of discussion, we are going to compare the minimax level of limit functionals $I_{\mathcal{P}}$ and I_∞ with the minimax level of the energy functional I associated with Eq. $(\mathcal{H}_s)$. Some arguments used to prove this result of comparison are used in the proof of Theorem 3.2.3.

Proposition 3.8.1. *Assume that $f(x,t)$ satisfies either (f_1)–(f_3) or (f_3)–(f_6); and additionally (f_7). Moreover, suppose that $a(x)$ and $f(x,t)$ satisfies either one of the following conditions,*

(i) $b(x) \equiv 0$, (V_1)–(V_2), (f_8);

(ii) $V(x) \geqslant 0$, $b(x)$ *has compact support,* (V_2)–(V_4), (f_9);

Then $c(I) \leqslant c(I_{\mathcal{P}})$ and $c(I) \leqslant c(I_\infty)$, respectively. Moreover, under these conditions, if we assume $(\mathscr{H})$, then (f_{10}) and (f_{10}') holds true respectively for each considered case.

Proof. (i). Let $u \in H^s_V(\mathbb{R}^N)$ be a non-negative (see Remark 3.2.7) non-trivial weak solution for the equation

$$(-\Delta)^s u + V(x)u = f_{\mathcal{P}}(x,u),$$

at the mountain pass level for $I_{\mathcal{P}}$, that is, $I_{\mathcal{P}}(u) = c(I_{\mathcal{P}})$. For each k, we define the path

$$\zeta_k(t) = tu(\cdot - y_k), \quad t \geqslant 0.$$

where $(y_k) \subset \mathbb{Z}^N$ is taken such that $|y_k| \to \infty$. The idea is to prove that

$$c(I) \leqslant \lim_{k\to\infty} \max_{t\geqslant 0} I(\zeta_k(t)) \leqslant \max_{t\geqslant 0} I_{\mathcal{P}}(tu) = c(I_{\mathcal{P}}). \tag{3.8.1}$$

In fact, taking into account that Φ and $\Phi_{\mathcal{P}}$ are locally Lipschitz in $H^s_V(\mathbb{R}^N)$ (they are C^1 in $H^s_V(\mathbb{R}^N)$) and the following estimate

$$|I(\zeta_k(t)) - I_{\mathcal{P}}(tu)| \leqslant \int_{\mathbb{R}^N} |F(x+y_k,tu) - F_{\mathcal{P}}(x+y_k,tu)|\,\mathrm{d}x,$$

by using a density argument we get that

$$\lim_{k\to\infty} I(\zeta_k(t)) = I_{\mathcal{P}}(tu), \quad \text{uniformly in compact sets of } \mathbb{R}.$$

Consequently we may proceed as in Proposition 2.6.1. First note that

$$\lim_{k\to\infty} \int_{\mathbb{R}^N} F(x+y_k, tu)\,\mathrm{d}x = \int_{\mathbb{R}^N} F_{\mathcal{P}}(x,tu)\,\mathrm{d}x, \quad \forall\, t>0.$$

In particular,

$$\int_{\mathbb{R}^N} F(x+y_k, u)\,\mathrm{d}x > 0, \quad \text{for } k \text{ large enough.}$$

Thus, using the uniformity in x of the considered conditions (f_1)–(f_3) or (f_3)–(f_6) and the arguments of Remark 3.3.4–(iv), we see that ζ_k belongs to Γ_I, for k large enough. As a consequence, there exist $t_k > 0$ such that

$$I(\zeta_k(t_k)) = \max_{t\geqslant 0} I(\zeta_k(t)) > 0.$$

We claim that the sequence (t_k) is bounded. In fact, suppose contrary to our claim that $t_k \to \infty$, up to subsequence. Thus, by the uniformity in x and the arguments of Remark 3.3.4–(iv), we get

$$I(\zeta_k(t_k)) = \frac{t_k^2}{2}\|u\|_V^2 - \int_{\mathbb{R}^N} F(x+y_k, t_k u)\,\mathrm{d}x \to -\infty, \text{ as } t\to\infty,$$

which leads to a contradiction with the fact that $I(\zeta_k(t_k)) > 0$ for all k. Therefore, up to subsequence, $t_k \to t_0$, and we have that

$$\lim_{k\to\infty} \max_{t\geqslant 0} I(\zeta_k(t_k)) = I_{\mathcal{P}}(t_0 u),$$

which leads to (3.8.1).

(ii). The second case is proved in a similar way. Let $w \in H^s_V(\mathbb{R}^N) = H^s(\mathbb{R}^N)$ be a non-trivial weak solution for the equation

$$(-\Delta)^s w + V_\infty w = f_\infty(w),$$

at the mountain pass level, more precisely, $I_\infty(w) = c(I_\infty)$. For each k, define the path

$$\lambda_k(t) = w\left(\frac{\cdot - y_k}{t}\right), \quad t \geqslant 0.$$

where (y_k) is chosen in a such way that $|y_k| \to \infty$. As before, we consider the estimate

$$\begin{aligned} &|I(\lambda_k(t)) - I_\infty(w(\cdot/t))| \\ &\qquad \leqslant \frac{1}{2} t^N \int_{\mathbb{R}^N} |(V(tx+y_k) - b(tx+y_k)) - V_\infty|\, w^2\,\mathrm{d}x \\ &\qquad\qquad + t^N \int_{\mathbb{R}^N} |F(tx+y_k, w) - F_\infty(w)|\,\mathrm{d}x, \end{aligned}$$

and the fact that the following functionals,

$$\Phi,\ \Phi_\infty,\ Q(u)=\int_{\mathbb{R}^N} V(x)u^2\,\mathrm{d}x \quad \text{and} \quad B(u)=\int_{\mathbb{R}^N} b(x)u^2\,\mathrm{d}x,$$

are locally Lipschitz in $H^s(\mathbb{R}^N)$ to obtain, by a density argument, that

$$\lim_{k\to\infty} I(\lambda_k(t)) = I_\infty(w(\cdot/t)), \quad \text{uniformly in compact sets of } \mathbb{R}.$$

We also have that the path λ_k belongs to Γ_I, for k large enough. In fact, assuming the contrary, we would obtain k_0 and a sequence $t_n\to\infty$ such that $I(\lambda_{k_0}(t_n))>0$, for all n. On the other hand, we have that

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} F(t_nx+y_{k_0},w)-\frac{1}{2}\left[V(t_nx+y_{k_0})-b(t_nx+y_{k_0})\right]w^2\,\mathrm{d}x$$
$$=\int_{\mathbb{R}^N} F_\infty(w)-\frac{1}{2}V_\infty w^2\,\mathrm{d}x,$$

which, by taking n large enough, leads to the contradiction $I(\lambda_{k_0}(t_n))<0$. Let $t_k>0$ such that

$$I(\lambda_k(t_k)) = \max_{t\geqslant 0} I(\lambda_k(t))>0.$$

Once again we claim that the sequence (t_k) is bounded. On the contrary, there is a subsequence (t_{n_k}) that implies in the following contradiction

$$\begin{aligned}0&<I(\lambda_k(t_{n_k}))\\ &=\frac{1}{2}t_{n_k}^{N-2s}[w]_s^2 - t_{n_k}^N\left[\int_{\mathbb{R}^N} F(t_{n_k}x+y_k,w)-\frac{1}{2}(V(t_{n_k}x+y_k)-b(t_{n_k}x+y_k)w^2\,\mathrm{d}x\right]\\ &\to -\infty,\ \text{as } k\to\infty.\end{aligned}$$

Thus, up to subsequence, $t_k\to t_0$ and we obtain that

$$\lim_{k\to\infty}\max_{t\geqslant 0} I(\lambda_k(t)) = I_\infty(w(\cdot/t_0)).$$

As a consequence we conclude that

$$c(I)\leqslant \lim_{k\to\infty}\max_{t\geqslant 0} I(\lambda_k(t)) \leqslant \max_{t\geqslant 0} I_\infty(w(\cdot/t)) = c(I_\infty),$$

where we have used Corollary 3.5.3 to induce that $t=1$ is the unique critical point of $I_\infty(w(\cdot/t))$.

Now assume $(\mathscr{H})$. Considering the above discussion, for each case, we have

$$\begin{cases} c(I)\leqslant \max_{t\geqslant 0} I(\zeta_k(t)) = I(t_ku(\cdot-y_k)) < I_{\mathcal{P}}(t_ku)\leqslant \max_{t\geqslant 0} I_{\mathcal{P}}(tu) = c(I_{\mathcal{P}}),\\ c(I)\leqslant \max_{t\geqslant 0} I(\lambda_k(t)) = I(u((\cdot-y_k)/t_k)) < I_\infty(u(\cdot/t_k))\leqslant \max_{t\geqslant 0} I_\infty(u(\cdot/t)) = c(I_\infty),\end{cases}$$

where k is taken large enough. ■

Similarly as it is made in Chapter 2, to prove our existence result without the compactness condition (f_{10}) and (f_{10}'), we use a similar argument as made in [33, proof of Theorem 1.2]. Thus we use Theorem 0.6.4 (see Remark 3.3.4–(i)).

Proof of Theorem 3.2.3 completed. From Lemma 3.3.3 and Proposition 3.3.5 we know about the existence of a bounded sequence (u_k) such that $I(u_k) \to c(I)$ and $I'(u_k) \to 0$, in both considered cases. Let be the sequences $(w^{(n)})$ and $(y_k^{(n)})$ provided by the Theorem 1.1.2 for the sequence (u_k). The underlying main idea to proof the concentration-compactness of Theorem 3.2.3 follows the same one of Theorem 2.2.4 and is the following: we prove that $w^{(n)} = 0$ for all $n \geqslant 2$, which by assertions (1.1.5), (1.1.8) and Proposition 3.4.1 implies that $u_k \to w^{(1)}$ in $H_V^s(\mathbb{R}^N)$, up to subsequence. In order to prove that, we argue by contradiction and assume the existence of at least one $w^{(n_0)} \neq 0$, $n_0 \geqslant 2$.

(i) In view of Remark 3.3.2–(ii), by Proposition 3.4.2 and estimate (1.1.7), up to subsequence, we have

$$c(I) = \lim_{k\to\infty} \left[\frac{1}{2}\|u_k\|_V^2 - \int_{\mathbb{R}^N} F(x, u_k)\,\mathrm{d}x\right] \geqslant I(w^{(1)}) + \sum_{n\in\mathbb{N}_0, n>1} I_{\mathcal{P}}(w^{(n)}), \tag{3.8.2}$$

where each term of the right-hand side of (3.8.2) is non-negative. In fact, following as in the proof of Theorem 3.2.1 we notice that $w^{(1)}$ and $w^{(n)}$, $n \geqslant 2$, are critical points for I and $I_{\mathcal{P}}$, respectively. In view of that, it is clear that (f_2) or (f_5) implies that $I(w^{(1)}) \geqslant 0$ and $I_{\mathcal{P}}(w^{(n)}) \geqslant 0$, $n \geqslant 2$, respectively. On the other hand, Remark 3.3.4–(iv) guarantees that the path $\zeta^{(n_0)}(t) = tw^{(n_0)}$ belongs to $\Gamma_{I_{\mathcal{P}}}$ and $c(I_{\mathcal{P}}) \leqslant I_{\mathcal{P}}(w^{(n_0)})$. This, together with (3.8.2) and (f_{10}) leads to a contradiction.

(ii) Following the proof of Theorem 1.1.2 it is clear that we can replace $\|\cdot\|$ by the equivalent norm $\|\cdot\|_{V_\infty}$ in assertions (1.1.5)–(1.1.8). Consequently, by estimate (1.1.7), Propositions 3.4.2 and 3.4.5, up to subsequence, we have

$$\begin{aligned} c(I) &= \lim_{k\to\infty} \left[\frac{1}{2}\|u_k\|_V^2 - \int_{\mathbb{R}^N} b(x)u_k^2\,\mathrm{d}x - \int_{\mathbb{R}^N} F(x, u_k)\,\mathrm{d}x\right] \\ &\geqslant I(w^{(1)}) + \sum_{n\in\mathbb{N}_0, n>1} I_\infty(w^{(n)}). \end{aligned} \tag{3.8.3}$$

Thus, it suffices to prove that the right-hand side of (3.8.3) is non-negative and $I_\infty(w^{(n)}) \geqslant c(I_\infty)$ for all $n \geqslant 2$. In fact, in this case, we have $c(I) \geqslant I(w^{(n_0)}) \geqslant c(I_\infty)$, which leads to a contradiction with (f_{10}). To do this, we prove that $w^{(1)}$ and $w^{(n)}$, $n \geqslant 2$, are critical points for I and I_∞, respectively. Let φ in $C_0^\infty(\mathbb{R}^N)$ and $h^{(n)} \in L^{2_s^*-1}(\mathrm{supp}(\varphi))$ as in (3.6.2). By (V_4) and (1.1.6), there exists $k_0 = k_0(\varphi)$ such that

$$V(x + y_k^{(n)}) < 1 + V_\infty, \quad \forall\, k > k_0, \quad x \in \mathrm{supp}(\varphi) \quad \text{and} \quad n \geqslant 2.$$

Thus,

$$\begin{cases} |V(x+y_k^{(n)})u_k(x+y_k^{(n)})\varphi(x)| \leqslant (\varepsilon + V_\infty)h^{(n)}(x)|\varphi(x)| \in L^1(\mathrm{supp}(\varphi)), \text{ for } k > k_0, \\ V(x+y_k^{(n)})u_k(x+y_k^{(n)})\varphi(x) \to V_\infty w^{(n)}(x)\varphi(x) \quad \text{a.e. in } \mathbb{R}^N. \end{cases}$$

This allow us to use Dominated Convergence Theorem to obtain

$$\begin{aligned} \lim_{k\to\infty}(u_k, \varphi(\cdot - y_k^{(n)}))_V &= \lim_{k\to\infty}\left[[u_k(\cdot + y_k^{(n)}), \varphi]_s + \int_{\mathbb{R}^N} V(x+y_k^{(n)})u_k(\cdot + y_k^{(n)})\varphi(x)\,\mathrm{d}x\right] \\ &= [w^{(n)}, \varphi]_s + \int_{\mathbb{R}^N} V_\infty w^{(n)}(x)\varphi(x)\,\mathrm{d}x. \end{aligned}$$

And for the same reason,

$$\lim_{k\to\infty}\int_{\mathbb{R}^N} f(x+y_k^{(n)}, u_k(\cdot + y_k^{(n)}))\varphi\,\mathrm{d}x = \int_{\mathbb{R}^N} f_\infty(w^{(n)})\varphi\,\mathrm{d}x.$$

Consequently, taking the limit in

$$I'(u_k)\cdot\varphi(\cdot - y_k^{(n)}) = (u_k, \varphi(\cdot - y_k^{(n)}))_V - \int_{\mathbb{R}^N} f(x+y_k^{(n)}, u_k(\cdot + y_k^{(n)}))\varphi\,\mathrm{d}x,$$

we deduce that $I'(w^{(1)}) = 0$ and $I'_\infty(w^{(n)}) = 0$, $n \geqslant 2$. Using (f_2) or (f_5) we also get that $I(w^{(1)}) \geqslant 0$ and $I_\infty(w^{(n)}) \geqslant 0$, $n \geqslant 2$. Finally, define the path $\lambda^{(n_0)}(t) = w^{(n_0)}(\cdot/t)$, $t \geqslant 0$. By Corollary 3.5.3 we have that

$$I_\infty(\lambda^{(n_0)}(t)) = \frac{1}{2}t^{N-2s}[w^{(n_0)}]_s^2 - t^N\left[\int_{\mathbb{R}^N} F_\infty(w^{(n_0)}) - \frac{V_\infty}{2}|w^{(n_0)}|^2\,\mathrm{d}x\right] \to -\infty, \text{ as } t \to \infty,$$

which, by Remark 3.3.2, allow us to conclude that $\lambda^{(n_0)}$ belongs to Γ_{I_∞}. Corollary 3.5.3 also implies that $t = 1$ is the unique critical point of $I_\infty(\lambda^{(n_0)}(t))$. Consequently,

$$c(I_\infty) < \max_{t\geqslant 0} I_\infty(\lambda^{(n_0)}(t)) = I_\infty(w^{(n_0)}),$$

which implies the aforementioned contradiction.

(iii) Finally, assume condition (3.2.3) instead of (f_{10}) and (f'_{10}). Consider the existence of $w^{(n_0)} \neq 0$, $n_0 \in \mathbb{N}_0$, and the paths $\zeta^{(n_0)}$ and $\lambda^{(n_0)}$ as above. Taking into account the above discussion, by estimates (3.8.2) and (3.8.3), for each case we have

$$\begin{cases} c(I) \leqslant \max_{t\geqslant 0} I(\zeta^{(n_0)}(t)) \leqslant \max_{t\geqslant 0} I_{\mathcal{P}}(\zeta^{(n_0)}(t)) = I_{\mathcal{P}}(w^{(n_0)}) \leqslant c(I), \\ c(I) \leqslant \max_{t\geqslant 0} I(\lambda^{(n_0)}(t)) \leqslant \max_{t\geqslant 0} I_\infty(\lambda^{(n_0)}(t)) = I_\infty(w^{(n_0)}) \leqslant c(I), \end{cases}$$

where we have used condition (3.2.3) to ensure that the paths $\zeta^{(n_0)}$ and $\lambda^{(n_0)}$ belongs to Γ_I. Thus, we have that the minimax level $c(I)$ is attained and we can apply Theorem 0.6.4 to obtain the existence of a critical point u for I_λ with $I_\lambda(u) = c(I_\lambda)$. If there is no $w^{(n)} \neq 0$, $n \in \mathbb{N}_0$, (which is the case where strict inequalities occurs) we can argue as above and obtain that $u_k \to w^{(1)}$, up to subsequence. ■

3.9 Proof of Theorem 3.2.4

Proof. The proof will be divided into three steps. Our argument follows the proof of Theorem 3.2.3 and [27, Theorem 5.2]. We first assume the case where $V(x)$ and $f(x,t)$ satisfies $(\mathscr{H}^*)$.

(i) Arguing in a similar way as in the proof of Lemma 3.3.3, we see that the functional I_* has the mountain pass geometry, which guarantees the existence of a sequence (u_k) in $\mathcal{D}^{s,2}(\mathbb{R}^N)$ such that $I_*(u_k) \to c(I_*) > 0$ and $I'_*(u_k) \to 0$. Let $(w^{(n)})$, $(y_k^{(n)})$, $(j_k^{(n)})$ the sequences provided by Theorem 1.1.1 and define the set

$$\mathbb{N}_\sharp = \left\{ n \in \mathbb{N}_* \backslash \{1\} : |\gamma^{j_k^{(n)}} y_k^{(n)}| \text{ is bounded} \right\}.$$

Passing to a subsequence and using a diagonal argument if necessary, we may assume that each sequence $(\gamma^{j_k^{(n)}} y_k^{(n)})$, $n \in \mathbb{N}_\sharp$, is convergent and we denote

$$a^{(n)} = \lim_{k\to\infty} \gamma^{j_k^{(n)}} y_k^{(n)}, \quad n \in \mathbb{N}_\sharp.$$

(ii) Now we shall prove the following estimate,

$$\begin{aligned}\limsup_k \|u_k\|_V^2 \geqslant \|w^{(1)}\|_V^2 &+ \sum_{n\in\mathbb{N}_*\backslash\mathbb{N}_\sharp} [w^{(n)}]_s^2 \\ &+ \sum_{n\in\mathbb{N}_+\cap\mathbb{N}_\sharp} \|w^{(n)}\|^2_{V_+(\cdot+a^{(n)}-a_*)} + \sum_{n\in\mathbb{N}_-\cap\mathbb{N}_\sharp} \|w^{(n)}\|^2_{V_-(\cdot+a^{(n)}-a_*)}, \end{aligned} \tag{3.9.1}$$

passing to a subsequence of (u_k) if necessary. For each $n \in \mathbb{N}_*$, let $(\varphi_j^{(n)})$ in $C_0^\infty(\mathbb{R}^N)$ such that $\varphi_j^{(n)} \to w^{(n)}$ in $\mathcal{D}^{s,2}(\mathbb{R}^N)$. Evaluating

$$\left\| u_k - \sum_{n\in M_*} d_k^{(n)} \varphi_j^{(n)} \right\|_V^2 \geqslant 0,$$

in a finite subset $M_* = \{1, \ldots, M\}$ of $\mathbb{N}_*$, we have

$$\|u_k\|_V^2 \geqslant 2 \sum_{n\in M_*} (u_k, d_k^{(n)}\varphi_j^{(n)})_V - \sum_{n\in M_*} \|d_k^{(n)}\varphi_j^{(n)}\|_V^2. \tag{3.9.2}$$

We are now going to study the limit in inequality (3.9.2). Let

$$v_k^{(n)} := d_k^{(n)} u_k = \gamma^{-\frac{N-2s}{2} j_k^{(n)}} u_k(\gamma^{-j_k^{(n)}} \cdot + y_k^{(n)}).$$

Notice that

$$\begin{aligned}(u_k, d_k^{(n)}\varphi_j^{(n)})_V &= [v_k^{(n)}, \varphi_j^{(n)}]_s \\ &+ \int_{\mathbb{R}^N} \gamma^{-2sj_k^{(n)}} V(\gamma^{-j_k^{(n)}}((x + y_k^{(n)}) + a_*)) v_k^{(n)}(\cdot + a_*)\varphi_j^{(n)}(\cdot + a_*)\, dx,\end{aligned}$$

and

$$\|d_k^{(n)}\varphi_j^{(n)}\|_V^2 = [\varphi_j^{(n)}]_s^2 + \int_{\mathbb{R}^N} \gamma^{-2sj_k^{(n)}} V(\gamma^{-j_k^{(n)}}((x+y_k^{(n)})+a_*))|\varphi_j^{(n)}(\cdot + a_*)|^2 \,\mathrm{d}x.$$

Fixed j, we can use condition (V_3^*) to conclude, up to a subsequence that

$$\lim_{k\to\infty}(u_k, d_k^{(n)}\varphi_j^{(n)})_V = [w^{(n)}, \varphi_j^{(n)}]_s \quad \text{and} \quad \lim_{k\to\infty}\|d_k^{(n)}\varphi_j^{(n)}\|_V^2 = [\varphi_j^{(n)}]_s^2, \tag{3.9.3}$$

provided that $n \notin \mathbb{N}_\sharp$ (this is the case when $n \in \mathbb{N}_0$). Similarly, up to a subsequence, by assumption (V_2^*) we have

$$\lim_{k\to\infty}(u_k, d_k^{(n)}\varphi_j^{(n)})_V = (w^{(n)}, \varphi_j^{(n)})_{V_\kappa(\cdot+a^{(n)}-a_*)}$$
$$\text{and} \quad \lim_{k\to\infty}\|d_k^{(n)}\varphi_j^{(n)}\|_V^2 = \|\varphi_j^{(n)}\|^2_{V_\kappa(\cdot+a^{(n)}-a_*)}, \tag{3.9.4}$$

where $\kappa = +, -$, whenever $n \in \mathbb{N}_+ \cap \mathbb{N}_\sharp$ or $\mathbb{N}_- \cap \mathbb{N}_\sharp$, respectively. Since

$$\mathbb{N}_*\backslash\{1\} = (\mathbb{N}_*\backslash\mathbb{N}_\sharp)\,\dot\cup\,[(\mathbb{N}_+ \cap \mathbb{N}_\sharp)\dot\cup(\mathbb{N}_- \cap \mathbb{N}_\sharp)]\,,$$

up to subsequence, we can apply the limits (3.9.3) and (3.9.4) in inequality (3.9.2) to get

$$\begin{aligned}\limsup_k \|u_k\|_V^2 \geqslant \|w^{(1)}\|_V^2 &+ \sum_{n\in M_*\cap\mathbb{N}_+\cap\mathbb{N}_\sharp} 2(w^{(n)},\varphi_j^{(n)})_{V_+(\cdot+a^{(n)}-a_*)} - \|\varphi_j^{(n)}\|^2_{V_+(\cdot+a^{(n)}-a_*)}\\ &+ \sum_{n\in M_*\cap\mathbb{N}_-\cap\mathbb{N}_\sharp} 2(w^{(n)},\varphi_j^{(n)})_{V_-(\cdot+a^{(n)}-a_*)} - \|\varphi_j^{(n)}\|^2_{V_-(\cdot+a^{(n)}-a_*)}\\ &+ \sum_{n\in M_*\backslash\mathbb{N}_\sharp} 2[w^{(n)},\varphi_j^{(n)}]_s - [\varphi_j^{(n)}]_s^2.\end{aligned} \tag{3.9.5}$$

Since the norms $\|\cdot\|_{V_+}$ and $\|\cdot\|_{V_-}$ are equivalent to the norm $[\,\cdot\,]_s$ in $\mathcal{D}^{s,2}(\mathbb{R}^N)$ we can take the limit in j in inequality (3.9.5) and use the arbitrariness of choice for M to obtain (3.9.1).

(iii) If $w^{(n)} = 0$ for all $n \geqslant 2$, then $u_k \to w^{(1)}$ in $\mathcal{D}^{s,2}(\mathbb{R}^N)$, with $w^{(1)}$ being a critical point of I_*. Let us argue by contradiction and assume the existence of $w^{(n_0)} \neq 0$, with $n_0 \geqslant 2$. By Proposition 2.4.1 and estimate (3.9.1), up to subsequence, we have that

$$c(I_*) \geqslant I_*(w^{(1)}) + \sum_{n\in\mathbb{N}_*\backslash\mathbb{N}_\sharp} I_0(w^{(n)}) + \sum_{n\in\mathbb{N}_+\cap\mathbb{N}_\sharp} I_+^{(n)}(w^{(n)}) + \sum_{n\in\mathbb{N}_-\cap\mathbb{N}_\sharp} I_-^{(n)}(w^{(n)}), \tag{3.9.6}$$

where

$$I_\pm^{(n)}(u) = \frac{1}{2}\|u\|^2_{V_\pm(\cdot+a^{(n)}-a_*)} - \int_{\mathbb{R}^N} F_\pm(u)\,\mathrm{d}x$$
$$\text{and} \quad I_0(u) = \frac{1}{2}[u]_s^2 - \int_{\mathbb{R}^N} F_0(u)\,\mathrm{d}x, \quad u \in \mathcal{D}^{s,2}(\mathbb{R}^N).$$

As before, we prove that each $w^{(n)}$ is a critical point for the functionals in the respective index of the sums in (3.9.6), and as a consequence of (f_2), the right-hand side of (3.9.6) is non-negative. In the next step we obtain that $c(I_*) < I_\kappa^{(n)}(w^{(n)})$ in the correspondent index, which leads to a contradiction with estimate (3.9.6). In fact, given φ in $C_0^\infty(\mathbb{R}^N)$, by reasoning as in the proof of (3.9.1), we get that

$$\lim_{k\to\infty}(u_k, d_k^{(n)}\varphi)_V = [w^{(n)}, \varphi]_s \quad \text{and} \quad \lim_{k\to\infty}(u_k, d_k^{(n)}\varphi)_V = (w^{(n)}, \varphi)_{V_\pm(\cdot + a^{(n)} - a_*)},$$

provided $n \in \mathbb{N}_*\backslash\mathbb{N}_\sharp$ and $n \in \mathbb{N}_\pm \cap \mathbb{N}_\sharp$, respectively. Since,

$$\left|\gamma^{-\frac{N+2s}{2}j_k^{(n)}} f\left(\gamma^{-j_k^{(n)}}x + y_k^{(n)}, \gamma^{\frac{N-2s}{2}j_k^{(n)}}t\right)\varphi\right| \leqslant C|t|^{2_s^*-1}, \text{ a.e. } x \in \mathbb{R}^N, \ \forall\, k, n \text{ and } t,$$

thanks to the Dominated Convergence Theorem, up to a subsequence, we may pass the limit in k in the following identity

$$I_*'(u_k)\cdot(d_k^{(n)}\varphi) = (v_k^{(n)}, \varphi)_V - \int_{\mathbb{R}^N} \gamma^{-\frac{N+2s}{2}j_k^{(n)}} f\left(\gamma^{-j_k^{(n)}}x + y_k^{(n)}, \gamma^{\frac{N-2s}{2}j_k^{(n)}}v_k^{(n)}\right)\varphi\,\mathrm{d}x,$$

to conclude that $I_*'(w^{(1)}) = (I_\pm^{(n)})'(w^{(n)}) = I_0'(w^{(n)}) = 0$, in the corresponding index.

(iv) To conclude the proof, we prove now that $c(I_*) < I_\pm^{(n_0)}(w^{(n_0)})$ or $c(I_*) < I_\pm^{(n_0)}(w^{(n_0)})$, which depends on the sets $\mathbb{N}_*\backslash\mathbb{N}_\sharp$ or $\mathbb{N}_\pm \cap \mathbb{N}_\sharp$ that n_0 may belong. Define the path

$$\begin{cases} \zeta^{(n_0)}(t) = tw^{(n_0)}, \quad t \geqslant 0, \quad \text{if} \quad n_0 \in \mathbb{N}_*\backslash\mathbb{N}_\sharp. \\ \zeta^{(n_0)}(t) = tw^{(n_0)}(\cdot + a_* - a^{(n)}), \quad t \geqslant 0, \quad \text{if} \quad n_0 \in \mathbb{N}_\pm \cap \mathbb{N}_\sharp. \end{cases}$$

By condition $(\mathscr{H}^*)$ and Remark 3.3.4–(iv) we have that $\zeta^{(n_0)}$ belongs to Γ_I with

$$\begin{cases} c(I_*) \leqslant \max\limits_{t\geqslant 0} I_*(\zeta^{(n_0)}(t)) < I_0(\zeta^{(n_0)}(\bar{t})) \leqslant \max\limits_{t\geqslant 0} I_0(\zeta^{(n_0)}(t)) = I_0(w^{(n_0)}), \text{ if } n_0 \in \mathbb{N}_*\backslash\mathbb{N}_\sharp. \\ c(I_*) \leqslant \max\limits_{t\geqslant 0} I_*(\zeta^{(n_0)}(t)) < I_\pm^{(n)}(\zeta^{(n_0)}(\bar{t})) \\ \qquad\qquad \leqslant \max\limits_{t\geqslant 0} I_\pm^{(n)}(\zeta^{(n_0)}(t)) = I_\pm^{(n)}(w^{(n_0)}), \text{ if } n_0 \in \mathbb{N}_\pm \cap \mathbb{N}_\sharp, \end{cases}$$

where $\bar{t}$ is the maximum of $I_*(\zeta^{(n_0)}(t))$. This together with the estimate (3.9.6) leads to a aforementioned contradiction.

(v) We now assume only conditions (3.1.3) and (3.1.4), instead of $(\mathscr{H}^*)$. Arguing in a similar way as in the proof of Theorem 3.2.3 (iii), we get that

$$u_k \to w^{(1)} \text{ in a subsequence} \quad \text{or} \quad c(I_*) = \max_{t\geqslant} I_*(\zeta^{(n_0)}(t)).$$

If the minimax level $c(I_*)$ is attained then we apply Theorem 0.6.4 to obtain the existence of a critical point $u \in \zeta^{(n_0)}([0,\infty))$ such that $I_*(u) = c(I_*)$. ■

Chapter 4

Existence and non-existence results for a class of nonlocal Schrödinger-Poisson systems with critical growth

In this chapter, we are concerned with the existence of non-trivial weak and ground state solutions for the following class of nonlinear fractional Schrödinger–Poisson System

$$\begin{cases} (-\Delta)^s u + a(x)u + \lambda K(x)\phi u = f(x,u) + g(x,u) & \text{in} \quad \mathbb{R}^3, \\ (-\Delta)^\alpha \phi = K(x)u^2 & \text{in} \quad \mathbb{R}^3, \end{cases} \tag{$\mathcal{SP}$}$$

where the nonlinearities $f(x,t)$ and $g(x,t)$ has oscillatory subcritical and critical growth respectively, $a(x)$ is not necessarily bounded away from zero and $K(x) \geqslant 0$ belongs to a suitable Lebesgue space. Here we follow the ideas developed in the previous chapters.

Outline. The chapter is organized as follows. In Sect. 4.3, we provide a suitable variational settings to prove our main results, more precisely, we describe the limit under the profile decomposition of the Palais-Smale sequence at the mountain pass level of the energy functional associated with ($\mathcal{SP}$) and we prove the aforementioned Pohozaev type identity. Moreover, we estimate the minimax level for the functional associated with ($\mathcal{SP}$). In Sect. 4.6 we study the behavior of the minimax levels of the considerate functionals. Sections 4.4, 4.5, 4.7, 4.8 and 4.9 are dedicated to the proof of Theorems 4.2.1, 4.2.2, 4.2.3, 4.2.4 and 4.2.5 respectively.

4.1 Hypothesis

In order to describe our results in a more precisely way, next we state the main assumptions on the weight $K(x)$, the potential $a(x)$ and the nonlinearities $f(x,t)$ and $g(x,t)$ respectively. We always assume $\lambda > 0$, $0 < s < 1$, $0 < \alpha < 1$ and $2\alpha + 4s > 3$.

- Assumptions on $K(x)$.

(K_1) $0 \leqslant K(x) \in L^r(\mathbb{R}^3) \cup L^\infty(\mathbb{R}^3)$, for some $r > 6/(2\alpha + 4s - 3)$.

(K_2) There exists $K_{\mathcal{P}}(x) \in L^r(\mathbb{R}^3) \cup L^\infty(\mathbb{R}^3)$, $1-$periodic in x_i, $i = 1, 2, 3$, in a such way that $\lim_{|x|\to\infty} |K(x) - K_{\mathcal{P}}(x)| = 0$.

(K_3) $K(x) - K_{\mathcal{P}}(x) \in L^\infty_{\text{loc}}(\mathbb{R}^3)$ and $K_{\mathcal{P}}(x)$ is continuous at 0.

- Assumptions on $a(x) = V(x) - b(x)$.

(V_1) $V(x) \in L^\sigma_{\text{loc}}(\mathbb{R}^3)$, for some $\sigma > 3/2s$.

(V_2) The following infimum

$$\mathcal{C}_V = \inf_{u \in C_0^\infty(\mathbb{R}^3), \|u\|_2 = 1} \int_{\mathbb{R}^3} |(-\Delta)^{s/2} u|^2 + V(x) u^2 \, \mathrm{d}x$$

is positive and $V(x) \geqslant -\mathcal{B}$ a. e. $x \in \mathbb{R}^3$, for some $\mathcal{B} > 0$.

(V_3) There exists $V_{\mathcal{P}}(x) \in L^{\sigma_{\mathcal{P}}}_{\text{loc}}(\mathbb{R}^3)$, $\sigma_{\mathcal{P}} > 3/2s$, $1-$periodic in x_i, $i = 1, 2, 3$, that satisfies (V_2) such that $\lim_{|x|\to\infty} |V(x) - V_{\mathcal{P}}(x)| = 0$.

(V_4) $0 \leqslant b(x) \in L^\beta(\mathbb{R}^3)$, for some $\beta > 3/2s$, and $\|b(x)\|_\beta < \mathcal{C}_V^{(\beta)}$, where

$$\mathcal{C}_V^{(\beta)} = \inf_{u \in H_V^s(\mathbb{R}^3), \|u\|_{2\beta'} = 1} \int_{\mathbb{R}^3} |(-\Delta)^{s/2} u|^2 + V(x) u^2 \, \mathrm{d}x, \quad \beta' = \beta/(\beta - 1).$$

- Assumptions on $f(x,t)$.

(f_1) $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function. Moreover, for every $\varepsilon > 0$ there exists $p_\varepsilon \in (2, 2_s^*)$ and $C_\varepsilon > 0$ such that

$$|f(x,t)| \leqslant \varepsilon(|t| + |t|^{2_s^* - 1}) + C_\varepsilon |t|^{p_\varepsilon - 1}, \quad \text{a.e. } x \in \mathbb{R}^3 \text{ and } \forall\, t \in \mathbb{R},$$

where $2_s^* = 6/(3 - 2s)$.

(f_2) There exists $\mu > 2$ such that,

$$\mu F(x,t) := \mu \int_0^t f(x,\tau)\,\mathrm{d}\tau \leqslant f(x,t)t \quad \text{a.e. } x \in \mathbb{R}^3 \text{ and } \forall\, t \in \mathbb{R}.$$

(f_3) There exists $R > 0$, $t_0 > 0$, $x_0 \in \mathbb{R}^3$ such that

$$|B_R| \inf_{B_R(x_0)} F(x,t_0) + |B_{R+1} \backslash B_R| \inf_{(x,t) \in (B_{R+1}(x_0) \backslash B_R(x_0)) \times [0,t_0]} F(x,t) > 0.$$

In the autonomous case, we consider the following variant of assumption (f_3).

(f_3') There exists $t_0 > 0$ such that $F(t_0) > 0$.

(f_4) The following limit are uniform in x,

$$\lim_{|t| \to \infty} \frac{F(x,t)}{t^4} = \infty.$$

(f_5) There exists a function $f_{\mathcal{P}}(x,t)$ that is $1-$periodic in x_i, $i = 1,2,3.$ such that the following limit exists and is uniformly convergent in compact sets in t

$$\lim_{|x| \to \infty} |f(x,t) - f_{\mathcal{P}}(x,t)| = 0.$$

Moreover, $f_{\mathcal{P}}(x,t)$ satisfies (f_1) and either (f_2), (f_3) with $\mu > 4$ or (f_4).

(f_6) For a.e. $x \in \mathbb{R}^3$ the function

$$t \mapsto \frac{f_{\mathcal{P}}(x,t)}{|t|} \quad \text{is strict increasing in } \mathbb{R}.$$

(f_7) There exists $c_0 > 0$ and $4 < p_0 < 2_s^*$ such that

$$F_{\mathcal{P}}(x,t) \geqslant c_0 |t|^{p_0}, \quad \text{a.e. } x \in \mathbb{R}^3 \text{ and } \forall\, t \in \mathbb{R}.$$

Notice that in condition (f_7) it is implicit that $s > 3/4$.

- Assumptions on $g(x,t)$.

(g_1) $g : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ satisfies the Carathéodory conditions. Moreover, there exists a positive constant $\mathcal{C}_*$ such that

$$|g(x,t)| \leqslant \mathcal{C}_* |t|^{2_s^*-1}, \quad \text{a.e. } x \in \mathbb{R}^3 \text{ and } \forall\, t \in \mathbb{R}.$$

(g_2) There exists $\mu_* > 2$ such that,

$$0 \leqslant \mu_* G(x,t) := \mu_* \int_0^t g(x,\tau)\,\mathrm{d}\tau \leqslant g(x,t)t, \qquad \text{a.e. } x \in \mathbb{R}^3 \text{ and } \forall\, t \in \mathbb{R}.$$

(g_3) For each real numbers $a_1, \ldots, a_M$, there exist $C = C(M) > 0$ such that

$$\left| G\left(x, \sum_{n=1}^{M} a_n\right) - \sum_{n=1}^{M} G(x, a_n) \right| \leqslant C(M) \sum_{m \neq n \in \{1,\ldots,M\}} |a_n|^{2_s^*-1}|a_m|, \text{ a.e. } x \in \mathbb{R}^3.$$

(g_4) There exists $\gamma > 1$, such that the following limits exist and are uniformly convergent in x and in compact sets for t

$$\begin{aligned} g_\infty(t) &:= \lim_{|x|\to\infty} g(x,t), \\ g_+(t) &:= \lim_{j\in\mathbb{Z}, j\to+\infty} \gamma^{-\frac{3+2s}{2}j} g\left(\gamma^{-j}x, \gamma^{\frac{3-2s}{2}j}t\right). \end{aligned}$$

(g_5) The function $g_\infty(t)$ is self-similar,

$$G_\infty(t) = \gamma^{-3j} G_\infty\left(\gamma^{\frac{3-2s}{2}j}t\right), \quad \forall\, t \in \mathbb{R} \text{ and } j \in \mathbb{Z}.$$

(g_6) The function

$$t \mapsto \frac{g_\infty(t)}{|t|}, \qquad \text{is strict increasing in } \mathbb{R}.$$

(g_7) There exists $c_* > 0$ such that

$$G_\infty(t) \geqslant c_*|t|^{2_s^*}, \quad \forall\, t \in \mathbb{R}, \quad \text{where } G_\infty(t) := \int_0^t g_\infty(\tau)d\tau.$$

(g_8) $g_+(t) \in C^1(\mathbb{R})$ and there is a positive constant c_+ such that

$$G_+(t) \geqslant c_+|t|^{2_s^*}, \quad \forall\, t \in \mathbb{R}, \quad \text{where } G_+(t) := \int_0^t g_+(\tau)d\tau.$$

Moreover, $c_+ \geqslant \mathcal{C}_*$.

The functional associated with $(\mathcal{SP})$

As mentioned earlier, we compare the minimax level of the associated functional of System $(\mathcal{SP})$ and the one of the following limit problem

$$\begin{cases} (-\Delta)^s u + a_{\mathcal{P}}(x)u + \lambda K_{\mathcal{P}}(x)\phi u = f_{\mathcal{P}}(x,u) + g_\infty(u) & \text{in } \mathbb{R}^3, \\ (-\Delta)^\alpha \phi = K_{\mathcal{P}}(x)u^2 & \text{in } \mathbb{R}^3. \end{cases} \tag{4.1.1}$$

In a similar fashion, we consider $H^s_V(\mathbb{R}^3)$ as the completion of $C^\infty_0(\mathbb{R}^3)$ with respect to the norm

$$\|u\|^2_V := \int_{\mathbb{R}^3} |(-\Delta)^{s/2}u|^2 + V(x)u^2 \,\mathrm{d}x.$$

By Proposition 3.3.1 if we assume that $V(x)$ belongs to $L^1_{\text{loc}}(\mathbb{R}^3)$ and satisfies (V_2), then $H^s_V(\mathbb{R}^N)$ is well defined and we have the following continuous embedding

$$H^s_V(\mathbb{R}^3) \hookrightarrow H^s(\mathbb{R}^3). \tag{4.1.2}$$

Next, to make our discussion clearer, we define the notion of weak solution.

Definition 4.1.1. We say that a pair $(u, \phi) \in H^s_V(\mathbb{R}^3) \times \mathcal{D}^{\alpha,2}(\mathbb{R}^3)$ is a weak solution of the System $(\mathcal{SP})$ when

$$\int_{\mathbb{R}^3} (-\Delta)^{s/2}u(-\Delta)^{s/2}v + (a(x) + \lambda K(x)\phi)uv \,\mathrm{d}x = \int_{\mathbb{R}^3} (f(x,u) + g(x,u))v \,\mathrm{d}x, \text{ and}$$
$$\int_{\mathbb{R}^3} (-\Delta)^{\alpha/2}\phi(-\Delta)^{\alpha/2}v \,\mathrm{d}x = \int_{\mathbb{R}^3} K(x)u^2 v \,\mathrm{d}x,$$

for all $v \in C^\infty_0(\mathbb{R}^3)$ and the above integrals are finite.

Given $u \in H^s(\mathbb{R}^3)$ we consider the linear operator $\mathcal{P}_u : \mathcal{D}^{\alpha,2}(\mathbb{R}^3) \to \mathbb{R}$ defined as

$$\mathcal{P}_u(v) = \int_{\mathbb{R}^3} K(x)u^2 v \,\mathrm{d}x.$$

If we assume that $K(x)$ satisfies condition (K_1) then, by Holder inequality, this operator is continuous (see (4.3.1) estimate below). Thus, by Riesz Theorem, there exists a unique $\phi_\alpha[u]$ in $\mathcal{D}^{\alpha,2}(\mathbb{R}^3)$ that solves $(-\Delta)^\alpha \phi_\alpha[u] = K(x)u^2$ in the weak sense, that is,

$$\int_{\mathbb{R}^3} (-\Delta)^{\alpha/2}(\phi_\alpha[u])(-\Delta)^{\alpha/2}v \,\mathrm{d}x = \int_{\mathbb{R}^3} K(x)u^2 v \,\mathrm{d}x, \quad \forall\, v \in \mathcal{D}^{\alpha,2}(\mathbb{R}^3). \tag{4.1.3}$$

Replacing $\phi = \phi_\alpha[u]$ in the first equation of $(\mathcal{SP})$, we obtain the following nonlinear fractional Schrödinger equation with a nonlocal term,

$$(-\Delta)^s u + a(x)u + \lambda K(x)\phi_\alpha[u]u = f(x,u) + g(x,u). \tag{$\mathcal{S}_{NL}$}$$

We consider associated with Eq. $(\mathcal{S}_{NL})$, the functional $I_\lambda : H^s_V(\mathbb{R}^3) \to \mathbb{R}$ given by

$$I_\lambda(u) = \frac{1}{2}\|u\|^2_V - \frac{1}{2}\int_{\mathbb{R}^3} b(x)u^2 \,\mathrm{d}x$$
$$+ \frac{\lambda}{4}\int_{\mathbb{R}^3} K(x)\phi_\alpha[u]u^2 \,\mathrm{d}x - \int_{\mathbb{R}^3} F(x,u)\,\mathrm{d}x - \int_{\mathbb{R}^3} G(x,u)\,\mathrm{d}x.$$

Thus, if $f(x,t)$ satisfies (f_1), $a(x)$ the assumptions (V_2), (V_4) and $g(x,t)$ the assumption (g_1), then $I \in C^1(H^s_V(\mathbb{R}^3))$ (see Proposition 4.3.3–(i)). Furthermore

$$I'_\lambda(u)\cdot v = \int_{\mathbb{R}^3} (-\Delta)^{s/2}u(-\Delta)^{s/2}v + (V(x)-b(x))uv\,\mathrm{d}x + \lambda\int_{\mathbb{R}^3} K(x)\phi_\alpha[u]uv\,\mathrm{d}x - \int_{\mathbb{R}^3}(f(x,u)+g(x,u))v\,\mathrm{d}x,\ \ u,v\in H^s_V(\mathbb{R}^3).$$

Consequently critical points of I_λ correspond to weak solutions of $(\mathcal{SP})$ and conversely. Regarding the minimax level, we put

$$c(I_\lambda) = \inf_{\gamma\in\Gamma_{I_\lambda}} \sup_{t\geqslant 0} I_\lambda(\gamma(t)). \tag{4.1.4}$$

where

$$\Gamma_{I_\lambda} = \left\{\gamma\in C([0,\infty),H^s_V(\mathbb{R}^3)) : \gamma(0)=0,\ \lim_{t\to\infty} I_\lambda(\gamma(t)) = -\infty\right\}. \tag{4.1.5}$$

Proceeding in a similar way, we can use the same argument based on Riesz theorem with System (4.1.1) to obtain the following equation

$$(-\Delta)^s u + a_{\mathcal{P}}(x)u + \lambda K_{\mathcal{P}}(x)\phi^{\mathcal{P}}_\alpha[u]u = f_{\mathcal{P}}(x,u) + g_\infty(u), \tag{$\mathcal{S}^{\mathcal{P}}_{NL}$}$$

and the corresponding C^1 functional associated with $(\mathcal{S}^{\mathcal{P}}_{NL})$,

$$I^{\mathcal{P}}_\lambda(u) := \frac{1}{2}\|u\|^2_{V_{\mathcal{P}}} + \frac{\lambda}{4}\int_{\mathbb{R}^3} K_{\mathcal{P}}(x)\phi^{\mathcal{P}}_\alpha[u]u^2\,\mathrm{d}x - \int_{\mathbb{R}^3} F_{\mathcal{P}}(x,u)\,\mathrm{d}x - \int_{\mathbb{R}^3} G_\infty(u)\,\mathrm{d}x, u\in H^s_V(\mathbb{R}^3);$$

where

$$\|u\|^2_{V_{\mathcal{P}}} := \int_{\mathbb{R}^3} \left|(-\Delta)^{s/2}u\right|^2 + V_{\mathcal{P}}(x)u^2\,\mathrm{d}x,$$

and $\phi^{\mathcal{P}}_\alpha[u]\in\mathcal{D}^{\alpha,2}(\mathbb{R}^3)$, is the unique weak solution of $(-\Delta)^\alpha v = K_{\mathcal{P}}(x)u^2$. Similarly, we consider $c(I^{\mathcal{P}}_\lambda)$ defined in the same way as in (4.1.4) and (4.1.5).

Next we finally state the assumption relative to the minimax level of the considered problems, which allow us to state the main results of the chapter.

$(\mathscr{C})$ $c(I_\lambda) < c(I^{\mathcal{P}}_\lambda),\ \forall\lambda > 0.$

4.2 Statement of the main results

We first state our results concerning existence of ground states solutions for the System $(\mathcal{SP})$. By a ground state solution for $(\mathcal{S}_{NL})$, we consider a solution u that satisfies $I_\lambda(u) \leqslant I_\lambda(v)$ for any other weak solution v for Eq. $(\mathcal{S}_{NL})$ in the same considered space of functions.

Theorem 4.2.1 (Periodic subcritical case). *Suppose that* $g(x,t) \equiv 0$ *and* $K(x)$, $a(x) \equiv V(x)$ *and* $f(x,t)$ *satisfy conditions* (K_1)–(K_2), (V_1)–(V_3), (f_1), (f_2), (f_5) *with* $K(x) \equiv K_{\mathcal{P}}(x)$, $V(x) \equiv V_{\mathcal{P}}(x)$ *and* $f(x,t) \equiv f_{\mathcal{P}}(x,t)$, *respectively. If we assume that either* $\mu > 4$ *and* (f_3) *or* $\mu = 4$ *and* (f_4), *then Eq.* $(\mathcal{S}_{NL})$ *has a ground state solution* $u \in H^s_{V_{\mathcal{P}}}(\mathbb{R}^3)$. *If additionally we have* (f_6), *then* u *is non-negative and* $I^{\mathcal{P}}_\lambda(u) = c(I^{\mathcal{P}}_\lambda)$.

Observe that Theorem 4.2.1 deal with the case where $(\mathscr{C})$ does not hold. Moreover, the potential $a(x)$ can change sign. It is worth to mention that this result complement the ones in [93] for the case where $s > 3/4$. Our next result is about existence of solutions in the case where $f(x,t)$ has 4-superlinear growth.

Theorem 4.2.2 (Nonautonomous subcritical case). *Suppose that* $K(x)$, $a(x) = V(x) - b(x)$ *and* $f(x,t)$ *satisfy assumptions* (K_1)–(K_2), (V_1)–(V_4), (f_1), (f_2), (f_5), (f_6) *and that* $g(x,t) \equiv 0$. *In addition, assume either one of the following conditions holds,*

(i) $V(x) \equiv V_{\mathcal{P}}(x)$, $b(x) \equiv 0$ *and* $(\mathscr{C})$; *or*

(ii) $V(x) \geqslant 0$, *for a. e.* $x \in \mathbb{R}^3$, $b(x)$ *has compact support and* $(\mathscr{C})$; *or*

(iii) Replace conditions $(\mathscr{C})$ *in the aboves items by*

$$I_\lambda(u) \leqslant I^{\mathcal{P}}_\lambda(u), \quad \forall\, u \in H^s_V(\mathbb{R}^3); \tag{4.2.1}$$

If we assume that either $\mu > 4$ *and* (f_3) *or* $\mu = 4$ *and* (f_4), *then Eq.* $(\mathcal{S}_{NL})$ *possess a non-trivial weak solution* u *in* $H^s_V(\mathbb{R}^3)$ *at the mountain pass level, that is,* $I_\lambda(u) = c(I_\lambda)$. *Moreover, under the assumptions of items (i) and (ii), any sequence* (u_k) *in* $H^s_V(\mathbb{R}^3)$ *such that* $I(u_k) \to c(I)$ *and* $I'(u_k) \to 0$ *has a convergent subsequence.*

Note that Theorem 4.2.2 also provides concentration-compactness of the Palais-Smale sequences at the Mountain-Pass level. Moreover, in this result the potential $a(x)$ can change sign in two different ways. Consequently, it complements some results of [105] and extend them to the fractional framework. Theorem 4.2.2 is inspired by Theorem 3.2.3. Our following result deals with existence of solution for Eq. $(\mathcal{S}_{NL})$ when the nonlinearity has general oscillatory critical growth.

Theorem 4.2.3 (Nonautonomous case with critical perturbation). *Assume that* $K(x)$, $a(x) = V(x) - b(x)$, $f(x,t)$ *and* $g(x,t)$ *satisfy conditions* (K_1)–(K_3), (V_2)–(V_4), (f_1), (f_2), (f_5)–(f_7), (g_1)–(g_7), *respectively, with* $0 \leqslant V(x), V_{\mathcal{P}}(x) \in L^\infty(\mathbb{R}^3)$ *and* $\mu \leqslant \mu_*$. *Moreover, that the following inequality holds,*

$$\frac{\mathcal{C}_*}{2^*_s c_*} \leqslant \left[\left(\frac{2^*_s}{\mu_*}\right)\frac{\mu_* - 2}{2^*_s - 2}\right]^{\frac{2^*_s - 2}{2}}. \tag{4.2.2}$$

Suppose also that either one of the following conditions hold,

(i) $b(x)$ *belongs to* $L^\infty(\mathbb{R}^3)$ *and has compact support,* (g_8) *and estimate* $(\mathscr{C})$*; or*

(ii) Replace condition $(\mathscr{C})$ *and* (g_8) *in the above item by* (4.2.1).

If we assume that $f(x,t)$ *satisfies either* $\mu > 4$ *and* (f_3) *or* $\mu = 4$ *and* (f_4)*, then there exists a non-trivial weak solution* $u \in H^s_V(\mathbb{R}^3)$ *for Eq.* $(\mathcal{S}_{NL})$*, such that* $I_\lambda(u) = c(I_\lambda)$*. Moreover, under condition* $(\mathscr{C})$ *of item (i), any sequence* (u_k) *in* $H^s_V(\mathbb{R}^3)$ *such that* $I_\lambda(u_k) \to c(I_\lambda)$ *and* $I'_\lambda(u_k) \to 0$ *has a convergent subsequence.*

To the best of our knowledge, Theorem 4.2.3 is the first result about existence of solution for System $(\mathcal{SP})$ with general critical nonlinearity $g(x,t)$, therefore, it complements the results of [94]. Moreover, it extends and improve the result of [107] about existence of solution. Note also that in Theorem 4.2.3 gives the compactness for Palais-Smale sequence at the mountain pass level. Next, we state our result which treats the case where nonlinearities does not depends on x.

Theorem 4.2.4 (Autonomous case)**.** *Assume that* $K(x) \equiv K_0 > 0$, $a(x) \equiv V_0$, $b(x) \equiv 0$, $f(x,t) \equiv f(t)$ *satisfies* (f_1), (f_2) *and* $g(x,t) \equiv g(t)$*. Moreover,* $\alpha > 3/4$ *and either one of the following conditions holds,*

(i) (f_7)*,* $g(t)$ *is self-similar,* (g_2), (g_8), $\mu_* \geqslant 3$, (4.2.2) *and*

$$\begin{cases} f(t)t \leqslant (\mu_* + 1)F(t), \\ g(t)t \leqslant (\mu_* + 1)G(t), \end{cases} \quad \forall\, t \in \mathbb{R}; \tag{4.2.3}$$

(ii) (f'_3)*, and* (4.2.3) *with* $\mu_* = \mu$ *and* $g(t) \equiv 0$.

Then there exists $0 < \lambda_* \leqslant \infty$ *such that, for any* $\lambda \in (0, \lambda_*)$*, there exists a non-trivial radial weak solution* u_λ *in* $H^s_{\mathrm{rad}}(\mathbb{R}^3)$ *for Eq.* $(\mathcal{S}_{NL})$ *satisfying* $I_\lambda(u_\lambda) = c(I_\lambda)$.

It seems for us that Theorem 4.2.4 is the first result concerning existence of solutions for autonomous nonlinearities with critical growth satisfying condition (f_2) for any value of μ. In fact, it is common in the present literature to use a Pohozaev type identity in order to apply L. Jeanjean Theorem [58, Theorem 1.1] to construct a bounded Palais-Smale sequence at the mountain pass level.

Nevertheless we prove a improved version of a Pohozaev type identity given in [93] and improve the non-existence result of [93, Theorem 1.6].

Theorem 4.2.5 (Non-existence)**.** *Suppose that* $K(x) \equiv K_0 > 0$, $f(x,t) \equiv f(t) \in C^1(\mathbb{R}^3)$, $g(x,t) \equiv 0$ *and either* *(i)* $a(x) \in C^1(\mathbb{R}^3)$, $2sa(x) + \langle \nabla a(x), x \rangle > 0$ *for all* x *in a non-zero measure domain and* $f(t)t \geqslant 2_s^* F(t)$, *for all* $t \in \mathbb{R}$; *or*

(ii) $a(x) \in C^1(\mathbb{R}^3)$, $a(x) > 0$, $\langle \nabla a(x), x \rangle < 0$ *for all* x *in a non-zero measure domain and there exists* $0 < \delta \leqslant 2$, *such that* $\delta F(t) \geqslant f(t)t$, *for all* $t \in \mathbb{R}$; *or*

(iii) $a(x) \equiv a_0 > 0$, *there exists* $0 \leqslant \delta \leqslant 2s/(3-2s)$, *in a such way that* $f(t)t + \delta a_0 t^2 \geqslant 2_s^* F(t)$, *for all* $t \in \mathbb{R}$; *or*

(iv) $a(x) \equiv a_0 > 0$, $\lambda \geqslant 1/4$, $\alpha = s$ *and* $|f(t)| \leqslant \mathcal{A}|t|^{p-1}$, *for all* $t \in \mathbb{R}$, *where* $2 < p \leqslant 3 \leqslant 2_s^*$ *and* $0 < \mathcal{A} \leqslant \min\{K_0, a_0\}$.

(v) $a(x) \equiv 0$ *and there exists* $0 < p < 2_s^*$ *such that* $pF(t) \geqslant f(t)t$ *for all* $t \in \mathbb{R}$.

If $(u, \phi) \in H^s(\mathbb{R}^3) \times \mathcal{D}^{\alpha,2}(\mathbb{R}^3)$ *is a weak solution of the System* $(\mathcal{SP})$, *such that* $F(u)$, $f(u)u$, $a(x)u^2$, $\langle \nabla a(x), x \rangle u^2$, ϕu^2 *belongs to* $L^1(\mathbb{R}^3)$ *and* $f(u)/(1+|u|)$ *belongs to* $L^{3/2s}_{\text{loc}}(\mathbb{R}^3)$, *then* $u = 0$.

Corollary 4.2.6. *Assume that* $K(x) \equiv K_0 > 0$, $a(x) \equiv a_0 > 0$, $f(x,t) = |t|^{p-2}t$ *and* $g(x,t) \equiv 0$. *Moreover, suppose that either one of the following conditions hold,*

(i) $p = 2_s^*$;

(ii) $p \leqslant 2$; *or*

(iii) $s \geqslant 1/2$, $2 < p \leqslant 3$ *and* $\lambda \geqslant 1/4$.

If $(u, \phi) \in H^s(\mathbb{R}^3) \times \mathcal{D}^{\alpha,2}(\mathbb{R}^3)$ *is a weak solution of the System* $(\mathcal{SP})$, *then* $u = \phi = 0$.

Remarks on the assumptions and in the main results

Remark 4.2.7. Some comments on our assumptions are in order.

(i) To the best of our knowledge, it seems that our results are the first concerning a general potential $K(x)$ for the System $(\mathcal{SP})$.

(ii) In order to get that I_λ has the mountain pass geometry, we use Ambrosetti-Rabinowitz condition (f_2). In this case the presence of the nonlocal term $\mathcal{N}_\alpha$ in Eq. $(\mathcal{S}_{NL})$ imposes that $\mu > 4$ in our argument. Despise this, in the general case of a non-autonomous linearity, we consider $\mu = 4$ and to overcome the associated difficulty we ask for assumption (f_4). The general case that $\mu > 2$ is considered in Theorem 4.2.4.

(iii) In our approach to study existence of weak solutions for Eq. $(\mathcal{S}_{NL})$ we use assumption (f_5), unlike the aforementioned papers, where the authors impose the more tight condition

$$|f(x,t) - f_{\mathcal{P}}(x,t)| \leqslant h(x)|t|^{q-1} \quad \text{a.e } x \text{ in } \mathbb{R}^3 \text{ and all } t \text{ in } \mathbb{R},$$

where $h(x)$ belongs to the class of functions in $C(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ such that for every $\varepsilon > 0$ the set $\{x \in \mathbb{R}^3 : |h(x)| \geqslant \varepsilon\}$ has finite Lebesgue measure.

(iv) The function $g_+(t)$ is self-similar. Moreover, if $g(x,t) \equiv g(t)$ is self-similar then $g(t) \equiv g_+(t)$.

(v) Once the limits in (V_3), (f_5) or (g_4) exist, to obtain compactness of Palais-Smale sequences at the minimax levels we need to require the additional condition over the minimax level given in assumption $(\mathscr{C})$. In fact, we do not believe that it is possible, in general, to achieve the compactness described in Theorems 4.2.2 and 4.2.3 without these conditions. We mention that this kind of approach was introduced by P.-L. Lions in [65–68].

(vi) We also consider the case when $(\mathscr{C})$ do not hold. Precisely, when it is allowed $c(I_\lambda) = c(I_\lambda^{\mathcal{P}})$. In this case, the concentration-compactness argument at the mountain pass level cannot be used. We apply [63, Theorem 2.3] to overcome this difficulty and prove existence of solution at the mountain pass level.

Remark 4.2.8. Under the assumptions (V_3), (f_5) and (g_4) we describe next conditions which guarantee that $(\mathscr{C})$ is satisfied.

$(\mathscr{C}')$ The following inequalities holds,

$$V(x) \leqslant V_{\mathcal{P}}(x), \quad \text{a. e. } x \in \mathbb{R}^3. \tag{4.2.4}$$

$$K(x) \leqslant K_{\mathcal{P}}(x), \quad \text{a. e. } x \in \mathbb{R}^3. \tag{4.2.5}$$

$$F_{\mathcal{P}}(x,t) + G_\infty(t) \leqslant F(x,t) + G(x,t), \quad \text{a. e. } x \in \mathbb{R}^3 \text{ and } t \in \mathbb{R}, \tag{4.2.6}$$

Moreover, at least one of the next conditions is true,

(i) The inequality (4.2.4) strict in a non-zero measure domain.

(ii) The inequality (4.2.5) strict in a non-zero measure domain.

(iii) There exists $\delta > 0$ such that the inequality (4.2.6) is strict for all $t \in (-\delta, \delta)$ and a. e. $x \in \mathbb{R}^3$.

In Sect. 4.6, under suitable conditions, we obtained the following estimate for the minimax levels: $c(I) \leqslant c(I_{\mathcal{P}})$. Moreover, we proved that under condition $(\mathscr{C}')$ we have that $(\mathscr{C})$ holds. We observe that on the corresponding assumption of Theorem 4.2.2, it is easy to see that inequalities (4.2.4), (4.2.5) and (4.2.6) imply that (4.2.1) is satisfied.

Remark 4.2.9. Using the same argument of Remark 3.2.7 it can be proved the existence of non-negative weak solutions of $(\mathcal{S}_{NL})$. In fact, assume that $h(x,t) := f(x,t) + g(x,t) \geqslant 0$ for all $t \geqslant 0$ and a. e. x in $\mathbb{R}^3$, and consider the truncation

$$\bar{h}(x,t) = \begin{cases} f(x,t) + g(x,t), & \text{if } t \geqslant 0, \\ 0, & \text{if } t < 0. \end{cases}$$

Assume that $a(x) \in L^1_{\text{loc}}(\mathbb{R}^3)$ and that conditions (f_1), (g_1) and (V_2) holds true with $b(x) \equiv 0$; if u is a weak solution for $(\mathcal{S}_{NL})$, with $f(x,t) + g(x,t)$ and replaced by $\bar{h}(x,t)$ then u is also a weak solution for $(\mathcal{S}_{NL})$ and $u \geqslant 0$. To see that, let $\xi \in C_0^\infty(\mathbb{R} : [0,1])$ such that

$$\xi(t) = \begin{cases} 1, & \text{if} \quad t \in [-1,1] \\ 0, & \text{if} \quad |t| \geqslant 2 \end{cases} \qquad \text{and} \qquad |\xi'(t)| \leqslant C \quad \forall\, t \in \mathbb{R},$$

for some C positive constant. For each $n \in \mathbb{N}$, define $\xi_n : \mathbb{R}^4 \to \mathbb{R}$ by $\xi_n(z) = \xi(|z|^2/n^2)$. Then $\xi_n \in C_0^\infty(\mathbb{R}^4)$ and verifies

$$|\nabla \xi_n(z)| \leqslant C \qquad \text{and} \qquad |z||\nabla \xi_n(z)| \leqslant C \quad \forall\, z \in \mathbb{R}^4.$$

By a density argument, we can take $\varphi = \xi_n w_-$ in (0.3.4). Since $w_-(z) = E_s(u_-)$, we have that

$$\int_{\mathbb{R}^4} y^{1-2s}\xi_n |\nabla w_-|^2 + y^{1-2s}\xi_n \left\langle \nabla w^+, \nabla w_- \right\rangle + y^{1-2s} \left\langle \nabla w^+ + \nabla w_-, w_- \nabla \xi_n \right\rangle \mathrm{d}x\mathrm{d}y$$
$$= \kappa_s \int_{\mathbb{R}^3} (\bar{h}(x,u) - V(x)u - \lambda K(x)\phi_\alpha[u]u)\xi_n(\cdot,0)u_- \,\mathrm{d}x,$$

and we may apply the Dominated Convergence Theorem and (0.3.2) to get

$$\|u_-\|_V^2 + \lambda \int K(x)\phi_\alpha[u]|u_-|^2 \,\mathrm{d}x = \int_{\mathbb{R}^3} \bar{h}(x,u)u_- \,\mathrm{d}x,$$

which implies that $u_- = 0$.

Example 4.2.10. Our approach include the following classes of potentials

(i) For the weight $K(x)$ that fulfills assumptions (K_1)–(K_3) we may consider

$$K(x) = Q(x)K_{\mathcal{P}}(x), \quad \text{where } 0 \leqslant Q(x) \leqslant Q_\infty := \lim_{|x|\to\infty} Q(x),$$

belongs to $C(\mathbb{R}^3)$ and $K_{\mathcal{P}}(x)$ is any function $1-$periodic in x_1, x_2, x_3 in $L^\beta(\mathbb{R}^3) \cap L^\infty_{\text{loc}}(\mathbb{R}^3)$ that is continuous at zero.

(ii) For potential $a(x) = V(x) - b(x)$ satisfying conditions (V_1)–(V_4) we set

$$V(x) = \left(\frac{1}{1+|x|^2}\right) V_{\mathcal{P}}(x)$$

where $V_{\mathcal{P}}(x) \geqslant 0$ is any function that is 1−periodic in x_1, x_2, x_3, belongs to $C(\mathbb{R}^3)$; and

$$b(x) = \|V(x)\|_\infty \eta(x),$$

where $\eta(x) \in C_0^\infty(\mathbb{R}^3)$ is chosen in a such way that $\|\eta(x)\|_\beta < \mathcal{C}_V^{(\beta)}/\|V(x)\|_\infty$.

Example 4.2.11. Note that the hypotheses of Theorems 4.2.1–4.2.4 are for example satisfied by nonlinearities of the following forms:

(i) For a nonlinearity fulfilling assumptions (f_1)–(f_7) we can chose

$$f(x,t) = k(x)|t|^{p-2}t + \exp\{k_0(x)(\sin(\ln|t|)+2)\}\left[k_0(x)\cos(\ln|t|)+p\right]|t|^{p-2}t,$$

where $f(x,0) := 0$, $s > 3/4$, $4 < p < 2_s^*$, $k(x) \in C(\mathbb{R}^3)$,

$$0 < k_\infty := \lim_{|x|\to\infty} k(x) \leqslant k(x),\ \forall\, x \in \mathbb{R}^3 \quad \text{and} \quad \lim_{|x|\to\infty} k_0(x) = 0.$$

Moreover, $k_0(x) \in C(\mathbb{R}^3)$, $\sup_{x\in\mathbb{R}^3} k_0(x) \leqslant p - \mu$ and $\mu < p$.

(ii) For a nonlinearity fulfilling conditions (f_1), (f_2) with $\mu = 4$ and (f_4)–(f_6) we take

$$h(x,t) = k(x)\left[t^3\ln(1+t) + (1+\cos(t))t^4 + 4(t+\sin(t))t^2t\right],\ t \geqslant 0,\ s > 1/2,$$

where $k(x)$ is taken as above, and consider

$$f(x,t) = \begin{cases} h(x,t), & \text{for } t \geqslant 0,\\ -h(x,-t), & \text{for } t < 0.\end{cases}$$

(iii) For a nonlinearity satisfying the conditions of Theorem 4.2.1 we can take

$$f(x,t) = c_0(x)\left[\varrho'(t)(\ln|t|t) + \varrho(t)\right]|t|^{\varrho(t)-2}t, \quad f(x,0) := 0,$$

where $0 \leqslant c_0(x) \in L^\infty(\mathbb{R}^3)$ is 1−periodic in x_1, x_2, x_3 and $\varrho(t)$ can be taken as in Remark 3.2.5–(i) with $\inf_{t\in\mathbb{R}} \varrho(t) > 4$.

(iv) Let $0 \leqslant c(x)$ be a continuous 1−periodic in x_i, $i = 1,2,3$, and consider

$$f(x,t) = c(x)\left[ph_\varepsilon(t) + h_\varepsilon'(t)t\right]|t|^{p-1}, \quad 4 < p < 2_s^*,$$

where $h_\varepsilon(t) \in C^\infty(\mathbb{R})$ is given in Remark 3.2.9–(iii). We empathize the fact that $F(x,t)$ changes sign.

(v) For a nonlinearity with critical growth satisfying the assumptions of Theorem 4.2.3 we may take

$$g(x,t) = \exp\{c_1(x)(\sin(\ln|t|)+2)\}\left[c_1(x)\cos(\ln|t|) + 2_s^*\right]|t|^{2_s^*-2}t, \quad g(x,0) := 0,$$

where $0 \leqslant c_1(x) \in C(\mathbb{R}^3)$, $\lim_{|x|\to\infty} c_1(x) = 0$ and

$$\mathcal{K} := \sup_{x\in\mathbb{R}^3} c_1(x) < 2_s^* - \mu_*, \quad \text{for some} \quad \mu_* \in (2, 2_s^*).$$

For $s > 3/4$ we choose $\mu_* \geqslant 4$ and $\mathcal{K}$ such that

$$\exp\{3\mathcal{K}\}(\mathcal{K} + 2_s^*) \leqslant 2_s^* \left[\left(\frac{2_s^*}{\mu_*} \right) \frac{\mu_* - 2}{2_s^* - 2} \right]^{\frac{2_s^*-2}{2}}.$$

(vi) For nonlinearities satisfying the assumptions of Theorem 4.2.4 we may take

$$f(t) = \exp\{c_0(\sin(\ln|t|) + 2)\}\,[c_0\cos(\ln|t|) + p]\,|t|^{p-2}t,\ 2 < p < 2_s^*,\ f(0) := 0,$$
$$g(t) = \exp\{c_1(\sin(\ln|t|) + 2)\}\,[c_1\cos(\ln|t|) + 2_s^*]\,|t|^{2_s^*-2}t,\ g(0) := 0,$$

where $s > 3/4$, $2_s^* - 1 < \mu \leqslant \mu_* < p$, $0 < c_0 \leqslant \min\{p - \mu, 1 - (p - \mu)\}$, $0 < c_1 \leqslant \min\{2_s^* - \mu_*, 1 - (2_s^* - \mu_*)\}$ and

$$\exp\{3c_1\}(c_1 + 2_s^*) \leqslant 2_s^* \left[\left(\frac{2_s^*}{\mu_*} \right) \frac{\mu_* - 2}{2_s^* - 2} \right]^{\frac{2_s^*-2}{2}}.$$

4.3 Variational settings

In this section we describe the variational settings that we use in this chapter.

Remark 4.3.1. Assume that conditions (V_1)–(V_3) holds true. Then $H_V^s(\mathbb{R}^3) = H_{V_\mathcal{P}}^s(\mathbb{R}^3)$ and the norms $\|\cdot\|_V$ and $\|\cdot\|_{V_\mathcal{P}}$ are equivalents. Indeed, let $R_1 > 0$ such that

$$|V(x) - V_\mathcal{P}(x)| < 1, \quad \forall\,|x| > R_1.$$

Given $u \in C_0^\infty(\mathbb{R}^3)$, by Hölder inequality we have

$$\int_{\mathbb{R}^3} V(x)u^2\,\mathrm{d}x \leqslant \left(\int_{B_{R_1}} |V(x)|^\sigma\,\mathrm{d}x \right)^{1/\sigma} \|u\|^2_{2\frac{\sigma}{\sigma-1}} + \int_{\mathbb{R}^3\setminus B_{R_1}} u^2\,\mathrm{d}x + \int_{\mathbb{R}^3\setminus B_{R_1}} V_\mathcal{P}(x)u^2\,\mathrm{d}x,$$

and similarly,

$$\int_{\mathbb{R}^3} V_\mathcal{P}(x)u^2\,\mathrm{d}x \leqslant \left(\int_{B_{R_1}} |V_\mathcal{P}(x)|^{\sigma_\mathcal{P}}\,\mathrm{d}x \right)^{1/\sigma_\mathcal{P}} \|u\|^2_{2\frac{\sigma_\mathcal{P}}{\sigma_\mathcal{P}-1}} + \int_{\mathbb{R}^3\setminus B_{R_1}} u^2\,\mathrm{d}x + \int_{\mathbb{R}^3\setminus B_{R_1}} V(x)u^2\,\mathrm{d}x.$$

Since $2 < 2\sigma/(\sigma-1), 2\sigma_\mathcal{P}/(\sigma_\mathcal{P}-1) < 2_s^*$ we can use embedding (4.1.2) in the inequalities above to obtain the desired result.

4.3.1 Study of the nonlocal term

We now pass to the study of the nonlocal term of Eq. $(\mathcal{S}_{NL})$ and start by revisiting the definition of the nonlocal term in Eq. $(\mathcal{S}_{NL})$. Assume that (K_1) holds true and let $\mathcal{P}_u : \mathcal{D}^{\alpha,2}(\mathbb{R}^3) \to \mathbb{R}$ given by,

$$\mathcal{P}_u(v) = \int_{\mathbb{R}^3} K(x)u^2 v \,\mathrm{d}x, \quad u \in H^s(\mathbb{R}^3),\ v \in \mathcal{D}^{\alpha,2}(\mathbb{R}^3).$$

Fixed $u \in H^s(\mathbb{R}^3)$, we have that

$$|\mathcal{P}_u(v)| \leqslant \begin{cases} \|K(x)\|_\infty \|u\|_{r_\infty}^2 \|v\|_{2_\alpha^*}, & r_\infty := 22_\alpha^*/(2_\alpha^* - 1), \\ \|K(x)\|_r \|u\|_{r_\alpha}^2 \|v\|_{2_\alpha^*}, & r_\alpha := 12r/((3+2\alpha)r - 6), \end{cases} \qquad v \in \mathcal{D}^{\alpha,2}(\mathbb{R}^3), \quad (4.3.1)$$

provided that $K(x)$ belongs to $L^\infty(\mathbb{R}^3)$ and $L^r(\mathbb{R}^3)$, respectively.

It is expected that for our general class of $K(x)$ the unique weak solution $\phi_\alpha[u] \in \mathcal{D}^{\alpha,2}(\mathbb{R}^3)$ for the equation $(-\Delta)^\alpha \phi_\alpha[u] = K(x)u^2$ can be characterized in terms of Riesz potential. In what follows, we give a briefly proof of that fact.

Proposition 4.3.2. *Suppose that $K(x)$ satisfies assumption (K_1) and let $u \in H^s(\mathbb{R}^3)$. Then*

$$\phi_\alpha[u](x) = c_\alpha \int_{\mathbb{R}^3} K(y)u^2(y)|x-y|^{2\alpha-3} \,\mathrm{d}y, \quad a.\ e.\ \mathbb{R}^3.$$

Proof. Denote $Q(u) = K(x)u^2$. Then, by Hölder inequality and condition (K_1), $Q(u)$ belongs to $L^p(\mathbb{R}^3)$, for $p = 2_\alpha^*/(2_\alpha^* - 1)$. Let (φ_k) a sequence in $C_0^\infty(\mathbb{R}^3)$ such that $\varphi_k \to Q(u)$ in $L^p(\mathbb{R}^3)$. If $v \in \mathcal{D}^{\alpha,2}(\mathbb{R}^3)$, then

$$\begin{aligned}\int_{\mathbb{R}^3} (-\Delta)^{\alpha/2}(\mathcal{I}_\alpha[\varphi_k - \varphi_l])(-\Delta)^{\alpha/2} v \,\mathrm{d}x &= \int_{\mathbb{R}^3} (-\Delta)^\alpha(\mathcal{I}_\alpha[\varphi_k - \varphi_l]) v \,\mathrm{d}x \\ &= \int_{\mathbb{R}^3} (\varphi_k - \varphi_l) v \,\mathrm{d}x \to 0 \quad as \quad k,l \to \infty,\end{aligned}$$

where we used the well know identity,

$$(-\Delta)^\alpha(\mathcal{I}_\alpha[\varphi]) = \varphi, \quad \forall\, \varphi \in \mathscr{S}(\mathbb{R}^3).$$

Consequently, $(\mathcal{I}_\alpha[\varphi_k])$ is a weakly Cauchy sequence in $\mathcal{D}^{\alpha,2}(\mathbb{R}^3)$, and must weakly converge in $\mathcal{D}^{\alpha,2}(\mathbb{R}^3)$ for some v_α. On the other hand, by Proposition 0.1.6, the sequence $(\mathcal{I}_\alpha[\varphi_k])$ converges to $\mathcal{I}_\alpha[Q(u)]$ in $L^{2_\alpha^*}(\mathbb{R}^3)$. This implies that $v_\alpha = \mathcal{I}_\alpha[Q(u)]$, a. e., and moreover, given $\varphi \in C_0^\infty(\mathbb{R}^3)$, we conclude that

$$\begin{aligned}\int_{\mathbb{R}^3} (-\Delta)^{\alpha/2}(\mathcal{I}_\alpha[Q(u)])(-\Delta)^{\alpha/2}\varphi \,\mathrm{d}x &= \lim_{k\to\infty} \int_{\mathbb{R}^3} (-\Delta)^{\alpha/2}(\mathcal{I}_\alpha[\varphi_k])(-\Delta)^{\alpha/2}\varphi \,\mathrm{d}x \\ &= \lim_{k\to\infty} \int_{\mathbb{R}^3} \varphi_k \varphi \,\mathrm{d}x = \int_{\mathbb{R}^3} Q(u)\varphi \,\mathrm{d}x.\end{aligned}$$

By unicity, $\phi_\alpha[u] = \mathcal{I}_\alpha[Q(u)]$ a. e. in $\mathbb{R}^3$. ∎

We now set the nonlocal functionals

$$\begin{aligned} \mathcal{N}_\alpha(u) &= \int_{\mathbb{R}^3} K(x)\phi_\alpha[u]u^2\,\mathrm{d}x, \qquad u \in H^s(\mathbb{R}^3), \\ \mathcal{N}_\alpha^{\mathcal{P}}(u) &= \int_{\mathbb{R}^3} K_{\mathcal{P}}(x)\phi_\alpha^{\mathcal{P}}[u]u^2\,\mathrm{d}x, \quad u \in H^s(\mathbb{R}^3), \end{aligned} \tag{4.3.2}$$

and summarize their basic properties.

Proposition 4.3.3. *Suppose that* (K_1) *holds true and let* $u \in H^s(\mathbb{R}^3)$. *Then*

(i) $\mathcal{N}_\alpha$ *belongs to* $C^1(H^s(\mathbb{R}^3))$ *and*

$$\mathcal{N}_\alpha'(u)\cdot v = 4\int_{\mathbb{R}^3} K(x)\phi_\alpha[u]uv\,\mathrm{d}x, \quad u,v \in H^s(\mathbb{R}^3);$$

(ii) If (u_k) *and* (v_k) *are bounded sequences in* $H^s(\mathbb{R}^3)$, *with* $u_k - v_k \to 0$ *in* $L^p(\mathbb{R}^3)$, *for some* $p \in (2, 2_s^*)$, *then* $\mathcal{N}_\alpha(u_k) - \mathcal{N}_\alpha(v_k) \to 0$;

(iii) $\phi_\alpha : H^s(\mathbb{R}^3) \to \mathcal{D}^{\alpha,2}(\mathbb{R}^3)$ *is continuous and maps bounded sets into bounded sets;*

(iv) $\phi_\alpha[u] \geqslant 0$ *and* $\mathcal{N}_\alpha(u) \leqslant C_\alpha\|u\|_q^4$, *where* $q = r_\infty$ *or* $q = r_\alpha$ *provided that* $K(x) \in L^\infty(\mathbb{R}^3)$ *or* $K(x) \in L^r(\mathbb{R}^3)$, *respectively;*

(v) $\mathcal{N}_\alpha(tu) = t^4\mathcal{N}_\alpha(u)$, *and if* $K(x) \equiv K_0 > 0$, *then* $\mathcal{N}_\alpha(u(\cdot/t)) = t^{3+2\alpha}\mathcal{N}_\alpha(u)$, *for all* $t > 0$;

(vi) If $u_k \rightharpoonup u$ *in* $H^s(\mathbb{R}^3)$, *then* $\phi_\alpha[u_k] \rightharpoonup \phi_\alpha[u]$ *in* $\mathcal{D}^{s,2}(\mathbb{R}^3)$;

Proof. (i) Follows by standard arguments as in the local where it is used Fubini Theorem (for instance, see [30]).

(ii) We assume that $K(x)$ belongs to $L^r(\mathbb{R}^3)$, since $L^\infty(\mathbb{R}^3)$ can be proved in a similar way. Observe first that, by an interpolation inequality $u_k - v_k \to 0$ in $L^p(\mathbb{R}^3)$, for all $p \in (2, 2_s^*)$. Next, we write

$$\begin{aligned} |\mathcal{N}_\alpha(u_k) - \mathcal{N}_\alpha(v_k)| \leqslant &\int_{\mathbb{R}^3} \left|K(x)\phi_\alpha[u_k](u_k^2 - v_k^2)\right|\mathrm{d}x \\ &+ \int_{\mathbb{R}^3} \left|K(x)(\phi_\alpha[u_k] - \phi_\alpha[v_k])v_k^2\right|\mathrm{d}x. \end{aligned} \tag{4.3.3}$$

The first integral in the right-hand side of (4.3.3) can be estimated by using estimate (4.3.1). In fact we have

$$\begin{aligned} |\mathcal{N}_\alpha(u_k) - \mathcal{N}_\alpha(v_k)| &\leqslant \int_{\mathbb{R}^3} \left|K(x)\phi_\alpha[u_k](u_k^2 - v_k^2)\right|\mathrm{d}x \\ &\leqslant \|K(x)\|_r\|(u_k - v_k)(u_k + v_k)\|_{r_\alpha/2}\|\phi_\alpha[u_k]\|_{2_\alpha^*} \to 0, \quad \text{as } k \to \infty, \end{aligned}$$

because $2 < r_\alpha < 2_s^*$, and where we used Proposition 0.1.6, with $p = 2_\alpha^*/(2_\alpha^* - 1)$ and $q = 2_\alpha^*$, to guarantee that $(\phi_\alpha[u_k])$ is bounded in $L^{2_\alpha^*}(\mathbb{R}^3)$. To estimate the second integral in the right-hand of (4.3.3) we notice first that $\phi_\alpha[u_k] - \phi_\alpha[v_k] = \phi_\alpha[u_k^2 - v_k^2]$. Thus,

$$\begin{aligned}\int_{\mathbb{R}^3} \left|K(x)(\phi_\alpha[u_k] - \phi_\alpha[v_k])v_k^2\right| \,\mathrm{d}x &\leqslant \|K(x)\|_r \|v_k\|_{r_\alpha}^2 \|\phi_\alpha[u_k^2 - v_k^2]\|_{2_\alpha^*} \\ &\leqslant \|K(x)\|_r^2 \|v_k\|_{r_\alpha}^2 \|u_k - v_k\|_{r_\alpha} \|u_k + v_k\|_{r_\alpha} \to 0, \quad \text{as } k \to \infty,\end{aligned}$$

where it is used Proposition 0.1.6 again with $p = 2_\alpha^*/(2_\alpha^* - 1)$ and $q = 2_\alpha^*$, to obtain the second inequality.

(iii) Let $u_k \to u$ in $H^s(\mathbb{R}^3)$. Define the functional

$$\mathcal{P}_k(v) = \int_{\mathbb{R}^3} K(x) u_k^2 v \,\mathrm{d}x, \quad v \in H^s(\mathbb{R}^3).$$

In order to prove that $\phi_\alpha[u_k] \to \phi_\alpha[u]$, it suffices to prove that $\mathcal{P}_k \to \mathcal{P}_u$ in the dual of $H^s(\mathbb{R}^3)$. This actually follows by (4.3.1) and using similar arguments as above.

(iv)–(vi) can be proved by using the definition of $\phi_\alpha[u]$ and the estimate (4.3.1). ■

Next we establish the behaviour of the weak convergence for the functional (4.3.2) under the profile decomposition for bounded sequences.

Proposition 4.3.4. *Assume that* (K_1)–(K_2) *holds true. Let* (u_k) *be a bounded sequence in* $H^s(\mathbb{R}^3)$ *and* $(w^{(n)})_{n\in\mathbb{N}_0}$ *given by Theorem 1.1.2. Then, up to subsequence,*

$$\lim_{k\to\infty} \mathcal{N}_\alpha(u_k) = \mathcal{N}_\alpha(w^{(1)}) + \sum_{n\in\mathbb{N}_0, n>1} \mathcal{N}_\alpha^{\mathcal{P}}(w^{(n)}). \tag{4.3.4}$$

Proof. By convergence (1.1.8) and Proposition 4.3.3 we have that

$$\lim_{k\to\infty} \left[\mathcal{N}_\alpha(u_k) - \mathcal{N}_\alpha\left(\sum_{n\in\mathbb{N}_0} w^{(n)}(\cdot - y_k^{(n)})\right)\right] = 0.$$

The uniform convergence in (1.1.8) allows us to reduce to prove that (up to subsequence),

$$\lim_{k\to\infty} \left[\mathcal{N}_\alpha\left(\sum_{n=1}^{M} w^{(n)}(\cdot - y_k^{(n)})\right) - \mathcal{N}_\alpha(w^{(1)}) - \sum_{n=2}^{M} \mathcal{N}_\alpha^{\mathcal{P}}(w^{(n)})\right] = 0, \ \forall\, M \in \mathbb{N}. \tag{4.3.5}$$

Since $\mathcal{N}_\alpha$ is C^1, by density, we may assume that $w^{(n)}$ belongs to $C_0^\infty(\mathbb{R}^3)$, for $n = 1, \ldots, M$. By condition (1.1.6), there exists k_0 such that $\operatorname{supp}(w^{(m)}(\cdot - y_k^{(m)})) \cap$

$\operatorname{supp}(w^{(n)}(\cdot - y_k^{(n)})) = \varnothing$, for all $m \neq n$ and $k > k_0$. Consequently,

$$\mathcal{N}_\alpha\left(\sum_{m=1}^{M} w^{(m)}(\cdot - y_k^{(m)})\right)$$
$$= \sum_{m=1}^{M}\sum_{n=1}^{M}\int_{\mathbb{R}^3} K(x + y_k^{(m)})|w^{(m)}|^2\phi_\alpha[w^{(n)}(\cdot - y_k^{(n)})](x + y_k^{(m)})\,\mathrm{d}x. \quad (4.3.6)$$

For a. e. x in $\mathbb{R}^3$ and $n \geqslant 2$, we obtain that

$$\lim_{k\to\infty}\phi_\alpha[w^{(n)}(\cdot - y_k^{(n)})](x + y_k^{(n)})$$
$$= \lim_{k\to\infty}\int_{\operatorname{supp}(w^{(n)})} K(y + y_k^{(n)})|w^{(n)}(y)|^2|x-y|^{2\alpha-3}\,\mathrm{d}y$$
$$= \int_{\operatorname{supp}(w^{(n)})} K_{\mathcal{P}}(y)|w^{(n)}(y)|^2|x-y|^{2\alpha-3}\,\mathrm{d}y = \phi_\alpha^{\mathcal{P}}[w^{(n)}](x),$$

in fact, this convergence follows by Lebesgue Theorem, once we take account that

- $K(y + y_k^{(n)})|w^{(n)}(y)|^2|x-y|^{2\alpha-3} \to K_{\mathcal{P}}(y)|w^{(n)}(y)|^2|x-y|^{2\alpha-3}$ a.e y in $\mathbb{R}^3$ and
- $K(y+y_k^{(n)}) < 1+|K_{\mathcal{P}}(y+y_k^{(n)})| = 1+|K_{\mathcal{P}}(y)| \in L^1(\operatorname{supp}(w))$, for k large enough.

By a similar argument we conclude, for $n \geqslant 2$, that

$$\lim_{k\to\infty}\int_{\mathbb{R}^3} K(x + y_k^{(n)})|w^{(n)}|^2\phi_\alpha[w^{(n)}(\cdot - y_k^{(n)})](x + y_k^{(n)})\,\mathrm{d}x$$
$$= \int_{\mathbb{R}^3} K_{\mathcal{P}}(x)|w^{(n)}|^2\phi_\alpha^{\mathcal{P}}[w^{(n)}]\,\mathrm{d}x. \quad (4.3.7)$$

Moreover, the same argument above together with condition (1.1.6) leads to

$$\lim_{k\to\infty}\int_{\mathbb{R}^3} K(x + y_k^{(m)})|w^{(m)}|^2\phi_\alpha[w^{(n)}(\cdot - y_k^{(n)})](x + y_k^{(m)})\,\mathrm{d}x = 0. \quad (4.3.8)$$

Convergence (4.3.5) follows by (4.3.6), (4.3.7) and (4.3.8). ■

Corollary 4.3.5. *Under the same assumptions of Proposition 4.3.4, we have*

$$\lim_{k\to\infty}\left[\mathcal{N}_\alpha(u_k) - \mathcal{N}_\alpha(w^{(1)}) - \mathcal{N}_\alpha(u_k - w^{(1)})\right] = 0.$$

Proof. It is easy to see that $\tilde{w}^{(1)} = 0$ and $\tilde{w}^{(n)} = w^{(n)}$, for $n \geqslant 2$ corresponds to a profile decomposition for the sequence $v_k = u_k - w^{(1)}$. Thus applying Proposition 4.3.4 to this sequence we obtain

$$\lim_{k\to\infty}\mathcal{N}_\alpha(u_k - w^{(1)}) = \sum_{n\in\mathbb{N}_0, n>1}\mathcal{N}_\alpha^{\mathcal{P}}(w^{(n)}). \quad (4.3.9)$$

The result follows by taking the difference between (4.3.4) and (4.3.9) ■

The next result gives the behavior of the derivative of $\mathcal{N}_\alpha$ under the profile decomposition described in Theorem 1.1.1, and gives the link necessary to treat the critical case.

Lemma 4.3.6. *Assume that $K(x)$ fulfills (K_1)–(K_3). Let (u_k) be a bounded sequence in $H^s(\mathbb{R}^3)$, and $(w^{(n)})_{\mathbb{N}_*}$, $(j_k^{(n)})$ and $(y_k^{(n)})$ given by Theorem 1.1.1. Then*

$$\lim_{k\to\infty} \mathcal{N}'_\alpha(u_k)\cdot d_k^{(n)}\varphi = \begin{cases} (\mathcal{N}_\alpha^{\mathcal{P}})'(w^{(n)})\cdot\varphi, & \text{if } n\in\mathbb{N}_0,\\ 0, & \text{if } n\in\mathbb{N}_+. \end{cases} \tag{4.3.10}$$

Proof. Next we use the fact that $\mathbb{N}_* = \mathbb{N}_0\cup\mathbb{N}_+$ (Proposition 1.4.4). We prove (4.3.10) by using the Dominated Convergence Theorem. We have

$$\mathcal{N}'_\alpha(u_k)\cdot d_k^{(n)}\varphi = \int_{\mathbb{R}^3} \gamma^{-2sj_k^{(n)}} K(\gamma^{-j_k^{(n)}}x + y_k^{(n)})\phi_\alpha[u_k](\gamma^{-j_k^{(n)}}\cdot + y_k^{(n)}) v_k^{(n)}\varphi\,\mathrm{d}x, \tag{4.3.11}$$

where

$$v_k^{(n)} = \gamma^{-\frac{3-2s}{2}j_k^{(n)}} u_k(\gamma^{-j_k^{(n)}}\cdot + y_k^{(n)}). \tag{4.3.12}$$

Notice that

$$\left[\phi_k^{(n)}[u_k]\right]_\alpha^2 := \left[\gamma^{-2sj_k^{(n)}}\phi_\alpha[u_k](\gamma^{-j_k^{(n)}}\cdot + y_k^{(n)})\right]_\alpha^2 = \gamma^{(3-2\alpha-4s)j_k^{(n)}}\left[\phi_\alpha[u_k]\right]_\alpha^2.$$

Thus, by Proposition 4.3.3, $(\phi_k^{(n)}[u_k])$ converges to zero in $\mathcal{D}^{\alpha,2}(\mathbb{R}^3)$, if $n\in\mathbb{N}_+$; and converges weakly, up to subsequence, for some $\xi^{(n)}$ in $\mathcal{D}^{\alpha,2}(\mathbb{R}^3)$, if $n\in\mathbb{N}_0$. Moreover, by the Dominated Convergence Theorem we have

$$\begin{aligned}\left[\xi^{(n)},\varphi\right]_\alpha = \lim_{k\to\infty}\left[\phi_k^{(n)}[u_k],\varphi\right]_\alpha &= \lim_{k\to\infty}\int_{\mathbb{R}^3} K(x+y_k^{(n)})|v_k^{(n)}|^2\varphi\,\mathrm{d}x\\ &= \int_{\mathbb{R}^3} K_{\mathcal{P}}(x)|w^{(n)}|^2\varphi\,\mathrm{d}x, \quad \text{for } n\in\mathbb{N}_0.\end{aligned}$$

That is, the definition of $\phi_\alpha^{\mathcal{P}}$ implies $\xi^{(n)} = \phi_\alpha^{\mathcal{P}}[w^{(n)}]$, $n\in\mathbb{N}_0$. We write identity (4.3.11) as

$$\begin{aligned}\mathcal{N}'_\alpha(u_k)\cdot d_k^{(n)}\varphi = &\int_{\mathbb{R}^3}\left[K(\gamma^{-j_k^{(n)}}x + y_k^{(n)}) - K_{\mathcal{P}}(\gamma^{-j_k^{(n)}}x + y_k^{(n)})\right]\phi_k^{(n)}[u_k]v_k^{(n)}\varphi\,\mathrm{d}x\\ &+ \int_{\mathbb{R}^3} K_{\mathcal{P}}(\gamma^{-j_k^{(n)}}x)\phi_k^{(n)}[u_k]v_k^{(n)}\varphi\,\mathrm{d}x.\end{aligned}$$

Using condition (K_3), it is easy to see that

$$K_{\mathcal{P}}(\gamma^{-j_k^{(n)}}x) < K(0)+1, \quad \forall\, x\in\operatorname{supp}(\varphi), \quad k \text{ large enough} \quad \text{and} \quad n\in\mathbb{N}_+.$$

Therefore, we can apply the Dominated Convergence Theorem again to obtain (4.3.10) ■

4.3.2 Mountain Pass Settings

In the following result we prove that our functional I_λ has the Mountain Pass Geometry.

Lemma 4.3.7. *Suppose that $K(x)$, $a(x) = V(x) - b(x)$, $f(x,t)$ and $g(x,t)$ satisfy (K_1), (V_1), (V_2), (V_4), (f_1), (g_1), (g_2), respectively. Moreover, assume either (f_2), (f_3) with $\mu > 4$ or (f_4). Then the functional I_λ possess the mountain pass geometry. Precisely,*

(i) $I_\lambda(0) = 0$;

(ii) There exists $r, b > 0$ such that $I_\lambda(u) \geqslant b$, whenever $\|u\|_V = r$;

(iii) There is $e_\lambda \in H^s_V(\mathbb{R}^3)$ with $\|e_\lambda\|_V > r$ and $I_\lambda(e_\lambda) < 0$;

In particular, $b < c(I_\lambda) < \infty$.

Proof. We follow a similar analysis to the one made in the proof of Lemma 3.3.3. Indeed, since $b(x) \in L^\beta(\mathbb{R}^3)$,

$$\int_{\mathbb{R}^3} b(x)u^2\,\mathrm{d}x \leqslant \left(\int_{\mathbb{R}^3} |b(x)|^\beta\,\mathrm{d}x\right)^{1/\beta}\left(\int_{\mathbb{R}^3} |u|^{2\beta/(\beta-1)}\,\mathrm{d}x\right)^{(\beta-1)/\beta}, \ \forall\, u \in H^s_V(\mathbb{R}^3),$$

with $2 < 2\beta/(\beta-1) < 2^*_s$, by conditions (f_1), (g_1) and (V_4), for any ε we get that

$$I_\lambda(u) \geqslant \left[\frac{1}{2}\left(1 - \frac{\|b(x)\|_\beta}{C_V^{(\beta)}} - 2\varepsilon\mathcal{C}_2\right) - (\varepsilon + \mathcal{C}_*)\mathcal{C}_{2^*_s}\|u\|_V^{2^*_s-2} - C_\varepsilon\mathcal{C}_{p_\varepsilon}\|u\|_V^{p_\varepsilon-2}\right]\|u\|_V^2,$$

for all $u \in H^s_V(\mathbb{R}^3)$, where $\mathcal{C}_2$, $\mathcal{C}_{2^*_s}$ and $\mathcal{C}_{p_\varepsilon}$ are positive constants provided by the embedding described in Proposition 3.3. This allow us to choose ε in a such way that the first term in the right-hand side of the above inequality is positive, once $\|u\|_V$ is taken small enough. Hence there exists $r > 0$ such that $I_\lambda(u) > 0$ provided that $\|u\|_V < r$. Let us assume first that conditions (f_2), (f_3) holds true. Let $\xi_R \in C_0^\infty(\mathbb{R})$, $R > 0$, such that $0 \leqslant \xi_R(t) \leqslant t_0$ and

$$\xi_R(t) = \begin{cases} t_0, & \text{if } |t| \leqslant R, \\ 0, & \text{if } |t| > R+1. \end{cases}$$

Set $v(x) := \xi_R(|x - x_0|)$. Then $v \in H^s_V(\mathbb{R}^3)$ and

$$\begin{aligned}\int_{\mathbb{R}^3} F(x,v)\,\mathrm{d}x &= \int_{B_R(x_0)} F(x,t_0)\,\mathrm{d}x + \int_{B_{R+1}(x_0)\backslash B_R(x_0)} F(x,v)\,\mathrm{d}x \\ &\geqslant |B_R| \inf_{B_R(x_0)} F(x,t_0) + |B_{R+1}\backslash B_R| \inf_{(x,t)\in(B_{R+1}(x_0)\backslash B_R(x_0))\times[0,t_0]} F(x,t) > 0.\end{aligned}$$

Since (f_2) and (g_2) are is equivalent to $d/dt(F(x,t)|t|^{-\mu}), d/dt(G(x,t)|t|^{-\mu_*}) \geqslant 0$, for $t > 0$, we have

$$\begin{cases} \displaystyle\int_{\mathbb{R}^3} F(x,tv)\,\mathrm{d}x \geqslant t^\mu \int_{\mathbb{R}^3} F(x,v)\,\mathrm{d}x, \\ \displaystyle\int_{\mathbb{R}^3} G(x,tv)\,\mathrm{d}x \geqslant t^{\mu_*} \int_{\mathbb{R}^3} G(x,v)\,\mathrm{d}x, \end{cases} \quad \text{whenever } t > 1.$$

Hence

$$\begin{aligned}
I_\lambda(tv) &= \frac{1}{2}t^2\|v\|_V^2 - \frac{1}{2}t^2\int_{\mathbb{R}^3} b(x)u^2\,\mathrm{d}x + \frac{\lambda}{4}t^4\mathcal{N}_\alpha(v) - \int_{\mathbb{R}^3} F(x,tv) + G(x,tv)\,\mathrm{d}x\\
&\leqslant \frac{1}{2}t^2\|v\|_V^2 + \frac{\lambda}{4}t^4\mathcal{N}_\alpha(v) - t^\mu\int_{\mathbb{R}^3} F(x,v)\,\mathrm{d}x - t^{\mu *}\int_{\mathbb{R}^3} G(x,v)\,\mathrm{d}x\\
&\quad \to -\infty,\ \text{as } t\to\infty.
\end{aligned}$$

Now suppose that assumption (f_4) holds true. For any given $R>0$, there exists $t_R>0$ such that

$$F(x,t) > Rt^4, \quad \forall\,|t|>t_R,\ \text{and}\ \forall\,x\in\mathbb{R}^3.$$

Let be $A(R,t) := \{x\in\mathbb{R}^3 : t|v(x)|>t_R\}$, for $t>0$. We have that

$$\begin{aligned}
\int_{\mathbb{R}^3} F(x,tv)\,\mathrm{d}x &= \int_{K_t} F(x,tv)\,\mathrm{d}x + \int_{A(R,t)} F(x,tv)\,\mathrm{d}x\\
&\geqslant \int_{K_t} F(x,tv)\,\mathrm{d}x + Rt^4\int_{A(R,t)} v^4\,\mathrm{d}x, \qquad (4.3.13)
\end{aligned}$$

where $K_t = (\mathbb{R}^3\backslash A(R,t))\cap \operatorname{supp}(v)$. Using condition (f_1), for each $t>0$, we get that

$$|F(x,tv)|\leqslant C, \quad \text{for a. e. } x\in K_t,$$

where C is a positive constant that does not depend in x and t. Consequently,

$$F(x,tv)\mathcal{X}_{K_t}(x)\to 0, \quad x\in\operatorname{supp}(v), \quad \text{as } t\to\infty,$$

where used that

$$\mathcal{X}_{\mathbb{R}^3\backslash A(R,t)}(x)\to\mathcal{X}_{\mathbb{R}^3\backslash\operatorname{supp}(v)}(x) = 0, \quad x\in\operatorname{supp}(v), \quad \text{as } t\to\infty.$$

Thus Lebesgue Convergence Theorem implies that the first integral in the right-hand side of (4.3.13) goes to zero as t goes to infinity. By the same reason, we also have that

$$\lim_{t\to\infty}\int_{A(R,t)} v^4\,\mathrm{d}x = \lim_{t\to\infty}\int_{\mathbb{R}^3} v^4\mathcal{X}_{A(R,t)}\,\mathrm{d}x = \int_{\mathbb{R}^3} v^4\mathcal{X}_{\{v\neq 0\}}\,\mathrm{d}x = \int_{\mathbb{R}^3} v^4\,\mathrm{d}x.$$

In particular, there exists a positive number $t_{0,R}$ such that

$$\frac{1}{4}\int_{\mathbb{R}^3} v^4\,\mathrm{d}x < \int_{A(R,t)} v^4\,\mathrm{d}x, \quad \forall\,t>t_{0,R}. \qquad (4.3.14)$$

Replacing (4.3.14) in (4.3.13) we obtain that

$$\begin{aligned}
I_\lambda(tv) &= \frac{t^2}{2}\|v\|_V^2 - \frac{t^2}{2}\int_{\mathbb{R}^3} b(x)v^2\,\mathrm{d}x + \frac{t^4}{4}\lambda\mathcal{N}_\alpha(v) - \int_{\mathbb{R}^3} F(x,tv) + G(x,tv)\,\mathrm{d}x\\
&\leqslant \frac{t^2}{2}\|v\|_V^2 + \frac{t^4}{4}(\lambda\mathcal{N}_\alpha(v) - R\|v\|_4^4) - \int_{K_t} F(x,tv)\,\mathrm{d}x - t^{\mu *}\int_{\mathbb{R}^3} G(x,v)\,\mathrm{d}x < 0,\\
&\quad \text{for } t>t_{0,R}.
\end{aligned}$$

provided R is chosen large enough. ■

Remark 4.3.8. (i) In addition to the assumptions of Lemma 4.3.7, assume that $F(x,t) > 0$ for a. e. $x \in \mathbb{R}^3$ and $t \neq 0$. Then, for any $u \in H^s_V(\mathbb{R}^3)\backslash\{0\}$, the path defined by $\zeta(t) = tu$ belongs to Γ_I. In fact, we make the following modification in the proof of Lemma 4.3.7, replacing v by u and using the same notation. We have that

$$\begin{cases} \displaystyle\int_{\mathbb{R}^3} F(x,tu)\,\mathrm{d}x \geqslant Rt^4 \int_{A(R,t)} u^4\,\mathrm{d}x, \\ \displaystyle\lim_{t\to\infty} \int_{A(R,t)} u^4\,\mathrm{d}x = \lim_{t\to\infty} \int_{\mathbb{R}^3} u^4 \mathcal{X}_{A(R,t)}\,\mathrm{d}x = \int_{\mathbb{R}^3} u^4 \mathcal{X}_{\{u\neq 0\}}\,\mathrm{d}x = \int_{\mathbb{R}^3} u^4\,\mathrm{d}x, \end{cases}$$

which enable us to proceed as in (4.3.14) to get

$$\varphi(t) := I_\lambda(tu) \leqslant \frac{t^2}{2}\|u\|_V^2 + \frac{t^4}{4}(\lambda\mathcal{N}_\alpha(u) - R\|u\|_4^4) \to -\infty, \text{ as } t \to \infty,$$

provided that R is taken large enough. Moreover, suppose that $h(x,t) = f(x,t) + g(x,t)$ satisfies the following condition: for a. e. $x \in \mathbb{R}^3$ the function

$$t \mapsto \frac{h(x,t)}{|t|}, \qquad \text{is strict increasing in } \mathbb{R}.$$

Then, taking into account that

$$\varphi'(t) = t\left[\|u\|_V^2 + t^2\lambda\mathcal{N}_\alpha(u) - \int_{\mathbb{R}^3} \frac{f(x,tu)}{t} u\,\mathrm{d}x - \int_{\mathbb{R}^3} \frac{g(x,tu)}{t} u\,\mathrm{d}x\right], \quad t > 0,$$

we infer that φ has a unique critical point.

(ii) In view of Lemma 4.3.7, we define the set

$$\Gamma^1_{I_\lambda} = \left\{\gamma \in C([0,1], H^s_V(\mathbb{R}^N)) : \gamma(0) = 0,\ \|\gamma(1)\| > r,\ I_\lambda(\gamma(1)) < 0\right\},$$

and

$$c_1(I_\lambda) = \inf_{\gamma\in\Gamma^1_{I_\lambda}} \sup_{t\in[0,1]} I_\lambda(\gamma(t)),$$

as the usual minimax level. We have that $c_1(I_\lambda) = c(I_\lambda)$.

(iii) Using the same arguments of the previous chapters we can see that when $f(x,t) \equiv f(t)$, the mountain pass geometry can be obtained by replacing (f_3) with (f_3'). In fact, let ξ_R as in the proof of Lemma 4.3.7 and define $\eta_R(x) = \xi_R(|x|)$. Then,

$$\int_{\mathbb{R}^3} F(\eta_R)\,\mathrm{d}x > 0,$$

provided that R is chosen large enough.

Proposition 4.3.9. *Suppose that $K(x)$, $a(x) = V(x) - b(x)$, $f(x,t)$ and $g(x,t)$ satisfy (K_1), (V_1), (V_2), (V_4), (f_1), (f_2), (g_1), (g_2), respectively, with $\mu \leqslant \mu_*$. Moreover, assume either $\mu > 4$ and (f_3) or $\mu = 4$ and (f_4). Then there exists a bounded sequence (u_k) such that $I_\lambda(u_k) \to c(I_\lambda)$ and $I'_\lambda(u_k) \to 0$, in the dual of $H^s_V(\mathbb{R}^3)$.*

Proof. In both cases, by Lemma 4.3.7, we may apply the standard Mountain Pass Theorem [16] in order to find a sequence (u_k) in $H^s_V(\mathbb{R}^3)$ such that $I_\lambda(u_k) \to c(I)$ and $I'_\lambda(u_k) \to 0$.
Assume first $\mu > 4$ and (f_3). For large k, we have

$$
\begin{aligned}
&c(I_\lambda) + 1 + \|u_k\|_V \\
&\quad \geqslant I_\lambda(u_k) - \frac{1}{\mu} I'_\lambda(u_k)\cdot u_k \\
&\quad = \left(\frac{1}{2} - \frac{1}{\mu}\right)\left(1 - \frac{\|b(x)\|_\beta}{\mathcal{C}_V^{(\beta)}}\right)\|u_k\|_V^2 + \lambda\left(\frac{1}{4} - \frac{1}{\mu}\right)\mathcal{N}_\alpha(u_k) \\
&\qquad\qquad - \int_{\mathbb{R}^3} F(x,u_k) - \frac{1}{\mu} f(x,u_k)u_k \,\mathrm{d}x - \int_{\mathbb{R}^3} G(x,u_k) - \frac{1}{\mu} g(x,u_k)u_k \,\mathrm{d}x \\
&\quad \geqslant \left(\frac{1}{2} - \frac{1}{\mu}\right)\left(1 - \frac{\|b(x)\|_\beta}{\mathcal{C}_V^{(\beta)}}\right)\|u_k\|_V^2,
\end{aligned}
$$

which implies that (u_k) is bounded. The case where $f(x,t)$ satisfies (f_2) with $\mu = 4$ and (f_4) follows by taking $\mu = 4$ in the above inequality. ■

It is worth to mention here the following complement of Proposition 3.4.5, which the proof follows the same argument.

Proposition 4.3.10. *Suppose that $V(x)$ satisfies (V_1)–(V_3) and $V(x) \geqslant 0$. Let (u_k) be a bounded sequence in $H^s(\mathbb{R}^3)$ and $(w^{(n)})_{n\in\mathbb{N}_0}$ provided by Theorem 1.1.2. Then*

$$\liminf_{k\to\infty} \int_{\mathbb{R}^3} V(x)u_k^2 \,\mathrm{d}x \geqslant \int_{\mathbb{R}^3} V(x)|w^{(1)}|^2 \,\mathrm{d}x + \sum_{n\in\mathbb{N}_0, n>1} \int_{\mathbb{R}^3} V_{\mathcal{P}}(x)|w^{(n)}|^2 \,\mathrm{d}x.$$

4.3.3 Estimate of the minimax level

Following the arguments of [42] we prove a estimate for the minimax level of the associated functionals. This is needed in order to prove Theorems 4.6.1 and 4.2.3. As might be seen in [28], the following infimum

$$\mathcal{S}_*(s) := \inf_{\substack{u\in\mathcal{D}^{s,2}(\mathbb{R}^3)\\ u\neq 0}} \left[\frac{\left(\int_{\mathbb{R}^3} |(-\Delta)^{s/2}u|^2 \,\mathrm{d}x\right)^{1/2}}{\left(\int_{\mathbb{R}^3} |u|^{2_s^*} \,\mathrm{d}x\right)^{1/2_s^*}}\right], \tag{4.3.15}$$

is attained by the following class of functions

$$u_\varepsilon(x) = \frac{\varepsilon^{\frac{3-2s}{2}}}{(|x|^2+\varepsilon^2)^{\frac{3-2s}{2}}}, \quad \varepsilon > 0,$$

where

$$\mathcal{S}_{*,s} = \left[2^{-2s}\pi^{-s}\frac{\Gamma\left(\frac{3-2s}{2}\right)}{\Gamma\left(\frac{3+2s}{2}\right)}\left(\frac{\Gamma(3)}{\Gamma(3/2)}\right)^{2s/3}\right]^{-1/2}.$$

Furthermore, consider $\xi \in C_0^\infty(\mathbb{R} : [0,1])$ a non-increasing cut-off such that

$$\xi(t) = \begin{cases} 1, & \text{if} \quad t \in [-1/2, 1/2] \\ 0, & \text{if} \quad |t| \geqslant 1 \end{cases} \qquad \text{and} \qquad |\xi'(t)| \leqslant C \quad \forall\, t \in \mathbb{R},$$

and define $\eta(x) = \xi(|x|)$. Then $\eta u_\varepsilon \in H^s_V(\mathbb{R}^3)$, provided that $V(x) \in L^\infty(\mathbb{R}^3)$ and (V_2) holds. Moreover, we have the following result.

Lemma C. *[42, Lemma 2.4] Let $\eta_\varepsilon = \eta u_\varepsilon/\|\eta u_\varepsilon\|_{2_s^*}$, then*

$$[\eta_\varepsilon]_s^2 \leqslant \mathcal{S}_*(s)^2 + \mathcal{O}(\varepsilon^{3-2s}),$$

$$\|\eta_\varepsilon\|_2^2 = \begin{cases} \mathcal{O}(\varepsilon^{2s}), & \textit{if} \quad 3 > 4s, \\ \mathcal{O}(\varepsilon^{2s}\log(1/\varepsilon)), & \textit{if} \quad 3 = 4s, \\ \mathcal{O}(\varepsilon^{3-2s}), & \textit{if} \quad 3 < 4s, \end{cases}$$

and

$$\|\eta_\varepsilon\|_p^p = \begin{cases} \mathcal{O}(\varepsilon^{\frac{6-(3-2s)p}{2}}), & \textit{if} \quad p \geqslant 3/(3-2s), \\ \mathcal{O}(\varepsilon^{\frac{(3-2s)p}{2}}), & \textit{if} \quad p \leqslant 3/(3-2s). \end{cases}$$

Here the notation $a_\varepsilon = \mathcal{O}(b_\varepsilon)$ means that $a_\varepsilon/b_\varepsilon$ is uniformly bounded with respect to ε, precisely, there exists positive constants c_1 and c_2 such that $c_1 < a_\varepsilon/b_\varepsilon < c_2$, for all ε.

Proposition 4.3.11. *Assume that (K_1) and (V_2) holds true with $V(x) \in L^\infty(\mathbb{R}^3)$. Moreover, suppose that $f(x,t) \equiv f_{\mathcal{P}}(x,t)$ and $g(x,t) \equiv g_\infty(t)$ satisfies (f_1), (f_7) and (g_1), (g_7), respectively. Then*

$$c(I_\lambda^{\mathcal{P}}) < \frac{s}{3}\left[\frac{\mathcal{S}_*(s)}{(2_s^* c_*)^{1/2_s^*}}\right]^{3/s}. \tag{4.3.16}$$

Proof. Define $\eta_\varepsilon^* = \eta_\varepsilon/(2_s^* c_*)^{1/2_s^*}$. It is easy to see that the path $t \mapsto t\eta_\varepsilon^*$ belongs to Γ_{I_λ}. We are going to prove that

$$\sup_{t \geqslant 0} I_\lambda(t\eta_\varepsilon^*) < \frac{s}{3}\left[\frac{\mathcal{S}_*(s)}{(2_s^* c_*)^{1/2_s^*}}\right]^{3/s}, \quad \text{for } \varepsilon \text{ small enough.}$$

By Proposition 4.3.3 we have that

$$I_\lambda(t\eta_\varepsilon^*) \leqslant \psi_\varepsilon(t) := \frac{1}{2}\|\eta_\varepsilon^*\|_V^2 t^2 + \frac{1}{4}C_\lambda\|\eta_\varepsilon^*\|_q^4 t^4 - c_0\|\eta_\varepsilon^*\|_{p_0}^{p_0}|t|^{p_0} - \frac{1}{2_s^*}|t|^{2_s^*}, \quad \forall\, t \geqslant 0,$$

where $C_\lambda = \lambda C_\alpha$. Since $\psi_\varepsilon(t) \to -\infty$ as $t \to +\infty$ and $\psi_\varepsilon(t) > 0$ for t close to zero, there exists $t_\varepsilon > 0$ so that $\sup_{t \geqslant 0} \psi_\varepsilon(t) = \psi_\varepsilon(t_\varepsilon)$. Thus $\psi'_\varepsilon(t_\varepsilon) = 0$ and as consequence,

$$\|\eta^*_\varepsilon\|^2_V + C^2_\lambda \|\eta^*_\varepsilon\|^4_q t^2_\varepsilon - c_0 \|\eta^*_\varepsilon\|^{p_0}_{p_0} |t_\varepsilon|^{p_0 - 2} = |t_\varepsilon|^{2^*_s - 2}. \tag{4.3.17}$$

Moreover, the above identity (4.3.17) implies that $0 < c_1 \leqslant t_\varepsilon \leqslant c_2$ for $\varepsilon < 1$, and some positive constants c_1 and c_2. Now let

$$\varphi_\varepsilon(t) = \frac{1}{2}\|\eta^*_\varepsilon\|^2_V t^2 - \frac{1}{2^*_s}|t|^{2^*_s},$$

which has a maximum point $\tilde{t}_\varepsilon = (\|\eta^*_\varepsilon\|^2_V)^{1/(2^*_s - 2)}$. We have

$$\begin{aligned}
\sup_{t \geqslant 0} \psi_\varepsilon(t) &\leqslant \sup_{t \geqslant 0} \varphi_\varepsilon(t) + \frac{1}{4} C_\lambda c_2^4 \|\eta^*_\varepsilon\|^4_q - c_0 c_1^{p_0} \|\eta^*_\varepsilon\|^{p_0}_{p_0} \\
&= \frac{s}{3}\|\eta^*_\varepsilon\|^{3/s}_V + \frac{1}{4} C_\lambda c_2^4 \|\eta^*_\varepsilon\|^4_q - c_0 c_1^{p_0} \|\eta^*_\varepsilon\|^{p_0}_{p_0}, \quad \varepsilon < 1.
\end{aligned} \tag{4.3.18}$$

Using Lemma C, we now pass to estimate each term in (4.3.18). Also, in what follows we use the inequality $(a + b)^\alpha \leqslant a^\alpha + \alpha(a + b)^{\alpha - 1} b$, $\alpha \geqslant 1$ and $a, b > 0$, and always consider $\varepsilon < 1$.

- For the first term we have

$$\begin{aligned}
\left(\|\eta^*_\varepsilon\|^2_V\right)^{3/2s} &\leqslant \left([\eta^*_\varepsilon]^2_s + \|V(x)\|_\infty \|\eta^*_\varepsilon\|^2_2\right)^{3/2s} \\
&\leqslant \left(\frac{\mathcal{S}_*(s)}{2^*_s c_*}\right)^{3/s} + c_3 \|\eta^*_\varepsilon\|^2_2 + \mathcal{O}(\varepsilon^{3-2s}), \quad \text{for some positive constant } c_3, \\
&= \begin{cases}
\left(\frac{\mathcal{S}_*(s)}{2^*_s c_*}\right)^{3/s} + \mathcal{O}(\varepsilon^{3-2s}) + \mathcal{O}(\varepsilon^{2s}), & \text{if} \quad 3 > 4s, \\
\left(\frac{\mathcal{S}_*(s)}{2^*_s c_*}\right)^{3/s} + \mathcal{O}(\varepsilon^{3-2s}) + \mathcal{O}(\varepsilon^{2s}\log(1/\varepsilon)), & \text{if} \quad 3 = 4s, \\
\left(\frac{\mathcal{S}_*(s)}{2^*_s c_*}\right)^{3/s} + \mathcal{O}(\varepsilon^{3-2s}), & \text{if} \quad 3 < 4s.
\end{cases}
\end{aligned}$$

- For the second term, we have

$$\left(\|\eta^*_\varepsilon\|^q_q\right)^{4/q} = \begin{cases}
\mathcal{O}(\varepsilon^{\frac{12 - 2(3-2s)q}{q}}), & \text{if} \quad q \geqslant 3/(3-2s), \\
\mathcal{O}(\varepsilon^{2(3-2s)}), & \text{if} \quad q \leqslant 3/(3-2s).
\end{cases}$$

- For the third term, we have $\|\eta^*_\varepsilon\|^{p_0}_{p_0} = \mathcal{O}(\varepsilon^{\frac{6 - (3-2s)p_0}{2}})$, since $p_0 > 4$ implies that $p_0 > 3/(3-2s)$.

Summing up, we get the following.

- For the case $q \geqslant 3/(3-2s)$, we get

$$\sup_{t\geqslant 0} I_\lambda(t\eta_\varepsilon^*) \leqslant \sup_{t\geqslant 0}\psi_\varepsilon(t) \leqslant$$
$$\begin{cases} \left(\dfrac{\mathcal{S}_*(s)}{2_s^* c_*}\right)^{3/s} + \mathcal{O}(\varepsilon^{3-2s}) + \mathcal{O}(\varepsilon^{2s}) + \mathcal{O}(\varepsilon^{\frac{12-2(3-2s)q}{q}}) - \mathcal{O}(\varepsilon^{\frac{6-(3-2s)p_0}{2}}), & \text{if } 3 > 4s, \\ \left(\dfrac{\mathcal{S}_*(s)}{2_s^* c_*}\right)^{3/s} + \mathcal{O}(\varepsilon^{3-2s}) + \mathcal{O}(\varepsilon^{2s}\log(1/\varepsilon)) & \\ \qquad\qquad + \mathcal{O}(\varepsilon^{\frac{12-2(3-2s)q}{q}}) - \mathcal{O}(\varepsilon^{\frac{6-(3-2s)p_0}{2}}), & \text{if } 3 = 4s, \\ \left(\dfrac{\mathcal{S}_*(s)}{2_s^* c_*}\right)^{3/s} + \mathcal{O}(\varepsilon^{3-2s}) + \mathcal{O}(\varepsilon^{\frac{12-2(3-2s)q}{q}}) - \mathcal{O}(\varepsilon^{\frac{6-(3-2s)p_0}{2}}), & \text{if } 3 < 4s. \end{cases}$$

- For the case $q < 3/(3-2s)$,

$$\sup_{t\geqslant 0} I_\lambda(t\eta_\varepsilon^*) \leqslant \sup_{t\geqslant 0}\psi_\varepsilon(t) \leqslant$$
$$\begin{cases} \left(\dfrac{\mathcal{S}_*(s)}{2_s^* c_*}\right)^{3/s} + \mathcal{O}(\varepsilon^{3-2s}) + \mathcal{O}(\varepsilon^{2s}) + \mathcal{O}(\varepsilon^{2(3-2s)}) - \mathcal{O}(\varepsilon^{\frac{6-(3-2s)p_0}{2}}), & \text{if } 3 > 4s, \\ \left(\dfrac{\mathcal{S}_*(s)}{2_s^* c_*}\right)^{3/s} + \mathcal{O}(\varepsilon^{3-2s}) + \mathcal{O}(\varepsilon^{2s}\log(1/\varepsilon)) & \\ \qquad\qquad + \mathcal{O}(\varepsilon^{2(3-2s)}) - \mathcal{O}(\varepsilon^{\frac{6-(3-2s)p_0}{2}}), & \text{if } 3 = 4s, \\ \left(\dfrac{\mathcal{S}_*(s)}{2_s^* c_*}\right)^{3/s} + \mathcal{O}(\varepsilon^{3-2s}) + \mathcal{O}(\varepsilon^{2(3-2s)}) - \mathcal{O}(\varepsilon^{\frac{6-(3-2s)p_0}{2}}), & \text{if } 3 < 4s. \end{cases}$$

Since the following inequalities are always true

$$\begin{cases} \dfrac{6-(3-2s)p_0}{2} < 2s < \dfrac{12-2(3-2s)q}{q}, \\ \dfrac{6-(3-2s)p_0}{2} < 3-2s, \end{cases} \qquad 0 < s < 1$$

and

$$\lim_{\varepsilon\to 0}\frac{\varepsilon^{\frac{6-(3-2s)p_0}{2}}}{\varepsilon^{2s}\log(1/\varepsilon)} = +\infty,$$

the term involving the power $\varepsilon^{\frac{6-(3-2s)p_0}{2}}$ grows faster near zero than any other terms involving ε in each considered case. Thus we obtain estimate (4.3.16) by taking ε small enough. ■

4.3.4 Pohozaev identity

We now prove that weak solutions for Eq. $(\mathcal{S}_{NL})$ satisfy a Pohozaev type identity. The proof follows similar arguments used in Proposition 3.5.1 with additional caution, since we have to consider the nonlocal term in Eq. $(\mathcal{S}_{NL})$. This is the reason why we ask an smoothness C^1 in the potential $a(x)$ instead the one in Proposition 3.5.1, that $a(x)$ may have finite points of discontinuity.

Proposition 4.3.12. *Assume that* $f(x,t) = f(t) \in C^1(\mathbb{R})$, $g(x,t) \equiv 0$, $K(x) \equiv K_0 > 0$ *and* $a(x) \in C^1(\mathbb{R}^3)$. *Let* $u \in H^s(\mathbb{R}^3)$ *be a weak solution of Eq.* $(\mathcal{S}_{NL})$ *such that* $f(u)/(1+|u|)$ *belongs to* $L^{N/2s}_{\rm loc}(\mathbb{R}^3)$. *If* $F(u)$, $f(u)u$, $a(x)u^2$ *and* $\langle \nabla a(x), x\rangle u^2$ *belongs to* $L^1(\mathbb{R}^3)$, *then* $u \in C^1(\mathbb{R}^3)$ *and*

$$\frac{3-2s}{2}\int_{\mathbb{R}^3} |(-\Delta)^{s/2}u|^2\,\mathrm{d}x + \frac{3}{2}\int_{\mathbb{R}^3} a(x)u^2\,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^3}\langle \nabla a(x), x\rangle u^2\,\mathrm{d}x + \frac{3+2\alpha}{4}\lambda K_0\int_{\mathbb{R}^3}\phi_\alpha[u]u^2\,\mathrm{d}x = 3\int_{\mathbb{R}^3} F(u)\,\mathrm{d}x. \quad (4.3.19)$$

Proof. We divide our proof in two steps. In what follows we assume without loss of generality that $\lambda = K_0 = 1$ and use that $w = E_s(u)$ is a weak solution of problem (0.3.3), where $h(x,u) := f(u) - (a(x) + \phi_\alpha[u])u$.

First step (Regularity). We shall prove first that u belongs to $L^r_{\rm loc}(\mathbb{R}^3)$ for all $r \geqslant 1$. In order to prove this, observe that, given $p > 1$, by Proposition 0.1.6, $\phi_\alpha[u]$ belongs to $L^q(\mathbb{R}^3)$, for some $q > 3/2s$ if, and only,

$$q = \frac{3p}{3-2\alpha p} > \frac{3}{2s}, \quad \text{that is,} \quad p > \frac{3}{2s+2\alpha}. \quad (4.3.20)$$

As seen in Proposition 4.3.2, we know that $p = 2^*_\alpha/(2^*_\alpha - 1)$, hence (4.3.20) holds true. Furthermore, we have that

$$\frac{|h(u)|}{1+|u|} \in L^{3/2s}_{\rm loc}(\mathbb{R}^3).$$

Thus, from now on we can follow exactly the same lines as in Proposition 2.3.1 to conclude that $u \in L^r_{\rm loc}(\mathbb{R}^3)$. Moreover, since $\phi_\alpha[u]$ is a weak solution of $(-\Delta)^\alpha\phi_\alpha[u] = K_0u^2$ the same conclusion follows for $\phi_\alpha[u]$, once is known that u belongs to $L^r_{\rm loc}(\mathbb{R}^3)$ for all $r \geqslant 1$. Writing

$$h(u) = \left[\frac{f(u)}{1+|u|}\operatorname{sgn}(u) - a(x) - \phi_\alpha[u]\right]u + \frac{f(u)}{1+|u|},$$

we see that the regularity follows by applying the results of [59] and proceeding as described in Proposition 0.4.1. Thus, for any $R > 0$ there exists $0 < y_0, r < R$ with $B^3_r \times [0, y_0] \subset B^+_R$ and $0 < \mu < 1$ such that

$$E_s(u),\ \nabla_x E_s(u),\ y^{1-2s}E_s(u),\ E_\alpha(\phi_\alpha),\ \nabla_x E_\alpha(\phi_\alpha),\ y^{1-2s}E_\alpha(\phi_\alpha) \in C^{0,\mu}(B^3_r \times [0, y_0]). \quad (4.3.21)$$

Second step (Local computation). Let ξ_n as in Remark 4.2.9. As before, we have

$$\begin{aligned}\operatorname{div}&(y^{1-2s}\nabla w)\langle z,\nabla w\rangle\xi_n\\ &=\operatorname{div}\left[y^{1-2s}\xi_n\left(\langle z,\nabla w\rangle\nabla w-\frac{|\nabla w|^2}{2}z\right)\right]+\frac{N-2s}{2}y^{1-2s}|\nabla w|^2\xi_n\\ &\qquad+y^{1-2s}\frac{|\nabla w|^2}{2}\langle z,\nabla\xi_n\rangle-y^{1-2s}\langle\nabla w,z\rangle\langle\nabla w,\nabla\xi_n\rangle.\end{aligned}\tag{4.3.22}$$

Note that $\partial B_{\sqrt{2}n,\delta}=F^1_{\sqrt{2}n,\delta}\cup F^2_{\sqrt{2}n,\delta}$. Let $\eta(z)=(0,0,-1)$ be the unit outward normal vector of $B_{\sqrt{2}n,\delta}$ on $F^1_{\sqrt{2}n,\delta}$. Since $\xi_n=0$ on $F^2_{\sqrt{2}n,\delta}$, by condition (0.3.2), identity (4.3.22) and Divergence Theorem we get

$$\begin{aligned}0&=\int_{B_{\sqrt{2}n,\delta}}\operatorname{div}(y^{1-2s}\nabla w)\langle z,\nabla w\rangle\xi_n\,\mathrm{d}x\mathrm{d}y\\ &=\int_{F^1_{\sqrt{2}n,\delta}}y^{1-2s}\xi_n\left[\langle z,\nabla w\rangle\langle\nabla w,\eta\rangle-\frac{|\nabla w|^2}{2}\langle z,\eta\rangle\right]\mathrm{d}x\mathrm{d}y+\theta_{n,\delta}\\ &=\int_{F^1_{\sqrt{2}n,\delta}}\xi_n\langle x,\nabla_x w\rangle(-y^{1-2s}w_y)\,\mathrm{d}x\\ &\qquad-\int_{F^1_{\sqrt{2}n,\delta}}y^{1-2s}\xi_n w_y^2 y\,\mathrm{d}x+\int_{F^1_{\sqrt{2}n,\delta}}y^{1-2s}\xi_n\frac{|\nabla w|^2}{2}y\,\mathrm{d}x+\theta_{n,\delta}\\ &=I^1_{n,\delta}+I^2_{n,\delta}+I^3_{n,\delta}+\theta_{n,\delta},\end{aligned}$$

where

$$\begin{aligned}\theta_{n,\delta}=&\int_{B_{\sqrt{2}n,\delta}}\frac{N-2s}{2}y^{1-2s}|\nabla w|^2\xi_n\,\mathrm{d}x\mathrm{d}y\\ &+\int_{B_{\sqrt{2}n,\delta}}y^{1-2s}\frac{|\nabla w|^2}{2}\langle z,\nabla\xi_n\rangle-y^{1-2s}\langle\nabla w,z\rangle\langle\nabla w,\nabla\xi_n\rangle\,\mathrm{d}x\mathrm{d}y.\end{aligned}$$

We know that there exists a sequence $\delta_k\to 0$ such that

$$I^2_{n,\delta_k}+I^3_{n,\delta_k}\to 0,\ \text{ as } k\to\infty.$$

Some computations leads to

$$\begin{aligned}\xi_n&(x,0)\langle x,\nabla u\rangle(f(u)-(a(x)+\phi_\alpha[u])u)\\ &=\operatorname{div}\left[\xi_n(x,0)\left(F(u)-\frac{1}{2}(a(x)+\phi_\alpha[u])u^2\right)x\right]\\ &\quad-\langle\nabla\xi_n(x,0),x\rangle F(u)-3\xi_n(x,0)F(u)+\frac{1}{2}\langle\nabla\xi_n(x,0),x\rangle(a(x)+\phi_\alpha[u])u^2\\ &\quad+\frac{1}{2}\xi_n(x,0)\langle\nabla(a(x)+\phi_\alpha[u]),x\rangle u^2+\frac{3}{2}\xi_n(x,0)(a(x)+\phi_\alpha[u])u^2.\end{aligned}$$

Thus, by Remark 0.4.3, conditions (0.3.2), (4.3.21) and the Divergence Theorem we have

$$
\begin{aligned}
\lim_{k\to\infty} I^1_{n,\delta_k} &= \kappa_s \int_{B^3_{\sqrt{2}n}} \xi_n(x,0)\,\langle x, \nabla u\rangle\,(f(u) - (a(x) + \phi_\alpha[u])u)\,\mathrm{d}x \\
&= -\kappa_s \int_{B^3_{\sqrt{2}n}} \langle \nabla \xi_n, x\rangle\, F(u) + 3\xi_n F(u)\,\mathrm{d}x \\
&\quad + \frac{\kappa_s}{2}\int_{B^3_{\sqrt{2}n}} \langle \nabla \xi_n, x\rangle\,(a(x) + \phi_\alpha[u])u^2\,\mathrm{d}x \\
&\quad + \frac{\kappa_s}{2}\int_{B^3_{\sqrt{2}n}} \xi_n\,\langle \nabla(a(x) + \phi_\alpha[u]), x\rangle\, u^2 + 3\xi_n(a(x) + \phi_\alpha[u])u^2\,\mathrm{d}x.
\end{aligned}
$$

Summing up, we get

$$
\begin{aligned}
0 &= \lim_{k\to\infty} \left[I^1_{n,\delta_k} + I^2_{n,\delta_k} + I^3_{n,\delta_k} + \theta_{n,\delta_k} \right] \\
&= -\kappa_s \int_{B^3_{\sqrt{2}n}} \langle \nabla \xi_n, x\rangle\, F(u) + 3\xi_n F(u)\,\mathrm{d}x \\
&\quad + \frac{\kappa_s}{2}\int_{B^3_{\sqrt{2}n}} \langle \nabla \xi_n, x\rangle\,(a(x) + \phi_\alpha[u])u^2\,\mathrm{d}x \\
&\quad + \frac{\kappa_s}{2}\int_{B^3_{\sqrt{2}n}} \xi_n\,\langle \nabla(a(x) + \phi_\alpha[u]), x\rangle\, u^2 + 3\xi_n(a(x) + \phi_\alpha[u])u^2\,\mathrm{d}x \\
&\quad + \int_{B^+_{\sqrt{2}n}} \frac{3-2s}{2} y^{1-2s} |\nabla w|^2 \xi_n\,\mathrm{d}x \\
&\quad + \int_{B^+_{\sqrt{2}n}} \frac{1}{2} y^{1-2s} |\nabla w|^2 \langle z, \nabla \xi_n\rangle - y^{1-2s}\langle \nabla w, z\rangle \langle \nabla w, \nabla \xi_n\rangle\,\mathrm{d}x. \qquad (4.3.23)
\end{aligned}
$$

Since $v = \phi_\alpha[u]$ is a weak solution for the equation $(-\Delta)^\alpha v = u^2$, similar arguments also can be applied to $w_\alpha = E_\alpha(\phi_\alpha[u])$. Hence,

$$
\begin{aligned}
0 = -\kappa_\alpha &\int_{B^3_{\sqrt{2}n}} 2\xi_n \phi_\alpha[u] u \langle \nabla u, x\rangle + 3\xi_n \phi_\alpha[u] u^2\,\mathrm{d}x \\
&+ \int_{B^+_{\sqrt{2}n}} \frac{3-2\alpha}{2} y^{1-2\alpha} |\nabla w_\alpha|^2 \xi_n + \frac{1}{2} y^{1-2\alpha} |\nabla w_\alpha|^2 \langle z, \nabla \xi_n\rangle\,\mathrm{d}x \\
&\qquad - \int_{B^+_{\sqrt{2}n}} y^{1-2\alpha}\langle \nabla w_\alpha, z\rangle \langle \nabla w_\alpha, \nabla \xi_n\rangle\,\mathrm{d}x. \qquad (4.3.24)
\end{aligned}
$$

On the other hand, integrating by parts, we have

$$
\begin{aligned}
&\int_{B^3_{\sqrt{2}n}} \xi_n \langle \nabla \phi_\alpha[u], x\rangle\, u^2\,\mathrm{d}x = \\
&\quad - \int_{B^3_{\sqrt{2}n}} u^2 \phi_\alpha[u] \langle \nabla \xi_n, x\rangle\,\mathrm{d}x - \int_{B^3_{\sqrt{2}n}} 2\xi_n u \phi_\alpha[u] \langle \nabla u, x\rangle + 3\xi_n u^2 \phi_\alpha[u]\,\mathrm{d}x. \qquad (4.3.25)
\end{aligned}
$$

Using identity (4.3.24) in (4.3.25) we obtain

$$\begin{aligned}
&\frac{\kappa_s}{2}\int_{B^3_{\sqrt{2}n}} \xi_n \left\langle \nabla\phi_\alpha[u], x\right\rangle u^2\,\mathrm{d}x = \\
&\quad -\frac{\kappa_s}{2}\int_{B^3_{\sqrt{2}n}} \left\langle \nabla\xi_n, x\right\rangle u^2\phi_\alpha[u]\,\mathrm{d}x - \frac{(3-2\alpha)\kappa_s}{4\kappa_\alpha}\int_{B^+_{\sqrt{2}n}} y^{1-2\alpha}|\nabla w_\alpha|^2\xi_n\,\mathrm{d}x \\
&\quad -\frac{\kappa_s}{2\kappa_\alpha}\left[\int_{B^+_{\sqrt{2}n}} \frac{1}{2}y^{1-2\alpha}|\nabla w_\alpha|^2\left\langle z,\nabla\xi_n\right\rangle - y^{1-2\alpha}\left\langle\nabla w_\alpha, z\right\rangle\left\langle\nabla w_\alpha,\nabla\xi_n\right\rangle\,\mathrm{d}x\right].
\end{aligned} \tag{4.3.26}$$

On the other hand, by a density argument, we can choose $w_\phi\xi_n$ as a test function in definition (0.3.4) and get as consequence,

$$\begin{aligned}
&\int_{B^+_{\sqrt{2}n}} y^{1-2\alpha}|\nabla w_\phi|^2\xi_n\,\mathrm{d}x\mathrm{d}y \\
&\qquad = \kappa_\alpha\int_{B^3_{\sqrt{2}n}} \xi_n\phi_\alpha[u]u^2\,\mathrm{d}x - \int_{B^+_{\sqrt{2}n}} y^{1-2\alpha}w_\phi\left\langle\nabla w_\phi,\nabla\xi_n\right\rangle\,\mathrm{d}x\mathrm{d}y.
\end{aligned} \tag{4.3.27}$$

It follows, replacing identity (4.3.27) in (4.3.26), that

$$\begin{aligned}
&\frac{\kappa_s}{2}\int_{B^3_{\sqrt{2}n}} \xi_n u^2\left\langle \nabla\phi_\alpha[u], x\right\rangle\,\mathrm{d}x = \\
&-\frac{\kappa_s}{2}\int_{B^3_{\sqrt{2}n}} \left\langle \nabla\xi_n, x\right\rangle u^2\phi_\alpha[u]\,\mathrm{d}x \\
&-\frac{(3-2\alpha)\kappa_s}{4\kappa_\alpha}\left[\kappa_\alpha\int_{B^3_{\sqrt{2}n}} \xi_n\phi_\alpha[u]u^2\,\mathrm{d}x - \int_{B^+_{\sqrt{2}n}} y^{1-2\alpha}w_\phi\left\langle\nabla w_\phi,\nabla\xi_n\right\rangle\,\mathrm{d}x\mathrm{d}y\right] \\
&-\frac{\kappa_s}{2\kappa_\alpha}\left[\int_{B^+_{\sqrt{2}n}} \frac{1}{2}y^{1-2\alpha}|\nabla w_\alpha|^2\left\langle z,\nabla\xi_n\right\rangle - y^{1-2\alpha}\left\langle\nabla w_\alpha, z\right\rangle\left\langle\nabla w_\alpha,\nabla\xi_n\right\rangle\,\mathrm{d}x\mathrm{d}y\right].
\end{aligned} \tag{4.3.28}$$

Finally, replacing expression (4.3.28) in (4.3.23) we obtain

$$\begin{aligned}
0 = &-\kappa_s\int_{B^3_{\sqrt{2}n}} \left\langle\nabla\xi_n, x\right\rangle F(u) + 3\xi_n F(u)\,\mathrm{d}x + \frac{3+2\alpha}{4}\kappa_s\int_{B^3_{\sqrt{2}n}} \xi_n\phi_\alpha[u]u^2\,\mathrm{d}x \\
&+\frac{\kappa_s}{2}\int_{B^3_{\sqrt{2}n}} \left\langle\nabla\xi_n, x\right\rangle a(x)u^2 + \xi_n\left\langle\nabla a(x), x\right\rangle u^2 + \frac{3}{2}\xi_n a(x)u^2\,\mathrm{d}x \\
&-\frac{\kappa_s}{2\kappa_\alpha}\left[\int_{B^+_{\sqrt{2}n}} \frac{1}{2}y^{1-2\alpha}|\nabla w_\alpha|^2\left\langle z,\nabla\xi_n\right\rangle - y^{1-2\alpha}\left\langle\nabla w_\alpha, z\right\rangle\left\langle\nabla w_\alpha,\nabla\xi_n\right\rangle\,\mathrm{d}x\mathrm{d}y\right] \\
&+\int_{B^+_{\sqrt{2}n}} \frac{3-2s}{2}y^{1-2s}|\nabla w|^2\xi_n\,\mathrm{d}x \\
&+\int_{B^+_{\sqrt{2}n}} \frac{1}{2}y^{1-2s}|\nabla w|^2\left\langle z,\nabla\xi_n\right\rangle - y^{1-2s}\left\langle\nabla w, z\right\rangle\left\langle\nabla w,\nabla\xi_n\right\rangle\,\mathrm{d}x \\
&+\frac{(3-2\alpha)}{4}\kappa_s\int_{B^+_{\sqrt{2}n}} y^{1-2\alpha}w_\phi\left\langle\nabla w_\phi,\nabla\xi_n\right\rangle\,\mathrm{d}x\mathrm{d}y.
\end{aligned} \tag{4.3.29}$$

Thus the identity (4.3.19) follows from (4.3.29) by applying the Dominated Convergence Theorem and using (0.3.2). ■

4.4 Proof of Theorem 4.2.1

Proof. Our argument follows the same one in the proof of Theorem 3.2.1–(i). For the reader convenience we divide the proof in several steps.

(i) By Proposition 4.3.9 we known of the existence of a bounded sequence (u_k) in a such way that $I_\lambda^{\mathcal{P}}(u_k) \to c(I_\lambda^{\mathcal{P}})$ and $(I_\lambda^{\mathcal{P}})'(u_k) \to 0$. Since it is bounded, it has a profile decomposition provided by Theorem 1.1.2. If we have $w^{(n)} = 0$ for all $n \in \mathbb{N}_0$, then by assertion (1.1.8), $u_k \to 0$ in $L^p(\mathbb{R}^3)$, for any $2 < p < 2_s^*$ and by convergence (1.1.5) $u_k \rightharpoonup 0$ in $H^s_{V_{\mathcal{P}}}(\mathbb{R}^3)$, in a subsequence. Consequently, by Propositions 3.4.1, 3.4.2 and 4.3.4, we have

$$\begin{cases} o(1) + c(I_\lambda^{\mathcal{P}}) = I_\lambda^{\mathcal{P}}(u_k) = \dfrac{1}{2}\|u_k\|^2_{V_{\mathcal{P}}} + \dfrac{\lambda}{4}\mathcal{N}_\alpha^{\mathcal{P}}(u_k) - \displaystyle\int_{\mathbb{R}^3} F_{\mathcal{P}}(x,u_k)\,\mathrm{d}x \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad = \dfrac{1}{2}\|u_k\|^2_{V_{\mathcal{P}}} + o(1), \\ o(1) = (I_\lambda^{\mathcal{P}})'(u_k)\cdot u_k = \|u_k\|^2_{V_{\mathcal{P}}} + \lambda\mathcal{N}_\alpha^{\mathcal{P}}(u_k) - \displaystyle\int_{\mathbb{R}^3} f_{\mathcal{P}}(x,u_k)u_k\,\mathrm{d}x \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad = \|u_k\|^2_{V_{\mathcal{P}}} + o(1), \end{cases} \tag{4.4.1}$$

a contradiction, since $c(I_\lambda^{\mathcal{P}}) > 0$. Thus, there must be at least one nonzero $w^{(n)}$.

(ii) Moreover, we have that each $w^{(n)}$ is a critical point of $I_\lambda^{\mathcal{P}}$. In fact, it is well known that, up to subsequence, we can take $h^{(n)}$ in $L^{\sigma'}(\mathrm{supp}(\varphi))$, $n \in \mathbb{N}_0$, such that

$$|u_k(x + y_k^{(n)})| \leqslant h^{(n)}(x) \quad \text{a. e. } x \in \mathrm{supp}(\varphi), \tag{4.4.2}$$

where $\sigma' = \sigma/(\sigma-1)$ and $\varphi \in C_0^\infty(\mathbb{R}^3)$, which can be done thanks to Proposition 3.3.1. Thus, for a.e. $x \in \mathbb{R}^3$, we have

$$\begin{cases} |V_{\mathcal{P}}(x+y_k^{(n)})u_k(x+y_k^{(n)})\varphi(x)| \leqslant h^{(n)}(x)|V_{\mathcal{P}}(x)\varphi(x)| \in L^1(\mathrm{supp}(\varphi)) \\ V_{\mathcal{P}}(x+y_k^{(n)})u_k(x+y_k^{(n)})\varphi(x) = V_{\mathcal{P}}(x)u_k(x+y_k^{(n)})\varphi(x) \to V_{\mathcal{P}}(x)w^{(n)}(x)\varphi(x), \end{cases}$$

which by the Lebesgue Convergence Theorem leads to

$$\begin{aligned} \lim_{k\to\infty}(u_k, \varphi(\cdot - y_k^{(n)}))_{V_{\mathcal{P}}} &= \lim_{k\to\infty}\left[[u_k(\cdot + y_k^{(n)}),\varphi]_s + \int_{\mathbb{R}^3} V_{\mathcal{P}}(x+y_k^{(n)})u_k(\cdot+y_k^{(n)})\varphi(x)\,\mathrm{d}x\right] \\ &= [w^{(n)},\varphi]_s + \int_{\mathbb{R}^3} V_{\mathcal{P}}(x)w^{(n)}\varphi\,\mathrm{d}x. \end{aligned}$$

By the same reason and (f_1), up to subsequence we have,

$$\lim_{k\to\infty}\int_{\mathbb{R}^3} f_{\mathcal{P}}(\cdot + y_k^{(n)}, u_k(\cdot + y_k^{(n)}))\varphi\,\mathrm{d}x = \int_{\mathbb{R}^3} f_{\mathcal{P}}(x, w^{(n)})\varphi\,\mathrm{d}x.$$

Consequently, by Lemma 4.3.6, we may pass the limit in

$$(I_\lambda^{\mathcal{P}})'(u_k)\cdot\varphi(\cdot - y_k^{(n)}) = (u_k, \varphi(\cdot - y_k^{(n)}))_{V_{\mathcal{P}}} + \lambda(\mathcal{N}_\alpha^{\mathcal{P}})'(u_k)\cdot\varphi(\cdot - y_k^{(n)}) - \int_{\mathbb{R}^3} f_{\mathcal{P}}(\cdot + y_k^{(n)}, u_k(\cdot + y_k^{(n)}))\varphi\,\mathrm{d}x,$$

to conclude that $(I_\lambda^{\mathcal{P}})'(w^{(n)}) = 0$, for all $n\in\mathbb{N}_0$.

(iii) In particular, we get that

$$\mathcal{G}_{\mathcal{S}} = \inf\left\{I_\lambda^{\mathcal{P}}(u) : u\in H^s_{V_{\mathcal{P}}}(\mathbb{R}^3)\backslash\{0\},\ (I_\lambda^{\mathcal{P}})'(u) = 0\right\} \geqslant 0,$$

We are going to prove that is $\mathcal{G}_{\mathcal{S}}$ is attained and is positive. Let (u_k) be a minimizing sequence of $\mathcal{G}_{\mathcal{S}}$, that is $I_\lambda^{\mathcal{P}}(u_k)\to\mathcal{G}_{\mathcal{S}}$ and $(I_\lambda^{\mathcal{P}})'(u_k) = 0$. Arguing as in Proposition 4.3.9 we obtain that (u_k) is bounded. We argue again by contradiction and assume that $w^{(n)} = 0$ for all $n\in\mathbb{N}_0$. In this case we actually have that $\mathcal{G}_{\mathcal{S}} > 0$, because on the contrary, if $\mathcal{G}_{\mathcal{S}} = 0$, then using (4.4.1) we would conclude that $\|u_k\|_{V_{\mathcal{P}}} = o(1)$, and at the same time,

$$\|u_k\|^2_{V_{\mathcal{P}}} \leqslant \|u_k\|^2_{V_{\mathcal{P}}} + \lambda\mathcal{N}_\alpha^{\mathcal{P}}(u_k) = \int_{\mathbb{R}^3} f_{\mathcal{P}}(u_k)u_k\,\mathrm{d}x \leqslant \varepsilon(C_2\|u_k\|^2_{V_{\mathcal{P}}} + C_*\|u_k\|^{2_s^*}_{V_{\mathcal{P}}}) + C_\varepsilon\|u_k\|^{p_\varepsilon}_{V_{\mathcal{P}}},$$

where $\mathcal{C}_2$, $\mathcal{C}_{2_s^*}$ and $\mathcal{C}_{p_\varepsilon}$ are positive constant obtained by applying the embedding described in Proposition 3.3.1. In particular,

$$(1-\varepsilon\mathcal{C}_2) \leqslant \varepsilon\mathcal{C}_{2_s^*}\|u_k\|^{2_s^*-2}_{V_{\mathcal{P}}} + \mathcal{C}_{p_\varepsilon}\|u_k\|^{p_\varepsilon-2}_{V_{\mathcal{P}}}, \quad \forall\, k\in\mathbb{N},$$

which, by taking ε small enough, would lead to a contradiction with the fact that $\|u_k\|_{V_{\mathcal{P}}} = o(1)$. In view of that, in any case, we can argue as above to conclude that there must be a nonzero $w^{(n_0)}$ that is a critical point of $I_\lambda^{\mathcal{P}}$.

(iv) Let us denote

$$\mathcal{F}(x,t) = \frac{1}{4}f(x,t)t - F(x,t), \quad x\in\mathbb{R}^3 \text{ and } t\in\mathbb{R}.$$

We know from convergence (1.1.5) that $u_k(x + y_k^{(n_0)}) \to w^{(n_0)}(x)$ a. e. in $\mathbb{R}^3$, up to subsequence, which allows us to apply Fatou Lemma to get

$$\begin{aligned}
\mathcal{G}_{\mathcal{S}} &= \lim_{k\to\infty}\left[\frac{1}{4}\|u_k(\cdot + y_k^{(n_0)})\|^2_{V_{\mathcal{P}}} + \int_{\mathbb{R}^3}\mathcal{F}(x, u_k(\cdot + y_k^{(n_0)}))\,\mathrm{d}x\right]\\
&\geqslant \liminf_{k\to\infty}\frac{1}{4}\|u_k(\cdot + y_k^{(n_0)})\|^2_{V_{\mathcal{P}}} + \liminf_{k\to\infty}\int_{\mathbb{R}^3}\mathcal{F}(x, u_k(\cdot + y_k^{(n_0)}))\,\mathrm{d}x\\
&\geqslant \frac{1}{4}\|w^{(n_0)}\|^2_{V_{\mathcal{P}}} + \int_{\mathbb{R}^3}\mathcal{F}(x, w^{(n_0)})\,\mathrm{d}x = I_\lambda^{\mathcal{P}}(w^{(n_0)}),
\end{aligned}$$

where we used (f_2) to ensure that $\mathcal{F}(x, u_k(\cdot + y_k^{(n_0)})) = \mathcal{F}(x, u_k) \geqslant 0$ a. e. in $\mathbb{R}^3$. Thus, once again by (f_2), we have $\mathcal{G}_{\mathcal{S}} = I_\lambda^{\mathcal{P}}(w^{(n_0)}) > 0$.

(v) Now assume in addition that $f_{\mathcal{P}}(x,t)$ satisfies (f_6), then $\mathcal{G}_{\mathcal{S}} = c(I_\lambda^{\mathcal{P}}) = I_\lambda^{\mathcal{P}}(w^{(n_0)})$, and $w^{(n_0)}$ is non-negative. Indeed, the truncation given in Remark 4.2.9 satisfies the assumptions of Theorem 4.2.1, and we can apply the same arguments of this remark to conclude that the ground state $w^{(n_0)}$ is non-negative. Furthermore, Remark 4.3.8–(i) guarantees that the path $\zeta(t) = tw^{(n_0)}$, $t \geqslant 0$, belongs to $\Gamma_{I_\lambda^{\mathcal{P}}}$ and $c(I_\lambda^{\mathcal{P}}) \leqslant I_\lambda^{\mathcal{P}}(w^{(n_0)})$. On the other hand, considering (u_k) the above sequence, by Remark 3.3.2–(ii), Propositions 3.4.2 and 4.3.4 and estimate (1.1.7), up to subsequence, we have

$$c(I_\lambda^{\mathcal{P}}) = \lim_{k\to\infty}\left[\frac{1}{2}\|u_k\|_{V_{\mathcal{P}}}^2 + \frac{\lambda}{4}\mathcal{N}_\alpha^{\mathcal{P}}(u_k) - \int_{\mathbb{R}^3} F_{\mathcal{P}}(x,u_k)\,\mathrm{d}x\right] \geqslant \sum_{n\in\mathbb{N}_0} I_\lambda^{\mathcal{P}}(w^{(n)}).$$

Consequently, using (f_2) we can guarantee that each $I_\lambda^{\mathcal{P}}(w^{(n)})$ is non-negative and conclude that $c(I_\lambda^{\mathcal{P}}) = \mathcal{G}_{\mathcal{S}}$. ∎

4.5 Proof of Theorem 4.2.2

In order to prove our existence result without the compactness condition $(\mathscr{C})$, once again we use a similar argument as made in the previous chapters. Thus we need Theorem 0.6.4, which states that the existence of a critical point of I is guaranteed whenever the minimax level (4.1.4) is attained (see Remark 4.3.8–(ii)).

Proof of Theorem 4.2.2 completed. From Lemma 4.3.7 and Proposition 4.3.9 we know about the existence of a bounded sequence (u_k) such that $I_\lambda(u_k) \to c(I_\lambda)$ and $I_\lambda'(u_k) \to 0$, in all considered cases. Let be the sequences $(w^{(n)})$ and $(y_k^{(n)})$ provided by Theorem 1.1.2 for the sequence (u_k). The underlying main idea to proof the concentration-compactness of Theorem 4.2.2 follows the same one of Theorem 2.2.4 and 3.2.3 and is the following: we prove that $w^{(n)} = 0$ for all $n \geqslant 2$, which by assertions (1.1.5), (1.1.8) and Propositions 3.4.2 and 4.3.4 implies that $u_k \to w^{(1)}$ in $H_V^s(\mathbb{R}^3)$, up to subsequence. In order to prove that, we argue by contradiction and assume the existence of at least one $w^{(n_0)} \neq 0$, $n_0 \geqslant 2$.

(i) In view of Remark 3.3.2–(ii), estimate (1.1.7), Propositions 3.4.2 and 4.3.4, up to subsequence, we have

$$\begin{aligned} c(I_\lambda) &= \lim_{k\to\infty}\left[\frac{1}{2}\|u_k\|_V^2 + \frac{\lambda}{4}\mathcal{N}_\alpha(u_k) - \int_{\mathbb{R}^3} F(x,u_k)\,\mathrm{d}x\right] \\ &\geqslant I_\lambda(w^{(1)}) + \sum_{n\in\mathbb{N}_0, n>1} I_\lambda^{\mathcal{P}}(w^{(n)}). \end{aligned} \tag{4.5.1}$$

Each term of the right-hand side of (4.5.1) is non-negative. In fact, following as in the proof of Theorem 4.2.1 (using Lemma 4.3.6) we notice that $w^{(1)}$ and $w^{(n)}$, $n \geqslant 2$, are critical points for I_λ and $I_\lambda^{\mathcal{P}}$, respectively. In view of that, it is clear that (f_2) implies that $I_\lambda(w^{(1)}) \geqslant 0$ and $I_\lambda^{\mathcal{P}}(w^{(n)}) \geqslant 0$, $n \geqslant 2$. On the other hand, Remark 4.3.8–(i) guarantees that the path $\zeta^{(n_0)}(t) = tw^{(n_0)}$ belongs to $\Gamma_{I_\lambda^{\mathcal{P}}}$ and $c(I_\lambda^{\mathcal{P}}) \leqslant I_\lambda^{\mathcal{P}}(w^{(n_0)})$. This, together with (4.5.1) and $(\mathscr{C})$ leads to a contradiction.

(ii) In this case we follow a similar argument, we apply estimate (1.1.7), Propositions 4.3.4, 3.4.2 and 4.3.10, to get up to subsequence, that

$$\begin{aligned} c(I_\lambda) &= \lim_{k\to\infty} \left[\frac{1}{2}\|u_k\|_V^2 - \frac{1}{2}\int_{\mathbb{R}^3} b(x)u_k^2 + \frac{\lambda}{4}\mathcal{N}_\alpha(u_k) - \int_{\mathbb{R}^3} F(x,u_k)\,\mathrm{d}x \right] \\ &\geqslant I_\lambda(w^{(1)}) + \sum_{n\in\mathbb{N}_0, n>1} I_\lambda^{\mathcal{P}}(w^{(n)}). \end{aligned} \tag{4.5.2}$$

Reasoning as in (4.4.2), we see again that $w^{(1)}$ and $w^{(n)}$, $n \geqslant 2$, are critical points for I_λ and $I_\lambda^{\mathcal{P}}$, respectively. In fact, by assertion (1.1.6), there exists $k_0 = k_0(\varphi)$ such that

$$\begin{cases} |V(x+y_k^{(n)})| < 1 + |V_{\mathcal{P}}(x)|, \quad \forall\, k > k_0,\ x \in \operatorname{supp}(\varphi) \ \text{ and } \ n \geqslant 2. \\ V(x+y_k^{(n)}) = \Big(V(x+y_k^{(n)}) - V_{\mathcal{P}}(x+y_k^{(n)})\Big) + V_{\mathcal{P}}(x) \to V_{\mathcal{P}}(x), \quad \text{a.e. in } \operatorname{supp}(\varphi). \end{cases}$$

Thus, using again (f_2) together with (V_4), we can guarantee that $I_\lambda(w^{(1)}) \geqslant 0$ and $I_\lambda^{\mathcal{P}}(w^{(n)}) \geqslant 0$, $n \geqslant 2$. Once more, Remark 4.3.8–(i) guarantees that the path $\zeta^{(n_0)}(t) = tw^{(n_0)}$ belongs to $\Gamma_{I_\lambda^{\mathcal{P}}}$ and $c(I_{\mathcal{P}}) \leqslant I_\lambda^{\mathcal{P}}(w^{(n_0)})$. This, together with (4.5.2) and $(\mathscr{C})$ leads to a contradiction.

(iii) Finally, assume that inequality (4.2.1) holds true instead condition $(\mathscr{C})$ in the items (i) and (ii). If there exists $w^{(n_0)} \neq 0$, $n_0 \in \mathbb{N}_0$, then

$$c(I_\lambda) \leqslant \max_{t\geqslant 0} I_\lambda(\zeta^{(n_0)}(t)) \leqslant \max_{t\geqslant 0} I_\lambda^{\mathcal{P}}(\zeta^{(n_0)}(t)) = I_\lambda^{\mathcal{P}}(w^{(n_0)}) \leqslant c(I_\lambda), \quad n_0 \in \mathbb{N}_0,$$

where we used condition (4.2.1) to ensure that the paths $\zeta^{(n_0)}$ belongs to Γ_{I_λ}. Thus, we have that the minimax level $c(I_\lambda)$ is attained by the path $t \mapsto tw^{(n_0)}$ and we can apply Theorem 0.6.4 to obtain the existence of a critical point u for I_λ with $I_\lambda(u) = c(I_\lambda)$. If there is no $w^{(n)} \neq 0$, $n \in \mathbb{N}_0$, (which is the case where strict inequalities occurs) we can argue as above and obtain that $u_k \to w^{(1)}$, up to subsequence. ∎

4.6 Study of the asymptotic problem

In order to prove Theorem 4.2.3 we first need to study the existence of weak solutions for the limiting problem (4.1.1). This provides a way to compare the minimax level of the functionals associated with systems $(\mathcal{SP})$ and (4.1.1), as mentioned in Remark 4.2.8.

Theorem 4.6.1 (Periodic case with critical perturbation)**.** *Assume that* (K_1)–(K_3), (V_1)–(V_3), (f_1)–(f_5), (f_7) (g_1)–(g_5), (g_7) *hold true and* $0 \leqslant V(x) \in L^\infty(\mathbb{R}^3)$. *Moreover,* *Suppose that $K(x) \equiv K_{\mathcal{P}}(x)$, $V(x) \equiv V_{\mathcal{P}}(x)$, $b(x) \equiv 0$, $f(x,t) \equiv f_{\mathcal{P}}(x,t)$ and $g(x,t) \equiv g_\infty(t)$. If we assume that either $\mu > 4$ and (f_3) or $\mu = 4$ and (f_4) then Eq. $(\mathcal{S}_{NL})$ possess a non-trivial weak solution u in $H^s_{V_{\mathcal{P}}}(\mathbb{R}^3)$. Furthermore, if additionally we have (f_6) and (g_6), then $I^{\mathcal{P}}_\lambda(u) = c(I^{\mathcal{P}}_\lambda)$.*

Proof. We divide the proof in several steps.

(i) By Lemma 4.3.7 and Proposition 4.3.9 we get the existence of a bounded sequence (u_k) such that $I^{\mathcal{P}}_\lambda(u_k) \to c(I^{\mathcal{P}}_\lambda)$ and $(I^{\mathcal{P}}_\lambda)'(u_k) \to 0$, in all considered cases. Let be the sequences $(w^{(n)})$, $(y^{(n)}_k)$ and $(j^{(n)}_k)$ provided by the Theorem 1.1.1 for the sequence (u_k).

(ii) If there is some $w^{(n_0)} \neq 0$, for some $n_0 \in \mathbb{N}_0$, then, as proved in Theorem 4.2.1, $w^{(n_0)}$ is a critical point of $I^{\mathcal{P}}_\lambda$. Let us assume, by contradiction, that $w^{(n)} = 0$ for all $n \in \mathbb{N}_0$. Thus, by Propositions 4.3.4, 3.4.1 and 3.4.2 we have,

$$\begin{cases} c(I^{\mathcal{P}}_\lambda) = \dfrac{1}{2}\|u_k\|^2_{V_{\mathcal{P}}} - \displaystyle\int_{\mathbb{R}^3} G_\infty(u_k)\,\mathrm{d}x + o(1), \\ \qquad 0 = \|u_k\|^2_{V_{\mathcal{P}}} - \displaystyle\int_{\mathbb{R}^3} g_\infty(u_k)u_k\,\mathrm{d}x + o(1), \end{cases} \tag{4.6.1}$$

In particular, up to subsequence,

$$b_0 := \limsup_{k\to\infty} \|u_k\|^2_{V_{\mathcal{P}}} = \limsup_{k\to\infty} \int_{\mathbb{R}^3} g_\infty(u_k)u_k\,\mathrm{d}x,$$

which combined with (g_1) and (4.3.15) leads to

$$b_0 \geqslant \left(\mathcal{C}_*(\mathcal{S}_*(s))^{-2^*_s}\right)^{-\frac{2}{2^*_s-2}}.$$

Consequently from (4.6.1) and (g_2) we can conclude that

$$c(I^{\mathcal{P}}_\lambda) \geqslant \frac{\mu_* - 2}{2\mu_*}\left(\mathcal{C}_*(\mathcal{S}_*(s))^{-2^*_s}\right)^{-\frac{2}{2^*_s-2}}, \tag{4.6.2}$$

a contradiction, because condition (4.2.2) do not allows that (4.3.16) and (4.6.2) holds simultaneously.

(iii) Assume additionally (f_6) and (g_6). By estimate (1.1.3), Propositions 1.4.4, 2.4.1, 3.4.2 and 4.3.4 we get

$$\begin{aligned} c(I^{\mathcal{P}}_\lambda) &= \lim_{k\to\infty}\left[\frac{1}{2}\|u_k\|^2_{V_{\mathcal{P}}} + \frac{\lambda}{4}\mathcal{N}^{\mathcal{P}}_\alpha(u_k) - \int_{\mathbb{R}^3} F_{\mathcal{P}}(x,u_k)\,\mathrm{d}x - \int_{\mathbb{R}^3} G_\infty(u_k)\,\mathrm{d}x\right] \\ &\geqslant \sum_{n\in\mathbb{N}_0} I^{\mathcal{P}}_\lambda(w^{(n)}) + \sum_{n\in\mathbb{N}_+} J_\infty(w^{(n)}), \end{aligned}$$

where J_∞ is the following C^1 functional in $\mathcal{D}^{s,2}(\mathbb{R}^3)$

$$J_\infty(u) = \frac{1}{2}\int_{\mathbb{R}^3} |(-\Delta)^{s/2}u|^2\,\mathrm{d}x - \int_{\mathbb{R}^3} G_\infty(u)\,\mathrm{d}x, \quad u \in \mathcal{D}^{s,2}(\mathbb{R}^3).$$

(iv) Since $g_\infty(t)$ is self-similar, each $w^{(n)}$, $n \in \mathbb{N}_+$, is a critical point of J_∞. In fact, let φ in $C_0^\infty(\mathbb{R}^3)$. It is easy to see that $(d_k^{(n)}\varphi)$ is bounded in $H^s_{V_\mathcal{P}}(\mathbb{R}^3)$. Applying the Dominated Convergence Theorem we get

$$\lim_{k\to\infty}\int_{\mathbb{R}^3} V_\mathcal{P}(x)u_k d_k^{(n)}\varphi\,\mathrm{d}x = \lim_{k\to\infty}\left[\gamma^{-2sj_k^{(n)}}\int_{\mathbb{R}^3} V_\mathcal{P}(\gamma^{-j_k^{(n)}}x)v_k^{(n)}\varphi\,\mathrm{d}x\right] = 0, \quad n \in \mathbb{N}_+,$$

where $v_k^{(n)}$ is defined in (4.3.12). Also, given $\varepsilon > 0$ we can use (f_1) to get the following estimate,

$$\limsup_{k\to\infty}\left|\int_{\mathbb{R}^3}\gamma^{-\frac{3+2s}{2}j_k^{(n)}} f_\mathcal{P}(\gamma^{-j_k^{(n)}}x + y_k^{(n)}, \gamma^{\frac{3-2s}{2}}v_k^{(n)})\varphi\,\mathrm{d}x\right| \leqslant \varepsilon, \quad n \in \mathbb{N}_+.$$

Therefore, by Lemma 4.3.6, it follows that

$$0 = \lim_{k\to\infty}\left[(I_\lambda^\mathcal{P})'(u_k)\cdot(d_k^{(n)}\varphi)\right] = J'_\infty(w^{(n)})\cdot\varphi, \quad n \in \mathbb{N}_+.$$

(v) Hence, by assumption (f_2), we obtain that $J_\infty(w^{(n)}) \geqslant 0$, $n \in \mathbb{N}_+$. This allows to conclude that $I_\lambda^\mathcal{P}(w^{(n_0)}) \leqslant c(I_\lambda^\mathcal{P})$. On the other hand, considering Remark 4.3.8–(i), assumptions (f_6) and (g_6) implies that $c(I_\lambda^\mathcal{P}) \leqslant I_\lambda^\mathcal{P}(w^{(n_0)})$. ∎

In what follows, we prove what is stated in Remark 4.2.8. It is also worth to mention that we use the next proposition to prove Theorem 4.2.3.

Proposition 4.6.2. *Assume that $K(x)$, $a(x) = V(x) - b(x)$, $f(x,t)$ and $g(x,t)$ satisfies either*

- *(i) (K_1)–(K_2), (V_1)–(V_4), (f_1), (f_2), (f_5), (f_6) and that $g(x,t) \equiv 0$. Moreover, suppose either $\mu > 4$ and (f_3) or $\mu = 4$ and (f_4).*
- *(ii) (K_1)–(K_3), (V_1)–(V_4), (f_1), (f_2), (f_5)–(f_7), (g_1)–(g_7), respectively, with $V_\mathcal{P}(x) \in L^\infty(\mathbb{R}^3)$ and $\mu \leqslant \mu_*$. Furthermore, suppose either $\mu > 4$ and (f_3) or $\mu = 4$ and (f_4). Also that the inequality (4.2.2) holds.*

Then $c(I_\lambda) \leqslant c(I_\lambda^\mathcal{P})$ respectively. Moreover, under these assumptions, $(\mathscr{C}')$ implies $(\mathscr{C})$. In addition, consider the following C^1 functional in $\mathcal{D}^{s,2}(\mathbb{R}^3)$

$$J_+(u) = \frac{1}{2}\int_{\mathbb{R}^3}|(-\Delta)^{s/2}u|^2\,\mathrm{d}x - \int_{\mathbb{R}^3} G_+(u)\,\mathrm{d}x, \quad u \in \mathcal{D}^{s,2}(\mathbb{R}^3). \tag{4.6.3}$$

If the following conditions are true,

- *(iii) (K_1)–(K_3), (V_1), (V_2), (V_4), (f_1), (f_2), (g_1)–(g_4), (g_8),*

then $c(I_\lambda) \leqslant c(J_+)$, where

$$c(J_+) = \inf_{\gamma\in\Gamma_{J_+}}\sup_{t\geqslant 0} J_+(\gamma(t)).$$

and

$$\Gamma_{J_+} = \left\{\gamma \in C([0,\infty), \mathcal{D}^{s,2}(\mathbb{R}^3)) : \gamma(0) = 0,\ \lim_{t\to\infty} J_+(\gamma(t)) = -\infty\right\}.$$

Proof. (i). Let u in $H^s_{V_{\mathcal{P}}}(\mathbb{R}^3)$ be a non-negative (see Theorem 4.2.1) non-trivial weak solution for the equation

$$(-\Delta)^s u + V_{\mathcal{P}}(x)u + \lambda K_{\mathcal{P}}(x)\phi_\alpha^{\mathcal{P}}[u]u = f_{\mathcal{P}}(x,u),$$

at the mountain pass level for $I_\lambda^{\mathcal{P}}$, that is, $I_\lambda^{\mathcal{P}}(u) = c(I_\lambda^{\mathcal{P}})$. For each k, we define the path

$$\zeta_k(t) = tu(\cdot - y_k), \quad t \geqslant 0,$$

where (y_k) is taken such that $|y_k| \to \infty$. The idea is to prove that

$$c(I_\lambda) \leqslant \lim_{k\to\infty} \max_{t\geqslant 0} I_\lambda(\zeta_k(t)) \leqslant \max_{t\geqslant 0} I_\lambda^{\mathcal{P}}(tu) = c(I_\lambda^{\mathcal{P}}). \tag{4.6.4}$$

In fact, taking account that the following functionals,

$$\Phi,\ \Phi_{\mathcal{P}},\ \mathcal{N}_\alpha,\ \mathcal{N}_\alpha^{\mathcal{P}},$$
$$Q(u) = \int_{\mathbb{R}^N} V(x)u^2\,\mathrm{d}x,\ Q_{\mathcal{P}}(u) = \int_{\mathbb{R}^N} V_{\mathcal{P}}(x)u^2\,\mathrm{d}x \text{ and } B(u) = \int_{\mathbb{R}^N} b(x)u^2\,\mathrm{d}x,$$

are locally Lipschitz in $H^s_V(\mathbb{R}^3)$ (they are C^1 in $H^s_V(\mathbb{R}^3)$) and the following estimate

$$\begin{aligned}\left|I_\lambda(\zeta_k(t)) - I_\lambda^{\mathcal{P}}(tu)\right| \leqslant{}& \frac{t^2}{2}\int_{\mathbb{R}^3} |V(x+y_k) - V_{\mathcal{P}}(x+y_k)|\,u^2\,\mathrm{d}x + \frac{t^2}{2}\int_{\mathbb{R}^3} b(x+y_k)u^2\,\mathrm{d}x\\ &+ \frac{\lambda}{4}t^4\left|\mathcal{N}_\alpha(u(\cdot - y_k)) - \mathcal{N}_\alpha^{\mathcal{P}}(u)\right|\\ &+ \int_{\mathbb{R}^3} |F(x+y_k,tu) - F_{\mathcal{P}}(tu)|\,\mathrm{d}x,\end{aligned}$$

by using a density argument we get that

$$\lim_{k\to\infty} I_\lambda(\zeta_k(t)) = I_\lambda^{\mathcal{P}}(tu), \quad \text{uniformly in compact sets of } \mathbb{R}.$$

Consequently we may proceed as in Proposition 3.8.1. Before that, notice first that

$$\lim_{k\to\infty}\int_{\mathbb{R}^3} F(x+y_k,tu)\,\mathrm{d}x = \int_{\mathbb{R}^3} F_{\mathcal{P}}(x,tu)\,\mathrm{d}x, \quad \text{for each } t>0.$$

In particular,

$$\int_{\mathbb{R}^3} F(x+y_k,u)\,\mathrm{d}x > 0, \quad \text{for } k \text{ large enough}.$$

Thus, using the arguments of Remark 4.3.8–(i), we see that ζ_k belongs to Γ_{I_λ}, for k large enough. As a consequence, for each k that is large enough, there exist $t_k > 0$ such that

$$I_\lambda(\zeta_k(t_k)) = \max_{t\geqslant 0} I_\lambda(\zeta_k(t)) > 0.$$

The sequence (t_k) is bounded. In fact, on the contrary, up to subsequence, we have the following contradiction

$$\begin{aligned}0 < I_\lambda(\zeta_k(t_k)) \leqslant \frac{1}{2}t_k^2\|u\|^2_{V(\cdot+y_k)} + \frac{\lambda}{4}t_k^4\mathcal{N}_\alpha(u(\cdot - y_k)) - \int_{\mathbb{R}^3} F(x+y_k,t_ku)\,\mathrm{d}x&\\ \to -\infty, \text{ as } k\to\infty,&\end{aligned}$$

where, as above, we used a density argument and the arguments used to prove Lemma 4.3.7. Therefore, up to subsequence, $t_k \to t_0$, and we have that

$$\lim_{k\to\infty} \max_{t\geqslant 0} I_\lambda(\zeta_k(t_k)) = I_\lambda^{\mathcal{P}}(t_0 u),$$

which leads to (4.6.4).

(ii) The second case is proved in a similar way, since the existence of a solution for the equation

$$(-\Delta)^s u + V_{\mathcal{P}}(x)u + \lambda K_{\mathcal{P}}(x)\phi_\alpha^{\mathcal{P}}[u]u = f_{\mathcal{P}}(x,u) + g_\infty(u),$$

at the Mountain Pass level is guaranteed by Theorem 4.6.1. Now assume that condition $(\mathscr{C}')$ holds true. Considering the above discussion, we have that

$$c(I_\lambda) \leqslant \max_{t\geqslant 0} I_\lambda(\zeta_k(t)) < \max_{t\geqslant 0} I_\lambda^{\mathcal{P}}(\zeta_k(t)) = \max_{t\geqslant 0} I_\lambda^{\mathcal{P}}(tu) = c(I_\lambda^{\mathcal{P}}),$$

where we used that ζ_k belongs to Γ_{I_λ} for k large enough.

(iii). Let u_0 in $\mathcal{D}^{s,2}(\mathbb{R}^3)$ be a non-negative weak solution for the equation

$$(-\Delta)^s u_0 = g_+(u_0),$$

at the mountain pass level, more precisely, $J_+(u_0) = c(J_+)$. We refer to one of the existence results in Sect. 2.2 and Remark 2.2.7 about the existence of such u_0. Define the sequence $u_n = u_0\xi_n(\cdot,0)$, where ξ_n is given by Remark 4.2.9. For each k, we consider the path

$$\lambda_k^n(t) = \gamma^{\frac{3-2s}{2}j_k} u_n(\gamma^{j_k}\cdot)t, \quad t \geqslant 0,$$

where (j_k) is a sequence in $\mathbb{Z}$ chosen in a such way that $j_k \to \infty$. Now observe that

$$\lim_{n\to\infty} \int_{\mathbb{R}^3} G_+(u_n)\,\mathrm{d}x = \int_{\mathbb{R}^3} G_+(u)\,\mathrm{d}x > 0,$$

where the positivity of the right-hand side of this limit is guaranteed by a Pohozaev type identity. On the other hand,

$$\lim_{k\to\infty} \int_{\mathbb{R}^3} F(x,\lambda_k^n(t)) + G(x,\lambda_k^n(t))\,\mathrm{d}x = \int_{\mathbb{R}^3} G_+(tu_n)\,\mathrm{d}x, \quad \forall\, t \geqslant 0.$$

Therefore, arguing as in Remark 4.3.8–(i), we can fix first n_0 large enough and conclude that λ_k^n belongs to Γ_I, for k large enough. Moreover, using the same density argument as above and the estimate

$$\begin{aligned}|I_\lambda(\lambda_k^{n_0}(t)) - J_+(tu_{n_0})| \leqslant{}& \frac{1}{2}t^2 \int_{\mathbb{R}^3} |\lambda_k^{n_0}(1)|^2\,\mathrm{d}x + \frac{\lambda}{4}t^4 \mathcal{N}_\alpha(\lambda_k^{n_0}(1))\\ &+ \int_{\mathbb{R}^3} |F(x,\lambda_k^{n_0}(t))|\,\mathrm{d}x + \int_{\mathbb{R}^3} |G(x,\lambda_k^{n_0}(t) - G_+(tu_{n_0}))|\,\mathrm{d}x,\end{aligned}$$

we may conclude that

$$\lim_{k\to\infty} I_\lambda(\lambda_k^{n_0}(t)) = J_+(tu_{n_0}), \quad \text{uniformly in compact sets of } \mathbb{R}.$$

Moreover, for each k that is large enough, there exist $t_k^{n_0} > 0$ such that

$$I_\lambda(\lambda_k^{n_0}(t_k^{n_0})) = \max_{t\geqslant 0} I_\lambda(\lambda_k^{n_0}(t)) > 0.$$

This sequence $(t_k^{n_0})$ is bounded, because on the contrary, up to subsequence, if $|t_k^{n_0}| \to \infty$ then

$$\begin{aligned} 0 < I_\lambda(\lambda_k^{n_0}(t_k)) &\leqslant \frac{1}{2}|t_k^{n_0}|^2\|\lambda_k^{n_0}(1)\|_V^2 + \frac{\lambda}{4}|t_k^{n_0}|^4\mathcal{N}_\alpha(\lambda_k^{n_0}(1)) \\ &- |t_k^{n_0}|^\mu \int_{\mathbb{R}^3} F(x,\lambda_k^{n_0}(1))\,\mathrm{d}x - |t_k^{n_0}|^{\mu_*}\int_{\mathbb{R}^3} G(x,\lambda_k^{n_0}(1))\,\mathrm{d}x \to -\infty \text{ as } k\to\infty, \end{aligned}$$

a contradiction. Consequently, up to subsequence $t_k^{n_0} \to a_{n_0}$ and we have

$$c(I_\lambda) \leqslant \lim_{k\to\infty}\max_{t\geqslant 0} I_\lambda(\lambda_k^{n_0}(t)) = J_+(a_{n_0}u_{n_0}) \leqslant \max_{t\geqslant 0} J_+(tu_{n_0}) = J_+(t_{n_0}u_{n_0}).$$

If $t_{n_0} \to \infty$ as $n_0 \to \infty$, then we get

$$0 < J_+(t_{n_0}u_{n_0}) = \frac{1}{2}t_{n_0}^2[u_{n_0}]_s^2 - t_{n_0}^\mu \int_{\mathbb{R}^3} G_+(u_{n_0})\,\mathrm{d}x \to -\infty \text{ as } n_0\to\infty,$$

where we used Proposition 2.3.2 and that

$$\begin{cases} \displaystyle\lim_{n_0\to\infty}[u_{n_0}]_s^2 = \lim_{n_0\to\infty}\left[\kappa_s\|\xi_{n_0}\nabla u + u\nabla\xi_{n_0}\|^2_{L^2(\mathbb{R}^4_+,y^{1-2s})}\right] = [u]_s^2, \\ \displaystyle\lim_{n_0\to\infty}\int_{\mathbb{R}^3} G_+(u_{n_0})\,\mathrm{d}x = \int_{\mathbb{R}^3} G_+(u)\,\mathrm{d}x. \end{cases}$$

Therefore, the sequence (t_{n_0}) is bounded and it converges, up to subsequence, that $t_{n_0} \to b_0$. Thus,

$$\begin{aligned} c(I_\lambda) \leqslant \lim_{n_0\to\infty}[J_+(t_{n_0}u_{n_0})] &= \lim_{n_0\to\infty}\left[\frac{1}{2}t_{n_0}^{3-2s}[u_{n_0}]_s^2 - \int_{\mathbb{R}^3} G_+(t_{n_0}u_{n_0})\,\mathrm{d}x\right] \\ &\leqslant J_+(b_0u_0) \leqslant \max_{t\geqslant 0} J_+(tu_0) = J_+(u_0) = c(J_+), \end{aligned}$$

where we used condition (g_8) to guarantee that $t = 1$ is a maximum point of the function $\varphi(t) = J_+(tu_0)$. In fact, it suffices to prove that

$$\varphi(t) \leqslant \frac{1}{2}t^2[u_0]_s^2 - c_+t^{2_s^*}\int_{\mathbb{R}^3} u_0^{2_s^*}\,\mathrm{d}x \leqslant \frac{1}{2}[u_0]_s^2 - \int_{\mathbb{R}^3} G_+(u_0)\,\mathrm{d}x, \quad \forall\, t\geqslant 0. \tag{4.6.5}$$

Using the Pohozaev type identity (Proposition 2.3.2) we observe that the second inequality (4.6.5) is equivalent to

$$\left[\frac{2_s^*}{2}(t^2-1)+1\right]\int_{\mathbb{R}^3} G_+(u_0)\,\mathrm{d}x \leqslant c_+t^{2_s^*}\int_{\mathbb{R}^3} u_0^{2_s^*}\,\mathrm{d}x, \quad \forall\, t\geqslant 0. \tag{4.6.6}$$

Since $c_+ \geqslant \mathcal{C}_*$ we have that

$$\left[\frac{2_s^*}{2}(t^2-1)+1\right]\mathcal{C}_* \leqslant c_+t^{2_s^*}, \quad \forall\, t\geqslant 0,$$

which ensures the validity of inequality (4.6.6). ■

4.7 Proof of Theorem 4.2.3

Proof. The proof uses similar arguments as the one used in Theorem 4.6.1 and we repeat some of them for completeness. Once again we may apply Lemma 4.3.7 and Proposition 4.3.9 in order to get the existence of a bounded sequence (u_k) such that $I_\lambda(u_k) \to c(I_\lambda)$ and $I'_\lambda(u_k) \to 0$, in all considered cases. Let $(w^{(n)})$, $(y_k^{(n)})$ and $(j_k^{(n)})$ be the sequences given by the Theorem 1.1.1 for the sequence (u_k).

(i) We start by noticing that assumption $(\mathscr{C})$ guarantees that $w^{(n)} = 0$, for all $n \in \mathbb{N}_0\backslash\{1\}$. Indeed, assume by contradiction that there exists with $w^{(n_0)} \neq 0$, with $n_0 \in \mathbb{N}_0\backslash\{1\}$. Using estimate (1.1.3), Propositions 1.4.4, 2.4.1, 3.4.2 and 4.3.4 we get

$$\begin{aligned} c(I_\lambda) &= \lim_{k\to\infty}\left[\frac{1}{2}\|u_k\|_V^2 - \frac{1}{2}\int_{\mathbb{R}^3} b(x)u^2 + \frac{\lambda}{4}\mathcal{N}_\alpha(u_k) - \int_{\mathbb{R}^3} F(x,u_k)\,\mathrm{d}x - \int_{\mathbb{R}^3} G(x,u_k)\,\mathrm{d}x\right] \\ &\geqslant I_\lambda(w^{(1)}) + \sum_{n\in\mathbb{N}_0, n>1} I_\lambda^{\mathcal{P}}(w^{(n)}) + \sum_{n\in\mathbb{N}_+} J_+(w^{(n)}). \end{aligned} \tag{4.7.1}$$

where J_+ is the C^1 functional in $\mathcal{D}^{s,2}(\mathbb{R}^3)$ given by (4.6.3). Similarly as argued before, each term in (4.7.1) is non-negative, because $w^{(1)}$ is a critical point of I_λ; $w^{(n)}$, $n \in \mathbb{N}_0\backslash\{1\}$ is of $I_\lambda^{\mathcal{P}}$ and $w^{(n)}$, $n \in \mathbb{N}_+$ is of J_+. In fact, let φ in $C_0^\infty(\mathbb{R}^3)$. Since $V(x)$ belongs to $L^\infty(\mathbb{R}^3)$, it is easy to see that $(d_k^{(n)}\varphi)$ is bounded in $H_V^s(\mathbb{R}^3)$. Moreover, up to subsequence, applying the Dominated Convergence Theorem we get

$$\lim_{k\to\infty}\int_{\mathbb{R}^3} V(x)u_k(d_k^{(n)}\varphi)\,\mathrm{d}x = \lim_{k\to\infty}\left[\gamma^{-2sj_k^{(n)}}\int_{\mathbb{R}^3} V(\gamma^{-j_k^{(n)}}x + y_k^{(n)})v_k^{(n)}\varphi\,\mathrm{d}x\right] = 0,\ n \in \mathbb{N}_+.$$

Also, given $\varepsilon > 0$ we can use (f_1) to get the following estimate,

$$\limsup_{k\to\infty}\left|\int_{\mathbb{R}^3}\gamma^{-\frac{3+2s}{2}j_k^{(n)}} f(\gamma^{-j_k^{(n)}}x + y_k^{(n)}, \gamma^{\frac{3-2s}{2}}v_k^{(n)})\varphi\,\mathrm{d}x\right| \leqslant \varepsilon, \quad n \in \mathbb{N}_+.$$

Therefore, by Lemma 4.3.6, it follows that

$$0 = \lim_{k\to\infty}\left[I'_\lambda(u_k)\cdot(d_k^{(n)}\varphi)\right] = J'_+(w^{(n)})\cdot\varphi, \quad n \in \mathbb{N}_+,$$

where $v_k^{(n)}$ is taken as in (4.3.12). Hence, using assumption (g_2), we obtain that $J_+(w^{(n)}) \geqslant 0$, $n \in \mathbb{N}_+$. Furthermore, as argued in the proof of Theorem 4.2.2, by condition (f_2) we have that $I_\lambda^{\mathcal{P}}(w^{(n)}) \geqslant 0$, $n \in \mathbb{N}_0$. Thus estimate (4.7.1) and Remark 4.3.8–(i) implies that $c(I_\lambda) \geqslant c(I_\lambda^{\mathcal{P}})$, a contradiction with assumption $(\mathscr{C})$.

Let us argue by contradiction and suppose that $w^{(1)} = 0$. By Propositions 4.3.4, 3.4.1 and 3.4.2 we have,

$$\begin{cases} c(I_\lambda) = \dfrac{1}{2}\|u_k\|_V^2 - \displaystyle\int_{\mathbb{R}^3} G(x,u_k)\,\mathrm{d}x + o(1), \\ \quad\ \ 0 = \|u_k\|_V^2 - \displaystyle\int_{\mathbb{R}^3} g(x,u_k)u_k\,\mathrm{d}x + o(1), \end{cases} \tag{4.7.2}$$

In particular, up to subsequence,

$$b_0 := \limsup_{k\to\infty} \|u_k\|_V^2 = \limsup_{k\to\infty} \int_{\mathbb{R}^3} g(x,u_k)u_k \,\mathrm{d}x,$$

which combined with (g_1) and (4.3.15) leads to

$$b_0 \geqslant \left(\mathcal{C}_*(\mathcal{S}_*(s))^{-2_s^*}\right)^{-\frac{2}{2_s^*-2}}. \tag{4.7.3}$$

Using inequality (4.7.3) in (4.7.2) and condition (g_2) we obtain the following estimate for the minimax level

$$c(I_\lambda) \geqslant \frac{\mu_* - 2}{2\mu_*}\left(\mathcal{C}_*(\mathcal{S}_*(s))^{-2_s^*}\right)^{-\frac{2}{2_s^*-2}}. \tag{4.7.4}$$

This leads to a contradiction with Proposition 4.3.11, because condition (4.2.2) together with Proposition 4.6.2 do not allows that (4.3.16) and (4.7.4) holds simultaneously.

We are going to prove now that $w^{(n)} = 0$, for all $n \in \mathbb{N}_+$. In order to do this, we argue by contradiction again and we assume the existence of $w^{(n_0)} \neq 0$, with $n_0 \in \mathbb{N}_+$. In fact, considering the path $t \mapsto w^{(n)}(\cdot/t)$, $t \geqslant 0$, it is easy to see, applying the Pohozaev identity Proposition 2.3.2, that $c(J_+) \leqslant J_+(w^{(n_0)})$. By Proposition 4.6.2 and estimate (4.7.1) we can conclude that

$$c(J_+) = I_\lambda(w^{(1)}) + \sum_{n\in\mathbb{N}_+} J_+(w^{(n)}).$$

This leads to the contradiction that $J_+(w^{(n_0)}) < c(J_+)$. The convergence $u_k \to w^{(1)}$ in $H_V^s(\mathbb{R}^3)$ now follows by applying Propositions 3.4.1, 3.4.2 and 4.3.4.

(ii) Assume now that inequality (4.2.1) holds true instead condition $(\mathscr{C})$. As discussed above, taking account the existence of $w^{(n_0)} \neq 0$, $n_0 \in \mathbb{N}_0$, we have

$$c(I_\lambda) \leqslant \max_{t\geqslant 0} I_\lambda(tw^{(n_0)}) \leqslant \max_{t\geqslant 0} I_\lambda^{\mathcal{P}}(tw^{(n_0)}) = I_\lambda^{\mathcal{P}}(w^{(n_0)}) \leqslant c(I_\lambda),$$

where we used estimate (4.7.1) to obtain the last inequality and condition (4.2.1) to ensure that the path $\zeta^{(n_0)} = tw^{(n_0)}$ belongs to Γ_{I_λ}. Thus, we have that the minimax level $c(I_\lambda)$ is attained by the path $t \mapsto tw^{(n_0)}$ and we can apply Theorem 0.6.4 to obtain the existence of a critical point u for I_λ with $I_\lambda(u) = c(I_\lambda)$. If there is no $w^{(n)} \neq 0$, $n \in \mathbb{N}_0$, (which is the case where strict inequalities occurs) we can argue as above and obtain that $u_k \to w^{(1)}$, up to subsequence. ■

4.8 Proof of Theorem 4.2.4

In this section we always consider that the assumptions of Theorem 4.2.4 holds true. Moreover, we are going to assume that $2 < \mu < 4$ since the case $\mu \geqslant 4$ is treated in Theorems 4.2.2 and 4.2.3. We restrict the energy functional I to the closed subspace $H^s_{\mathrm{rad}}(\mathbb{R}^3)$.

Lemma 4.8.1 (Geometry)**.** *There exists* $0 < \lambda_* \leqslant \infty$ *and a positive constant* b*, such that* $c_\lambda > b$*, for all* $\lambda \in (0, \lambda_*)$.

Proof. We only prove case (i), since case (ii) follows the same argument. Also we start assuming that $\mu > 3$. Consider the radial function $v \in C_0^\infty(\mathbb{R}^3)$ given by Remark 4.3.8–(iii) and define the paths

$$\zeta_{\varepsilon,\theta}(t) = t^\varepsilon v(t^\theta \cdot), \quad \text{where } \varepsilon, \theta \text{ are positive constant to be determined.}$$

A simple computation shows that

$$\mathcal{N}_\alpha(\zeta_{\varepsilon,\theta}(t)) = t^{4\varepsilon - 2\alpha\theta - 3\theta}\mathcal{N}_\alpha(v), \quad \forall\, t \geqslant 0,$$

Since $\alpha > 3/4$, we may select ε and θ in such way that

$$3/2 < \varepsilon/\theta < 2\alpha/(4-\mu)$$

in order to get that for each $\lambda > 0$, we have

$$\begin{aligned} I_\lambda(\zeta_{\varepsilon,\theta}(t)) \leqslant \frac{1}{2}t^{2\varepsilon-\theta(3-2\alpha)}[v]_s^2 + \frac{1}{2}V_0 t^{2\varepsilon - 3\theta}\|v\|_2^2 + \frac{\lambda}{4}t^{4\varepsilon-\theta(3-2\alpha)}\mathcal{N}_\alpha(v) \\ - t^{\mu\varepsilon-3\theta}\int_{\mathbb{R}^3} F(v)\,\mathrm{d}x - t^{\mu_*\varepsilon-3\theta}\int_{\mathbb{R}^3} G(v)\,\mathrm{d}x < 0, \quad \forall\, t \text{ large enough.} \end{aligned}$$

On the other hand, arguing as in the proof of Lemma 4.3.7, we obtain the existence of $b, r > 0$, which does not depend in the parameters λ, such that

$$b < \frac{1}{2}\|u\|_{V_0}^2 + \frac{\lambda}{4}\mathcal{N}_\alpha(u) - \int_{\mathbb{R}^3} F(u) + G(u)\,\mathrm{d}x \leqslant I_\lambda(u), \quad \text{for } \|u\|_{V_0} = r \text{ and } \lambda > 0.$$

Therefore for $\mu > 3$, we chose $\lambda_* = \infty$ in order to get that $c_\lambda \geqslant b$.

Finally, assume that $2 < \mu \leqslant 3$. We know that there is v_1 in $H^s_{\mathrm{rad}}(\mathbb{R}^3)$ such that $I_0(v_1) < 0$. Since $\lambda \mapsto I_\lambda(v_1)$ is continuous, there exists $\lambda_* > 0$ such that $I_\lambda(v_1) < 0$ for all $\lambda \in (0, \lambda_*)$. ■

Hence, from the Mountain Pass Theorem, for each $\lambda \in (0, \lambda_*)$ there exists a sequence (u_k^λ) in $H^s_{\mathrm{rad}}(\mathbb{R}^3)$ such that $I_\lambda(u_k^\lambda) \to c_\lambda$ and $I'_\lambda(u_k^\lambda) \to 0$ in the dual of $H^s_{\mathrm{rad}}(\mathbb{R}^3)$.

Lemma 4.8.2. *(Boundedness) The sequence* (u_k^λ) *is bounded in* $H^s_{\mathrm{rad}}(\mathbb{R}^3)$.

Proof. In fact, considering first the case (i) we use condition (4.2.3) to get that

$$\begin{aligned}
&(\mu_* + 1)(c(I_\lambda) + 1) + \|u_k^\lambda\|_{V_0} \\
&\qquad \geqslant (\mu_* + 1)I_\lambda(u_k^\lambda) - I'_\lambda(u_k^\lambda)\cdot(u_k^\lambda) \\
&\qquad = \frac{\mu_* - 2}{2}\|u_k^\lambda\|_{V_0}^2 + \lambda\frac{\mu_* - 3}{4}\mathcal{N}_\alpha(u_k^\lambda) \\
&\qquad\quad + \int_{\mathbb{R}^3}(\mu_* + 1)F(u_k^\lambda) - f(u_k^\lambda)u_k^\lambda \,\mathrm{d}x \\
&\qquad\quad + \int_{\mathbb{R}^3}(\mu_* + 1)G(u_k^\lambda) - g(u_k^\lambda)u_k^\lambda \,\mathrm{d}x \\
&\qquad \geqslant \frac{\mu_* - 2}{2}\|u_k^\lambda\|_{V_0}^2, \quad \text{for } k \text{ large enough.}
\end{aligned}$$

For the case (ii), we take any $\mu_* = \mu$ in the previous estimate. ■

Proof of Theorem 4.2.4 completed. In view of the results of this chapter, the proof of (i) follows the same argument as the one used in the proof of Theorem 4.2.3. In fact, let $(w^{(n)})$, $(y_k^{(n)})$ and $(j_k^{(n)})$ be the sequences given by the Theorem 1.1.1 for the sequence (u_k).

- In view of Corollary 1.4.2, we have that $w^{(n)} = 0$ for all $n \in \mathbb{N}_0\backslash\{1\}$.

- $w^{(1)}$ is a critical point of I_λ.

- If $w^{(1)} = 0$ we use the same argument of the proof of Theorem 4.2.3: condition (4.2.2) leads to a contradiction with Proposition 4.3.11.

- We now use estimate (1.1.3), Propositions 1.4.4, 2.4.1, 3.4.2 and 4.3.4 to obtain the following estimate
$$c(J) \geqslant c(I_\lambda) \geqslant I_\lambda(w^{(1)}) + \sum_{n\in\mathbb{N}_+} J_\lambda(w^{(n)}),$$
where J is the following C^1 functional in $\mathcal{D}^{s,2}(\mathbb{R}^3)$
$$J(u) = \frac{1}{2}\int_{\mathbb{R}^3}|(-\Delta)^{s/2}u|^2 \,\mathrm{d}x - \int_{\mathbb{R}^3}G(u)\,\mathrm{d}x, \quad u \in \mathcal{D}^{s,2}(\mathbb{R}^3).$$

- $w^{(n)}$, $n \in \mathbb{N}_+$, is a critical point of J.

- We use condition (4.2.3) to get that
$$(\mu_* + 1)I_\lambda(w^{(1)}) = (\mu_* + 1)I_\lambda(w^{(1)}) - I'_\lambda(w^{(1)})\cdot(w^{(1)}) \geqslant 0,$$
and by a Pohozaev type identity $J(w^{(n)}) \geqslant 0$, $n \in \mathbb{N}_+$.

- If there exists $w^{(n_0)} \neq 0$, $n_0 \in \mathbb{N}_+$, then

$$c(J) = I_\lambda(w^{(1)}) + \sum_{n \in \mathbb{N}_+} J_\lambda(w^{(n)}), \tag{4.8.1}$$

where we used that $c(J) \leqslant J(w^{n_0})$. Identity (4.8.1) leads to the contradiction $c(J) > J_\lambda(w^{(n_0)})$.

- Convergence $u_k \to w^{(1)}$ in $H^s_{\mathrm{rad}}(\mathbb{R}^3)$ now follows by applying Propositions 3.4.1, 3.4.2 and 4.3.4.

- The case (ii) is proved by taking the profiles in Theorem 1.1.2 and using the fact (given by Corollary 1.4.2) that $w^{(n)} = 0$ for all $n \in \mathbb{N}_0 \backslash \{1\}$. In particular, the convergence $u_k \to w^{(1)}$ in $H^s_{\mathrm{rad}}(\mathbb{R}^3)$ follows. ■

4.9 Proof of Theorem 4.2.5

Proof. (i) Applying Proposition 4.3.12, we get

$$\int_{\mathbb{R}^3} |(-\Delta)^{s/2} u|^2 \,\mathrm{d}x + \frac{3}{3-2s} \int_{\mathbb{R}^3} a(x) u^2 \,\mathrm{d}x \\ + \frac{1}{3-2s} \int_{\mathbb{R}^3} \langle \nabla a(x), x \rangle u^2 \,\mathrm{d}x + \frac{3+2\alpha}{2(3-2s)} \lambda \mathcal{N}_\alpha(u) \leqslant \int_{\mathbb{R}^3} f(u) u \,\mathrm{d}x,$$

from which, using that $I'(u) \cdot u = 0$, we obtain

$$\int_{\mathbb{R}^3} (2sa(x) + \langle \nabla a(x), x \rangle) u^2 \,\mathrm{d}x + \frac{1}{2}(2\alpha + 4s - 3) \lambda \mathcal{N}_\alpha(u) \leqslant 0,$$

which leads to $u = 0$.

(ii) Using Proposition 4.3.12 again we get that

$$\frac{3-2s}{6} \delta \int_{\mathbb{R}^3} |(-\Delta)^{s/2} u|^2 \,\mathrm{d}x + \frac{\delta}{2} \int_{\mathbb{R}^3} a(x) u^2 \,\mathrm{d}x + \frac{\delta}{6} \int_{\mathbb{R}^3} \langle \nabla a(x), x \rangle u^2 \,\mathrm{d}x \geqslant \int_{\mathbb{R}^3} f(u) u \,\mathrm{d}x,$$

and we can derive that $u \equiv 0$, because

$$\left(1 - \frac{3-2s}{6} \delta\right) \int_{\mathbb{R}^3} |(-\Delta)^{s/2} u|^2 \,\mathrm{d}x + \left(1 - \frac{\delta}{2}\right) \int_{\mathbb{R}^3} a(x) u^2 \,\mathrm{d}x \\ - \frac{\delta}{6} \int_{\mathbb{R}^3} \langle \nabla a(x), x \rangle u^2 \,\mathrm{d}x + \left(1 - \frac{3+2\alpha}{12} \delta\right) \lambda \mathcal{N}_\alpha(u) \leqslant 0.$$

(iii) Applying identity (4.3.19), we obtain

$$\int_{\mathbb{R}^3} |(-\Delta)^{s/2} u|^2 \,\mathrm{d}x \\ + \frac{3}{3-2s} a_0 \int_{\mathbb{R}^3} u^2 \,\mathrm{d}x + \frac{3+2\alpha}{2(3-2s)} \lambda \mathcal{N}_\alpha(u) \leqslant \int_{\mathbb{R}^3} f(u) u \,\mathrm{d}x + \delta a_0 \int_{\mathbb{R}^3} u^2 \,\mathrm{d}x,$$

which leads to

$$(2s + \delta(2s-3))\, a_0 \int_{\mathbb{R}^3} u^2 \,\mathrm{d}x + \frac{1}{2}\,(2\alpha + 4s - 3)\,\lambda \mathcal{N}_\alpha(u) \leqslant 0,$$

that implies $u = 0$.

(iv) Since $\alpha = s$, we can choose $v = |u|$ as test function in definition (4.1.3) to get

$$K_0 \int_{\mathbb{R}^3} |u|^3 \,\mathrm{d}x \leqslant \frac{1}{4} \int_{\mathbb{R}^3} |(-\Delta)^{\alpha/2}(\phi_\alpha[u])|^2 \,\mathrm{d}x + \int_{\mathbb{R}^3} |(-\Delta)^{\alpha/2} u|^2 \,\mathrm{d}x, \tag{4.9.1}$$

where we used Cauchy inequality (with $\varepsilon = 1$). Now taking $v = \phi_\alpha[u]$ in definition (4.1.3) it follows that $\mathcal{N}_\alpha(u) = [\phi_\alpha[u]]_\alpha^2$. Moreover, using that $\lambda \geqslant 1/4$ in (4.9.1) we have

$$\lambda \mathcal{N}_\alpha(u) \geqslant K_0 \int_{\mathbb{R}^3} |u|^3 \,\mathrm{d}x - \int_{\mathbb{R}^3} |(-\Delta)^{\alpha/2} u|^2 \,\mathrm{d}x. \tag{4.9.2}$$

Using estimate (4.9.2) in the equation $I'(u) \cdot u = 0$ we obtain

$$\int_{\mathbb{R}^3} a_0 u^2 + K_0 |u|^3 - \mathcal{A}|u|^p \,\mathrm{d}x \leqslant 0,$$

which implies that $u \equiv 0$, since the function $t \mapsto V_0 t^2 + K_0 t^3 - \mathcal{A} t^p$, $t \geqslant 0$, is non-negative.

(v) Following the same above arguments we have

$$\int_{\mathbb{R}^3} |(-\Delta)^{s/2} u|^2 \,\mathrm{d}x + \frac{3 + 2\alpha}{2(3-2s)} \lambda \mathcal{N}_\alpha(u) \geqslant \frac{2_s^*}{p} \int_{\mathbb{R}^3} f(u) u \,\mathrm{d}x$$

which yields $u \equiv 0$, because

$$\left(\frac{2_s^*}{p} - 1\right) \int_{\mathbb{R}^3} |(-\Delta)^{s/2} u|^2 \,\mathrm{d}x + \left(\frac{2_s^*}{p} - \frac{3 + 2\alpha}{2(3-2s)}\right) \lambda K_0 \mathcal{N}_\alpha(u) \leqslant 0. \qquad \blacksquare$$

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