Sobolev spaces and embedding theorems

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Contents

1. Introductory remarks

In this initial part of the lecture an auxiliary material needed in the main body will be presented. The following notions will be discussed:

- Classes of domains with boundaries satisfying *cone condition, Lipschitz condition* or *of the class* C^k. Also an *extension property* allowing to restrict most proofs to the case of the whole of R^n will be discussed.
- Generalized (Sobolev, weak) derivatives. Their properties, comparison with distributional derivatives.
- Basic properties of L^p spaces and the space L^1_{loc}. Inequalities.
- Poincaré inequality and *interpolation inequality*.

The theory of Sobolev spaces has been originated by Russian mathematician S.L. Sobolev around 1938 [SO]. These spaces were not introduced for some theoretical purposes, but for the need of the theory of partial differential equations. They are closely connected with the theory of distributions, since elements of such spaces are special classes of distributions.

In order to discuss the theory of Sobolev spaces we shall start with some simple basic notions that are necessary for introducing and studying these spaces. The first object that we need to discuss is the domain in R^n and the possible classes of the domains that are considered in the theory of Sobolev spaces. This is important, since elements of such spaces are functions defined on the domains in R^n with, say, real values (or complex values).

1.1. **Domains.** By a *domain* in R^n we understand an open set in $n-$dimensional real Euclidean space R^n. There are several classes of domains, described in terms of the "smoothness" of their boundary $\partial\Omega$ considered in this theory. We will concentrate our description on the case of *bounded* domains Ω, when the definitions are simpler pointing e.g. to [AD] for the case of not necessarily bounded Ω.

The three classes of domains are most often considered;

- Domains $\Omega \subset R^n$ having the *cone property*.
- Domains having the *local Lipschitz property*.
- Domains having the $C^m - regularity\ property$.

We start with the description of the first property. Given a point $x \in R^n$ an open ball B_1 with center in x and an open ball B_2 not containing x the set $C_x = B_1 \cap \{x + \lambda(y - x) : y \in B_2, \lambda > 0\}$ is called a *finite cone* in R^n having vertex at x. By $x + C_0 = x + y : y \in C_0$ we will denote the finite cone with vertex in x obtained by parallel translation of a finite cone C_0 with vertex at 0.

Definition 1. *Ω has the cone property if there exists a finite cone C such that each point $x \in \Omega$ is the vertex of a finite cone C_x contained in Ω and congruent to C. (That means, C_x is obtained from C by a rigid motion).*

Definition 2. *Ω has a locally Lipschitz boundary, if each point x on the boundary of Ω has a neighbourhood U_x such that bdry $\Omega \cap U_x$ is the graph of a Lipschitz continuous function.*

Definition 3. *A bounded domain $\Omega \subset R^n$ is called a domain of class C^k, $0 \le k \le \infty$, provided that the following conditions are satisfied:*

- *There is a finite open cover of the boundary $\partial\Omega$,*

$$\partial\Omega \subset \sum_{r=1}^{N} V_r \tag{1}$$

such that the intersection $\partial\Omega \cap V_r$ can be described as: $x_n = g_r(x')$ where $g_r \in C^k(\bar{\Delta})$ and Δ is a cube $\{|x_j| < a, j = 1, \cdots, n-1\}$.

- *There is real $b > 0$, such that:*
$$\{x : g_r(x') - b < x_n < g_r(x'), x' \in \Delta\} \subset R^n \setminus \bar{\Omega} \tag{2}$$
also
$$\{x : g_r(x') < x_n < g_r(x') + b,\ x' \in \Delta\} \subset \Omega. \tag{3}$$

With the exception of the cone property all the other properties require Ω to lie on one side of its boundary. It can be shown, that the following connections between these conditions are valid:

$$\left\{ \begin{array}{l} C^m - \text{regularity \ property } (m \geq 1) \Rightarrow \\ \text{local Lipschitz \ property} \Rightarrow \\ \text{cone property.} \end{array} \right. \tag{4}$$

Most of the important properties of the Sobolev spaces require only the assumption that $\partial\Omega$ satisfies the cone condition. One more property of the domains is very important in the considerations. This is the, so called, *extension property*. This property will be described below.

We set the definition first:

Definition 4. *Let Ω be a domain in R^n. For given numbers m and p a linear operator E mapping $W^{m,p}(\Omega)$ into $W^{m,p}(R^n)$ is called an (m,p) -extension operator for Ω provided that:*

$$\left\{ \begin{array}{l} Eu(x) = u(x) \ a.e. \ in \ \Omega, \\ \forall_{u \in W^{m,p}(\Omega)} \exists_{K(m,p)} \ \|Eu\|_{W^{m,p}(R^n)} \leq K \|u\|_{W^{m,p}(\Omega)}. \end{array} \right. \tag{5}$$

The existence of an (m,p)-extension operator for a domain Ω guarantees that $W^{m,p}(\Omega)$ inherits many nice properties possessed by $W^{m,p}(R^n)$. If, for example, the embedding $W^{m,p}(R^n) \subset L^q(R^n)$ is known to hold, similar property will be true for the spaces over Ω. We will quote below a theorem justifying existence of such extension operator:

Theorem 1. *Let Ω be either a half-space in R^n or a bounded domain in R^n having the C^m-regularity property, then for any positive integer m there exists an extension operator E for Ω.*

The proof of this result can be found in [AD], p.84 and it uses the notion of the *partition of unity.*

1.2. **Generalized derivatives.** By a space $L^1_{loc}(\Omega)$ we understand the set of all Lebesgue measurable in Ω functions having absolute value integrable on each compact subset of the set Ω. By a *multi-index* α we understand a vector $(\alpha_1, \cdots, \alpha_n)$ having natural components α_i. We set $|\alpha| = \alpha_1 + \cdots + \alpha_n$. We also define the partial derivative

$$D^\alpha \phi = \frac{\partial^{|\alpha|} \phi}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}. \tag{6}$$

We are now able to define the notion of *weak derivative.*

Definition 5. *A locally integrable function v (element of $L^1_{loc}(\Omega)$) is called the α-th weak derivative of $u \in L^1_{loc}(\Omega)$, if it satisfies*

$$\int_\Omega \phi v dx = (-1)^{|\alpha|} \int_\Omega u D^\alpha \phi dx \ \textit{for all} \ \phi \in C_0^{|\alpha|}(\Omega). \tag{7}$$

Note that the weak derivative is uniquely determined up to the set of measure zero. We call a function *weakly differentiable* if all its partial derivatives of first order exist and *k times weakly differentiable* if all its weak derivatives exist for orders up to and including k. Denote the linear space of k times weakly differentiable functions by $W^k(\Omega)$. Clearly $C^k(\Omega) \subset W^k(\Omega)$. The concept of weak derivatives then extends the concept of the classical derivatives.

There is no time here to introduce the (more general) notion of the *distributional derivative*. The weak derivative is a special kind of the distributional derivative.

1.3. L^p **spaces.** Let Ω be a bounded domain in R^n. By a *measurable function* we shall mean an equivalent class of measurable functions on Ω which differ only on a subset of measure zero. The supremum and infimum of a measurable function will be understood as the essential supremum or essential infimum respectively.

Let $p \geq 1$, by $L^p(\Omega)$ we denote the Banach space consisting of all measurable functions on Ω thats $p-$powers are integrable. The norm in $L^p(\Omega)$ is defined by:

$$\|u\|_{L^p(\Omega)} = \left(\int_\Omega |u|^p dx\right)^{\frac{1}{p}}. \tag{8}$$

When $p = \infty$, we set:

$$\|u\|_{L^\infty(\Omega)} = esssup|u|. \tag{9}$$

There are a few elementary inequalities of the importance for the future that we shall quote now. The first is called *Young inequality*:

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}, \tag{10}$$

which holds for positive reals a, b, p, q that satisfy additionally

$$\frac{1}{p} + \frac{1}{q} = 1 \tag{11}$$

(the, so called, *condition of Hölder conjugacy*). With the same exponents as above the *Hölder inequality* is satisfied:

$$\int_\Omega uvdx \leq \|u\|_{L^p(\Omega)}\|v\|_{L^q(\Omega)}. \tag{12}$$

It is a consequence of the *Young's inequality*. As a consequence of the Hölder inequality the *interpolation inequality for L^p spaces* is satisfied:

$$\|u\|_{L^q(\Omega)} \leq \|u\|^\lambda_{L^p(\Omega)}\|u\|^{1-\lambda}_{L^r(\Omega)}, \tag{13}$$

valid for $u \in L^r(\Omega)$ with $p \leq q \leq r$ and $\frac{1}{q} = \frac{\lambda}{p} + \frac{(1-\lambda)}{r}$.

The further and more involved inequality we can face is the, so called, *Poincaré inequality*. Several versions of it are known, so we can quote only one of the best known among them.

Lemma 1. *Let Ω be a domain with continuous boundary. Let $1 \leq p < \infty$. Then for any $u \in W^{1,p}(\Omega)$:*

$$\int_\Omega \left|u(x) - \frac{1}{|\Omega|}\int_\Omega u(y)dy\right|^p dx \leq c\int_\Omega \sum_{i=1}^n \left|\frac{\partial u}{\partial x_i}\right|^p dx. \tag{14}$$

Proof. The proof is equivalent with showing that:

$$\int_\Omega |u(x)|^p dx \leq c\int_\Omega \sum_{i=1}^n \left|\frac{\partial u}{\partial x_i}\right|^p dx, \tag{15}$$

for the $W^{1,p}(\Omega)$ functions having zero main value.

Assume, at contrary, that there exists a sequence $\{u_k\} \subset W^{1,p}(\Omega)$, such that $\|u_k\|_{W^{1,p}(\Omega)} = 1$ and

$$1 = \|u_k\|^p_{W^{1,p}(\Omega)} \geq \int_\Omega |u_k|^p dx > k\int_\Omega \sum_{i=1}^n \left|\frac{\partial u_k}{\partial x_i}\right|^p dx. \tag{16}$$

Equivalently, this can be written as:

$$\int_\Omega \sum_{i=1}^n \left|\frac{\partial u_k}{\partial x_i}\right|^p dx < \frac{1}{k}. \tag{17}$$

Using the result of compactness of the embedding $W^{1,p}(\Omega) \subset L^p(\Omega)$ (that can be proved later), we are thus able to find a subsequence $\{u_{k_l}\}$ convergent in $L^p(\Omega)$ to u. Thanks to (17) we have also, $u_{k_l} \to u$ in $W^{1,p}(\Omega)$. But this same estimate shows also, that $\frac{\partial u}{\partial x_i} = 0$ a.e. in Ω. It can be shown, that $u = const.$ in Ω. Because of the zero mean (the property preserved from the sequence) u must be equal zero (a.e. in Ω). This contradicts the property $\|u\|_{W^{1,p}(\Omega)} = 1$. □

Similar estimate, even beter known, is valid for the functions equal zero on $\partial\Omega$, more precisely belonging to the space $W_0^{1,p}(\Omega)$. For more general versions of the Poincaé inequality consult [ZI], p.182.

Finally we present an elementary proof of a version of the *interpolation inequality*. We have:

Theorem 2. *Let $u \in W_0^{k,p}(\Omega)$. Then, for any $\varepsilon > 0, 0 < |\beta| < k$*

$$\|D^\beta u\|_{L^p(\Omega)} \leq \varepsilon \|u\|_{W^{k,p}(\Omega)} + C\varepsilon^{\frac{|\beta|}{|\beta|-k}} \|u\|_{L^p(\Omega)}, \tag{18}$$

where $C = C(k)$.

Proof. We establish (18) for the case $|\beta| = 1, k = 2$. The rest can be obtained through a suitable induction argument.

Assume at first that $u \in C_0^2(R)$ and consider an interval (a,b) of the length $b - a = \varepsilon$. For $x' \in (a, a + \frac{\varepsilon}{3}), x'' \in (b - \frac{\varepsilon}{3}, b)$, by the mean value theorem,

$$|u'(\bar{x})| = \left|\frac{u(x') - u(x'')}{x' - x''}\right| \leq \frac{3}{\varepsilon}(|u(x')| + |u(x'')|), \tag{19}$$

for some $\bar{x} \in (a,b)$. Consequently, for any $x \in (a,b)$, $u'(x) = u'(\bar{x}) + \int_{\bar{x}}^x u"(s)ds$, so that

$$|u'(x)| \leq \frac{3}{\varepsilon}(|u(x')| + |u(x'')|) + \int_a^b |u''|dx. \tag{20}$$

Integrating the above with respect to x' and x'' over the intervals $(a, a + \frac{\varepsilon}{3}), (b - \frac{\varepsilon}{3}, b)$ respectively, we get

$$\frac{\epsilon}{3}|u'(x)| \leq \frac{3}{\epsilon}\left(\int_a^{a+\frac{\epsilon}{3}} |u(x')|dx' + \frac{\epsilon}{3}|u(x")|\right) + \frac{\epsilon}{3}\int_a^b |u"|dx,$$

$$\begin{aligned}\left(\tfrac{\epsilon}{3}\right)^2 |u'(x)| &\leq \int_a^{a+\frac{\epsilon}{3}} |u(x')|dx' + \int_{b-\frac{\epsilon}{3}}^b |u(x")|dx" + \left(\tfrac{\epsilon}{3}\right)^2 \int_a^b |u"|dx \\ &\leq 2\int_a^b |u(x)|dx + \left(\tfrac{\epsilon}{3}\right)^2 \int_a^b |u"|dx\end{aligned}$$

giving

$$|u'(x)| \leq \int_a^b |u''|dx + \frac{18}{\varepsilon^2}\int_a^b |u|dx. \tag{21}$$

By Hölder inequality,

$$|u'(x)|^p \leq 2^{p-1}\left\{\varepsilon^{p-1}\int_a^b |u''|^p dx + \frac{(18)^p}{\varepsilon^{p+1}}\int_a^b |u|^p dx\right\}. \tag{22}$$

Integrating with respect to x over (a,b), we obtain:

$$\int_a^b |u'(x)|^p dx \le 2^{p-1}\left\{\varepsilon^p \int_a^b |u''|^p + \left(\frac{18}{\varepsilon}\right)^p \int_a^b |u|^p dx\right\}. \tag{23}$$

Dividing R into intervals of length ε, adding such inequalities, we obtain:

$$\int_R |u'|^p dx \le 2^{p-1}\left\{\varepsilon^p \int_R |u''|^p dx + \left(\frac{18}{\varepsilon}\right)^p \int_R |u|^p dx\right\} \tag{24}$$

Which is the desired result in the one-dimensional case. To extend to the higher dimensions fix $i, 1 \le i \le n$ and apply the above estimate to $u \in C_0^2(\Omega)$ regarded as a function of x_i only. Integrating successively over the remaining variables we obtain

$$\int_{R^n} |D_i u|^p dx_1 \cdots dx_n \le 2^{p-1}\left\{\varepsilon^p \int_{R^n} |D_{ii} u|^p dx_1 \cdots dx_n + \frac{C}{\varepsilon}^p \|u\|_{L^p(\Omega)}\right\}, \tag{25}$$

so that

$$\|D_i u\|_{L^p(\Omega)} \le \varepsilon \|D_{ii} u\|_{L^p(\Omega)} + \frac{C}{\varepsilon}\|u\|_{L^p(\Omega)}, \tag{26}$$

with $C = 36$. □

We call [AD], [G-T], [TA] for further versions of such theorems. We quote here a version formulated in [G-T], p. 173:

Theorem 3. *Let Ω be a C^2 domain in R^n and $u \in W^{k,p}(\Omega)$. Then, for any $\varepsilon > 0, 0 < |\beta| < k$,*

$$\|D^\beta u\|_{L^p(\Omega)} \le \varepsilon \|u\|_{W^{k,p}(\Omega)} + C\varepsilon^{\frac{|\beta|}{|\beta|-k}} \|u\|_{L^p(\Omega)}, \tag{27}$$

where $C = C(k,\Omega)$.

2. Sobolev spaces

The main body of the lecture will be given in this part, including:

- Definitions of Sobolev spaces and their basic properties.
- Dense subsets and approximation by smooth functions of elements of Sobolev spaces.

2.1. **Definition of the Sobolev spaces.**

Define a functional $\|u\|_{W^{m,p}(\Omega)}$, where m is a nonnegative integer and $1 \le p \le \infty$, as follows:

$$\begin{cases} \|u\|_{W^{m,p}(\Omega)} = \left\{\sum_{0\le|\alpha|\le m} \|D^\alpha u\|^p_{L^p(\Omega)}\right\}^{\frac{1}{p}}, \text{if } 1 \le p < \infty, \\ \|u\|_{W^{m,\infty}(\Omega)} = max_{0\le|\alpha|\le m}\|D^\alpha u\|_{L^\infty(\Omega)}. \end{cases} \tag{28}$$

We are now able to define the Sobolev spaces as:

$$W^{m,p}(\Omega) = \{u \in L^p(\Omega) : D^\alpha u \in L^p(\Omega),\ 0 \le |\alpha| \le m\}. \tag{29}$$

Further, we set,

$$W_0^{m,p}(\Omega) = \text{the closure of } C_0^\infty(\Omega) \text{ in } W^{m,p}(\Omega). \tag{30}$$

Equipped with the above defined norms, $W^{m,p}(\Omega)$ are called Sobolev spaces. Clearly also, $W^{0,p}(\Omega) = L^p(\Omega)$, and for $1 \le p < \infty$, $W_0^{0,p} = L^p(\Omega)$. Moreover, for any m,

$$W_0^{m,p}(\Omega) \subset W^{m,p}(\Omega) \subset L^p(\Omega). \tag{31}$$

The spaces $W^{m,p}(\Omega)$ has been introduced by S.L. Sobolev [SO], now there are many similar, but not necessarily coinciding spaces known in the literature.

Several important properties of the Sobolev spaces are most easily obtained by regarding $W^{m,p}(\Omega)$ as a closed subspace of the Cartesian product of the spaces $L^p(\Omega)$. Let $N = \sum_{0\leq|\alpha|\leq m} 1$ be the number of multi-indices α satisfying $0 \leq |\alpha| \leq m$. For $1 \leq p \leq m$, let $L^p_N = \prod_{j=1}^N L^p(\Omega)$, the norm of $u = (u_1, \cdots, u_N)$ in L^p_N being given by:

$$\|u\|_{L^p_N} = \left(\sum_{j=1}^N \|u_j\|^p_{L^p(\Omega)}\right)^{\frac{1}{p}}, \tag{32}$$

for $1 \leq p < \infty$ and, for $p = \infty$,

$$\|u\|_{L^\infty_N} = max_{1\leq j\leq N}\|u_j\|_{L^\infty(\Omega)}. \tag{33}$$

Known properties of the spaces $L^p(\Omega)$ allow, through this characterization, to show that the spaces $W^{m,p}(\Omega)$ are Banach spaces, they are *separable* when $1 \leq p < \infty$ also *reflexive* and *uniformly convex* when $1 < p < \infty$. When $p = 2$ then $W^{2,p}(\Omega)$ are Hilbert spaces equipped with the scalar product:

$$(u, v)_m = \sum_{0\leq|\alpha|\leq m} (D^\alpha u, D^\alpha v), \tag{34}$$

where $(u, v) = \int_\Omega u(x)\bar{v}(x)dx$ is the scalar product in $L^2(\Omega)$.

2.2. **Dense subsets and approximation in Sobolev spaces.** Typically dense subsets of the Sobolev spaces are constructed using the idea of approximation with *mollifiers*. Let ρ be a non-negative $C^\infty(R^n)$ function that vanishes outside a unit ball. We also assume that $\int \rho(x)dx = 1$. Such a function is called a *mollifier*. An example of the mollifier is given by:

$$\rho(x) = c\exp\left(\frac{1}{|x|^2 - 1}\right), \text{ when } |x| < 1, \tag{35}$$

extended with zero outside a unit ball. The constant c is choosen just to normalize the integral of ρ to one. For any element $u \in L^1_{loc}(R^n)$ and $h > 0$ the *regularization* of u is defined as:

$$u_h(x) = h^{-n}\int_\Omega \rho\left(\frac{x-y}{h}\right)u(y)dy, \tag{36}$$

when $h < dist(x, \partial\Omega)$. Thanks to the properties of the convolutions, $u_h \in C^\infty(\Omega')$ in any subdomain Ω' strictly included in the domain Ω. What we need to see now is that the functions u_h nicely approximate u as h tends to 0 (in various possible norms). We have:

Lemma 2. *For $u \in C^0(\Omega)$, the corresponding u_h converges to u uniformly on any subdomain Ω' strictly included in Ω.*

Proof. Changing the variable, we obtain:

$$u_h(x) = h^{-n}\int_{|x-y|\leq h} \rho\left(\frac{x-y}{h}\right)u(y)dy = \int_{|z|\leq 1} \rho(z)u(x - hz)dz. \tag{37}$$

Hence, for a strictly included subdomain $\Omega' \subset \Omega$ with $2h < dist(\Omega', \partial\Omega)$,

$$\begin{aligned} &sup_{\Omega'}|u - u_h| \leq sup_{x\in\Omega'}\int_{|z|<1} \rho(z)|u(x) - u(x - hz)|dz \\ &\leq sup_{x\in\Omega'}sup_{|z|\leq 1}|u(x) - u(x - hz)|. \end{aligned} \tag{38}$$

Now, the uniform continuity of u over the set

$$B_h(\Omega') = \{x : \mathrm{dist}(x, \Omega') < h\}, \tag{39}$$

shows the u_h tends to u uniformly over Ω'. □

A similar easy proof shows that when $u \in L^p(\Omega)$ then its mollifier u_h converges to u in $L^p(\Omega)$. For the proof see [G-T], p.148.

We can state now a generalizatin of this lemma concerning approximation in the $W^{m,p}(\Omega)$ sense (see [AD], Chapter III). We have:

Lemma 3. *Let $u \in W^{m,p}(\Omega)$ and $1 \le p < \infty$. Then, for any strict subdomain $\Omega' \subset \Omega$ holds; $u_h \to u$ in $W^{m,p}(\Omega')$.*

Proof. Define $\rho_h = h^{-n}\rho(\frac{x}{h})$. Let $\varepsilon < dist(\Omega', \partial\Omega)$ and $h < \varepsilon$. For any $\phi \in C_0^\infty(\Omega')$

$$\begin{aligned}\int_{\Omega'} \rho_h * u(x) D^\alpha \phi(x) dx &= \int_{R^n}\int_{R^n} \bar{u}(x-y)\rho_h(y) D^\alpha\phi(x) dxdy \\ &= (-1)^{|\alpha|}\int_{R^n}\int_{R^n} D_x^\alpha \bar{u}(x-y)\rho_h(y)\phi(x)dxdy \\ &= (-1)^{|\alpha|}\int_{\Omega'} \rho_h * D^\alpha u(x)\phi(x)dx,\end{aligned} \tag{40}$$

where $\bar{u}$ denotes the zero extension of u outside Ω. Therefore, $D^\alpha \rho_h * u = \rho_h * D^\alpha u$, in the sense of distributions in Ω'. Since $D^\alpha u \in L^p(\Omega)$ for $0 \le |\alpha| \le m$, by the previous lemma

$$lim_{h\to 0}\|D^\alpha \rho_h * u - D^\alpha u\|_{L^p(\Omega')} = lim_{h\to 0}\|\rho_h * D^\alpha u - D^\alpha u\|_{L^p(\Omega')} = 0. \tag{41}$$

Thus $lim_{h\to 0}\|\rho_h * u - u\|_{W^{m,p}(\Omega')} = 0.$ □

For more informations concerning dense subsets of Sobolev spaces consult [AD], Chapter III.

3. Embeddings of Sobolev spaces

The importance of Sobolev spaces lies in their connections with the spaces of continuous and uniformly continuous functions. This is expressed in *embedding theorems*:

- Continuous embeddings between Sobolev spaces and the spaces of continuous and Hölder continuous functions.
- Compact embeddings of Sobolev spaces. Rellich-Kondrachov theorem.

3.1. Continuous embeddings of Sobolev spaces.

We start this section with the proof of the, classical today, embedding result which is due to S.L. Sobolev. Recall first a generalization of the Hölder inequality which is due to E. Gagliardo (see [NE 1]).

Lemma 4. *Let $C = (-1,1)^n, C' = (-1,1)^{n-1}$ and let*

$$f_i(x') = f_i(x_1,, x_2, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n) \in L^{n-1}(C').$$

Put $f_i(x) = f_i(x')$ in C. Then:

$$\int_C \prod_{i=1}^n |f_i| dx \le \prod_{i=1}^n \left(\int_{C'} |f_i|^{n-1} dx'\right)^{\frac{1}{n-1}}. \tag{42}$$

We will not include the proof here.

We now have:

Theorem 4. *Let Ω be a Lipschitz domain. Let $1 \le p < n$ and $\frac{1}{q} = \frac{1}{p} - \frac{1}{n}$. Then $W^{1,p}(\Omega) \subset L^q(\Omega)$, i.e. the identity mapping from $W^{1,p}(\Omega)$ to $L^q(\Omega)$ is bounded.*

Proof. It can be shown, that suffices to prove the theorem when Ω is a cube. Consider first $p > 1$. Let $u \in W^{1,p}(C)$ and C be the cube from the previous lemma. There exists a sequence $\{u_k\} \subset C^1(\bar{C})$ convergent to u in $W^{1,p}(C)$ and $u_k(x', 1) = 0$. Consider a power:

$$|u_k(x)|^{\frac{np-p}{n-p}}.$$

This function is absolutely continuous on all the parallels to the axis and almost everywhere on such parallel:

$$\left|\frac{\partial}{\partial x_i}\left(|u_k|^{\frac{np-p}{n-p}}\right)\right| = \frac{np-p}{n-p}|u_k|^{\frac{np-n}{n-p}}\left|\frac{\partial u_k}{\partial x_i}\right|. \tag{43}$$

So, we have next:

$$\max_{-1\le x_i\le 1}|u_k(x)|^{\frac{np-p}{n-p}} \le \frac{np-n}{n-p}\int_0^1 |u_k|^{\frac{np-p}{n-p}}\left|\frac{\partial u_k}{\partial x_i}\right| dx_i. \tag{44}$$

Hence

$$\begin{aligned} &\int_{C'} \max_{-1\le x_i\le 1}|u_k(x)|^{\frac{np-p}{n-p}}\,dx' \\ &\le \frac{np-p}{n-p}\left(\int_C |u_k|^{\frac{np}{n-p}}dx\right)^{\frac{p-1}{p}}\left(\int_C \left|\frac{\partial u_k}{\partial x_i}\right|^p dx\right)^{\frac{1}{p}}. \end{aligned} \tag{45}$$

It thus follows from the previous lemma, that:

$$\begin{aligned} &\int_C |u_k(x)|^{\frac{np}{n-p}}dx \le \int_C \left(\prod_{i=1}^n \max_{-1\le x_i\le 1}|u_k(x)|^{\frac{p}{n-p}}\right)dx \\ &\le \prod_{i=1}^n\left(\int_{C'}\max_{-1\le x_i\le 1}|u_k(x)|^{\frac{np-p}{n-p}}dx'\right)^{\frac{1}{n-1}} \\ &\le \left(\frac{np-p}{n-p}\right)^{\frac{n}{n-1}}\left(\int_C |u_k|^{\frac{np}{n-p}}dx\right)^{\frac{np-n}{np-p}}\|u_k\|_{W^{1,p}(C)}^{\frac{n}{n-1}}, \end{aligned} \tag{46}$$

therefore

$$\left(\int_C |u_k(x)|^{\frac{np}{n-p}}dx\right)^{\frac{n-p}{np}} \le \frac{np-p}{n-p}\|u_k\|_{W^{1,p}(C)}, \tag{47}$$

and, as $k \to \infty$, we get the result for $p > 1$. If $p = 1$, it sufficies to pass with p to 1. The proof is completed. □

Consider a subspace $C^{k,\mu}(\bar{\Omega})$ of $C^k(\Omega)$, consisting of all such functions, whose k-th partial derivatives are μ-Hölder continuous. The norm in this space is introduced through the formula:

$$\|u\|_{C^{k,\mu}(\bar{\Omega})} = \|u\|_{C^k(\Omega)} + \sum_{|\alpha|=k} sup_{x\neq y}\frac{|D^\alpha u(x) - D^\alpha u(y)|}{|x-y|^\mu}. \tag{48}$$

The following continuous embedding of the Sobolev spaces is now valid:

Theorem 5. *Let Ω be a domain with Lipschitz boundary. Let $p > n$ and $\mu = 1 - \frac{n}{p}$. Then $W^{1,p}(\Omega) \subset C^{0,\mu}(\Omega)$.*

For the proof see e.g. [NE], p. 28.

Let's only quote a general theorem, the two above embeddings are only special cases of which.

Proposition 1. *If the domain Ω in R^n has the cone property, the following embeddings are continuous:*

$$W^{j+m,p}(\Omega) \subset W^{j,q}(\Omega), \text{ when } p \leq q \leq \frac{np}{n-mp}. \tag{49}$$

If Ω is a Lipschitz domain, then:

$$W^{j+m,p}(\Omega) \subset C^{j,\lambda}(\bar{\Omega}), \text{ for } 0 < \lambda \leq m - \frac{n}{p}. \tag{50}$$

For the proof of the above, see [AD], p. 97.

3.2. **Compact embeddings of Sobolev spaces.** In the applications of Sobolev spaces to partial differential equations it is often important to know whether the embeddings of Sobolev spaces are compact. We present below an example of such a result, quoting here also a different proof in [NE], p. 29. We start with two lemmas.

Lemma 5. *Let Q be a cube $\{0 \leq x_i \leq \sigma\} \subset R^n$. Let u be a real function in $C^1(Q)$. Then:*

$$\|u\|^2_{L^2(Q)} \leq \frac{1}{\sigma^n}\left(\int_Q u(x)dx\right)^2 + \frac{n}{2}\sigma^2\|u\|^2_{H^1(Q)}. \tag{51}$$

Proof. For any $x, y \in Q$,

$$\begin{aligned} u(x_1,\cdots,x_n) - u(y_1,\cdots,y_n) &= \int_{y_1}^{x_1} \frac{\partial}{\partial \xi_1} u(\xi_1, x_2, \cdots, x_n) d\xi_1 \\ + \int_{y_2}^{x_2} \frac{\partial}{\partial \xi_2} u(y_1, \xi_2, \cdots, x_n) d\xi_2 &+ \cdots + \int_{y_n}^{x_n} \frac{\partial}{\partial \xi_n} u(y_1, \cdots, y_{n-1}, \xi_n) d\xi_n. \end{aligned} \tag{52}$$

Taking squares and using the Schwarz inequality,

$$\begin{aligned} u^2(x) + u^2(y) - 2u(x)u(y) &\leq n\sigma \int_0^\sigma \left(\frac{\partial}{\partial \xi_i} u(\xi_1, x_2, \cdots, x_n)\right)^2 d\xi_1 \\ + n\sigma \int_0^\sigma \left(\frac{\partial}{\partial \xi_2} u(y_1, \xi_2, \cdots, x_n)\right)^2 d\xi_2 &+ \cdots + n\sigma \int_0^\sigma \left(\frac{\partial}{\partial \xi_n} u(y_1, \cdots, y_{n-1}, \xi_n)\right)^2 d\xi_n. \end{aligned} \tag{53}$$

Integrating with respect to $x_1, \cdots, x_n, y_1, \cdots, y_n$, we get:

$$2\sigma^n \int_Q u^2(x)dx - 2\left(\int_Q u(x)dx\right)^2 \leq n\sigma^{n+2} \sum_{i=1}^n \int_Q \left(\frac{\partial u(x)}{\partial x_i}\right)^2 dx, \tag{54}$$

implying (51). □

We have next the following *Friedrich's Inequality*:

Lemma 6. *For any $\varepsilon > 0$ there exists an integer $M > 0$ and a real-valued functions $w_1, \cdots, w_M$ in $L^2(\Omega)$, the domain Ω bounded, that $\|w_j\|_{L^2(\Omega)} = 1$, and for any real-valued function $u \in H_0^1(\Omega)$:*

$$\|u\|^2_{L^2(\Omega)} \leq \epsilon\|u\|^2_{H^1(\Omega)} + \sum_{j=1}^M (u, w_j)^2. \tag{55}$$

Proof. It suffices to prove the lemma for $u \in C_0^1(\Omega)$, the validity for all $u \in H_0^1(\Omega)$ then follow by completion. Extend u outside Ω with 0 and let Q be a cube containing $\bar{\Omega}$ having edges parallel to the coordinate axes. Let σ_0 be the length of each edge. Divide Q into cubes by introducing hyperplanes $x_i = y_{m_i}$, so that $y_{m+1,i} - y_{m,i} = \sigma$ for all $m, i; \sigma$ is such that $\frac{\sigma_0}{\sigma}$ is an integer. Denote by $Q_1, \cdots, Q_M$ ($M = \left(\frac{\sigma_0}{\sigma}\right)^2$) the cubes thus obtained.

By the previous lemma,

$$\left(\|u\|_{L^2(Q_j)}\right)^2 \le \frac{1}{\sigma^n}\left(\int_{Q_j} u dx\right)^2 + \frac{n}{2}\sigma^2\|u\|^2_{H^1(Q_j)}. \tag{56}$$

Summing over j and introducing the functions:

$$w_j = \begin{cases} \sigma^{-\frac{n}{2}} & \text{in } Q_j \\ 0 & \text{outside } Q_j \end{cases}$$

we get the result provided σ is such that $\frac{n\sigma^2}{2} \le \varepsilon$. □

We are now able to formulate and prove the famous *Rellich's Theorem*:

Theorem 6. *Let $\{u_m\}$ be a sequence of functions in $H^1_0(\Omega)$ (Ω bounded) such that $\|u_m\|_{H^1(\Omega)} \le const. < \infty$. There exists a subsequence $\{u_{m'}\}$ convergent in $L^2(\Omega)$.*

Proof. Let $h = 1, 2, \cdots$, and take for any $\varepsilon = \frac{1}{h}$ a finite sequence w_{jh} $(j = 1, \cdots, M(h))$ as in the Friedrich's Inequality. Let $\{u_{m_1}\}$ be a subsequence of $\{u_m\}$ such that $\{(u_{m_1}, w_{j_1})_{L^2}\}$ are convergent sequences for $j = 1, 2, \cdots, M(1)$. Similarly we define $\{u_{m,h+1}\}$, inductively, to be a subsequence of $\{u_{mh}\}$ such that $\{(u_{m,h+1}, w_{j,h+1})_{L^2}\}$ are convergent sequences for $j = 1, 2, \cdots, M(h+1)$. Let $\{u_{m'}\}$ be the diagonal sequence $\{u_{mm}\}$. Then $(u_{m'}, w_{jh})_{L^2}$ is convergent as $m' \to \infty$, for any h, j.

Given $\varepsilon = \frac{1}{h}$ choose m'_0 such that

$$\sum_{j=1}^{M(h)} (u_{m'} - u_{k'}, w_{jk})^2_{L^2} < \varepsilon \text{ if } m' \ge m'_0, k' \ge m'_0. \tag{57}$$

Then, by the Friedrich's Inequality,

$$\|u_{m'} - u_{k'}\|^2_{L^2} \le \varepsilon + \varepsilon\left(\|u_{m'}\|_{H^1} + \|u_{k'}\|_{H^1}\right)^2 \le (1 + 4K^2)\varepsilon, \tag{58}$$

where $K = sup\|u_m\|_{H^1}$. Since ε can be made arbitrarily small, the proof is completed. □

Following [AD] we will now quote the general result concerning compact embeddings of Sobolev spaces. We have:

Proposition 2. *Let Ω be a bounded domain in R^n having the cone property. Then the following embeddings are compact:*

$$W^{j+m,p}(\Omega) \subset W^{j,q}(\Omega) \text{ if } 0 < n - mp \text{ and } j + m - \frac{n}{p} \ge j - \frac{n}{q}, \tag{59}$$

also,

$$W^{j+m,p}(\Omega) \subset C^j(\bar{\Omega}) \text{ if } mp > n. \tag{60}$$

We send to [AD] for the proofs.

4. Applications of Sobolev spaces

Sobolev spaces were introduced mostly for the use of the theory of partial differential equations. Differential operators are often closable in such spaces. The advantage of using Sobolev spaces will be briefly discussed in this part of the lecture.

Of course, Sobolev spaces being examples of Banach or, sometimes, Hilbert spaces are interesting object for themselves. But their importance is connected with the fact that the theory of partial differential equations can be, and even most easily, developed just in such a spaces. The reason is because partial differential operators are very well situated in Sobolev spaces.

Simultaneously, the spaces of continuous (or of class C^k) functions is not very suitable for the studies of partial differential equations. Instead we need to consider Hölder continuous or Hölder continuous together with the derivatives; $C^{k,\mu}$ classes of functions. These are the other, competitive with the Sobolev spaces, classes of functions in which p.d.e. can be studied. Why spaces of (only) continuous functions are not very suitable? The answer is connected with the following observation ([NI], Chapt. 2.5).

Namely, for every $k \in N$ the Laplace operator

$$\Delta : C^{k+2} \to C^k,\ u = 0 \text{ on } \partial\Omega, \tag{61}$$

is continuous, but its image is not closed in $C^k(\bar{\Omega})$. In particular, for continuous right hand side $f \in C^0(\bar{\Omega})$ the solution of the equation:

$$\Delta u = f,\ u = 0 \text{ on } \partial\Omega, \tag{62}$$

in general must not be a $C^2(\bar{\Omega})$ function. Similar situation we face for other elliptic operators.

4.1. **Closedness of differential operators in Sobolev spaces.** Besides the property, that elliptic operators are closed (or, closable) in the Sobolev spaces, another important property of such spaces are their nice and reach connections with other function spaces expressed in the previously discussed embedding theorems. Working with solutions of partial differential equations lying in smooth Sobolev spaces (when the parameters m, p describing the space are large) it is very easy to check that these solutions are directly elements of the, say, C^k.

We shall describe below the problem of closedness of differential operators in Sobolev spaces more carefully. We start with the definition:

Definition 6. *Let T be a linear operator from a linear vector space X into a linear vector space Y, having a domain D_T. The graph G_T of T is the set of points (x, Tx) in $X \times Y$, where $x \in D_T$. If G_T is closed in the Cartesian product $X \times Y$, we say that T is a closed operator.*

We next have:

Definition 7. *Let T be a linear operator from a linear vector space X into a linear vector space Y, having domain D_T. A linear operator S from X into Y is called an extension of T if $D_T \subset D_S$ and $Tx = Sx$ for all $x \in D_T$.*

If, for a linear operator T there is an operator S which is closed, linear and extends T and, for any operator S' having these two properties S' needs to be an extension of S, then S is called the *closure* of T. It will be denoted by $\bar{T}$.

The following is a well known equivalent condition for T to have a closure:

Lemma 7. *Let T be a linear operator from a linear subspace D_T of a Banach space X into a Banach space Y. T has a closure $\bar{T}$ if and only if the following condition is satisfied:*

$$x_n \in D_T,\ x_n \to 0,\ Tx_n \to y \ \textit{imply}\ y = 0. \tag{63}$$

We will not prove it here.

Consider now a partial differential operator with constant coefficients a_α:

$$P(D) = \sum_{|\alpha| \le m} a_\alpha D^\alpha. \tag{64}$$

It can be considered as an operator from C^∞ into itself, or as an operator from $C^k(\Omega)$ into $C^{k-m}(\Omega)$, for any $m \le k$. But we prefer to define it as:

$$[P(D)u](x) = \sum_{|\alpha| \le m} a_\alpha D^\alpha u(x), \tag{65}$$

on the linear subspace $\hat{C}^m(\Omega)$ consisting of all the functions in $C^m(\Omega)$ that have a finite norm:

$$\|u\|_{W^{m,p}(\Omega)} = \left\{ \sum_{|\alpha|\le m} \int_\Omega |D^\alpha u|^p dx \right\}^{\frac{1}{p}}. \tag{66}$$

It is often desirable to extend $P(D)$ to a closed linear operator in $L^p(\Omega)$, $(1 \le p < \infty)$. We shall prove, this is possible.

Theorem 7. *The operator $P(D)$ from $L^p(\Omega)$ $p \in [1,\infty)$ into itself with a domain $\hat{C}^m(\Omega)$ has a closure.*

Proof. In view of the necessary condition above it suffices to show that if $u_k \in \hat{C}^m(\Omega)$, $\|u_k\|_{W^{0,p}(\Omega)} \to 0, \|P(D)u_k - v\|_{W^{0,p}(\Omega)} \to 0$, then $v = 0$. Let $\phi \in C_0^\infty(\Omega)$. Integration by parts gives:

$$\int_\Omega P(D)u_k \phi dx = \int_\Omega u_k P(-D)\phi dx. \tag{67}$$

As $k \to \infty$, the integrals on the right converge to zero whereas the integrals on the left converge to $\int_\Omega v\phi dx$. Therefore,

$$\int_\Omega v\phi dx = 0 \tag{68}$$

for all $\phi \in C_0^\infty(\Omega)$. Thus, by density, we conclude that $v = 0$. The proof is completed. □

This result show that the Sobolev spaces are natural for the studies of differential operators. One can also choose the spaces of Hölder continuous functions as has been done in [FR].

4.2. **The Lax-Milgram lemma.** Another advantage of the Sobolev spaces is connected with the fact that the study of elliptic p.d.e. becames very simple and elegant within these spaces. We quote here one such very known result. It justifies weak solvability of elliptic equations in Hilbert approach. The following abstract result known as the *Lax-Milgram lemma* is used in this proof.

Assume that Ω is a bounded domain in R^n. Then we have

Theorem 8. *Let $B[x,y]$ be a bilinear form (that is, linear in x and antilinear in y) in a Hilbert space H with norm $\| \ \|$ and scalar product $(\ , \ )$ and let $B[x,y]$ be bounded;*

$$|B[x,y]| \le const.\|x\|\|y\| \text{ for } x,y \in H. \tag{69}$$

Suppose further that

$$|B[x,x]| \ge c\|x\|^2 \text{ for all } x \in H \tag{70}$$

for some positive constant c. Then every bounded linear functional $F(x)$ in H can be represented in the form:

$$F(x) = B[x,v] = \overline{B[w,x]} \tag{71}$$

for some elements $v,w \in H$ that are uniquely determined by F.

Proof. For fixed v, $B[x,v]$ is a bounded linear functional. Hence there exists a unique y such that:

$$B[x,v] = (x,y). \tag{72}$$

Set $y = Av$. This A is a bounded linear operator in H. Since:

$$c\|v\|^2 \le |B[v,v]| \le |(v,x)| \le \|v\|\|y\|, \tag{73}$$

or $\|v\| \le \frac{\|y\|}{c}$;$A$ has a bounded inverse. Hence the range of A, $R(A)$ is a closed linear subspace of H. We claim that $R(A) = H$. Indeed, if $R(A) \ne H$ then there is an element $z \ne 0$ orthogonal to $R(A)$; $(z, Av) = 0$ for all $v \in H$. This implies that $B[z,v] = (z,Av) = 0$. Taking $v = z$, we get $B[z,z] = 0$. Hence, by (70) $z = 0$. A contradiction.

Consider now the functional $F(x)$. By the theorem of Riesz (in Hilbert spaces) there is an element b in H such that $F(x) = (x, b)$ for $x \in H$. Since $R(A) = H$, there is an element v such that $Av = b$. Hence

$$F(x) = (x, b) = (x, Av) = B[x, v], \tag{74}$$

and v is uniquely determined. Indeed, if $B[x, v'] = F(x)$ for some $v' \in H$ and for all $x \in H$, then $B[x, v - v'] = 0$ for all x. Taking $x = v - v'$ and using (70) we get $v = v'$.

The representation $F(x) = \overline{B[w, x]}$ follows by applying the previous result to the bilinear form $\overline{B[y, x]}$. □

Consider now a differential operator of order $2m$ in a bounded domain Ω

$$Lu = \sum_{|\alpha| \le 2m} a_\alpha(x) D^\alpha u. \tag{75}$$

having the coefficients a_α in $C^{|\alpha|-m}$ for $m < |\alpha| \le 2m$; then it can be rewritten in the *divergence form*:

$$Lu = \sum_{0 \le |\rho|, |\sigma| \le m} (-1)^{|\rho|} D^\rho (a^{\rho\sigma}(x) D^\sigma u). \tag{76}$$

The condition of *strong ellipticity* has now the form:

$$Re \left\{ \sum_{|\mu| = |\sigma| = m} \xi^\rho a^{\rho\sigma}(x) \xi^\sigma \right\} \ge c_0 |\xi|^{2m} \ (c_0 > 0), \tag{77}$$

for any real vector ξ. We associate with L the bilinear form:

$$B[u, v] = \sum_{0 \le |\rho|, |\sigma| \le m} (a^{\rho\sigma} D^\rho v, D^\sigma u). \tag{78}$$

Under the following assumptions:

- L is strongly elliptic in Ω with a module of strong ellipticity c_0 independent of x in Ω,
- The coefficients $a^{\rho\sigma}$ are bounded in Ω by a constant c_1,
- The principal coefficients of L have modulus of continuity $c_2(t)$;

$$|a^{\rho\sigma}(x) - a^{\rho\sigma}(y)| \le c_2(|x - y|) \text{ if } |\rho| = |\sigma| = m, \ x \in \Omega, y \in \Omega, \tag{79}$$

  and $c_2(t) \to 0$ if $t \to 0$,

we have the following theorem:

Theorem 9. *Let L satisfies the assumptions above and let the associated bilinear form $B[\phi, u]$ satisfies*

$$ReB[\phi, \phi] \ge c\|\phi\|^2_{H^m(\Omega)} \text{ for all } \phi \in H_0^m(\Omega), \tag{80}$$

with some $c > 0$. Then there exists a unique solution of the Dirichlet problem

$$\begin{cases} Lu = f \ \text{ in } \Omega, \\ u = 0 \ \text{ on } \partial\Omega. \end{cases} \tag{81}$$

Proof. Evidently the bilinear form $B[\phi, u]$ and the functional $F(\psi) = (\psi, f)$ satisfy the assumptions of the Lax-Milgram lemma with $H = H_0^m(\Omega)$. There exists thus a unique $u \in H_0^m$ that

$$(\psi, f) = B[\psi, u] \text{ for all } \psi \in H_0^m, \tag{82}$$

and this proves the theorem, since such a u is the generalized solution to the above Dirichlet problem. □

We leave here the discussion of the further possible applications of the Sobolev spaces in the theory of partial differential equations. Of course there are much more important points that can be discussed there.

There is a huge literature devoted to the theory of Sobolev spaces. Usually in any larger monograph devoted to partial differential equations some elements of this theory can be found. There are also some monographs devoted specially to Sobolev Spaces like [AD], [MR], [MA]. A short list of the references is presented in order. Of course, in presentation of the lecture only a small part of the material given in monographs below has been given.

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